2007

On the Calibration of the Libor Market Model

U Lagunzad DEMELINDA
Singapore Management University, demelinda.2005@smu.edu.sg

On the Calibration of the LIBOR Market Model

Demelinda U. Lagunzad

Under the supervision of Prof. Lim Kian-Guan

A Thesis submitted

for the degree of Master of Science in Finance (by Research)

Office of Research

Singapore Management University

TABLE OF CONTENTS

ACKNOWLEDGEMENT

A most sincere thank you to my supervisor, Prof. Lim Kian Guan, for his:

- *clever way of guiding a student's research work such that learning is not just superficial, but profound and rewarding;*
- *reasonable strictness coupled with nurturing compassion that can greatly motivate a student to work hard; and*
- *generosity that can rekindle ones hope amidst life's hard blows.*

Thanks again Professor Lim for being such a great mentor and professor!

I would like to express my deep gratitude to my home university, the University of the Philippines, for always supporting my endeavors to improve and build on my academic career.

Thanks to Dr. Cesar G. Saldana for the helpful pieces of advice.

My classmates – Christine, Rachel, Cindy, Fernando, Daniel, and Simon, a million thanks to all of you! You all made this one-year journey an exciting one. Perhaps, the best treasure one could always carry along is friendship.

Thanks to my parents and siblings for the prayers and support. Thanks for being such a great inspiration that keeps me going.

Most importantly, I thank the Lord Jesus for always keeping me in the palm of His gentle hand.

ABSTRACT

This thesis presents a study of LIBOR[1] market model calibration. In particular, the study builds on the prevailing calibration methodologies in an attempt to find a method that simultaneously recovers implied volatility and forward rate correlations structures from market prices of plain vanilla options. In order to ensure that complex derivative pricing and hedging requirements are jointly addressed, the study extends the performance analysis of calibration methods from a static level of goodness-of-fit with market prices test, to a dynamic level of approximation to next period's LIBOR dynamics when tested on a series of market prices.

Among the methodologies considered, the results show that for caplets, full calibration results in least pricing error when tested on an intra-day pricing prediction, and generates a stable evolution of day-to-day implied volatility. For swaptions, analytic approximation provides better estimate on an intra-day pricing but Monte Carlo simulation with parametrized correlations matrix provides a stable evolution of volatility and correlation (or covariance). This approach for swaptions calibration outperforms the other methods used despite the modifications made in volatility and initial thetas[2] specifications.

All together, the results suggest that the Monte Carlo method with parametrized correlations appear to be superior as it provides smooth evolution of covariance of forward rates that is desired in complex derivative pricing and hedging.

---

[1] LIBOR stands from London Interbank Offer Rate. LIBOR has become a standard term for the quoted interest rate at which a particular bank is willing to make a large wholesale deposit.

[2] Parameters for the correlations matrix based on Rebonato's volatility specification expressed in spherical coordinates.

1 INTRODUCTION

Perhaps a most interesting development in finance is the emergence of exotic options and structured products that paved the way for more advanced risk management techniques. Further, the growing liquidity of plain vanilla derivatives has elevated hedging techniques to a level that goes beyond the assumed linear relationship between an option and its underlying asset (delta hedging). These developments have spawned a two-fold challenge in the finance industry – that of pricing and hedging.

The pricing of exotic interest rate products hinges on interest rate process with parameters that are calibrated according to existing market prices of related interest rate products. This thesis improves on prevailing studies by extracting information on implied volatilities as well as implied correlations from frequently traded caps and swaptions. In addition, in order to ensure that pricing and hedging requirements are satisfied, we extend the performance analysis to a dynamic level by investigating the goodness-of-fit when tested on a one-month time series of market prices. We motivate this research as follows.

Among the financial markets, the interest rate market and currency market are the most dynamic in terms of exotic derivative trading; and among derivatives, pricing interest rate exotics remains the most challenging. This is due to the fact that interest rates underlie the concepts used in assets or future claims valuation. Hence, how interest rates will behave in the future remains a widely-researched topic in finance. Despite the extensively researched literature, the notion of interest rate dynamics modeling continuously evolves in response to the changing needs of the markets. Recently, the use of models by complex derivative traders for pricing and hedging has shifted the emphasis in modeling from one that simply accounts for the features of the underlying variable to one that effectively recovers the prices of plain vanilla options.

The challenge of complex derivatives pricing is brought about by the need of models to formulate arbitrage-free prices. Prevailing pricing practice among traders hinges on the notion of market efficiency such that inputs to models that are implied from liquidly traded options are preferred over statistically estimated parameters. In addition, emerging complex derivatives rely on the joint realization of underlying rates such that models that allow for changes in the shape of the underlying yield curve are preferred. Traders' pricing needs have heightened their interest in the ability of a model to accurately describe the behavior of the parameters as implied by market prices of standard options.

Aside from pricing, equally important are the practices of joint delta-gamma and vega hedging triggered by the increased volatility of plain vanilla options, specifically caps and swaptions in interest rate markets. Ease in trading these options has encouraged traders to include them in the set of hedging instruments. The challenge in hedging lies in the appropriate choice of model inputs such that today's market prices are recovered while tomorrow's model dynamics (i.e. volatility and correlation) will produce plain vanilla prices as close as possible to the market prices. This implies that the best model for hedging purposes does not only recover current prices but likewise approximates tomorrow's model's inputs. The widely-accepted Black-Scholes model in the market is inadequate since vega hedging involves neutralizing the sensitivity of an instrument to fluctuations in volatility that Black-Scholes model assumes to be constant.

Because of the joint industry practice of complex derivative pricing and hedging, traders have resorted to the so-called "relative approach" in creating arbitrage-free prices and correct hedging position. Relative pricing approach entails estimating the dynamics (i.e. volatility and correlation) of the underlying financial variables that influence the prices of an instrument. The challenge therefore is to choose the best model and employ a calibration methodology such that the objectives of pricing and hedging are jointly addressed.

Among the models on term structure of interest rates, the LIBOR market model is built within a framework appropriate for the above pricing and hedging issues. Being a Heath-Jarrow-Merton type of model, LMM is defined by the volatilities imposed on various rates. Further, LMM affords a methodology that is built around market observable variables and is consistent with the Black formula in pricing two standard interest rate options – caps and swaptions. Thus, after a rigorous derivation of the Black formula based on interest rate dynamics, LMM provides a model by which market information on the behavior of interest rate dynamics can be extracted. These features of LMM earned the model a "market model" title.

The first step when using the LMM is to appropriately choose the volatilities of the interest rates. Since LMM is formulated under a forward risk neutral world, the volatility as given by the model is a forward volatility. Hence, the volatilities as given by LMM define the evolution or dynamics of market volatilities of interest rates. In addition, an important characteristic of LMM is that it helps achieve decorrelation[3] among forward rates by finding a most effective way of redistributing the variance of forward rates over time (Brigo and Mercurio, 2001). LMM improves on the short rate models[4] which imply perfectly correlated forward rate dynamics. Thus, fundamental in using LMM is the determination of these two structures: volatility and correlation. This process is called calibration. These parameters allow traders to introduce changes in the shape of the yield curve that are useful in complex derivatives pricing and hedging.

With the benefits afforded by LMM, complex derivatives pricing and hedging problem simply boils down to choosing a calibration methodology such that the desired characteristics of the dynamics of interest rates are recovered. Several calibration methodologies have been proposed in an attempt to address the above issues. However, most of the studies are limited to model calibration to caplets and an exogenously defined correlations matrix. Further, prevailing methodologies were evaluated on their static performance as the fit is mainly tested on today's prices. For pricing and hedging purposes, especially for vega hedging, this is insufficient since future re-estimation would have to be done to ensure that vega hedge position, given today's volatilities, is correct. This thesis therefore attempts to address this gap in the literature of calibration.

---

[3] This means lowering the correlation of forward rates.
[4] Examples of short rate models are the Hull-White and Black-Karasinski models.

1.1 Background

The main challenge in the calibration of the LIBOR market model lies in the correct specification of the instantaneous volatilities of forward rates such that decorrelation among forward rates is achieved.

Several specifications from non-parametric to parametric approach have been formulated in an attempt to effectively capture the market observed dynamics as discussed in Brigo and Mercurio (2001). Although the non-parametric approach is preferred due to its high degree of freedom, the choice of volatility specification is greatly influenced by a trader's beliefs on how the market behaves as well as the prices of the standard interest rate options. Non-parametric approach also poses minimization problems as the number of unknown parameters becomes large and impossible to estimate given that the number of forward rates alive may not be sufficient.

Due to the dimensionality problem, Pedersen (1999) and Sidenius (2000) proposed the use of principal component analysis as rank-reduction technique. However, this technique implies that the rank of the covariance matrix must be less than the number of factors being used. In order to address this problem, Rebonato (1999) proposed an elegant parametrization using hypersphere decomposition before a reduced-rank minimization is performed. This technique, however, generates an infinite number of solutions. Correlation matrices can be arbitrary and highly depends on the constraints imposed on the angles that determine the correlation matrix entries.

Brigo and Morini (2004) proposed a calibration methodology via parametrization that establishes a one-to-one correspondence between the instantaneous covariance parameters and swaptions volatilities. Their proposed cascade calibration attempts to generate the general piecewise instantaneous volatilities structure that exactly recovers swaptions prices, but not caps prices. In addition, although the method eliminates the need for optimization and simulation processes, smoothing must first be applied on the swaptions matrix to ensure that resulting volatility estimates are all positive. Wu (2004) developed a calibration approach using Lagrange multipliers. However, the approach hinges on an exogenously determined correlation matrix. For pricing and hedging purposes, it is deemed more appropriate to use information on volatilities and correlation as implied by market prices of caps and swaptions. This is consistent with the fundamental notion on risk-neutral probability measure underlying pricing, in contrast to a physical probability measure-based exogenous correlation information. Further, existing methodologies in the literature were evaluated on their static performance as the fit is mainly tested on today's prices, which is deemed insufficient for hedging purposes.

This inadequacy in the literature of calibration is the primary motivation of this study. The study builds on the prevailing methodologies and improves on the evaluation criteria by extending the analysis to a dynamic level of goodness-of-fit by testing the method on a series of market prices of plain vanilla options. Results of the study will significantly contribute in modeling arbitrage-free prices and evaluating correct hedging positions.

1.2 Objectives of the Study

The primary objective of this thesis is to find a calibration methodology that simultaneously recovers volatility and correlations as implied by the market prices of plain vanilla derivatives. Under this umbrella objective are the following supporting goals:

i) Present a comprehensive review of the theory of the LIBOR market model (LMM).
ii) Identify alternative calibration methodologies.
iii) Apply the methodologies identified in (ii) to a set of market prices. In this study, South Korean caps and swaptions are used. This is a dynamic and interesting Asian market that is scarcely studied in the literature.

iv) Evaluate the performance of the identified calibration methodologies based on a set of criteria consistent with the joint practice of complex derivative pricing and hedging requirements.

1.3 Structure of the Thesis

The thesis proceeds as follows: Chapter 2 presents a comprehensive discussion of the theory of the LIBOR market model. It discusses the fundamental pricing principles that underlie the model. Chapter 3 contains the discussion of relevant calibration concepts and methodologies. Relevant theoretical results are presented to facilitate understanding of and show the appropriate approach toward the calibration proposed in this thesis. It should be stated outfront that the theoretical results in the form of theorems are not original but are reproduced form from existing literature. Chapter 4 analyzes the results of the empirical work. Chapter 5 concludes the thesis with a summary and recommendations for future research. In addition, it discusses the limitations of this study.

2 DISCUSSION ON THE THEORY OF LIBOR MARKET MODEL[5]

The LIBOR market model (LMM)[6], introduced and developed by Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Mutsiela and Rutkowski (1997), is a tool for pricing and hedging interest rate derivatives. The LMM models LIBOR forward rates and expresses the expected payoffs of the derivative products in terms of these rates under some LIBOR measure. The forward rates are assumed to follow a geometric Brownian motion. They modeled the forward rates such that the Black formula-based price of a European style option is recovered.

However, forward rates are not directly traded in the market. In order to achieve consistency with the no-arbitrage pricing theory, LMM used bonds as the underlying tradable assets and expressed forward rates in terms of bond prices. Thus, the dynamics of bond prices forms the foundation of LMM pricing. Since the primary objective of LMM is to recover exactly the prices of liquidly traded plain vanilla options as priced by the Black model, its framework recognizes the fact that certain interest-rate derivative products depend on the joint realization of a finite number of rates at pre-specified times. Hence, pricing under the LMM depends on the evolution of forward rates. Because the rates are interrelated, the model defines a class of no-arbitrage specifications of the yield curve dynamics that hinges on the covariance structure among the rates.

This feature of the LMM has spawned a variety of versions attempting to define the appropriate structure of the volatility of the forward rate. Brace, Gatarek, and Musiela (1995) and Musiela and Rutkowski (1997) developed the FRA-based LIBOR market model by assuming that the volatilities of the forward rates are deterministic over time. Jamishidian (1997) introduced the swap-rate version of the model that assumes deterministic volatility on a vector of forward swap rates.

[5] Theoretical discussions in this section are mainly based on the books of Brigo and Mercurio (2001) and Hull and White (2006).

[6] LMM is also commonly called the BGM, referring to the authors of the first paper which gave a rigorous discussion of the model.

These two versions of the model however are not consistent. While the first model affords exact recovery of the Black generated prices for caplets (but not for swaptions prices), the second version produces swaptions prices consistent with Black prices. This is due to the fact that forward rates and swap rates cannot be simultaneously lognormal. This study however will focus more on the first version of the LMM – also called the BGM model.

Because LMM was formulated with the aim to price plain vanilla interest rate options, the next sections will walk through the fundamental concepts that led to the formulation of the model.

2.1 Fundamental Interest Rates Concepts

Since LMM is about forward rates, it is imperative that one has a good grasp of the fundamental concepts of interest rates to ensure correct implementation of the model. Hence, in this section, we briefly define interest rate terms that are relevant in the next sections of this thesis.

Definition 1

The *n-year zero coupon* interest rate is the rate of interest earned on an investment that starts at time t and lasts for n years.

It is important to note that an investment placed on a zero-coupon bond has no intermediate or coupon payments, hence the term "zero-coupon". Zero rates can be extracted either from market prices of coupon-bearing bonds (for Treasury zero rates) or interest rate swaps (for LIBOR zero rates). It can also be found from STRIP prices.

Definition 2

Forward rates are interest rates implied by current zero rates for periods of time in the future. Forward rates are characterized by three time instants - t at which the rate is considered, T is the expiry of the agreement and S is the end of the interest accrual – where $t \le T \le S$.

Another way to define forward rates is to relate it to forward rate agreements (FRA). An FRA is an agreement that give the holder an interest payment between the expiry T and maturity S. The agreed rate, K, at time t is the fixed rate that must be exchanged against a floating payment at maturity date. In practice, K is quoted such that the value of the FRA is 0. Such K is referred to as a forward rate.

Definition 3

Let K be the fixed interest rate, τ_i, $i = \alpha + 1, ..., \beta$, be the time interval between the pre-specified future dates $T_{\alpha+1}, ..., T_\beta$, P be the notional amount, $L(T_{i-1}, T_i)$ be the observed LIBOR resetting at the previous instant T_{i-1} for the maturity given by the current payment instant T_i. Given T_α, an *interest rate swaps (IRS)* is a contract that exchanges payments between the fixed leg, $P\tau_i K$, and the floating leg, $P\tau_i L(T_{i-1}, T_i)$.

Given the above definition, the discounted payoff of an IRS for the counterparty receiving the fixed rate can be written as

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) P\tau_i (K - L(T_{i-1}, T_i)) \tag{1}$$

where $D(t, T_i)$ is the discount factor.

Definition 4

The rate K that makes the value of an IRS in Equation (1) zero is called the forward swap rate, $S_{\alpha,\beta}(t)$. If $L_j(t) = L(t; T_{j-1}, T_j)$ is the forward LIBOR at time t for the time instant between T_{j-1} and T_j, then $S_{\alpha,\beta}(t)$ can be written as

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1 + \tau_j L_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1 + \tau_j L_j(t)}} \tag{2}$$

These definitions will be used in the succeeding sections.

2.2 Theoretical framework for pricing derivatives[7]

LMM is built around the forward LIBOR rates such that payoffs of contingent claims are expressed in these rates. By a change-of-numeraire technique, LMM affords a no-arbitrage pricing technique that is consistent with prevailing term structure of interest rates. Fundamental in the development of LMM is the underlying economic assumption of arbitrage-free markets and pricing. Hence, in this section, we present a concise discussion of this underlying concept that led to the formulation of the LMM.

Consider a market $\boldsymbol{M}$ with $N+1$ assets traded continuously in a compact time interval $[0, T]$. The future prices of these assets are uncertain and assumed to follow a geometric Brownian motion on a probability space (Ω, F, P). Define a filtration $\mathbf{F} = \{F_t\}_{t\in[0,T]}$ as the augmentation of the natural filtration generated by the Brownian motion, i.e. $F(t)$ is the σ-field generated by $\sigma(W(s): 0 \le s \le t)$ and the null sets of F. For $i = 1,...,N$, let $B_i(\cdot)$ be the price of asset i. $B_i(\cdot)$ is assumed to be a positive Itô diffusion, i.e. $B_i(\cdot)$ is assumed to satisfy the following stochastic differential equation

$$\begin{aligned} \frac{dB_i(t)}{B_i(t)} &= \mu_i(t)dt + \beta_i(t)\cdot dW(t) & \\ &= \mu_i(t)dt + \sum_{j=1}^{d} \beta_{ij}\ dW_j(t) & ,\ 0 \le t \le T \qquad (3) \\ B_i(0) &= b_{0,i} & ,\ i = 1,...,N \end{aligned}$$

where $b_{0,i}$ is the price of asset i at time zero. The processes $\mu_i : [0, T] \times \Omega \to R$ and $\beta_i : [0, T] \times \Omega \to R^d$ may be stochastic and are assumed to be locally bounded and previsible. Further, assume that there are an infinite number of investors in the market and hold portfolios of the assets such that no individual investor can significantly affect market prices.

[7] Based on the discussion of Brigo and Mercurio (2001).

Definition 5

(i) A portfolio process π, $\pi : [0, T] \times \Omega \to R^N$ is any locally bounded F-previsible process.

(ii) The value process defined on a portfolio π is the process $V^\pi : [0, T] \times \Omega \to R$ such that

$$V^\pi(t) \equiv \sum_{i=1}^{N} \pi_i(t)\, B_i(t) \qquad , \quad 0 \le t \le T \tag{4}$$

(iii) A portfolio π is said to be self-financing if its value process V^π follows the process given by

$$dV^\pi(t) = \sum_{i=1}^{N} \pi_i(t)\, dB_t(t) \qquad , \quad 0 \le t \le T \tag{5}$$

(iv) A self-financing portfolio π is called admissible in the market M if the corresponding value process V^π is bounded from below almost surely (a.s.), i.e. there exists a real number $K < \infty$ such that

$$V^\pi(t) \ge -K \qquad \forall\, t \qquad , \quad 0 \le t \le T \quad \text{a.s.} \tag{6}$$

A portfolio π at any given time $t \in [0, T]$ holds $\pi_i(t)$ amount of security i, $i = 1, 2, \ldots, N$. An admissible self-financing portfolio π is tradable in the market $\boldsymbol{M}$ at the price $V^\pi(t)$ at time $t \in [0, T]$. Note that $\pi_i(t)$ is allowed to take negative values which amounts to short-selling asset i. Condition (iv) of Definition 1 excludes portfolios with doubling-up strategies, which make almost sure profits starting with zero value.

Definition 6

i) An arbitrage portfolio π is a self-financing portfolio that has zero value at time 0 and that has a non-negative value at time T a.s., with positive probability of the value being strictly positive at time T.

ii) A market $\boldsymbol{M}$ is said to be arbitrage-free if there does not exist an arbitrage portfolio in $\boldsymbol{M}$ at any given time $t \in [0, T]$.

iii) An equivalent martingale probability measure Q of market $\boldsymbol{M}$ is a probability measure on (Ω, F) equivalent to P, such that all assets are martingales under Q.

Theorem 1 (Absence of arbitrage)

If an equivalent martingale measure exists for the market $\boldsymbol{M}$, then $\boldsymbol{M}$ is arbitrage-free.

Proof:

Suppose that there exists π an admissible arbitrage portfolio in $\boldsymbol{M}$. Then by Definition 6.(iii), V^{π} is a martingale under Q. By the martingale property, it follows that

$$E^{Q}\left[V^{\pi}\left(T\right)\right]=V^{\pi}\left(0\right)=0 \tag{7}$$

But $V^{\pi}\left(T\right)\geq 0$ a.s. by Definition 6.(i). Hence, a contradiction. Thus, $\boldsymbol{M}$ is arbitrage-free. ■

In this study, we assume that markets are arbitrage-free and any portfolio is self-financing and admissible.

In asset pricing, prices of assets are expressed in values relative to the prices of a traded asset – referred to as a numeraire in finance literature. Detailed discussion of numeraires in relation to LMM is tackled in Section 2.4 of this chapter.

Suppose B is a numeraire, then any asset B_i, where $i=1,2,...,N$; $\left\{\frac{B_1}{B}, \frac{B_2}{B}, ..., \frac{B_N}{B}\right\}$ together with the probability space $\left(\Omega, F, P, \mathbf{F}\right)$ constitute a market $\boldsymbol{M}$ where the prices are expressed in units of the numeraire, B. Such a transformation of markets is referred to as *change of numeraire.*

Let $\mathbf{X}$ be a set of $F\left(t\right)$-measurable random variables on the probability space$\left(\Omega, F, P, \mathbf{F}\right)$. Denote by $X\in\mathbf{X}$ a contingent T-claim which pays out a random amount X at time T.

Definition 7

i) A portfolio π is said to hedge against a claim X if

$$V^{\pi}(T) = X \quad \text{a.s.} \tag{8}$$

Claim X is said to be attainable in the market $\boldsymbol{M}$.

ii) If for every $X \in \mathbf{X}$, X is attainable, then $\boldsymbol{M}$ is said to be complete with respect to $\mathbf{X}$.

iii) The price of a claim X is the smallest value x such that there exists a portfolio π that hedges against X and that $V^{\pi}(0) = x$.

iv) A hedging portfolio of the claim X is the minimal cost portfolio that hedges against a claim X.

It is easy to show then that if π is a hedging portfolio of a claim X at a price x, then $-\pi$ is a hedging portfolio of the claim $-X$ at a price $-x$. This implies that the price of a hedge is the same whether the position of an investor is short or long.

Theorem 2 (Completeness)

If there exists an equivalent martingale measure $\mathbf{Q}$ for the market $\boldsymbol{M}$ and if such a measure is unique, then every claim $X \in \mathbf{X}$ is attainable in the market $\boldsymbol{M}$.

The proof of the above theorem can be found in Karatzas and Shreve (1991) and uses the Brownian-martingale integral representation theorem.

Theorem 3

Suppose there exists an equivalent martingale measure $\mathbf{Q}$ for the market $\boldsymbol{M}$. Let $X \in \mathbf{X}$ be attainable in $\boldsymbol{M}$. Then the price of the claim X at time t, $0 \le t \le T$ is given by $E^{\mathbf{Q}}[X \mid F(t)]$. In particular, if $\tilde{\mathbf{Q}}$ is an equivalent martingale measure for a market $\tilde{\boldsymbol{M}}$ that is obtained from $\boldsymbol{M}$ under a change of numeraire B, then the price of the claim X at time t, $0 \le t \le T$, is given by

$$B(t)\, E^{\tilde{\mathbf{Q}}}\left[\frac{X}{B(T)} \mid F(t) \right]. \tag{9}$$

Proof:

Since the value process is a martingale under $\mathbf{Q}$, then

$$V^{\pi}(t) = E^{\mathbf{Q}}\left[V^{\pi}(T) \mid F(t)\right]$$

Using Definition 7.iii that the value of X at time t is equal to the value process of its hedging portfolio, then we have the following

$$V^{\pi}(t) = E^{\mathbf{Q}}\left[V^{\pi}(T) \mid F(t)\right] = E^{\mathbf{Q}}\left[X \mid F(t)\right] \qquad , 0 \le t \le T.$$

It follows by change of numeraire technique that

$$E^{\tilde{Q}}\left[\left.\frac{X}{B(T)}\right| F(t)\right] = \frac{X}{B(t)}$$

Hence, under the equivalent martingale measure $\tilde{Q}$ the price of the claim X is given by

$$X = B(t)\, E^{\tilde{\mathbf{Q}}}\left[\left.\frac{X}{B(T)}\right| F(t)\right] \qquad \blacksquare$$

2.3 The LIBOR Market Model

Given the above market framework, we now limit our discussion to the LIBOR market wherein LMM operates.

