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Synthetic Collateral Debt Obligation Pricing

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SYTHETIC COLLATERAL DEBT OBLIGATION PRICING

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To my parents and my sister for their

Love and Support

Notation

$B(t)$: Risk free discount factor at time t

$\tilde{B}(t)$: Defaultable discount factor at time t associate with corporate bonds

λ: Default intensity / hazard rate of defaultable asset

y: Tranche Spread

τ Default time

$1\{\tau \le t\}$: Indicate function of default

R Recovery rate

$P\left(\tau < T \middle| M = m\right)$: Conditional on M=m, the probability of no default before time T

Accumulated Loss Function in percentage, equally weighted credit portfolio:

$$l(t) \triangleq \sum_{i=1}^{N} \frac{(1-R_i)}{N} 1(\tau_i \le t),$$ N is the asset number in portfolio

pdf: Probability distribution function

cdf: Cumulative distribution function

LHP: Large Homogeneous Portfolio

$N(0,t)$ The normal distribution with mean 0 and variance t

Abstract

Portfolio credit products, such as CDO and Single Tranche CDO (STCDO) have gained their popularity in financial industry. The key problem facing by the financial engineers is how to price these portfolio credit derivatives, especially how to model the dependent default structure. Copula model proposed by Li (2000) is widely used in practice. Comparing with simulation, factor copula model and conditional independent framework provide good balance between accuracy and computational efficiency, but it is hard to achieve good performance if sticking to normal distribution. There are a few ways to improve it: introducing Levy distributions, using generic copula functions, and the semi parametric estimation. In this paper the Levy distribution and conditional independent factor copula model are examined. The flexibility and accuracy improvement comes from calibrating the skewness and heavy tail of Levy distribution for the underlying marginal distributions. The simulation result and short period prediction result are discussed too.

One of the other benefits of this model is that once calibrating to the standard market tranches spreads, the model can handle the customized CDO, e.g. Single Tranche CDO,.

JEL classification: G13

Key words: factor copula model, portfolio credit derivatives, Levy process

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1. INTRODUCTION

Synthetic Collateralized Debt Obligations (CDO) has gained great attention from both industry and academy due to the increasing traded CDO and the difficulty to price them. CDO is basket credit derivative based on Credit Default Swap (CDS). By pooling and tranching some CDSs, CDO transfers the credit risk of the reference credit portfolio to the investors with different seniority in tranches.

The risk of credit default losses on the reference credit portfolio is divided into tranches of increasing seniority. Losses will first affect the equity or first tranche, next the mezzanine tranches, and finally the senior tranches.

The investors receive the premium payment every 3 month as the compensation to bear the losses for outstanding tranche value incurred by credit defaults. The spread of premium for different tranches is determined by CDO pricing models.

Dow Jones iTraxx Indices

This index composites of 125 equally weighted entities of CDS based on investment grade European bonds according to some liquid and diversification criterion. The pool's outstanding amount will reduce upon default events. And the pool is tranched and sold to the investors quoted by spread[1]. The payment is made every 3 month and the payment is based on the outstanding amount for each tranche at payment tenor.

There is a total index spread which is the average of CDS spread in the pool, since the CDS are equally chosen in the CDO.

The most liquid synthetic CDOs are based on iTraxx index.
The attachment and detachment points for the DJ iTraxx European CDO are 3%, 6%, 9%, 12%, 22%. They are called equity tranche, junior mezzanine tranche, senior mezzanine

[1] The equity tranched is quoted as upfront payment plus 5% annual payment.

tranche, senior tranche and most senior tranche. The tranche 22-100% is not quoted in market, but its spread is implied by the index and the other five tranches.

The CDO pricing models are calibrated by these five tranches spreads and CDS spreads.

DJ iTraxx European CDO 5 year is the chosen instrument since it is most liquid with the least bid ask difference of quoted spreads.

The Economic of CDO

The CDO can help investors to hedge or speculate according to their own risk attitude and perspective on default risk and default correlation.

Here are some benefits gained from CDO trading:

- Some single name credit derivatives are not liquid, the bid ask spread for some singe name CDS could be prohibitive high for the potential investors. CDO provides the liquidity and diversity for the portfolio credit derivative market.
- For some institutes, such as commercial banks and insurance companies, the loan and insurance asset are not tradable. And they can diversify their portfolio to reduce the systematical risk by CDO products.
- The CDO also meets some investors' individual risk aversion attitude for credit risk, which can hardly meet without CDO products.
- The market for credit risks is not complete, and CDO make some credit investment chance possible.
- Facing the new capital requirement in Basel II, the institutes such as banks or insurance companies may find that it is more profitable to use CDO to transfer the asset with credit risk in their balance sheets.
- The CDO indices provide the transparence and liquidity of credit risk market.

2. LITERAUTRE REVIEW

The benchmark of synthetic CDO model is the factor model (factor copula model) and conditional independence concept, which means that conditional on the factors, individual defaults are independent. Since there are 125 CDSs in the Dow Jones iTraxx tranched CDO, this factor copula model can substantially reduce the pricing dimension and make the CDO pricing model tractable.

Homogenous portfolio assumption further simplifies the factor copula model. The building block is copula method introduced by Li (2000). However the Gaussian copula factor model does not yield a unique correlation through all the tranches. Default events are rare and happen in the tail range of the distribution; however, the Gaussian distribution has a very thin tail to capture the dependent defaults. Other copula functions and other distributions are proposed to improve the CDO model's pricing capability.

Among factor copula models, Gaussian copula and student *t* copula model are popular, see [Hull (2004, 2005)]. And [Laurent (2005)] provides a good survey on comparison among the different models. The main advantage in these Gaussian copula and student *t* copula factor models is that the correlation coefficients have economic meaning, hence can be communicated and interpreted as dependence on common market factor or industry section factor. The student t distribution is conditional normal[2] with heavier tail than Gaussian distribution.

However, the Gaussian copula factor model fails to catch the dependent structure of the rare default events; it results in correlation smile for the quoted CDO spreads among different tranches. In this thesis I try to apply other heavy tail asymmetric distribution to price synthetic CDO while still keep the factor model and conditional independence framework for tractability.

[2] Conditional on Chi-Square variable, student *t* distribution is normal.

There are some further extended models beyond conditional independent factor models which are not so tractable, such as double student *t* [Hull & White (2004)], Clayton copula model [Schönbucher (2001)], random factor loadings [Andersen & Sidenius (2004)] and CDO pricing with term structure of default intensity [Schönbucher (2005)]. In addition, Fast Fourier Transformation (FFT) and Inverse FFT are proposed to facilitate computational implementation in the CDO modeling.

3. CREDIT MODELS

There are mainly two kinds of credit models: intensity model and structural model. And sometimes they are both used as a hybrid model.

Intensity Model

Intensity model is also called reduced form model. It models the default hazard rate.

Let $\lambda(t)$ be the hazard rate (default intensity) function given no default up to current time. Let τ be the default time, $S(t)$ be the survival probability from time *0*, and $p(t)$ be the default probability[3]. Then:

$$\frac{d(S(t))}{S(t)} = -\lambda(t) \Rightarrow$$

$$\Pr(\tau > t) = S(t) = \exp\left(-\int_0^t \lambda(s)\,ds\right)$$

$$p(t) = \Pr(\tau < t) = 1 - S(t) = 1 - \exp\left(-\int_0^t \lambda(s)\,ds\right)$$

Structural Model

Structural model originates from Merton's framework of valuing corporate equity as an option. The structural model assumes complete market and risk neutral measure. Let the firm's value $V(t)$ follows the process of:

$$dV(t) = V(t)\left[r(t)dt + \sigma(t)dW(t)\right]$$

$r(t)$ is interest rate ; $\sigma(t)$ is volatility of the firm; $W(t) \sim N(0,t)$ is the standard Brownian motion.

$N(0,t)$ is the normal distribution with mean 0 and variance *t*.

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[3] The probability measure used in this thesis is a risk neutral measure.

By solving this SDE,

$$V(t)=V(0)\exp\left(\int_0^t\left[\left(r(s)-\frac{1}{2}\sigma(s)^2\right)ds+\sigma(s)dW(s)\right]\right)$$

If $r(t)$ and $\sigma(t)$ are constant:

$$V(t)=V(0)\exp\left(rt-\frac{1}{2}\sigma^2 t+\sigma W(t)\right)$$

In the factor copula model, assume $W^i(t)$ be the random variable for the i^{th} company.

Assume all the variables $W^i(t)$ are correlated by depending on a common factor $M(t)$:

$$W^i(t)=\rho M(t)+\sqrt{1-\rho^2}B^i(t)$$

where ρ is correlation coefficient to the common factor $M(t)$, $B^i(t)$ is idiosyncratic factor[4]. $M(t)\sim N(0,t)$, $B(t)^i\sim N(0,t)$. Assume that $M(t)$ and $B(t)$ are independent, $W^i(t)$ is also normal distribution.

It is worth to note that conditional on the common factor $M=m$, given $i\neq j$ the conditional variables $W^i(t)|M=m$ and $W^j(t)|M=m$ are independent.

For simple, I drop off the index *i*.

If $M(t)$ is joint normal distributed vector, $B(t)$ is another independent normal variable,

$$W(t)=\rho M(t)+\sqrt{1-\rho^2}B(t)$$

Since $M(t)$ and $B(t)$ are the normal univariables $N(0,t)$, and scale them by $\frac{1}{\sqrt{t}}$, let:

$$W(t)=\sqrt{t}\cdot W\text{ ; } M(t)=\sqrt{t}\cdot M\text{ ; and } B(t)=\sqrt{t}\cdot\varepsilon$$

So W, M and ε are independent normal variables with standard normal distribution:

$$W(t),M(t),\varepsilon(t)\sim N(0,1)$$

The above formula becomes:

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[4] If $B^i(t)$ $i=1,\cdots,N$ have the same distribution, ρ is same for all $W^i(t)$.

$$W = \rho M + \sqrt{1-\rho^2}\varepsilon$$

M is common factor and ε is individual variable. If they follow other distributions, the above expression is similar. For instance, W, ε and M have the cumulative distribution function of F_1, F_2 and F_3 respectively.

The structural model assumes that the default happens when the firm value first time drops below an exogenous threshold determined by the firm's debt value. For long term bond the threshold is exponential increase function of time t. For short term, the simplest structural model assumes the default happens if the firm's value is less than its debt value D when debt is mature, namely:

$$V(0)\exp\left(\left(r-\frac{1}{2}\sigma^2\right)t+\sigma Wt\right)<D$$

The unconditional default for name i occurs if:

$$W < K \triangleq -\frac{\log\left(V(0)/D\right)+\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

K is called the default threshold.

The conditional default probability becomes:

$$\begin{aligned}
&p(t \mid M=m)=\Pr(\tau<t \mid M=m)\\
&=\Pr\{W<K\}\\
&=\Pr\left\{\rho m+\sqrt{1-\rho^2}\varepsilon<K\right\}\\
&=\Phi\left(\frac{(K-\rho m)}{\sqrt{1-\rho^2}}\right)
\end{aligned}$$

Φ is normal cumulative distribution function.

In general case, assume the unconditional cumulative distribution function of a CDS default is $F_1(x)$ and the default probability $p_i(t)$ at time t, the default threshold is:

$$K = F_1^{-1}(p(t))$$

Recall:

$$p(t) = 1 - \exp(-\lambda t)$$

Conditional on *M=m*, the default happens if:

$$W = \left(\rho m + \sqrt{1-\rho^2}\,\varepsilon\right) < K$$

$$\Rightarrow \varepsilon < \frac{(K - \rho m)}{\sqrt{1-\rho^2}}$$

The conditional default probability is:

$$P\left(\tau < t \middle| M = m\right) = F_2\left(\frac{(K - \rho m)}{\sqrt{1-\rho^2}}\right)$$

Where τ is the default time for the firm. F_2 is the cumulative distribution function of individual variable. In simple case, it follows standard normal distribution Φ, $N(0,1)$.

The conditional default independent model is built upon structured model, while the default probability is from intensity model. The above conclusions play important role in later synthetic CDO pricing model.

Drawbacks of Structural model

The major drawback of the structural model is that the default probability reduces to zero as time horizon approaching zero. The structural model calculates very small spread for corresponding short time horizon. However this violates the observed market data. Since there is some concern of credit event happens when the firm value drops suddenly in a short period of time. On the other hand, the intensity model allows default jump in a very short time horizon, and it is popular in the industry.

In CDO pricing model, the conditional default probability is based on structural model, and the unconditional default probability $p(t)$ at time t is derived from intensity model.

Credit Default Swap

First the defaultable bond price is investigated in the intensity model. Let $B(t)$ be risk free discount factor at time *t*. If interest rate r is stochastic, then:

$$B(t) = E\left[\exp\left(-\int_0^t r(s)ds\right)\right]$$

However, stochastic interest rate r has little effect on the result of synthetic CDO model in this thesis, deterministic interest rate r is used. Then $B(t)$ becomes:

$$B(t) = \exp\left(-\int_0^t r(s)ds\right)$$

If the bond recovery rate is 0, the corresponding defaultable discount factor $\tilde{B}(t)$ is:

$$\tilde{B}(t) = B(t)S(t) = \exp\left(-\int_0^t \left(r(s)+\lambda(s)\right)ds\right)$$

$\lambda(t)$ is the default intensity and $S(t)$ is the cumulative survival probability at time *t*.

Assume that the bond recovery rate R is based on par value, then $\tilde{B}(t)$ becomes:

$$\tilde{B}(t) = \exp\left\{-\int_0^t \left(r(s)+\lambda(s)\right)ds\right\} + R\int_0^t \lambda(s)\cdot S(s)\cdot \exp\left(-\int_0^s r(\tau)d\tau\right)ds$$

$$= \exp\left\{-\int_0^t \left(r(s)+\lambda(s)\right)ds\right\} + R\int_0^t \exp\left(-\int_0^s \left(r(\tau)+\lambda(\tau)\right)d\tau\right)\lambda(s)\,ds$$

In above expression, the first term is the payment if there is no default during the CDS contract life time. The second term is the expected recovery amount of the default bond if the default happens during the CDS contract life time.

The above formula shows that $\tilde{B}(t)$ is not equal to $B(t)$ even when $R=1$. This is due to the recovery payment is par once bond defaults while the corresponding zero coupon bond value is less than par before maturity. So when $R=1$, $\tilde{B}(t)$ is slightly larger than $B(t)$; assuming R=1, $\tilde{B}(t)$ equals $B(t)$ only if risk free interest rate is zero.

