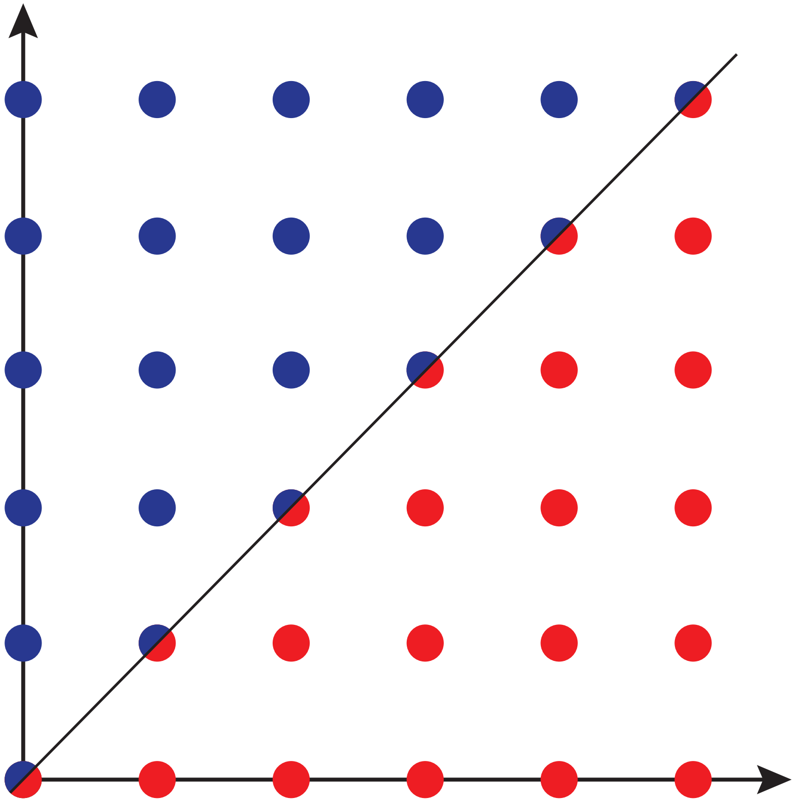
Essays on Mathematical Economics

Uuganbaatar Ninjbat



Essays on Mathematical Economics

Uuganbaatar Ninjbat

Akademisk avhandling

som för avläggande av ekonomie doktorsexamen
vid Handelshögskolan i Stockholm
framläggs för offentlig granskning
tisdagen den 23 oktober 2012, kl 15.15
i Peter Wallenberg Salen, Handelshögskolan,
Sveavägen 65, Stockholm

ESSAYS ON MATHEMATICAL ECONOMICS

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Uuganbaatar Ninjbat

Dissertation for the Degree of Doctor of Philosophy, Ph.D.
Stockholm School of Economics 2012

Keywords: Optimal schedules, FCFS, The Leontief preferences, Approval Voting, Young's theorem, May's theorem, Strategy-proofness, Options sets, Impossibility theorem

ESSAYS ON MATHEMATICAL ECONOMICS

ISBN 978-91-7258-871-4

Printed in Sweden by:
Ineko, Göteborg 2012

Distributed by:
The Research Office
Stockholm School of Economics
Box 6501, SE-113 83 Stockholm, Sweden
www.hhs.se

Preface

This report is a result of a research project carried out at the department of Economics at the Stockholm School of Economics (SSE).

This volume is submitted as a doctor's thesis at SSE. The author has been entirely free to conduct and present his research in his own ways as an expression of his own ideas.

SSE is grateful for the financial support which has made it possible to fulfill the project.

Göran Lindqvist
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Contents

Acknowledgements

During my Ph.D studies at SSE, I have been under academic and administrative support of two persons: Mark Voorneveld and Richard Friberg. The journey started with Mark back in 2009. After it took off with him we did the landing with Richard. Without their support thesis would not be here at this final stage. So, thanks to both of them.

I am also thankful to Magnus Johannesson, who had two important administrative duties during this time: initially as our Ph.D program coordinator, and later as our department chair. In 2009 Jörgen Weibull asked me to be his TA for the Math-1 course. I think that it was a rewarding experience for me and I am thankful to him for giving me this opportunity.

Our administrative secretaries, Ritva Kiviharju, Carin Blanksvärd and Lilian Öberg, were very kind and helpful whenever I knocked their door for different reasons ranging from preparation of a certain official documentation to matters related to the kitchen at the 8'th floor. I am thankful to them for their kindness.

My fellow students were very helpful and willing to share their own experiences whenever I turned to them on more personal matters related to local and regional issues. Especially, Karin and Anna were the most reliable sources when it comes to the arrangements of all such local matters.

Finally, financial support from the Wallander-Hedelius foundation is gratefully acknowledged.

Chapter 1

Introduction

This thesis consists of six single authored research papers. Five out of the six papers are published as journal articles:

1. Optimality of first-come-first-served: a unified approach, *Mongolian Mathematical Journal*, 2011, **15**: 45-53. (included in Chapter 2)

2. An axiomatization of the Leontief preferences, to appear in *Finnish Economic Papers*, 2012, **25** (1). (included in Chapter 3)

3. Remarks on Young's theorem, *Economics Bulletin*, 2012, **32** (1): 706-714. (included in Chapter 4)

4. Approval voting without faithfulness, 2012. (included in Chapter 5)

5. Another direct proof for the Gibbard-Satterthwaite theorem, *Economics Letters*, 2012, **116** (3): 418-421. (included in Chapter 6)

6. Symmetry vs. complexity in proving the Muller-Satterthwaite theorem, *Economics Bulletin*, 2012, **32** (2): 1434-1441. (included in Chapter 7)

The first paper is on queues and schedules. Queueing theory mathematically analyzes certain aspects of congestion situations that often arise in service and manufacturing industries as well as in communication networks. It is a subfield of Operations Research and Industrial and Systems Engineering studies. A related field to queueing theory is scheduling theory, which studies the problem of assigning, over time, a certain number of jobs to a certain number machines so that the jobs get processed by the machines according to this assignment.

The main difference between these two fields is that the former has a stochastic setting, while the latter usually has a deterministic setting. However, the problem of finding the best allocation of jobs (or customers) to machines (or servers) arises in both disciplines. In particular, the main theme of scheduling theory is, under given objectives and constraints, to find the optimal such allocations.

Within this context, among other problems, the problems of investigating the optimality properties of the well known schedules (or queue disciplines), such as first-come-first-served (FCFS), last-come-first-served and shortest-remaining-processing-time, are studied. Such problems are usually solved with the techniques of optimization theory, i.e. Linear, Integer and Dynamic Programming methods.

In our paper, we suggest an alternative approach to investigate optimality properties of FCFS schedule. We demonstrate our approach in a single and identical m parallel machine scheduling settings. Using our approach we verify that FCFS is optimal for the following "bottleneck" problems: *max* of completion time, *max* of flow time and *max* of waiting time. The underlying idea of our approach is *to compare schedules over the last busy period before the system reaches to its peak under FCFS*. Then, the optimality proof pins down to comparison of sums of real numbers and hence, our approach leads to simple, direct and self-contained proofs for the optimality results related to FCFS.

On the other hand, there are two standard approaches used in scheduling literature to solve these problems. The first one uses the "interchange argument," which is a method based on the following reasoning: an optimal schedule is characterized by the property that interchanging two jobs does not improve the value of the objective function. The second method refers to a classic result by Hardy, Littlewood and Polya when solving this problem (see for instance, Section 5.3 in Pinedo, 2008).

The former of these two is perhaps more general. It can be used for solving other scheduling problems, and can also be applied to a stochastic setting. The key aspect of this method is that it is related to a general principle in optimization theory: "optimality can be studied through variation." The latter is, on the other hand, more delicate and it gives the result immediately. It is also related to another common and powerful technique in optimization theory, the Majorization technique (see for instance, Marshall, et al., 2009). In comparison, according to our opinion, our approach is more direct than the other two.

Given that scheduling problems are renowned for their complexities (see Brucker, 2007; Brucker and Knust, 2011), a new approach to investigate such problems can be useful. Moreover, in a computer environment, applying the interchange argument can be very costly as it is based on binary comparisons. In such cases, a more direct approach is preferable. In particular, if the problem is to determine whether FCFS is better than some other given schedule, our approach is certainly more efficient than the others.

The second paper is on choice theory. We give an axiomatic characterization of weak preferences as the Leontief preferences (or the *maximin* preferences). The Leontief preferences frequently appear in economic literature. In consumer choice theory they are the most useful tool to analyze the consumer choice when commodities are complements, i.e. when the value of each commodity positively depends on the relative abundance of the others. In social choice theory they appear in the formalization of Rawlsian theory of justice (see Rawls, 1971).

It is also closely related to the well known *minimax* decision criteria in statistical decision theory. A decision maker whose objective is to minimize the maximum loss (or regret) follows such a decision rule. This rule has received wide attention after the works of A.Wald and L.J.Savage (for a short overview, see Stoye, 2009). On the other hand, the same decision maker, when her objective is to maximize the minimum gain instead, follows the *maximin* decision rule, i.e. she has the *maximin* preferences. Hence, as long as "gain" and "loss" are related, so are these decision rules.

The key axiom in our axiomatization is *upper consistency*, which is based on the following trinary comparison: *given two alternatives, under which condition would a decision maker weakly prefer a third alternative to at least one of the two?* Then, along with two other rather common axioms (symmetry and local non-satiation), specifying such a condition leads us to the *maximin* preferences. Hence, our analysis suggests that, despite of its rich philosophical and decision theoretical content, the *maximin* preferences have a simple structure in terms of preferences.

The last four papers are on voting and social choice theory. One of the main themes of this field is to find out which choice aggregation procedures have which properties. Along this line there are both positive, in the sense of possibility or existence of certain procedures satisfying certain properties, and negative, in the sense of impossibility or non-existence of such procedures, results in this literature. We will consider both kinds of results in our papers (see Chapters 4 to 7).

The third paper is on axiomatization of social choice scoring rules. Scoring rules constitute a broad class of voting methods which share the following common structure: voters rank the alternatives arbitrarily, and each alternative, depending on its position in each voter's ranking, receives a score (a real number), and the alternatives that collect the highest score are chosen. A case in point is the well known *Borda rule*, which is a scoring rule with the scores of $0, 1, ..., m-1$, where m is the number of alternatives, and the i'th best alternative receives $m-i$ score.

In a variable electorate setting, i.e. when the number of voters is not fixed, Smith (1973) and Young (1975) axiomatized scoring rules with four axioms: anonymity, neutrality, continuity and consistency. The first two axioms represent the idea that the voters and the alternatives are to be treated equally. However, the key axioms in their characterization are the last two, continuity and consistency. The *continuity* axiom is a kind of "domination by large numbers" principle and it says that, when a subgroup of voters is replicated sufficiently many times the enlarged group eventually chooses the alternatives which are in the subgroup's choice. The *consistency* axiom says that, when there are two disjoint groups whose choices overlap or agree to some extend, and when these groups come together, the new group's choice must be those alternatives which both groups had in common in their choices (for more discussions, see Chapter 9 in Moulin, 1988).

In our paper, we provide some additional results related to the axiomatization of scoring rules. We first show that when there are two alternatives, the continuity axiom in the above characterization is unnecessary. We then provide a rather complete analysis of the two alternative case by showing that,

(a) the smaller set of axioms (i.e. anonymity, neutrality and consistency) characterize scoring rules even if voters are allowed to be indifferent between the alternatives,

(b) one can obtain a variant May's theorem (which gives an axiomatic characterization of majority rule) from the result in (a), and

(c) in each of these results, the axioms of neutrality and cancellation can be used interchangeably.

The fourth paper is on Approval Voting (AV). AV is considered as a simple yet powerful voting method. Under AV, each voter divides alternatives into two classes, those she approves and those she does not, and the approved alternatives are casted in her ballot. After collecting all ballots, the most approved alternatives are chosen as the winners.

Among its advantages perhaps the most notable one is that it reduces voters' incentives to vote insincerely, in part, by giving them rich and flexible ballot choices (see Brams and Fishburn, 1978). Moreover, according to Brams and Fishburn (2005), the other advantages of AV are

- it helps to elect the strongest candidate, in the sense of *Condorcet winner*: a candidate who would win against each of the other candidates in a two candidate election under majority rule,

- it reduces negative campaigning,
- it increases voter turnout, and
- it gives minority candidates their proper due.

Fishburn (1978) showed that AV is the only ballot aggregation function satisfying the axioms of neutrality, consistency, cancellation and faithfulness. Notice that the first two axioms already appeared in the axiomatization of scoring rules discussed above. *Cancellation* requires that when all alternatives receive the same number of votes, the ballot aggregation function should choose all of them. *Faithfulness* requires that when group's profile consists of a single voter's ballot, then those alternatives in her ballot are to be chosen at that profile.

In our paper, we investigate the implications of dropping the axiom of faithfulness in Fishburn (1978)'s axiomatization. We show that if one drops it, then there are only three ballot aggregation functions satisfying the remaining axioms (neutrality, consistency and cancellation), namely,

- AV, and
- a function that chooses the least approved alternatives, which we call as Inverse Approval Voting, and
- a function that chooses the whole set of alternatives at all profiles.

Hence, our finding suggests that the primary role of faithfulness in Fishburn (1978)'s axiomatization is to distinguish AV from two other functions, one of which has a trivial structure, while the other has a similar but opposite structure as AV. Finally, it is worth to mention that there is an interesting similarity between our result and Wilson's impossibility theorem (see Wilson, 1972), which is obtained as a consequence of dropping the Pareto axiom in Arrow's impossibility theorem.

The last two papers are on social choice impossibility theorems. Impossibility theorems are special kind of axiomatization results which show that a set of (desirable) axioms are incompatible when they put together. The most well known example of such result is Arrow's Impossibility Theorem, which states that, when there are at least three alternatives, there is no social welfare function satisfying the axioms of independence of irrelevant alternatives, Pareto efficiency and non-dictatorship. The other well known impossibility theorems are the Gibbard-Satterthwaite theorem and the Muller-Satterthwaite theorem. The former of the two shows that there is no social choice function satisfying the axioms of unanimity, strategy-proofness (non-manipulability) and non-dictatorship, while the latter is a similar result obtained by replacing strategy-proofness with (Maskin) monotonicity.

These theorems received much attention since their initial appearances. For example, several extensions and generalizations are obtained, their interconnections are analyzed, as well as several ways of proving these kind of theorems are discovered (for a survey see, Campbell and Kelly 2002; Barberà, 2011). However, they are often mentioned as "paradoxes," which indicates that more work needs to be done on these results. Indeed, some authors already suggested that any amount of mystery in these results needs to be "demystified" (see Saari, 2008).

In the literature, impossibility theorems are usually presented as characterization theorems of the following type: a social choice function satisfies all but the last axiom (non-dictatorship) if and only if it is a *dictatorial* social choice function (i.e. a function whose choice coincides with a certain individual's choice at all the time), which is then inconsistent with the remaining axiom. Moreover, a common approach to prove such results works as follows. In order to prove that a function satisfying a set of axioms is dictatorial, first, a subset of the set of individuals containing a *dictator* (a distinguished individual who determines the group's choice) is defined or identified (e.g., decisive coalition, pivotal voter etc.). Then, certain properties of that set are investigated, and eventually it is shown that this set is a singleton. A classic example of such approach is Sen's proof of Arrow's Impossibility Theorem using the so called Field Extension Lemma and Group Contraction Lemma (see Sen, 1986). The other well known cases of this common approach are Barberà's pivotal voters approach (Barberà 1980, 1983), and Geanakoplos' extremely pivotal voters approach (Geanakoplos, 2005).

In the fifth paper we propose an alternative approach to prove this kind of results. We consider the case of the Gibbard-Satterthwaite (impossibility) theorem. The essence of our approach is that, in contrast to the common approach described above, we focus on the individuals who are not candidates for a dictator. More specifically, it consists of two steps. First, we show that when a social choice function is strategy-proof and unanimous, at any given profile, if an individual's most preferred alternative differs from the social choice, then she can not change the social choice at that profile by changing her preferences. Then, in the second step we deduce the Gibbard-Satterthwaite theorem from this result.

We believe that our approach can be seen as a complementary (or dual) approach to the other approaches that focus on the potential candidates of a dictator of a social choice function. Moreover, in our opinion, it provides some additional insights on social choice impossibility results.

The last paper is on the Muller-Satterthwaite (impossibility) theorem. We provide an induction proof of the theorem. We first prove it in the baseline case of two persons and three alternatives. Then, we show that it actually suffices to prove this result for the case of three alternatives with arbitrary N individuals, as it then can easily be extended to the general case of finite but more than three alternatives. We then complete our proof of the Muller-Satterthwaite theorem by showing that the result holds for the decisive case of three alternatives by induction on N.

In our proof, we explicitly use the symmetry property (or the neutrality axiom) which is hidden in the definition of a social choice function. This allows us to see the underlying structure of the Muller-Satterthwaite theorem more clearly. In particular, the monotonicity axiom, which is central in this result, is above all an order theoretical notion: it defines, with respect to each alternative, a *preorder* on the set of preference profiles. In this respect, our paper is a step toward putting the order theoretical aspects of the Muller-Satterthwaite theorem in front, and more work along this direction is a subject to further research.

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Chapter 2

Optimality of First-Come-First-Served: A Unified Approach[1]

Abstract: This paper provides a unified approach that can directly verify the following results related to First-Come-First-Served (FCFS): **(a)** in the case of a single server system, FCFS is optimal for *max* of C (completion time) and *max* of F (flow time), **(b)** in the case of a multi server system with identical servers, when customers have the equal processing time, any optimal discipline for the *total (sum)* of C, F and W (waiting time) has the same service starting times as FCFS, and **(c)** in the latter case, FCFS is optimal for *max* of C, *max* of F and *max* of W.

Keywords: Optimality of FCFS, Optimal non-preemptive queue disciplines, Parallel machine scheduling with equal processing time.

2.1 Introduction

In queueing models, FCFS is often assumed to be the queue discipline. Moreover, FCFS is very common in real life situations such as the grocery stores. This paper aims to (re)investigate certain optimality properties of this commonly used queue discipline. There are several results on the optimality of FCFS in queueing literature (see Gittins [6], Foss [4], Doshi and Lipper [3], Daley [2], Wolf [12], Righter and Shanthikumar [9], Liu and Towsley [8], Foss and Chernova [5]).

[1]Published in *Mongolian Mathematical Journal*, 2011, **15**: 45-53.

Generality of these results necessarily depends on their setting: properties of the queueing system, class of disciplines among which the comparison is made and the optimality criterion that is considered. For instance, Gittins [6] shows that in $(GI/GI/m)$ queues, if the processing times are *i.i.d* across customers, then the *expected waiting time* for a typical customer in steady state is minimum under FCFS among all non-preemptive queue disciplines.[2] More recent works provide rather general results: in $(G/GI/m)$ queues, if the processing times are *i.i.d* across customers, the expected value of any Schur convex function of *customer waiting times* and *total workload after arrival of each customer*, and of any symmetric and convex function of *customer flow times* are minimum under FCFS, among all non-preemptive queue disciplines (see Foss [4], Daley [2], Liu and Towsley [8], Towsley [11], Foss and Chernova [5]).

In this paper we treat the discipline design problem as a scheduling problem. Accordingly, we use performance measures used in scheduling theory to evaluate different queue disciplines: *total* and *max* of *completion time*, *flow time* and *waiting time* (C, F, W), and all of our results can be interpreted in the context of (n/m) parallel machine scheduling problem. Our main findings are as follows. First, in Theorem 2.2 in Section 2.3 we show that, in the case of a single server system, FCFS minimizes *max* of C and *max* of F. We then consider systems with identical customers, i.e. with equal processing times. However, all performance measures that we consider are convex and symmetric (hence, Schur convex) and the equal processing time case is a special case of that being *i.i.d.* But in Theorem 2.3 in Section 2.3 we show that, when all customers have the same processing time, not only FCFS is optimal for the the *sum* of C, F, W, but any optimal schedule has the same service starting times as FCFS. Then, in Theorem 2.5 in Section 2.3 we also show that, in that case FCFS is optimal for *max* of W. Finally, in Theorem 2.7 and 2.8 in Section 2.4, we extend results in Theorem 2.3 and 2.5 in Section 2.3 to the case of a multi server system with identical servers.

Most of our results are known in scheduling literature. For instance, results in Theorem 2.2 are well known in scheduling theory (see Lawler [7], Baker and Trietsch [1]). Results in Theorem 2.3 **(a)** and 2.7 follow from "*optimality of greedy schedules*," whereas results in Theorem 2.5 and 2.8 follow from a more general result in Simmons [10]. However, our proof technique is based on a recursive reasoning and it is different than the other popular techniques: interchange argument, forward and backward induction, majorization

[2] In queueing theory, so called Kendall's notation, $A/B/m$, is often used to describe queueing systems. A describes the arrival time distribution, B describes the service times distribution and the last entry, m describes the number of service channels. G stands for general and GI stands for general independent distributions.

argument and linear and dynamic programming. It is based on a simple observation that "*in order to show optimality of FCFS for max (bottleneck)-problems, it suffices to compare feasible schedules over the last busy period before system reaches its peak under FCFS*," and provides a direct, self-contained and unified approach, used thoroughly in proving Theorem 2.2, 2.5, 2.8 in Section 2.3 and 2.4.

In the next section we introduce our notation and main definitions. In Section 2.3, we consider single server systems and in Section 2.4 we consider multi server systems with identical servers and the last section concludes.

2.2 The set up

Let there be $n \in \mathbb{N}$ customers $J = \{1, 2, ..., n\}$ and a single server. Each customer $i \in J$ has a processing time $p_i \geq 0$ and an arrival time $t_i \geq 0$ and let $p = (p_1, ..., p_n) \in \mathbb{R}^n_+$ and $t = (t_1, ..., t_n) \in \mathbb{R}^n_+$ be the corresponding vectors. Without loss of generality we may assume that $0 \leq t_1 \leq \cdots \leq t_n$. A **schedule** $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$ for a given $(p, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ assigns to each customer $i \in J$ a starting time $s_i \geq 0$ when the server begins to process it. It is **feasible** if processing of a customer does not start before his arrival, $s_i \geq t_i, i \in J$, and the server does not process more than one customer simultaneously, $\forall i, j \in J$ with $i \neq j$, $[s_i, s_i + p_i) \cap [s_j, s_j + p_j) = \emptyset$. **First-come-first-served-schedule** (FCFSS) is a schedule s^* that processes all jobs in the order of their arrival and does so as soon as possible: $s^*_1 = t_1$ and $s^*_i = \max\{s^*_{i-1} + p_{i-1}, t_i\}$ for $i = 2, \ldots, n$. Note that, according to the above definition we only consider permutation schedules without preemption, but we allow the server to stay idle when there are customers available for the service. From now on we only consider feasible schedules and we first verify that the FCFSS is feasible.