Consider an arbitrage-free LIBOR market $\boldsymbol{M}$ with $N+1$ zero-coupon bonds. Assume that the bonds are driven by a *d*-dimensional Weiner process with maturities $\{T_i\}_{i=1}^{N+1}$, $0 < T_1 < \ldots < T_{N+1}$ for each of the i^{th} bond. Let $T_0 \equiv 0$.

Define the maturity date T_N of the N^{th} zero-coupon bond as the horizon time T of the LMM. Denote by $B_i(\cdot)$ as the price process of the i^{th} bond such that

$$\frac{dB_i(t)}{B_i(t)} = \mu_i(t)\,dt + \beta_i(t)\,dW(t)$$

$$= \mu_i(t)dt + \sum_{j=1}^{i} \beta_{ij}(t)\,dW_j(t) \qquad , 0 \le t \le T_i$$

(10)

$$B_i(0) = b_{0,i}^{Market} \qquad , i = 1,2,\ldots, N+1$$

where $b_{0,i}^{Market}$ is the bond price observed in the market at time 0.

LMM is expressed in forward rates, which makes it easy for traders to work with. Since these forward rates are rates foreseen in future transactions, investors normally hedge positions from fluctuating rates by entering into forward rate agreements (FRAs). Normally, FRAs are entered into with the underlying assumption that the applicable borrowing or lending rate is the prevailing LIBOR.

A LIBOR FRA at T_i $(i = 1, 2, \ldots, N)$ is a contract to borrow or lend 1 unit of currency from time T_i until time T_{i+1} at a fixed rate r. The contract gives the holder an interest rate payment within the accrual period $\delta_i = T_{i+1} - T_i$, $i = 1, 2, \ldots, N$. The fixed rate agreed upon is called the forward LIBOR for lending/borrowing in the period from T_i to T_{i+1}. It is formally defined as the return, L_i, at time T_{i+1} of 1 unit of currency borrowed at T_i, where $L_i : \lfloor 0, T_i \rfloor \times \Omega \to R$ and

$$1 + \delta_i L_i = \frac{B_i(t)}{B_{i+1}(t)} \qquad , \qquad 0 \le t \le T_i \ , \ i = 1, 2,.., N . \qquad (11)$$

In LMM, one has to specify the instantaneous volatility of the forward LIBOR rates. Suppose $\sigma_i : [0, T_i] \times \Omega \to R^d$ are locally bounded previsible processes for $i = 1,2,.., N$, the bond price process is defined such that

$$\frac{dL_i(t)}{L_i(t)} = \ldots + \sigma_i(t) \cdot dW(t) \qquad , 0 \le t \le T \ , \ i = 1,2,\ldots,N \qquad (12)$$

This entails establishing a set of conditions on β_i , $i = 1, 2, ..., N+1$ such that the bond price process is defined as in Equation (10). Define a process $s_i : [0, T_i] \times \Omega \to R^d$ by $s_i(t) = L_i(t)\sigma_i(t)$ where $0 \le t \le T_i$, $j = 1, 2, ..., d$; $i = 1, 2, 3, ..., N$. Then, Equation (12) becomes

$$dL_i(t) = ... + s_i(t) \cdot dW(t) \qquad , 0 \le t \le T \ , \ i = 1,2,..., N \tag{13}$$

Taking the derivative of Equation (11) and applying to Equation (13), then Equation (13) can be written as

$$\begin{aligned} dL_i(t) &= \frac{1}{\delta_i} d\left(\frac{B_i(t)}{B_{i+1}(t)}\right) \\ &\overset{(*)}{=} \frac{1}{\delta_i} \frac{B_i(t)}{B_{i+1}(t)} \left(\left[\mu_i(t) - \mu_{i+1}(t) - (\beta_i(t) - \beta_{i+1}(t)) \cdot \beta_{i+1}(t)\right] dt + \left[\beta_i(t) - \beta_{i+1}(t)\right] \cdot dW(t) \right) \end{aligned} \tag{14}$$

where $0 \le t \le T$, $i = 1, 2, ...N$

(*) is by the results in stochastic differential of the quotient of two processes. Comparing Equations (13) and (14), it follows that the necessary condition for β_i is

$$\beta_i(t) - \beta_{i+1}(t) = \frac{\delta_i}{1 + \delta_i L_i(t)} s_i(t) \qquad , 0 \le t \le T \ , \ i = 1,2,..., N \tag{15}$$

At this point, we need to define a new index for the bond price.

Definition 8

For $t \in [0, T]$, define $i(t)$ as the index of the bond which expires at time t where

$$T_{i-1} < t < T_i .$$

We then can express Equation (15) as

$$\begin{aligned} \beta_i(t) - \beta_{i+1}(t) &= \sum_{j=i(t)}^{i} \left(\beta_j(t) - \beta_{j+1}(t)\right) \\ &= \sum_{j=i(t)}^{i} \frac{\delta_j}{1 + \delta_j L_j(t)} s_j(t) \quad , i = i(t), ..., N(t) \ , \ 0 \le t \le T \end{aligned} \tag{16}$$

where $\delta_j = t_{j+1} - t_j$.

Let $\beta : [0, T] \times \Omega \to R^d$ be any locally bounded F-previsible process, continuous on (T_i, T_{i+1}), $i = 1, 2, ..., N+1$. Then, it can easily be shown that if β_i, for every i, satisfies

$$\beta_i(t) = \begin{cases} \beta(t) - \sum_{j=i(t)}^{i-1} \dfrac{\delta_j}{1 + \delta_j L_j(t)} s_j(t) & , \quad 0 \le t \le T_{i-1} \\ \beta(t) & , \quad T_{i-1} < t \le T_i \end{cases} \tag{17}$$

then Equation (10) is satisfied. This ensures that the price process defined is self-financing.

2.4 No arbitrage assumption

Following is the no arbitrage assumption of the LMM for the drift term μ.

Assumption 1

Assume that there exists a locally bounded F-previsible process $\varphi^{MPR} : [0, T] \times \Omega \to R^d$ [8] such that

$$\mu_i(t) = \beta_i(t) \cdot \varphi^{MPR}(t) \tag{18}$$

for all $i = 1, 2, ..., N+1$, $0 \le t \le T_i$. This assumption about the existence of φ^{MPR} is used in the construction of an equivalent martingale measure for the LIBOR market model. Two measures developed under the LMM framework will be discussed in the next section. The existence of these equivalent martingale measures ensures absence of arbitrage in the forward LIBOR-based market. If the process φ^{MPR} is almost surely uniquely defined at all times, then the LIBOR market is said to be complete under Theorem 2.

[8] MPR stands for "market price of risk". It is measured as the quotient of expected rate of return over the amount of uncertainty.

Note that component j of the process $\varphi^{MPR}(t)$ refers to the market price of risk due to the uncertainty of the process W_j at time $t \in [0, T]$, $j = 1, 2, ..., d$. Thus, Equation (18) only implies that the market price of risk per factor at a particular point in time is the same for all bonds i, $i = 1, 2, ..., N+1$.

2.5 Measures and Numeraires

Definition 9

A *numeraire* is any non-dividend paying asset with price positive almost surely.

Alternatively, numeraire can also be defined as a reference asset that is chosen to normalize all other asset prices with respect to it. This means that if Z is the chosen numeraire, then the relative prices S^k/Z, $k = 1,2,..,N$ are considered instead of the original prices themselves. It was shown in the previous section that the resulting relative price is a martingale given a measure for the market price of risk.

Several asset prices can be chosen as numeraire that results in a more convenient calculation of contingent claims prices. Different numeraire has different resulting martingale measures. This section presents two types of numeraire as well as their martingale measures that work under the LMM. A thorough understanding of how LMM defines the dynamics of the forward LIBOR is important in pricing caplets and swaptions. Specifically, this plays a significant role in defining a discrete process for the dynamics of the forward LIBOR in Monte Carlo simulation.

2.5.1 Spot LIBOR measure

Under this measure, the spot LIBOR portfolio is assumed to consist of bonds with the following investment strategy:

(i) At $t = 0$, with an initial investment of \$1, buy $1/B_1(0)$ T_1 – bonds.

(ii) At $T_1 = 1$, reinvest the proceeds of $\$1/B_1(0)$ in $\frac{1}{B_1(0)}\Big/B_2(T_1)$ T_2 – bonds.

(iii) At $T_2 = 2$, reinvest again the proceeds of $\$\,1/B_1(0)\,B_2(0)$ in $[1/B_1(0)B_2(0)]/B_3(T_2)$ T_3- bonds.

(iv) And so on…

This type of numeraire is also referred to as a "rolling cd." Under this measure, between times T_i and T_{i+1}, the spot LIBOR portfolio holds an amount of $1/\Pi_{j=1}^{i+1} B_j(T_{j-1})$ of T_{i+1}-bonds. Hence, the value $B(t)$ at time t, $0 \le t \le T$, of the spot LIBOR portfolio is given by

$$B(t) = \frac{B_{i+1}(t)}{\Pi_{j=1}^{i+1}\, B_j(T_{j-1})} \qquad , T_i \le t \le T_{i+1} \tag{19}$$

Note that if we take the first order derivative of Equation (19), it follows that the spot LIBOR portfolio is self-financing. This implies that the stochastic differential equation of the spot LIBOR process can be written as follows

$$\frac{dB(t)}{B(t)} = \mu_B(t)dt + \beta_B(t)\cdot dW(t) \qquad , 0 \le t \le T \tag{20}$$

where $\mu_B(t)$ and $\beta_B(t)$ are linear combinations of $\mu_i(t)$ and $\beta_i(t)$, respectively; where $\mu_{B,}\,\mu_i : [0,T] \times \Omega \to R$; $\beta_B,\ \beta_i : [0, T] \times \Omega \to R^d$ and $i = 1, \ldots, N+1$. Since under the spot LIBOR measure quotients of price processes have to become martingales, then each component of the portfolio in Equation (19) must be a

$$\begin{aligned}\frac{dB_i(t)/B(t)}{B_i(t)/B(t)} &= [\mu_B(t) - \mu_i(t) - (\beta_B(t) - \beta_i(t))\cdot\beta_i(t)]\cdot dt \\ &\quad + [\beta_B(t) - \beta_i(t)]\cdot\, dW(t)\end{aligned} \tag{21}$$

for $0 \le t \le T$, $i = 1,\ 2,\ \ldots,\ N+1$.

Now, suppose we incorporate the no-arbitrage assumption in Assumption 1. Suppose there exists a process $\varphi^{Spot} : [0, T] \times \Omega \rightarrow R^d$ such that

$$\varphi^{Spot}(t) \equiv \varphi^{MPR}(t) - \beta_i(t) \quad , \qquad 0 \le t \le T \tag{22}$$

where φ^{MPR} is the process defined in Assumption 1.

Because the spot LIBOR portfolio is self-financing, then it follows that if V_1 and V_2 are LIBOR portfolios, the following can be obtained from Equation (22)

$$\begin{aligned}\mu_{V1}(t) - \mu_{V2}(t) &= [\beta_{V1}(t) - \beta_{V2}(t)] \cdot \varphi^{MPR}(t) \\ &= [\beta_{V1}(t) - \beta_{V2}(t)] \cdot [\varphi^{Spot}(t) + \beta_i(t)]\end{aligned}$$

Simplifying the above expresssion, we get

$$\mu_{V1}(t) - \mu_{V2}(t) - [\beta_{V1}(t) - \beta_{V2}(t)] \cdot \beta_i(t) = [\beta_{V1}(t) - \beta_{V2}(t)] \cdot \varphi^{Spot}(t) \tag{23}$$

Define a local martingale $M : [0, T] \times \Omega \rightarrow R$ by

$$M(t) \equiv \int_0^t \varphi^{Spot}(s) \cdot dW(s) \qquad , 0 \le t \le T \tag{24}$$

and define the process $W^{Q,Spot} : [0, T] \times \Omega \rightarrow R^d$ by

$$\begin{aligned}W^{Q,Spot} &\equiv W(t) + \langle W, M \rangle (t) \\ &= W(t) + \int_0^T \varphi^{Spot}(s)\, ds \quad , \qquad 0 \le t \le T\end{aligned} \tag{25}$$

which follows from Kunita-Watanabe results. Applying the results from Girsanov's theorem, it follows that $W^{Q,Spot}$ is a local martingale under the measure Q_{Spot} as determined by its Radon-Nikodym derivative as follows[9]

[9] See stochastic calculus books such as those by Karatzas and Shreve (1991).

$$\begin{aligned}\frac{dQ_{Spot}}{dP}(t) &\equiv e^{M(t)-\frac{1}{2}\langle M\rangle(t)} \\ &= e^{\int_0^t \varphi^{Spot}(s)\cdot dW(s)-\frac{1}{2}\int_0^t \left\|\varphi^{Spot}(s)\right\|^2 ds} \quad , \quad 0 \le t \le T\end{aligned} \tag{26}$$

[10]Note that $\int_0^t \varphi(s)\,ds$ is a finite variation process. Thus, $W^{Q,Spot}$ has quadratic variation structure similar to a Brownian motion. In addition, $W^{Q,Spot}$ is a local martingale under Q_{Spot}.[11]

Re-writing Equation (21) in terms of the Q_{Spot} − Brownian motion $W^{Q,Spot}$ and applying the result in Equation (23), we obtain

$$\begin{aligned}\frac{dB_i(t)/B(t)}{B_i(t)/B(t)} &= \left[\mu_B(t)-\mu_i(t)-(\beta_B(t)-\beta_i(t))\cdot\beta_i(t)\right]\cdot dt \\ &\quad +\left[\beta_B(t)-\beta_i(t)\right]\cdot\left[dW^{Q,Spot}(t)-\varphi^{Spot}(t)\cdot dt\right] \\ &= \left[\beta_B(t)-\beta_i(t)\right]\cdot dW^{Q,Spot}(t)\end{aligned} \tag{27}$$

for $0 \le t \le T$, $i=1, \ldots, N+1$. Equation (27) only shows that quotient of price processes are martingales under the measure Q_{Spot}, and we refer to this measure as the spot LIBOR.

If we substitute Equation (25) into Equation (14) and using the results in Equations (15) and (16), we derive the SDE for the LIBOR forward rates under the spot LIBOR measure as follows

[10] The norm $\|\ \|$ used in Equation 25 denotes the L^2 norm unless explicitly stated otherwise. P is the probability measure of the probability space we defined earlier for the geometric Brownian motion.

[11] See stochastic calculus books such as those by by Karatzas and Shreve (1991).

$$dL_i(t) = \frac{1+\delta_i L_i(t)}{\delta_i} \cdot \left[(\beta_i(t) - \beta_{i+1}(t)) \cdot (\beta_B(t) - \beta_{i+1}(t)) \cdot dt + (\beta_i(t) - \beta_{i+1}(t)) \cdot dW^{Q,Spot}(t) \right]$$

$$= \sum_{j=(B,i)}^{i} \frac{\delta_j s_j(t) \cdot s_i(t)}{1+\delta_j L_j(t)} dt + s_i(t) \cdot dW^{Spot,Q}(t) \qquad (28)$$

for $0 \le t \le T$, $i = 1,2,...,N$. Recall that $\sigma_i(\cdot) \equiv L_i(\cdot) s_i(\cdot)$. So we can re-write Equation (28) to obtain the following final result for the dynamics of the forward LIBOR

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=(B,i)}^{i} \frac{\delta_j L_j(t)\, \sigma_j(t) \cdot \sigma_i(t)}{1+\delta_j L_j(t)} dt + \sigma_i(t) \cdot dW^{Spot,Q}(t) \qquad (29)$$

for $0 \le t \le T$, $i = 1, 2, ..., N$.

2.5.2 Terminal LIBOR Measure

Under this measure, the numeraire is one of the bonds, say B_{n+1} for some $n \in \{1, 2, ..., N\}$. This portfolio with only one bond is automatically self-financing.

For an asset price process to be a martingale under this measure, it has to be expressed as a quotient over the bond price process. Specifically, B_n / B_{n+1} is a martingale. It then follows that an n^{th} LIBOR forward rate which can be expressed as an affine transformation of B_n / B_{n+1} is a martingale under this measure.

Following the same lines of argument as in the spot LIBOR measure, it can then be shown that under the terminal measure, the dynamics of the forward LIBOR is given by

$$\frac{dL_i(t)}{L_i(t)} = -\sum_{j=i+1}^{n} \frac{\delta_j L_j(t)\, \sigma_j(t) \cdot \sigma_i(t)}{1+\delta_j L_j(t)} dt + \sigma_i(t) \cdot dW^{Terminal}(t) \qquad (30)$$

for $0 \le t \le \min(T_i, T_{n+1})$, $i = 1, 2, ..., N$.

3 CALIBRATION

Calibration, in general, is the process of estimating the parameters of a model consistent with market information as implied by quoted prices of liquid instruments. In this study, we focus on the calibration of the LIBOR market model (LMM). Hence, throughout this thesis, "calibration" will always refer to LMM calibration.

In LMM calibration, recall from Equations (29) and (30) that the dynamics of forward LIBOR is defined in terms of two important parameters: volatility and covariance among rates. Normally, covariance is normalized and correlation is used instead to measure the same relationship between variables. Hence, the objective in LMM calibration is to estimate the parameters σ_i and ρ_{ij}, $i, j = 1,2,..., N$, such that model-derived prices match as close as possible to market-observed prices of liquidly traded instruments, specifically caplets and swaptions. The art of LMM calibration heavily relies on the specification of the volatility structure of LIBOR forward rates. This can be determined in several ways depending on some specific targets such as to recover the market prices of some (or a subset of) liquid standard options, to reflect traders' beliefs about the behavior of interest rate volatilities, or to match historical information. In pricing and hedging, however, the first target is deemed most relevant.

The liquid standard options used in the calibration are called calibration instruments. In interest rate markets, these are the caps and swaptions. Although historical forward rate correlation matrix is sometimes used, in this study, we instead extract implied correlations to be consistent with the forward risk neutrality assumption.

This section tackles the details of the calibration process. Specifically, this section shows why caps and swaptions are good calibration instruments and presents the details of the calibration methodologies used.

3.1 Calibration Instruments: Caps and Swaptions[12]

In the interest rate markets, caps and swaptions are the frequently traded standard options. This section presents the features of these derivative instruments.

3.1.1 Caps

Interest rate caps are popular over-the-counter instruments offered by financial institutions. It is designed to insure the holder from increases in interest rates above a certain level called the cap rate. It can be shown that its payoff has a call option-like feature. Hence, it is usually referred to as a call option on the LIBOR in the literature. Hence, just like other options, it is quoted in the market at Black implied volatilities.

A cap is usually tied with a floating rate note such that interest rate is periodically reset equal to the LIBOR. The time between resets is called the tenor and is usually equal to three months. At each reset date over the life of the cap, the observed LIBOR determines the amount of payments that must be made. In other words, if the life of the cap is T , principal is P , the cap rate is K and the reset dates are $t_1, t_2, ..., t_n$ such that $t_{n+1} = T$, the cap leads to a payoff at time t_{j+1} , $(j = 1, 2, ..., n)$

$$P\delta_j \max\left(L_j - K_j, 0\right) \tag{31}$$

where $\delta_j = t_{j+1} - t_j$ and L_j is the LIBOR for the period t_j and t_{j+1}. Notice that Equation (31) resembles a call option feature with the forward LIBOR as the underlying. Each of such call option is called a caplet. Because of its option-like feature, its price can be expressed under the Black formula, as follows

$$Caplet_j^{Black}(\sigma) = P\delta_j e^{-rt_{j+1}}\left(L_j N(d_1) - K_j N(d_2)\right) \tag{32}$$

where

[12] This thesis focuses on European-style exercise features of options.

$$d_1 = \frac{\ln\left(L_j / K_j\right) + \frac{1}{2}\sigma^2 t_j}{\sigma\sqrt{t_j}}$$

$$d_2 = \frac{\ln\left(L_j / K_j\right) - \frac{1}{2}\sigma^2 t_j}{\sigma\sqrt{t_j}} = d_1 - \sigma\sqrt{t_j}$$

where $N : R \to [0, 1]$ is the standard normal distribution function. The Black implied volatility of a caplet is the volatility with which the Black formula returns the market quoted price of the caplet price.

A cap can therefore be viewed as a portfolio of *n* such options with price given by

$$\begin{aligned} Cap &= \sum_{j=1}^{n} P\delta_j \; E^j\left[\max\left(L_j - K_j, 0\right)\right] \\ &= \sum_{j=1}^{n} P\delta_j e^{-rt_{j+1}}\left(L_j N(d_1) - K_j N(d_2)\right) \end{aligned} \qquad (33)$$

where d_1 and d_2 are as in Equation (32).

Moving on to the LIBOR market, LMM assumes that the LIBOR forward rates are lognormally distributed. Hence, it follows that the Black implied volatility for the j^{th} caplet is some average of the instantaneous volatility $\sigma_n(\cdot)$. This can easily be proven as shown in the next paragraphs.

To compute the price of a caplet, $Caplet_j^{LMM}\left(t_j, K_j\right)$ of the j^{th} caplet within the LMM, $j \in \{1, 2, ..., n\}$, the j^{th} terminal measure $Q_{t_{j+1}}$ will be used. Under this measure, the j^{th} LIBOR forward rate is a martingale,

$$\frac{dL_j(t)}{L_j(t)} = \sigma_j(t) \cdot d\mathbf{W}^{\mathbf{Q}_{T_{j+1}}}(t) \qquad , \quad 0 \le t \le T \qquad (34)$$

Hence, the LMM price of the j^{th} caplet is given by

$$Caplet_j^{LMM}\left(t_j, K\right) = P\delta_j B_{j+1}(0)\ E^{\mathbf{Q}_{t_{j+1}}}\left[\frac{\left(L_j\left(t_j\right) - K\right)^+}{B_{j+1}\left(t_{j+1}\right)}\right] \tag{35}$$
$$= P\delta_j B_{j+1}(0)\, E^{\mathbf{Q}_{t_{j+1}}}\left(L_j\left(t_j\right) - K\right)^+$$

since $B_{j+1}\left(t_{j+1}\right) = 1$. Application of integral calculus on the above expectation leads to the following pricing expressions for the j^{th} caplet

$$Caplet_j^{LMM}\left(t_j,\ K\right) = P\delta_j B_{j+1}(0)\left[L_j(0)\, N(d_1) - K\, N(d_2)\right] \tag{36}$$

where

$$d_1 = \frac{\log\left(\frac{L_j(0)}{K}\right) + \frac{1}{2}\tau^2}{\tau}$$

$$d_2 = \frac{\log\left(\frac{L_j(0)}{K}\right) - \frac{1}{2}\tau^2}{\tau}$$

and

$$\tau^2 \equiv \int_0^T \left\|\sigma_j(s)\right\|^2 ds$$

Notice that Equation (36), if compared with the Black formula for the caplet price in Equation (32), the following relation can be derived:

Theorem 4

The Black implied volatility of the j^{th} caplet, $j \in \{1, 2, ..., N\}$ under the LIBOR market model framework is given by

$$\sigma_j^{Black,LMM} = \sqrt{\frac{1}{t_j}\int_0^{t_j}\left\|\sigma_j(s)\right\|^2 ds} \tag{37}$$

Because of the ability of LMM to exactly match the market price of caplets, caplets therefore become a source of information on forward LIBOR volatility as implied by the market prices.

3.1.2 Swaptions

Swap options or swaptions are a popular type of option on interest rate swaps. This type of option gives the holder the right to enter into a certain swap agreement at a certain time in the future. A swap agreement is a contract between two parties to swap fixed for floating interest rate payments on some notional loan amount. Typically there are several exchanges of interest payments over the life of the loan, so a swap can be decomposed into swaplets. Each swaplet prescribes the swap of fixed for floating interest rate over each accrual period of the loan contract. Actual payments happen at the end of each accrual period. The rate of the fixed leg that makes the value of the swap agreement equal to zero is called the swap rate.

In order to understand the valuation of a swaption, we first review the mechanics of a swap agreement. Consider a swap agreement composed of a number of swaplets. Suppose that the first swaplet is set at time T_i with T_α a given date and first payment is on $T_{\alpha+1}$ and the last swaplet is set at time $T_{\beta-1}$ with payment on T_β, for some $\alpha < \beta$, $\alpha \in \{1,2,..., N\}$, $\beta \in \{1, 2,...., N+1\}$. Hence, a swap agreement consists of $\beta - \alpha$ swaplets. Denote by $S_{\alpha,\beta}$ the pre-negotiated fixed swap rate, which can be defined as follows

$$S_{\alpha,\beta} = \frac{B_\alpha(t) - B_\beta(t)}{\sum_{k=\alpha+1}^{\beta} \delta_k B_{k+1}(t)} \qquad , \; 0 \le t \le T_i \quad ^{13}$$

(38)

Now consider an option on this swap agreement. Suppose that the strike rate is K and the swaptions expiry is T_α. The cash flow from the swaptions at time T_k, $k = \alpha+1,...,\beta$ can be expressed in the following manner

$$P\delta_k \left(S_{\alpha,\beta}(T_i) - K \right)^+ \tag{39}$$

[13] This equivalent to the definition of a swap rate as shown in Equation (2) if we apply the value the spot LIBOR portfolio shown in Equation (19).