For CDS with 5 years to maturity or less, we can assume the interest rate and default intensity are constant, the above formula becomes:

$$\tilde{B}(t) = \exp\left\{-(r+\lambda)t\right\} + \lambda \cdot R \cdot \int_0^t e^{-(r+\lambda)\tau} d\tau$$

Under no arbitrage opportunity assumption, at the starting time of CDS, the premium payment spread is determined to ensure the expected premium leg value equals to the expected default leg value. The CDS premium pays every three months.

The following is how to calculate the expected premium leg value

Assuming y is the CDS premium spread, then:

$$y\left(\sum_{j=1}^{m}\left(t_j - t_{j-1}\right)\exp\left(-\lambda t_j\right)B\left(t_j\right) + accrumed\right)$$

If default occurs, the CDS accrued interest is paid with the same spread rate for the interval between default and last premium payment date.

The expected accrued payment value approximates to:

$$accrumed = \sum_{j=1}^{m} B\left(\frac{t_j + t_{j-1}}{2}\right)\left\{\exp\left(-\lambda t_{j-1}\right) - \exp\left(-\lambda t_j\right)\right\}\left(\frac{t_j - t_{j-1}}{2}\right)$$

Adding the above two terms together, the following integration is a good approximation for the expected premium leg value:

$$y\int_0^t \exp(-\lambda s)\, B(s)\, ds$$

On the other side, the expected default leg value is:

$$\int_0^t (1-R)\lambda e^{-\lambda s} B(s)\, ds$$

Further if the CDS term is within 5 year, the creditworthiness is relatively stable. So we can assume that the hazard rate λ and recovery rate, R, are constant.

At the beginning of the CDS contact, the values for two legs are equal, so:

$$\int_0^t (1-R)\lambda e^{-\lambda s} B(s)\, ds = y\int_0^T e^{-\lambda s} B(s)\, ds$$

In this simplest case, hazard rate λ is determined by credit default y spread and recovery rate R:

$$\lambda = \frac{y}{(1-R)}$$

From the CDS spread, the hazard rate can be derived as above; hence the default probability at any given time is obtained.

4. FACTOR COPULAE FUNCTIONS

The most commonly used copula functions are Gaussian and student *t*. Other copula functions include double *t* copula, exponential copula, Archimedean copula and Clayton copula function, for details see appendix.

Common Copula functions

Gaussian Copula

As shown in previous chapter, conditional on common factor M, the default probability is:

$$P(\tau < t | M = m) = \Phi\left(\frac{(K - \rho \cdot m)}{\sqrt{1-\rho^2}} \right)$$

$$K = \Phi^{-1}(p(t))$$

Where Φ is normal cdf; M and ρ are either scalar ρ or vector, where ρ^2 becomes $\|\rho\|^2$.

Student t Distribution

The student distribution is the quotation of a normal variable and square root of a Chi-Square distribution scaled by its degree of freedom, namely:

$$\frac{X}{\sqrt{Z/n}} \sim t_n$$

$$\text{Where } \begin{cases} X \sim N(0,1) \text{ normal distribution} \\ Z \sim \chi^2_n \end{cases}$$

This is a symmetric *t* distribution, it has similar bell shape curve as normal distribution, but with heavier tail.

Conditional on each implementation of the random variable Z , the conditional variable $\left(\frac{X}{\sqrt{Z/n}} \middle| Z = z \right)$ follows normal distribution. So the above conditional independent Gaussian copula expression is applicable.

Double *t* Copula:

In double *t* copula, the distributions change from normal distribution to student *t* distribution.

As shown in previous chapter, assume the individual firm random variable is correlated with the common random variable:

$$W = \rho \left(\frac{n_1 - 2}{n_1} \right)^{\frac{1}{2}} \xi_1 + \sqrt{1-\rho^2} \left(\frac{n_2 - 2}{n_2} \right)^{\frac{1}{2}} \xi_2$$

Where t_1 and t_2 are independent *t* distribution with degree of freedom n_1 and n_2 . ξ_1 is the market factor.

The random variable is normalized with unit variance to get a unique expression.

$$P\left(\tau_i < t \middle| \xi_1 = m\right) = t_{n_2} \left(\sqrt{\left(\frac{n_2}{n_2 - 2} \right)} \frac{\left(\bar{K}(m) - \rho M \sqrt{\left(\frac{n_1}{n_1 - 2} \right)} \right)}{\sqrt{1-\rho^2}} \right)$$

$$\bar{K}(M) = F_*^{-1}\left(p(t) \right)$$

F_* is the cumulative distribution function for W .

The double student t copula overcomes the thin tail problem in the Gaussian copula. By adjusting the degree freedom parameters, the double student t copula is capable to catch the correlated default structure more accurate than the Gaussian copula. However the determinant of the degree of freedom is a new problem. And the student t distribution is not stable under convolution[5] hence the double student t copula is computationally costly.

Generalization for Levy copula:

First I will generalize the expression:

$$W = \rho\left(\frac{n_1-2}{n_1}\right)^{\frac{1}{2}}\xi_1 + \sqrt{1-\rho^2}\left(\frac{n_2-2}{n_2}\right)^{\frac{1}{2}}\xi_2$$

If we define $\left(\frac{n_1-2}{n_1}\right)^{\frac{1}{2}}\xi_1$ and $\left(\frac{n_2-2}{n_2}\right)^{\frac{1}{2}}\xi_2$ as new random variables: common factor M and individual variable ε. If the cumulative distribution function for individual variable ε is F_2, then the conditional individual default probability is:

$$P\left(\tau_i < t \middle| X_1 = m\right) = F_2\left(\frac{\left(F_*^{-1}\left(p(t)\right) - \rho m\right)}{\sqrt{1-\rho^2}}\right)$$

This generic formula is also held in Levy one factor copula. For example, common factor M and individual variable ε follow Variance Gamma distribution [Luciano (2005)] or Normal Inverse Gaussian distribution [Kalemanova (2005)].

The distribution parameters in these Levy factor models are calibrated to the quoted market tranches spreads. This calibration is associated with the model in the following chapters on synthetic CDO pricing.

[5] The distribution of sum of random variables is implemented by convolution.

5. SYNTHETIC CDO PRICING

The general problem of pricing synthetic CDO is how to calculate the dynamic loss distribution of the reference portfolio over different time horizon under some specified default correlation structure.

The factor copula used in the synthetic CDO pricing model in this thesis is based on conditional independence; namely conditional on some common factor, the conditional defaults are independent.

The justification for the factor copula model is that the correlation coefficients in the model have economy interpretation and are easy to commute. Similar methods have been adopted: see [JPMorgan (2004)] on base correlation and [Elizalde (2005)] for general review on different CDO pricing models.

The frame work of pricing synthetic CDO in this thesis is:

1. Find out the marginal default distribution under the risk neutral measure
2. Identity default dependent structure
3. Discount the default loss distribution to calculate the expected default loss
4. Calculate the expected premium based on marginal default distribution and default dependent structure
5. Find out premium payment spread for each tranche by dividing expected default loss with expected premium value

The essential components in CDO pricing model are: individual marginal default distribution and default correlation structure.

If the default correlation increases, the CDO’s equity tranche spread decreases, while the senior tranches spreads increase. The relationship between the spreads and correlation for the mezzanine tranches is more complicated.

Since the 5-year synthetic CDO tranches' spreads are not sensitive to interest rate, the deterministic Euro interest rate swap is used. Most synthetic CDOs are financed by interest rate swap, so the swap rate is the proper choice for discounting.
And the recovery rate is assumed constant.

Model Specification

I choose the conditional independent copula model by adjusting marginal default distribution to Levy distribution. First a very generic *t* copula model is discussed briefly and then conditional independent factor copula model is elaborated. The conditional independence approach is more parsimonious than the generic copula model. So it speeds up the computational time and has clear economic interpretation.

A Generic *t* Copula Model

Consider a portfolio with N different names of CDS; this generic CDO model does not required equally weighted CDS pool.

Let the random variable τ_i represents the random default time for each name.
According to intensity model, the risk neutral default probability for each name is:

$$p_i(t) = \Pr(\tau_i \le t) = 1 - \exp\left(-\int_0^t \lambda_i(u)du\right)$$

The joint default time distribution can be expressed in the following generic formula:

$$\begin{aligned}&\Pr(\tau \le T) = \Pr(\tau_1 \le T_{1,} \cdots \tau_i \le T_{i,} \cdots \tau_N \le T_N)\\ &= \Phi_N\left(\phi_1^{-1}\left(p_1(T_1)\right), \phi_2^{-1}\left(p_2(T_2)\right), \cdots \phi_N^{-1}\left(p_N(T_N)\right)\right)\end{aligned}$$

T is the time vector; Φ_N is a multi dimension copula function with covariance matrix Σ and ϕ_i^{-1} is the inverse of marginal cumulative distribution function.

The above copula function is an *n*-dimension cumulative distribution function. Under some loose conditions, differentiating this *n*-dimension joint cumulative distribution function uniquely determines an *n*-dimension joint probability distribution function $f(\tau \le T)$.[6]

Here $\tau \le T$ means $\tau_1 \le T_{1,} \cdots \tau_i \le T_{i,} \cdots \tau_N \le T_N$

For multi dimension Gaussian or *t* copula function, there is explicit expression on this joint default distribution function. For example, *t* copula will give the following result on the multi dimension probability distribution function: [7]

$$f(\tau \le T) = C_{N,v} \bullet \left(1 + \frac{z^T \Sigma^{-1} z}{v}\right)^{-\frac{N+v}{2}} \prod_{k=1}^{N} \lambda_k (T_i)(1 - p_k(T_k)) \bullet \frac{1}{C_{1,v}} \bullet \left(1 + \frac{z^T \Sigma^{-1} z}{v}\right)^{\frac{1+v}{2}}$$

$$\text{Here } C_{N,v} = \frac{\Gamma\left(\frac{v+N}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{|\Sigma|(v\pi)^N}},$$

$\Gamma(x)$ is Gamma function.

Here $z_k \triangleq t_{1,v}^{-1}(p_k(T_k))$, $t_{1,v}$ is the cumulative distribution function of one dimension *t* distribution with *v* degree of freedom; $t_{1,v}^{-1}$ is the corresponding inverse function.

After specifying the expressions of premium and default value for the tranche in synthetic CDO with attachment point α and the detachment point β, we can use this joint default distribution to compute the expected premium value and default value for that tranche.

This approach is also applicable to cash CDO given the cash flow structure.

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[6] See appendix on Sklar' theorem
[7] For details, see [Andersen 2003]

Conditional Independent Model for synthetics CDO Pricing

In this model, the synthetic CDO pool consists of N equally weighted names of CDS. Since the synthetic CDO is composite of equally weighted CDS, the conditional default loss for each CDS is assumed homogeneous, hence exchangeable.

The conditional independent model can reduce the dimension of covariance matrix Σ in the previous generic t copula model and speed up the computation.

The conditional independent model is also referred as semi analytical parametric model on dependent defaults, because there is an explicit expression for the conditional cumulative default loss function of the CDS pool.

The loss function for each individual CDS is:

$$\frac{(1-R)}{N}1(\tau_i \le t).$$

Here R is recovery rate; random variable τ_i represents the random default time for each name; and it is scaled by the equal weight $\frac{1}{N}$ for each CDS.

Define: at time t, the cumulative portfolio percentage loss function as

$$l(t) \triangleq \sum_{i=1}^{N} \frac{(1-R)}{N}1(\tau_i \le t)$$

The function $l(t)$ is standardized to range from 0 to 1. It takes the discrete values of

$$k\frac{(1-R)}{N}, k = 0,1,\cdots,N.$$

It is worthy to notice that:

$$E\left[l(t)\right] = \sum_{i=1}^{N} \frac{(1-R)}{N} E\left[1(\tau_i \le t)\right] = \sum_{i=1}^{N} \frac{(1-R)}{N} \Pr(\tau_i \le t)$$

And

$$\Pr(\tau_i \le t) = 1 - \exp(\lambda_i t)$$

This shows the linkage between the indication function and the individual default probability, which is used to calculate the expected cumulative portfolio percentage loss.

Recalled that the conditional default probability for i[th] CDS is:

$$p_i\left(t\mid m\right) \triangleq \Pr\left(\tau_i < t \middle| M = m\right) = F_{2,\,i}\left(\frac{\left(K_i - \rho_i m\right)}{\sqrt{1-\rho_i^{\,2}}}\right)$$

Where $F_{2,\,i}$ is the cumulative distribution function of the i[th] individual variable ε_i ,

$$K_i = F_{1,\,i}^{-1}\left(1-\exp\left(-\int_0^t \lambda\left(s\right)ds\right)\right)$$

$F_{1,i}$ is the unconditional cumulative distribution function of the i[th] CDS default probability.

Since the CDSs are equally weighted, the homogeneity in the CDS pool is assumed, then the subscript *i* is omitted.

The cumulative portfolio percentage loss function $l\left(t\right) = \sum_{i=1}^{N} \frac{\left(1-R\right)}{N} 1\left(\tau_i \le t\right)$ takes the discrete values $k\frac{\left(1-R\right)}{N}, k = 0,1,\cdots,N$.

Notice that conditional on the common factor, the individual defaults are independent. Then the probability of cumulative portfolio percentage loss being $k\frac{\left(1-R\right)}{N}$ is:

$$\Pr\left(l\left(t\right) = \frac{k\left(1-R\right)}{N} \middle| M = m\right) = \binom{N}{k}\left(p\left(t\middle|m\right)\right)^k\left(1-p\left(t\middle|m\right)\right)^{N-k}$$

The conditional default loss for each CDS is a binary random variable; only two possible states are possible:

$$\begin{cases} \frac{\left(1-R\right)}{N}, & \text{with probability } p_i\left(t\middle|m\right) \\ 0, & \text{with probability } 1-p_i\left(t\middle|m\right) \end{cases}$$

And the CDS is assumed homogenous, so the conditional default loss for each CDS is exchangeable. Thus the cumulative portfolio percentage loss follows the above binomial expression.