Proposition 2.1 *s^* is feasible for any $(p, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$.*

Proof. We can express the feasibility condition above as follows: $s_i \geq t_i, i \in J$, and $\forall i, j \in J$ such that $i \neq j$, if $s_i < s_j$, then $s_i + p_i \leq s_j$. Then, by definition s^* satisfies both of these conditions. □

A queue discipline is defined as a complete contingent plan of schedules. More formally, a **queue discipline** is a mapping $q : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ which assigns to each $(p, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ a feasible schedule s. The **FCFS** is a queue discipline q^* such that $\forall (p, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $q^*(p, t) = s^*$. The following performance measures are commonly used in both queueing and scheduling theory. The **flow time**, the **completion time** and the **waiting time** for customer $i \in J$ under schedule s are the time that he spends in the queueing system, the time that is needed before he gets the service completed and the time that he waits in the queue until he gets the service, respectively. The corresponding formulas are: $F_i(s) = s_i + p_i - t_i$, $C_i(s) = s_i + p_i$ and $W_i(s) = s_i - t_i$. The **sum (total)** and the **maximum** of these measures are defined as usual: $F(s) = \sum_1^n F_i(s)$, $F_{\max}(s) = \max\{F_1(s), ..., F_n(s)\}$; $C(s) = \sum_1^n C_i(s)$, $C_{\max}(s) = \max\{C_1(s), ..., C_n(s)\}$; $W(s) = \sum_1^n W_i(s)$, $W_{\max}(s) = \max\{W_1(s), ..., W_n(s)\}$.

A schedule s is an **optimal schedule** for the performance measure M if there is no other schedule s' such that $M(s') < M(s)$ and a queue **discipline** q **is optimal** for M if $\forall(p,t) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, $q(p,t)$ is optimal for M. Finally, the following permutation defined for any set of finitely many real numbers is very useful in our proofs. Let $\alpha = (\alpha_{k+1}, ..., \alpha_{k+m}) \in \mathbb{R}_+^m$ be a set of m nonnegative real numbers and let $\alpha^\pi = (\alpha_{\pi_1}, ..., \alpha_{\pi_m})$ a permutation of α. Then we call α^π as **ranking** of α if $\alpha_{\pi_i} \leq \alpha_{\pi_{i+1}}$ for $i = 1, ..., m-1$.

2.3 Single server systems

Theorem 2.2 *Consider a single server system and let $(p,t) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ be arbitrary. Then q^* is optimal for $C_{\max}$ and $F_{\max}$.*

Proof. For the first claim, we need to prove that, given $(p,t) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, for any schedule s, $C_{\max}(s) \geq C_{\max}(s^*)$. By definition, $s_i^* = \max\{s_{i-1}^* + p_{i-1}, t_i\}$ for $i = 2, ..., n$, which implies that $s_i^* + p_i \geq s_{i-1}^* + p_{i-1}$, for $i = 2, ..., n$ and hence, $C_{\max}(s^*) = s_n^* + p_n$.

Let us define $j = \max\{i : 1 \leq i \leq n, s_i^* = t_i\}$. So, j is the the last customer in J who gets the service at his arrival under s^*. Note that j is well defined since $s_1^* = t_1$. For an arbitrary schedule s, consider $\alpha = (s_j, ..., s_n)$ and its ranking $\alpha^\pi = (s_{\pi_1}, ..., s_{\pi_{n-j+1}})$. By feasibility, $s_i \geq t_i \geq t_j$ for $i = j, ..., n$, and $s_{\pi_{n-j+1}} + p_{\pi_{n-j+1}} \geq s_{\pi_{n-j}} + p_{\pi_{n-j}} + p_{\pi_{n-j+1}} \geq ... \geq t_j + \sum_{i=j}^n p_i$. By definition, $C_{\max}(s) \geq s_{\pi_{n-j+1}} + p_{\pi_{n-j+1}}$. But since $s_i^* = s_{i-1}^* + p_{i-1}$ for $i = j+1, ..., n$ and $s_j^* = t_j$, $C_{\max}(s^*) = t_j + \sum_{i=j}^n p_i$. Hence, $C_{\max}(s) \geq C_{\max}(s^*)$. This proves the first claim.

For the second claim, we need to prove that, given $(p,t) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, for any schedule s, $F_{\max}(s) \geq F_{\max}(s^*)$. By definition, $F_1(s^*) = p_1$ and for $i = 2, ..., n$, $F_i(s^*) = s_i^* + p_i - t_i = \max\{s_{i-1}^* + p_{i-1}, t_i\} - t_i + p_i = \max\{s_{i-1}^* + p_{i-1} - t_i, 0\} + p_i$. Let $k \in J$ be such that $F_{\max}(s^*) = F_k(s^*)$. Let us define $j = \max\{i : 1 \leq i \leq k, s_i^* = t_i\}$. So, j is the the last customer in $\{1, ..., k\}$ who gets the service at his arrival under s^*. Note that j is well defined since $s_1^* = t_1$. Then by definition, $F_k(s^*) = s_k^* + p_k - t_k = t_j + \sum_{i=j}^k p_i - t_k$.

For an arbitrary schedule $s = (s_1, ..., s_n)$, consider $\alpha = (s_j, ..., s_k)$ and its ranking $\alpha^\pi = (s_{\pi_1}, ..., s_{\pi_{k-j+1}})$. Note that $t_j \leq t_{\pi_1} \leq s_{\pi_1}$ and $t_{\pi_{k-j+1}} \leq t_k$ and by feasibility we conclude that $F_{\pi_{k-j+1}}(s) = s_{\pi_{k-j+1}} + p_{\pi_{k-j+1}} - t_{\pi_{k-j+1}} \geq s_{\pi_{k-j}} + p_{\pi_{k-j}} + p_{\pi_{k-j+1}} - t_{\pi_{k-j+1}} \geq$ $... \geq s_{\pi_1} + \sum_{i=j}^{k} p_i - t_{\pi_{k-j+1}} \geq t_j + \sum_{i=j}^{k} p_i - t_k = F_k(s^*)$. Since $F_{\max}(s) \geq F_{\pi_{k-j+1}}(s)$, this completes the proof. □

Results in Theorem 2.2 show that s^* is always optimal for $C_{\max}$ and $F_{\max}$. The following example shows that for the other performance measures, such a general result does not hold.

Example 2.1 Let $n = 2$ and $t = (0, 1)$ and $p = (10, 1)$. Consider schedule $s = (2, 1)$. Then $F(s) = 13 < F(s^*) = 20$; $C(s) = 14 < C(s^*) = 21$; $W(s) = W_{\max}(s) = 2 < 9 = W(s^*) = W_{\max}(s^*)$. ◁

However, if all customers have the equal processing time, s^* is optimal for all of these performance measures.

Theorem 2.3 *Let all customers have the equal processing time, $p_i = p_0 \in \mathbb{R}_+$, $i \in J$. Then,*

(a) *schedule s is optimal for $F(s)$,$W(s)$and $C(s)$ if and only if $s_{\pi_i} = s_i^*$ for all $i \in J$, where $s^\pi = (s_{\pi_1}, ..., s_{\pi_n})$ is the ranking of s, and*

(b) *s^* is optimal for the performance measures in **(a)**. Moreover, s^* is the unique optimal schedule if and only if $t \in \mathbb{R}_+^n$ is such that $t_i + p_0 < t_{i+2}$, for $i = 1, ..., n-2$.*

Proof. **(a)** Note that the objectives differ by a constant, hence the optimal schedules coincide. Each objective take its minimum value whenever $\sum s_i$ is at its minimum. Let s be an arbitrary schedule.

For the if part, it suffices to show that $\sum s_i^* \leq \sum s_i$. Consider the ranking $s^\pi = (s_{\pi_1}, ..., s_{\pi_n})$ of s. Since s^π is a permutation of s, $\sum s_{\pi_i} = \sum s_i$. Note that by feasibility, $s_{\pi_j} \geq t_j$ for all $j = 1, ..., n$ since there must be at least j customers have arrived in order π_j to be the $j'th$ customer to be served. In particular, $s_{\pi_1} \geq t_1 = s_1^*$. For $2 \leq i \leq n$, if $s_{\pi_{i-1}} \geq s_{i-1}^*$, then it is also true that $s_{\pi_i} \geq s_i^*$ since $s_{\pi_i} \geq s_{\pi_{i-1}} + p_0 \geq s_{i-1}^* + p_0$ and $s_{\pi_i} \geq t_i$. But since $s_{\pi_1} \geq s_1^*$, we conclude that $s_{\pi_i} \geq s_i^*$, $i \in J$. Hence, $\sum s_{\pi_i} \geq \sum s_i^*$.

For the only if part, suppose s is optimal. Then it must be the case that $\sum s_{\pi_i} = \sum s_i^*$ and since it is also true that $s_{\pi_i} \geq s_i^*$, $i \in J$, the equality of the two sums is possible only if each term in the sum is equal. This completes the proof.

(b) The optimality of s^* is trivial from **(a)**. Note that, if s is an optimal schedule, then $s_{\pi_1} = s^*_1 = t_1$, which implies that $\pi_1 = 1$ since it is possible to assign t_1 only to the customer 1. For the if part, suppose $t_1 + p_0 < t_3$. Since $t_2 < t_3$, we conclude that $t_3 > s^*_2$. But for an optimal schedule s, the only customer that can be scheduled at $s_{\pi_2} = s^*_2$ is the customer 2 since no other customer is available at s^*_2. Hence, for any optimal schedule s, $\pi_2 = 2$. Similarly, we conclude that $\pi_i = i$ for $i = 3, ..., n$. Hence, if s is optimal, then $s = s^*$.

For the only if part, let s^* be the only optimal schedule and let there be $j \in \{1, ..., n-2\}$ such that $t_j + p_0 > t_{j+2}$. Consider schedule s such that it agrees with s^* in all positions but $(j+1)'th$ and $(j+2)'th$: $s_{\pi_i} = s^*_i$ for $i \in J$ and $\pi_i = i$ for $i \in J \backslash \{j+1, j+2\}$ and $\pi_{j+1} = j+2$ and $\pi_{j+2} = j+1$. Then by construction s is optimal and $s \neq s^*$, which contradicts to the uniqueness of s^*. $\square$

The theorem above can be interpreted as: *when all customers have the same processing time, the optimal schedule is characterized by the starting times of* s^*. Moreover, from this exact characterization, for any $t \in \mathbb{R}^n_+$, one can fully describe the set of corresponding optimal schedules:

Corollary 2.4 *Let all customers have the equal processing time, $p_i = p_0 \in \mathbb{R}_+$, $i \in J$, and let $t \in \mathbb{R}^n_+$ be the vector of arrival times. Let us define $L_1, ..., L_n$ and $i_1, ..., i_n$ as follows: $L_k = \{i \in \mathbb{N} : 1 \leq i \leq n, s^*_k \geq t_i\} = \{1, 2, ..., i_k\}$. Then,*

(a) *$L_1 = 1, L_n = \{1, ..., n\}$ and $k \leq i_k \leq i_{k+1} \leq n$, for $k = 1, ..., n$, and*

(b) *There are $\varphi = \prod_{k=1}^{n} (i_k - k + 1)$ many distinct optimal schedules and every such schedule can be generated by the following procedure:*

Step 1: *Assign for the first position of the schedule $\pi_1 = 1$ and update L_k into L^1_k by deleting all the first entries of L_k, for $k = 2, ..., n$,*

Step j for $2 \leq j \leq n$**:** *Assign for the $j'th$ position of the schedule any $\pi_j \in L^{j-1}_j$ and update L^{j-1}_k into L^j_k by deleting all the first entries of L^{j-1}_k and replacing all π_j in L^{j-1}_k by the deleted first entry, for $k = j+1, ..., n$.*

Proof. **(a)** By definition $L_1 = 1, L_n = \{1, ..., n\}$ and $k \leq i_k \leq n$, for $k = 1, ..., n$. Note that if $i_k \in L_k$, then $i_k \in L_{k+1}$ for $k = 1, ..., n$ since $s^*_{k+1} > s^*_k \geq t_{i_k}$. Hence, $i_k \leq i_{k+1}$ for $k = 1, ..., n$.

(b) Note that every optimal schedule uniquely corresponds to an assignment

$$\{\pi_1, ..., \pi_n : \pi_k \in L_k, \pi_k \neq \pi_j \text{ for } k \neq j\}$$

and by construction the procedure above gives all such assignments for any given $t \in \mathbb{R}^n_+$. Consider *Step* j: any customer in L_j^{j-1} can be assigned to the $j'th$ position and those are the only possible choices for that position. Note that, by construction there are $(i_j - j + 1)$ elements in L_j^{j-1}. Hence, there are φ distinct optimal schedules. $\square$

Let us demonstrate procedure in Corollary 2.4 with an example:

Let $n = 6$ and $t \in \mathbb{R}^6_+$ and $p_0 \in \mathbb{R}_+$ be such that $L_1 = \{1\}, L_2 = \{1, 2\}, L_i = \{1, 2, 3, 4, 5\}$, for $3 \leq i \leq 5$, and $L_6 = \{1, 2, ..., 6\}$. Let us construct a matrix M with L_i in its $i'th$ row: $M = \begin{bmatrix} 1 & & & & & \\ 1 & 2 & & & & \\ 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$.

In Step 1 we assign $\pi_1 = 1$ and obtain an updated matrix: $M_1 = \begin{bmatrix} \mathbf{1} & & & & & \\ . & 2 & & & & \\ . & 2 & 3 & 4 & 5 & \\ . & 2 & 3 & 4 & 5 & \\ . & 2 & 3 & 4 & 5 & \\ . & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$.

In Step 2 we assign $\pi_2 = 2$ since that is the only feasible assignment and update M_1:

$$M_2 = \begin{bmatrix} \mathbf{1} & & & & & \\ . & \mathbf{2} & & & & \\ . & . & 3 & 4 & 5 & \\ . & . & 3 & 4 & 5 & \\ . & . & 3 & 4 & 5 & \\ . & . & 3 & 4 & 5 & 6 \end{bmatrix}.$$

In Step 3 we may assign any of $\{3, 4, 5\}$ for π_3 and let $\pi_3 = 4$, we update M_2:

$$M_3 = \begin{bmatrix} \mathbf{1} & & & & & \\ . & \mathbf{2} & & & & \\ . & . & 3 & \mathbf{4} & 5 & \\ . & . & . & 3 & 5 & \\ . & . & . & 3 & 5 & \\ . & . & . & 3 & 5 & 6 \end{bmatrix}.$$

In Step 4 we can assign any of $\{3, 5\}$ for π_4 and let $\pi_4 = 5$, we update M_3:

$$M_4 = \begin{bmatrix} \mathbf{1} & & & & & \\ . & \mathbf{2} & & & & \\ . & . & 3 & \mathbf{4} & 5 & \\ . & . & . & . & \mathbf{5} & \\ . & . & . & . & 3 & \\ . & . & . & . & 3 & 6 \end{bmatrix}.$$

Then, for π_5 the only feasible assignment is 3 and for π_6 its 6: $M_6 = \begin{bmatrix} \mathbf{1} & & & & & \\ . & \mathbf{2} & & & & \\ . & . & 3 & \mathbf{4} & 5 & \\ . & . & . & . & \mathbf{5} & \\ . & . & . & . & \mathbf{3} & \\ . & . & . & . & . & \mathbf{6} \end{bmatrix}.$

Theorem 2.5 *When all customers have the equal processing time, $p_i = p_0 \in \mathbb{R}_+, i \in J$, s^* is optimal for $W_{\max}$.*

Proof. By definition, $W_1(s^*) = 0$ and for $i = 2, ..., n$,

$$W_i(s^*) = s_i^* - t_i = \max\{s_{i-1}^* + p_0, t_i\} - t_i = \max\{s_{i-1}^* + p_0 - t_i, 0\}.$$

Let $k \in J$ be such that $W_{\max}(s^*) = W_k(s^*)$. Let us define $j = \max\{i : 1 \leq i \leq k, s_i^* = t_i\}$. So, j is the the last customer in $\{1, ..., k\}$ who gets the service at his arrival under s^*. Note that j is well defined since $s_1^* = t_1$. Then by definition, $W_k(s^*) = s_k^* - t_k = t_j + (k-j) \cdot p_0 - t_k$.

For an arbitrary schedule s, consider $\alpha = (s_j, ..., s_k)$ and its ranking $\alpha^\pi = (s_{\pi_1}, ..., s_{\pi_{k-j+1}})$. Note that $t_j \leq t_{\pi_1} \leq s_{\pi_1}$ and $t_{\pi_{k-j+1}} \leq t_k$ and by feasibility we conclude that $W_{\pi_{k-j+1}}(s) = s_{\pi_{k-j+1}} - t_{\pi_{k-j+1}} \geq s_{\pi_1} + (k-j) \cdot p_0 - t_{\pi_{k-j+1}} \geq t_j + (k-j) \cdot p_0 - t_k = W_k(s^*)$. Since $W_{\max}(s) \geq W_{\pi_{k-j+1}}(s)$, this completes the proof. $\square$

2.4 Multi server systems with equal processing time

In this section we extend the main results in Section 2.3 to a multi server system. Before we state the main results, we shall introduce some more notations and extend some of the main definitions to the new setting.

Let there be $k \in \mathbb{N}$ identical servers, $M = \{m_0, m_1, ..., m_{k-1}\}$. For $i \in J$, let $z(i), y(i) \in \mathbb{Z}_+$ be such that $i = z(i) \cdot k + y(i)$ with $y(i) < k$, i.e. $i \equiv y \mod k$. Each customer has a processing time $p \geq 0$ and let $t = (t_1, ..., t_n) \in \mathbb{R}^n_+$ be the vector of customer arrival times. As before we may assume that $0 \leq t_1 \leq \cdots \leq t_n$. We redefine the notions of **schedule**, **feasibility** and **FCFSS**, and the other notions are defined same as in Section 2.2. A **schedule** $[s] = [(s_1, 1_M), \ldots, (s_n, n_M)]$ for a given $t \in \mathbb{R}^n_+$ assigns to each customer $i \in J$ a pair of (s_i, i_M) where $s_i \geq 0$ is the starting time for i gets the service and $i_M \in M$ is the corresponding server. It is **feasible** if processing of a customer does not start before his arrival: $s_i \geq t_i, \forall i \in J$ and none of the servers processes more than one job simultaneously: $\forall m \in M$, $\forall i, j \in J$ with $i \neq j$, if $i_M = j_M = m$, then $[s_i, s_i + p) \cap [s_j, s_j + p) = \emptyset$. **First-come-first-served-schedule** (FCFSS) is a schedule $[s^*] = [(s^*_1, 1^*_M), \ldots, (s^*_n, n^*_M)]$ that processes all jobs in the order of their arrival and does so as soon as possible: for $1 \leq i < k$, $(s^*_i, i^*_M) = (t_i, m_i)$; for $i = k$, $(s^*_k, k^*_M) = (t_k, m_0)$; and for $k < i \leq n$, $(s^*_i, i^*_M) = (\max\{s^*_{(z(i)-1)\cdot k + y(i)} + p, t_i\}, m_{y(i)})$.

Note that FCFSS can be defined up to an arbitrary assignment of the initial k arrivals to the servers. Here we have chosen a particular one, $i'th$ arrival is assigned to $i'th$ server. Since our enumeration of the servers was arbitrary, none of the results that follow depend on this particular choice. Before we state the main results of this section we prove the following lemma.

Lemma 2.6 *Let $[s]$ be any schedule and $s = (s_1, ..., s_n)$ be the vector of the starting times of $[s]$. Consider $s^\pi = (s_{\pi_1}, ..., s_{\pi_n})$, the ranking of s. For any $(k+1)$ sequence $(s_{\pi_l}, s_{\pi_{l+1}}, ..., s_{\pi_{l+k}})$, $\exists j \in \{0, ..., k-1\}$ such that $s_{\pi_{l+k}} \geq s_{\pi_{l+j}} + p$.*

Proof. Since there are k servers, there are at least two customers π_q, π_i with $q < i$ among $(\pi_l, ..., \pi_{l+k})$ who assigned to the same server, by the pigeonhole principle. Then the result follows by feasibility: $s_{\pi_{l+k}} \geq s_{\pi_i} \geq s_{\pi_q} + p$. □

The following results are extensions of Theorem 2.3 and 2.5, subsequently.

Theorem 2.7 *Schedule $[s]$ is optimal for $F(s)$, $W(s)$ and $C(s)$ if and only if $s_{\pi_i} = s^*_i$ for all $i \in J$ where $s^\pi = (s_{\pi_1}, ..., s_{\pi_n})$ is the ranking of $s = (s_1, ..., s_n)$, the vector of starting times of $[s]$.*

Proof. Note that all of the measures in the theorem take their minimum value whenever $\sum s_i$ is at its minimum. Let $[s]$ be an arbitrary schedule and s be the vector of starting time of $[s]$.

For the if part, it suffices to show that $\sum s_i^* \leq \sum s_i$. Consider the ranking $s^\pi = (s_{\pi_1}, ..., s_{\pi_n})$ of s. Since s^π is a permutation of s, $\sum s_{\pi_i} = \sum s_i$. Note that by feasibility, $s_{\pi_j} \geq t_j$ for all $j \in J$ since there must be at least j customers have arrived in order π_j be the $j'th$ customer to be served. In particular, for $1 \leq i \leq k, s_{\pi_i} \geq s_i^* = t_i$. For $k < i \leq n$, if $s_{\pi_{(z(i)-1)\cdot k+y(i)}} \geq s^*_{(z(i)-1)\cdot k+y(i)}$, then it is also true that $s_{\pi_i} \geq s_i^*$ since $s_{\pi_i} \geq t_i$ and by Lemma 2.6, $s_{\pi_i} \geq s_{\pi_{(z(i)-1)\cdot k+y(i)}} + p \geq s^*_{(z(i)-1)\cdot k+y(i)} + p$. But since $s_{\pi_i} \geq s_i^* = t_i$ for $1 \leq i \leq k$, we conclude that $s_{\pi_j} \geq s_j^*$ for all $j \in J$. Hence, $\sum s_{\pi_i} \geq \sum s_i^*$.

For the only if part, suppose $[s]$ is optimal. Then it must be the case that $\sum s_{\pi_i} = \sum s_i^*$ and since it is also true that $s_{\pi_i} \geq s_i^*$ for all $i \in J$, the equality of the two sums is possible only if each term in the sum is equal. This completes our proof. □

Theorem 2.8 $[s^*]$ *is optimal for* $W_{\max}, C_{\max}$ *and* $F_{\max}$.