Applying Equation (2) that relates swap rates and forward rates, Equation (39) can be expressed as follows:

$$\left(\sum_{k=\alpha+1}^{\beta} P \delta_k \left(L_{\alpha,\beta}(T_i) - K \right) \right)^+ \tag{40}$$

Notice that the expression in Equation (40) cannot be decomposed additively, unlike the price formula for a cap in Equation (33). Equation (40) implies that the joint distribution of the forward LIBOR within the life of a swaption is necessary to compute for its price. Thus, correlation among forward LIBOR is fundamental in pricing swaptions.

Swaptions prices, similar to other options, are quoted in the market at implied volatilities. Under the Black framework, swap rates are assumed to be log-normally distributed with constant volatility. By Black formula, the above described swaptions with instantaneous volatility $\sigma(t)$, $0 \le t \le T$, $\sigma : [0, T_i] \to [0, \infty)$ is priced as follows

$$P \sum_{k=\alpha+1}^{\beta} \delta_k B_k(0) \left[S_{\alpha,\beta}(0) N(d_1) - K N(d_2) \right] \tag{41}$$

where

$$d_1 = \frac{\log\left(\frac{S_{\alpha,\beta}(0)}{K} \right) + \frac{1}{2} \int_0^{T_\alpha} \sigma^2(s) ds}{\sqrt{\int_0^{T_\alpha} \sigma^2(s) ds}}$$

$$d_2 = \frac{\log\left(\frac{S_{\alpha,\beta}(0)}{K} \right) - \frac{1}{2} \int_0^{T_\alpha} \sigma^2(s) ds}{\sqrt{\int_0^{T_\alpha} \sigma^2(s) ds}} = d_1 - \sqrt{\int_0^{T_\alpha} \sigma^2(s) ds}$$

In order to price swaptions under the LMM framework, swap rates volatility must be expressed in terms of forward rates volatility. The following Existing Result[14] shows the equivalence between the swap rates and forward rates volatilities:

Theorem 5

[14] We only show a sketch of the proof. A detailed proof can be found in Hull and White (2000).

Within the LMM, the swap rate $S_{\alpha,\beta}$ for $\alpha < \beta, \alpha \in \{1,2,..., N\}$, $\beta \in \{1, 2, ..., N+1\}$, satisfies the following stochastic differential equation

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} = ... + \sigma_{\alpha,\beta}(t) \cdot dW(t) \qquad , \; 0 \le t \le T_i$$

where $\sigma_{\alpha,\beta} : [0, T_i] \times \Omega \to R^d$ is defined by

$$\sigma_{\alpha,\beta}(t) \equiv \sum_{k=\alpha+1}^{\beta} \frac{\delta_k L_k(t) \gamma_k^{\alpha,\beta}(t)}{1+\delta_k L_k} \sigma_k(t) \tag{42}$$

and $\gamma_k^{\alpha,\beta} : [0, T_i] \times \Omega \to R$

$$\gamma_k^{\alpha,\beta}(t) \equiv \frac{\prod_{l=\alpha+1}^{\beta}(1+\delta_l L_l(t))}{\left(\prod_{l=\alpha+1}^{\beta}(1+\delta_l L_l(t))\right)-1} - \frac{\sum_{l=\alpha+1}^{k-1} \delta_l \prod_{m=l+1}^{\beta}(1+\delta_m L_m(t))}{\sum_{l=\alpha+1}^{\beta} \delta_l \prod_{m=\alpha+1}^{\beta}(1+\delta_m L_m(t))}$$

for t, $0 \le t \le T_i$, $k = i,..., j-1$.

Proof:

Recall from Equation (2) that the forward swap rate can be written as follows

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j L_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j L_j(t)}}$$

which can also be expressed as

$$S_{\alpha,\beta}(t) = \frac{\prod_{j=\alpha+1}^{\beta}[1+\tau_j L_j(t)] - 1}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i}[1+\tau_j L_j(t)]}.$$

Taking the natural logarithm of the above equation, we obtain

$$\ln S_{\alpha,\beta}(t) = \ln\left[\prod_{j=\alpha+1}^{\beta}[1+\tau_j L_j(t)] - 1\right] - \ln\left[\sum_{i=\alpha+1}^{\beta} \prod_{j=\alpha+1}^{i}[1+\tau_j L_j(t)]\right]$$

such that

$$\frac{1}{S_{\alpha,\beta}(t)} \frac{\partial S_{\alpha,\beta}(t)}{\partial L_k(t)} = \frac{\delta_k \gamma_k(t)}{1+\delta_k L_k(t)}$$

where

$$\gamma_k^{\alpha,\beta}(t) \equiv \frac{\prod_{j=\alpha+1}^{\beta}(1+\delta_l L_l(t))}{\prod_{j=\alpha+1}^{\beta}(1+\delta_l L_l(t))-1} - \frac{\sum_{i=\alpha+1}^{k-1} \delta_i \prod_{j=l+1}^{\beta}(1+\delta_j L_j(t))}{\sum_{i=\alpha+1}^{\beta} \delta_i \prod_{j=\alpha+1}^{i}(1+\delta_j L_j(t))}$$

Applying Ito's Lemma, the volatility, we therefore, obtain the volatility of the forward swap rate as follows

$$\sigma_{\alpha,\beta}(t) \equiv \sum_{k=\alpha+1}^{\beta} \frac{\delta_k L_k(t)\gamma_k^{\alpha,\beta}(t)}{1+\delta_k L_k} \sigma_k(t) \blacksquare$$

3.2 Some Specifications of Instantaneous Volatility[15]

The diffusion term $\sigma_i(t):[0, T_{n+1}] \rightarrow R^d$ in the dynamics of the forward LIBOR $L_i(t)$ is referred to as the instantaneous volatility of the forward LIBOR. If we take the set of unit vectors $\{e_1, e_2, ..., e_M\}$ spanning the R^d, then every volatility structure can be decomposed into

$$\sigma_i(t) = \gamma_i(t)\, e_i(t) \qquad , e_i \in R^d \;,\; i = 1,2,..., M \qquad (43)$$

where $\gamma_i : [0, T_{n+1}] \rightarrow R_+$.

Hence, because of the above decomposition, instantaneous volatility structure can be specified in several ways depending on the belief of the trader. This also enables separate calibration method to swaptions since only γ_i influences the prices of caplets while choice of e_i will determine the correlation structure.

---

[15] Based on the discussion in Brigo and Mercurio (2001).

Some of the possible specifications of the volatility decomposition are as follows:

i) Piecewise-constant instantaneous volatility

Under this assumption, the instantaneous volatility of $L_i\left(t\right)$ is constant in each "expiry-maturity" time interval t, $T_{i-1} < t \leq T_i$, i.e. This specification entails parameter estimates equivalent to the number of expiry-maturity time intervals defined or $M(M+1)/2$. This poses problems on estimation since normally the number of forward rates alive is less than the number of time intervals specified.

ii) Time-to-maturity dependent volatilities

An alternative specification of the volatility structure is to assume that the forward LIBOR diffusion depends only on the time-to-maturity. This formulation reduces the number of volatility parameters to M.

iii) Constant instantaneous volatility

This formulation assumes that the volatilities of the forward LIBOR are constant regardless of t.

iv) Separable Piecewise Constant

In this formulation, each instantaneous volatility is expressed as a product of two factors: a) time-to-maturity dependent factor and b) maturity-dependent factor.

Following Theorem 4, it is easy to show that for caplets, time in-homogenous volatility function exactly fits caplet prices under the LMM framework. This is evidenced by the humped volatility structure that is observed in the data.

Regardless of the volatility specification used, a result shown by Rebonato (1999) lends useful insight in the calibration process. He showed that expressing the instantaneous volatilities in spherical coordinates enables independent minimization scheme for volatility and correlation in the calibration process. This is discussed in detail in the next section.

3.3 Spherical Coordinates

Rebonato (1999) proposed a methodology that allows for independent calibration of volatility and correlation by expressing instantaneous volatilities in terms of spherical coordinates. Using hypersphere decomposition, his study shows that use of spherical coordinates in the specification of instantaneous volatility allows for a more robust minimization scheme. The calibration method reduces to a methodology similar to principal component analysis. Herein we discuss the results of Rebonato's study and show how to employ the method in a comparative calibration exercise to a dynamic setting.

Definition 10

Let $(\theta_1, \theta_2, \ldots, \theta_{d-1}) \in R^{d-1}$ be a vector of angles. Define a mapping $f : R^{d-1} \to S^{d-1}$ where $S^{d-1} = \{x \in R^d : \| x \| = 1\}$ and f is given by

$$f_j(\theta) \equiv \begin{cases} \cos(\theta_j) \prod_{k=1}^{j-1} \sin(\theta_k) & \text{if } j = 1,\ldots,d-1 \\ \prod_{k=1}^{d-1} \sin(\theta_k) & \text{if } j = d \end{cases} \tag{44}$$

Definition 11

Let $\Sigma_i : [0, T_i] \to [0, \infty)$ and $\theta_i : [0, T_i] \to R^{d-1}$ be functions. The instantaneous volatility structure $\sigma_i : [0, T_i] \to R^d$, $i = 1,2,\ldots, N$ is said to be written in spherical coordinates if

$$\sigma_{ij}(t) = \Sigma_i(t)\, f_j(\theta_i(t)) \quad , j = 1,\ldots,d \;;\; 0 \le t \le T_i \;, i = 1,2,\ldots, N \tag{45}$$

Σ_i is referred to as the total instantaneous volatility of the i^{th} forward.

Note that if we perform some simple manipulation of the above expression in Equation (45), we obtain the following relationships:

$$\begin{aligned} \Sigma_i^2(t) &= \sum_{j=1}^{d} \sigma_{ij}^2(t) \\ f_j(\theta_i(t)) &= \frac{\sigma_{ij}}{\Sigma_i(t)} \end{aligned} \tag{46}$$

where $j = 1,..,d$; $0 \le t \le T_i$; $i = 1,2,...,N$.

Recall the formula for caplet volatility shown in Equation (36). Notice that the expressions in Equations (45) and (46) imply that caplet prices depend only on $\Sigma(\cdot)$ while correlation among forward rates depend only on $\theta_{.}(\cdot)$.

These established relations between the spherical coordinates of instantaneous volatilities and the caplets volatilities and forward rate correlations afford a separate minimization strategy when volatility and correlation structures are calibrated. Hence, in the calibration process, separate fit to market prices can be performed to obtain information on instantaneous volatility and correlations. In particular, implied volatility of forward LIBOR is obtained by a fit to caps while implied correlations among rates are obtained by a fit to swaptions.

Another important consequence of the above is that it affords parametrization of the correlations structure of forward LIBOR under the LMM (Rebonato, 1999). Applying hypersphere decomposition, correlations calibration is reduced to specifying arbitrary thetas, $\theta_{.}(\cdot)$, such that the model correlation matrix matches the market implied correlations. This addresses the problem caused by insufficient forward LIBOR alive at each time step to establish the covariance relationship among rates since the number of factors or thetas can be exogenously specified. For instance, for a two-factor model, Rebonato (1999) proved that the correlation between rates i and j is given by

$$\rho_{ij} = \cos(\theta_{i1} - \theta_{j1}) \tag{47}$$

At this point, it is important to emphasize that the focus in the calibration process is to estimate the instantaneous correlation matrix. Instantaneous correlation matrix summarizes the degree of dependence between the changes of forward rates as seen at instant time, say t. Information on instantaneous correlation may be obtained from historical data or from prices of swap options. For pricing and hedging, the latter is deemed more appropriate and consistent with the forward risk neutrality assumption underlying LMM.

3.4 Term Structure of Volatilities and Terminal Correlations

As mentioned in the earlier part of this study, a significant contribution of LMM is that it allows one to extract information on the dynamics of interest rates once calibrated to the prices of liquid plain vanilla options, i.e. caps and swaptions. In particular, once instantaneous volatilities and correlations are obtained from the prices of caps and swaptions, one can then plot the term structure of volatilities, and define the correlation of forward rates at future times, or the so-called "terminal correlations".

The term structure of volatility at time t_j is the graph of points

$$\left\{\left(t_{j+1}, V\left(t_{j,} t_{j+1}\right)\right), \left(t_{j+2}, V\left(t_j, t_{j+2}\right), ..., \left(t_{M-1}, V\left(t_j, t_{M-1}\right)\right)\right)\right\}$$

where

$$V^2\left(t_j, t_{h-1}\right) = \frac{1}{t_{h-1} - t_j} \int_{t_j}^{t_{h-1}} \sigma_h^2\left(t_j\right) dt_j \tag{48}$$

for $h > j+1$. Note that different specifications of σ_h imply different evolutions of the term structure of volatilities. Since this evolution is deterministic in LMM, it is generally perceived as smooth and qualitatively stable (Brigo, 2001).

Similarly, once LMM is calibrated, correlation of forward rates at future time instants can be analytically computed as follows

$$Corr\left(F_i\left(t_\alpha\right), F_j\left(t_\alpha\right)\right) = \frac{\exp\left(\int_0^{t_\alpha} \sigma_i\left(t\right) \cdot \sigma_j\left(t\right) \cdot \rho_{ij}\ dt\right) - 1}{\sqrt{\exp\left(\int_0^{t_\alpha} \sigma_i^2\left(t\right) dt\right) - 1} \cdot \sqrt{\exp\left(\int_0^{t_\alpha} \sigma_j^2\left(t\right) dt\right) - 1}} \tag{49}$$

where $t_\alpha > t$ (Brigo 2001, Rebonato 1999). An alternative way is to perform Monte Carlo simulations to evolve forward rates F_i and F_j as implied by a calibrated LMM at time t under a forward measure Q^γ as follows

$$Corr^{\gamma}\left(F_i(t_\alpha), F_j(t_\alpha)\right) = \frac{E^{\gamma}\left[\left(F_i(t_\alpha) - E^{\gamma}(F_i(t_\alpha))\right)\cdot\left(F_j(t_\alpha) - E\gamma\left(F_j(t_\alpha)\right)\right)\right]}{\sqrt{E^{\gamma}\left[F_i(t_\alpha) - E^{\gamma}(F_i(t_\alpha))\right)^2\right]}\cdot\sqrt{E^{\gamma}\left[F_j(t_\alpha) - E^{\gamma}\left(F_j(t_\alpha)\right)^2\right]}} \quad (50)$$

However, traders prefer to use the analytic formula rather than perform Monte Carlo simulation due to time constraints. Notice that terminal correlation between forward rates is influenced not only by the instantaneous correlation but the by the volatility specification as well.

Terminal correlation plays a significant role in complex derivative pricing and hedging. It is a mechanism that allows for a stochastic evolution of the underlying forward rates (Rebonato, 2004).

3.5 Calibration Methodologies

As mentioned in the preceding section, LMM calibration involves estimation of volatility and correlations parameters such that market prices of caps and swaptions are recovered. This section discusses the details of the methodologies employed in this study.

3.5.1 Preliminary Steps

In the calibration process, preliminary steps have to be done since some necessary information are not readily available in the market. The following summarizes the initial steps before actual calibration is performed:

i) Bootstrapping of the IRS to extend the LIBOR zero curve.
ii) Computation of forward LIBOR rates.

LIBOR rates are typically quoted only for maturities out to 12 months. Traders normally use swap rates to extend the LIBOR zero curve. This hinges on the notion that newly issued swaps are quoted at prevailing swap rates such that the value of a fixed-rate bond underlying the swap equals the value of a floating-rate bond underlying the same swap. Thus, applying the bootstrap method, the LIBOR zero curve is constructed out to ten years using the quoted interest rate swaps (IRS)[16].

The LIBOR zero-curve is then used to compute the forward rates and to discount payoffs in options valuation. Although for the latter purpose risk-free rates or Treasury bills rates are more appropriate, in this study we follow the usual practice in the industry. Traders prefer to use the LIBOR zero rates over Treasury rates as discount rates for some reasons. Among those reasons is that the increased demand for Treasury bills and bonds that drives prices up and yield down is highly motivated by regulatory requirements that must be complied with by some institutions (Hull, 2006).

In this study, in order to achieve consistency with the day count convention in the South Korean interest markets, we express the bootstrapped LIBOR zero curves in quarterly compounding. For discounting purposes, we express the LIBOR zero rates in continuous compounding consistent with the Black assumption.

Once LIBOR zero curve is determined, we compute the forward rates for each LIBOR rate and the forward swap rates using Equation (2) defined in Chapter 2. Forward swap rates are necessary for swaption valuation and calibration to swaption prices.

After all the preliminary steps are done, actual calibration process can be performed. The details of the methodologies are discussed in the subsequent sections.

3.5.2 Volatility Calibration

[16] Linear interpolation is used to determine the rates at intervening periods (quarterly periods) since IRS are quoted at annual rates.

As discussed in Section 3.3, when written in spherical coordinates, caplets depend entirely on the total instantaneous volatility of the i^{th} forward. Hence, to back out market information about volatility of forward rates, caplets are a good source of information. Further, as shown in Section 3.1.1, caplets, which are quoted at Black prices in the market, can be exactly recovered by the LMM. Hence, calibration to caplets is automatic. One can simply input the caplet implied volatilities in the LMM.

However, caplets prices are not directly observable in the market. Instead, markets adopt the convention of quoting the cap implied volatility and assumes flat volatility among the caplets forming the cap. Hence, given the strike and the expiry date, the associated implied volatility is the single number that must be plugged into the Black formula for all the corresponding caplets such that the cap price is simply the sum of the resulting caplet prices. Thus, caplets must be bootstrapped from the quoted caps implied volatilities. In this study, we follow the piecewise linear method described below.

1. Given the caps quoted implied volatilities $\sigma_T^{Market}(t)$, where $T = 1,2,...,N$ maturities, perform linear interpolation to get the flat volatility for a cap that matures on the caplet quarterly maturities, i.e. if $\tau = 0.25$, for every $T = 1,2,..., N-1$, interpolate $\sigma_{T+b\tau}^{Market}$, $b = 1,2,3$. For caps with maturities less than one year, use the one-year cap implied volatility.

2. Using the interpolated flat volatilities, compute the fair value of the cap using the Black formula shown in Equation (35). Note that quoted caps are at-the-money, hence, the strike rate is the prevailing swap rate with the same maturity as the caps.

3. Compute the fair value of the intervening caplet prices by taking the difference of the adjacent cap prices.

$$Caplet_T^{Black} = Cap_{T+b_{i+1}\tau}^{Black} - Cap_{T+b_i\tau}^{Black} \quad b = 1, 2, 3 \ , \ \tau = 0.25 \ , T = 0,1,..., N-1 \quad (51)$$

4. Once the set of caplet prices has been computed using Equation (36), compute the implied instantaneous volatilities using the Black formula in Equation (37).

3.5.3 Correlations Calibration

Correlations calibration is more delicate compared with volatilities calibration due to insufficient forward rates alive and the dimensionality problem.

Calibration of the LMM to swaptions is analogous to modeling correlations among forward rates. Thus, one must ensure that the following properties of the correlation matrix are preserved:

i) Symmetry: $\rho_{ij} = \rho_{ji} \quad , \ \forall i, j$

ii) Positive semi-definiteness: $x'\rho \ x \geq 0 \ , \ \forall \ x \in R^M$

iii) Unitary diagonal: $\rho_{ii} = 1 \quad , \ \forall \ i$

iv) Normalized entries: $\left|\rho_{ij}\right| \leq 1 \quad , \ \forall \ i, \ j$

In addition to the above properties, forward rate correlations bear additional properties based on both intuition and empirical observation. The first desired characteristic is termed in the literature as decorrelation. Decorrelation means that the correlation decreases as the distance between maturities increases. This means that column entries decrease when moving away from the main diagonal. This feature is typically observed in the interest rate market. Another important feature is the increase in interdependency between equally spaced forward rates as their maturities increase. This means that the sub-diagonals of the correlations matrix are increasing when moving south-eastwards. This feature can be justified with the well-observed behavior of the zero curve dynamics that “flattens” for large maturities. (Brigo and Morini, 2003)

In this section, we discuss in detail how we extract implied correlations from swaptions and caplets.

Method A: Joint Calibration using Rebonato's Approximation

This method hinges on the analytical approximation to Black swaptions implied volatilities derived by Rebonato (Rebonato, 2004).

The objective in this methodology is to find the best-fitting parameters ψ_i and θ_i with initial guesses of $\psi_i = 1$ and $\theta_i = \pi/2$, $\alpha = i,...,N$, $\beta = i+1,...,N+1$, $i, j = 1,...,N$ such that

$$\underset{\psi_i}{Min} \sum \left(\sigma_i^{Market} - \sigma_i^{LMM} \left(t \right) \right)^2 \tag{52}$$

and
$$\underset{\theta_i}{Min} \left(\frac{\left| \sigma_{\alpha,\beta}^{Market} - \sigma_{\alpha,\beta}^{LMM} \right|}{\sigma_{\alpha,\beta}^{Market}} \times 100 \right) \tag{53}$$

where

$$\sigma_i^{\mathrm{LMM}}\left(t \right) \equiv \phi_i \ \psi_{i-(j(t)-1)} \tag{54}$$

$$\phi_i^2 = \frac{\left(Caplet_i^{Market} \right)^2}{\sum_{k=1}^{i} \tau_{k-2,k-1} \psi_{i-k+1}^2} \tag{55}$$

$$\left(\sigma_{\alpha i,\beta}^{LMM} \right)^2 = \sum_{i,j=\alpha+1}^{\beta} \psi_{i-j+1}^2 \rho_{ij} \int_0^{T_\alpha} \sigma_i\left(t \right) \sigma_j\left(t \right) dt$$
(56)

$$\rho_{ij} = \cos\left(\theta_i - \theta_j \right) \tag{57}$$

Equation (54) specifies the volatility structure. Equation (55) follows from Equation (54), given the equivalence between LMM and Black's caplet prices as shown in Theorem 4. Equation (56) is the Rebonato approximation to Black's implied volatilities for swaptions. Equation (57) follows Rebonato's 2-factor parametrization of the correlation structures for forward rates using hypersphere decomposition (Rebonato, 1999).

This approach uses a two-stage minimization. First minimization to obtain the optimum Σ_i s is performed using MS Excel Solver. Second minimization for optimum θ_i is performed using MS Excel Visual Basic Applications.

Method B: Monte Carlo Simulation with parametrized correlations matrix

This method recovers the prices of swaptions by Monte Carlo simulation with parametrized correlations matrix. The correlations matrix is based on Rebonato's hypersphere decomposition model. The algorithm for this methodology is described below:

1. Simulate *n* forward rates (L_{ij}, $i, j = 1,2,...N$.) using Milstein scheme:

$$\ln L_{ij}(t+\Delta t) = \ln L_{ij}(t) + \sigma_j(t)\sum_{k=i+1}^{j} \frac{\rho_{kj}\,\tau_k\,\sigma_k(t) L_{ik}(t)}{1+\tau_k L_{ik}(t)}\Delta t - \frac{\sigma_j^2(t)}{2}\Delta t + \sigma_j(t)\left[Z_j(t+\Delta t) - Z_j(t)\right] \quad (58)$$

Using trigonometric identities, the correlation matrix is constructed via the following functions:

$$b_{ik}(t) = \begin{cases} \cos(\theta_{ik}(t)) \prod_{j=1,k-1} \sin(\theta_{ij}(t)) & \text{for } k = 1, s-1 \\ \prod_{j=1,k-1} \sin(\theta_{ij}(t)) & \text{for } k = s \end{cases} \quad (59)$$

$$[\rho_{ij}]_N \equiv [b_{ij}]^T [b_{ij}] \quad (60)$$

where s refers to the number of factors in Rebonato's hypersphere parametrization of the correlations matrix and $\theta_{ik} \in [0, \pi]$. In this method, s is determined as the number of non-zero eigenvalues when principal component analysis is performed on the historical forward rate correlations matrix as follows

$$\hat{\mu}_i = \frac{1}{m}\sum_{k=0}^{m-1} \ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right) \quad (61)$$

$$\hat{V}_{i,j} = \frac{1}{m}\sum_{k=0}^{m-1}\left[\left(\ln\left(\frac{F_i\left(t_{k+1}\right)}{F_i\left(t_k\right)}\right)-\mu_i\right)\cdot\left(\ln\left(\frac{F_j\left(t_{k+1}\right)}{F_j\left(t_k\right)}\right)-\mu_j\right)\right] \tag{62}$$

2. Compute swaption prices using Equation (40), assuming the notional amount $P = 1$, given $\alpha = 1,...,N$, $\beta = i+1,...,N+1$

$$swaption_{\alpha,\beta}^{Simulated} = \left(\sum_{k=\alpha+1}^{\beta}\delta_k\left(L_{\alpha,\beta}\left(T_i\right)-K\right)\right)^{+} \tag{63}$$

3. Compute swaption prices using Black formula in Equation (41), given the market implied volatilities, $\sigma_{\alpha,\beta}^{Market}$.