The unconditional probability of cumulative portfolio percentage loss being $k\frac{(1-R)}{N}$ is obtained by integrating the product of the above conditional expression and probability density function of common *M* over the value range of common factor *M*:

$$\Pr\left(l(t)=\frac{k(1-R)}{N}\right)=\int_{-\infty}^{\infty}\binom{N}{k}\left(p(t|m)\right)^{k}\left(1-p(t|m)\right)^{N-k}dF_3(m)$$

Here, $F_3(m)$ is the cumulative distribution function for common factor *M* at time *t*.

Now consider the probability that the cumulative portfolio percentage loss $l(t)\triangleq\sum_{i=1}^{N}\frac{(1-R)}{N}1(\tau_i\le t)$ does not exceed $x\in[0,1]$. x is percentage loss of the portfolio.

Define $F(x)$ as the probability that cumulative portfolio percentage loss doesn't exceed *x*:

$$F(x)=\Pr\left(l(t)\le x\right),\quad x\in[0,1]$$

From here $F(x)$ is called cumulative portfolio percentage loss probability function or just cumulative loss probability for simple.

F(x) is the cumulative distribution function of portfolio percentage loss, both x and function value range between 0 and 1.

Notice that $F(x)$ is a function of time *t*, and $F(x)$ should be written as $F(x,t)$, however if there is no confusion in the context, I omit *t* in the expression.

Specifically, in the above discrete setting for portfolio default loss percentage is:

$$F(x) = \sum_{k=0}^{\lfloor xN \rfloor} \Pr\left(l(t) = \frac{k(1-R)}{N} \right)$$

Here $\lfloor xN \rfloor$ is the maximum integral less or equal to xN

Similarly, conditional cumulative probability of portfolio default loss percentage not exceeding $x \in [0,1]$ is defined as:

$$F(x \mid m) = \Pr\left(l(t) \le x \mid M = m \right), \quad x \in [0,1]$$

Specifically,

$$F(x \mid m) = \sum_{k=0}^{\lfloor xN \rfloor} \Pr\left(l(t) = \frac{k(1-R)}{N} \mid M = m \right)$$

Generally, once we specify this cumulative loss probability, the portfolio's joint default distribution is determined. So the tranches spreads in synthetic CDO are determined. The following is the details of the procedure to calculate the tranche spreads.

According to the CDO tranche structure, at time *t*, the premium payment is the product of the spread and outstanding value for each tranche.

At time *t*, the outstanding value of tranche (α, β) takes the form of following function:

$$\mathrm{H}[\mathrm{t}] \triangleq: \left(\beta - l(t)\right)^{+} - \left(\alpha - l(t)\right)^{+}$$

Similarly at time *t*, the tranche loss value of tranche (α, β) is defined in this function:

$$\mathrm{Q}(t) \triangleq: (\beta - \alpha) - \mathrm{H}(t) = \left(l(t) - \alpha\right)^{+} - \left(l(t) - \beta\right)^{+}$$

Notice the fact that at any time *t* the sum of tranche outstanding value and tranche loss value is the initial tranche value:

$$\mathrm{Q}(t) + H(t) = (\beta - \alpha)$$

Since the portfolio percentage loss $l(t)$ takes discrete values, so the functions of $\mathrm{Q}(t)$ and $H(t)$ are discrete increasing functions; the increment is triggered y default occurrence.

Define $dQ(t)$ is as the increment of $Q(t)$ at time *t*. $dQ(t)$ is positive when default happens. This is for the later integration.

Let me further investigate the relationship between the values of $\mathrm{Q}(t)$ and $l(t)$:

$$
\begin{aligned}
\mathrm{Q}(t) &= \left(l(t)-\alpha\right)^{+} - \left(l(t)-\beta\right)^{+} \\
&= \begin{cases} l(t)-\alpha, & l(t)\in[\alpha,\beta] \\ \beta-\alpha, & l(t)\in[\beta,1] \\ 0, & l(t)\in[0,\alpha] \end{cases}
\end{aligned}
$$

Apply this result, the expected value for tranche loss value $\mathrm{Q}(t)$ can be rewritten as:

$$
\begin{aligned}
E\left[\mathrm{Q}(t)\right] &= E\left[\left(l(t)-\alpha\right)^{+} - \left(l(t)-\beta\right)^{+}\right] \\
&= \int_{\alpha}^{\beta}(x-\alpha)\,dF(x) + \int_{\beta}^{1}(\beta-\alpha)\,dF(x) \\
&= \left(F(x)(x-\alpha)\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta}F(x)\,dx\right) + (\beta-\alpha)F(x)\Big|_{\beta}^{1} \\
&= F(\beta)(\beta-\alpha) - \int_{\alpha}^{\beta}F(x)\,dx + (\beta-\alpha)\left(F(1)-F(\beta)\right) \\
&= F(1)(\beta-\alpha) - \int_{\alpha}^{\beta}F(x)\,dx \\
&= \int_{\alpha}^{\beta}\left(1-F(x)\right)\cdot dx
\end{aligned}
$$

The third line is derived by integrating by parts.

The cumulative portfolio percentage loss probability function $F(x)$ is defined above as the probability of cumulative portfolio percentage loss not exceeding *x*:

$$
F(x) = \Pr\left(l(t)\le x\right), \quad x\in[0,1]
$$

The last line is due to the fact that $F(1)=1$ for any given time *t*, which means the probability of percentage loss less than 1 is definitely for sure, with probability 1.

And the expectation of the outstanding tranche value can be calculated according to this:

$$
\begin{aligned}
E\left[\mathrm{H}(t)\right] &= E\left[(\beta-\alpha)-\mathrm{Q}(t)\right] \\
&= (\beta-\alpha)-E\left[\mathrm{Q}(t)\right] \\
&= \int_{\alpha}^{\beta} F(x)\,dx
\end{aligned}
$$

These two expectations expression hold over different time through the CDO life time.

The premium leg value $PL(t)$ is the present value of all spread payments made based on outstanding tranche value over the payment period of time:

$$PL(t) = y\sum_{k=1}^{K} \Delta t_k H(t_k) B(t_k)$$

And the expectation of premium leg value $PL(t)$ is:

$$E\left[PL(t)\right] = y\sum_{k=1}^{K} \Delta t_k E\left[H(t_k)\right] B(t_k)$$

Here t_k, $k = 1,\cdots,K$ are the premium spread payment dates, and $t_K = T$ is the maturity date of the synthetic CDO. Let $t_0 = 0$; $\Delta t = t_k - t_{k-1}$ denotes the time interval between each payment; y is CDO spread rate; $B(t_i)$ is the discount factor. The expectation is calculated under risk neutral measure.

If the accrued premium payment is considered for the defaults between payment times, the more accurate premium expectation of premium leg value should be:

$$
\begin{aligned}
&E\left[PL(t)\right] = \\
&y\sum_{k=1}^{K} \Delta t_k E\left[H(t_k)\right] B(t_k) + y\sum_{k=1}^{K}\Big(E\left[H(t_k)\right] - E\left[H(t_{k-1})\right]\Big)\frac{(t_k - t_{k-1})}{2} B\left(\frac{(t_k - t_{k-1})}{2}\right)
\end{aligned}
$$

The second term is the accrued premium payment, assuming the defaults happen at the middle points of each time interval. It is a first order approximation. If higher order approximation is applied, then the expectation of premium leg value asymmetrically approaches to the following expression:

$$E\left[PL(t)\right] = y\int_0^T B(t)E\left[H(t)\right]dt$$

In the programming the first order approximation is used. But for conciseness, in the following part of the thesis, I use the first expression without accrued payments.

Rephrase the definition of $dQ(t)$ as the increment of tranche loss value $Q(t)$ at time *t*. $dQ(t)$ is positive when default happens.

The default leg value is the summery of product of $dQ(t)$ and the corresponding discount factor at each default time, namely:

$$DL = \int_0^T B(t)\,dQ(t)$$

This is Riemann-Stieltjes integration since $\mathrm{Q}(t)$ is a discrete increasing function.

And the expectation of default leg value is:

$$E\left[DL\right] = E\left[\int_0^T B(t)\,dQ(t)\right]$$

Then simplify the DL expression.

First integrate the default leg value DL by parts:

$$\begin{aligned} DL &= \int_0^T B(t)\,dQ(t) \\ &= B(t)Q(t)\Big|_{t=0}^{t=T} - \int_t^T dB(t)Q(t) \end{aligned}$$

Then apply the fact that:

$$B(t) = \exp\left(-\int_0^t f(s)\,ds\right);$$

$f(t)$ is the instantaneous forward rate. So:

$$dB(t) = -f(t)B(t).$$

Then the above default leg value DL expression becomes:

$$
\begin{aligned}
DL &= \int_0^T B(t)\,dQ(t) \\
&= B(t)Q(t)\Big|_{t=0}^{t=T} - \int_0^T f(t)B(t)Q(t)\,dt \\
&= B(T)Q(T) - \int_0^T f(t)B(t)Q(t)\,dt
\end{aligned}
$$

The third line is because that no default occurs at t=0, hence $Q(t=0)=0$.

And the expectation of default leg value *DL* becomes:

$$
\begin{aligned}
E[DL] &= E\left[\int_0^T B(t)\,dQ(t)\right] \\
&= E\left[B(T)Q(T) - \int_0^T f(t)B(t)Q(t)\,dt\right] \\
&= B(T)E\left[Q(T)\right] + \int_0^T E\left[Q(t)\right]f(t)B(t)\,dt
\end{aligned}
$$

Here the expectation and integration is assumed exchangeable.

When synthetic CDO starts, y for the tranche (α,β) the expectations of default leg value and premium leg value are set equal:

$$
E[DL] = E[PL].
$$

Since $E\left[PL(t)\right] = y\sum_{k=1}^{K}\Delta t_k E\left[H(t_k)\right]B(t_k)$, then the spread y of the tranche (α,β) is:

$$
y = \frac{B(T)E\left[Q(T)\right] + \int_0^T E\left[Q(t)\right]f(t)B(t)\,dt}{\sum_{k=1}^{K}\Delta t_k E\left[H(t_k)\right]B(t_k)}
$$

If the forward rate is a constant *r*, the expectation of default leg value $E[DL]$ becomes:

$$
E[DL] = E\left[\int_0^T B(t)\,dQ(t)\right] = B(T)E\left[Q(T)\right] + \int_0^T E\left[Q(t)\right]f(t)B(t)\,dt
$$

$$
= \exp(-rT)E\left[Q(T)\right] + r\int_0^T \exp(-rt)E\left[Q(t)\right]dt
$$

Then the spread *y* is:

$$y = \frac{\exp(-rT)E(Q(T)) + r\int_0^T \exp(-rt)E[Q(t)]dt}{\sum_{k=1}^{K} \Delta t_k E[H(t_k)]B(t_k)}$$

More details on calculating $E(H(t))$ and $E(Q(t))$ will be conducted in the next section.

Cumulative Loss Distribution Function of Homogenous Portfolio

Since the main problem in synthetic CDO pricing model is to derive the cumulative loss distribution of the correlated defaults, in this section details on conditional independent factor copula model is presented. Since the synthetic CDO is composite of equally weighted CDS, the conditional default loss for each CDS is assumed homogeneous, hence exchangeable. .

Remind that the default leg value is:

$$E[DL] = B(T)E[Q(T)] + \int_0^T E[Q(t)]f(t)B(t)dt$$

Since $E[Q(t)]$ is the only unknown, the following shows how to calculate $E[Q(t)]$.

Remind that for tranche (α, β), the tranche loss value function is:

$$Q(t) = (l(t)-\alpha)^+ - (l(t)-\beta)^+ = \begin{cases} 0, & l(t) \in [0,\alpha] \\ l(t)-\alpha, & l(t) \in [\alpha,\beta] \\ \beta-\alpha, & l(t) \in [\beta,1] \end{cases}$$

Remind the probability that the cumulative loss function $l(t)$ equals to $\frac{k(1-R)}{N}$ is:

$$\Pr\left(l(t) = \frac{k(1-R)}{N}\right) = \int_{-\infty}^{\infty} \binom{N}{k} (p(t|m))^k (1-p(t|m))^{N-k} dF(m)$$

Since $l(t)$ takes the discrete values of $\frac{k(1-R)}{N}, k = 1, \cdots, N$, the expectation of tranche loss value $E[Q(t)]$ becomes:

$$E\left[Q(t)\right] =$$

$$(\beta-\alpha)\sum_{k=\lfloor \beta N\rfloor+1}^{N}\Pr\left(l(t)=\frac{k(1-R)}{N}\right)+\sum_{k=\lfloor \alpha N\rfloor+1}^{\lfloor \beta N\rfloor}\left(l(t)-\alpha\right)\bullet\Pr\left(l(t)=\frac{k(1-R)}{N}\right)$$

In there are one summation and one integration in this expression of $E\left[Q(t)\right]$, since $\Pr\left(l(t)=\frac{k(1-R)}{N}\right)=\int_{-\infty}^{\infty}\binom{N}{k}\left(p(t|m)\right)^{k}\left(1-p(t|m)\right)^{N-k}dF(m)$ requires integration over the range of common factor.

So in the expression $E[DL]=B(T)E\left[Q(T)\right]+\int_0^T E\left[Q(t)\right]f(t)B(t)dt$, there are two integrations and one summation. The additional integration is over the time horizon.

The integrations and summation are assumed exchangeable. In the numerical implementation, it is more convenient to make the integration over time before the summation in computing $E\left[Q(t)\right]$.

And the expectation of tranche outstanding value can be derived by:

$$E\left[H(t)\right]=(\beta-\alpha)-E\left[Q(t)\right]$$

Once $E\left[Q(t)\right]$ and $E\left[H(t)\right]$ are calculated, the tranche spread is delivered by:

$$y=\frac{B(T)E\left[Q(T)\right]+\int_0^T E\left[Q(t)\right]f(t)B(t)dt}{\sum_{k=1}^{K}\Delta t_k E\left[H(t_k)\right]B(t_k)}$$

The main difficulty and variety of the synthetic CDO pricing model lie in how to compute cumulative percentage loss probability function $F(x)=\Pr\left(l(t)\le x\right),\quad x\in[0,1]$. This crucial function can be asymmetric Levy distributions providing more flexibility and accuracy. But the parameters in Levy distributions need to be calibrated to the market quoted spreads *y*.

Asymptotic Large Homogenous Portfolio Approximation

The idea of asymptotic large homogenous portfolio approximation is from [Vasicek, (1987), (1991)]. According to the law of large number, when the number of CDS in the portfolio is very large, the distribution of portfolio percentage loss approaches to the individual default loss. However since the accurate expression on cumulative portfolio percentage loss is shown above, this asymptotic approximation is no more necessary. But, this model can be modified based on Levy processes, see [Albrecher (2006)], [Baxter (2006)] and [Moosbrucker (2006a, 2006b)], especially useful for risk management[8].