Proof. We prove the result for $W_{\max}$ and essentially the same procedure works for $C_{\max}$ and $F_{\max}$. By definition, for $1 \leq i \leq k, W_i(s^*) = 0$ and for $k < i \leq n$, $W_i(s^*) = s_i^* - t_i = \max\{s^*_{(z(i)-1)\cdot k+y(i)} + p, t_i\} - t_i = \max\{s^*_{(z(i)-1)\cdot k+y(i)} + p - t_i, 0\}$. Let $r \in \{1, ..., n\}$ be such that $W_{\max}(s^*) = W_r(s^*)$.

Let us define $j = \max\{i : 1 \leq i \leq r, i = z(i) \cdot k + y(r), s_i^* = t_i\}$. So, j is the the last customer in $\{y(r), k + y(r), 2 \cdot k + y(r), ..., r\}$ (here we identify $0'th$ customer with $k'th$) who gets the service at his arrival under $[s^*]$. Note that j is well defined since $s^*_{y(r)} = t_{y(r)}$.

Then by definition, $W_r(s^*) = s_r^* - t_r = t_j + (z(r) - z(j)) \cdot p - t_r$. For an arbitrary schedule $[s]$ with the vector of starting times s, consider $\alpha = (s_j, s_{j+1}, ..., s_r)$ and its ranking $\alpha^\pi = (s_{\pi_1}, ..., s_{\pi_{r-j+1}})$. Note that $t_j \leq t_{\pi_1} \leq s_{\pi_1}$ and $t_{\pi_{r-j+1}} \leq t_r$ and by Lemma 2.6 we conclude that $W_{\pi_{r-j+1}}(s) = s_{\pi_{r-j+1}} - t_{\pi_{r-j+1}} \geq s_{\pi_1} + (z(r) - z(j)) \cdot p - t_{\pi_{r-j+1}} \geq t_j + (z(r) - z(j)) \cdot p - t_r = W_r(s^*)$. Since $W_{\max}(s) \geq W_{\pi_{r-j+1}}(s)$, this completes our proof. □

2.5 Conclusion

Since FCFS is commonly used both in theory and in applications, its optimality properties receive considerable attention. This paper provides a technique that can be used to investigate optimality properties related to FCFS in a single and (identical) multi server settings. The underlying idea of our technique is to *compare schedules (queue disciplines) over the last busy period before system reaches to its peak under FCFS*. Then, *max* - objectives can be expressed as recursive sums over that period and optimality proofs pin down to simple comparisons of sums of real numbers. Hence, our approach provides simple, unified and self-contained proofs for optimality results related to FCFS.

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Chapter 3

An Axiomatization of the Leontief Preferences [1]

Abstract: An axiomatic characterization of the well known Leontief preferences is given. The key axiom is *upper consistency*, which states that for any two bundles, another bundle is weakly preferred to at least one of them if and only if it is weakly preferred to the bundle that contains the least amount of each commodity in them.

JEL: D01, D11
Keywords: Leontief preferences, Maximin social welfare ordering

3.1 Introduction

In economic literature, it is often assumed that a decision maker, either as an individual or as a group, has the following preferences: alternative $x \in \mathbb{R}^n_+$ is weakly preferred to alternative $y \in \mathbb{R}^n_+$ if and only if $\min_i\{a_i x_i\} \geq \min_i\{a_i y_i\}$, where $x, y \in \mathbb{R}^n_+$ can be either consumption bundles, as in the consumer choice literature, or utility profiles as in the social choice literature. Such preferences are known as the Leontief preferences.

The Leontief preferences are representable by the Leontief utility function, which is one of the standard functional forms used in economics. Moreover, in consumer choice theory, it is the most useful tool to demonstrate the idea of complementarity of economic

[1]To appear in *Finnish Economic Papers*, 2012, **25** (1). I am thankful to Mark Voorneveld for suggesting this research topic, to Tore Ellingsen for pointing out a potential connection between the Leontief preferences and the Leontief production technology and to the editor and an anonymous referee of this journal for their helpful comments.

goods, often attributed to the case of right and left shoes. We do not have any clear reference on its entrance to this field. However, its analogy in production theory, the Leontief production technology used in input-output analysis was developed as early as 1933 by Wassily Leontief (see Dorfman (2008)). The standard reference in this respect is Leontief (1951) (see for instance, Chap. 9 in Dorfman et al., (1958)).

This paper contributes to the economic literature on choice theory by axiomatizing this prototypical preferences. In particular, we give three basic axioms that fully characterize weak preferences on $\mathbb{R}^n_+$ as the Leontief preferences (Theorem 3.1 in Section 3.3). Some of the earlier works related to ours are as follows. Segal and Sobel (2002) provides a joint axiomatization of *min*, *max* and *sum* utility functions, defined respectively as, for all $x \in \mathbb{R}^n$ with $n \geq 3$, $u(x) = \min_i\{x_i\}$, $u(x) = \max_i\{x_i\}$ and $u(x) = \sum x_i$, with five axioms: continuity, monotonicity, symmetry, linearity and partial separability (see Theorem 2 in Segal and Sobel (2002)). However, since they do not directly characterize the Leontief utility function, which corresponds to $u(x) = \min_i\{x_i\}$, one needs additional axiom(s) in order to obtain such characterization from their result. Moreover, they only consider the unweighted *min*, *max* and *sum* utility functions whereas the standard form of the Leontief utility function involves positive weights.

In the social choice literature, the *maximin* social welfare ordering defined on the utility profiles of the society members has the same form as the Leontief preferences. It can be defined for societies with finite or countably infinite members: see for instance Bosmans and Ooghe (2006) and Miyagishima (2010) for the former, and Lauwers (1997) and Chambers (2009) for the latter. The former is more relevant to us since our setting is restricted to a finite dimensional Euclidean space. In that case, Bosmans and Ooghe (2006) characterizes the *maximin* social welfare ordering with four axioms: anonymity, continuity, weak Pareto and Hammond equity, and Miyagishima (2010) shows that one can drop anonymity and modify Hammond equity into a weighted Hammond equity to characterize the *weighted maximin* social welfare ordering. In contrast to these characterizations, we do not use any of the continuity, weak Pareto and Hammond equity, but only use a counterpart of anonymity (see **A.2** in Section 3.2) in our characterization.

The next section introduces the main definitions, Section 3.3 gives the main result, the characterization theorem, and the last section concludes.

3.2 The preliminaries

Define preferences on a set X in terms of a binary relation $\succsim$ ("weakly preferred to") which is:

complete: for all $x, y \in X, x \succsim y, y \succsim x$ or both;

transitive: for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

We call $\succsim$ as a weak order on X. As usual, $x \succ y$ means $x \succsim y$, but not $y \succsim x$, whereas $x \sim y$ means both $x \succsim y$ and $y \succsim x$. For any $x \in X$, let $\mathbf{U}(x) = \{y \in X : y \succsim x\} \subseteq X$ be the set of alternatives that are at least as good as $x \in X$, and $\mathbf{I}(x) = \{y \in X : y \sim x\} \subseteq X$ be the set of alternatives that are indifferent to $x \in X$, according to $\succsim$. From now on we take X as $\mathbb{R}^n_+$. A weak order $\succsim$ on $\mathbb{R}^n_+$ is the Leontief preferences if

$$\forall x, y \in \mathbb{R}^n_+, x \succsim y \Leftrightarrow \min\{x_1, ..., x_n\} \geq \min\{y_1, ..., y_n\},$$

and it is an $\mathbf{a} = (a_1, ..., a_n)$-weighted Leontief preferences with $a_i > 0$ for $i = 1, ..., n$, if

$$\forall x, y \in \mathbb{R}^n_+, x \succsim y \Leftrightarrow \min\{a_1 x_1, ..., a_n x_n\} \geq \min\{a_1 y_1, ..., a_n y_n\}.$$

Note that the former is a special case of the latter with $a_i = 1$, $i = 1, ..., n$. For any $x, y \in \mathbb{R}^n_+$, let $\min\{x, y\} = (\min\{x_1, y_1\}, ..., \min\{x_n, y_n\}) \in \mathbb{R}^n_+$. For any $\varepsilon > 0$, an ε-ball around $x \in \mathbb{R}^n_+$ is $B_\varepsilon(x) = \{y \in \mathbb{R}^n_+ : \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} < \varepsilon\}$. For any positive numbers $a_1, ..., a_n$, let $l(a_1, ..., a_n) \subset \mathbb{R}^n_+$ be a line with

$$l(a_1, ..., a_n) = \{x \in \mathbb{R}^n_+ : a_1 x_1 = ... = a_n x_n\}$$

and when $a_i = 1$ for $i = 1, ..., n$, we write l instead of $l(1, ..., 1)$. Finally, for any $x \in \mathbb{R}^n_+$ and for $i, j = 1, ..., n$, let $\pi_{i,j}(x) \in \mathbb{R}^n_+$ be a vector obtained from $x \in \mathbb{R}^n_+$ by interchanging its $i'th$ and $j'th$ components, and for any $x, y \in \mathbb{R}^n_+$ let $x * y \in \mathbb{R}^n_+$ be defined as
$x * y = (x_1 y_1, ..., x_n y_n)$.

We say that $\succsim$ on $\mathbb{R}^n_+$ is

A.1: *Upper consistent* if $\forall x, y \in \mathbb{R}^n_+, \mathbf{U}(\min\{x, y\}) = \mathbf{U}(x) \cup \mathbf{U}(y)$.

A.2: *Symmetric* (or *Neutral*) with respect to l if whenever $x \sim y$, we have $\pi_{i,j}(x) \sim \pi_{i,j}(y)$, for all $i, j = 1, ..., n$.

A.2': *Symmetric* (or *Neutral*) with respect to $l(a_1, ..., a_n)$ if whenever $x \sim y$, we have

$$(\frac{1}{a_1}, ..., \frac{1}{a_n}) * \pi_{i,j}(a_1x_1, ..., a_nx_n) \sim (\frac{1}{a_1}, ..., \frac{1}{a_n}) * \pi_{i,j}(a_1y_1, ..., a_ny_n)$$

for all $i, j = 1, ..., n$.

A.3: *Locally non-satiable* if $\forall \varepsilon > 0$ and $\forall x \in \mathbb{R}^n_+, \exists y \in B_\varepsilon(x)$ with $y \succ x$.

A.1 says that for any two bundles, another bundle is weakly preferred to at least one of them if and only if it is weakly preferred to the bundle that contains the least amount of each commodity in them. **A.2** is a way of saying that goods are of equal importance, i.e. it does not matter if one exchanges the roles of right and left shoes. More precisely, the indifference relation induced by $\succsim$ is unaffected by renaming of the commodities. $\mathbf{A.2'}$ is a variant of **A.2** after rescaling of coordinates with $\mathbf{a} \in \mathbb{R}^n_{++}$. **A.3** is a standard axiom in microeconomics and it rules out thick indifference curves.

3.3 The characterization theorem

We now state and prove a characterization theorem for $\mathbb{R}^2_+$ since the main idea is best illustrated in that case. However, we remark here that the result holds in the general domain of $\mathbb{R}^n_+$ (see Theorem 3.3 in Appendix 3.5).

Theorem 3.1 *Let $\succsim$ be a weak order on $\mathbb{R}^2_+$. Then,*

(a) *$\succsim$ satisfies $\mathbf{A.1}, \mathbf{A.2}$ and $\mathbf{A.3}$ if and only if it is the Leontief preferences, and*

(b) *$\succsim$ satisfies $\mathbf{A.1}, \mathbf{A.2'}$ and $\mathbf{A.3}$ if and only if it is an (a_1, a_2)-weighted Leontief preferences.*

Proof. Since IF parts are easy to check, we prove ONLY IF parts.

(a) Suppose $\succsim$ satisfies $\mathbf{A.1} - \mathbf{A.3}$. Consider $x^* \in l$, i.e. $x^*_1 = x^*_2$. Let

$$L(x^*) = \{x \in \mathbb{R}^2_+ : x_i \geq x^*_i, i = 1, 2\}.$$

We proceed in 3 steps.

Step 1: We claim that $\mathbf{U}(x^*) = L(x^*)$. First, note that for any $x \in L(x^*)$, $\min\{x^*, x\} = x^*$ and then by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(x^*)$. In particular, $x \in \mathbf{U}(x^*)$. Hence, $L(x^*) \subseteq \mathbf{U}(x^*)$. For the other inclusion, suppose $\exists y \in \mathbf{U}(x^*)$ such that $y \notin L(x^*)$. Since

$y \in \mathbf{U}(x^*)$, by transitivity of $\succsim$, we conclude that $\mathbf{U}(y) \subseteq \mathbf{U}(x^*)$. Consider $x^1 = \min\{y, x^*\}$. Suppose that $x^1 \neq y$. By **A.1** and our last conclusion, $\mathbf{U}(x^1) = \mathbf{U}(x^*)$. Hence, $x^1 \sim x^*$. Let $x^{\alpha_1} = \alpha_1 x^1 + (1-\alpha_1)x^*$, for $\alpha_1 \in [0,1]$. Note that by **A.1**, $\mathbf{U}(x^*) \subseteq \mathbf{U}(x^\alpha) \subseteq \mathbf{U}(x^1)$. But since $\mathbf{U}(x^1) = \mathbf{U}(x^*)$, we conclude that $\mathbf{U}(x^1) = \mathbf{U}(x^{\alpha_1}) = \mathbf{U}(x^*)$, which implies that $x^1 \sim x^{\alpha_1} \sim x^*$ for $\alpha_1 \in [0,1]$. Let $x^2 = \pi_{1,2}(x^1)$ be the symmetric image of $x^1 : x^2 = (x_2^1, x_1^1)$. Then by **A.2**, $x^2 \sim x^{\alpha_2} \sim x^*$ where $x^{\alpha_2} = \alpha_2 x^2 + (1-\alpha_2)x^*$, for $\alpha_2 \in [0,1]$. For any $\alpha_1, \alpha_2 \in [0,1]$, let $x^{\alpha_1,\alpha_2} = \min\{x^{\alpha_1}, x^{\alpha_2}\}$. Then by **A.1**, $x^{\alpha_1,\alpha_2} \sim x^{\alpha_1} \sim x^{\alpha_2} \sim x^*$, which implies that alternatives in a square with vertices at $\{x^1, x^*, x^2, \min\{x^1, x^2\}\}$ (see Fig. 1) are indifferent to each others. But that contradicts **A.3**.

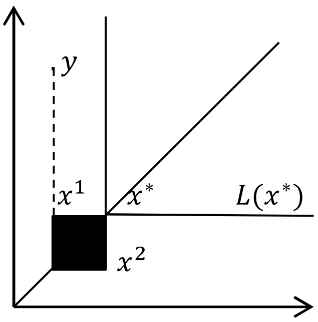

Figure 3.1: $x^1 \neq y$

Let's consider the other case. Suppose $x^1 = y$. Then by repeating the same argument we conclude that, alternatives in a quadrilateral with vertices at $\{y,\ x^*,\ y',\ \min\{y, y'\}\}$ where $y' = (y_2, y_1)$ is the symmetric image of y (see Fig. 2) are indifferent to each others, which contradicts **A.3**. Hence, we conclude that $\nexists y \in \mathbf{U}(x^*)$ such that $y \notin L(x^*)$. So, $\mathbf{U}(x^*) \subseteq L(x^*)$ and the claim is established.

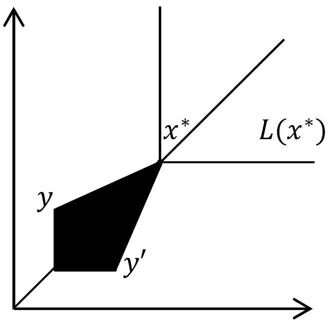

Figure 3.2: $x^1 = y$

Step 2: Let $\partial(L(x^*)) = \{x \in \mathbb{R}_+^2 : x_i \geq x_i^*, i = 1,2, x_j = x_j^* \text{ for some } j = 1,2\}$. We claim that $\mathbf{I}(x^*) = \partial(L(x^*))$. Take any $x \in \partial(L(x^*))$. Note that by **A.1**,

$\mathbf{U}(x) \subseteq \mathbf{U}(x^*)$ since $\min\{x, x^*\} = x^*$. Hence, $x \gtrsim x^*$. Suppose $\exists x \in \partial(L(x^*))$ such that $x \succ x^*$. Consider $x' \in \partial(L(x^*))$ which is the symmetric image of x. Then, by **A.1**, $\mathbf{U}(x^*) = \mathbf{U}(x) \cup \mathbf{U}(x')$, since $x^* = \min\{x, x'\}$. By Step **1**, then $L(x^*) = \mathbf{U}(x) \cup \mathbf{U}(x')$. Since $x^* \notin \mathbf{U}(x)$, the last equality implies that $x^* \in \mathbf{U}(x')$. But since $x' \in \mathbf{U}(x^*) = L(x^*)$, we conclude that $x' \sim x^*$. But that contradicts **A.2**. So $\nexists x \in \partial(L(x^*))$ such that $x \succ x^*$ and we conclude that $\forall x \in \partial(L(x^*)), x^* \gtrsim x$. Hence, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$. For the other inclusion, suppose $\exists x \in \mathbf{I}(x^*)$ such that $x \notin \partial(L(x^*))$. But since by definition, $\mathbf{I}(x^*) \subseteq \mathbf{U}(x^*)$ and by Step **1**, $\mathbf{U}(x^*) = L(x^*)$, it must be the case that $x \in L(x^*)$. Hence, $\min\{x^*, x\} = x^*$. Let $x' \in \partial(L(x^*))$ be such that $x_i = x'_i$ for some $i = 1, 2$, i.e. projection of x into $\partial(L(x^*))$. Note that $x \sim x^* \sim x'$ since the first conclusion is by definition and the second is by the statement just shown, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$, which implies that $\mathbf{U}(x) = \mathbf{U}(x') = \mathbf{U}(x^*)$. Consider $x^{\alpha'} = \alpha' x + (1 - \alpha')x'$ and $x^{\alpha^*} = \alpha^* x + (1 - \alpha^*)x^*$ for $\alpha', \alpha^* \in [0, 1]$. Note that by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(x^{\alpha'}) \subseteq \mathbf{U}(x')$ and $\mathbf{U}(x) \subseteq \mathbf{U}(x^{\alpha^*}) \subseteq \mathbf{U}(x^*)$, which implies that $\mathbf{U}(x) = \mathbf{U}(x^{\alpha'}) = \mathbf{U}(x') = \mathbf{U}(x^{\alpha^*}) = \mathbf{U}(x^*)$, hence $x \sim x^{\alpha'} \sim x' \sim x^{\alpha^*} \sim x^*$ for $\alpha', \alpha^* \in [0, 1]$. For $\alpha', \alpha^* \in [0, 1]$, let $x^{\alpha', \alpha^*} = \min\{x^{\alpha'}, x^{\alpha^*}\}$. Then by **A.1**, $\forall \alpha', \alpha^* \in [0, 1]$, $x^{\alpha', \alpha^*} \sim x^{\alpha'} \sim x^{\alpha^*} \sim x^*$, which implies that alternatives in a triangle with vertices at $\{x, x', x^*\}$ (see Fig. 3) are indifferent to each others. But that contradicts **A.3**. Hence, we conclude that $\nexists x \in \mathbf{I}(x^*)$ such that $x \notin \partial(L(x^*))$ and $\mathbf{I}(x^*) \subseteq \partial(L(x^*))$.

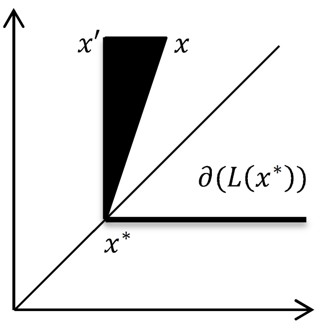

Figure 3.3: $x \in \mathbf{I}(x^*), x \notin \partial(L(x^*))$

Step 3: Suppose $y, z \in \mathbb{R}^2_+$ are such that $y \gtrsim z$ and let $y^*, z^* \in \mathbb{R}^2_+$ be such that

$$y^* = (\min\{y_1, y_2\}, \min\{y_1, y_2\})$$

and

$$z^* = (\min\{z_1, z_2\}, \min\{z_1, z_2\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By Step **2**, $y \sim y^*$ and $z \sim z^*$ which implies $y^* \succsim z^*$. By Step **1**, $y^* \succsim z^* \Leftrightarrow \min\{y_1, y_2\} \geq \min\{z_1, z_2\}$ and hence, $\succsim$ is the Leontief preferences.

(b) Starting with a point $x^* \in l(a_1, a_2)$ and repeating the same arguments as above one can show that (**1**) $\mathbf{U}(x^*) = L(x^*)$ and (**2**) $\mathbf{I}(x^*) = \partial(L(x^*))$. Suppose $y, z \in \mathbb{R}^2_+$ are such that $y \succsim z$ and let $y^*, z^* \in l(a_1, a_2)$ be such that

$$y^* = (\frac{1}{a_1}\min\{a_1y_1, a_2y_2\}, \frac{1}{a_2}\min\{a_1y_1, a_2y_2\})$$

and

$$z^* = (\frac{1}{a_1}\min\{a_1z_1, a_2z_2\}, \frac{1}{a_2}\min\{a_1z_1, a_2z_2\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By (**2**), $y \sim y^*, z \sim z^*$ which implies that $y^* \succsim z^*$. By (**1**), $y^* \succsim z^* \Leftrightarrow \min\{a_1y_1, a_2y_2\} \geq \min\{a_1z_1, a_2z_2\}$ and hence, $\succsim$ is an (a_1, a_2)-weighted Leontief preferences. $\square$

There are two commonly accepted criterion for the validity of an axiomatization result: consistency and logical independence (see also discussions in Chap. 1 in Kreps (1988)). Consistency of **A**.**1**, **A**.**2** (or **A**.**2**$'$) and **A**.**3** is established by the IF part of the characterization theorem above, and their independence can easily be verified:

Example 3.1 A preference satisfying **A**.**1** and **A**.**2** but not **A**.**3** is as follows (the arrow indicates the direction of utility increase):

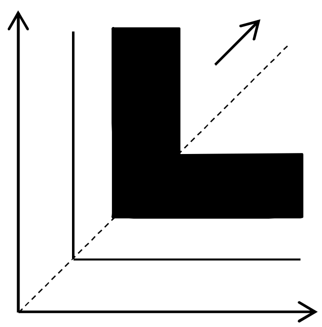

Figure 3.4: **A**.**3** is not satisfied

$\triangleleft$

Example 3.2 A preference satisfying **A**.**1** and **A**.**3** but not **A**.**2** is as follows:

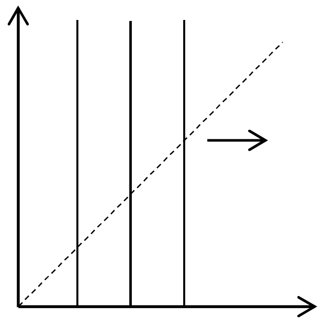

Figure 3.5: **A**.**2** is not satisfied

◁

Example 3.3 A preference satisfying **A**.**2** and **A**.**3** but not **A**.**1** is as follows:

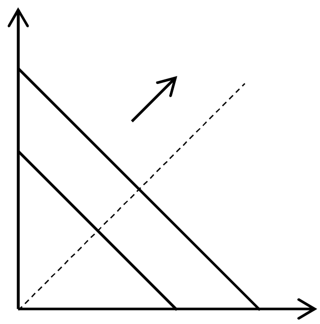

Figure 3.6: **A**.**1** is not satisfied

◁

3.4 Conclusions

This paper axiomatizes weak preferences on $\mathbb{R}^n_+$ as the Leontief preferences with three basic axioms: upper consistency, neutrality (or symmetry) and local non-satiation. Among the axioms, local non-satiation is standard in economic literature while neutrality (or symmetry) is also used, especially in the context of social choice (see for instance, *anonymity* in Lauwers (1997); and *symmetry* in Segal and Sobel (2002)). However, the upper consistency axiom is, to our best knowledge, new to the field.