4. Solve the following optimization equation:

$$\underset{\theta_{ij}}{Min} = \frac{\left|swaption_{\alpha,\beta}^{Market} - swaption_{\alpha,\beta}^{Simulated}\right|}{swaption_{\alpha,\beta}^{Market}} \tag{64}$$

The above algorithm is implemented in MS Excel Visual Basic Applications.

Method C: Monte Carlo Simulation, non-parametric correlations matrix.

This method recovers the prices of swaptions by Monte Carlo simulation. The correlations between forward rates are randomly chosen from $\left[0,1\right]$ such that adjacent swaptions prices are recovered. The problem that has to be addressed in this method is that typically, the number of available swaptions are not sufficient to completely calibrate the correlations matrix. For instance, in the South Korean market, while the swap lengths run from one year to ten years, the options expiries available are only from one year to five years.

In order to complete the entries in the correlation matrix, this study approximates the instantaneous correlation of forward rates with missing option expiry series by assuming that the terminal correlations are solely determined by the chosen volatility specification. Hence, in the calibration process, we expect to get mostly ones for correlations at the fourth quadrant of the matrix.

The algorithm for this methodology is described below:

1. Simulate n forward rates (L_{ij}, $i, j = 1,2,...N$.) using Milstein scheme shown in Equation (58):

   The correlation matrix ρ_{ij} is randomly drawn from the interval $\left(0, 1\right]$, except for the terminal correlations, which are estimated using Equation (50).

2. Compute swaption prices using Equation (40), assuming that the notional amount $P = 1$, given $\alpha = 1,..., N$, $\beta = i + 1,..., N + 1$

$$swaption_{\alpha,\beta}^{Simulated} = \left(\sum_{k=\alpha+1}^{\beta} \delta_k \left(L_{\alpha,\beta}\left(T_i\right) - K \right) \right)^+ \tag{65}$$

3. Compute swaption prices using Black formula in Equation (41), given the market implied volatilities, $\sigma_{\alpha,\beta}^{Market}$.

4. Solve the following optimization equation:

$$\underset{\rho_{ij}}{Min} = \frac{\left| swaption_{\alpha,\beta}^{Market} - swaption_{\alpha,\beta}^{Simulated} \right|}{swaption_{\alpha,\beta}^{Market}} \tag{66}$$

5. Compute for the terminal correlations using Equation (50).

The above algorithm is implemented in MS Excel Visual Basic Applications.

4 EMPIRICAL ANALYSIS

This section presents an application to the observed market data of the methodologies discussed in Chapter 3. We use quoted caps and swaptions prices in the South Korean market.

Emphasis on South Korean interest rate market is motivated by the remarkable growth of the South Korean economy that landed the country in the 12th spot in GDP ranking in 2005. Moreover, its incumbent government is determined to make South Korea a full-blown financial hub in East Asia by 2020. South Korea, aside from Japan, emerged as the only Asian country which has substantial market for securitization, as the country attempts to restructure huge amounts of distressed assets which are offshoots of the 1997-1998 Asian crisis. This led to the development of interest-rate based securities and active trading of interest rate options to hedge positions in credit instruments. Financial liberalization has also led to the development of innovative financing options in the country.

4.1 Description of the data

The data set consists of South Korean caps and swaptions implied volatilities quoted within the period February 1, 2006 until February 28, 2006. Within this period, there were 19 trading days. On each trading day, the following at-the-money implied volatilities were available: options for exercise of cap on into 1 year, 2 years, 3 years, 4 years, 5 years, 7 years and 10 years swap rates. As for swaptions, on each trading day, at-the-money implied volatilities were obtained: 1y, 2y, 3y, 4y, and 5y option into 1y, 2y, 3y, 5y, 7y, 10y swap rates. The data were obtained from Bloomberg.

4.2 Evaluation Criteria

There is no definitive set of criteria for a good model calibration. After several investigations of calibration methodologies, Brigo (2001) enumerated the following desired features of a well-calibrated LMM:

1. A small calibration error, where the error, ε_i is computed as follows:

$$\varepsilon_{\alpha,\beta} = 100 \times \frac{\left| swaption_{\alpha,\beta}^{Market} - swaption_{\alpha,\beta}^{Simulated} \right|}{swaption_{\alpha,\beta}^{Market}} \tag{67}$$

2. Regular instantaneous correlations or decorrelation is observable in the correlations matrix. This means that a monotonically decreasing pattern when moving away from a diagonal term of the matrix along the related row or column is deemed desirable.
3. Regular terminal correlations.
4. Smooth and qualitatively stable evolution of the term structure of caplet volatilities over time.

In addition to the above, this thesis extends the evaluation to a dynamic level by investigating the performance of calibrated parameters over a period. This is done by computing the average calibration changes, ξ_{ij}, of a series of calibrated volatilities and correlations (covariance), i.e.

$$\xi_{ij} = \frac{1}{T} \sum_{t=1}^{T} \left| \sigma_i(t)\sigma_j(t)\rho_{ij}(t) - \sigma_i(t-1)\sigma_j(t-1)\rho_{ij}(t-1) \right| \tag{68}$$

for $i, j = 1,..., N$

Small ξ_{ij} means that current estimates of volatility and correlations from a chosen calibration methodology closely approximate tomorrow's volatility and correlations. This indeed is a desirable quality as it implies less need for very frequent re-estimation of model parameters which may be too costly or impractical to do so.

4.3 Analysis of Results

For a comprehensive analysis in line with the previously discussed evaluation criteria, we first present individual analysis for each parameter, volatility and correlation, and then the joint analysis for both parameters.

4.3.1 Volatility Calibration

As discussed in Chapter 3 Section 2, instantaneous volatility of forward LIBOR can be specified in several ways. Although calibration of volatilities to caplets is automatic, the evolution of the term structure of volatilities differs depending on the chosen volatility specification.

Empirical observations show that a humped shape term structure of volatilities is a desirable quality (i.e. Rebonato, 1998). Moreover, Rebonato (2004) described such quality as normal or low volatility state of the instantaneous forward rates in contrast to a monotonically decreasing shape which he described as an excited state. Figure 1 shows selected instantaneous forward LIBOR curves bootstrapped from caps market prices. Notice that the term structures of South Korean forward LIBOR show a monotonically decreasing shape on February 1 and maintained more humped shape towards the end of the month. This could be attributed to the fact the South Korean interest rate markets is in its development stage, hence, the growing liquidity of caps contribute to the evolution of the term structure of forward rates.

Figures 2a and 2b show the implied evolution of forward LIBOR rates under different specifications. As discussed in Section 3.2, instantaneous volatility of forward rates can be specified in many ways depending on the belief of the trader on how the forward rates are expected to behave in the future. These differences in assumptions or beliefs lead to different evolution of the term structure of volatilities. Based on the set of criteria discussed in Section 2 of this Chapter, a smooth and stable evolution is preferred and signifies a well-calibrated set of implied volatilities.

Notice that specifications i) - constant time-to-maturity volatility – and ii) – constant instantaneous volatility - yield smooth sets of curves.

4.3.2 Correlations Calibration

For the correlations structure, Tables 1 and 2 and Figure 2 show the results in the calibration to South Korean swaptions. Notice that none of the correlations matrices exhibit the desired qualities mentioned in the previous section. However, among the three methods, Method B shows less fluctuation among the forward LIBOR. Perhaps the behavior of the correlations matrices could be attributed to the less liquid swaptions, especially the into 4-year option expiry swaptions, as the South Korean interest rate market is still in its development process. Frequent trading of South Korean swaptions for all options expiry and swap maturities began only in the last quarter of year 2005.

In terms of market fit and convergence time, Table 3a summarizes the results for the three methods. Among the three methods, Method A requires less time due to the fact that swaptions volatility uses the analytic approximation proposed by Rebonato (Rebonato, 2004). However, Method A's market fit is the worst. Method B with parametrized correlations matrix has the longest simulation/optimization time. This is due to the additional step of parametrization in establishing the correlations matrix. Method C has the least error in estimation, hence assures good market fit. However, it must be noted that the lower portion (in the fourth quadrant) of the correlations matrix is only estimated due to the unavailability of sufficient swaptions to calibrate with.

4.3.3 Joint analysis on the calibrated volatilities and correlations

Recall in the previous sections we mentioned that the best calibration methodology must not only recover today's prices but must closely estimate tomorrow's interest rates dynamics as well. Hence, the best way to test whether a method is able to satisfy this criterion is to examine its performance when applied not only to a one-day set of prices but to a time series of prices. This criterion ensures that the simultaneously calibrated volatilities and correlations minimize the need for future re-estimation of LMM parameters for hedging purposes.

In the empirical analysis, Table 3b shows the performance of Methods A, B and C. Notice that among the methods used, Method B shows the minimum calibration errors. This implies that a parametrized correlations matrix provides better estimates for correlations matrix not only for today's dynamics but for the next period's as well. This result is evident in Figure 4. Figure 4 graphs the volatilities as jointly implied by the prices of caps and swaptions. Indeed, Method B shows less fluctuation among volatilities in the period February 1 – 28, 2006.

4.3.4 Alternatives to Method B

Since Method B appears to be a promising approach given the results from the empirical analyses, we investigate some alternatives to improve the implied correlations structure as well as the convergence rate.

In order to ensure the robustness of the results of Method B and explore possibilities of improved convergence, we employ different volatility specifications that assumes different distribution of volatilites among forward rates at different maturities and initial thetas in the parametrization of the correlation matrix. We implement three alternatives to Method B. In the first alternative, we use as initial values the thetas obtained form the analytic-based calibration of correlations matrix. This is to investigate if convergence is further improved. We also employed different distribution of forward rate volatilities such as constant time-to-maturity volatility and constant instantaneous volatility specification as discussed in Chapter 3, Section 3.2. The objective of this modification is to ensure that the results obtained in Method B will hold regardless of the beliefs on the behavior of the volatility structure that the traders may follow. Summary of descriptions of the alternative specifications for Method B is found in Table 5a.

Results in Table 5a show that not much improvement is achieved. However, looking at the correlations structure in Figure 6, Method B3 appears to generate a smoothly decorrelated matrix except at the 4-year option expiry, which could be due to the illiquidity of 4-year option expiry series. Further, Table 5b shows the volatilities of forward LIBOR as implied by caps and swaptions. Among the alternatives, Method B3 has the least average calibration error. Notice that all the Method B alternatives show average calibration errors that are lower than the errors under Methods A and C.

The results in this section of the study only confirm that, despite the modified specifications for volatility and initial thetas, a parametrized correlations matrix provides better estimates of the LIBOR dynamics.

4.4 Further Analysis: Intra-day pricing

Following De Jong, Driessen and Pelsser (2001), we employ an intra-day analysis of pricing errors of the calibration methods identified in this study to further verify the robustness of the results in Section 4.3.

In the intra-day analysis, we segment the daily at-the-money prices of caps and swaptions into two sets, A and B. At each trading day, Set A , which is 20% of the entire sample set, contains one swaption price for each option expiry and swap maturity[17]. The remaining swaption prices comprise the Set B (80%). Set B is used in the calibration process and set A is used for pricing comparison between the market prices and the calibrated LMM prices. Since after the data is segmented, set B has incomplete option expiry-swap maturity series, we perform linear interpolation to complete the option expiry and swap length series. This is consistent with the method we used earlier to fill in missing swaptions maturity series.

Then, we follow similar calibration algorithm for Methods A, B and C on the new dataset B. Once the calibrated parameters are generated, we price dataset A using the LMM. Pricing results are compared with quoted market prices. Errors

[17] Extremes of the series (i.e. 1y-1y and 5y-5y) however are retained in Set B (calibration set) so as to enable linear interpolation for missing series and form a complete set of swaptions prices for calibration purposes.

are expressed in percentage as well as in implied volatility points[18]. Results are shown in Table 6a.

Results show that Methods B and C which use constant time to maturity volatility specification result in least pricing error for caps compared with Method A which is based on a separable piecewise instantaneous volatility function. This result is intuitively correct since cap prices are quoted in Black implied volatility that assumes constant volatility. At intra-day pricing, constant time to maturity volatility specification of Methods Band C would simply equal to the constant instantaneous volatility. Errors could be due to the less liquidity of the set of caps used in intra-day pricing analysis.

For swaptions, the result shows that the analytic approximation (Method A) is superior compared with the Monte-Carlo based methods (Methods A and B). Again, this result is consistent with the analytic approximation used in the method that is directly from the Black swaptions formula by Rebonato (Rebonato, 1999). Hence, on an intra-day basis, Method A will result in prices that are close to the quoted Black swaptions prices.

Examination of the prices further shows that LMM generally overprices swaptions under Methods B and C (see Table 6b). However, under Method A, LMM results are generally lower than Black prices. Results further show that instances of overpricing decreases towards the end of February. This could be attributed to the increasing liquidity of swaptions in the South Korean market.

[18] This will enable us to compare the results obtained by De Jong, et. al. (2001).

5 CONCLUSION

In this study, we attempt to address the prevailing challenge in the joint practice of complex derivative pricing and hedging observed in the interest rate market. The key to address this challenge is to choose a calibration methodology for the LMM such that prices of plain vanilla options are recovered while decorrelation among forward rates at future time instants is achieved.

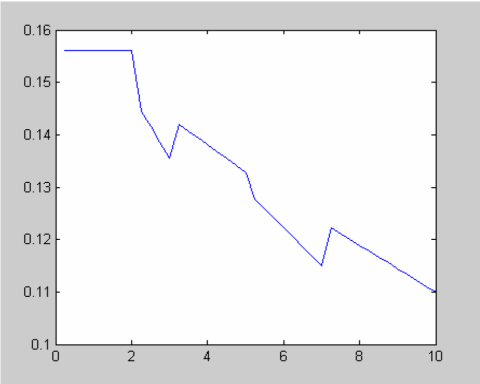
Hence the thesis identified and tested several calibration methods that jointly recover caps and swaptions prices. When tested on a series of options prices, results show that Monte Carlo simulation with parametrized correlations matrix (Method B) seems a promising approach in LMM volatility and correlations calibration. Not only is the approach able to generate good market fit with current prices of caps and swaptions, it is also able to closely estimate next period's LIBOR dynamics as evidenced by low average calibration changes error. Furthermore, Method B remains superior compared with the other calibration methods despite the changes made in the volatility specification consistent with prevailing market beliefs.

However, when intra-day pricing test was done, joint analytic calibration (Method A) proved better compared with the Monte Carlo based methods (B and C) for swaptions pricing. This is due to Method A's swaptions approximation that is directly derived from the Black formula. For caplets, methods B and C generated better results due to the consistency of the volatility specification with the Black assumption of constant volatility. For complex derivative pricing and trading, we deem that intra-day pricing test is not enough since the prime objective is to extract the dynamics of forward rates such that evolution of term structure of interest rates is smooth and decorrelation among rates is regular.

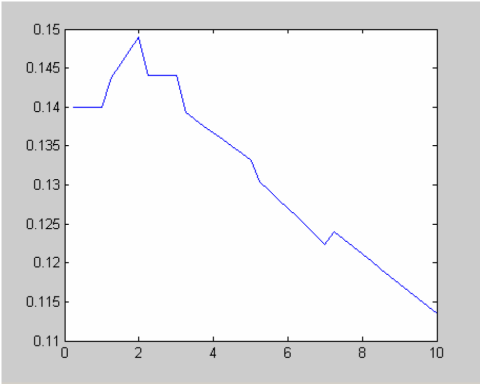
Given these results, Method B seems superior for the joint practice of complex derivative pricing and hedging as it provides smooth evolution of covariance of forward rates that is desired in complex derivative pricing and hedging.

Directions for future research include the use of more sophisticated optimization method or low discrepancy numbers for improved and fast convergence. Smoothing can also be used on the dataset to ensure that market noises that could distort the results are excluded. In addition, the sample size could be extended to capture more movements in the prices of caps and swaptions that could affect the performance of the calibration methodologies identified in the study.

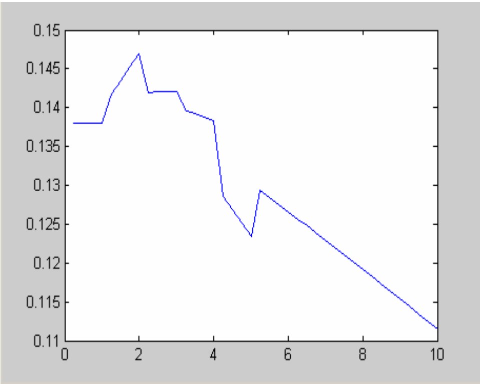
Figure 1. Selected volatility term structure from South Korean Won caplets



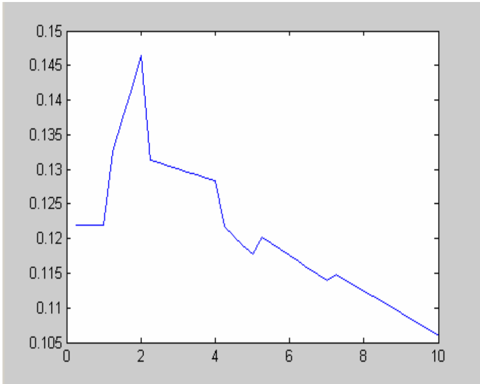
a. February 1, 2006



b. February 14, 2006



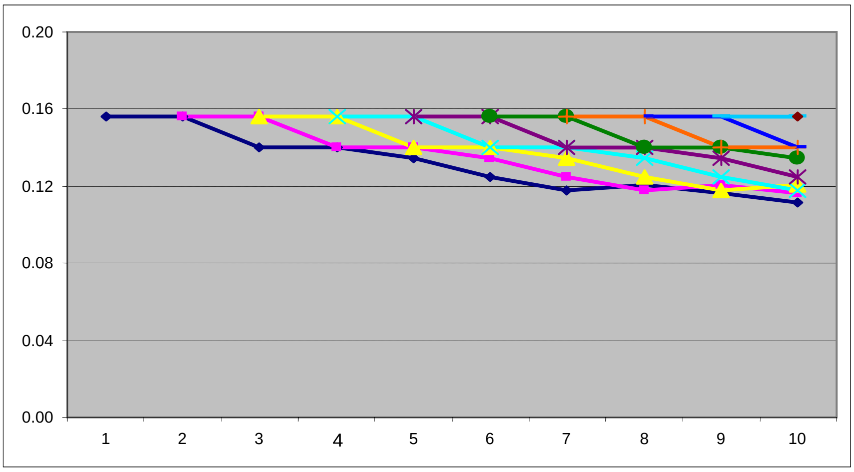
c. February 20, 2006



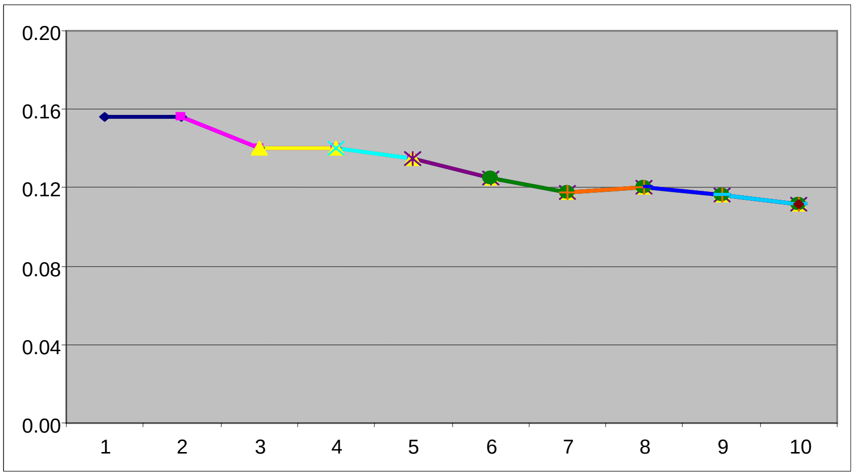
d. February 28, 2006

Note: This figure shows the term structure of volatilities at selected dates. These volatilities are bootstrapped from South Korean caps quoted at implied volatilities. The x-axis represents the term of the caplet (in years) while the y-axis shows the volatility values.

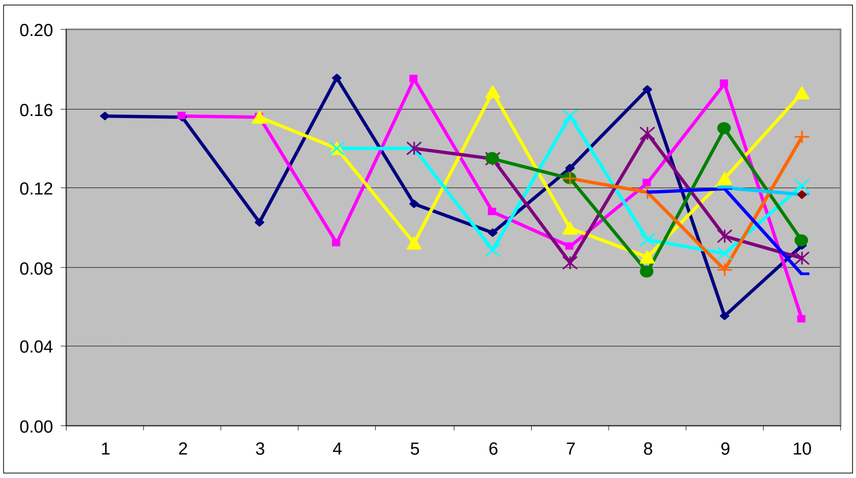
Figure 2a. Evolution of the term structure of volatilities under different volatility specifications, February 1, 2006.



i. Constant time-to-maturity specification



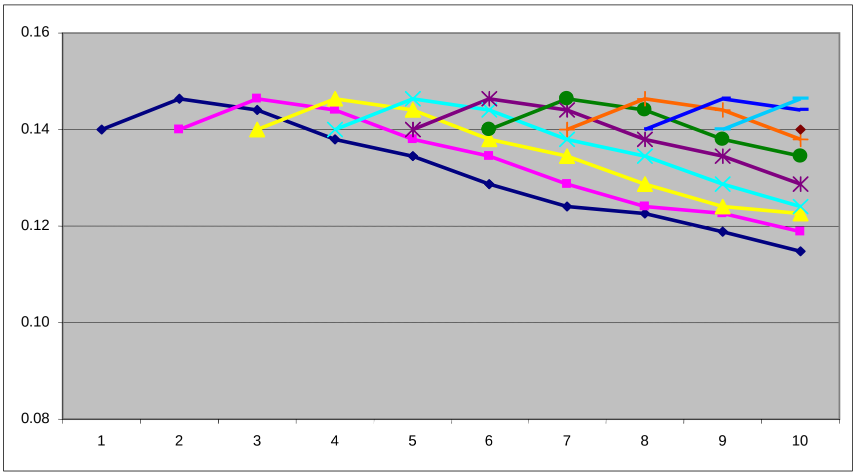
ii. Constant instantaneous volatility specification



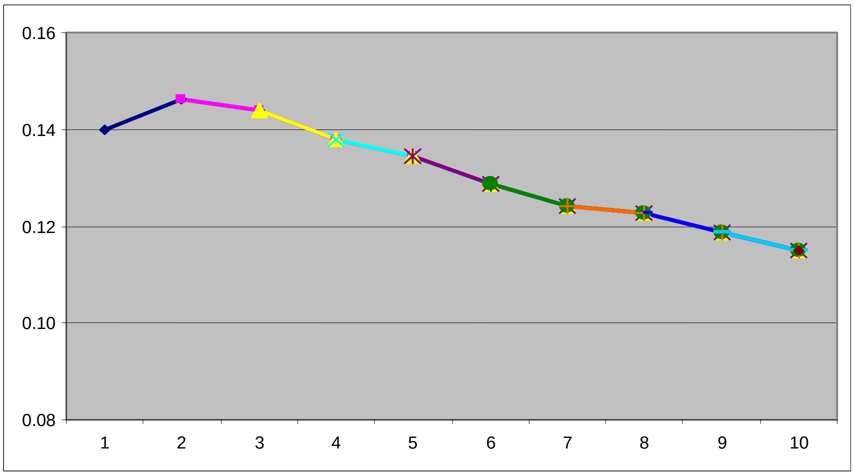
iii. Separable piecewise constant instantaneous

Note: The figures above show the evolution of the term structure of volatilities under different specification of volatility as discussed in Section3.2. The x-axis represents term (in years) while the y-axis shows the volatility values.