The Levy process in the model compensates the inaccuracy incurred by the asymptotic approximation. The adjusted parameters in Levy distribution provide the capability and flexibility to fit the market quoted spreads.

Remind the expectation of tranche loss value function:

$$E\left(\mathrm{Q}(t)\right)=\int_{\alpha}^{\beta}\left(1-F\left(x\right)\right)\cdot dx$$

The cumulative portfolio percentage loss probability function $F\left(x\right)$ is defined above as the probability of cumulative portfolio percentage loss not exceeding *x*:

$$F\left(x\right)=\Pr\left(l\left(t\right)\leq x\right),\quad x\in[0,1]$$

Here the value of $F\left(x\right)$ is the only unknown. As discussed before, $F\left(x\right)$ is a function of time *t* and should be expressed as $F\left(x,t\right)$. Some researchers propose that in the fully diversified portfolio of many equally weighted CDSs, the homogeneity is assumed. Then by the law of large number, the portfolio loss distribution $l\left(t\right)$ converges to the individual default probability: $p\left(t\right)=\Pr\left(\tau\leq t\right)=1-\exp\left(-\int_0^t\lambda(u)du\right)$

[8] The Basel Benchmark Risk Weight is based this asymptotic approximation. For detail , see appendix 4

One of the reprehensive papers is [Albrecher et al. (2006)]: [9]

$$F(t,x) = F_{Portfolio\ Loss}(x) \triangleq 1 - F_M\left(\frac{C - \sqrt{1-\rho^2}\ F_Z^{-1}(x)}{\rho}\right)$$

This is a function of *t* since *C* is a function of *t*.

In that paper Variance Gamma distribution is proposed for $F_M(x)$ and $F_{Portfolio\ Loss}(x)$ based on [Schoutens (2003)] and they also checked Normal Inverse Gaussian (NIG) distribution under asymptotic large homogeneous approximation.

In this model, if *F(x, t)* is symmetric, then:

$$F_{Portfolio\ Loss}(x) = F_M\left(\frac{\sqrt{1-\rho^2}\ F_Z^{-1}(x) - C}{\rho}\right)$$

Some authors give this result without claiming symmetric assumption.

These loss distribution functions F_Z, F_M and $F_{Portfolio\ Loss}(x)$ can be different asymmetric Levy distributions such as Variance Gamma or NIG.

However, since exact expression on pricing synthetic CDO is derived, I use the conditional dependent copula factor model for the following computations.

[9] See appendix 2 for the conduction of asymptotic large homogenous portfolio approximation.

First Result from Conditional Independent Factor Model

Settings:

There are $N = 125$ CDS with five years to maturity in the synthetic CDO. The hypothetic tranches are 0-3%, 3-14% and 14-100%.

The correlation coefficient is defined as ρ^2.

Parameters: $r = 0.05$, $R = 0.4$, correlation coefficient $\rho^2 = 0.3$, default intensity $\lambda = 0.03$.

Normal distribution $N(0,1)$ is assumed as underlying distribution in this first attempt.

The following table shows the tranches spreads obtained from conditional independent factor copula model shown in previous context. This is exact number and later I will compare the spreads values in this table with the simulated spreads values.

Table 1 Tranche Spreads
from conditional independent factor model

Tranche	0%-3%	3%-14%	14%-100%
Spread	41.48%	9.685%	0.34754%[10]

For the equity tranche 0-3%, the upfront payment based on a 5% annual spread is quoted; this upfront payment effort minimizes the counterparty risk for the equity tranche. And the corresponding spread for the equity tranche is 67.32%.
The upfront payment is used in all the following examples for equity tranche 0-3%.

The following three figures depict conditional expectation of premium leg value, individual default probability and conditional cumulative portfolio default probability.

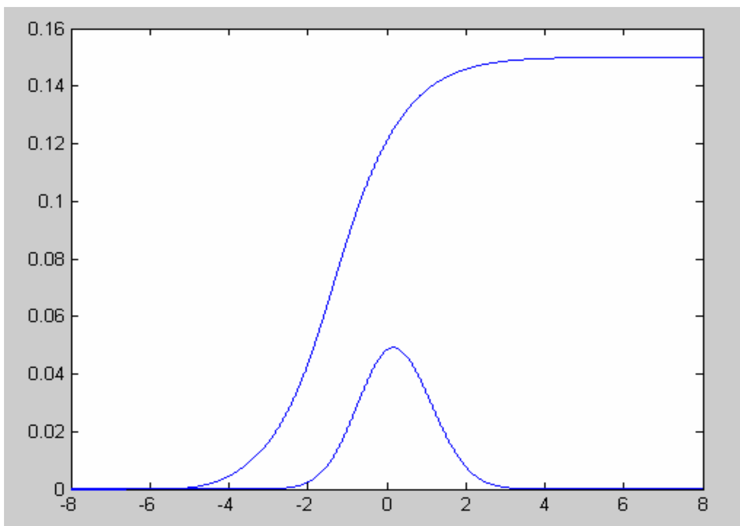
[10] Namely 34.754 bp, 1bp=0.0001

There two curves in Figure 1. The upper curve shows the conditional premium leg value for the 0-3% equity tranche conditional on common factor $M = m$. Namely, the value of:

$$E\left[PL \mid M = m\right] = \sum_{k=1}^{K} \Delta t_k E\left(H\left(t_k\right) \mid M = m\right) B\left(t_k\right)$$

Figure 1
Conditional Expectation of Premium Leg Value
with respect to common factor M
$E\left[PL \mid M = m\right]$

$E\left[PL \mid M = m\right]$
(Conditional Expectation of Premium Leg Value)



Common Factor m

Upper curve (curve 1): $E\left[PL \mid M = m\right]$

Lower curve (curve 2): $E\left[PL \mid M = m\right] \bullet pdf\left(m\right)$

Area below curve 2: unconditional expectation of premium leg value

The x-coordination is common factor value M=m.

The lower bell shape curve (curve 2) is the product of upper curve and the probability density function of common factor *M*. Normal distribution $N\left(0,1\right)$ is assumed as underlying distribution in this first simple attempt. Although $N\left(0,1\right)$ is symmetric, curve 2 is asymmetric, skews to the right.

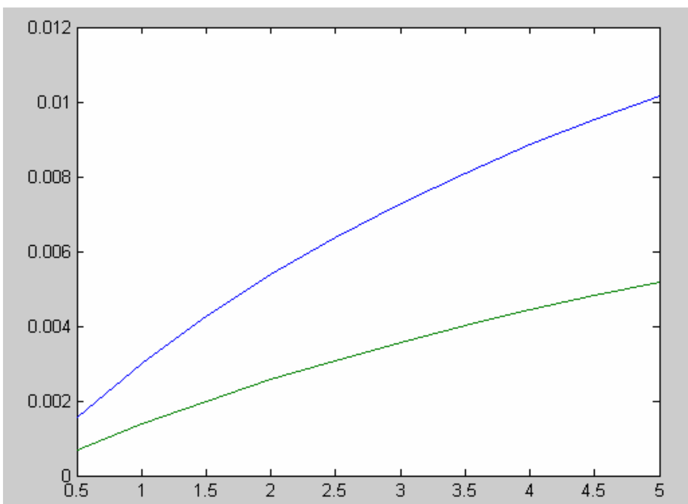
The expectation of premium leg value is the area below the bell shape curve 2. This expectation is computed by integrating $E\left[PL \mid M=m\right] \bullet pdf\left(m\right)$ numerical over the possible range of common factor *M*:

$$\int E\left[PL \mid M=m\right] \bullet pdf\left(m\right) \bullet dm$$

If the common factor M=m follows other asymmetric Levy distribution, in the above formula on the expectation of premium leg value, the distribution density function $pdf\left(m\right)$ will change accordingly. But these two curves still have similar shape.

Figure 2
Individual Default Probability
with respect to time *t*

Individual Default Probability



***t* in year**

These two curves depict individual default probability with different default intensity.

The above curve has higher default intensity $\lambda = 0.03$, and 0.015 for lower curve.

The time ranges from 0 to 5 years.

Figure 3 shows the conditional cumulative loss probability for the CDO portfolio percentage loss; it is conditional on common factor *M*= 0.

The common factor M is set as M=0.

The x-coordination is the portfolio percentage loss.

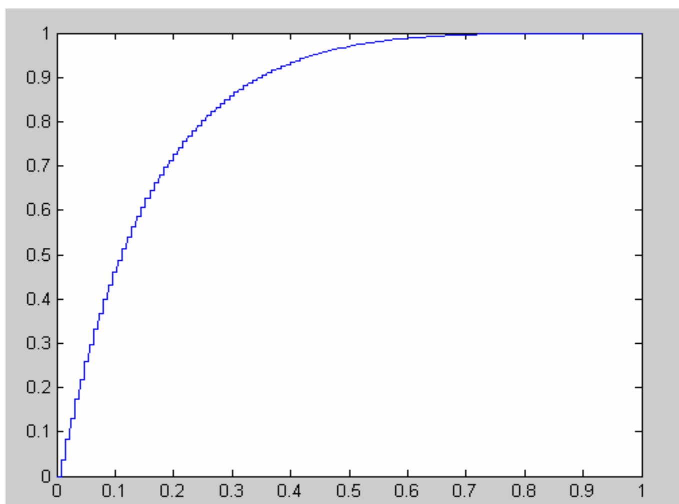
$F(x \mid M = m)$ is defined above as the conditional probability of cumulative portfolio percentage loss not exceeding *x*:

$$F(x) = \Pr(l(t) \leq x), \quad x \in [0,1]$$

It is called conditional cumulative percentage loss probability function.

Figure 3
Conditional Cumulative Percentage Loss Probability
with respect to portfolio percentage loss x
$F(x \mid M = 0)$

$F(x \mid M = 0)$



x, (percentage loss)

Since *F(x)* is the cumulative distribution function of portfolio percentage loss, both x and function value range from 0 to 1.

Spread Sensitivity Analysis

The following tables and figures show the sensitivity of CDO tranches spreads with respective to the different parameters in the conditional independent factor copula model; the parameters include correlation coefficient, default intensity, maturity, interest rate and recovery rate.

1. Spread Sensitivity of Correlation Coefficient ρ^2 for 0-3% tranche

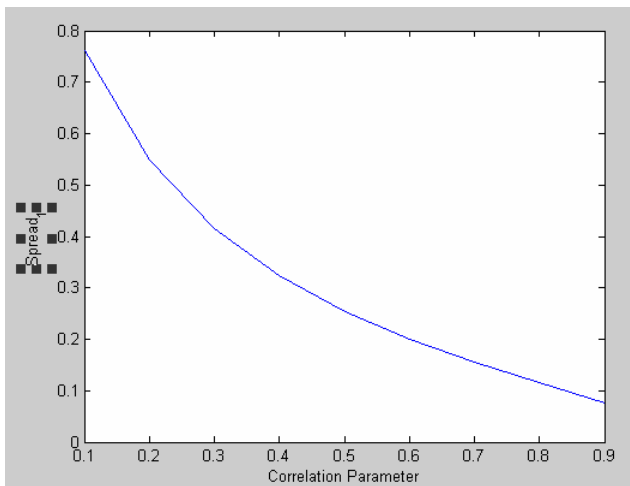
Table 2 Spread Sensitivity of Correlation Coefficient ρ^2 for 0-3% tranche

Correlation Coefficient	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Upfront Payment	76.19%	54.77%	41.48%	32.30%	25.45%	20.06%	15.57%	11.62%	7.69%

For tranche 0-3%, the higher correlation coefficient ρ^2 , the lower the upfront payment.

Figure 4 Spread Sensitivity of Correlation Coefficient ρ^2 for 0-3% tranche

Spread
0-3%
tranche



Correlation Coefficient

The higher coefficient assigns higher spread for senior tranche and lower spread or upfront payment to the equity tranche, vice versa. The relationship between the mezzanine tranche spread and correlation coefficient is more complicate. It is not monotonic function, increasing with correlation coefficient when coefficient is at low level, but decreasing when coefficient is high.

2. Spread Sensitivity of Default Intensity λ for 0-3% tranche

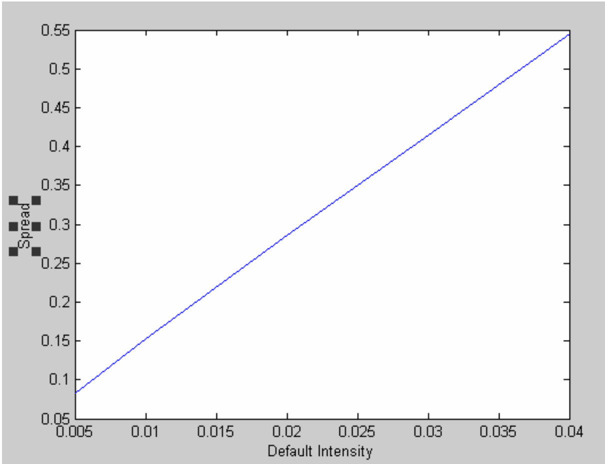
Table 3 Spread Sensitivity of Default Intensity λ for 0-3% tranche

Default Intensity	0.005	0.01	0.02	0.03	0.04
Spread	0.08239	0.1533	0.2856	0.4148	0.5451

The higher default intensity results in higher premium spread; but the marginal increment decreases.