Then one could ask whether upper consistency is related to the other axioms, especially to those used in the axiomatization of the *maximin* social welfare ordering, mentioned above. In this respect, it can be shown that upper consistency and local non-satiation

together imply monotonicity (or weak Pareto) (see Lemma 3.2 in Appendix **A**). Also, an example of preferences that satisfy upper consistency, but not Hammond equity can be given (see Example 2 in Section 3.3), and it can easily be checked that the following preferences satisfy Hammond equity but not upper consistency:
$\forall x \in \mathbb{R}^2_+, u(x) = -\max\{x_1, x_2\}$.

3.5 Appendix

For $x, y \in \mathbb{R}^n_+$, we write $x >> y$ if $x_i > y_i$ for all $i = 1, ..., n$. We say that $\gtrsim$ on $\mathbb{R}^n_+$ is

A.4: *Monotonic* if whenever $x, y \in \mathbb{R}^n_+$ are such that $x >> y$, we have $x \succ y$.

Lemma 3.2 *If $\gtrsim$ on $\mathbb{R}^n_+$ satisfies* **A.1** *and* **A.3***, then it satisfies* **A.4***.*

Proof. Consider $y \in \mathbb{R}^n_+$ such that $x >> y$. Then, $\min\{x, y\} = y$ and by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(y)$, which implies that $x \gtrsim y$. Suppose $x \sim y$ and consider the following n -dimensional box $\mathbf{B}(x, y) = \{z \in \mathbb{R}^n_+ : x_i \geq z_i \geq y_i, i = 1, ..., n\}$. Then, for any $z \in \mathbf{B}(x, y)$, $\min\{x, z\} = z$ and $\min\{z, y\} = y$ and by **A.1**, $x \gtrsim z$ and $z \gtrsim y$. Since $x \sim y$, we then conclude that $x \sim z \sim y$. But that contradicts **A.3**. Hence, $x \succ y$. $\square$

Theorem 3.3 *Let $\gtrsim$ be a weak order on $\mathbb{R}^n_+$. Then,*

(a) *$\gtrsim$ satisfies* $\mathbf{A.1}, \mathbf{A.2}$ *and* **A.3** *if and only if it is the Leontief preferences, and*

(b) *$\gtrsim$ satisfies* $\mathbf{A.1}, \mathbf{A.2'}$ *and* **A.3** *if and only if it is an* **a***-weighted Leontief preferences.*

Proof. Since IF parts are easy to check, we prove ONLY IF parts. By Lemma 3.2 we may assume that $\gtrsim$ satisfies **A.4**.

(a) Suppose $\gtrsim$ satisfies **A.1** – **A.4**. Consider $x^* \in l$. Let

$$L(x^*) = \{x \in \mathbb{R}^n_+ : x_i \geq x^*_i, i = 1, ..., n\}.$$

We proceed in 3 steps.

Step 1(a): We claim that $\mathbf{U}(x^*) = L(x^*)$. First, note that for any $x \in L(x^*)$, $\min\{x, x^*\} = x^*$. Then by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(x^*)$. In particular, $x \in \mathbf{U}(x^*)$. Hence, $L(x^*) \subseteq \mathbf{U}(x^*)$. For the other inclusion, suppose $\exists y \in \mathbf{U}(x^*)$ such that $y \notin L(x^*)$. Since $y \in \mathbf{U}(x^*)$, by transitivity of $\gtrsim$, we conclude that $\mathbf{U}(y) \subseteq \mathbf{U}(x^*)$. Consider $x = \min\{y, x^*\}$. By

A.**1** and by our last conclusion, $\mathbf{U}(x) = \mathbf{U}(x^*)$ and hence, $x \sim x^*$. Since $y \notin L(x^*)$, it is the case that $x \notin L(x^*)$ and hence $\exists j \in \{1, ..., n\}$ such that $x_j < x_j^*$. For $i = 1, ..., n$, let $x^i = \pi_{i,j}(x)$. Then by **A**.**2**, $x^i \sim x^*$, $i = 1, ..., n$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2}, ..., \min\{x^3, \min\{x^2, x^1\}\}...\}\}\}.$$

Then by repeated use of **A**.**1** and by our conclusion that $x^i \sim x^*$, $i = 1, ..., n$, we conclude that $x^{\min} \sim x^*$. Note that by construction, $x_i^{\min} \leq x_j$, $i = 1, ..., n$ which implies that $x^{\min} << x^*$. But that contradicts **A**.**4**. Hence, we conclude that $\mathbf{U}(x^*) \subseteq L(x^*)$.

Step 2(a): Let $\partial(L(x^*)) = \{x \in \mathbb{R}^n_+ : x_i \geq x_i^*, i = 1, ..., n, x_j = x_j^* \text{ for some } j = 1, ..., n\}$. We claim that $\mathbf{I}(x^*) = \partial(L(x^*))$. Since by definition, $\mathbf{I}(x^*) \subseteq \mathbf{U}(x^*)$, by Step **1(a)** we conclude that $\mathbf{I}(x^*) \subseteq L(x^*)$. Then by **A**.**4**, $\mathbf{I}(x^*) \subseteq \partial(L(x^*))$. For the other inclusion, consider any $x \in \partial(L(x^*))$. Then by Step **1(a)**, $x \gtrsim x^*$. Suppose $\exists x \in \partial(L(x^*))$ such that $x \succ x^*$. Since $x \in \partial(L(x^*))$, $\exists j \in \{1, ..., n\}$ such that $x_j = x_j^*$. For $i = 1, ..., n$, let $x^i = \pi_{i,j}(x)$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2}, ..., \min\{x^3, \min\{x^2, x^1\}\}...\}\}\}.$$

Note that by construction, $x^{\min} = x^*$. Then, by repeated use of **A**.**1** we conclude that $\mathbf{U}(x^*) = \mathbf{U}(x^1) \cup ... \cup \mathbf{U}(x^n)$. Hence, $\exists k \in \{1, ..., n\}$ such that $x^* \in \mathbf{U}(x^k)$, which implies that $x^* \sim x^k$. Then by **A**.**2**, $x^* \sim x^j = x$, which is a contradiction. Hence, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$.

Step 3(a): Suppose $y, z \in \mathbb{R}^n_+$ are such that $y \gtrsim z$ and let $y^*, z^* \in l$ be such that

$$y^* = (\min\{y_1, ..., y_n\}, ..., \min\{y_1, ..., y_n\})$$

and

$$z^* = (\min\{z_1, ..., z_n\}, ..., \min\{z_1, ..., z_n\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By Step **2(a)**, $y \sim y^*$ and $z \sim z^*$ which implies that $y^* \gtrsim z^*$. Then, by Step **1(a)**,

$$y^* \gtrsim z^* \Leftrightarrow \min\{y_1, ..., y_2\} \geq \min\{z_1, ..., z_n\}$$

and hence, $\gtrsim$ is the Leontief preferences.

(b) Consider $x^* \in l(a_1, ..., a_n)$. Note that $\forall i, j = 1, ..., n$,

$$(\frac{1}{a_1}, ..., \frac{1}{a_n}) * \pi_{i,j}(a_1 x_1^*, ..., a_n x_n^*) = x^*$$

i.e. the neutral (symmetric) image of x^* is itself, since $a_i x_i^* = a_j x_j^* \Leftrightarrow \frac{1}{a_i} a_j x_j^* = x_i^*$. We proceed in 3 steps.

Step 1(b): We claim that $\mathbf{U}(x^*) = L(x^*)$. First, by repeating the same argument as in Step **1(a)** we conclude that $L(x^*) \subseteq \mathbf{U}(x^*)$. For the other inclusion, suppose $\exists y \in \mathbf{U}(x^*)$ such that $y \notin L(x^*)$. Since $y \in \mathbf{U}(x^*)$, by transitivity of $\gtrsim$, we conclude that $\mathbf{U}(y) \subseteq \mathbf{U}(x^*)$. Consider $x = \min\{y, x^*\}$. By **A**.**1** and by our last conclusion, $\mathbf{U}(x) = \mathbf{U}(x^*)$ and hence, $x \sim x^*$. Since $y \notin L(x^*)$, $x \notin L(x^*)$ and $\exists j \in \{1, ..., n\}$ such that $x_j < x_j^*$. For $i = 1, ..., n$, let $x^i = (\frac{1}{a_1}, ..., \frac{1}{a_n}) * \pi_{i,j}(a_1 x_1, ..., a_n x_n)$. Then by **A**.**2**$'$, $x^i \sim x^*$, $i = 1, ..., n$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2}, ..., \min\{x^3, \min\{x^2, x^1\}\}...\}\}\}.$$

Then by repeated use of **A**.**1** and by our conclusion that $x^i \sim x^*$, $i = 1, ..., n$, we conclude that $x^{\min} \sim x^*$. Note that by construction, the $i'th$ component of x^i is $x_i^i = \frac{1}{a_i} a_j x_j$, for $i = 1, ..., n$. Then, $x_i^i < x_i^*$, for $i = 1, ..., n$ since $\frac{1}{a_i} a_j x_j < \frac{1}{a_i} a_j x_j^* = \frac{1}{a_i} a_i x_i^* = x_i^*$, which implies that $x^{\min} << x^*$. But that contradicts **A**.**4**. Hence, we conclude that $\mathbf{U}(x^*) \subseteq L(x^*)$.

Step 2(b): We claim that $\mathbf{I}(x^*) = \partial(L(x^*))$. First, by repeating the same argument as in Step **2(a)** we conclude that $\mathbf{I}(x^*) \subseteq L(x^*)$. Then by **A**.**4**, $\mathbf{I}(x^*) \subseteq \partial(L(x^*))$. For the other inclusion, consider any $x \in \partial(L(x^*))$. Then by Step **1(b)**, $x \gtrsim x^*$. Suppose $\exists x \in \partial(L(x^*))$ such that $x \succ x^*$. Since $x \in \partial(L(x^*))$, $\exists j \in \{1, ..., n\}$ such that $x_j = x_j^*$. For $i = 1, ..., n$, let $x^i = (\frac{1}{a_1}, ..., \frac{1}{a_n}) * \pi_{i,j}(a_1 x_1, ..., a_n x_n)$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2}, ..., \min\{x^3, \min\{x^2, x^1\}\}...\}\}\}.$$

Note that by construction, $\forall i = 1, ..., n$, the $i'th$ component of x^i is $x_i^i = \frac{1}{a_i} a_j x_j = \frac{1}{a_i} a_j x_j^* = \frac{1}{a_i} a_i x_i^* = x_i^*$, and when $i \neq j$, $\forall k \in \{1, ..., n\} \backslash \{i\}$, the $i'th$ component of x^k is $x_i^k = x_i \geq x_i^*$ (recall that $x \in \partial(L(x^*))$), and when $i = j$, $\forall k \in \{1, ..., n\} \backslash \{j\}$, the $j'th$ component of x^k is $x_j^k = \frac{1}{a_j} a_k x_k \geq \frac{1}{a_j} a_k x_k^* = \frac{1}{a_j} a_j x_j^* = x_j^*$. This implies that $x^{\min} = x^*$. Then, by repeated use of **A**.**1** we conclude that

$$\mathbf{U}(x^*) = \mathbf{U}(x^1) \cup ... \cup \mathbf{U}(x^n).$$

Hence, $\exists q \in \{1, ..., n\}$ such that $x^* \in \mathbf{U}(x^q)$, which implies that $x^* \sim x^q$. Then by $\mathbf{A.2'}$, $x^* \sim x^j = x$, which is a contradiction. Hence, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$.

Step 3(b): Suppose $y, z \in \mathbb{R}^n_+$ are such that $y \succsim z$ and let $y^*, z^* \in l(a_1, ..., a_n)$ be such that

$$y^* = (\frac{1}{a_1}\min\{a_1y_1, ..., a_ny_n\}, ..., \frac{1}{a_n}\min\{a_1y_1, ..., a_ny_n\})$$

and

$$z^* = (\frac{1}{a_1}\min\{a_1z_1, ..., a_nz_n\}, ..., \frac{1}{a_n}\min\{a_1z_1, ..., a_nz_n\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By Step **2(b)**, $y \sim y^*$ and $z \sim z^*$ which implies that $y^* \succsim z^*$. Then, by Step **1(b)**,

$$y^* \succsim z^* \Leftrightarrow \min\{a_1y_1, ..., a_ny_n\} \geq \min\{a_1z_1, ..., a_nz_n\}$$

and hence, $\succsim$ is an **a**-weighted Leontief preferences.

□

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Chapter 4

Remarks on Young's Theorem [1]

Abstract: In this paper we analyze the simple case of voting over two alternatives with variable electorate. Our main findings are **(a)** the axiom of continuity is redundant in the axiomatization of scoring rules in Young (1975), SIAM J. Appl. Math. 28: 824-838, **(b)** the smaller set of axioms characterize the scoring rules when indifferences are allowed in voter's preferences, **(c)** a version of May's theorem can be derived from this last result and finally, **(d)** in each of these results, the axioms of neutrality and cancellation can be used interchangeably.

JEL: D71, D72
Keywords: Scoring rules, Young's theorem, May's theorem

4.1 Introduction

In this paper we reconsider the problem of axiomatizing scoring rules. Early results on this problem are due to Smith (1973) and Young (1975). They characterized social welfare and social choice functions, respectively, as scoring rules with four basic axioms: anonymity, neutrality, consistency (or separability, or reinforcement) and continuity (or Archimedian, or overwhelming majority). Following them, Myerson (1995) showed that essentially the same set of axioms characterize scoring rules even if some of the assumptions of Smith (1973) and Young (1975) are weakened.

Our objective in this paper is to point out an important detail that has seemingly been ignored in this literature: in the special case of two alternatives, the continuity axiom in

[1]Published in *Economics Bulletin*, 2012, **32** (1): 706-714. I am thankful to the associate editor Jordi Massó and two anonymous referees for their helpful comments.

Young (1975) is redundant in the axiomatization of scoring rules. Hence, our main result is (Theorem 4.3 in Section 4.3.1) "When there are two alternatives, a social choice function is anonymous, neutral and consistent if and only if it is a simple scoring function." We also show that the same result holds, i.e. the smaller set of axioms characterize this voting rule, when indifferences are allowed in the voters' preferences (Theorem 4.5 in Section 4.3.2). Moreover, from this result we derive another result (Theorem 4.6 in Section 4.3.2) that can be seen as a variant of May's theorem in May (1952), and hence establish a formal connection between the two classic results, Young's Theorem and May's Theorem. Finally, we also show that in each of our results, axioms of neutrality and cancellation property can be used interchangeably (Proposition 4.4 and 4.7).

In the next section we introduce our notation and the main definitions. Section 4.3 gives the main results and the last section concludes.

4.2 The preliminaries

Let $\mathbb{R}^n$ denote the set of all n-tuples of real numbers and let $\mathbb{R}^n_+$ be its nonnegative orthant. The notions of weak (and associated indifference relation) and strict preferences over a set B are defined as usual and when $a \in B$ is weakly, strictly preferred and indifferent to $b \in B$, we write $a \succsim b, a \succ b$ and $a \sim b$, respectively. A transposition on set $B = \{a, b\}$, that is a permutation that exchanges the roles of a and b, is denoted as $a \leftrightarrows b$.

Our main setting follows closely that of Young (1975). Let $A = \{a_1, ..., a_m\}$ be the set of alternatives. Let $\mathbb{N}$ denote the set of nonnegative integers which constitute names for the voters and let $\mathbb{P}$ be the set of all preference orders (strict) on A. For any finite $V \subset \mathbb{N}$, a *profile* is a function from V to $\mathbb{P}$ and a *social choice function* (SCF) is a function from set X of all profiles to the family of non-empty subsets of $A, 2^A \backslash \{\emptyset\}$. A SCF is said to be *anonymous* if it depends only on the number of voters associated with each preference order. We can represent the domain of an anonymous SCF by $\mathbb{N}^{m!}$, i.e. the set of all $m!$-tuples with nonnegative integer coordinates, indexed by $\mathbb{P}$, where for any $x \in \mathbb{N}^{m!}$ and any $p \in \mathbb{P}, x_p$ represents the number of voters having preference order p. Let S_m be the group of permutations of the index set $\{1, 2, ..., m\}$. Each $\sigma \in S_m$ induces permutations of the alternatives (which we also denote by σ), and hence profiles, in the natural way. We say that SCF is *neutral* if $f \circ \sigma = \sigma \circ f$ for all $\sigma \in S_m$. An anonymous SCF f is *consistent* if $\forall x', x'' \in \mathbb{N}^{m!}$ such that $f(x') \cap f(x'') \neq \emptyset$, $f(x' + x'') = f(x') \cap f(x'')$, and it is *continuous* if whenever $f(x) = \{a_i\}$, $\forall y \in \mathbb{N}^{m!}$ there is a sufficiently large integer n such that $f(y + n'x) = \{a_i\}$ for $n' \geq n$.

We say that a SCF f has the *cancellation property* if whenever x is a profile such that the number of voters preferring a_i to a_j equals that of preferring a_j to a_i for all pairs $a_i \neq a_j$, then $f(x) = A$. A SCF is a *simple scoring function*, denoted by f^s, if there is a vector $s = (s_1, ..., s_m) \in \mathbb{R}^m$ of scores such that for any given profile, it assigns a score of s_i to each voter's $i'th$ most preferred alternative and chooses the alternative(s) with the highest total score at that profile.[2] A SCF is *trivial* (f^*) if $\forall x \in \mathbb{N}^{m!}, f^*(x) = A$. Note that f^* is a simple scoring function with $s = (0, ..., 0)$.

The following result is known as Young's Theorem (Theorem 1.(ii) in Young, 1975):

Theorem 4.1 *Let f be a SCF. Then, f is anonymous, neutral, consistent and continuous if and only if it is a simple scoring function.*

4.3 The axioms for scoring rules when $m = 2$

4.3.1 Two remarks on Young's theorem

Before we consider the case of $m = 2$, we prove the following Lemma which holds for any finite m.

Lemma 4.2 *Suppose f is an anonymous, neutral and consistent SCF. Then,*

(a) *f is either trivial or it contains all the singletons of $2^A \backslash \{\emptyset\}$ in its range:* $\forall a_i \in A, \exists x \in \mathbb{N}^{m!}$ *such that* $f(x) = \{a_i\}$.

(b) *Let $e = (1, ..., 1) \in \mathbb{N}^{m!}$. Then, $\forall n \in \mathbb{N}$, $f(ne) = A$.*

Proof. (a) Note that when $m = 1$ the result is trivial. Suppose $m \geq 2$. We show that $R(f)$, the range of f, includes at least one singleton $\{a_i\}$. Then, the result follows by neutrality. When $m = 2$, the claim is trivial since $f \neq f^*$ immediately implies that $\exists x \in \mathbb{N}^{2!}$ such that $f(x) = \{a_i\}$ for some $a_i \in \{a_1, a_2\}$. Let $m \geq 3$ and suppose $R(f)$ does not contain any singleton. Then, we claim that it can't have any 2-element sets, 3 -element sets,..., $(m-1)$-element sets. Because if $R(f)$ has a 2-element set $\{a_i, a_j\}$, then by neutrality it has another 2-element set $\{a_i, a_k\}$. Then for $x' \in f^{-1}(\{a_i, a_j\})$ and $x'' \in f^{-1}(\{a_i, a_k\})$, consistency implies that $f(x' + x'') = \{a_i\}$, which is a singleton. Hence, we reach to a contradiction. Similarly, we conclude that $R(f)$ can't have any k-element sets, for $3 \leq k < m$. But then, $f = f^*$.

[2]Note that scoring rules constitute rather general class of voting procedures. In particular, the possibility of assigning lower scores to more preferred alternatives is allowed.

(b) We show that $f(e) = A$. Then the result follows by consistency. Note that $e \in \mathbb{N}^{m!}$ is invariant under all permutations in S_m. Then $f(e)$ must be so by neutrality. But in $2^A \backslash \{\emptyset\}$, the only set with that property is A. □

We remark here that Lemma 4.2 **(b)** is already established in Young (1975). Let us now consider the case of $m = 2$. Our main result is the following:

Theorem 4.3 *Let $m = 2$ and let f be a SCF. Then, f is anonymous, neutral and consistent if and only if it is a simple scoring function.*

Proof. Since the IF part is easy to verify we prove the ONLY IF part. For any $x = (x_1, x_2) \in \mathbb{N}^2 \subset \mathbb{R}^2_+$, let x_1 be the number of voters with preference $p_1 : a_1 \succ a_2$ and let x_2 be that with $p_2 : a_2 \succ a_1$. We shall partition $\mathbb{N}^2 \subset \mathbb{R}^2_+$ as follows:

$$\begin{aligned} D &= \{x \in \mathbb{N}^2 : x_1 = x_2\}; \\ D' &= \{x \in \mathbb{N}^2 : x_1 > x_2\}; \\ D'' &= \{x \in \mathbb{N}^2 : x_1 < x_2\}. \end{aligned}$$

Then by Lemma 4.2 **(b)**, $\forall x \in D, f(x) = \{a_1, a_2\}$. We claim that $\forall x' \in D', f(x') = f(1, 0)$. Suppose $\exists x' \in D'$ such that $f(x') = f(1, 0)$. Then, by consistency

$$f(x' + (1, 0)) = f(x'_1 + 1, x'_2) = f(1, 0)$$

and

$$f(x' + (1, 1)) = f(x'_1 + 1, x'_2 + 1) = f(1, 0).$$

Hence, for any such $x' \in D'$, its two immediate neighbors, one on the right side and one on the upper right side, take the same value. Since $(1, 0) \in D'$, this proves our claim. By neutrality, then $\forall x'' \in D'', f(x'') = f(0, 1)$. Then by Lemma 4.2 **(a)**, f is either f^*, or f_1:

$$f_1(x) = \begin{cases} A \text{ if } x \in D \\ \{a_1\} \text{ if } x \in D' \\ \{a_2\} \text{ if } x \in D'' \end{cases},$$

or f_2:

$$f_2(x) = \begin{cases} A \text{ if } x \in D \\ \{a_2\} \text{ if } x \in D' \\ \{a_1\} \text{ if } x \in D'' \end{cases}.$$

Since f_1 and f_2 correspond to f^s with $s_1 > s_2$ and $s_1 < s_2$ respectively, the proof is completed. $\square$

Let us show that one can use the axioms of neutrality and cancellation property interchangeably in Theorem 4.3.