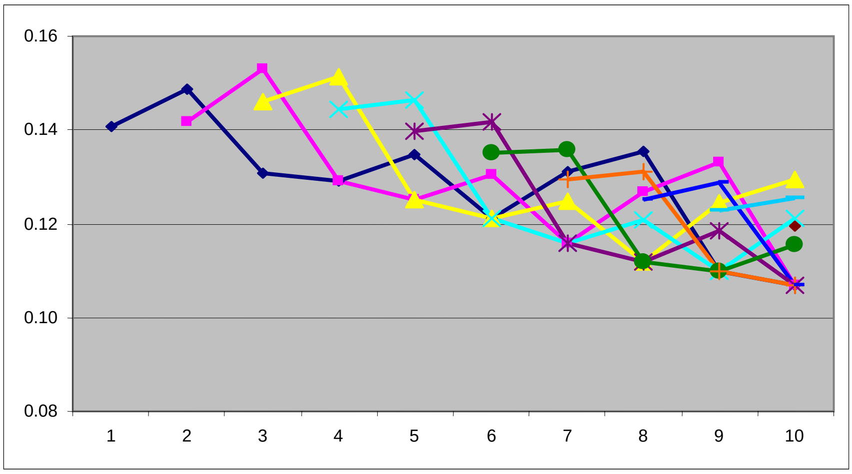
Figure 2b. Evolution of the term structure of volatilities under different volatility specifications, February 14, 2006**.**



i. Constant time-to-maturity specification



ii. Constant instantaneous volatility specification



iii. Separable piecewise constant instantaneous

Note: The figures above show the evolution of the term structure of volatilities under different specification of volatility as discussed in Section3.2. The x-axis represents term (in years) while the y-axis shows the volatility values.

Tables 1a-c. Correlation matrices

February 1, 2006

	1	2	3	4	5	6	7	8	9	10
1	1.000	0.401	0.898	0.885	0.394	0.245	0.908	0.281	0.331	0.322
2	0.401	1.000	0.764	0.782	1.000	0.986	0.747	0.992	0.997	0.738
3	0.898	0.764	1.000	1.000	0.758	0.647	1.000	0.675	0.713	0.128
4	0.885	0.782	1.000	1.000	0.776	0.668	0.999	0.695	0.733	0.156
5	0.394	1.000	0.758	0.776	1.000	0.988	0.742	0.993	0.998	0.744
6	0.245	0.986	0.647	0.668	0.988	1.000	0.628	0.999	0.996	0.839
7	0.908	0.747	1.000	0.999	0.742	0.628	1.000	0.656	0.695	0.103
8	0.281	0.992	0.675	0.695	0.993	0.999	0.656	1.000	0.999	0.818
9	0.331	0.997	0.713	0.733	0.998	0.996	0.695	0.999	1.000	0.787
10	0.322	0.738	0.128	0.156	0.744	0.839	0.103	0.818	0.787	1.000

a. Method A

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1.0000	0.9999	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.9999	1.0000	1.0000	0.9998	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999
3	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0.9999	0.9998	0.9999	1.0000	0.9999	1.0000	0.9998	1.0000	0.9998	1.0000
5	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9999	1.0000
7	1.0000	1.0000	1.0000	0.9998	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999
8	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9999	1.0000
9	1.0000	1.0000	1.0000	0.9998	1.0000	0.9999	1.0000	0.9999	1.0000	0.9999
10	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9999	1.0000

b. Method B

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1.0000	0.1391	0.4298	0.7643	0.7650	0.4312	0.8567	0.2117	0.0246	0.8897
2	0.1391	1.0000	0.0948	0.7913	0.9012	0.6162	0.9494	0.2918	0.1969	0.5727
3	0.4298	0.0948	1.0000	0.9647	0.1797	0.9147	0.5264	0.9461	0.6604	0.0465
4	0.7643	0.7913	0.9647	1.0000	0.0263	0.9233	0.4785	0.9551	0.2093	0.9648
5	0.7650	0.9012	0.1797	0.0263	1.0000	0.1227	0.9141	0.9600	0.4587	0.6653
6	0.4312	0.6162	0.9147	0.9233	0.1227	1.0000	1.0000	1.0000	1.0000	1.0000
7	0.8567	0.9494	0.5264	0.4785	0.9141	1.0000	1.0000	1.0000	1.0000	1.0000
8	0.2117	0.2918	0.9461	0.9551	0.9600	1.0000	1.0000	1.0000	1.0000	1.0000
9	0.0246	0.1969	0.6604	0.2093	0.4587	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.8897	0.5727	0.0465	0.9648	0.6653	1.0000	1.0000	1.0000	1.0000	1.0000

c. *Method C*

Note: Each entry of the above matrices corresponds to ρ_{ij}, *where* i *(rows) is the index for time at future instant (i.e.* t_i *) and* j *(columns) is the index for the maturity/length in time of the forward rate (i.e.* t_j*)* , $i, j = 1,...,10$.

Tables 2a-c. Correlation matrices

February 14, 2006

	1	2	3	4	5	6	7	8	9	10
1	1.000	0.554	0.739	0.547	0.976	0.908	0.534	0.309	0.263	0.765
2	0.554	1.000	0.151	0.394	0.361	0.155	0.408	0.620	0.949	0.960
3	0.739	0.151	1.000	0.968	0.867	0.953	0.964	0.869	0.456	0.132
4	0.547	0.394	0.968	1.000	0.715	0.847	1.000	0.965	0.664	0.120
5	0.976	0.361	0.867	0.715	1.000	0.977	0.704	0.507	0.048	0.608
6	0.908	0.155	0.953	0.847	0.977	1.000	0.839	0.679	0.165	0.426
7	0.534	0.408	0.964	1.000	0.704	0.839	1.000	0.969	0.676	0.136
8	0.309	0.620	0.869	0.965	0.507	0.679	0.969	1.000	0.836	0.375
9	0.263	0.949	0.456	0.664	0.048	0.165	0.676	0.836	1.000	0.822
10	0.765	0.960	0.132	0.120	0.608	0.426	0.136	0.375	0.822	1.000

a. Method A

	1	2	3	4	5	6	7	8	9	10
1	1.0000	0.9916	0.9852	0.9749	0.9993	0.9955	0.9988	0.9707	0.9561	0.9720
2	0.9916	1.0000	0.9991	0.9953	0.9863	0.9994	0.9968	0.9933	0.9851	0.9940
3	0.9852	0.9991	1.0000	0.9986	0.9785	0.9969	0.9924	0.9974	0.9915	0.9978
4	0.9749	0.9953	0.9986	1.0000	0.9666	0.9914	0.9845	0.9998	0.9970	0.9999
5	0.9993	0.9863	0.9785	0.9666	1.0000	0.9914	0.9963	0.9618	0.9457	0.9633
6	0.9955	0.9994	0.9969	0.9914	0.9914	1.0000	0.9990	0.9888	0.9787	0.9896
7	0.9988	0.9968	0.9924	0.9845	0.9963	0.9990	1.0000	0.9811	0.9688	0.9822
8	0.9707	0.9933	0.9974	0.9998	0.9618	0.9888	0.9811	1.0000	0.9983	1.0000
9	0.9561	0.9851	0.9915	0.9970	0.9457	0.9787	0.9688	0.9983	1.0000	0.9980
10	0.9720	0.9940	0.9978	0.9999	0.9633	0.9896	0.9822	1.0000	0.9980	1.0000

b. Method B

	1	2	3	4	5	6	7	8	9	10
1	1.0000	0.9544	0.9492	0.9017	0.9919	0.9317	0.8567	0.2117	0.0246	0.8897
2	0.9544	1.0000	0.6546	0.9537	0.1122	0.9351	0.3686	0.2918	0.1969	0.5727
3	0.9492	0.6546	1.0000	0.9161	0.5299	0.7221	0.2936	0.9602	0.6604	0.0465
4	0.9017	0.9537	0.9161	1.0000	0.9992	0.9490	0.9600	0.9354	0.9007	0.9648
5	0.9919	0.1122	0.5299	0.9992	1.0000	0.7245	0.2981	0.9617	0.3811	0.9432
6	0.9317	0.9351	0.7221	0.9490	0.7245	1.0000	1.0000	1.0000	1.0000	1.0000
7	0.8567	0.3686	0.2936	0.9600	0.2981	1.0000	1.0000	1.0000	1.0000	1.0000
8	0.2117	0.2918	0.9602	0.9354	0.9617	1.0000	1.0000	1.0000	1.0000	1.0000
9	0.0246	0.1969	0.6604	0.9007	0.3811	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.8897	0.5727	0.0465	0.9648	0.9432	1.0000	1.0000	1.0000	1.0000	1.0000

c. Method C

Note: Each entry of the above matrices corresponds to ρ_{ij}, where i (rows) is the index for time at future instant (i.e. t_i) and j (columns) is the index for the maturity/length in time of the forward rate (i.e. t_j) , $i, j = 1,...,10$.

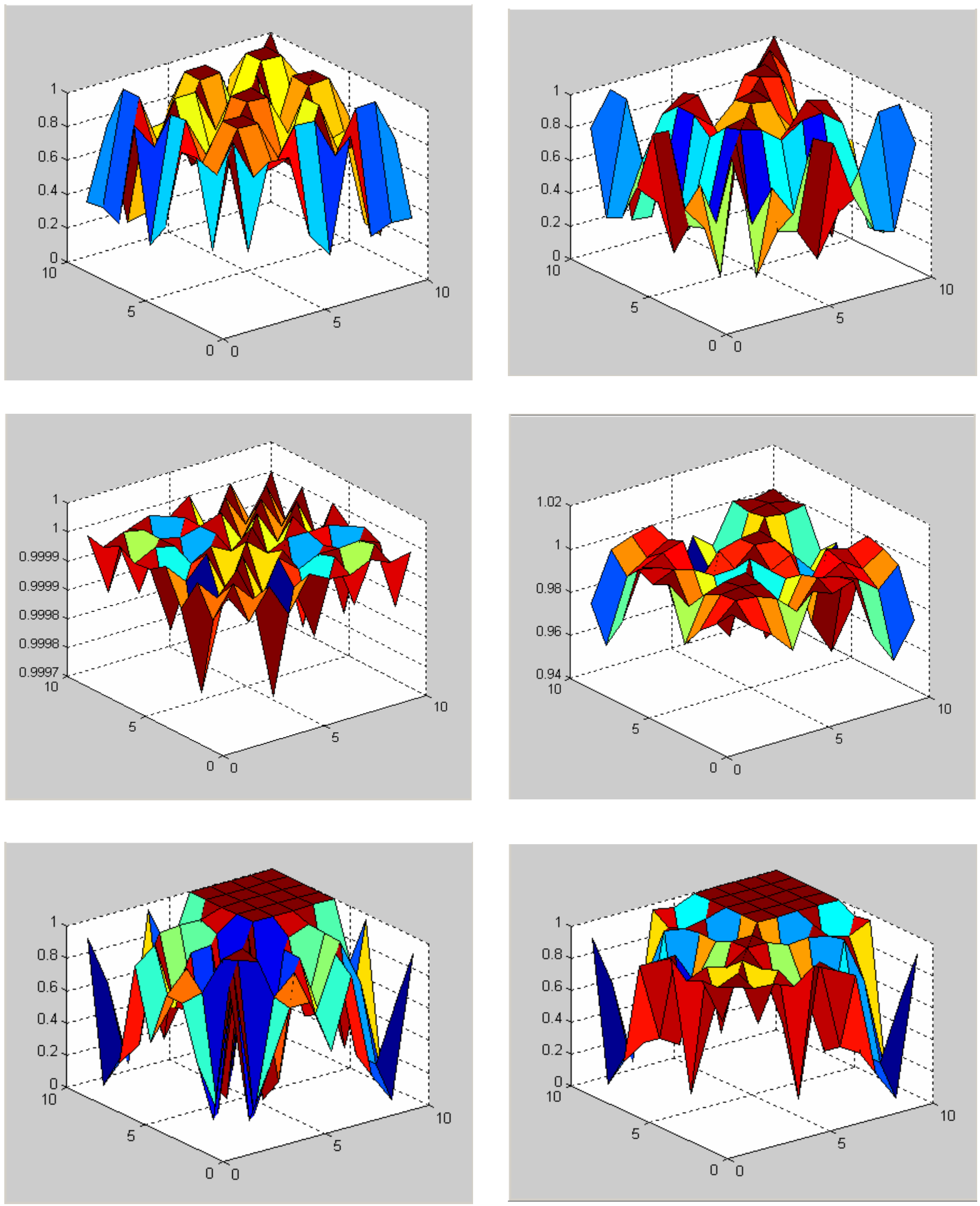
Figure 3. Correlation surfaces
February 1 and 14, 2006

Feb 1 **Feb 14**

A

B

C



Note: Each point of the above surface represents x-axis represents ρ_{ij} *where* i *is the index for time at future instant (i.e.* t_i *) and* j *is the index for the maturity/length in time of the forward rate (i.e.* t_j*) ,* $i, j = 1,...,10$*. x-axis represents* i *values, y-axis represents* j *values and z-axis represents the correlation values.*

Table 3a
Summary of methods

This table presents a summary of alternative calibration methods identified and described in Chapter 3, Section 4.

Method	Description	Correlation Specification	Average Calibration Time	Average Estimation Error (in %)
A	Joint Calibration using Rebonato's approximation	2-factor hypersphere decomposition	4 min, 29 sec	15.59%
B	Monte Carlo simulation with parametric correlation matrix	5-factor hypersphere decomposition	12 min, 15 sec	2. 21%
C	Monte Carlo Simulation with non-parametric correlation (exact calibration to swaption prices)	Randomly drawn from a sequence of low-discrepancy numbers; terminal correlations estimated through simulation	10 min, 58 sec	0.24%

Table 3b
Average calibration changes error

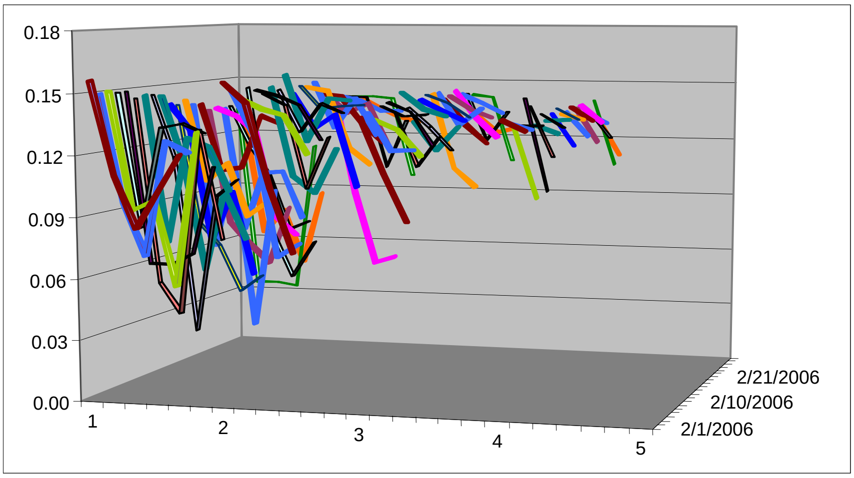
This table presents the averages from a 19-day set of volatility and correlations changes calibrated to South Korean caps and swaptions under each of the three methods described in Chapter 3 Section 4.

Method	Volatility Curve	Correlation Surface														
		1-1	1-2	1-3	1-4	1-5	2-2	2-3	2-4	2-5	3-3	3-4	3-5	4-4	4-5	5-5
A	0.0390	0.0024	0.0343	0.0275	0.0360	0.0079	0.0030	0.0367	0.0249	0.0379	0.0041	0.0393	0.0168	0.0037	0.0292	0.0047
B	0.0228	0.0023	0.0023	0.0027	0.0027	0.0034	0.0027	0.0023	0.0025	0.0039	0.0038	0.0025	0.0030	0.0037	0.0019	0.0044
C	0.0228	0.0023	0.0317	0.0369	0.0065	0.0338	0.0027	0.0441	0.0039	0.0035	0.0038	0.0117	0.0428	0.0037	0.0440	0.0044

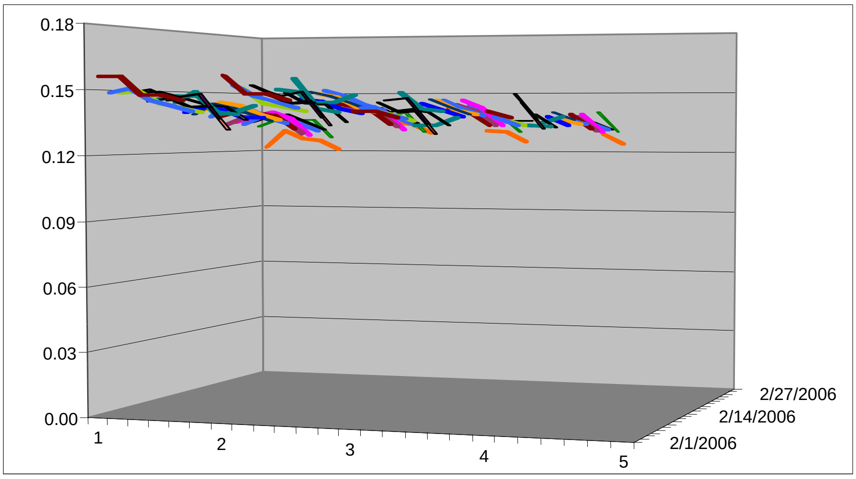
Note: Dataset includes all trading days within the period February 1 – 28, 2006.
Data Source: Bloomberg

Figure 4. Daily forward rates volatilities,
calibrated to caps and swaptions

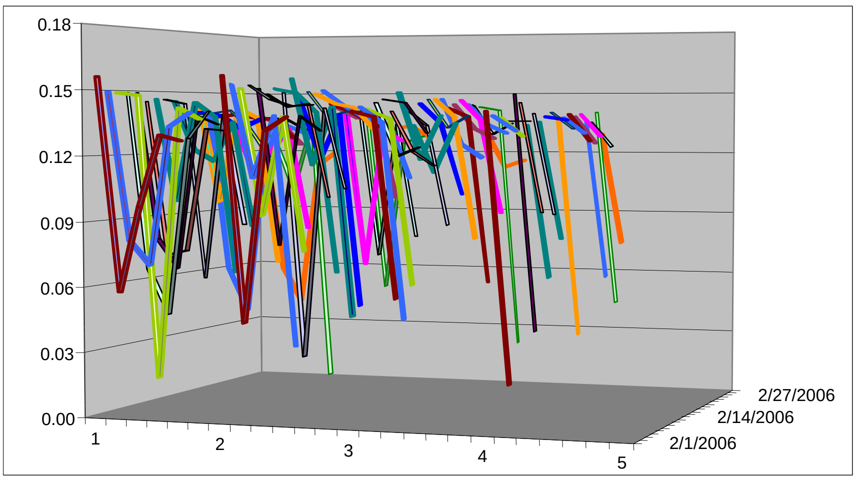
February 1 – 28, 2006.



a. Method A



b. Method B



c. Method C

Note: x-axis represents term in year; y-axis represents the forward rate volatilities; and the z-axis represents dates (daily interval).

Table 5a
Summary of Alternative Specifications for Method B

Method	Volatility Specification	Correlation Specification	Initial Thetas	Average Simulation Time	Average Estimation Error (in %)
B	Spherical Coordinates	5-factor, hypersphere decomposition	π and π/2 (increments are LD numbers)	12 min, 15 sec	2.21%
B1	Constant Time-to-Maturity	5-factor, hypersphere decomposition	Joint-calibration, using Rebonato approximation	6 min, 52 sec	2.52%
B2	Constant Instantaneous	5-factor, hypersphere decomposition	Joint-calibration, using Rebonato approximation	13 min, 3 sec	3.83%
B3	Constant Time-to-Maturity	5-factor, hypersphere decomposition	Low discrepancy (LD) numbers, within the interval $[-3.1416, 3.1416]$	10 min, 28 sec	2.34%

Table 5b
Average Calibration Changes Error

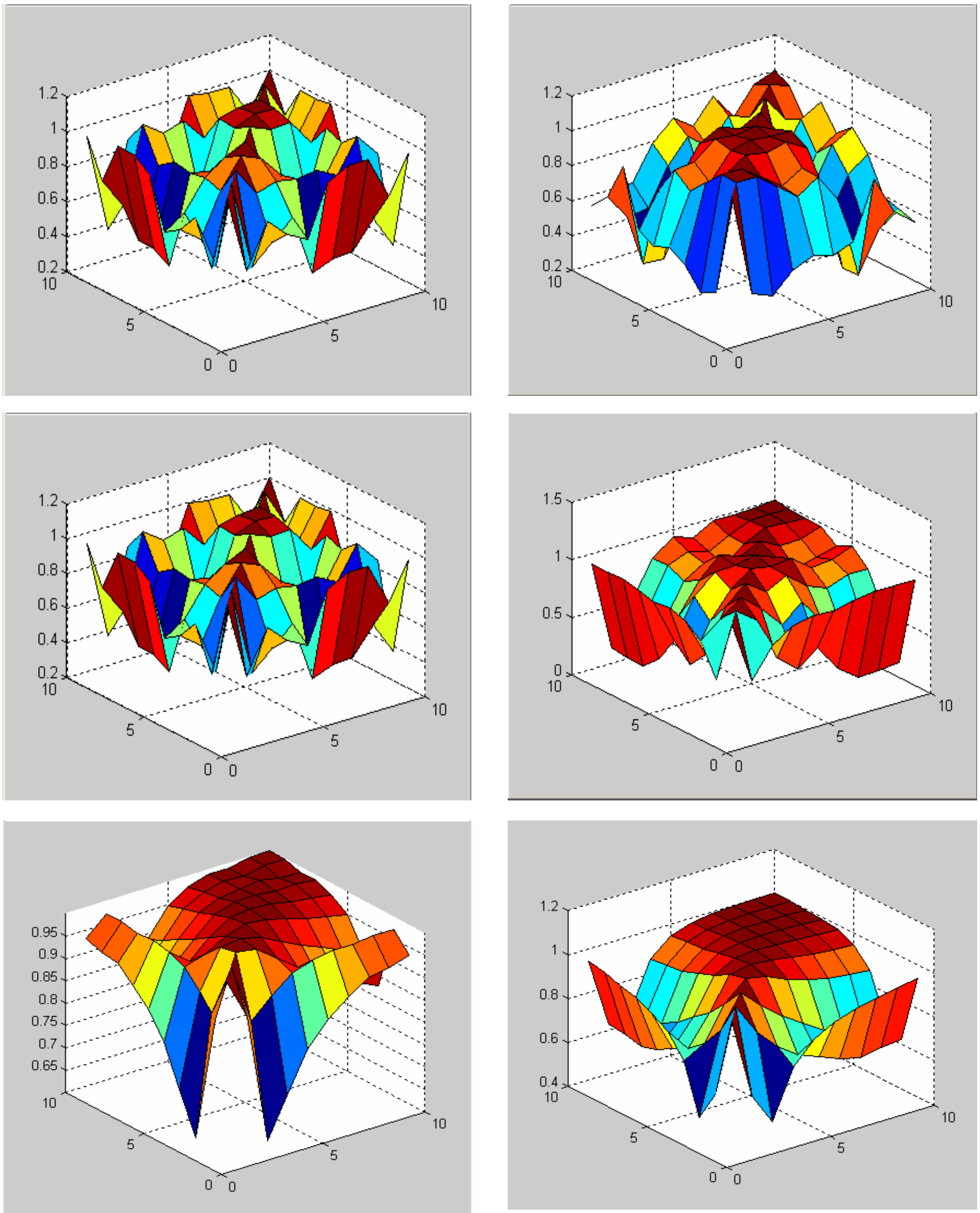
This table presents the averages from a 19-day set of volatility and correlations changes calibrated to South Korean caps and swaptions under each of the alternative specifications for method B described in Table2a.

Met-hod	Volatility Curve	Correlation Surface														
		1-1	1-2	1-3	1-4	1-5	2-2	2-3	2-4	2-5	3-3	3-4	3-5	4-4	4-5	5-5
B	**0.0228**	**0.0023**	**0.0023**	**0.0027**	**0.0027**	**0.0034**	**0.0027**	**0.0023**	**0.0025**	**0.0039**	**0.0038**	**0.0025**	**0.0030**	**0.0037**	**0.0019**	**0.0044**
B1	0.0228	0.0023	0.0304	0.0309	0.0361	0.0267	0.0027	0.0165	0.0298	0.0291	0.0038	0.0182	0.0198	0.0037	0.0111	0.0044
B2	0.0228	0.0023	0.0191	0.0223	0.0289	0.0208	0.0027	0.0128	0.0240	0.0237	0.0038	0.0108	0.0165	0.0038	0.0087	0.0044
B3	0.0028	0.0023	0.0240	0.0195	0.0105	0.0053	0.0027	0.0072	0.0099	0.0143	0.0038	0.0037	0.0036	0.0037	0.0020	0.0044

Note: Dataset includes all trading days within the period February 1 – 28, 2006.
Data Source: Bloomberg

Figure5. Correlation surfaces
Method B alternatives
February 1 and 14, 2006

	Feb 1	**Feb 14**
B1		
B2		
B3		



Note: Each point of the above surface represents x-axis represents ρ_{ij} where i is the index for time at future instant (i.e. t_i) and j is the index for the maturity/length in time of the forward rate (i.e. t_j) , $i, j = 1,...,10$. x-axis represents i values, y-axis represents j values and z-axis represents the correlation values.