Figure 5 Spread Sensitivity of Default Intensity λ for 0-3% tranche

Spread 0-3% tranche



Default Intensity λ

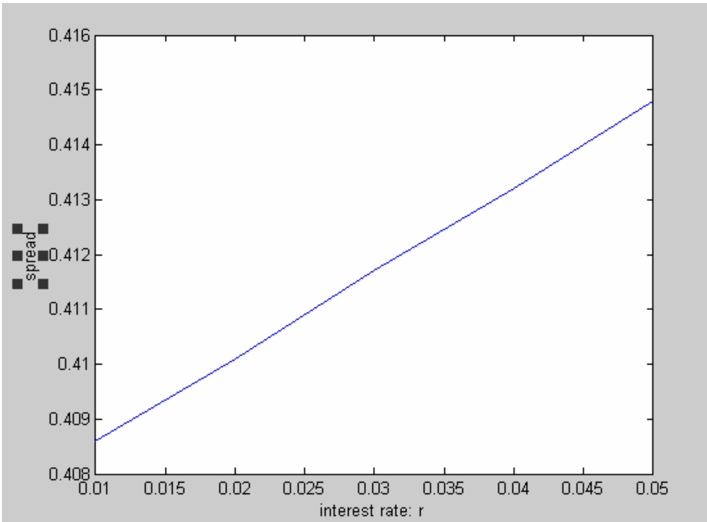
3. Spread Sensitivity of Interest Rate r for 0-3% tranche

Table 4 Spread Sensitivity of Interest Rate r for 0-3% tranche

Interest Rate	0.01	0.02	0.03	0.04	0.05
Tranche Spread	0.4086	0.4101	0.4117	0.4132	0.4148

Figure 6 Spread Sensitivity of Interest Rate r for 0-3% tranche

Spread 0-3% tranche



Interest Rate

There is another illustration on spread sensitivity of interest rate for the 9-12% tranche.

Table 5 Spread Sensitivity of Interest Rate r for 9-12% tranche

Interest rate	0%	1%	2%	3%	4%
Tranche Spread	45.21 bp	44.84 bp	44.46 bp	44.08 bp	43.71 bp

Here “bp” is base point, 1 bp = 0.01%.

These two tables demonstrate that the tranches spreads are not sensitive to the interest rate. Neither of then are sensitive to interest rate. The results for other tranches on interest rate sensitivity are similar. This justifies the usage of constant interest rate in synthetic CDO pricing model.

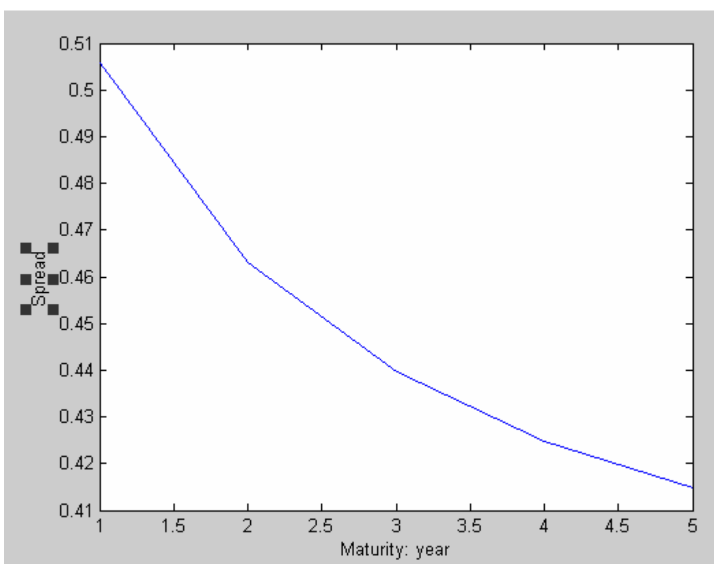
4. Spread Sensitivity of Maturity for 0-3% tranche

Table 6 Spread Sensitivity of Maturity for 0-3% tranche

Maturity (in year)	1	2	3	4	5
Spread	0.5058	0.4631	0.4397	0.4249	0.4148

Figure 7 Spread Sensitivity of Maturity for 0-3% tranche

Spread
0-3%
tranche
{Upfront}
{Payment}



Maturity (year)

From the model, based on 5% annual spread rate, as CDO contract time increases from 1 to 5 years, the upfront payment for equity tranche 0-3% decreases from 0.5058 to 0.4148.

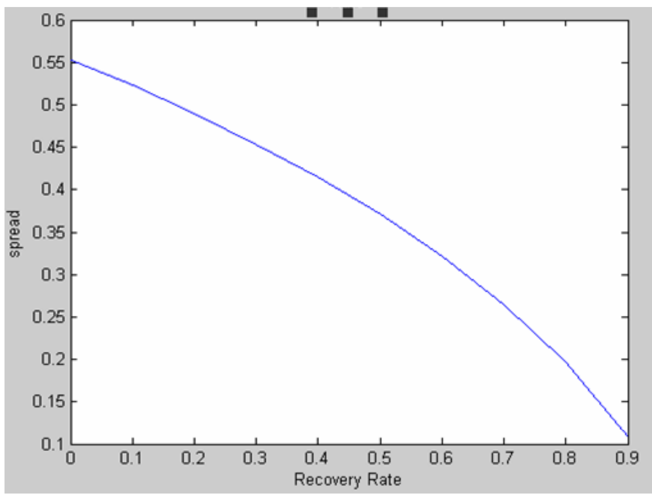
5. Spread Sensitivity of Recovery Rate for 0-3% tranche

Table 7 Spread Sensitivity of Recovery Rate for 0-3%Tranche

Recovery Rate	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Tranche Spread	0.5531	0.5234	0.4895	0.4538	0.4148	0.3710	0.3219	0.2650	0.1965	0 .1083

Figure 8 Spread Sensitivity of Recovery Rate for tranche 0-3%

Spread
0-3%
tranche



Maturity (year)

From the above figures, the spread of the equity tranche 0-3% is not sensitive to interest rate, but it is very sensitive to the correlation coefficient, default intensity and recovery rate. High correlation incurs a low the spread for equity tranche.

The spread for senior tranche is also sensitive to correlation coefficient, default intensity and recovery rate. But high correlation incurs a high spread for senior tranche, such as senior tranche 9-12%.

6. SIMULATION METHOD FOR PRCING SYNTHETIC CDO

Simulation Procedure

Monte Carlo simulation is convenient to implement. Since there is no imperative requirement for explicit expression, this method is very flexible, but at the cost of long computational time. Because convergence rate is quite slow for high dimension simulation on the joint default distribution, simulation time is long. There are some new simulation methods to reduce simulation time, such as Important Sampling and Sequential Monte Carlo. But they are not covered here, since the conditional independent factor model is implemented in the next chapter.

Model Settings:

Again consider a portfolio with N equally weighted CDS, each one with notional $\frac{1}{N}$.

At time t, given default time $\tau_i, i = 1...n$, the cumulative default loss function is:

$$l(t) = \frac{(1-R)}{N}\sum_{i=1}^{N} 1\{\tau_i \le t\}$$

And

$$E\left[l(t)\right] = \frac{(1-R)}{N}\sum_{i=1}^{N} \Pr\{\tau_i \le t\}$$

$$\Pr\{\tau_i \le t\} = 1 - \exp(\lambda_i t)$$

So the tranche loss function and remained outstanding are given as:

$$\mathrm{H}(s) = (\beta - l(s))^{+} - (\alpha - l(s))^{+}$$

$$\mathrm{Q}(s) = (\beta - \alpha) - \mathrm{H}(s) = (l(s) - \alpha)^{+} - (l(s) - \beta)^{+}$$

Let $t_1 < t_2 ... < t_n$ denote the payment dates.

For each payment date, calculate the expected[11] value of premium payment; then sum the product of these payment and risk free discounted factor over all dates to get premium

[11] Under risk neutral measure.

leg PL. Similar method is applied to for expected value of default leg DL. From default leg DL and premium leg PL, the spread y is derived.

The simulation is based on the following one factor and multi factor copula models.

One Factor Copula Model:

The one factor model assumes that the i^th individual variable W_i correlates with each other by depending on a common market factor M:

$$W_i = \rho_i M + \sqrt{1-\rho_i^2}\,\varepsilon_i$$

Where ε_i is idiosyncratic factor of firm i; it is independent of M.

Since ε_i and M are independent, covariance matrix of $W \triangleq (W_1, \cdots, W_N)^T$ is

$$\Sigma(i,j) = \begin{cases} 1, & i = j \\ \rho_i \rho_j, & i \neq j \end{cases}$$

Multi Factor Copula Model:

The multi factor model assumes that the i^th individual variable W_i is correlated with a few common market factors M_k:

$$W_i = \sum_{k=1}^{K} \rho_{i,k} M_k + \varepsilon_i \sqrt{1 - \sum_{k=1}^{K} \rho_{i,k}^2}$$

Where K is the number of common factors. $(\varepsilon_1, \cdots, \varepsilon_N)^T$ and $M \triangleq (M_1, \cdots, M_K)^T$ are independent. So conditional on the common factors M_k, W_i is independent with each other; the conditional default is independent.

In homogenous portfolio, the subscript i is dropped, hence:

$$W = \sum_{k=1}^{K} \rho_k M_k + \varepsilon \sqrt{1 - \sum_{k=1}^{K} \rho_k^2}$$

For example, if there are two common factors:

$$W_i = \rho_1 M_1 + \rho_2 M_2 + Z_i \sqrt{1 - \rho_1^2 - \rho_2^2}$$

The common factor M_1 is shared by all individual variables ε_i ; M_2 is divided by sectors. In each sector, ε_i has same values of M_1 and M_2 ; cross different sectors, only M_1 keeps unchanged; M_2 varies independently cross sectors.

In this two factor model, the covariance matrix of X_i becomes:

$$\Sigma(i, j) = \begin{cases} 1, & i = j \\ \rho_1^2 + \rho_2^2, & i \neq j \text{ name i , j in same section} \\ \rho_1^2, & \text{otherwise} \end{cases}$$

According to the single or two factor model, I simulate the joint random variables τ_i based on the distributions such as Normal or NIG. Then convert these random variables τ_i to the default times to compute synthetic CDO spreads, as shown following.

First generate the random variables according above description. The in each implementation, calculate the premium and default leg value, the spread is their quotient.

Here are the steps for Monte Carlo simulation:

- Generate a n joint random variables W_i as above formulae
- Calculate the default time by $\tau_i = -\frac{\ln(1-u_i)}{\lambda}$, $u_i = F(W_i)$, F is the cumulative distribution function of W_i. See following recap for details.
- Calculate the cumulative default loss function for each payment date based on default times
- Calculate the present values of premium and default payment
- Sum the present values to get the total value of premium leg payment PL and default leg payment DL

Repeat the above steps, calculate the average of both premium leg PL and default leg DL . Calculate tranche spread from these two values.

The second step is derived in the following recap:

Recap: $E\left[1\{\tau_i \le t\}\right] = \Pr(\tau_i \le t) = u_i$

Let the cumulative value $u_i = F(W_i)$ equal to default probability:

$$u_i = p_i(t) = \Pr(\tau_i \le t) = 1 - \exp(-\lambda_i t)$$

The default time is an exponential distribution. And it can be generated by:

$$\tau_i = -\frac{\ln(1-u_i)}{\lambda_i}$$

$$E\left[1\{\tau_i \le t\}\right] = \Pr(\tau_i \le t) = u_i$$

Simulation Result

Single Factor Copula Model Simulation Result

Factor Copula model setting:

$N = 125$, $r = 0.05$ and $R = 0.4$ correlation $\rho^2 = 0.3$ and default intensity $\lambda = 0.03$.

The tranches are 0-3%, 3-14% and 14-100%.

The simulation is based on Matlab software at a PC with 2.0G CPU.

Table 8 Simulation Result in One Factor Gaussian Copula Model

Iteration	500	5,000	25,000	50,000
Simulation Time	5.42 second	52.75 second	266.7 second	530.2 second
0-3%	41.03%	4.18%,	41.69%	41.98%
3-14%	9.406%	9.798%	9.769%	9.781%
14-100%	0.3552%	0.3482%	0.3519%	0.3515%

The equity tranche spread is quoted as the upfront payment based 5% annual premium rate. The absolute spreads will be 67.51 %, 67.99%, 67.84% and 67.63% correspondingly.

The following is simulation result comparison among one factor and two factor's model based on Gaussian and Normal Inverse Gaussian (NIG) distributions. For more details on NIG distribution property, see appendix 3.

Multi Factor Copula Model Simulation Result

In this multi factor model, two factor copula model is applied, the market factor and the section factor.

Settings: $N = 125$, $r = 0.05$, $R = 0.4$, default intensity $\lambda = 0.03$, correlation $\rho_1^2 = 0.3$, $\rho_2^2 = 0.1$, or $\rho_1 = \sqrt{0.3}$, $\rho_2 = \sqrt{0.1}$.

Table 9 Two Factor Gaussian Copula Model

Iteration	500	5,000	25,000	50,000
Computing Time (seconds)	4.6	55.4	262.1	481.1
0-3%	40.46%	39.48%	39.06%	39.06%
3-14%	9.775%	9.760%	9.551%	9.551%
14-100%	0.3841%	0.3806%	0.3780%	0.3755%

By comparing the results with one or two correlation factors, it is shown that with an additional factor the spread for equity tranche decrease and the spread for senior tranche increase. It is similar to the effect of increasing correlation coefficient.

Table 10 Simulation Result Comparison among Factor Copula Model
(Gaussian and NIG Factor Models)

In one fact model, $\rho^2 = 0.3$ (or $\rho = \sqrt{0.3}$)

In two fact model, $\rho_1^2 = 0.3$, $\rho_2^2 = 0.3$ (or $\rho_1 = \sqrt{0.3}$, $\rho_2 = \sqrt{0.1}$)

NIG distribution setting: $u = 0$ and $\delta = 1$

Copula Model	1 Factor NIG	2 Factor NIG	1 Factor Gaussian	2 Factor Gaussian
Iterations	5,000	5,000	50,000	50,000
Computing Time	100 minutes	100 minutes	6 minutes	6 minutes
0-3% (Spread)	76.36%	74.41%	67.63%	65.57%
3-14%	9.42%	10.00%	9.71%	9.58%
14-100%	0.43%	0.47%	0.35%	0.37%

The following table compares the simulation result with the result computed by the conditional independent copula factor model introduced in previous chapter.

Table 11 Comparison between spreads in one factor model and simulation

Tranche	0%-3%	3%-14%	14%-100%
Spread by Conditional Independent Factor Model	67.32% (41.48%) [12]	9.685%	0.34754%
Spread by Simulation Method 5000 iterations	67.63% (41.98%)	9.781%	0.3515%

This table illustrates the performance of simulation method and converge speed. The Monte Carlo simulation with 5000 iteration already provides a sufficient accuracy.

Recap:
From the above tables and analysis, there are two main advantages for the simulation methods: flexibility and easy to implement.

The simulation method based on other heavy tail distributions, such NIG distribution, can give a higher spread for the senior tranche while a lower spread for the equity tranche. But the computational speed is much slower than the Gaussian model.