Proposition 4.4 *Let $m = 2$ and let f be an anonymous and consistent SCF. Then, f has cancellation property if and only if it is neutral.*

Proof. IF: We have shown in Lemma 4.2 **(b)** that anonymity, consistency and neutrality imply that, $\forall x \in D, f(x) = \{a_1, a_2\}$. Since $m = 2$, any profile $x \in \mathbb{N}^2$ is such that the number of voters who prefers $a_1 \succ a_2$ is same as that of who prefers $a_2 \succ a_1$, if and only if $x \in D$. Hence, f satisfies the cancellation property.

ONLY IF: Suppose f satisfies anonymity, consistency and cancellation property, but not neutrality: $\exists y \in \mathbb{N}^2$ such that $f \circ \sigma'(y) \neq \sigma' \circ f(y)$ for some $\sigma' \in S_2$. The only candidate for such $\sigma' \in S_2$ is $\sigma' : a_1 \leftrightarrows a_2$. Note that if one of $f \circ \sigma'(y) \in 2^A \backslash \{\emptyset\}$ and $f(y) \in 2^A \backslash \{\emptyset\}$ is A, then consistency implies that they both must be A, since $\sigma'(y) + y \in D$ and by cancellation property, $f(\sigma'(y) + y) = A$. But then $f \circ \sigma'(y) = \sigma' \circ f(y) = A$ which contradicts to our assumption. Hence, none of $f \circ \sigma'(y) \in 2^A \backslash \{\emptyset\}$ and $f(y) \in 2^A \backslash \{\emptyset\}$ is A. Suppose $f \circ \sigma'(y) = \{a_i\}$ and $f(y) = \{a_j\}$ for $i, j \in \{1, 2\}, i \neq j$. Then $f \circ \sigma'(y) = \sigma' \circ f(y) = \{a_i\}$, which is a contradiction. Hence, the only possibility left is $f \circ \sigma'(y) = f(y) = \{a_i\}$ for some $i \in \{1, 2\}$. Then by consistency $f(\sigma'(y) + y) = \{a_i\}$, which contradicts to cancellation property. This completes our proof. $\square$

In our opinion, redundancy of the continuity axiom in Theorem 4.1 when $m = 2$ is not so obvious until one proves Theorem 4.3. However, one can also verify it directly from Theorem 1.(i) in Young (1975) which states that a SCF is anonymous, neutral and consistent if and only if it is a (*composite*) scoring function. Provided that Theorem 1.(i) is proven, it suffices to notice that when $m = 2$, a *composition* $g = f^{s^2} \circ f^{s^1}$ of two simple scoring functions f^{s^1}, f^{s^2}, defined as $g(x) = f^{s^1}(x)$ if $f^{s^1}(x) \subseteq A$ is a singleton set, otherwise apply f^{s^2} to break the ties in $f^{s^1}(x)$, is a simple scoring function. Note that there are two possibilities: either f^{s^1} is trivial or it is not. Suppose $f^{s^1} = f^*$. Then, since $g = f^{s^2} \circ f^* = f^{s^2}$, g is a simple scoring function. Now suppose $f^{s^1} \neq f^*$. Then, since when $m = 2, f^{s^1}(x)$ produces ties if and only if $x \in D$ and since $f^{s^2}(x) = A$ for $x \in D$, we conclude that $g = f^{s^1}$, hence g is a simple scoring function.

4.3.2 Allowing for indifferences in voters' preferences

Let us now change our initial setting by allowing indifferences in the individual's preferences, hence enlarging the domain of SCF. Sets A, S_m and $\mathbb{N}$ are defined as above in Section 4.2. Let $\mathbb{W}$ be the set off all weak preference orders on A. For any finite $V \subset \mathbb{N}$, an *extended profile* is a function from V to $\mathbb{W}$ and an *extended social choice function* (ESCF) is a function from set Y of all extended profiles to the set $2^A \backslash \{\emptyset\}$. An ESCF is said to be *anonymous* if it depends only on the number of voters associated with each preference order. We can represent the domain of an anonymous ESCF by $\mathbb{N}^{\#\mathbb{W}}$. The notions of neutrality, consistency and cancellation property for an ESCF are defined analogously to that for a SCF.

Since scoring rules are initially defined for profiles with strict preferences in Section 4.2, it needs to be generalized. As we are eventually interested in the case of $m = 2$, we impose a rather weak condition in our generalization: whenever alternatives are indifferent to each others at a given preference they must receive the same score (for a more specific generalization which applies to the case of any finite m, see Vorsatz, 2008). So when $m = 2$, an ESCF is a *simple scoring function*, denoted by F^s, if there is a vector $s = (s_1, s_2, s_3) \in \mathbb{R}^3$ of scores such that for any given profile, it assigns a score of s_i to each voter's $i'th$ strictly most preferred alternative, for $i = 1, 2$, and assigns a score of s_3 to each voter's indifferent alternatives, and chooses the alternative(s) with the highest total score at that profile. An ESCF is *trivial (F^*)* if $\forall x \in \mathbb{N}^{\#\mathbb{W}}, F^*(x) = A$. Note that when $m = 2$, F^* is a simple scoring function with $s = (0, 0, 0)$.

Theorem 4.5 *Let $m = 2$ and let F be an ESCF. Then, F is anonymous, neutral and consistent if and only if it is a simple scoring function.*

Proof. IF: F^s is clearly anonymous since the outcome of F^s depends only on the total scores and that in turn depends only on the number of voters associated with each preference. F^s is neutral since exchanging the roles of a_1 and a_2 is same as exchanging the total scores received by each. F^s is consistent since the total score received by a_i under $x + y \in \mathbb{N}^3$ is the sum of the scores received under each of $x, y \in \mathbb{N}^3$, for any $x, y \in \mathbb{N}^3$ and $i = 1, 2$.

ONLY IF: For any $x = (x_1, x_2, x_3) \in \mathbb{N}^3 \subset \mathbb{R}^3_+$, let x_1, x_2 and x_3 be the number of voters with the preferences $p_1 : a_1 \succ a_2, p_2 : a_2 \succ a_1$ and $p_3 : a_1 \sim a_2$, respectively. We shall partition $\mathbb{N}^3 \subset \mathbb{R}^3_+$ as follows:

$$\begin{aligned} D_3 &= \{x \in \mathbb{N}^3 : x_1 = x_2, x_3 \in \mathbb{N}\}; \\ D'_3 &= \{x \in \mathbb{N}^3 : x_1 > x_2, x_3 \in \mathbb{N}\}; \\ D''_3 &= \{x \in \mathbb{N}^3 : x_1 < x_2, x_3 \in \mathbb{N}\}. \end{aligned}$$

Firstly, note that $\forall x \in D_3, F(x) = \{a_1, a_2\}$ since $x \in D_3$ is invariant under all permutations in S_2 (recall that the indifference relation is symmetric) and by neutrality so must be $F(x)$. But the only set with that property in $\{a_1\}, \{a_2\}$ and $\{a_1, a_2\}$ is the very last one.

Let $P_n \subset \mathbb{N}^3$ be defined as follows: $\forall n \in \mathbb{N}, P_n = \{x \in \mathbb{N}^3 : x_1, x_2 \in \mathbb{N}, x_3 = n\}$. We claim that for $n \geq 1$, if $F(x) = F(1,0,0)$, $\forall x \in P_{n-1} \cap D'_3$, then $F(x) = F(1,0,0)$, $\forall x \in P_n \cap D'_3$. Suppose $y \in P_{n-1} \cap D'_3$ and $F(y) = F(1,0,0)$. Then by consistency, $F(y + (0,0,1)) = F(y) = F(1,0,0)$ since $(0,0,1) \in D_3$ and $F(0,0,1) = A$. But for $n \geq 1, P_n = \{x \in \mathbb{N}^3 : x = y + (0,0,1), y \in P_{n-1}\}$ and this proves our claim.

Recall that we already showed in the proof of Theorem 4.3 that $F(x) = F(1,0,0)$, $\forall x \in P_0 \cap D'_3$. Hence, we conclude that $\forall x \in D'_3, F(x) = F(1,0,0)$, and then by neutrality, $\forall x \in D''_3, F(x) = F(0,1,0)$. To complete the proof, suppose $F \neq F^*$, then F includes all singletons in its range since $F \neq F^*$ implies that $\exists x \in \mathbb{N}^3$ such that $F(x) = \{a_i\}$ for some $a_i \in A$ and hence by neutrality, $\exists x^i \in \mathbb{N}^3$ such that $F(x^i) = \{a_i\}$, for $i = 1, 2$. Combining our last observation with the above conclusions, we have established that F is either F^*, or F_1 :

$$F_1(x) = \begin{cases} A \text{ if } x \in D_3 \\ \{a_1\} \text{ if } x \in D'_3 \\ \{a_2\} \text{ if } x \in D''_3 \end{cases},$$

or F_2 :

$$F_2(x) = \begin{cases} A \text{ if } x \in D_3 \\ \{a_2\} \text{ if } x \in D'_3 \\ \{a_1\} \text{ if } x \in D''_3 \end{cases}.$$

Since F_1 and F_2 correspond to F^s with $s_1 > s_2$ and $s_1 < s_2$ respectively, the proof is completed. $\square$

Let us now derive a variant of May's theorem from Theorem 4.5 above. First, we need to introduce some more properties for anonymous ESCFs. For any extended profile $x \in \mathbb{N}^3$, let $N(a_i, x) \in \mathbb{N}$ be the number of voters who prefers (weakly) a_i to a_j at $x \in \mathbb{N}^3$, for $i, j \in \{1, 2\}$ and $i \neq j$. An anonymous ESCF is a *simple majority rule (F^M)* if

$$F^M(x) = \begin{cases} \{a_i\} \text{ if } N(a_i, x) > N(a_j, x) \\ \{a_1, a_2\} \text{ if } N(a_i, x) = N(a_j, x) \end{cases}$$

for $i, j \in \{1, 2\}$, $i \neq j$. It is an *inverse simple majority rule (F^{-M})* if

$$F^{-M}(x) = \begin{cases} \{a_i\} \text{ if } N(a_i, x) < N(a_j, x) \\ \{a_1, a_2\} \text{ if } N(a_i, x) = N(a_j, x) \end{cases}$$

for $i, j \in \{1, 2\}$, $i \neq j$. Finally, anonymous ESCF is *positive responsive to voter addition* (positive responsiveness) if whenever $a_i \in F(x)$ for $x \in \mathbb{N}^3$, and $y \in \mathbb{N}^3$ is obtained from $x \in \mathbb{N}^3$ by adding one more voter with preferences of $a_i \succ a_j$, we have $F(y) = \{a_i\}$, for $i, j \in \{1, 2\}$, $i \neq j$. The second part of the following result is a variant of May's Theorem (May (1952)):

Theorem 4.6 *Let $m = 2$ and let F be an ESCF. Then,*

(a) *F is anonymous, neutral and consistent if and only if it is either trivial, or a simple majority rule, or an inverse majority rule and*

(b) *F is anonymous, neutral and positive responsive if and only if it is a simple majority rule.*

Proof. Since the IF parts are easy we prove the ONLY IF parts.

(a) By definition, F_1 and F_2 in the proof of Theorem 4.5 correspond to simple majority rule and inverse simple majority rule, respectively.

(b) We present two proofs.

1. Let us show that positive responsiveness with anonymity and neutrality imply consistency. From the proof of Theorem 4.5, we know that anonymity and neutrality imply that, $\forall x \in D_3, F(x) = \{a_1, a_2\}$. For any $x = (x_1, x_2, x_3) \in \mathbb{N}^3$, let $x_{D_3} \in D_3$ be defined as $x_{D_3} = (\min\{x_1, x_2\}, \min\{x_1, x_2\}, x_3)$. We can write x as $x = x_{D_3} + (x - x_{D_3})$. We claim that $F(x) = F(x - x_{D_3})$. Note that if $x \in D_3$, then $F(x) = F(0) = A$, hence the claim is true. Suppose $x \notin D_3$. Then one can generate $x \in \mathbb{N}^3$ from $x_{D_3} \in D_3$ and $x - x_{D_3} \in \mathbb{N}^3$ from $0 \in D_3$ by one and the same procedure: if $x_1 > x_2$, adding $|x_1 - x_2| > 0$ voters, one at a time, who strictly prefers a_1 to a_2, if otherwise adding the same number of voters with the reverse preferences. Then, by positive responsiveness, $F(x) = F(x - x_{D_3})$ as we claimed, and moreover, $F(x) = A$ if and only if $x = x_{D_3} \in D_3$.

Suppose $x, y \in \mathbb{N}^3$ are such that $F(x) \cap F(y) \neq \emptyset$. We can express as $F(x) \cap F(y) = F(x - x_{D_3}) \cap F(y - y_{D_3})$. Let $z = x + y$ and notice that $z_{D_3} = x_{D_3} + y_{D_3}$, since the 'min' operator is additive. Hence, $F(z) = F(x_{D_3} + y_{D_3} + (x - x_{D_3}) + (y - y_{D_3})) = F((x - x_{D_3}) + (y - y_{D_3}))$. We claim that $F((x - x_{D_3}) + (y - y_{D_3})) = F(x - x_{D_3}) \cap F(y - y_{D_3})$. Notice that if at least one of x, y is in D_3, then the claim is established: if $x \in D_3$, then $x = x_{D_3}$ and $F(x - x_{D_3}) = A$. Suppose $x, y \notin D_3$. Then, $F(x - x_{D_3}) \cap F(y - y_{D_3}) \neq \emptyset$ implies $F(x - x_{D_3}) = F(y - y_{D_3}) = \{a_i\}$ for some $i \in \{1, 2\}$. By positive responsiveness, that is only possible if $\min\{x_1, x_2\} = x_j$ and $\min\{y_1, y_2\} = y_j$ for $j \in \{1, 2\}$ and $j \neq i$. Then by positive responsiveness, $F((x - x_{D_3}) + (y - y_{D_3})) = \{a_i\}$ since one can generate $(x - x_{D_3}) + (y - y_{D_3}) \in \mathbb{N}^3$ from $0 \in D_3$ by adding $(x_i - x_j) + (y_i - y_j)$ many voters with strict preferences of $a_i \succ a_j$. Hence, our second claim is established, which then implies that F is consistent.

Then, the result in part **(a)** implies that F is one of F^*, F_1 and F_2. But none of F^* and F_2 satisfies positive responsiveness. Hence, $F = F_1$ which is the simple majority rule.

2. The proof above is rather indirect and a more direct proof is as follows. We know that anonymity and neutrality imply that, $\forall x \in D_3, F(x) = \{a_1, a_2\}$. We claim that $\forall x' \in D_3', F(x') = \{a_1\}$. Suppose $x' = (x_1', x_2', x_3') \in D_3'$. Consider $x^* = (x_2', x_2', x_3') \in D_3$. Then, $F(x^*) = \{a_1, a_2\}$. We can generate x' from x^* by adding $(x_1' - x_2')$ voters, one at a time, with the preferences of $a_1 \succ a_2$. Then by positive responsiveness, $F(x') = \{a_1\}$. Hence, $\forall x' \in D_3', F(x') = \{a_1\}$ and by neutrality, $\forall x'' \in D_3'', F(x'') = \{a_2\}$, which imply that $F = F_1$, which is the simple majority rule. $\square$

May (1952) axiomatizes majority rule with anonymity, neutrality and strong monotonicity. The main difference between May's Theorem and Theorem 4.6 **(b)** is, May (1952) considers a fixed electorate setting while we consider a variable electorate setting. Then, the axiom of positive responsiveness to voter addition should be seen as a modification of the strong monotonicity axiom to the new setting.[3]

Let us show that one can use cancellation property axiom instead of neutrality in Theorem 4.5 and 4.6.

Proposition 4.7 *Let $m = 2$ and let F be an anonymous ESCF. Then,*

(a) *If F is consistent then it has cancellation property if and only if it is neutral, and*

[3]Strictly speaking, positive responsiveness to voter addition can not be stated in the original setting with fixed electorate. But it captures the underlying idea of the strong monotonicity axiom, and hence Theorem 4.6 **(b)** can be seen as a variant of May's Theorem.

(b) *If F has cancellation property and is positive responsive then it is consistent.*

Proof. **(a)** IF: We have shown in the proof of Theorem 4.5 that anonymity, consistency and neutrality imply that, $\forall x \in D_3, F(x) = A$. Since $m = 2$, any profile $x \in \mathbb{N}^3$ is such that the number of voters who prefers $a_1 \gtrsim a_2$ is same as that of who prefers $a_2 \gtrsim a_1$, if and only if $x \in D_3$. Hence, F satisfies the cancellation property.

ONLY IF: Let $F_{|P_0}$ be restriction of F into $P_0 = \{x \in \mathbb{N}^3 : x_1, x_2 \in \mathbb{N}, x_3 = 0\} \subset \mathbb{N}^3$. Note that $F_{|P_0}$ is a SCF and since F satisfies anonymity, consistency and cancellation property, so is $F_{|P_0}$. Then by Proposition 4.4, we conclude that $F_{|P_0}$ satisfies neutrality. Note also that by cancellation property, $F(0,0,1) = A$, which implies that $F(0,0,x_3) = A$ for all $x_3 \geq 1$, by consistency. Then, we conclude that $\forall x = (x_1, x_2, x_3) \in \mathbb{N}^3$ such that $x_3 \geq 1, F(x_1, x_2, x_3) = F(x_1, x_2, 0) = F_{|P_0}(x_1, x_2)$ since $F(x_1, x_2, x_3) = F(x_1, x_2, 0) + F(0, 0, x_3) = F(x_1, x_2, 0)$, where the last equality follows by consistency. Then, $\forall x \in \mathbb{N}^3, \forall \sigma \in S_2, F \circ \sigma(x) = F_{|P_0} \circ \sigma(x) = \sigma \circ F_{|P_0}(x_1, x_2) = \sigma \circ F(x)$ since the first and the last equality follows by our second conclusion, while the second equality follows by our first conclusion, and hence F is neutral.

(b) We prove the statement indirectly showing that anonymity, cancellation property and positive responsiveness imply that $F = F^M$. By definition, cancellation property implies that $\forall x \in D_3, F(x) = \{a_1, a_2\}$. Repeating the same argument as above in the second proof of Theorem 4.6 **(b)**, we conclude that, by positive responsiveness, $\forall x' \in D'_3, F(x') = \{a_1\}$. Let $x'' = (x''_1, x''_2, x''_3) \in D''_3$. Consider $x^* = (x''_1, x''_1, x''_3) \in D_3$. Then, $F(x^*) = \{a_1, a_2\}$. We can generate x'' from x^* by adding $(x''_2 - x''_1)$ voters, one at a time, with the preferences of $a_2 \succ a_1$. Then by positive responsiveness, $F(x'') = \{a_2\}$. Hence $\forall x'' \in D''_3, F(x'') = \{a_2\}$. So, $F = F^M$ and hence, it is consistent. $\square$

4.4 Final comments

When it is presented Young's Theorem is often accompanied by the following remark: "its proof is difficult and omitted" (see for instance, Chap. 9 in Moulin, 1988). However, the analysis above shows that in the special case of voting over two alternatives it can easily be proved. One may also wonder whether the axiom of continuity can be eliminated when there are more than two alternatives. The answer to this question is negative since the example of a (composite) scoring function satisfying the axioms of anonymity, neutrality and consistency but not continuity, given in Section 3 of Young (1975), can easily be extended to the case of any finite (but three or more) alternatives. This observation implies that any such elimination contradicts Theorem 1 in Young (1975).

It may seem that the setting with two alternatives is rather restrictive, especially in the context of scoring rules. However, note that the analysis of this simple case can shed a light on possible improvements of some of the axiomatization results in voting theory. For instance, as majority rule and approval voting (AV) coincide when $m = 2$, the result in Theorem 4.6 **(a)** is related to the axiomatization of AV in Fishburn (1978). In the simple case it is easy to see that the axioms of neutrality and cancellation can be used interchangeably (see Proposition 4.7). On the other hand, Alós-Ferrer (2006) shows that one can drop neutrality in the presence of anonymity, consistency, cancellation and faithfulness in Fishburn (1978)'s axiomatization of AV. Hence, Alós-Ferrer (2006)'s result can be seen as an extension of Theorem 4.6 **(a)**. Finally, note also that Theorem 4.6 **(a)** admits another extension: one can keep neutrality and drop cancellation in the axiomatization of AV, which is, to my best knowledge, an open question.

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Chapter 5

Approval Voting without Faithfulness

Abstract: In this short note, we analyze the implications of dropping the axiom of faithfulness in the axiomatization of Approval Voting, due to P.C. Fishburn. We show that a ballot aggregation function satisfies the remaining axioms (neutrality, consistency and cancellation) if and only if it is either a function that chooses the whole set of alternatives, or an Approval Voting, or a function that chooses the least approved alternatives.

JEL: D71, D72
Keywords: Approval voting, Faithfulness, Inverse approval voting

5.1 Introduction

There are a number studies related to axiomatization of Approval Voting (AV) in the literature (for a survey, see Hu, 2010). Early results on this problem are due to Fishburn (1978a, 1978b). Fishburn (1978a) shows that AV is the only ballot aggregation function (BAF) satisfying the axioms of neutrality, consistency, cancellation and faithfulness, while Fishburn (1978b) axiomatizes AV with the axioms of neutrality, consistency and disjoint equality. Alós-Ferrer (2006) shows that one can drop the axiom of neutrality in the axiomatization of Fishburn (1978a).