Table 6a
Intra-Day Pricing Results[19]

This table contains summary statistics on intra-day pricing errors. Daily sample of caps and swaptions was segmented into two sets: A and B. Set A was used as benchmark prices for LMM pricing while Set B was used for calibration. Results are presented for the three calibration methods described in the body of the thesis.

Method	Caplets		Swaptions	
	In % Price Error	In Implied Volatility Points Error	In % Price Error	In Implied Volatility Points Error
A	11.3%	1.95	14.9%	2.31
B	10.8%	2.50	20.0%	2.76
C	10.8%	2.50	16.1%	2.48

[19] Paper (by A. Pelsser, et.al. (2001)) reported pricing errors in the interval 0 - 3.26 implied volatility points for caplets and 0.6 – 2.23 implied volatility points for swaptions.

Table 6b
Selected Intra-Day Pricing Results

This table shows selected prices of swaptions when calibrated LMM dynamics are applied to Set A sample and the quoted Black swaptions prices. Swaptions selected are expressed in terms of option expiry (in years) – swap length (in years), i.e. 2y-4y means a swaption with 2-year option on a 4-year swap.

Method A

Date / Swaption	Prices in Implied Volatility Points		Prices in South Korean Won		% of Set A such that LMM > Black prices
	Black	LMM	Black	LMM	
Feb 1: (1y-2y)	0.1590	0.1771	0.0057	0.0063	40%
Feb 13: (2y-3y)	0.1390	0.1379	0.0102	0.0102	40%
Feb 20: (3y-4y)	0.1310	0.1240	0.0145	0.0146	40%
Feb 28 : (4y-5y)	0.1110	0.1155	0.0163	0.0177	40%

Method B

Date / Swaption	Prices in Implied Volatility Points		Prices in South Korean Won		% of Set A such that LMM > Black prices
	Black	LMM	Black	LMM	
Feb 1: (1y-2y)	0.1590	0.1777	0.0057	0.0064	80%
Feb 13: (2y-3y)	0.1380	0.1460	0.0101	0.0108	60%
Feb 20: (3y-4y)	0.1310	0.1308	0.0145	0.0144	40%
Feb 28 : (4y-5y)	0.1110	0.1260	0.0163	0.0180	40%

Method C

Date / Swaption	Prices in Implied Volatility Points		Prices in South Korean Won		% of Set A such that LMM > Black prices
	Black	LMM	Black	LMM	
Feb 1: (1y-2y)	0.1590	0.1785	0.0056	0.0063	80%
Feb 13: (2y-3y)	0.1380	0.1382	0.0101	0.0103	60%
Feb 20: (3y-4y)	0.1310	0.1205	0.0145	0.0143	60%
Feb 28 : (4y-5y)	0.1110	0.1131	0.0163	0.0177	40%

REFERENCES

Brace, A., Gatarek, D., Musiela, M. (1977). *The Market Model of Interest Rate Dynamics*. Mathematical Finance 7 (2), pp. 127-155.

Brigo, D. and Morini, M. (2004). E*mpirically Efficient Cascade Calibration of the LIBOR Market Model based only on Directly Quoted Swaptions data*. Working Paper.

Brigo, D., Mercurio, F. and Moroni, M. (2004). *The LIBOR model dynamics: Approximations, Calibration and Diagnostics*. European Journal of Operational Research 163, pp 30-51.

Brigo, D. and Mercurio, F. (2001). *Interest Rate Models: Theory and Practice*. Springer Finance.

Cox, J., Ingersoll, J. and Ross, S. (1985). *A Theory of the Term Structure of Interest Rates*. Econometrica, Volume 53 (2), pp. 385-408.

De Jong, F. , Driessen, J. and Pelsser, A. (2001). *LIBOR Market Models versus Swap Market Models for Pricing Interest Rate Derivatives: An Empirical Analysis*. European Finance Review 5, pp. 201-237.

Hull, John C. (2006). *Options, Futures, and Other Derivatives*, 6th ed. Prentice Hall.

Hull, J. C. and White A. (2000). *Forward Rate Volatilities, Swap Rate Volatilities, and Implementation of the LIBOR Market Model*. Journal of Fixed Income 4, pp. 46-62.

Jackel, P. (2002). *Monte Carlo Methods in Finance*. John Wiley & Sons Ltd.

Jackel, P. and Rebonato, R (2002). *The Link between Caplet and Swaption Volatilities in a BGM/J Framework: Approximate Solutions and Empirical Evidence*. Working Paper.

Jamishidian, F. (1977). *LIBOR and Swap Market Models and Measures*. Finance and Stochastics 1, pp. 293-330.

Karatzas, I., and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd edition. Berlin: Springer-Verlag.

Pedersen, M.B. (1999). *On the LIBOR Market Models*. Working Paper, SimCorp Financial Research.

Rebonato, R. (2004). V*olatility and Correlation: The Perfect Hedger and the Fox*, 2nd ed. John Wiley and Sons, Inc.

Rebonato, R. (2004). *A Two-regime, Stochastic Volatility Extension of the LIBOR Market Model.* International Journal of Theoretical and Applied Finance, Volume 7 (5), pp. 555-575.

Rebonato, R. (2003). *Term Structure Models: A Review*. Quantitative Research Center – Royal Bank of Scotland, Oxford University.

Rebonato, R. (1999). *On the Simultaneous Calibration of Multi-Factor Log-normal Assumption for the Cap and Swaption Markets*. Journal of Computational Finance, Volume 2, 4, pp. 5 - 27.

Rebonato, R. (1999). A*ccurate and Optimal Calibration to Co-Terminal European Swaptions in a FRA-based BGM Framework*, Working Paper.

Rutkowski, M. (1999). *Models of Forward LIBOR and Swap Rates*. Applied Mathematical Finance 6, pp. 29-60.

Sidennius, J. (2000). *LIBOR Market Models in Practice*. Journal of Computational Finance, Volume 3 (3), pp. 5-26.

Wu, L. (2002). *Fast At-the-money Calibration of the LIBOR Market Model through Lagrange Multipliers.* Journal of Computational Finance, Volume 6 (2), pp. 39-77.

APPENDIX A

Documentation of Thesis Workbook

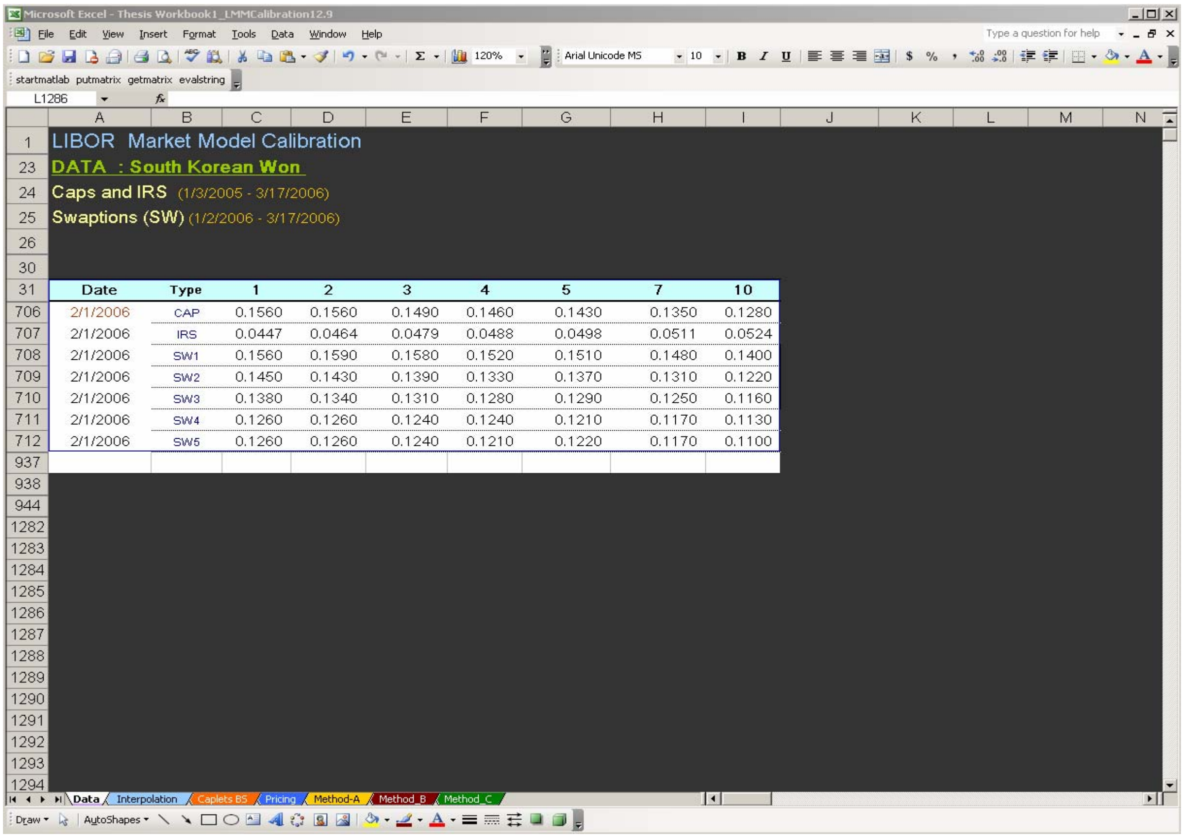
The thesis implementation used the Microsoft Excel Visual Basic program with MatLab 7.04 interface. This section discusses in detail the steps to run the calibration and pricing process within this workbook.

Thesis Workbook: LMM Calibration

This workbook runs the calibration process of the LIBOR market model under the different alternative methods described in the body of the thesis. It has seven (7) sheets described below:

Sheet Number	Name	Content
1	Data	- Daily quotes of South Korean Won caps and swaptions prices (February 1 – February 28, 2006) - Daily quotes of South Korean IRS rates (February 1 – February 28, 2006)
2	Interpolation	This section performs interpolation of interest rates depending on the compounding frequency chosen. In this study, we mainly use quarterly compounding as it is the convention observed in the South Korean interest rate market.
3	Caplets BS	This sheet performs the bootstrapping of caplet volatilities using the piecewise linear method. The returned values are the implied volatilities of caplets.
4	Pricing	In this sheet, swaptions implied volatilities are converted into prices using the Black formula. Interpolation is first done to complete the matrix of swaptions prices.
5	Method A	This sheet performs LMM calibration using Method A as described in the body of the thesis.
6	Method B	This sheet performs LMM calibration using Method B as described in the body of the thesis.
7	Method C	This sheet performs LMM calibration using Method C as described in the body of the thesis.

Sheet 1: Data



LIBOR Market Model Calibration

DATA : South Korean Won

Caps and IRS (1/3/2005 - 3/17/2006)

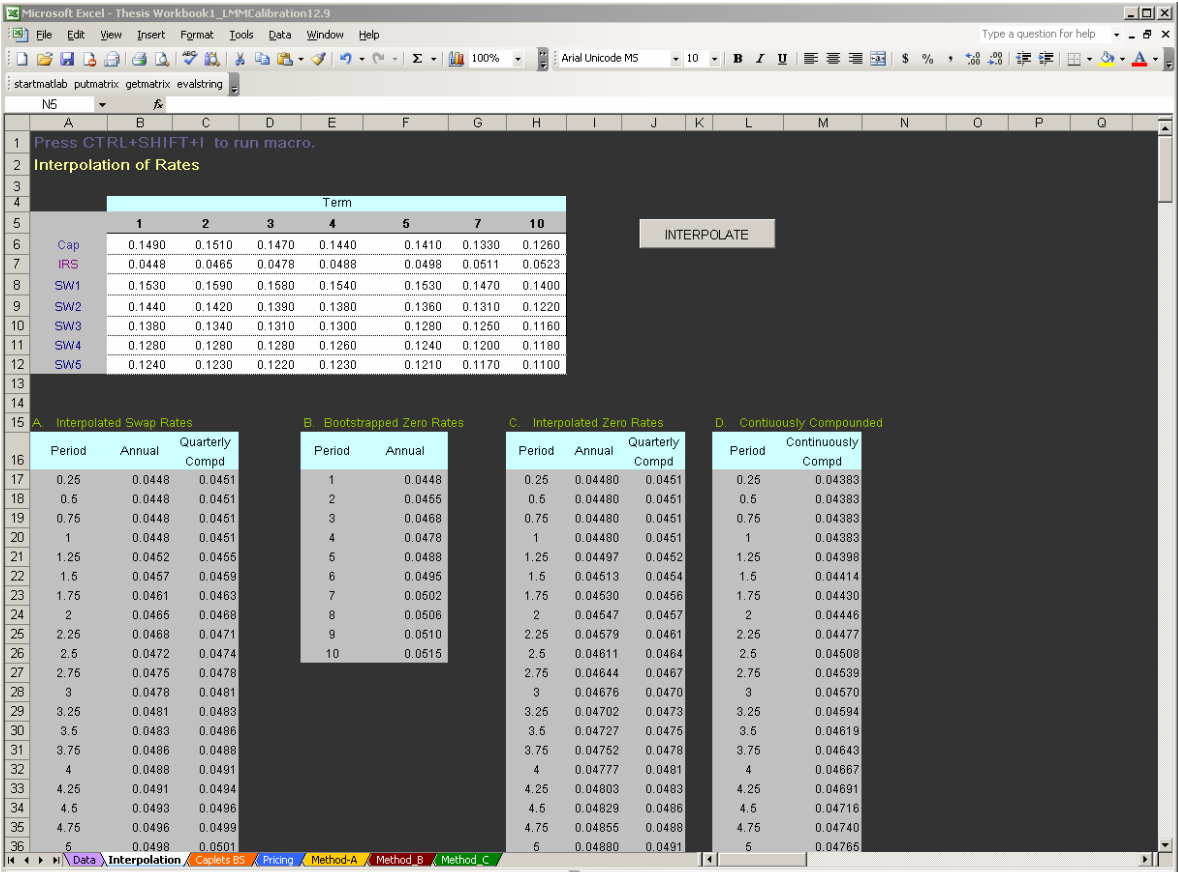
Swaptions (SW) (1/2/2006 - 3/17/2006)

Date	Type	1	2	3	4	5	7	10
2/1/2006	CAP	0.1560	0.1560	0.1490	0.1460	0.1430	0.1350	0.1280
2/1/2006	IRS	0.0447	0.0464	0.0479	0.0488	0.0498	0.0511	0.0524
2/1/2006	SW1	0.1560	0.1590	0.1580	0.1520	0.1510	0.1480	0.1400
2/1/2006	SW2	0.1450	0.1430	0.1390	0.1330	0.1370	0.1310	0.1220
2/1/2006	SW3	0.1380	0.1340	0.1310	0.1280	0.1290	0.1250	0.1160
2/1/2006	SW4	0.1260	0.1260	0.1240	0.1240	0.1210	0.1170	0.1130
2/1/2006	SW5	0.1260	0.1260	0.1240	0.1210	0.1220	0.1170	0.1100

Steps:

1. On Sheet 1 (Data), click on "Date" cell until the arrow tabs for sorting appears.
2. Select from the drop- down list the desired date of quoted prices to which LMM will be calibrated.
3. Highlight the cells containing the quoted prices.
4. Click on "Copy" icon or right-click your mouse and select "Copy".
5. Go to Sheet 2 (Interpolation).

Sheet 2: Interpolation



Microsoft Excel - Thesis Workbook1_LMMCalibration12.9

Press CTRL+SHIFT+I to run macro.

Interpolation of Rates

	Term						
	1	2	3	4	5	7	10
Cap	0.1490	0.1510	0.1470	0.1440	0.1410	0.1330	0.1260
IRS	0.0448	0.0465	0.0478	0.0488	0.0498	0.0511	0.0523
SW1	0.1530	0.1590	0.1580	0.1540	0.1530	0.1470	0.1400
SW2	0.1440	0.1420	0.1390	0.1380	0.1360	0.1310	0.1220
SW3	0.1380	0.1340	0.1310	0.1300	0.1280	0.1250	0.1160
SW4	0.1280	0.1280	0.1280	0.1260	0.1240	0.1200	0.1180
SW5	0.1240	0.1230	0.1220	0.1230	0.1210	0.1170	0.1100

INTERPOLATE

A. Interpolated Swap Rates

Period	Annual	Quarterly Compd
0.25	0.0448	0.0451
0.5	0.0448	0.0451
0.75	0.0448	0.0451
1	0.0448	0.0451
1.25	0.0452	0.0455
1.5	0.0457	0.0459
1.75	0.0461	0.0463
2	0.0465	0.0468
2.25	0.0468	0.0471
2.5	0.0472	0.0474
2.75	0.0475	0.0478
3	0.0478	0.0481
3.25	0.0481	0.0483
3.5	0.0483	0.0486
3.75	0.0486	0.0488
4	0.0488	0.0491
4.25	0.0491	0.0494
4.5	0.0493	0.0496
4.75	0.0496	0.0499
5	0.0498	0.0501

B. Bootstrapped Zero Rates

Period	Annual
1	0.0448
2	0.0455
3	0.0468
4	0.0478
5	0.0488
6	0.0495
7	0.0502
8	0.0506
9	0.0510
10	0.0515

C. Interpolated Zero Rates

Period	Annual	Quarterly Compd
0.25	0.04480	0.0451
0.5	0.04480	0.0451
0.75	0.04480	0.0451
1	0.04480	0.0451
1.25	0.04497	0.0452
1.5	0.04513	0.0454
1.75	0.04530	0.0456
2	0.04547	0.0457
2.25	0.04579	0.0461
2.5	0.04611	0.0464
2.75	0.04644	0.0467
3	0.04676	0.0470
3.25	0.04702	0.0473
3.5	0.04727	0.0475
3.75	0.04752	0.0478
4	0.04777	0.0481
4.25	0.04803	0.0483
4.5	0.04829	0.0486
4.75	0.04855	0.0488
5	0.04880	0.0491

D. Contiuously Compounded

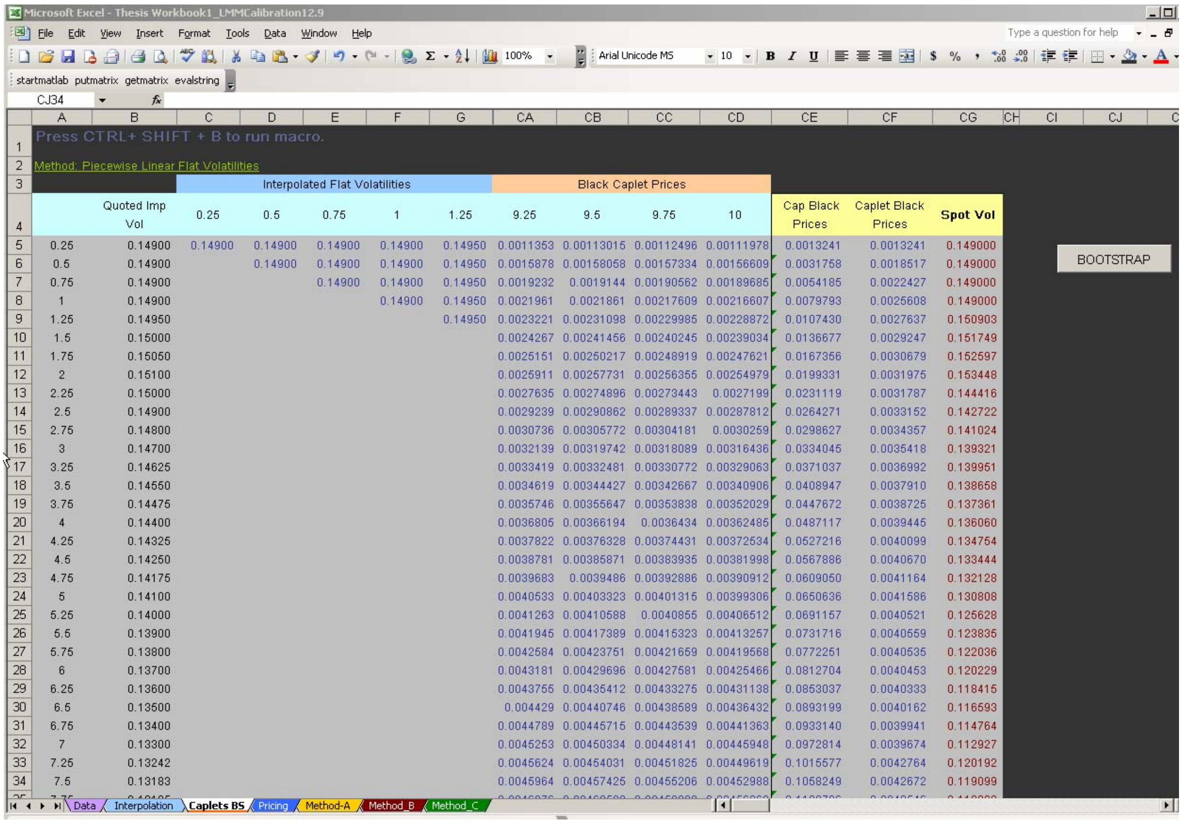
Period	Continuously Compd
0.25	0.04383
0.5	0.04383
0.75	0.04383
1	0.04383
1.25	0.04398
1.5	0.04414
1.75	0.04430
2	0.04446
2.25	0.04477
2.5	0.04508
2.75	0.04539
3	0.04570
3.25	0.04594
3.5	0.04619
3.75	0.04643
4	0.04667
4.25	0.04691
4.5	0.04716
4.75	0.04740
5	0.04765

Data | Interpolation | Caplets BS | Pricing | Method-A | Method_B | Method_C

Steps, cont.

6. On Sheet 2 (Interpolation), activate cell B6 (click on cell B6).
7. Right-click your mouse and select "Paste Special". If "Paste Special" command cannot be found from the drop-down list, go to "Edit" from the menu tab and select "Paste Special".
8. In the "Paste Special" dialogue box, click on "Values" in the set of options under the "Paste" category.
9. Click "OK".
10. Click the "Interpolate" button to run macro.
11. Go to Sheet 3 (Caplet BS) for caplet prices bootstrapping.