[12] The number in blanket is the upfront payment. In 0-3% tranche, the quotations are upfront payment with an additional 5% annual spread paid quarterly from the protection buyer to the protection seller. The benefit of upfront payment arrangement is to reduce the counterpart risk for the protection buyer.

7. NUMERIAL RESULT FOR MARKET DATA

Data Source: Bloomberg consolidate Market Data, Date: March 20th 2006

Underlying Index: iTraxx European CDS Tranched Index

iTraxx Series 3: based on 125 five years time maturity CDS

The five year CDS is the most liquid CDS in the market and series 3 can provide most trading information comparing with other series. It is effective from 2005 March. The date ranges from March 30th 2005 – July 25th 2006.

The average iTraxx Credit Index spread on that day is 39.917 bp. The average credit rating for iTraxx Credit Index is A-. There is an adjustment for the index's average spread due to the volatility among the CDS spreads.

The tranche detachment and attachment points are: 3%, 6%, 9%, 12% and 22%. The trading tranches are 0-3%, 3-6%, 6-9%, 9-12% and 12-22%. The most senior tranche 22-100% is not quoted in the market. But its spread can be determined by the all the tranche spread and the index spread.

The Euro interest swap rate is chosen as the interest rate. From the above analysis, the spread and upfront payment is not sensitive to the interest rate.

Implied Correlation Coefficients in Conditional Independent Model

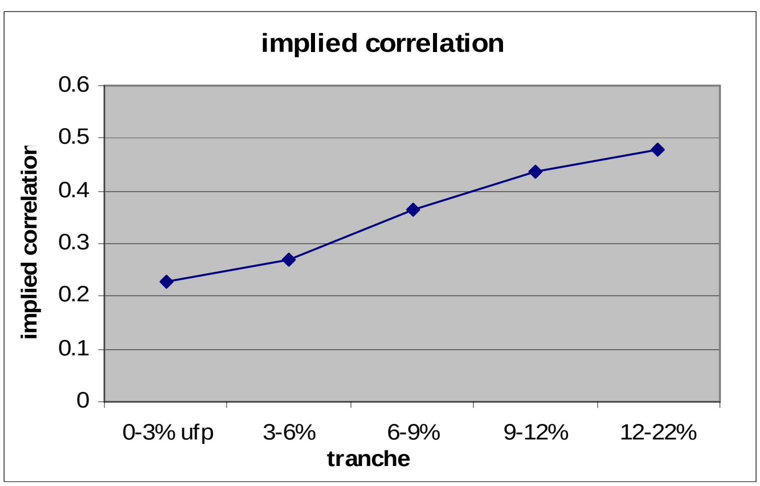
The normal and NIG one factor copula model is used.
The quoted tranches spreads at March 20th 2006 are chosen to calibrate the model. The implied coefficients in normal one factor copula model on different tranches are:

Table 9 Implied Correlation Coefficients ρ^2
(Normal one factor copula model)
Date: March 20th 2006

Tranche	Spread[13]	Implied Correlation Coefficient ρ^2
0-3% (upfront)	22.158%	0.229129
3-6%	60.323 bp	0.270185
6-9%	17.661 bp	0.36606
9-12%	10.431 bp	0.437607
12-22%	2.951 bp	0.477493

In this normal one factor copula model, the implied correlation coefficient ρ^2 increases as tranche getting senior. During the calibration, it is found that the spread for tranche 12-22% is very sensitive to the correlation coefficient and it is distinctively different from the implied correlation for equity tranche. It is impossible to give a flat correlation structure for all tranches, which results in correlation smile effects.

The following figure depicts the above correlation smile relationship between tranches and correlation coefficients.



[13] The "bp" is base point, which is 100th of 1%.

In Gaussian factor copula model, no unique correlation coefficient can fit all the quoted spreads simultaneously. This leads to correlation smile as above graph shows.

There are some reasons for the correlation smile in Gaussian factor copula model:

1. The tail of normal distribution is to thing to catch the dependent structure
2. The underlying distribution for the structural model is not symmetric
3. The dependent structure is more complex than the factor model describes

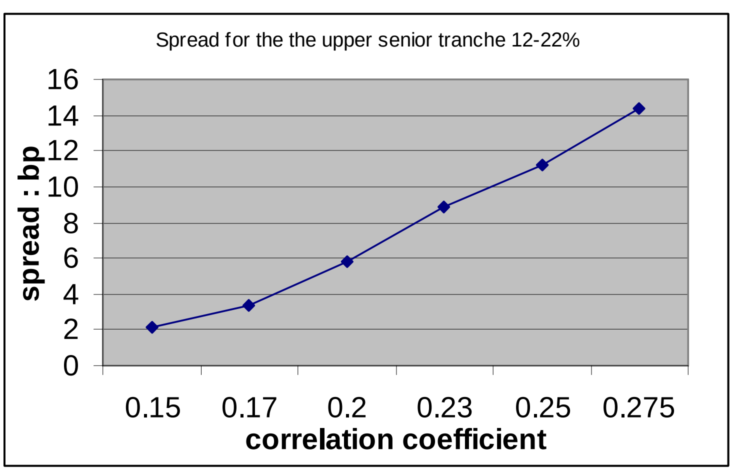
The asymmetric Levy distribution chosen in conditional independent factor copula model will overcome the first two problems here.

The following is the computed spread for tranche 12-22% in different correlation coefficient ρ^2 settings. The spread is strict increasing function with respect to coefficient for this tranche. Other tranches spreads may have different function forms.

Table 10 Tranche spread (12-22%) in different Correlation Coefficient ρ^2

Correlation coefficient ρ^2	0.15	0.17	0.2	0.23	0.25	0.275
Tranche Spread	2.15bp	3.38bp	5.81 bp	8.88bp	11.2bp	14.4 bp

Figure 9 Tranche spread (12-22%) in different Correlation Coefficient ρ^2



The correlation smile has been discussed above. The following model will assume a correlation coefficient for all tranches, and determine correlation coefficient value by minimizing the sum of absolute errors for all tranches.

The NIG distribution setting is $u = 0$ and $\delta = 1$. It has more parameters α, β to adjust during calibration process to fit better market data better. The parameters α and β decide the shape of NIG distribution, skewness and tail decay speed.

Table 11 Comparison between Gaussian and NIG Copula factor model
Date: April 29th 2005

Tranche	Market Quotation	Gaussian Copula factor Model	NIG factor model with two parameters
0-3%[14]	24.962%	24.88%	24.75%
3-6%	161.5 bp	285.1 bp	153.0 bp
6-9%	50.25 bp	105.5 bp	53.89 bp
9-12%	23.00 bp	44.31bp	28.72 bp
12-22%	13.75 bp	10.06 bp	12.75 bp
Correlation Coefficient	NA	0.24	0.232
α, β	NA	NA	$\alpha = 1.2$ $\beta = 0$

NIG distribution setting: $u = 0$ and $\delta = 1$.

The calibration criterion for NIG distribution $f(x; \alpha, \beta, \mu, \delta)$ is:

$$\min_{u,\delta} \left\{ \sum_{i=1}^{TR} | Spread_i - \overline{Spread_i} | \right\}$$

TR: the total tranche number;

$spread$: computed spread from conditional independent factor copula model;

$\overline{spread}$: quoted spread

[14] This is upfront payment based on 5% tranche spread for equity tranche 0-3%

The iTraxx CDS index for that day is 39.917, which is relatively higher than other days.

The table shows that the main drawback of the Gaussian factor model is failure to fit all the market quoted tranches spreads consistently with one correlation coefficient for all tranches. Especially the market quoted mezzanine tranche spread is much lower than the one fitted in Normal copula factor model. The required correlation coefficient is higher for mezzanine tranche than both equity and more senior tranches. In the NIG distribution, the two parameters α and β affect symmetric property and the shape and thickness of the distribution tail. The parameter α decides the skewness; when it is 0, the distribution is symmetric. Beta (β) decides the tails' shape in NIG distribution. Since Normal distribution is symmetric, I fixed Alpha at 0 to compare the results. The fitting result in NIG factor Copula model is even better if alpha is not fixed at 0. But it is hard to determine its value numerically.

Since the NIG model can improve the fitting to market data by choosing the distribution parameters, Alpha and Beta, it is more reliable to price customized Single Tranche CDO.

The factor copula models based on other Levy distributions can provide similar result. For details of the specific property of NIG, see [Kalemanova (2005)] and for details of general Levy processes see [Schoutens (2003)].

From optimization process of NIG factor model:

1. As the correlation coefficient increases, the upfront payment of equity tranche decreases, and the spreads of senior tranches increase. This is inline with all the factor copula models.
2. For the heavier tailed distribution, Beta generally determines the tail's thickness. The higher the value of Beta, the heavier the tail. When Beta value increases, the upfront payment will decrease and spreads senior tranches increase. This means the extreme events happen more frequently than Gaussian distribution, which is one drawback in Normal copula. So senior tranches have a higher spread.
3. Alpha is symmetric parameter, when its value is not 0, it is asymmetric.

Correlation Smile and Trading Strategy

As shown above, in Normal factor copula model, market quoted spreads of the mezzanine tranches (3-6% and 6-9%) is much smaller than the ones from the model. Due to this inconsistence; the market implied correlation coefficient for the mezzanine tranche is smaller than the implied correlation for equity and senior tranches. In the market, this inconsistence leads to a so called "bear-bull" trading strategy: the dealer holds the equity tranche and sells the mezzanine tranche to trade the correlation inconsistence.

The NIG factor copula model can cure this correlation smile effect by choosing optimal parameters. The NIG factor model fits tranches spreads better to market quoted spreads.

Short Term Prediction

This is forward prediction over short time horizon, such as one day. The short term prediction is also a test of model consistence. First fit the correlation coefficient in the Normal factor copula model to the tranches spreads at the prevailing day. Then use this correlation coefficient and next day's iTraxx Index data to predict the forward CDO tranches spreads. Finally compare the predicted spreads with the next day's spreads in the market. The following is the conclusion.

European iTraxx index based on 125 equally weighted CDS:
X coordinate is the number of trading days from the starting of the 3rd series iTrxx index based on 5 year CDS. Y coordinate is spread in base points.

Figure 10 iTraxx European Tranche Index

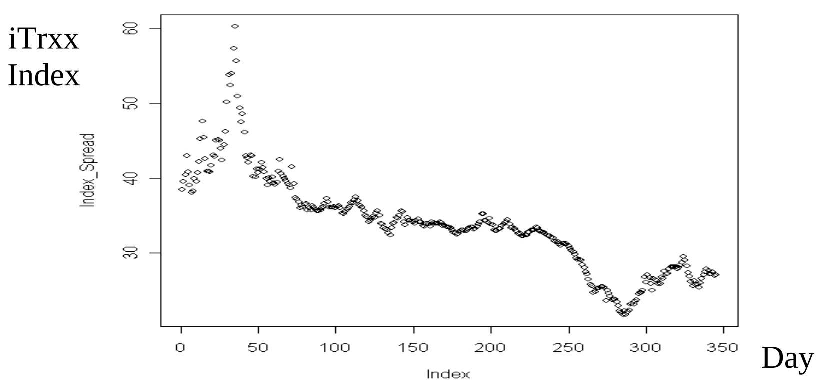
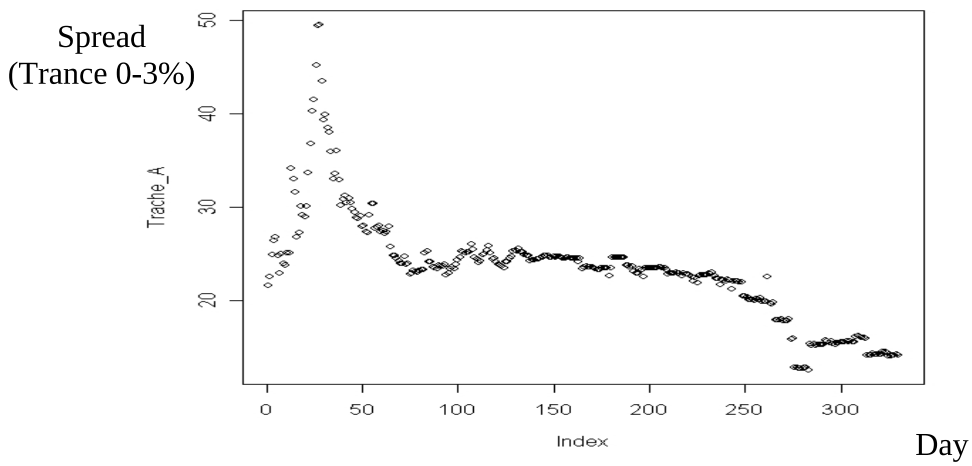


Figure 11 Spread for Tranche A 0-3%



The X and Y setting is same as above figure.