The primary objective of this short paper is to investigate the implications of dropping the axiom of faithfulness in the axiomatization of Fishburn (1978a), hence to analyze cutting power of this axiom. Our main finding is (Theorem 5.1 in Section 5.2), a BAF satisfies the axioms of neutrality, consistency and cancellation if and only if it is either a trivial function that chooses the whole set of alternatives at all profiles, or AV, or its inverse, that is a function that chooses the least approved alternatives.

5.2 Characterization

Let $\mathbb{N}$ denote the set of nonnegative integers. Let X be a (finite) set of alternatives and let S be the set of all permutations of X. For $\sigma \in S$ and $Y \subseteq X$, $\sigma(Y) \subseteq X$ is the image of Y under σ. A *ballot* B is a nonempty subset of X and let $\mathcal{B} = 2^X \backslash \{\emptyset\}$ be the set of all admissible ballots. Voters can cast any ballot, approving as many candidates as they want. A *voter response profile* is a function $\pi : \mathcal{B} \to \mathbb{N}$ such that $\pi(B)$ is the number of voters who cast ballot B. Let Π be the set of all possible voter response profiles, including the empty profile π^0 with $\pi^0(B) = 0$ for all $B \in \mathcal{B}$. A *ballot aggregation function* (BAF) is a correspondence f which assigns to every possible voter response profile $\pi \in \Pi$, a nonempty set of selected alternatives, $\emptyset \subsetneq f(\pi) \subseteq X$. A BAF is an *approval voting* (f^A) if

$$f^A(\pi) = \arg\max_{x \in X} \sum \{\pi(B) : x \in B \in \mathcal{B}\},$$

it is an *inverse approval voting* (f^{-A}) if

$$f^{-A}(\pi) = \arg\min_{x \in X} \sum \{\pi(B) : x \in B \in \mathcal{B}\},$$

and it is a *trivial* function (f^*) if $f^*(\pi) = X, \forall \pi \in \Pi$.

Given $x \in X$ and $\pi \in \Pi$, the number of voters who approve of x in π is given by

$$v(x, \pi) = \sum \{\pi(B) : x \in B \in \mathcal{B}\}.$$

For any $B \in \mathcal{B}$, let $\pi_B \in \Pi$ denote the voter response profile with $\pi_B(B) = 1$ and $\pi_B(B') = 0$ for all $B' \neq B$, i.e. π_B consists of only one ballot B. When $B \in \mathcal{B}$ consists of a single element $x \in X$, we write π_x instead of $\pi_{\{x\}}$. For any $\pi, \pi' \in \Pi$, $\pi + \pi' \in \Pi$ is a voter response profile with $(\pi + \pi')(B) = \pi(B) + \pi'(B)$, $\forall B \in \mathcal{B}$, and whenever $\pi, \pi' \in \Pi$ are such that $\pi(B) = \pi'(B)$,$\forall B \in \mathcal{B}$, we write $\pi = \pi'$.

A BAF satisfies

Neutrality: if $f(\pi \circ \sigma) = \sigma(f(\pi))$ for every $\sigma \in S$ and for every $\pi \in \Pi$, where $\pi \circ \sigma \in \Pi$ is defined as $(\pi \circ \sigma)(B) = \pi(\sigma(B))$, $\forall B \in \mathcal{B}$;

Faithfulness: if $f(\pi_B) = B$ for all $B \in \mathcal{B}$;

Consistency: if whenever $f(\pi) \cap f(\pi') \neq \emptyset$ for $\pi, \pi' \in \Pi$, we have $f(\pi+\pi') = f(\pi) \cap f(\pi')$;

Cancellation: if whenever $\pi \in \Pi$ satisfies $v(x, \pi) = v(y, \pi)$ for all $x, y \in X$, then $f(\pi) = X$.

For interpretations of the axioms, see Fishburn (1978a,b) and Hu (2010). We now state and prove our main result (of which part **(b)** is already established in Fishburn (1978a)):

Theorem 5.1 *Let f be a BAF. Then*

(a) *f satisfies neutrality, consistency and cancellation if and only if it is either a trivial function, or an approval voting, or an inverse approval voting, and*

(b) *in addition, such f is faithful if and only if it is an approval voting.*

Proof. Since the IF parts are easy to prove, we prove the ONLY IF parts.

(a) We proceed in 4 steps. We remark here that Steps $2, 3$ in our proof are the same as Steps $1, 2$ in the proof of Theorem 1 in Alós-Ferrer (2006).

Step 1: Let us prove that for all $B \subseteq X$, $f(\pi_B)$ is either B, or $X \backslash B$ or X, and similarly, $f(\sum_{x \in B} \pi_x)$ is either B, or $X \backslash B$ or X. To see this, note that both $\pi_B \in \Pi$ and $\sum_{x \in B} \pi_x \in \Pi$ are invariant under any permutation $\sigma_B \in S$ that permutes the elements of B and $X \backslash B$ in an arbitrary way, but does not interchange the elements of these two sets. That is, for any such $\sigma_B \in S$,

$$\pi_B \circ \sigma_B = \pi_B$$

and

$$\Big(\sum_{x \in B} \pi_x\Big) \circ \sigma_B = \sum_{x \in B} \pi_x.$$

Then, by neutrality so must be $f(\pi_B)$ and $f(\sum_{x \in B} \pi_x)$: for any such $\sigma_B \in S$,

$$\sigma_B(f(\pi_B)) = f(\pi_B)$$

and

$$\sigma_B\Big(f\Big(\sum_{x \in B} \pi_x\Big)\Big) = f\Big(\sum_{x \in B} \pi_x\Big).$$

But it is easy to check that the only sets in $2^X \backslash \{\emptyset\}$ with such property (invariant under any $\sigma_B \in S$) are B, $X \backslash B$ and X.

Step 2: Let us prove that for all $\pi \in \Pi$ and for all B', $B'' \subseteq X$ such that $B' \cap B'' = \emptyset$, we have

$$f(\pi + \pi_{B' \cup B''}) = f(\pi + \pi_{B'} + \pi_{B''}).$$

To see this, note that cancellation implies that

$$f(\pi_{B'} + \pi_{B''} + \pi_{X \backslash (B' \cup B'')}) = X.$$

Then, by consistency,

$$f(\pi + \pi_{B' \cup B''}) = f(\pi + \pi_{B' \cup B''} + \pi_{B'} + \pi_{B''} + \pi_{X \backslash (B' \cup B'')}).$$

Note also that by cancellation,

$$f(\pi_{B' \cup B''} + \pi_{X \backslash (B' \cup B'')}) = X$$

and then by consistency,

$$f(\pi + \pi_{B'} + \pi_{B''}) = f(\pi + \pi_{B'} + \pi_{B''} + \pi_{B' \cup B''} + \pi_{X \backslash (B' \cup B'')}).$$

Hence,

$$f(\pi + \pi_{B' \cup B''}) = f(\pi + \pi_{B'} + \pi_{B''}). \qquad [\mathbf{1}]$$

Step 3: Let $\pi \in \Pi$ be an arbitrary voter response profile. Let $\pi' \in \Pi$ be such that $v(x, \pi) = v(x, \pi')$ for all $x \in X$, but π' consists only of singleton ballots, i.e. $\pi'(B) > 0$ implies that $|B| = 1$. The profile π' is constructed from π by taking apart each ballot cast under π into separate, singleton ballots. Then iteration of [**1**], starting from $\pi^0 \in \Pi$, shows that $f(\pi) = f(\pi')$.

Step 4: Let $x \in X$ be any alternative and consider $\pi_x \in \Pi$. By Step **1**, $f(\pi_x)$ is either $\{x\}$, or $X \backslash \{x\}$, or X. We show that these possibilities correspond, respectively, to the cases of f being either f^A, or f^{-A}, or f^*.

Case 1: Let $f(\pi_x) = \{x\}$. By neutrality, $f(\pi_y) = \{y\}$ for any $y \in X$. We claim that for all $B \subseteq X$, $f(\pi_B) = B$, i.e. f is faithful. Since by cancellation (or by neutrality), $f(\pi_X) = X$, we may assume that $B \subsetneq X$. Note that by Step **1**, $f(\pi_B)$ is either B, or $X \backslash B$, or X. Suppose $f(\pi_B) \neq B$, then consistency implies that for $z \in X \backslash B$,

$$f(\pi_B + \pi_z) = f(\pi_B) \cap f(\pi_z) = \{z\}.$$

By [**1**],

$$f(\pi_B + \pi_z) = f(\pi_{B\cup\{z\}}),$$

which implies that $f(\pi_{B\cup\{z\}}) = \{z\}$. But that contradicts to Step **1**. So, $f(\pi_B) = B$ and the claim is established. Then, by repeating the same argument as in Step 3 of the proof of Theorem 1 in Alós-Ferrer (2006), we can conclude that $f = f^A$.

Case 2: Let $f(\pi_x) = X\backslash\{x\}$. By neutrality, $f(\pi_y) = X\backslash\{y\}$ for all $y \in X$. We claim that for all $B \subsetneq X$, $f(\pi_B) = X\backslash B$. Note that by [**1**],

$$f(\pi_B) = f(\sum_{z\in B} \pi_z),$$

and starting from any two elements, $z_1, z_2 \in B$, by repeated use of consistency,

$$f(\sum_{z\in B} \pi_z) = \bigcap_{z\in B} X\backslash\{z\} = X\backslash B$$

which implies that, $f(\pi_B) = X\backslash B$ and the claim is established. Let $\pi \in \Pi$ be a given profile. Let $K = \max\{v(x,\pi)\}$ and note that K is well defined since X is finite. For each $k = 0, ..., K$, we define $B_k = \{x \in X : v(x,\pi) = k\}$. Then, the sets B_k form a partition of X. Consider the profile

$$\pi^* = \pi_{B_K} + \pi_{B_K\cup B_{K-1}} + ... + \pi_{B_K\cup B_{K-1}\cup ...\cup B_1}.$$

Since for $B \subsetneq X$, $f(\pi_B) = X\backslash B$ and $f(\pi_X) = X$, consistency implies that

$$f(\pi^*) = X\backslash(B_K \cup B_{K-1} \cup ... \cup B_{j+1}) = B_j,$$

where $j = \min\{k : B_k \neq \emptyset\}$.[1] But iteration of [**1**] implies that $f(\pi^*) = f(\pi')$ and by Step **3**, we conclude that $f(\pi) = B_j$. Thus, $f = f^{-A}$.

Case 3: Let $f(\pi_x) = X$. Then by neutrality, $f(\pi_y) = X$ for all $y \in X$. We claim that for all $B \subseteq X$, $f(\pi_B) = X$. Since $f(\pi_X) = X$ by cancellation (or by neutrality), we can assume that $B \subsetneq X$. By consistency,

$$f(\pi_B + \sum_{z\in X\backslash B} \pi_z) = f(\pi_B) \cap \sum_{z\in X\backslash B} f(\pi_z) = f(\pi_B)$$

[1]Notice that by construction, $B_K \cup B_{K-1} \cup ... \cup B_j = X$, and hence for $0 \leq i \leq j$, $f(\pi_{B_K\cup ...\cup B_i}) = X$.

since $\forall z \in X \backslash B$, $f(\pi_z) = X$. But by cancellation,

$$f(\pi_B + \sum_{z \in X \backslash B} \pi_z) = X.$$

Hence, $f(\pi_B) = X$ and the claim is established. As in Case 2, let us consider $\pi^* \in \Pi$. Since $f(\pi_{B_k}) = X$, consistency implies that $f(\pi^*) = X$. But iteration of [**1**] implies that $f(\pi^*) = f(\pi')$ and by Step **3**, we conclude that $f(\pi) = X$. Thus, $f = f^*$.

(b) Clearly, none of f^{-A} and f^* is faithful. Hence, $f = f^A$. □

5.3 Final Remarks

From the outset, it may seem that the primary role of the axiom of faithfulness is to fix an orientation, i.e. to set the right direction. The analysis above clarifies that intuition: in the axiomatization of Approval Voting (AV) in Fishburn (1978a), faithfulness helps us to distinguish AV from a function that always chooses the whole set of alternatives, and a function that always chooses the least approved alternatives.

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Chapter 6

Another Direct Proof for the Gibbard-Satterthwaite Theorem [1]

Abstract: We prove the following result which is equivalent to the Gibbard- Satterthwaite Theorem: when there are at least 3 alternatives, for any unanimous and strategy-proof social choice function, at any given profile if an individual's top ranked alternative differs from the social choice, then she can not change the social choice at that profile by changing her ranking. Hence, proving it yields a new proof for the Gibbard-Satterthwaite Theorem.

JEL: D71, D72
Keywords: the Gibbard-Satterthwaite theorem, Strategy-proofness, Option sets

6.1 Introduction

There are a number of elegant proofs for the Gibbard-Satterthwaite (G-S) Theorem (Gibbard, 1973; Satterthwaite, 1975) in the literature: see for instance, Schmeidler and Sonnenschein (1978), Barberà (1983), Barberà and Peleg (1990), Benoit (2000), Reny (2001), Sen (2001), Cato (2009), and for a survey, see Section 3.3 in Barberà (2011).

This paper suggests yet another approach to prove this classic result. We prove it in two steps. First, in Theorem 6.3 in Section 6.3 we show that when there are at least 3 alternatives, if a social choice function is unanimous and strategy-proof, then for any given preference profile, if an individual is not a candidate for a *dictator* (an individual whose choice coincides or determines group's choice), then she can not change the social

[1]Published in *Economics Letters*, 2012, **116** (3): 418-421.

choice at that profile by unilateral deviation. Then, in the second step we verify that such social choice function is necessarily dictatorial, hence deduce the G-S Theorem from this result (Corollary 6.4).

Our approach differs from some of the existing proofs in the following sense: contrary to some of the existing proofs, our focus is not on the potential candidates for a dictator of a social choice function (e.g., *pivotal voter* in Barberà, 1983; stepwise procedures focusing on *extremely pivotal voter* used in Benoit, 2000 and Reny, 2001), but on the individuals who are not candidates for a dictator. It also implies that the result in Theorem 6.3 is equivalent to the G-S Theorem since one can easily deduce the former from the latter.

Among the existing proofs mentioned above, our approach is close to that of Barberà and Peleg (1990), which introduced the notion of (group) option set: under an N individual social choice function, an *option set of a group of* $N-1$ *individuals* at a given profile is the set of alternatives that they can achieve by changing their rankings collectively, keeping the other individual's ranking fixed (see Defn. 5.2 in Barberà and Peleg, 1990). They showed that if a social choice function is unanimous and strategy-proof, then the option sets are either a singleton or the full set of alternatives (see Lemma 2.9 and 5.7 in Barberà and Peleg, 1990). In contrast, our result in Theorem 6.3 shows that for any unanimous and strategy-proof social choice function, at any given profile if an individual is not a candidate for a dictator, then *her option set* (i.e. the set of alternatives that she can achieve by a unilateral deviation) is a singleton.

When $N = 2$, Barberà and Peleg (1990)'s group option set is the same as individual's option set and consequently, in that case, Theorem 6.3 in Section 6.3 is an immediate corollary of Lemma 2.9 in Barberà and Peleg (1990). However, when $N > 2$, the connection between our result in Theorem 6.3 and their result in Lemma 5.7 is less immediate: when the number of alternatives is finite in Barberà and Peleg (1990), both results are equivalent to the G-S Theorem, hence equivalent to each others, but none is an immediate corollary of the other.

In the next section, we introduce our notation and the main definitions. Section 6.3 gives the main result and our proof of the G-S Theorem and the last section concludes.

6.2 The preliminaries

Let A denote the set of alternatives with $n \in \mathbb{N}$ elements and let X denote the set of strict linear orders (strict rankings) on A. Let there be N individuals in the group $I = \{1, 2, ..., N\}$. A function $f : X^N \to A$ is called as a social choice function. A member

$x = (x_1, ..., x_N)$ of X^N is called a profile of rankings (or simply a profile) and its $i'th$ component, x_i, is called the individual $i's$ ranking. When a is ranked above b according to x_i we write $a \succ_{x_i} b$. $\forall x \in X^N$ and $\forall i \in I$, let $(x'_i, x_{-i}) \in X^N$ denote the profile that has $x'_i \in X$ in its $i'th$ component instead of $x_i \in X$, and otherwise same as $x \in X^N$. Under a social choice function f, an *option set* of individual $i \in I$ at a given profile $x \in X^N$ is the set of alternatives that i can achieve by a unilateral deviation:

$$O_i(x) = \{a \in A : \exists x'_i \in X \text{ s.t } f(x'_i, x_{-i}) = a\}.$$

We say that a social choice function $f : X^N \to A$ is **unanimous** (UNM) if $\forall a \in A$, whenever a is on top of x_i for $i = 1, ..., N$, then $f(x) = a$. It is **manipulable** (MNP) at $x \in X^N$ by $i \in I$ via $x'_i \in X$ if $f(x'_i, x_{-i}) \succ_{x_i} f(x)$. It is **strategy-proof** (STP) if it is not manipulable. Finally, it is **dictatorial** (DT) if $\exists i \in I$, referred to as the dictator, such that $\forall x \in X^N, \forall a \in A, f(x) = a$ if and only if a is at the top of x_i. The following result is known as the G-S Theorem (Gibbard 1973; Satterthwaite, 1975):

Theorem 6.1 *If $n \geq 3$, a social choice function $f : X^N \to A$ is UNM and STP if and only if it is DT.*

6.3 The main result and the proof

Throughout this section we assume that $n \geq 3$. Let $f : X^N \to A$ be a given social choice function and let $x \in X^N$ be a given profile. Let $G_1^f(x) \subseteq I$ be the set of individuals $i \in I$ who ranks $f(x)$ at the top of their rankings, x_i, and let $G_2^f(x) = I \backslash G_1^f(x)$ be its complement in I.

We first prove the following result which is a 2 individual case of our main result in Theorem 6.3.

Lemma 6.2 *Let $N = 2$ and let $f : X^2 \to A$ be a UNM and STP social choice function and let $x \in X^2$ be a given profile. Then,*

(a) *$G_1^f(x) \neq \emptyset$ and*

(b) *If $i \in G_2^f(x)$, then $O_i(x) = \{f(x)\}$.*

Proof. **(a)** Suppose $x \in X^2$ is such that $G_1^f(x) = \emptyset$. Let a be at the top of x_1 and b be at the top of x_2 and let $f(x) = c$ where c is distinct from a and b. By UNM, $a \neq b$. Consider $x'_1 = (a \succ b \succ c \succ ...)$ and $x'_2 = (b \succ a \succ c \succ ...)$. Then, $f(x'_1, x_2) = b$ since

$f(x_1', x_2) \neq a$ as otherwise f is MNP at $x \in X^2$ by 1 via x_1' and $f(x_1', x_2)$ can not be any alternative that is ranked below b in x_1' since otherwise f is MNP at $(x_1', x_2) \in X^2$ by 1 via any x_1'' ranks b at the top. Similarly, $f(x_1, x_2') = a$. Note that $f(x_1', x_2') = a$ since otherwise f is MNP at $(x_1', x_2') \in X^2$ by 1 via x_1. But then f is MNP at $(x_1', x_2') \in X^2$ by 2 via x_2. Hence a contradiction. We remark here that this claim is already established in Sen (2001).

(b) Without loss of generality we may assume that $2 \in G_2^f(x)$. By **(a)**, then $1 \in G_1^f(x)$ and let $f(x) = a$. If a is at the bottom of x_2 and $f(x_1, x_2') \neq a$ for some $x_2' \in X$, then f is MNP at $x \in X^2$ by 2 via x_2'. Hence, the statement is true. Let $x_2 = (b_1 \succ ... \succ b_{k-1} \succ a \succ c_{k+1} \succ ... \succ c_n)$ for some $k = 2, ..., n-1$. Then, $\nexists x_2' \in X$ such that $f(x_1, x_2') \in \{b_1, ..., b_{k-1}\}$ since otherwise f is MNP at $x \in X^2$ by 2 via x_2'. Suppose $\exists x_2' \in X$ such that $f(x_1, x_2') = c \in \{c_{k+1}, ..., c_n\}$. Then, the following cross examinations lead us to a contradiction. Let x_2'' be a ranking with $x_2'' = (b_1 \succ c \succ a \succ ...)$. Then, $f(x_1, x_2'') \neq b_1$ since otherwise f is MNP at $(x_1, x_2) \in X^2$ by 2 via x_2'', and $f(x_1, x_2'')$ can not be any alternative ranked below c in x_2'' since otherwise f is MNP at $(x_1, x_2'') \in X^2$ by 2 via x_2'. Hence, $f(x_1, x_2'') = c$.