Sheet 3: Caplet BS

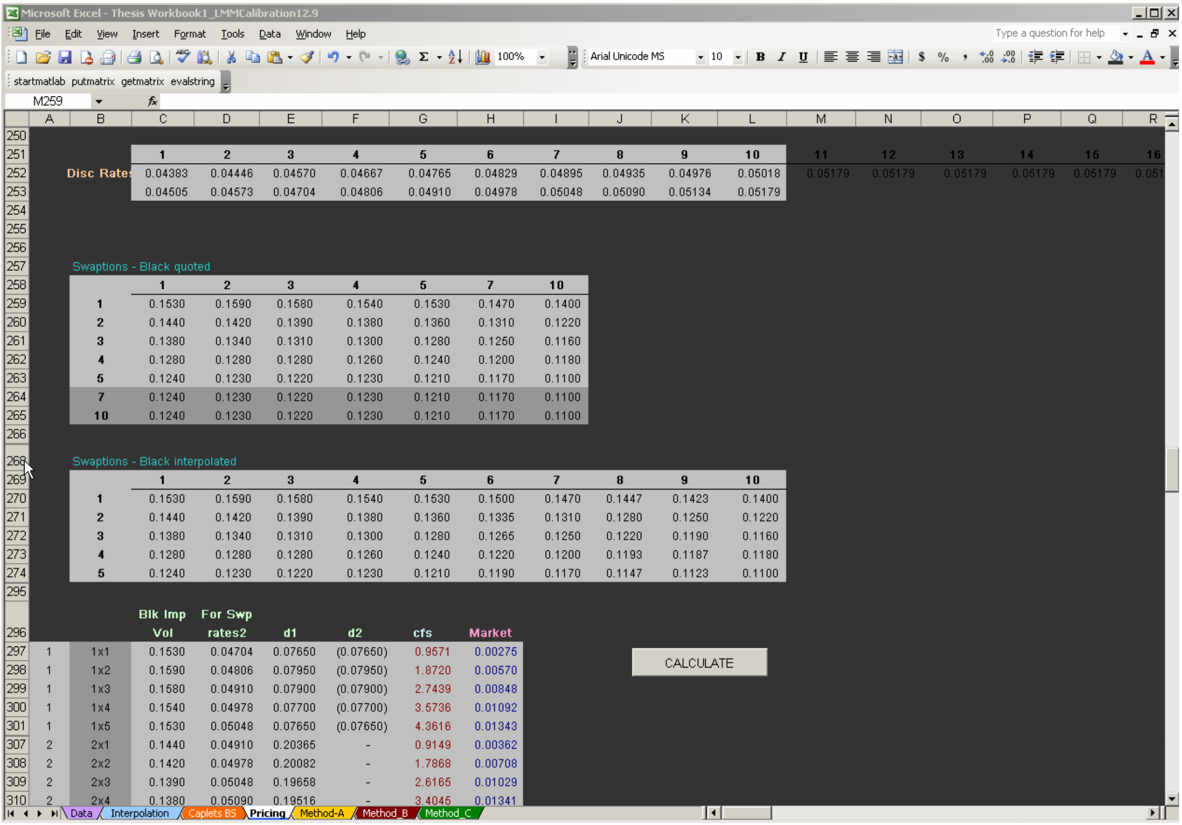


	A	B	C	D	E	F	G	CA	CB	CC	CD	CE	CF	CG
3			Interpolated Flat Volatilities					Black Caplet Prices						
4		Quoted Imp Vol	0.25	0.5	0.75	1	1.25	9.25	9.5	9.75	10	Cap Black Prices	Caplet Black Prices	Spot Vol
5	0.25	0.14900	0.14900	0.14900	0.14900	0.14900	0.14950	0.0011353	0.00113015	0.00112496	0.00111978	0.0013241	0.0013241	0.149000
6	0.5	0.14900		0.14900	0.14900	0.14900	0.14950	0.0015878	0.00158058	0.00157334	0.00156609	0.0031758	0.0018517	0.149000
7	0.75	0.14900			0.14900	0.14900	0.14950	0.0019232	0.0019144	0.00190562	0.00189685	0.0054185	0.0022427	0.149000
8	1	0.14900				0.14900	0.14950	0.0021961	0.0021861	0.00217609	0.00216607	0.0079793	0.0025608	0.149000
9	1.25	0.14950					0.14950	0.0023221	0.00231098	0.00229985	0.00228872	0.0107430	0.0027637	0.150903
10	1.5	0.15000						0.0024267	0.00241456	0.00240245	0.00239034	0.0136677	0.0029247	0.151749
11	1.75	0.15050						0.0025151	0.00250217	0.00248919	0.00247621	0.0167356	0.0030679	0.152597
12	2	0.15100						0.0025911	0.00257731	0.00256355	0.00254979	0.0199331	0.0031975	0.153448
13	2.25	0.15000						0.0027635	0.00274896	0.00273443	0.0027199	0.0231119	0.0031787	0.144416
14	2.5	0.14900						0.0029239	0.00290862	0.00289337	0.00287812	0.0264271	0.0033152	0.142722
15	2.75	0.14800						0.0030736	0.00305772	0.00304181	0.0030259	0.0298627	0.0034357	0.141024
16	3	0.14700						0.0032139	0.00319742	0.00318089	0.00316436	0.0334045	0.0035418	0.139321
17	3.25	0.14625						0.0033419	0.00332481	0.00330772	0.00329063	0.0371037	0.0036992	0.139951
18	3.5	0.14550						0.0034619	0.00344427	0.00342667	0.00340906	0.0408947	0.0037910	0.138658
19	3.75	0.14475						0.0035746	0.00355647	0.00353838	0.00352029	0.0447672	0.0038725	0.137361
20	4	0.14400						0.0036805	0.00366194	0.0036434	0.00362485	0.0487117	0.0039445	0.136060
21	4.25	0.14325						0.0037822	0.00376328	0.00374431	0.00372534	0.0527216	0.0040099	0.134754
22	4.5	0.14250						0.0038781	0.00385871	0.00383935	0.00381998	0.0567886	0.0040670	0.133444
23	4.75	0.14175						0.0039683	0.0039486	0.00392886	0.00390912	0.0609050	0.0041164	0.132128
24	5	0.14100						0.0040533	0.00403323	0.00401315	0.00399306	0.0650636	0.0041586	0.130808
25	5.25	0.14000						0.0041263	0.00410588	0.0040855	0.00406512	0.0691157	0.0040521	0.125628
26	5.5	0.13900						0.0041945	0.00417389	0.00415323	0.00413257	0.0731716	0.0040559	0.123835
27	5.75	0.13800						0.0042584	0.00423751	0.00421659	0.00419568	0.0772251	0.0040535	0.122036
28	6	0.13700						0.0043181	0.00429696	0.00427581	0.00425466	0.0812704	0.0040453	0.120229
29	6.25	0.13600						0.0043755	0.00435412	0.00433275	0.00431138	0.0853037	0.0040333	0.118415
30	6.5	0.13500						0.004429	0.00440746	0.00438589	0.00436432	0.0893199	0.0040162	0.116593
31	6.75	0.13400						0.0044789	0.00445715	0.00443539	0.00441363	0.0933140	0.0039941	0.114764
32	7	0.13300						0.0045253	0.00450334	0.00448141	0.00445948	0.0972814	0.0039674	0.112927
33	7.25	0.13242						0.0045624	0.00454031	0.00451825	0.00449619	0.1015577	0.0042764	0.120192
34	7.5	0.13183						0.0045964	0.00457425	0.00455206	0.00452988	0.1058249	0.0042672	0.119099

Steps:

12. On Sheet 3 (Caplet BS), click on the "Bootstrap" button to run macro.
13. The instantaneous volatilities as implied by caplet prices are shown under column "CG".
14. Go to Sheet 4 (Pricing).

Sheet 4: Pricing



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Disc Rate	0.04383	0.04446	0.04570	0.04667	0.04765	0.04829	0.04895	0.04935	0.04976	0.05018	0.05179	0.05179	0.05179	0.05179	0.05179
	0.04505	0.04573	0.04704	0.04806	0.04910	0.04978	0.05048	0.05090	0.05134	0.05179					

Swaptions - Black quoted

	1	2	3	4	5	7	10
1	0.1530	0.1590	0.1580	0.1540	0.1530	0.1470	0.1400
2	0.1440	0.1420	0.1390	0.1380	0.1360	0.1310	0.1220
3	0.1380	0.1340	0.1310	0.1300	0.1280	0.1250	0.1160
4	0.1280	0.1280	0.1280	0.1260	0.1240	0.1200	0.1180
5	0.1240	0.1230	0.1220	0.1230	0.1210	0.1170	0.1100
7	0.1240	0.1230	0.1220	0.1230	0.1210	0.1170	0.1100
10	0.1240	0.1230	0.1220	0.1230	0.1210	0.1170	0.1100

Swaptions - Black interpolated

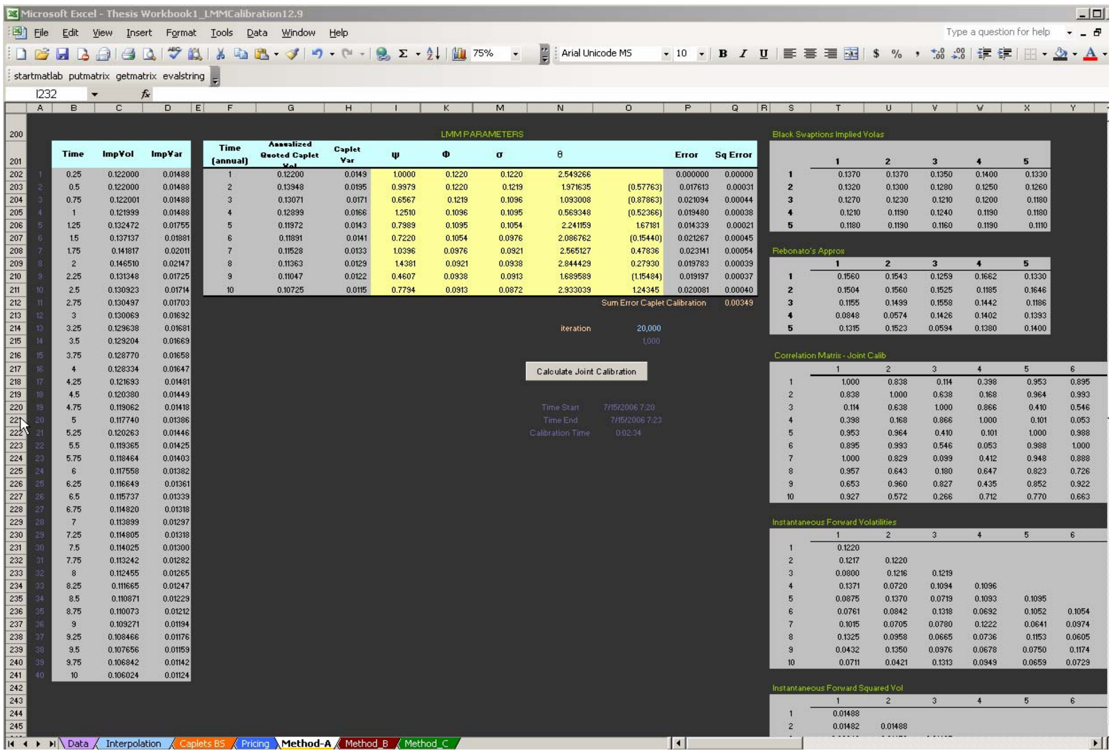
	1	2	3	4	5	6	7	8	9	10
1	0.1530	0.1590	0.1580	0.1540	0.1530	0.1500	0.1470	0.1447	0.1423	0.1400
2	0.1440	0.1420	0.1390	0.1380	0.1360	0.1335	0.1310	0.1280	0.1250	0.1220
3	0.1380	0.1340	0.1310	0.1300	0.1280	0.1265	0.1250	0.1220	0.1190	0.1160
4	0.1280	0.1280	0.1280	0.1260	0.1240	0.1220	0.1200	0.1193	0.1187	0.1180
5	0.1240	0.1230	0.1220	0.1230	0.1210	0.1190	0.1170	0.1147	0.1123	0.1100

		Blk Imp Vol	For Swp rates2	d1	d2	cfs	Market
1	1x1	0.1530	0.04704	0.07650	(0.07650)	0.9571	0.00275
1	1x2	0.1590	0.04806	0.07950	(0.07950)	1.8720	0.00570
1	1x3	0.1580	0.04910	0.07900	(0.07900)	2.7439	0.00848
1	1x4	0.1540	0.04978	0.07700	(0.07700)	3.5736	0.01092
1	1x5	0.1530	0.05048	0.07650	(0.07650)	4.3616	0.01343
2	2x1	0.1440	0.04910	0.20365	-	0.9149	0.00362
2	2x2	0.1420	0.04978	0.20082	-	1.7868	0.00708
2	2x3	0.1390	0.05048	0.19658	-	2.6165	0.01029
2	2x4	0.1380	0.05090	0.19516	-	3.4045	0.01341

Steps:

15. On Sheet 4 (Pricing), click on the "Calculate" button to run macro that converts swaptions implied volatilities to monetary prices.
16. Go to Sheet 5 (Method A) to perform calibration using Method A.

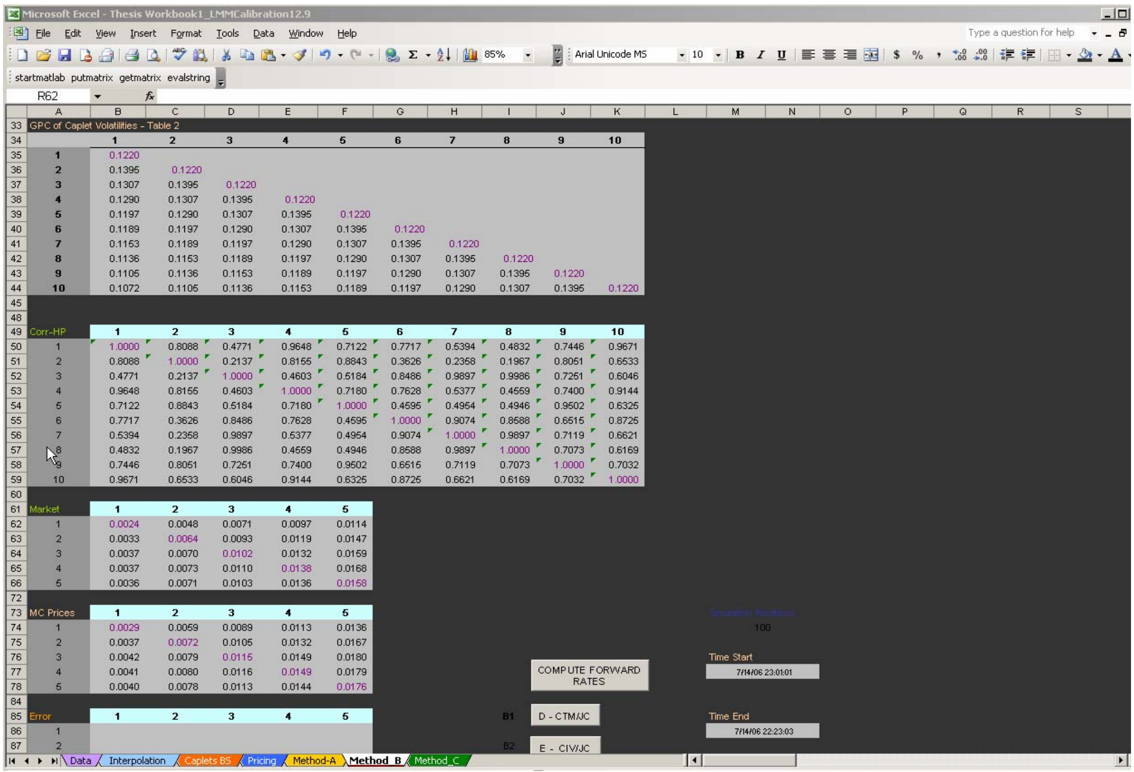
Sheet 5: Method A



Steps:

17. Input the desired number of iterations on cell O214.
18. Click on the “Calculate Joint Calibration” to run macro.
19. The calibrated volatilities are found on cells C231:AC240.
20. The calibrated correlations matrix is found on cells T218:AC227.

Sheet 6: Method B



GPC of Caplet Volatilities - Table 2

	1	2	3	4	5	6	7	8	9	10
1	0.1220									
2	0.1395	0.1220								
3	0.1307	0.1395	0.1220							
4	0.1290	0.1307	0.1395	0.1220						
5	0.1197	0.1290	0.1307	0.1395	0.1220					
6	0.1189	0.1197	0.1290	0.1307	0.1395	0.1220				
7	0.1153	0.1189	0.1197	0.1290	0.1307	0.1395	0.1220			
8	0.1136	0.1153	0.1189	0.1197	0.1290	0.1307	0.1395	0.1220		
9	0.1105	0.1136	0.1153	0.1189	0.1197	0.1290	0.1307	0.1395	0.1220	
10	0.1072	0.1105	0.1136	0.1153	0.1189	0.1197	0.1290	0.1307	0.1395	0.1220

Corr-HP	1	2	3	4	5	6	7	8	9	10
1	1.0000	0.8088	0.4771	0.9648	0.7122	0.7717	0.5394	0.4832	0.7446	0.9671
2	0.8088	1.0000	0.2137	0.8155	0.8843	0.3626	0.2358	0.1967	0.8051	0.6533
3	0.4771	0.2137	1.0000	0.4603	0.5184	0.8486	0.9897	0.9986	0.7251	0.6046
4	0.9648	0.8155	0.4603	1.0000	0.7180	0.7628	0.5377	0.4559	0.7400	0.9144
5	0.7122	0.8843	0.5184	0.7180	1.0000	0.4595	0.4954	0.4946	0.9502	0.6325
6	0.7717	0.3626	0.8486	0.7628	0.4595	1.0000	0.9074	0.8588	0.6515	0.8725
7	0.5394	0.2358	0.9897	0.5377	0.4954	0.9074	1.0000	0.9897	0.7119	0.6621
8	0.4832	0.1967	0.9986	0.4559	0.4946	0.8588	0.9897	1.0000	0.7073	0.6169
9	0.7446	0.8051	0.7251	0.7400	0.9502	0.6515	0.7119	0.7073	1.0000	0.7032
10	0.9671	0.6533	0.6046	0.9144	0.6325	0.8725	0.6621	0.6169	0.7032	1.0000

Market	1	2	3	4	5
1	0.0024	0.0048	0.0071	0.0097	0.0114
2	0.0033	0.0064	0.0093	0.0119	0.0147
3	0.0037	0.0070	0.0102	0.0132	0.0159
4	0.0037	0.0073	0.0110	0.0138	0.0168
5	0.0036	0.0071	0.0103	0.0136	0.0158

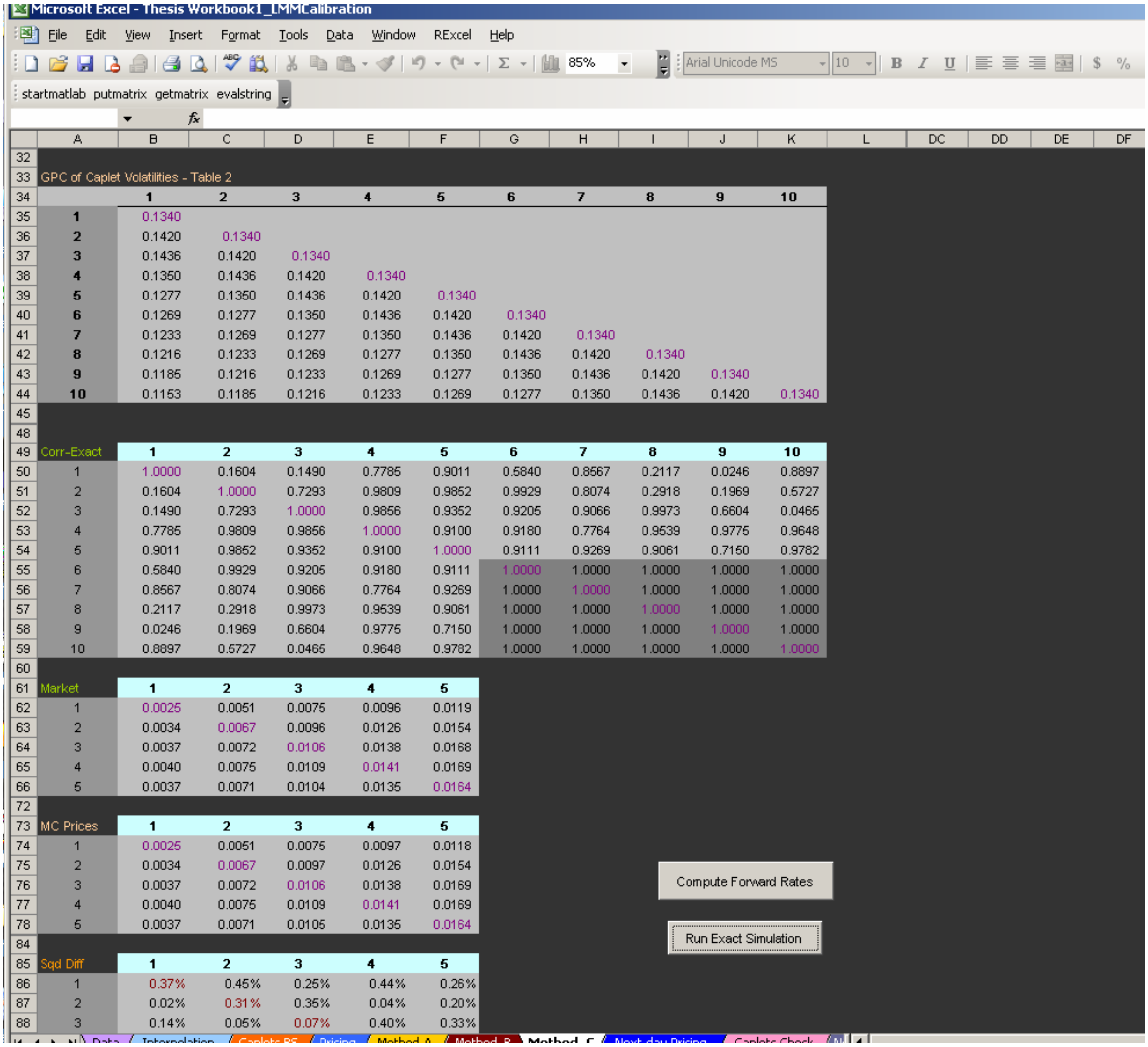
MC Prices	1	2	3	4	5
1	0.0029	0.0059	0.0089	0.0113	0.0136
2	0.0037	0.0072	0.0105	0.0132	0.0167
3	0.0042	0.0079	0.0115	0.0149	0.0180
4	0.0041	0.0080	0.0116	0.0149	0.0179
5	0.0040	0.0078	0.0113	0.0144	0.0176

Error	1	2	3	4	5
1					
2					

Steps:

21. Input the desired optimization iteration on cell M94.
22. Click on the button "Compute Forward Rates" to compute the initial rates.
23. Select the button indicating the desired volatility specification to run macro. This sheet includes the alternatives for Method B as described in Table 5a.
24. The calibrated volatilities are found on cells B35:K44.
25. The calibrated correlation matrix is found on cells B50:K59.

Sheet 7: Method C



GPC of Caplet Volatilities - Table 2

	1	2	3	4	5	6	7	8	9	10
1	0.1340									
2	0.1420	0.1340								
3	0.1436	0.1420	0.1340							
4	0.1350	0.1436	0.1420	0.1340						
5	0.1277	0.1350	0.1436	0.1420	0.1340					
6	0.1269	0.1277	0.1350	0.1436	0.1420	0.1340				
7	0.1233	0.1269	0.1277	0.1350	0.1436	0.1420	0.1340			
8	0.1216	0.1233	0.1269	0.1277	0.1350	0.1436	0.1420	0.1340		
9	0.1185	0.1216	0.1233	0.1269	0.1277	0.1350	0.1436	0.1420	0.1340	
10	0.1153	0.1185	0.1216	0.1233	0.1269	0.1277	0.1350	0.1436	0.1420	0.1340

Corr-Exact	1	2	3	4	5	6	7	8	9	10
1	1.0000	0.1604	0.1490	0.7785	0.9011	0.5840	0.8567	0.2117	0.0246	0.8897
2	0.1604	1.0000	0.7293	0.9809	0.9852	0.9929	0.8074	0.2918	0.1969	0.5727
3	0.1490	0.7293	1.0000	0.9856	0.9352	0.9205	0.9066	0.9973	0.6604	0.0465
4	0.7785	0.9809	0.9856	1.0000	0.9100	0.9180	0.7764	0.9539	0.9775	0.9648
5	0.9011	0.9852	0.9352	0.9100	1.0000	0.9111	0.9269	0.9061	0.7150	0.9782
6	0.5840	0.9929	0.9205	0.9180	0.9111	1.0000	1.0000	1.0000	1.0000	1.0000
7	0.8567	0.8074	0.9066	0.7764	0.9269	1.0000	1.0000	1.0000	1.0000	1.0000
8	0.2117	0.2918	0.9973	0.9539	0.9061	1.0000	1.0000	1.0000	1.0000	1.0000
9	0.0246	0.1969	0.6604	0.9775	0.7150	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.8897	0.5727	0.0465	0.9648	0.9782	1.0000	1.0000	1.0000	1.0000	1.0000

Market	1	2	3	4	5
1	0.0025	0.0051	0.0075	0.0096	0.0119
2	0.0034	0.0067	0.0096	0.0126	0.0154
3	0.0037	0.0072	0.0106	0.0138	0.0168
4	0.0040	0.0075	0.0109	0.0141	0.0169
5	0.0037	0.0071	0.0104	0.0135	0.0164

MC Prices	1	2	3	4	5
1	0.0025	0.0051	0.0075	0.0097	0.0118
2	0.0034	0.0067	0.0097	0.0126	0.0154
3	0.0037	0.0072	0.0106	0.0138	0.0169
4	0.0040	0.0075	0.0109	0.0141	0.0169
5	0.0037	0.0071	0.0105	0.0135	0.0164

Sqd Diff	1	2	3	4	5
1	0.37%	0.45%	0.25%	0.44%	0.26%
2	0.02%	0.31%	0.35%	0.04%	0.20%
3	0.14%	0.05%	0.07%	0.40%	0.33%

Steps:

26. Input the desired optimization iteration on cell M74.
27. Click on the button "Compute Forward Rates" to compute the initial rates.
28. Click on the "Run Exact Simulation" to run macro.
29. The calibrated volatilities are found on cells B35:K44.
30. The calibrated correlation matrix is found on cells B50:K59.