Figure 12 Spread for Tranche B 3-6%

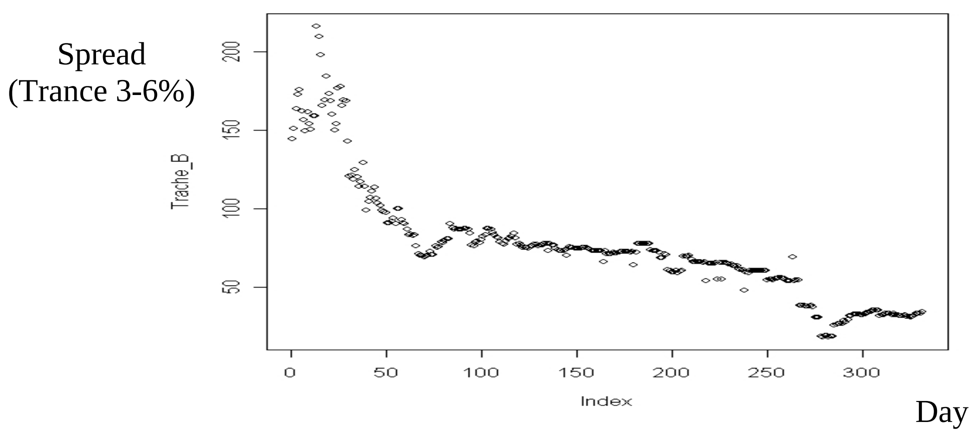


Figure 13 Spread for Tranche C 6-9%

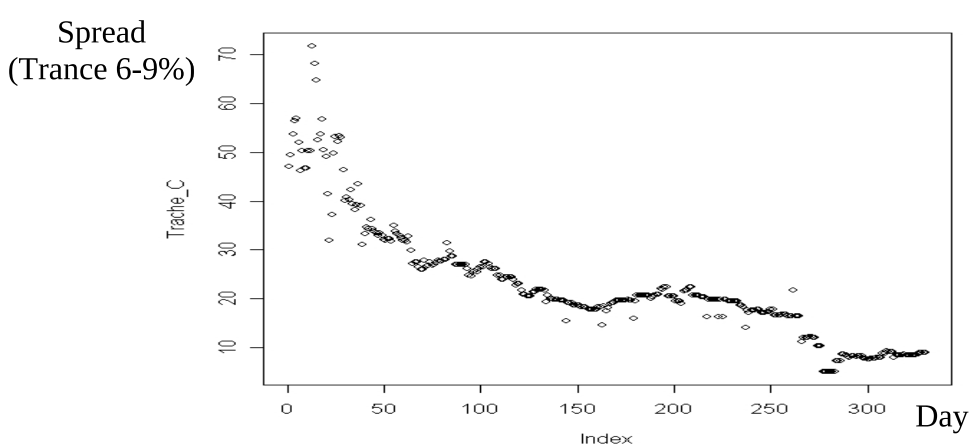


Figure 14 Spread for Tranche D 9-12%

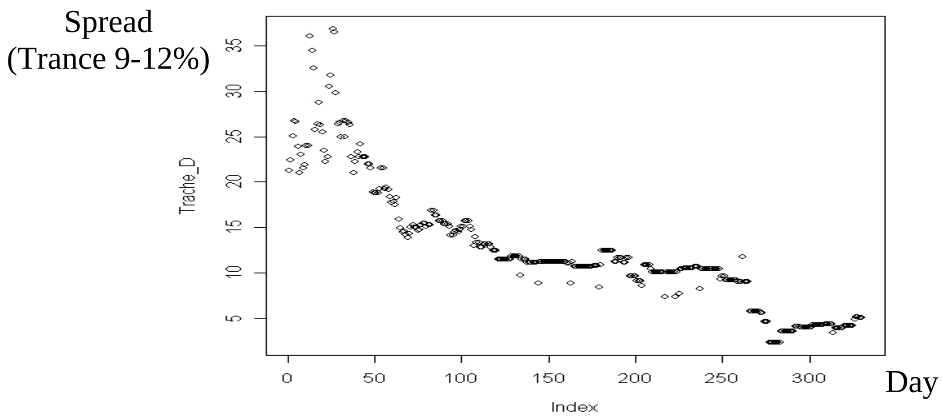
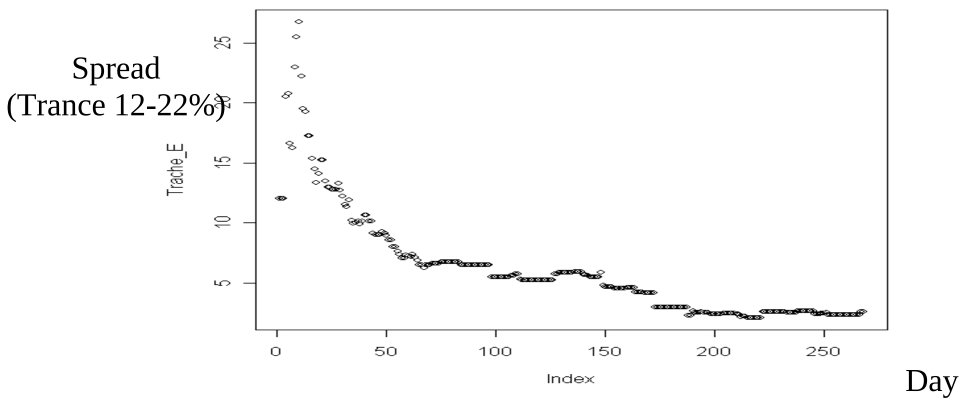


Figure 15 Spread for Tranche E 12-22%



These figures show similar pattern, the trading data for tranche E 12-22% started at April 2005 and has less trading days than other ones; it is not as liquid as other tranches.

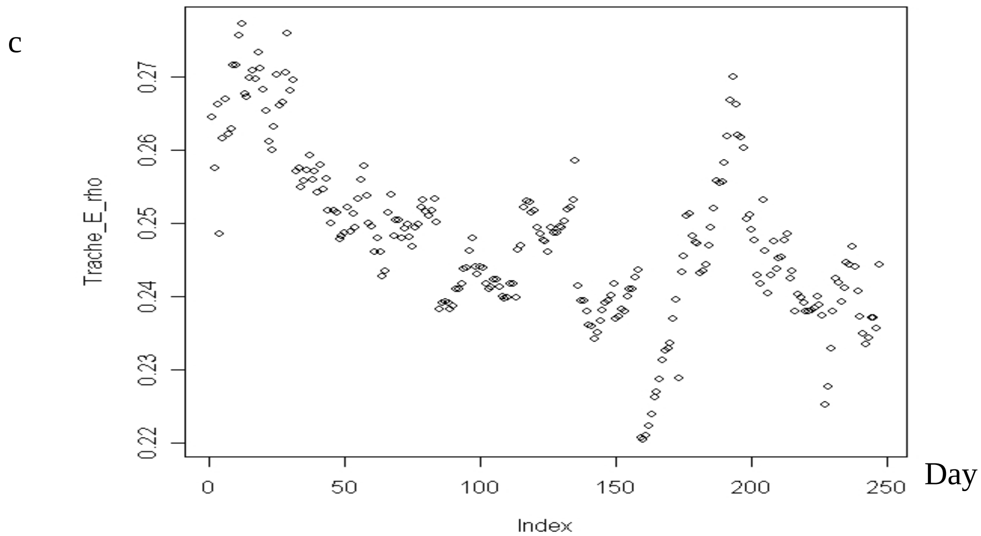
Some outliers in the above figures have been deleted. The period of forward prediction is from May 2005 to July 2006.

Prediction Results:

The following figure shows implied correlation coefficient of tranche 12-22%.

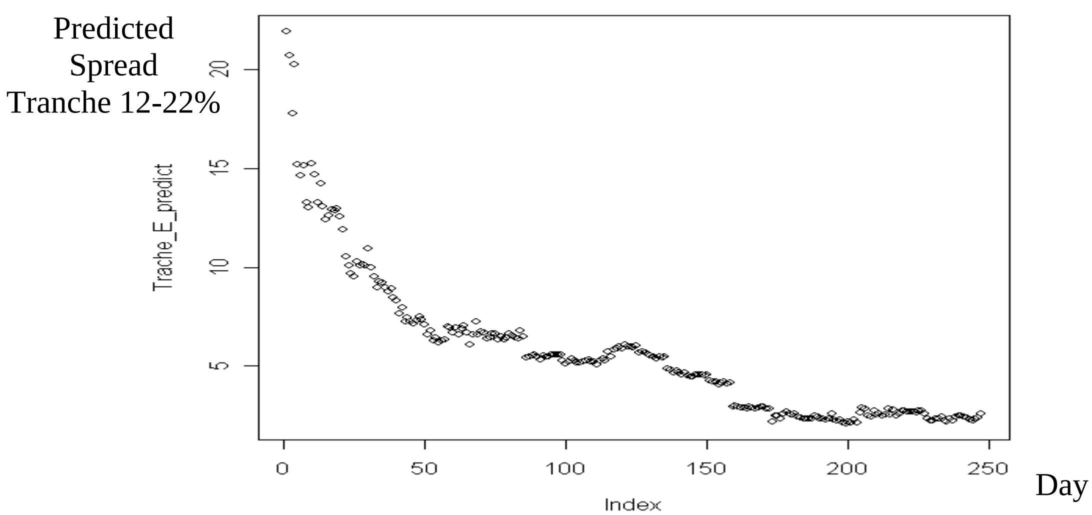
The correlation coefficient for this tranche fluctuates around 0.25 during this period of time. The fluctuation ranges from near 0.28 to 0.22.

Figure 16 Implied Correlation Coefficient from tranche E 12-22%



This is the implied correlation coefficient calibrated from super senior tranche 12-22%. It ranges from 0.22 to 0.28, but most points are around 0.25.

Figure 12 One Day Forward Prediction Result



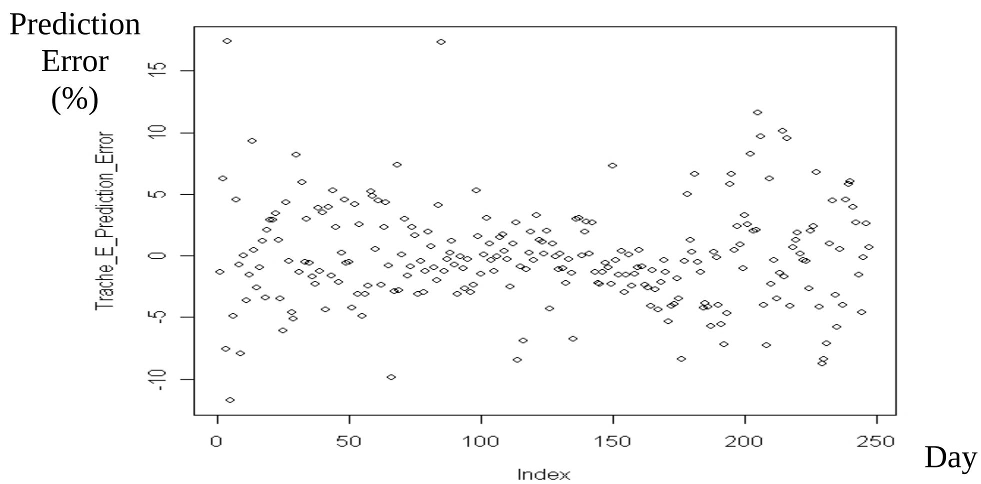
Check the predict error:

Define the prediction error as the ratio:

$$\frac{\text{Predicted Spread - Quoted Spread}}{\text{Quoted Spread}}$$

The following figure shows this prediction error.

Figure 17 Prediction Errors for Tranche E 12-22%



The Y coordinator is in percentage.

The prediction is based on conditional independent Normal factor copula model, since the NIG factor in the predicted date has unknown parameters which need to be calibrated by the market quoted spreads from the next predicting day.

From the figure, most of the one day forward prediction errors lay in the range between 5% and -5%. Given the fact that the bid ask spread in the credit derivative market is about the same level as percentage of tranche spreads, the prediction error is relative accurate.

The other tranches prediction results are similar.

8. CONCLUSION

The conditional independent factor copula provides an explicit and accurate solution for synthetic CDO pricing. Comparing with generic copula method by simulation, it is still very parsimonious and computational faster.

The model driven by Levy distribution fits the market quoted spreads better than Normal factor copula model. This model can give relevant accurate forward prediction for the spreads over short time horizon. But longer prediction requires modeling the dynamic of default intensity; this will be further development.

The factor model becomes popular due to its tractability and clear economic meaning. The correlation coefficient can explain most of the tranches spreads if a flexible Levy distribution is chosen and calibrated to the market quoted spreads. Since the spread of trance is a nonlinear function of both attachment and detachment points, the model's capability to price other bespoke and customized Single Tranche CDO depends on how accurately the model is calibrated to the market quoted tranches and how flexible the model is.

Further Development

The dynamic model which models the dynamic term structure of CDS spread and correlation simultaneously is one of the further research areas. Stochastic recovery rate is another one. Further contract, such as Option on CDO, Single Tranche CDO and forward starting CDO pricing model require modeling the dynamic of the default term structure, stochastic correlation coefficient or recovery rate; the model will be more reliable if it is calibrated to the whole default intensity curve, which is similar idea as the HJM model. See [Schönbucher (2006)] for term structure of default intensity curves, or transition matrix [Albanese et al. (2006)].

The further two dynamic CDO model approaches are:

1. Markov transition probability density or transition matrix: [Schönbucher (2006)], [Albanese (2006)], [Sidenius (2005)] and [Walker (2006)]
2. The Generic Factor Levy Copula CDO factor model: [Baxter (2006)], [Cariboniy & Schoutens (2006)], [Luciano & Schoutens (2005)] and [Moosbrucker (2006b)].

Hedging

The iTraxx index family becomes a popular hedge instrument for CDS and CDO. There are some sub indices for different industries in the iTraxx index family, such as Financial Industry, Automobile Industry or High Volatility sub iTraxx indices. The CDS in corresponding industry section can be hedged by using the sub indices.

Risk Management

The factor copula method is also applied for risk management to calculate new VaR in Basel II accord. See appendix 4 and Walker (2004) for details.

9. APPENDIX

1. The Theory of Copula Function

The theory of copula investigates the dependence structure of multi-dimensional random vectors given the marginal distribution. Copula is function that joins or “couples” multivariate distribution functions to their corresponding marginal distribution functions. A copula function itself is a multivariate distribution function with uniform margins on the interval [0, 1]. As a way of studying the dependence structure of an asset portfolio irrespective of its marginal asset-return distributions, copula is of interest in credit-risk management. It is a starting point for constructing multi-dimensional distributions for asset portfolios for simulation.

The common definition of the copula function is:

Definition: Let $C:\ [0,1]^n \to [0,1]$ be an n-dimensional distribution function on $[0,1]^n$.

Then C is called a copula if it has uniformly distributed margins on the interval [0, 1].

Example 1: Let F is the cumulative distribution function of random variable ε and define

$$X := F(\varepsilon).$$

For any $x \in [0,1]$, then:

$$\Pr(X \le x) = x$$

So X is unit random variable, the probability distribution function is:

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}.$$

The following theorem gives the foundation for a copula to inherit the dependence structure of a multi-dimensional distribution.

Theorem (Sklar's theorem): Let F be an n-dimensional distribution function with margins $F_1,...,F_n$. Then there exists a copula function C, such that for all $X \in R^n$

$$F\left(x_1,...,x_n\right)=C\left(F_1\left(x_1\right),...,F_n\left(x_n\right)\right)$$

If $F_1,...,F_n$ are all continuous, then C is unique; otherwise C is uniquely determined on $RanF\times\cdots\times RanF_n$. Conversely, if C is a copula and $F_1,...,F_n$ are distribution functions, then the function F defined above is an n-dimensional distribution function with margins $F_1,...,F_n$.

An immediate Corollary shows how one can obtain the copula of a multi-dimensional distribution function.