Consider $x_1' = (a \succ b_1 \succ c \succ ...)$. Then, $f(x_1', x_2) = a$ since otherwise f is MNP at $(x_1', x_2) \in X^2$ by 1 via x_1. We claim that $f(x_1', x_2'') = c$. Since $f(x_1, x_2'') = c$, $f(x_1', x_2'') \in \{a, b_1, c\}$ since otherwise f is MNP at $(x_1', x_2'') \in X^2$ by 1 via x_1. But $f(x_1', x_2'') \neq a$ since otherwise f is MNP at $(x_1, x_2'') \in X^2$ by 1 via x_1'. Note also that $f(x_1', x_2'') \neq b_1$ since otherwise f is MNP at $(x_1', x_2) \in X^2$ by 2 via x_2''. Hence, $f(x_1', x_2'') = c$. Consider now $x'' = (x_1'', x_2'') \in X^2$ such that b_1 is at the top of x_1''. Then, $f(x'') \neq b_1$ since otherwise f is MNP at $(x_1', x_2'') \in X^2$ by 1 via x_1''. But by UNM, $f(x'') = b_1$, hence a contradiction. $\square$

Our main result is as follows:

Theorem 6.3 *Let $N \geq 2$ and let $f : X^N \to A$ be a UNM and STP social choice function and let $x \in X^N$ be a given profile. Then,*

(a) *$G_1^f(x) \neq \emptyset$ and*

(b) *If $i \in G_2^f(x)$, then $O_i(x) = \{f(x)\}$.*

Proof. **(a)** Suppose $x \in X^N$ is such that $G_1^f(x) = \emptyset$. Let $a \in A$ be the top alternative in x_1 and let x_k be the next ranking whose top alternative differs from a, i.e.

$$k = \min\{1 < i \leq N : \text{the top of } x_k \neq a\}.$$

By UNM such x_k must exist. Let $b \in A$ be the top alternative of x_k and let $f(x) = c$. Consider $x'_1 = (a \succ b \succ c \succ ...)$ and $x'_k = (b \succ a \succ c \succ ...)$. Then, $f(x'_1, x_{-1}) \in \{b, c\}$ since $f(x'_1, x_{-1}) \neq a$ as otherwise f is MNP at $x \in X^N$ by 1 via x'_1, and $f(x'_1, x_{-1}) \in A$ can not be any alternative that is ranked below c in x'_1 since otherwise f is MNP at $(x'_1, x_{-1}) \in X^N$ by 1 via x_1. Similarly, $f(x'_k, x_{-k}) \in \{a, c\}$. There are 4 possible cases:

A. $f(x'_1, x_{-1}) = b$, $f(x'_k, x_{-k}) = a$,

B. $f(x'_1, x_{-1}) = b$, $f(x'_k, x_{-k}) = c$,

C. $f(x'_1, x_{-1}) = c$, $f(x'_k, x_{-k}) = a$,

D. $f(x'_1, x_{-1}) = c$, $f(x'_k, x_{-k}) = c$.

Suppose **A** is the case. Then, $f(x'_1, x_2, ..., x'_k, ..., x_N) = a$ since otherwise f is MNP at $(x'_1, x_2, ..., x'_k, ..., x_N) \in X^N$ by 1 via x_1. But then f is MNP at $(x'_1, x_2, ..., x'_k, ..., x_N) \in X^N$ by k via x_k. Hence a contradiction and **A** can not be the case. Suppose $f(x'_k, x_{-k}) = c$ as in the cases of **B** and **D**. Then,

$$f(x_1, ..., x_{k-1}, x'_1, ..., x_N) = c \qquad [1]$$

since $f(x_1, ..., x_{k-1}, x'_1, ..., x_N) \notin \{a, b\}$ as otherwise f is MNP at $(x'_k, x_{-k}) \in X^N$ by k via x'_1. Similarly, $f(x_1, ..., x_{k-1}, x'_1, ..., x_N) \in A$ can not be any alternative ranked below c in x'_1, since otherwise f is MNP at $(x_1, ..., x_{k-1}, x'_1, ..., x_N) \in X^N$ by k via x'_k. Note that by construction $a \in A$ is at the top of each first k rankings in $(x_1, x_2, ..., x'_1, ..., x_N) \in X^N$.

Suppose **C** is the case. We first show that $f(x'_1, x'_1, ..., x'_1, x_k, ..., x_N) = c$. Start with the second individual and change her ranking to x'_1. Then,

$$f(x'_1, x'_1, x_3, ..., x_{k-1}, x_k, ..., x_N) \in \{b, c\}$$

since $f(x'_1, x'_1, x_3, ..., x_{k-1}, x_k, ..., x_N) \neq a$ as otherwise f is MNP at $(x'_1, x_{-1}) \in X^N$ by 2 via x'_1. Similarly, $f(x'_1, x'_1, x_3, ..., x_{k-1}, x_k, ..., x_N) \in A$ can not be any alternative ranked below c in x'_1 as otherwise f is MNP at $(x'_1, x'_1, x_3, ..., x_{k-1}, x_k, ..., x_N) \in X^N$ by 2 via x_2. Suppose $f(x'_1, x'_1, x_3, ..., x_{k-1}, x_k, ..., x_N) = b$. Recall that $f(x'_k, x_{-k}) = a$. Then $f(x'_1, x_2, ..., x'_k, ..., x_N) = a$ as otherwise f is MNP at $(x'_1, x_2, ..., x'_k, ..., x_N) \in X^N$ by 1 via x_1. Then, $f(x'_1, x'_1, x_3, ..., x_{k-1}, x'_k, x_{k+1}, ..., x_N) = a$ as otherwise f is MNP at $(x'_1, x'_1, x_3, ..., x_{k-1}, x'_k, x_{k+1}, ..., x_N) \in X^N$ by 2 via x_2. But then f is MNP at $(x'_1, x'_1, x_3, ..., x_{k-1}, x'_k, x_{k+1}, ..., x_N) \in X^N$ by k via x_k. Hence a contradiction. So,

$f(x'_1, x'_1, x_3, ..., x_{k-1}, x_k, ..., x_N) = c$. Repeating the same argument for individuals 3 to $k-1$, we conclude that $f(x'_1, x'_1, ..., x'_1, x_k, ..., x_N) = c$.

Let us now change each of the first $k-1$ individuals' ranking to x'_k, one at a time starting with individual 1. Then, $f(x'_k, x'_1, ..., x'_1, x_k, ..., x_N) = c$ since

$$f(x'_k, x'_1, ..., x'_1, x_k, ..., x_N) \notin \{a, b\}$$

as otherwise f is MNP at $(x'_1, x'_1, ..., x'_1, x_k, ..., x_N) \in X^N$ by 1 via x'_k. Similarly,

$$f(x'_k, x'_1, ..., x'_1, x_k, ..., x_N) \in A$$

can not be any alternative ranked below c in x'_k since otherwise f is MNP at

$$(x'_k, x'_1, ..., x'_1, x_k, ..., x_N) \in X^N$$

by 1 via x'_1. Repeating the same argument for individuals 2 to $k-1$, we then conclude that

$$f(x'_k, x'_k, ..., x'_k, x_k, ..., x_N) = c. \qquad [2]$$

Notice that by construction $b \in A$ is at the top of each first k rankings in

$$(x'_k, x'_k, ..., x'_k, x_k, ..., x_N) \in X^N.$$

Then, in either of the possible cases (of **B**-**D**), we conclude that there is a profile $y \in X^N$ such that $c \in A$ is not ranked at the top of any ranking $y_i, i = 1, ..., N$ and the top alternatives of the first k rankings are the same, but $f(y) = c$ (see [1] and [2]). We then consider the next individual $j > k$ whose top element is different from that of the first k rankings at $y \in X^N$. By UNM such y_j exists. Repeating the same argument as above, we then conclude that there must be another profile $z \in X^N$ such that $c \in A$ is not ranked at the top of any ranking $z_i, i = 1, ..., N$ and the top alternatives of the first j rankings are the same, but $f(z) = c$. Since N is finite, eventually, we conclude that there is a profile $z' \in X^N$ such that the top alternatives of the first N rankings are the same (and distinct from c), but $f(z') = c$, which contradicts to UNM. This completes our proof.

(b) Let $f(x) = a$ and without loss of generality we may assume that $G_1^f(x) = \{1, ..., j-1\}$ and $G_2^f(x) = \{j, ..., N\}$. Suppose on contrary that individual $j \in G_2^f(x)$ is such that for some $x'_j \in X, f(x'_j, x_{-j}) \neq f(x)$. Let us construct $f^2 : X^2 \to A$ from $f : X^N \to A$ with the following reductions: combine all individuals' preferences in $G_1^f(x)$ into one and the same, i.e. individuals in the group are coalesced (same as in Sen, 2001), and fix everyone's preferences in $G_2^f(x)$, except $j's$, at x (same as in Cato, 2009). So, $\forall z \in X^2$ let $f^2(z_1, z_2) = f(z_1, ..., z_1, z_2, x_{j+1}, ..., x_N)$.

Claim 1: $f^2 : X^2 \to A$ is UNM.

Consider any $x_1'' \in X$ that ranks a on top. Suppose $f(x_1'', ..., x_1'', x_j, ..., x_N) \neq a$. Then, $f(x_1, x_1'', ..., x_1'', x_j, ..., x_N) \neq a$ since otherwise f is MNP at $(x_1'', ..., x_1'', x_j, ..., x_N) \in X^N$ by 1 via x_1. Similarly, $f(x_1, x_2, x_1'', ..., x_1'', x_j, ..., x_N) \neq a$ since otherwise f is MNP at $(x_1, x_1'', ..., x_1'', x_j, ..., x_N) \in X^N$ by 2 via x_2. Repeating the same argument, after $j-1$ steps we conclude that $f(x) \neq a$, which is a contradiction. So, $\forall x_1'' \in X$ such that a is ranked on top,

$$f(x_1'', ..., x_1'', x_j, ..., x_N) = a. \qquad [3]$$

Now let $x_j'' \in X$ be such that $a \in A$ is ranked on top. Then, $f(x_1'', ..., x_1'', x_j'', ..., x_N) = a$ since otherwise f is MNP at $(x_1'', ..., x_1'', x_j'', ..., x_N) \in X^N$ by j via x_j. So, for any $x_1'', x_j'' \in X$ such that $a \in A$ is ranked on top, $f(x_1'', ..., x_1'', x_j'', ..., x_N) = a$.

Let $b \in A$ be any other alternative and let $y_1 \in X$ be a ranking with $y_1 = (b \succ a \succ ...)$. We claim that $f(y_1, ..., y_1, x_j, ..., x_N) = b$. Note that $f(y_1, x_{-1}) \in \{a, b\}$ since otherwise f is MNP at $(y_1, x_{-1}) \in X^N$ by 1 via x_1. Similarly, $f(y_1, y_1, x_3, ..., x_N) \in \{a, b\}$ since otherwise f is MNP at $(y_1, y_1, x_3, .., x_N) \in X^N$ by 2 via x_2. Then, repeating the same argument, we conclude that $f(y_1, ..., y_1, x_j, ..., x_N) \in \{a, b\}$. But by Theorem 6.3 **(a)**, $f(y_1, ..., y_1, x_j, ..., x_N) \neq a$ since by construction $a \in A$ is not on the top of any ranking in $(y_1, ..., y_1, x_j, ..., x_N) \in X^N$. So, $f(y_1, ..., y_1, x_j, ..., x_N) = b$. Now by the same argument as for $a \in A$, we will then conclude that for any $y_1'', y_j'' \in X$ such that $b \in A$ is ranked on top, $f(y_1'', ..., y_1'', y_j'', x_{j+1}, ..., x_N) = b$. Then, by definition, $\forall b \in A$ and $\forall z \in X^2$ such that b is ranked on top of each z_i, $i = 1, 2$, $f^2(z) = b$. Hence, f^2 is UNM.

Claim 2: $f^2 : X^2 \to A$ is STP.

Clearly the second individual can not manipulate f^2 since otherwise individual j would manipulate f. Suppose f^2 is MNP at $y \in X^2$ by individual 1 via $y_1' \in X$: $f^2(y) = b$ and $f^2(y_1', y_{-1}) = c$ and $c \succ_{y_1} b$. Then by definition, $f(y_1, ..., y_1, y_2, x_{j+1}, ..., x_N) = b$ and $f(y_1', ..., y_1', y_2, x_{j+1}, ..., x_N) = c$. Let's change first $j-1$ individuals' preferences from y_1 to y_1' each at a time, starting with individual 1. Then, $f(y_1', y_1, ..., y_1, y_2, x_{j+1}, ..., x_N) = c_1 \in A$ is not ranked above b in y_1 since otherwise f is MNP at $(y_1, ..., y_1, y_2, x_{j+1}, ..., x_N) \in X^N$ by individual 1 via y_1'. Repeat the same argument for the second individual when changing her preferences from y_1 to y_1'. Then, $f(y_1', y_1', ..., y_1, y_2, x_{j+1}, ..., x_N) = c_2 \in A$ is not ranked above c_1 in y_1 since otherwise f is MNP at $(y_1', y_1, ..., y_1, y_2, x_{j+1}, ..., x_N) \in X^N$ by individual 2 via y_1'. Note that since y is transitive, c_2 is not ranked above b in y_1. After $j-1$ repetitions, we then conclude that $c_{j-1}(=c)$ is not ranked above b in y_1 which is a contradiction. Hence, f^2 is not MNP by individual 1 and the claim is established.

Claim 3: Recall that initially $x \in X^N$ is such that $f(x) = a$, and by assumption $\exists x_j' \in X$ such that $f(x_j', x_{-j}) \neq a$. Then, $f^2(x_1, x_j) = a$ and $f^2(x_1, x_j') \neq a$.

We have already established the first part of the claim (see [3]). For the second part, suppose on the contrary that $f^2(x_1, x_j') = a$. By definition, then $f(x_1, ..., x_1, x_j', ..., x_N) = a$. Suppose we change the second individual's preferences to $x_2 \in X$. Then,

$$f(x_1, x_2, x_1, ..., x_j', .., x_N) = a$$

since otherwise f is MNP at $(x_1, x_2, x_1, ..., x_j', .., x_N) \in X^N$ by 2 via x_1. Repeating the same argument for individuals 3 to $j-1$, we then conclude that $f(x_j', x_{-j}) = a$, which is a contradiction. Hence, the claim is established.

But then f^2 is a UNM and STP social choice function and at $x^* = (x_1, x_j) \in X^2$, $2 \in G_2^{f^2}(x^*)$ and for $x_j' \in X, f^2(x_1, x_j') \neq f^2(x^*)$, which contradicts Lemma 6.2 **(b)**. This completes our proof of Theorem 6.3. ☐

Corollary 6.4 *Theorem 6.1.*

Let us now prove the G-S Theorem. Let $x^0 \in X^N$ be such that $x_i^0 \in X$ is $x_i^0 = (a_1 \succ a_2 \succ a_3 \succ ...)$ for $i = 1, ..., N-1$, and $x_N^0 \in X$ is $x_N^0 = (a_2 \succ a_1 \succ a_3 \succ ...)$. Then, by Theorem 6.3 **(a)**, $f(x^0) \in \{a_1, a_2\}$. Suppose $f(x^0) = a_2$. Then, we show that N is a dictator. Note that since $1 \in G_2^f(x^0)$, by Theorem 6.3 **(b)** , $\forall x_1' \in X$, $f(x_1', x_{-1}^0) = a_2$. Consider any $x_2' \in X$. Note that $\forall x_1' \in X, 2 \in G_2^f(x_1', x_2^0, x_3^0, ..., x_N^0)$ and then by Theorem 6.3 **(b)**, $\forall x_1' \in X, \forall x_2' \in X, f(x_1', x_2', x_3^0, ..., x_N^0) = a_2$. Repeating the same argument we then conclude that $\forall x' \in X^N, f(x_N^0, x_{-N}') = a_2$. But then for any $x_N \in X$ such that a_2 is ranked at the top of x_N, $f(x_N, x_{-N}') = a_2$ since otherwise f is MNP at $(x_N, x_{-N}') \in X^N$ by N via x_N^0. Hence, we *conclude* that whenever $x \in X^N$ is such that a_2 is ranked at the top of x_N, $f(x) = a_2$. Now suppose $\exists y \in X^N$ such that $f(y) = a_k$ where a_k is not the top ranked alternative of x_N. Then, by definition $N \in G_2^f(y)$ and by our last *conclusion*, $a_k \neq a_2$. Then by Theorem 6.3 **(b)**, $f(x_N^0, y_{-N}) = a_k$, which is a contradiction. Hence, $N \in I$ is a dictator of f.

Suppose now $f(x^0) = a_1$. Then by definition, $N \in G_2^f(x^0)$ and by Theorem 6.3 **(b)**, $f(z, x_{-N}^0) = a_1$ for $z \in X$ such that $z = (a_3 \succ a_2 \succ a_1 \succ ...)$. Let us consider $x^1 \in X^N$

such that $x_i^1 = x_i^0, i = 1, ..., N-2$, $x_{N-1}^1 = x_N^0$ and $x_N^1 = z$. Then, by Theorem 6.3 **(a)**, $f(x^1) \in \{a_1, a_2, a_3\}$. But if $f(x^1) = a_3$ then f is MNP at $x^1 \in X^N$ by $N-1$ via x_{N-1}^0. So, $f(x^1) \in \{a_1, a_2\}$. If $f(x^1) = a_2$, we show that $N-1 \in I$ is a dictator with the same argument as above. If otherwise, $f(x^1) = a_1$, then we consider $x^2 \in X^N$ such that $x_i^2 = x_i^0, i = 1, ..., N-3$, $x_{N-2}^2 = x_N^0$ and $x_{N-1}^2 = x_N^2 = z$, and we repeat the same steps. Hence, whenever $x^j \in X^N$ is such that $f(x^j) = a_2$, then we show that individual $N-j \in I$ is a dictator of f by the same argument as above. But notice that $\exists j \in \{1, ..., N-1\}$ such that $f(x^j) = a_2$, since if $f(x^k) = a_1$ for all $k = 1, ..., N-2$, then in the last step, by Theorem 6.3 **(a)**, $f(x^{N-1}) \in \{a_2, a_3\}$, but $f(x^{N-1}) \neq a_3$ since otherwise f is MNP at $x^{N-1} \in X^N$ by 1 via x_1^0. This completes our proof.

6.4 Final remarks

A common approach to prove social choice impossibility theorems is as follows. First, to identify or define a subset of individuals that contains a dictator (e.g., decisive coalition, (extremely) pivotal voters, etc.). Then, to investigate some of the properties of that set (e.g., non-empty, etc.), and eventually to show that it is a singleton. The approach pursued in this paper differs from this standard technique by focusing on the individuals who are not candidates for a dictator. More specifically, our approach is built on the following result: if a social choice function satisfies the axioms of unanimity and strategy-proofness, at any given profile, if an individual is not a candidate for a dictator, then she can not change the social choice at that profile by unilateral deviation (Theorem 6.3, Section 6.3). We then show that this result is sufficient to verify that such social choice function is dictatorial (Corollary 6.4, Section 6.3).

It also shows that Theorem 6.3 is logically equivalent to the G-S Theorem: as already established above (Corollary 6.4, Section 6.3) it implies the G-S Theorem and it is also rather easy to deduce it from the G-S Theorem. This equivalence is of considerable interest: Theorem 6.3 addresses to a rather *local property* of a unanimous and strategy-proof social choice function while the property of being dictatorial in the G-S Theorem is a more *global* one.

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Chapter 7

Symmetry vs. Complexity in Proving the Muller-Satterthwaite Theorem [1]

Abstract: In this short note, we first provide two rather straightforward proofs for the Muller-Satterthwaite theorem in the baseline case of 2 person 3 alternatives, and 2 person $n \geq 3$ alternatives. We also show that it suffices to prove the result in the special case of 3 alternatives with arbitrary N individuals, as it then can easily be extended to the general case. We then prove the result in the decisive case of 3 alternatives with arbitrary N individuals by induction on N.

JEL: D71, D72
Keywords: the Muller-Satterthwaite theorem, Monotonicity

7.1 Introduction

An analogous result to Arrow's Impossibility Theorem (Arrow 1963) in the context of voting is the Gibbard-Satterthwaite (G-S) Theorem (Gibbard 1973; Satterthwaite 1975). The interconnection between these two results is a recurrent topic of study in social choice and voting theory. In this respect, it is known that

(a) one can prove each theorem with the help of the other (see Gibbard 1973; Satterthwaite 1975; Schmeidler and Sonnenschein 1978),

[1] Published in *Economics Bulletin*, 2012, **32** (2): 1434-1441.

(b) one can provide a more general result that implies both theorems (e.g., Miller 2009), and

(c) one can obtain a single proof for both theorems (see Reny 2001).

Moreover, it can be observed that the connection between these results is usually obtained through another result, the Muller-Satterthwaite (M-S) Theorem (Muller and Satterthwaite 1977). Hence, the latter constitutes a common ground for the former two. In particular, the fact that the monotonicity axiom in the M-S Theorem is analogous to the independence axiom in Arrow's Impossibility Theorem, and on the unrestricted domain of strict preferences, it is equivalent to the strategy-proofness in the G-S Theorem, allows one to easily obtain results mentioned above in **(a)** - **(c)**: see Reny (2001), Miller (2009) and Chap. 2 in Vohra (2011).

In this paper, we first provide two proofs of a variant of the M-S Theorem (Theorem 7.1, Section 7.2) in the baseline case of 2 person, 3 alternatives. Since it is well known that the M-S Theorem has the G-S Theorem as a corollary (see Reny 2001), we also prove the G-S Theorem in the baseline case. As Barberà (2011) notes, "the 2 person 3 alternative case contains all the essential elements of the (G-S) theorem, in a nutshell," in the sense that it is possible to prove the theorem in the general case by a double induction on the number of individuals and the number of alternatives, once it is proved in the baseline case (see Satterthwaite 1975; Schmeidler and Sonnenschein 1978).

The essence of our proofs is to directly verify the result in the baseline case. However, we reduce the complexity of the problem in two ways: (1) via explicit use of neutrality (symmetry), and (2) via tying up all reasoning on a monotone social choice function with full domain to that of a monotone social choice function with a smaller domain of 1 person society. Then, in Section 7.4 we show that one can easily prove the M-S Theorem in the general case, once it is proved for the case of 3 alternatives (Proposition 7.3). Such extension can be useful in inductive proofs of the M-S Theorem. We then complete the proof of the theorem by proving it in the decisive case of 3 alternatives (Proposition 7.4).

In the next section we introduce our main definitions and state the theorem to be proven. Section 7.3 gives the proofs of the M-S Theorem in the baseline case, while Section 7.4 shows how one can extend the M-S Theorem with 3 alternatives to the general case of arbitrary but finite alternatives. The last section concludes.

7.2 The preliminaries

Let $A = \{a_1, ..., a_n\}$ denote the set of alternatives with $n \in \mathbb{N}$ elements and let X denote the set of strict linear orders (strict rankings) on A. Let there be N individuals in the group. A function $f : X^N \to A$ is called a social choice function (SCF). A member $x = (x_1, ..., x_N)$ of X^N is called a profile of rankings (or simply a profile) and its *i'th* component, x_i, is called the individual *i's* ranking. We say that a SCF $f : X^N \to A$ is *Pareto efficient* (PE) if whenever alternative a is on top of x_i for $i = 1, ..., N$, then $f(x) = a$. It is *monotonic* (MT) if whenever $f(x) = a$ and for every individual i and every alternative b the ranking x'_i ranks a above b if x_i does, then $f(x') = a$. Finally, it is *dictatorial* (DT) if there is individual i such that $f(x) = a$ if and only if a is at the top of x_i, and we denote such function as f_d^i, for $i = 1, ..., N$.