APPENDIX B
Program Codes

Method A

```
Option Explicit
Option Base 1

'*******************************************************************
'  Returns  the  opt  theta  for  a  2-factor  HP  using  Rebonato
      approximation
'  in Joint calibration to caps and swaptions
'*******************************************************************

Sub Reb()

Dim i As Integer, j As Integer
Dim k As Integer, l As Integer, m As Integer
Dim iteration As Integer, n As Long
Dim forwardrates As Variant, ForwardSwaps As Variant
Dim sigma As Variant
Dim rho As Variant
Dim theta As Variant
Dim marketvolas As Variant

forwardrates = Range("AH244:AQ253")
ForwardSwaps = Range("AH231:AQ240")
rho = Range("T218:AC227")
sigma = Range("T231:AC240")
theta = Range("N202:N211")
marketvolas = Range("T202:X206")
iteration = Range("O214")

Range("O220").Value = Now()

Application.ActiveSheet.Calculate

Application.Range("N202:N211").ClearContents
Application.Range("T210:X214").ClearContents
Application.Range("AB202:AF206").ClearContents
Application.ActiveWindow.ScrollColumn = 14

For i = 1 To 5
For j = 1 To 5
ReDim swaptionsvola(i, j)
ReDim Errors(i, j)

Do
```

```
 For m = i To (i + j)

 For n = 1 To iteration

        ReDim LD2(1 To iteration) As Double
        ReDim randoms(1 To iteration)

         randoms(n) = MoroNormSInv(Rnd)

        If 0 <= theta(m, 1) < 3.1416 Then

        theta(m, 1) = 1.5708 + randoms(n)

        Else:
            If theta(m, 1) < 0 Then
                theta(m, 1) = theta(m, 1) + 1.5708
        Else:
                 theta(m, 1) = theta(m, 1) - 1.5708
            End If
        End If

If i < j Then
      For k = i + 1 To j
         For l = 1 To i
             ReDim Integral(i)
             Integral(i) = Integral(i)+sigma(i, l) * sigma(j, l)
         Next l

          swaptionsvola(i,   j)   =   Sqr(forwardrates(i,   k)   *
      forwardrates(j,   k)   *   rho(k,   j)   *   Integral(i)   /
      ForwardSwaps(i, j) ^ 2)
    Errors(i, j) = (Sqr((marketvolas(i, j) - swaptionsvola(i, j)) ^
      2)) / marketvolas(i, j)

      Next k
End If

If i > j Then
For k = j To i - 1
    For l = 1 To j
    ReDim Integral(j)
        Integral(j) = Integral(j)+ sigma(i, l) * sigma(j, l)
    Next l
    swaptionsvola(i, j) = Sqr(forwardrates(i, k) * forwardrates(j,
      k) * rho(k, j) * Integral(j) / ForwardSwaps(j, i) ^ 2)
    Errors(i, j) = (Sqr((marketvolas(i, j) - swaptionsvola(i, j)) ^
      2)) / marketvolas(i, j)

Next k
End If
```

```
If i = j Then
    For l = 1 To j
    ReDim Integral(j)
        Integral(j) = Integral(j) + sigma(i, l) * sigma(j, l)
    Next l
    swaptionsvola(i, j) = Sqr(Integral(j))
    Errors(i, j) = (Sqr((marketvolas(i, j) - swaptionsvola(i, j)) ^
      2)) / marketvolas(i, j)
End If

Application.ActiveSheet.Calculate
Cells(209 + i, 19 + j) = swaptionsvola(i, j)
Cells(201 + i, 27 + j) = Errors(i, j)
Cells(201 + m, 14) = theta(m, 1)
Cells(215, 15) = n

Next n
Next m

Loop While n < iteration

Next j
Next i

Application.ActiveSheet.Calculate
Range("O221").Value = Now()
Application.ActiveSheet.Calculate
MsgBox "Joint Calibration Complete!"

End Sub


'*******************************************************************
'  Returns the equivalent first sequence number of QMC
'  with base = 2
'*******************************************************************

Function Base2(n As Long) As Double

Dim c As Double, ib As Double
Dim i As Long, n1 As Long, n2 As Long

n1 = n
c = 0
ib = 1 / 2
Do While n1 > 0
    n2 = Int(n1 / 2)
    i = n1 - n2 * 2
    c = c + ib * i
    ib = ib / 2
    n1 = n2
```

```
Loop
Base2 = c
End Function

'******************************************************************
'   Returns the equivalent first sequence number used in Halton,
      Faure, Sobol
'  base = b, arbitrary/varible
'******************************************************************

Function Baseb(b As Long, n As Long) As Double
Dim c As Double, ib As Double
Dim i As Long, n1 As Long, n2 As Long

n1 = n
c = 0
ib = 1 / b

Do While n1 > 0
    n2 = Int(n1 / b)
    i = n1 - n2 * b
    c = c + ib * i
    ib = ib / b
    n1 = n2
Loop
Baseb = c
End Function

'******************************************************************
'  Calculates the standard normal numbers given u, the associated
'  uniform number (0,1)
'******************************************************************

Function MoroNormSInv(u As Double) As Double
Dim c1, c2, c3, c4, c5, c6, c7, c8, c9
Dim X As Double
Dim r As Double
Dim a As Variant
Dim b As Variant
a = Array(2.50662823884, -18.61500062529, 41.39119773534, -
      25.44106049637)
b = Array(-8.4735109309, 23.08336743743, -21.06224101826,
      3.13082909833)
c1 = 0.337475482272615
c2 = 0.976169019091719
c3 = 0.160797971491821
c4 = 2.76438810333863E-02
c5 = 3.8405729373609E-03
c6 = 3.951896511919E-04
c7 = 3.21767881768E-05
c8 = 2.888167364E-07
```

```
c9 = 3.960315187E-07
X = u - 0.5

If Abs(X) < 0.42 Then
    r = X ^ 2
    r = X * (((a(4) * r + a(3)) * r + a(2)) * r + a(1)) / ((((b(4)
      * r + b(3)) * r + b(2)) * r + b(1)) * r + 1)
Else
    If X > 0 Then r = Log(-Log(1 - u))
    If X <= 0 Then r = Log(-Log(u))

    r = c1 + r * (c2 + r * (c3 + r * (c4 + r * (c5 + r * (c6 + r *
      (c7 + r * (c8 + r * c9)))))))
    If X <= 0 Then r = -r

End If
MoroNormSInv = r
End Function
```

Method B

```
Option Explicit
Option Base 1

'*******************************************************************
'*Computation of Monte Carlo Simulation swaption prices-B
'*******************************************************************

Sub MC5by5_5factor()

Dim iteration As Long, n As Long, iter As Long, niter As Integer
Dim i As Integer, j As Integer, k As Integer, r As Integer, c As
      Integer
Dim l As Integer, a As Integer, b As Integer, u As Integer, v As
      Integer
Dim InFwd As Variant, sigma As Variant, Theta1 As Variant, Theta2
      As Variant, Expiry As Variant
Dim Tyr As Variant, DiscR As Variant, Interval As Variant, DiscRQ
      As Variant
Dim Strike As Variant, rho As Variant, vectorB As Variant, JCTheta
      As Variant
Dim diff As Variant, TDiff As Double, TPrice As Double, Tfwds As
      Double, Afwds As Double
Dim APrice As Double, ADiff As Double, inc As Double, thetadiff As
      Double
Dim Market As Variant, CountNo As Double

Range("M77").Value = Now()
Expiry = Range("A50:A59")
```

```
sigma = Range("B35:K44")    'assumes constant vol in each expiry-
      maturity time interval (Vol Table2)
InFwd = Range("AA21:AJ30")   'forward rates
Strike = Range("AA21:AJ30") 'quarterly compounded forward swap
      rates
rho = Range("b50:k59")       'using Hypershpere decomposition 3-
      factors
JCTheta = Range("O146:O155")   'from jc theta
Interval = Range("b2:k2")      'annual
Market = Range("B62:F78")
DiscR = Range("AA46:AV46")
DiscRQ = Range("Q2:BD2")
inc = Range("B157")
niter = Range("M94")
thetadiff = Range("B158")

Application.ActiveSheet.Calculate
Application.Range("B74:F78").ClearContents
Application.Range("B86:F90").ClearContents
Application.Range("Q211:Q4786").ClearContents

For u = 1 To 5
For v = 1 To 5

iter = 1

Do
    For a = 1 To 10
    ReDim factor1(a, 1)
    ReDim factor2(a, 1)
    ReDim factor3(a, 1)
    ReDim factor4(a, 1)
    ReDim factor5(a, 1)
    ReDim rho(a, a)
    ReDim Theta1(a, 1)
    ReDim Theta2(a, 1)
    ReDim Theta3(a, 1)
    ReDim Theta4(a, 1)
    ReDim LD2(1 To niter) As Double
    ReDim randoms2(1 To niter)

         LD2(iter) = Base2(iter)
         randoms2(iter) = MoroNormSInv(LD2(iter))

         Cells(210 + iter, 17) = randoms2(iter)

         Theta1(a, 1) = 3.1416 / a + randoms2(iter)
         Theta2(a, 1) = 1.5708 + randoms2(iter)
         Theta3(a, 1) = 3.1416 / a - randoms2(iter)
         Theta4(a, 1) = 1.5708 - Rnd
```

```
        If Abs(Theta1(a, 1)) <= 3.1416 Then
              Theta1(a, 1) = Theta1(a, 1)
         Else:  Theta1(a, 1) = Theta1(a, 1) - 3.1416
        End If

        If Abs(Theta2(a, 1)) <= 3.1416 Then
           Theta2(a, 1) = Theta2(a, 1)
        Else:  Theta2(a, 1) = Theta2(a, 1) - 3.1416
        End If

        If Abs(Theta3(a, 1)) <= 3.1416 Then
            Theta3(a, 1) = Theta3(a, 1)
        Else:  Theta3(a, 1) = Theta3(a, 1) - 3.1416
        End If

       If Abs(Theta4(a, 1) <= 3.1416) Then
           Theta4(a, 1) = Theta4(a, 1)
        Else:  Theta4(a, 1) = Theta4(a, 1) - 3.1416
        End If

        Cells(145 + a, 2) = Theta1(a, 1)
        Cells(145 + a, 3) = Theta2(a, 1)
        Cells(145 + a, 4) = Theta3(a, 1)
        Cells(145 + a, 5) = Theta4(a, 1)

       factor1(a, 1) = Abs(Cos(Theta1(a, 1)))
       factor2(a, 1) = Abs(Cos(Theta2(a, 1)) * Sin(Theta1(a, 1)))
       factor3(a, 1) = Abs(Cos(Theta3(a, 1)) * Sin(Theta1(a, 1)) *
     Sin(Theta2(a, 1)))
       factor4(a, 1) = Abs(Cos(Theta4(a, 1)) * Sin(Theta1(a, 1)) *
     Sin(Theta2(a, 1)) * Sin(Theta3(a, 1)))
       factor5(a, 1) = Abs(Sin(Theta1(a, 1)) * Sin(Theta2(a, 1)) *
     Sin(Theta3(a, 1)) * Sin(Theta4(a, 1)))


       Cells(145 + a, 7) = factor1(a, 1)
       Cells(145 + a, 8) = factor2(a, 1)
       Cells(145 + a, 9) = factor3(a, 1)
       Cells(145 + a, 10) = factor4(a, 1)
       Cells(145 + a, 11) = factor5(a, 1)

   Next a

iteration = Range("m74").Value
ReDim Price(iteration) As Double
ReDim forwards(iteration) As Double
ReDim Payoff(u, v)
ReDim LD(1 To iteration) As Double
ReDim randoms(1 To iteration)
```

```
TPrice = 0
CountNo = 0

For n = 1 To iteration

        randoms(n) = Rnd()
        Cells(210 + n, 3) = randoms(n)

        For l = 1 To v
        ReDim fwd(u, v) As Double
        ReDim summation(v) As Double

        If u < v Then

        For r = u + 1 To u + v

        summation(v) = summation(v) + rho(r, v) * sigma(r, u) *
      InFwd(u, r) * Interval(1, r) / (1 + Interval(1, r) * InFwd(u,
      r))

        Next r

        fwd(u, v) = Exp(Log(InFwd(u, v)) + sigma(u + l, u + l) *
      summation(v) * Interval(1, l) - 0.5 * (sigma(u + l, u + l) ^
      2) * Interval(1, l) + sigma(u + l, u + l) * randoms(n) *
      Sqr(Interval(1, l)))

        Payoff(u, v) = Payoff(u, v) + Exp(-DiscR(1, u + l + 1) * (u
      + l + 1)) * (fwd(u, v) - Strike(u, v))

        End If

        If u > v Then

        For r = v + 1 To u + v

        summation(v) = summation(v) + rho(r, v) * sigma(r, v) *
      InFwd(u, r) * Interval(1, l) / (1 + Interval(1, l) * InFwd(u,
      r))

        Next r

        fwd(u, v) = Exp(Log(InFwd(u, v)) - sigma(u + l, u + l) *
      summation(v) * Interval(1, l) - 0.5 * (sigma(u + l, u + l) ^
      2) * Interval(1, l) + sigma(u + l, u + l) * randoms(n) *
      Sqr(Interval(1, l)))

        Payoff(u, v) = Payoff(u, v) + Exp(-DiscR(1, l) * (l)) *
      (fwd(u, v) - Strike(u, v))

        End If
```

```
        If u = v Then

          fwd(u, v) = Exp(Log(InFwd(u, v)) - 0.5 * (sigma(v, v) ^ 2)
      * Interval(1, l) + sigma(v, v) * randoms(n) * Sqr(Interval(1,
      l)))

          Payoff(u, v) = Payoff(u, v) + Exp(-DiscR(1, l) * (l)) *
      (fwd(u, v) - Strike(u, v))

        End If

          If l = 1 Then
              forwards(n) = fwd(u, v)
              Tfwds = forwards(n)
          End If

          Next l

          Price(n) = Application.WorksheetFunction.Max(Payoff(u, v),
      0)
          TPrice = Price(n)
Next n

          iter = iter + 1
          Cells(93, 13) = iter
          Application.ActiveSheet.Calculate

Loop While Application.WorksheetFunction.And((Abs((Market(u, v) -
      TPrice / n)) / Market(u, v)) > 0.01, iter <= niter)

APrice = TPrice / (iteration)
ADiff = ((Market(u, v) - APrice) / Market(u, v))
Afwds = Tfwds * 10 / iteration
Cells(u + 73, v + 1) = APrice
Cells(u + 85, v + 1) = ADiff
Cells(20 + u, 38 + v) = Afwds

Next v
Next u

Range("M86").Value = Now()
Application.ActiveSheet.Calculate
MsgBox "Simulation Complete"

End Sub
```

Method C

```
Option Explicit
Option Base 1

'*******************************************************************
'*Computation of Monte Carlo Simulation swaption prices - EXACT
'*******************************************************************

Sub MC5by5_Exact()

Dim iteration As Long, n As Long, iter As Long, niter As Long
Dim i As Integer, j As Integer, k As Integer, r As Integer, c As
      Integer
Dim l As Integer, a As Integer, b As Long, u As Integer, v As Long
Dim InFwd As Variant, sigma As Variant, Theta1 As Variant, Theta2
      As Variant, Expiry As Variant
Dim Tyr As Variant, DiscR As Variant, Interval As Variant, DiscRQ
      As Variant
Dim Strike As Variant, rho As Variant, vectorB As Variant, JCTheta
      As Variant, Infwd2 As Variant
Dim diff As Variant, TDiff As Double, TPrice As Double, Tfwds As
      Double, Afwds As Double, TDiff2 As Double
Dim Afwds2 As Double, termcorr As Double, Afwds3 As Double, TDiff3
      As Double, Tfwds2 As Double
Dim APrice As Double, ADiff As Double, inc As Double, thetadiff As
      Double
Dim Market As Variant, CountNo As Double, f As Double

Range("M77").Value = Now()
Expiry = Range("A50:A59")
sigma = Range("B35:K44")     'assumes constant vol in each expiry-
      maturity time interval (Vol Table2)
InFwd = Range("AA21:AJ30")    'forward rates
Infwd2 = Range("B4:K4")       'rates for term corr
Strike = Range("AA21:AJ30")  'quarterly compounded forward swap
      rates
rho = Range("b50:k59")        'using Hypershpere decomposition 3-
      factors
Interval = Range("b2:k2")      'annual
Market = Range("B62:F78")
DiscR = Range("AA46:AV46")
DiscRQ = Range("Q2:BD2")
inc = Range("B157")
niter = Range("M94")
thetadiff = Range("B158")

Application.ActiveSheet.Calculate
Application.Range("P74:Y83").ClearContents
Application.Range("B86:K90").ClearContents
```

```
For u = 1 To 5
For v = 1 To 5

iter = 1

Do
    For a = u To v
    For b = a + 1 To u + v
    ReDim rhos(a, b)
    ReDim LD2(1 To niter) As Double
    ReDim randoms2(1 To niter)

         LD2(Application.WorksheetFunction.Min(iter * b, 30000)) =
      Base2(Application.WorksheetFunction.Min(iter * b, 30000))
         randoms2(Application.WorksheetFunction.Min(iter * b,
      30000)) =
      MoroNormSInv(LD2(Application.WorksheetFunction.Min(iter * b,
      30000)))               'using quasi-monte carlo

         rhos(a,b)= randoms2(Application.WorksheetFunction.Min(iter
      * b, 30000))

Do
    If Abs(rhos(a, b)) > 1 Then
       rhos(a, b) = Abs(rhos(a, b)) - 0.1
    Else:  rhos(a, b) = Abs(rhos(a, b))
    End If
    Cells(49 + a, b + 1) = rhos(a, b)

Loop While Abs(rhos(a, b)) > 1

Next b
Next a

iteration = Range("m74").Value
ReDim Price(iteration) As Double
ReDim forwards(iteration) As Double
ReDim forwards2(iteration) As Double
ReDim FwdsDiff(iteration) As Double
ReDim FwdsDiff2(iteration) As Double
ReDim FwdsDiff3(iteration) As Double
ReDim Payoff(u, v)
ReDim LD(1 To iteration) As Double
ReDim randoms(1 To iteration)

TPrice = 0
CountNo = 0
For n = 1 To iteration

        'LD(N) = Base2(N)
        'randoms(N) = MoroNormSInv(LD(N)  'using quasi-monte carlo
```

```
  randoms(n) = Rnd()

  For l = 1 To v
  ReDim fwd(u, v) As Double
  ReDim summation(v) As Double

  If u < v Then
      For r = u + 1 To u + v
      summation(v) = summation(v) + rho(r, v) * sigma(r, u) *
InFwd(u, r) * Interval(1, r) / (1 + Interval(1, r) * InFwd(u,
r))
      Next r

  fwd(u, v) = Exp(Log(InFwd(u, v)) + sigma(u + l, u + l) *
summation(v) * Interval(1, l) - 0.5 * (sigma(u + l, u + l) ^
2) * Interval(1, l) + sigma(u + l, u + l) * randoms(n) *
Sqr(Interval(1, l)))

  Payoff(u, v) = Payoff(u, v) + Exp(-DiscR(1, u + l + 1) * (u
+ l + 1)) * (fwd(u, v) - Strike(u, v))

  End If

  If u > v Then

  For r = v + 1 To u + v
  summation(v) = summation(v) + rho(r, v) * sigma(r, v) *
InFwd(u, r) * Interval(1, l) / (1 + Interval(1, l) * InFwd(u,
r))
   Next r

  fwd(u, v) = Exp(Log(InFwd(u, v)) - sigma(u + l, u + l) *
summation(v) * Interval(1, l) - 0.5 * (sigma(u + l, u + l) ^
2) * Interval(1, l) + sigma(u + l, u + l) * randoms(n) *
Sqr(Interval(1, l)))

  Payoff(u, v) = Payoff(u, v) + Exp(-DiscR(1, u + l) * (u +
l)) * (fwd(u, v) - Strike(u, v))

  End If

  If u = v Then
  fwd(u, v) = Exp(Log(InFwd(u, v)) - 0.5 * (sigma(v, v) ^ 2)
* Interval(1, l) + sigma(v, v) * randoms(n) * Sqr(Interval(1,
l)))

  Payoff(u, v) = Payoff(u, v) + Exp(-DiscR(1, l) * (l)) *
(fwd(u, v) - Strike(u, v))

 End If
```

```
    Next l

' to compute terminal correlations

        For a = u + 5 To u + 5
        For i = u + 6 To Application.WorksheetFunction.Min(10, u +
      6)
        For b = v + 5 To v + 5
        For j = v + 6 To Application.WorksheetFunction.Min(10, v +
      6)

        For k = 1 To j
        For l = 1 To b
        ReDim fwds(a, b) As Double
        ReDim fwds(i, j) As Double
        ReDim summation(b) As Double
        ReDim summation(j) As Double

        If a < b Then

        For r = a + 1 To Application.WorksheetFunction.Min(10, a +
      b)
         summation(b) = summation(b) + rho(r, b) * sigma(r, a) *
      Infwd2(1, r) * Interval(1, r) / (1 + Interval(1, r) *
      Infwd2(1, r))

        Next r

        fwds(a, b) = Exp(Log(InFwd(1, b)) + sigma(a, a) *
      summation(b) * Interval(1, l) - 0.5 * (sigma(a, a) ^ 2) *
      Interval(1, l) + sigma(a, a) * Rnd() * Sqr(Interval(1, l)))
        f = fwds(a, b)

        End If

        If a > b Then

        For r = b + 1 To Application.WorksheetFunction.Min(10, a +
      b)
        summation(b) = summation(b) + rho(r, b) * sigma(r, b) *
      Infwd2(1, r) * Interval(1, l) / (1 + Interval(1, l) *
      Infwd2(1, r))
         Next r

        fwds(a, b) = Exp(Log(Infwd2(1, b)) - sigma(b, b) *
      summation(b) * Interval(1, l) - 0.5 * (sigma(b, b) ^ 2) *
      Interval(1, l) + sigma(b, b) * randoms(n) * Sqr(Interval(1,
      l)))
        f = fwds(a, b)
        End If
```

```
  If a = b Then
  fwds(a, b) = Exp(Log(Infwd2(1, b)) - 0.5 * (sigma(b, b) ^
2) * Interval(1, l) + sigma(b, b) * Rnd() * Sqr(Interval(1,
l)))
   f = fwds(a, b)
  End If

  Next l

If i < j Then
  For r = i + 1 To Application.WorksheetFunction.Min(10, i +
j)
 summation(j) = summation(j) + rho(r, i) * sigma(r, i) *
Infwd2(1, r) * Interval(1, r) / (1 + Interval(1, r) *
Infwd2(1, r))
 Next r

  fwds(i, j) = Exp(Log(Infwd2(1, j)) + sigma(i, i) *
summation(j) * Interval(1, k) - 0.5 * (sigma(i, i) ^ 2) *
Interval(1, k) + sigma(i, i) * Rnd() * Sqr(Interval(1, k)))

  End If

If i > j Then

  For r = j + 1 To Application.WorksheetFunction.Min(10, i +
j)
  summation(j) = summation(j) + rho(r, j) * sigma(r, j) *
Infwd2(1, r) * Interval(1, k) / (1 + Interval(1, k) *
Infwd2(1, r))
  Next r

  fwds(i, j) = Exp(Log(Infwd2(1, j)) - sigma(j, j) *
summation(j) * Interval(1, k) - 0.5 * (sigma(j, j) ^ 2) *
Interval(1, k) + sigma(j, j) * randoms(n) * Sqr(Interval(1,
k)))
  End If

 f i = j Then
   fwds(i, j) = Exp(Log(Infwd2(1, j)) - 0.5 * (sigma(j, j) ^
2) * Interval(1, k) + sigma(k, k) * randoms(n) *
Sqr(Interval(1, k)))
  End If

  Next k
      forwards(n) = fwds(a, b)
      forwards2(n) = fwds(i, j)
      Tfwds = forwards(n)
      Tfwds2 = forwards2(n)
```

```
            FwdsDiff(n) = (fwds(a, b) - Tfwds / n) * (fwds(i, j) -
      Tfwds2 / n)
            FwdsDiff2(n) = (fwds(a, b) - Tfwds / n) ^ 2
            FwdsDiff3(n) = (fwds(i, j) - Tfwds2 / n) ^ 2
            TDiff3 = FwdsDiff3(n)
            TDiff2 = FwdsDiff2(n)
            TDiff = FwdsDiff(n)

        Next j
      Next b
    Next i
 Next a

'** ends computation of terminal correlations

        Price(n) = Application.WorksheetFunction.Max(Payoff(u, v),
      0)
        TPrice = Price(n)
Next n

iter = iter + 1
Cells(93, 13) = iter

Application.ActiveSheet.Calculate
APrice = TPrice / (iteration)
ADiff = (Abs(Market(u, v) - APrice) / Market(u, v))
Afwds = TDiff / iteration
Afwds2 = TDiff2 / iteration
Afwds3 = TDiff3 / iteration

Loop While ADiff > 0.005 And iter < niter

Cells(u + 73, v + 1) = APrice
Cells(u + 85, v + 1) = ADiff

Cells(55 + u - 1, 7 + u + v - 1) = Afwds / (Sqr(Afwds2) *
      Sqr(Afwds3))

Next v
Next u

Range("M86").Value = Now()
Application.ActiveSheet.Calculate
Application.Range("L55:P59").ClearContents
MsgBox "Simulation Complete"

End Sub
```