Corollary: Let F be an n-dimensional continuous distribution function with margins $F_1,...,F_n$. Then the corresponding copula C has representation

$$C\left(u_1,...,u_n\right)=F\left(F_1^{-1}\left(u_1\right),...,F_n^{-1}\left(u_n\right)\right)$$

Where $F^{-1}{}_1,...,F^{-1}{}_n$ denote the generalized inverse distribution functions of $F_1,...,F_n$, i.e. for all $u_1,...,u_n\in\left(0,1\right)$: $F_i^{-1}\left(u_i\right):=\inf\left\{x\in R\middle|F_i\left(x_i\right)\geq u_i\right\}, i=1,...n.$

Remark:

The copula is invariant while the margins may be changed at will, it follows that it is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. (almost surely) strictly increasing transformations. Thus the copula function represents the dependence structure of a multivariate random vector. Adding some more copula properties needed here:

- Independent distribution's copula:

$$C\left(u_1,...,u_n\right)=F\left(x_1,...,x_n\right)=\prod_{i=1}^{n}F\left(x_i\right)=\prod_{i=1}^{n}u_i$$

- Upper bound:

$$C(u_1,...,u_n) = C(F_1(x_1),...,F_n(x_n)) = F(x_1,...,x_n) \le F(1...,1,x_i,1,...,1) = x_i$$
$$C(u_1,...,u_n) \le \min(u_1,...,u_n)$$

- Lower bound:

$$C(u_1,...,u_n) = F(x_1,...,x_n) \ge 1 - \sum_{i=1}^{n}(1 - F(x_i)) = \sum_{i=1}^{n} u_i - (n-1)$$

- A copula is increasing in each component. In particular the partial derivatives

$$\partial C(u_i) / \partial u_i \text{ , i = 1 . . . n,}$$

  Exist almost everywhere.

- Consequently, the following conditional distributions of the form exist.

$$C(u_1,...,u_{j-1},u_{j+1},...,u_n | u_j) \text{ , j = 1, . . . , n,}$$

- A copula C is uniformly continuous on $[0,1]^n$

Tail Dependence

This can be interpreted as given one name defaults, what is the default probability for another name in the portfolio.

Definition (Upper Tail dependence coefficient)

Let $X = (X_1, X_2)'$ be a two-dimensional random vector. We say that X is tail dependent if

$$\lambda := \lim_{\nu \to 1^-} P\left(X_1 > F_1^{-1}(\nu) \middle| X_2 > F_2^{-1}(\nu)\right) > 0$$

Definition (Lower Tail dependence coefficient)

Let $X = (X_1, X_2)'$ be a two-dimensional random vector. We say that X is tail dependent if

$$\omega := \lim_{\nu \to 0^-} P\left(X_1 \le F_1^{-1}(\nu) \middle| X_2 \le F_2^{-1}(\nu)\right) > 0$$

Proposition:

Let X be a continuous bi variant random vector, then

$$\lambda = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}$$

Where C denotes the copula of X.

Analogous $\omega = \lim_{u \to 0^+} \frac{C(u,u)}{u}$ holds for the lower tail dependence coefficient.

Example:

If the two random variables are independent, then $C(u,u) = u^2$

$$\lambda = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1^-} (1-u) = 0$$

And

$$\omega = \lim_{u \to 0^+} \frac{C(u,u)}{u} = \lim_{u \to 0^+} u = 0$$

In fact, if

$$\lim_{u \to 1^-} C(u,u) = \lim_{u \to 1^-} \left(1 - 2(1-u) + o(1-u)\right),$$

Then

$$\lambda = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1^-} o(1-u) = 0$$ [15]

If

$$\lim_{u \to 0^+} C(u,u) = o(u).$$

Then

$$\omega = \lim_{u \to 0^+} \frac{C(u,u)}{u} = \lim_{u \to 0^+} \frac{o(u)}{u} = 0$$

These two cases are lack of tail dependence in upper right or bottom left corners. To deal with the dependent structure in finance and insurance, the copula functions are chosen with these two limits not equal to zero.

Even with the simple characterization for upper and lower tail dependence in this proposition, it will be still difficult and tedious to verify certain tail dependencies if the

[15] The proof can be deducted by definition of conditional probability.

copula is not a closed-form expression. Therefore, the following Theorem gives another approach calculating tail dependence. We restrict ourselves to the upper tail.

Let X be a bivariate random vector with differentiable copula C.
Then the upper tail dependence coefficient λ can be expressed using conditional probabilities if the following limit exists:

$$\lambda = \lim_{v \to 1^-} P\left(\left(U_1 > v\right) \middle| U_2 = v\right) + P\left(\left(U_2 > v\right) \middle| U_1 = v\right)$$

where $\left(U_1, U_2\right)$ are distributed according to the copula C of X.

If U_1, U_2 have some marginal distribution of Normal or student t, then:

$$\lambda = \lim_{v \to 1^-} P\left(\left(U_1 > v\right) \middle| U_2 = v\right) + P\left(\left(U_2 > v\right) \middle| U_1 = v\right) = 2 \lim_{v \to 1^-} P\left(\left(U_1 > v\right) \middle| U_2 = v\right)$$

The tail dependent coefficient for bi normal distribution is zero, except perfect linear correlation. So the Gaussian copula can not hand the required dependent structure for risk management or collateral obligations.

$$x = \rho y + \sqrt{1-\rho^2}\, z$$
$$-1 < \rho < 1$$

Where y, z are i.i.d normal distributions $N(0,\ 1)$

$$\lambda = \lim_{v \to 1^-} 2P\left(\left(U_1 > v\right) \middle| U_2 = v\right) = 0$$

Archimedean Copula

A bivariate Archimedean copula has the form $C\left(u, v\right) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for some continuous, strictly decreasing, and convex *generator* function $\varphi : [0,1] \to [0, \infty]$ such that $\varphi(1) = 0$ and the pseudo-inverse function $\varphi^{[-1]}$ is defined by:

$$\varphi^{[-1]}(x) = \begin{cases} \varphi^{-1}(x), & 0 \le x \le \varphi(0) \\ 0, & \varphi(0) < x \le \infty \end{cases}$$

From Cízek (2003), it can be shown:

- Upper tail-dependence implies, $\varphi'(1) = 0$ and $\lambda = 2 - (\varphi^{-1} \circ 2\varphi)'(1)$
- $\varphi'(1) < 0$ implies upper tail-independence,
- $\varphi'(0) > -\infty$ or $\varphi(0) < \infty$ implies lower tail-independence,
- lower tail-dependence implies $\varphi'(0) = -\infty$, $\varphi(0) = \infty$ and $\omega = (\varphi^{-1} \circ 2\varphi)'(0)$

Clayton is one of the popular Archimedean Copulae.

$$C(u,v) = \max\left\{\left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, 0\right\}$$

$$\theta \in [-1, \infty) \setminus \{0\}.$$

There is hardly any meaningful economics explanation for the parameters estimated in Archimedean copula models.

Exponential copula Semi Parametric Method

This method was introduced by econometricians and it is useful for risk management as well. The basic idea is to form the underlying distribution by a non parametric estimation and form the dependent structure in a parametric form. Once the data of daily traded CDO tranches spreads are large enough, the seized maximum likelihood estimation[16] could be conducted. This is a problem called "curse of dimensionality" in high dimension non parametric estimation methods, the estimation accuracy will decay quickly as dimension increases.

[16] In econometrics literature the copula function's parameters are estimated by Seize Max Likelihood method.

2. The Conduction of Asymptotic Large Homogenous Approximation

Define default function for each CDS as:

$$z_i(t) = 1_{\{\tau_i \le t\}}$$

Given $M = m$, by the law of large number, the portfolio loss percentage is:

$$l(t) = \frac{1}{N}\sum_{i=1}^{N} z_i(t) \xrightarrow{a.s.} E\left[z_i(t)\right], \quad (N \to \infty)$$

From central limit theorem,

$$\lim_{N\to\infty} \Pr\left(l(t) \le x \mid M = m\right) = 1_{\{E[z_i(t)] \le x\}}$$

So:

$$\Pr\left(l(t) \le x \mid M = m\right) \simeq 1_{\{E[z_i(t)] \le x\}}$$

Since,

$$\begin{aligned}
&E\left[z_i(t)\right] \le x \Leftrightarrow \Pr\left(\tau_i \le t\right) \\
&\Leftrightarrow \Pr\left\{X_i \le C \mid M = m\right\} \le x \\
&\Leftrightarrow \Pr\left\{z_i \le \frac{C - \rho m}{\sqrt{1-\rho^2}} \mid M = m\right\} \le x \\
&\Leftrightarrow F_z\left(\frac{C - \rho m}{\sqrt{1-\rho^2}} \mid M = m\right) \le x \\
&\Leftrightarrow \frac{C - \rho m}{\sqrt{1-\rho^2}} \le F_Z^{-1}(x) \\
&\Leftrightarrow m \ge \frac{C - \sqrt{1-\rho^2}\, F_Z^{-1}(x)}{\rho}
\end{aligned}$$

Hence,

$$\begin{aligned}
&F(x,t) \triangleq \Pr\left\{l(t) \le x\right\} = \int_{-\infty}^{+\infty} \Pr\left\{l(t) \le x \mid M = m\right\} dF_M(m) \\
&= \int_{-\infty}^{+\infty} 1_{\left\{m \ge \frac{C - \sqrt{1-\rho^2} F_Z^{-1}(x)}{\rho}\right\}} dF_M(m) \\
&= 1 - F_Z\left(\frac{C - \sqrt{1-\rho^2}\, F_Z^{-1}(x)}{\rho}\right)
\end{aligned}$$

3. NIG distribution and its properties

NIG is a mixture of normal and Inverse Gaussian distribution.[17]

The density function of Inverse Gaussian distribution is:

$$f_{IG}(x,\alpha,\beta)=\begin{cases}\frac{\alpha}{\sqrt{2\pi\beta}}y^{-3/2}\exp\left(-\frac{(\alpha-\beta y)^2}{2\beta y}\right), & y>0\\ 0, & y\le 0\end{cases}$$

A random variable X follows NIG distribution with parameters (α,β,u,δ) if:

$$X\,|\,Y=y\sim N(u+\beta y,y)$$
$$Y\sim f_{IG}\left(\delta\sqrt{\alpha^2-\beta^2},(\alpha^2-\beta^2)\right)$$

The NIG density function has the explicit expression as:

$$f_{NIG}(x;\alpha,\beta,u,\delta)=\frac{\delta\alpha\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-u)\right)}{\pi\sqrt{\delta^2-(x-u)^2}}K_1\left(\alpha\sqrt{\delta^2-(x-u)^2}\right)$$

Where K_1 is the modified Bessel function of third the type:

$$K_1(x):=\frac{1}{2}\int_0^\infty\exp\left(-\frac{1}{2}x\left(t+\frac{1}{t}\right)dt\right)$$

The NIG moment generating function $M(t)=E\left[\exp(tx)\right]$ is:

$$M_{NIG}(t;\alpha,\beta,\mu,\delta)=\exp(ut)\frac{\exp\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2-(\beta+t)^2}\right)}$$

The important scaling property for NIG distribution is:

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[17] For generic Levy distributions property, see Schoutens (2003)

$$X \sim NIG(\alpha, \beta, u, \delta)$$
$$\Rightarrow cX \sim NIG\left(\frac{\alpha}{c}, \frac{\beta}{c}, cu, c\delta\right)$$

And further more,

$$X \sim NIG(\alpha, \beta, u_1, \delta_2)$$
$$Y \sim NIG(\alpha, \beta, u_2, \delta_2)$$
$$\Rightarrow X + Y \sim NIG(\alpha, \beta, u_1 + u_2, \delta_1 + \delta_2)$$

The mean and variance of $X \sim NIG(\alpha, \beta, u_1, \delta_2)$ are:

$$E[x] = u + \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}}$$
$$V[x] = \delta \frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}^3}$$

Hence in the NIG distribution the parameter

α determines the distribution shape;

β determines the distribution skewness

u determines the distribution location

δ is a scaling parameter.

From these properties, the NIG distribution can capture the asymmetrical heavy tail distribution of the dependent defaults.

Further NIG distribution properties refer to [Schoutens, (2003)] and [Kalemanov (2005)].

4. Basel II Accord and Copula Function

The Basel II accord is based on Gaussian factor model to compute VaR, Walker (2004) and Rosen (2005).

Vasicek (1987, 1991) considers the fractional number of defaults in a portfolio of a large number of risky correlated loans. He finds that the probability that the fractional number of defaults is less that θ is

$$\left(\rho m + Z_i \sqrt{1-\rho^2}\right) < K_i = F_*^{-1}\left(p(t)\right)$$

$$W\left(\theta\right) = \Phi\left(\frac{\sqrt{1-\rho^2}\cdot\Phi^{-1}(\theta) - \Phi^{-1}(p\left(t\right))}{\rho}\right)$$

$p\left(t\right)$ $P(t)$ is default probability at time *t*, Φ is normal cumulative distribution function *N(0,1)*.

Define the "fractional defaults at risk," *FDaR*, to be the fractional number of defaults that will not be exceeded within a 99.9% confidence level. Then *W(FDaR)* = 0.999
Solving *FDaR gives:*

$$\text{FDaR } = \Phi\left(\frac{\Phi^{-1}(P(t)) - \rho\cdot\Phi^{-1}(0.999)}{\sqrt{1-\rho^2}}\right)$$

Using different distribution functions to set up the economic capital requirement is an interesting topic.

$$W\left(\theta\right) = 1 - F_1\left(\frac{F_*^{-1}(P(t)) - \sqrt{1-\rho^2}\cdot F_2^{-1}(\theta)}{\rho}\right)$$

Then

$$\text{FDaR} = F_2\left(\frac{F_*^{-1}(P(t)) - \rho \cdot F_1^{-1}(0.001)}{\sqrt{1-\rho^2}}\right)$$

Here F_1, F_2 and F_* can be different heavy tailed distribution, which allow a more accurate estimation of the Basel II formula for (corporate, etc.) risk-weighted assets:

Risk-Weighted Assets = 12.50 * FDaR * LGD * EAD *MatAd

where LGD is the loss given default, EAD is the exposure at default, and MatAd is a maturity adjustment. Taking 8% of the risk-weighted assets gives the maximum loss at the 99.9% confidence level, and the required buffer capital is set equal to this loss.

The formula for the correlation parameter given in Basel II, which has an empirical basis, can be simplified to

$$\rho^2 = 0.12[1 + \exp(-50PD)]$$

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