The following result is known as (a variant of) the M-S Theorem (see also Reny 2001):

Theorem 7.1 *If $n \geq 3$, a SCF $f : X^N \to A$ is PE and MT if and only if it is DT.*

7.3 The proofs for the baseline case

Let us introduce a binary relation R_{a_1} on X, called as the *monotonicity relation w.r.t a_1 on X*, defined as $\forall x, y \in X$, $xR_{a_1}y$, i.e. x is related to y according to R_{a_1}, if $x, y \in X$ are such that for any alternative a_j, if a_1 is ranked above a_j in x then so is in y. R_{a_i} for $i = 2, ..., n$ are defined analogously. Whenever $xR_{a_i}y$, we say y is a *successor* of x in R_{a_i}. We also introduce a similar binary relation $(R_{a_1}^N)$ on the full domain of X^N: $\forall x, y \in X^N$, $xR_{a_1}^N y$ if $x, y \in X^N$ are such that for any alternative a_j and any individual $i = 1, ..., N$, if a_1 is ranked above a_j in x_i, then so is in y_i. $R_{a_i}^N$ for $i = 2, ..., n$ are defined analogously.[2] Whenever $xR_{a_i}^N y$, we say y is a *successor* of x in $R_{a_i}^N$. Note that by definition, for any MT SCF $f : X^N \to A$, $xR_{a_i}^N y$ implies that if $f(x) = a_i$, then $f(y) = a_i$, for $i = 1, ..., n$.

$R_{a_i}^N$ for $i = 1, ..., n$ has the following properties:

Lemma 7.2 *For $i = 1, ..., n$,*

(a) *$\forall x, y \in X^N, xR_{a_i}^N y$ if and only if $x_j R_{a_i} y_j$, for $j = 1, ..., N$, and*

(b) *$R_{a_i}^N$ is a preorder (reflexive and transitive) on X^N.*

[2] See also the notion of *monotonic transformation* in Klaus and Bochet (2011).

Proof. **(a)** Both directions of the statement immediately follow from the definitions of R_{a_i} and $R^N_{a_i}$. **(b)** Observe that it is easy to verify that R_{a_i} is a preorder on X. The result follows combining this observation with **(a)**. $\square$

Lemma 7.2 **(a)** allows any reasoning on $R^N_{a_i}$ to be entirely based on R_{a_i}, for $i = 1, ..., n$, while Lemma 7.2 **(b)** allows us to reason recursively.

Let us now assume $N = 2$ and $n = 3$. We can code the elements of X as follows: $a_1 \succ a_2 \succ a_3 \equiv 123$; $a_1 \succ a_3 \succ a_2 \equiv 132$; $a_2 \succ a_1 \succ a_3 \equiv 213$; $a_2 \succ a_3 \succ a_1 \equiv 231$; $a_3 \succ a_1 \succ a_2 \equiv 312$; $a_3 \succ a_2 \succ a_1 \equiv 321$. Let's construct the following graph which represents R_{a_1}:

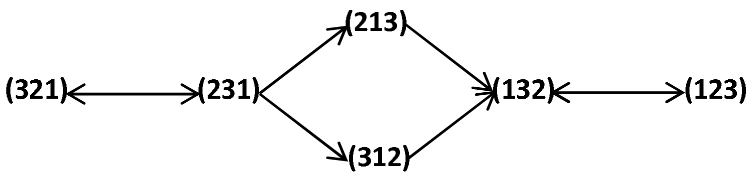

Figure 7.1: Γ_{a_1}

We call Γ_{a_1} as "Monotonicity Graph for a_1" and note that $\forall \alpha, \beta \in V(\Gamma_{a_1})$ (the set of vertices), $\alpha R_{a_1} \beta$ if and only if there is directed path from α to β (we assume that every node is connected to itself by a directed path). Note that if a node is assigned to a_1 under any MT SCF $g : X \to A$, then all of its successors in Γ_{a_1} must be assigned to a_1. Similarly, we can create 'monotonicity graphs' for a_2 and a_3:

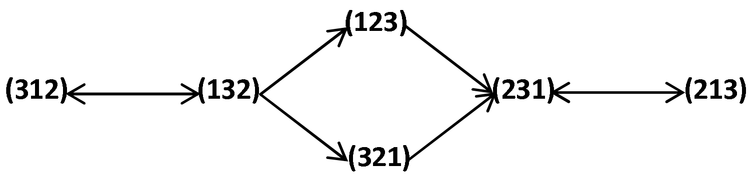

Figure 7.2: Γ_{a_2}

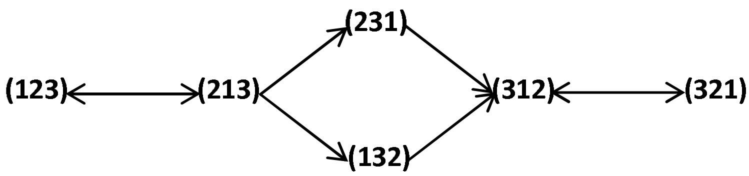

Figure 7.3: Γ_{a_3}

Proof 1: Let us now prove the M-S Theorem. Let $f : X^2 \to A$ be a MT and PE SCF and for $i = 1, 2, 3$, let D_i denote the set of profiles such that a_i is ranked at the top of each ranking:

$$D_1 = \{(123, 123), (123, 132), (132, 123), (132, 132)\},$$

$$D_2 = \{(213, 213), (213, 231), (231, 213), (231, 231)\},$$

$$D_3 = \{(312, 312), (312, 321), (321, 312), (321, 321)\}.$$

Note that none of the profiles in $D_2 \cup D_3$ can be assigned to a_1, by PE. Hence, none of their *predecessors* in $R^2_{a_1}$ (α is a predecessor of β if $\alpha R^2_{a_1} \beta$ and $\alpha \neq \beta$) can be assigned to a_1. So, every profile in

$$P(D_2 \cup D_3) = \{(213, 321), (321, 213), (231, 312), (312, 231), (231, 321), (321, 231)\}$$

needs to be assigned either to a_2 or a_3. Let $f(213, 321) = a_3$. Then, referring to $R^2_{a_3}$ (to Γ_{a_3}) we conclude that

$$\begin{aligned} \omega_1 \quad : \quad & f(213, 321) = f(231, 321) = f(231, 312) = f(213, 312) = \\ = \quad & f(123, 312) = f(132, 312) = f(123, 321) = f(132, 321) = a_3. \end{aligned}$$

Note that there is a complete *symmetry* among the elements of A in our renaming them as a_1, a_2 and a_3: any of the a_1, a_2, a_3 can equally represent any of the three alternatives in A. This *symmetry* is often called as the *neutrality axiom* and it is implicit in our definition of SCF. Because of the *symmetry* between a_3 and a_2 (exchanging the roles of a_3 and a_2), we can conclude that decisions in ω_1 are one and the same as the following decisions:

$$\begin{aligned} \omega_2 \quad : \quad & f(132, 213) = f(123, 213) = f(123, 231) = f(132, 231) = \\ = \quad & f(312, 231) = f(321, 231) = f(312, 213) = f(321, 213) = a_2. \end{aligned}$$

Similarly, since there is a *symmetry* between a_3 and a_1, they are are also one and the same as the following decisions:

$$\begin{aligned} \omega_3 \quad : \quad & f(231, 123) = f(213, 132) = f(213, 123) = f(231, 132) = \\ = \quad & f(321, 132) = f(312, 132) = f(321, 123) = f(312, 123) = a_1. \end{aligned}$$

Hence, **once the initial decision is made, all the other decisions follow** (recall that the profiles in D_i are assigned to a_i by PE, for $i = 1, 2, 3$). Alternatively, let $f(213, 321) = a_2$. Note that $f(213, 132) \neq a_3$, since otherwise referring to $R^2_{a_3}$ we conclude that $f(213, 321) = a_3$, which is a contradiction. Note also that $f(213, 132) \neq a_1$ since otherwise by the *symmetry* between a_1 and a_3, we conclude that $f(231, 312) = a_3$, which then implies that (referring to $R^2_{a_3}$) $f(231, 321) = a_3$. But $(231, 321)$ is a successor of $(213, 321)$ in $R^2_{a_2}$, hence $f(231, 321) = a_2$, which is a contradiction. So, $f(213, 132) = a_2$. Then referring to $R^2_{a_2}$, we can conclude that

$$\begin{aligned} \varphi_1 \quad &: \quad f(213,321) = f(231,321) = f(213,132) = f(213,123) = \\ &= \quad f(231,132) = f(231,123) = f(231,312) = f(213,312) = a_2. \end{aligned}$$

Because of *symmetry*, decisions in φ_1 are one and the same as the following decisions:

$$\begin{aligned} \varphi_2 \quad &: \quad f(123,312) = f(132,312) = f(123,231) = f(123,213) = \\ &= \quad f(132,231) = f(132,213) = f(132,321) = f(123,321) = a_1 \end{aligned}$$

and

$$\begin{aligned} \varphi_3 \quad &: \quad f(321,132) = f(312,132) = f(321,213) = f(321,231) = \\ &= \quad f(312,213) = f(312,231) = f(312,123) = f(321,123) = a_3. \end{aligned}$$

Hence, there are only two possible assignments, $\{\omega_1, \omega_2, \omega_3\}$ and $\{\varphi_1, \varphi_2, \varphi_3\}$, and each of them corresponds to a DT social choice function with one of the two individuals being a dictator. This completes the proof.

Proof 2: Suppose $f(213, 321) = a_3$. Note that (213) has 6 successors in Γ_{a_3} while (321) has 2. Since by Lemma 7.2 **(a)** any combination of successors of (213) and (321) in Γ_{a_3} is a successor of $(213, 321)$ in $R^2_{a_3}$, there are $12 = 6 \times 2$ (including $(213, 321)$) profiles to be assigned to a_3. By *symmetry*, then there are 12 profiles to be assigned to a_i, $i = 1, 2$. Since X^2 has 36 elements, once the initial decision is made all the other decisions follow i.e., there is a unique function $f : X^2 \to A$ which is PE, MT and $f(213, 321) = a_3$. On the other hand $f^2_d : X^2 \to A$ has these properties: it is PE, MT and $f^2_d(213, 321) = a_3$. Hence, $f = f^2_d$.

Alternatively, suppose $f(213, 321) = a_2$. Then, by the same argument as in Proof **1** we conclude that $f(213, 132) = a_2$. Then, repeating the same argument just used for the case of $f(213, 321) = a_3$, we conclude that there is a unique PE and MT function $f : X^2 \to A$ such that $f(213, 132) = a_2$. Since $f_d^1 : X^2 \to A$ has these properties, we then conclude that $f = f_d^1$. This completes our proof.

Proof 2 for $N = 2$, $n \geq 3$: Consider a profile $x \in X^2$ such that $x_1 = (a_1 \succ a_2 \succ ... \succ a_n)$ and $x_2 = (a_2 \succ a_3 \succ ... \succ a_n \succ a_1)$. Let $f : X^2 \to A$ be PE and MT. We claim that $f(x) \in \{a_1, a_2\}$. Suppose on the contrary that $f(x) = a_j \notin \{a_1, a_2\}$. Then, consider $x_1^* = (a_2 \succ a_1 \succ a_3 \succ ... \succ a_n)$. By MT, if $f(x) = a_j \notin \{a_1, a_2\}$, then $f(x_1^*, x_2) = a_j$ which then contradicts PE. Hence, the claim is established. Let $f(x) = a_1$. Since any ranking is a successor of x_2 in R_{a_1}, it has $n!$ successors, and since any ranking with a_1 ranked at the top is a successor of x_1 in R_{a_1}, it has $(n-1)!$ successors. Then by Lemma 7.2 **(a)**, any combination of successors of x_1 and x_2 in R_{a_1} is a successor of x in $R_{a_1}^2$, there are $n!(n-1)!$ profiles to be assigned to a_1 under f. By *symmetry*, then there are $n!(n-1)!$ profiles to be assigned to a_i, $i = 2, ..., n$. Since X^2 has $(n!)^2$ elements, once the initial decision is made all the other decisions follow. Hence, there is a unique PE and MT $f : X^2 \to A$ with $f(x) = a_1$. But since f_d^1 has these properties, we conclude that $f = f_d^1$.

Alternatively, let $f(x) = a_2$. Let $x' \in X^2$ be such that $x'_1 = x_1 = (a_1 \succ a_2 \succ ... \succ a_n)$ and $x'_2 = (a_2 \succ a_1 \succ a_3 \succ ... \succ a_n)$. Since $f(x) = a_2$, $f(x') = a_2$ by MT. Consider $x'' \in X^2$ such that $x''_1 = (a_1 \succ a_3 \succ ... \succ a_n \succ a_2)$ and $x''_2 = x'_2$. We claim that $f(x'') \in \{a_1, a_2\}$. Suppose on the contrary $f(x'') = a_j \notin \{a_1, a_2\}$. Consider $x_2^* = (a_1 \succ a_2 \succ a_3 \succ ... \succ a_n)$. If $f(x'') = a_j \notin \{a_1, a_2\}$, then $f(x'_1, x_2^*) = a_j$ by MT, which then contradicts PE. Hence, the claim is established. Note that $f(x'') \neq a_1$ as otherwise it would imply that $f(x') = a_1$ by MT. Hence, we conclude that $f(x'') = a_2$. Then by the same argument as above we can show that there is a unique PE and MT function with $f(x'') = a_2$, and since f_d^2 has these properties, we then conclude that $f = f_d^2$.

7.4 Sufficiency of proving the M-S Theorem for $n = 3$

The following result shows that it suffices to prove Theorem 7.1 when $n = 3$:

Proposition 7.3 *Suppose Theorem 7.1 holds when $n = 3$. Then it holds for any finite $n > 3$.*

Proof. Let $n > 3$ and let $f : X^N \to A$ be a MT and PE SCF. Let $A_3 = \{a_1, a_2, a_3\} \subset A$ and let $X_{A_3} \subset X^N$ be the set of all profiles $x \in X^N$ such that for each x_i, $i = 1, ..., N$, the top 3 alternatives of x_i are in A_3, and for $j = 4, ..., n$, the $j'th$ top alternative of x_i is $a_j \in A$. We claim that $\forall x \in X_{A_3}$, $f(x) \in A_3$. Suppose on the contrary that $\exists y \in X_{A_3}$ such that $f(y) = a_r$ with $r > 3$. By MT this implies that $\forall x \in X_{A_3}$, $f(x) = a_r$ which contradicts PE. Hence, the claim is established.

Let X_3 be the set of all strict rankings on A_3 and let us define $f^3 : X_3^N \to A_3$ as $\forall z \in X_3^N$, $f^3(z) = f(x^z)$ where $x^z \in X_{A_3}$ is a profile such that x_i^z and z_i coincide on A_3, i.e. $x_i^z = (z_i \succ a_4 \succ ... \succ a_n)$, for all $i = 1, ..., N$. Notice that for each $z \in X_3^N$ there is a unique such $x^z \in X_{A_3}$. Combining this with our claim we conclude that, f^3 is a well defined 3 alternative SCF. Moreover, since f is PE and MT, so is f^3. Hence, by our hypothesis f^3 must be DT.

Without loss of generality we may assume that 1 is the dictator of f^3. Let us then show that 1 is the dictator of f. Consider $z \in X_3^N$ such that $z_1 = (a_1 \succ a_2 \succ a_3)$ and $z_i = (a_2 \succ a_1 \succ a_3)$ for $i = 2, ..., N$. Since 1 is the dictator of f^3, $f^3(z) = f(x^z) = a_1$. Now let $x_i' = (a_2 \succ a_3 \succ ... \succ a_n \succ a_1)$ for $i = 2, ..., N$. We first claim that $f(x_1^z, x_2', x_3^z, ..., x_N^z) = a_1$. Note that $f(x_1^z, x_2', x_3^z, ..., x_N^z) \neq a_2$ as otherwise it would imply that $f(x^z) = a_2$ by MT, and also $f(x_1^z, x_2', x_3^z, ..., x_N^z) \notin \{a_3, ..., a_n\}$ since if $f(x_1^z, x_2', x_3^z, ..., x_N^z) = a_j$ for some $j \in \{3, ..., n\}$, then $f(x_1', x_2', x_3^z, ..., x_N^z) = a_j$ where $x_1' = (a_2 \succ a_1 \succ a_3 \succ ... \succ a_n)$ by MT, which then contradicts PE. Hence, $f(x_1^z, x_2', x_3^z, ..., x_N^z) = a_1$.

We can change rankings of individuals 3 to N from x_i^z to x_i', each at a time, and repeat the same argument to conclude that $f(x_1^z, x_2', ..., x_N') = a_1$. Notice that x_1^z has $(n-1)!$ successors in R_{a_1}, while x_i' has $n!$ successors in R_{a_1}. By Lemma 7.2 **(a)**, then $(x_1^z, x_2', x_3', ..., x_N') \in X^N$ has $(n-1)!(n!)^{N-1}$ successors in $R_{a_1}^N$. Hence, there are $(n-1)!(n!)^{N-1}$ profiles to be assigned to a_1 under f. By *symmetry*, then there are $(n-1)!(n!)^{N-1}$ profiles to be assigned to a_i, $i = 2, ..., n$. Since X^N has $(n!)^N$ elements, there is a unique PE and MT SCF such that $f(x_1^z, x_2', ..., x_N') = a_1$. But since $f_d^1 : X^N \to A$ has these properties, we conclude that $f = f_d^1$. □

For completeness, let us verify that Theorem 7.1 holds when $n = 3$.

Proposition 7.4 *Theorem 7.1 holds when $n = 3$.*

Proof. We use induction on N. As shown above in Proof **1** and **2**, the statement is true when $N = 2$. Suppose it is true when $N = k \geq 2$ and let us consider the case of $N = k + 1$. Let $f : X^{k+1} \to A$ be PE and MT SCF. Consider a profile $x \in X^{k+1}$ with $x_1 = (a_1 \succ a_2 \succ a_3)$ and $x_i = (a_2 \succ a_3 \succ a_1)$ for $i = 2, ..., k + 1$. Then, $f(x) \neq a_3$ since otherwise by MT $f(x^*) = a_3$ for $x^* \in X^{k+1}$ such that $x_1^* = (a_2 \succ a_1 \succ a_3)$ and $x_i^* = x_i$,

$i = 2, ..., k+1$, which then contradicts PE. Hence, $f(x) \in \{a_1, a_2\}$. Suppose $f(x) = a_1$. Notice that $x_1 \in X$ has 2 successors in R_{a_1} while $x_i \in X$ has 3! successors in R_{a_1}. By Lemma 7.2 **(a)**, then $x \in X^{k+1}$ has $2 \cdot (3!)^k$ successors in $R_{a_1}^{k+1}$, and there are $2 \cdot (3!)^k$ many profiles to be assigned to a_1 under f. By *symmetry*, then there are $2 \cdot (3!)^k$ many profiles to be assigned to a_i, $i = 2, 3$. Since X^{k+1} has $(3!)^{k+1}$ elements, there is a unique $f : X^{k+1} \to A$ which is PE, MT and satisfies $f(x) = a_1$. Since f_d^1 has these properties, we conclude that $f = f_d^1$.

Alternatively, suppose $f(x) = a_2$. Let us define $g : X^k \to A$ as $\forall y \in X^k$, $g(y) = f(x_1, y_2, ..., y_{k+1})$, i.e. we fix individual $1's$ ranking at x_1. Note that since f is MT, so is g. We claim that g is also PE. Notice that when $a_1 \in A$ is on top of each ranking y_i, $i = 2, ..., k+1$, $g(y) = a_1$, by PE of f. Note also that when $a_2 \in A$ is on top of each y_i, $i = 2, ..., k+1$, $g(y) = a_2$ by MT. Consider $x' \in X^{k+1}$ such that $x'_1 = (a_1 \succ a_3 \succ a_2)$ and $x'_i = (a_3 \succ a_2 \succ a_1)$ for $i = 2, ..., k+1$. Then, $f(x') \neq a_1$ since otherwise $f(x) = a_1$ by MT, which is a contradiction. Also $f(x') \neq a_2$ since otherwise $f(x^{**}) = a_2$ for $x^{**} \in X^{k+1}$ such that $x_1^{**} = (a_3 \succ a_1 \succ a_2)$ and $x_i^{**} = x'_i$, $i = 2, ..., k+1$, which then contradicts PE. Hence, $f(x') = a_3$. By MT, this implies that $f(x'') = a_3$ for $x'' \in X^{k+1}$ such that $x''_1 = x'_1$ and $x''_i = (a_3 \succ a_1 \succ a_2)$ for $i = 2, ..., k+1$.

Consider $x''' \in X^{k+1}$ such that $x'''_1 = x_1 = (a_1 \succ a_2 \succ a_3)$ and $x'''_i = x''_i$, $i = 2, ..., k+1$. Then, $f(x''') \neq a_1$ since otherwise $f(x'') = a_1$ by MT, which is a contradiction as we just concluded that $f(x'') = a_3$. Also $f(x''') \neq a_2$ since otherwise by MT $f(x^{***}) = a_2$ for $x^{***} \in X^{k+1}$ such that $x_1^{***} = x'''_1$ and $x_i^{***} = (a_1 \succ a_3 \succ a_2)$ for $i = 2, ..., k+1$, which then contradicts PE. Hence, we conclude that $f(x''') = a_3$. This implies that, for all $y \in X^k$ such that a_3 is ranked at the top of each y_i, $i = 2, ..., k+1$, $g(y) = f(x_1, y_2 ..., y_{k+1}) = a_3$. Hence, $g : X^k \to A$ is PE.

Then by our induction hypothesis, $g : X^k \to A$ is DT. Without loss of generality, we may assume that individual 2 is the dictator of g. We claim that 2 is also the dictator of f. Consider $x^{\tilde{}} \in X^{k+1}$ such that $x_2^{\tilde{}} = (a_3 \succ a_2 \succ a_1)$ and $x_i^{\tilde{}} = x_1$ for $i = 1, 3, ..., k+1$. Then, $f(x^{\tilde{}}) = g(x_2^{\tilde{}}, ..., x_{k+1}^{\tilde{}}) = a_3$. Repeating the same argument as in the first part of the proof, we can conclude that there is a unique $f : X^{k+1} \to A$ which is PE, MT and $f(x^{\tilde{}}) = a_3$. Since f_d^2 has these properties, we conclude that $f = f_d^2$. This completes our proof. $\square$

7.5 Final comments

In the first part of this paper (Section 7.3), we presented two rather straightforward proofs of the Muller-Satterthwaite (M-S) Theorem in the baseline case of 2 person and 3 alternatives. With a slight modification of the set up each approach can prove Arrow's Impossibility Theorem in that case. Moreover, in principle it is possible to prove the M-S Theorem in the general case using the same approach. In order to that, one needs to investigate more abstract properties of the binary relations introduced above. In particular, the fact that these relations can be defined recursively, starting with the simplest case of a single individual profile, indicates a possibility for such investigation.

In the second part (Section 7.4), we showed how one can extend the special case of the M-S Theorem with 3 alternatives to the general case of arbitrary but finite number of alternatives (in Proposition 7.3). Such extension can be relevant for inductive proofs of the M-S Theorem. In particular, it shows that for such proofs, using double induction on both number of alternatives and number individuals is unnecessary.

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Doctoral Dissertation
in Economics
Stockholm School of Economics
Sweden, 2012