Averaging and Singular Perturbation Methods for Analysis of Dynamical Systems with Disturbances

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Abstract

Stability and robust stability analysis of general nonlinear systems requires a range of tools; some of these tools conclude about robust stability of the original system via similar property of simpler auxiliary systems. Methods that are based on a time scale separation of state variables, such as the averaging and singular perturbations methods, play an important role in this context. Numerous results using averaging or singular perturbation techniques consider stability properties of the original nonlinear systems based on simpler approximated models. These classical techniques are invaluable in a range of engineering applications, such as power electronics and electro-mechanical systems. The current trends in technology and advances in control engineering lead to system models of increased complexity that necessitate various extensions of these classical results in a range of directions. This thesis contains several novel results that extend classical averaging and singular perturbation theory to new important classes of models, such as switched and hybrid systems.

We first consider a parameterized family of discrete-time systems, which may arise when an approximate discrete-time model of a sampled-data system with disturbances is used for controller design. This situation arises often in controller design for nonlinear sampled-data plants when the exact discrete-time model is not possible to obtain analytically. We adapt recently proposed notions of strong and weak averages to parameterized systems with disturbances. We show under appropriate conditions that the solutions of the time varying family of discrete-time systems with disturbances converge uniformly on compact time intervals to the solutions of the average family of discrete-time systems. Moreover, we show that input-to-state stability (ISS) of the strong average system implies semi-global practical ISS (SGP-ISS) of the actual family of systems. Furthermore, the actual family of systems are semi-globally practically derivative ISS (SGP-DISS) if their weak average is ISS.

We next consider stability of disturbed switched nonlinear and linear systems, for which the switching signal is rapidly time-varying. We show that the

appropriate notions of strong and weak averages play an important role in the context of switched systems. We show under appropriate conditions that ISS of the strong average implies SGP-ISS of the actual switched system. A similar result is shown to hold for weak averages but the conclusion is slightly weaker as we can only prove semi-global practical differential ISS (SGP-DISS) that requires the derivatives of disturbances to be bounded. We also introduce the definition of partial strong average, and provide stronger conclusions for the linear switched system. We show that exponential ISS of the strong or the partial strong average system with linear gain imply exponential ISS with linear gain of the actual switched system. Similarly, exponential ISS of the weak average guarantees an appropriate exponential derivative ISS property for the actual system. Moreover, using the Lyapunov method, we show that linear ISS gains of the actual system and its average converge to each other as the rate of the switching signal is increased.

After that, stability of a class of time-varying hybrid dynamical systems via averaging method is considered. Closeness of solutions of the time-varying system to solutions of its weak or strong average on compact time domains is given under the assumption of forward completeness for the average system. We also show that ISS of the strong average implies SGP-ISS of the actual system. In a similar fashion, ISS of the weak average implies SGP-DISS of the actual system. A pulse-width-modulated hybrid feedback control example is used to illustrate the results.

Finally, we consider a class of singularly perturbed hybrid dynamical systems without disturbances. The fast states are restricted to a compact set a priori. The continuous-time boundary layer dynamics produce solutions that are assumed to generate a well-defined average vector field for the slow dynamics. This average, the projection of the jump map in the direction of the slow states, and flow and jump sets from the original dynamics define the reduced, or average, hybrid dynamical system. Appropriate assumptions for the average system lead to conclusions about the original, higher-dimensional system. For example, forward pre-completeness for the average system leads to a result on closeness of solutions between the original and average system on compact time domains. In addition, global asymptotic stability for the average system implies semiglobal, practical asymptotic stability for the original system. We also give examples to illustrate the averaging concept and to relate it to classical singular perturbation results as well as to other singular perturbation results that have appeared recently in the literature.

Declaration

This is to certify that:

(i) the thesis comprises only my original work towards the PhD except where indicated,

(ii) due acknowledgement has been made in the text to all other material used,

(iii) the thesis is less than 100,000 words in length, exclusive of table, maps, bibliographies, appendices and footnotes.

Wei Wang

Publications

A number of papers published or submitted are listed below. Most results in these papers are included in the thesis except for the paper on path following that is not included.

Journal Paper

- **W. Wang and D. Nešić,***Input-to-state stability and averaging of linear fast switching systems*, IEEE Transactions on Automatic Control, Vol. 55, No. 5. 2010, pp. 1274-1279.

- **W. Wang and D. Nešić,** *Input-to-state stability analysis via averaging for parameterized discrete-time systems*, Dynamics of Continuous, Discrete and Impulsive Systems, Vol. 17, No. 6. 2010, pp. 765-787.

- **W. Wang, D. Nešić and A. R. Teel,** *Input-to-state stability for a class of hybrid dynamical systems via averaging*, Mathematics of Control, Signals, and Systems, submitted.

- **W. Wang, A. R. Teel and D. Nešić,** *Analysis for a class of singularly perturbed hybrid systems via averaging*, Automatica, accepted.

- **D. B. Dačić, D. Nešić, A. R. Teel and W. Wang,** *Path-Following for Nonlinear Systems with Unstable Zero Dynamics: an Averaging Solution*, IEEE Transactions on Automatic Control, Vol. 56, No. 4. 2011, pp. 880-886.

Conference Paper

- **W. Wang and D. Nešić,** *Input-to-state stability analysis via averaging for parameterized discrete-time systems*, Proceedings of the 48th IEEE Conference on Decision and Control held jointly with the 2009 28th Chinese Control Conference, pp. 1399-1404, 2009, Shanghai, P.R. China.

- **W. Wang and D. Nešić,** *A note on input-to-state stability and averaging of fast switching systems*, Proceedings of the 48th IEEE Conference on Decision and Control held jointly with the 2009 28th Chinese Control Conference, pp. 2168-2173, 2009, Shanghai, P.R. China.
- **W. Wang, A. R. Teel and D. Nešić,** *Averaging of a class of singularly perturbed hybrid systems*, Submitted to the 50th IEEE Conference on Decision and Control and European Control Conference, 2011, Hilton Orlando Bonnet Creek, USA.
- **W. Wang, D. Nešić, and A. R. Teel,** *Averaging of hybrid dynamical systems with disturbances*, Submitted to the Australian Control Conference, 2011, Melbourne, Australia.
- **W. Wang, A. R. Teel and D. Nešić,** *Averaging tools for hybrid systems with singular perturbations*, Submitted to the Australian Control Conference, 2011, Melbourne, Australia.

Book Chapter

- **W. Wang, A. R. Teel and D. Nešić,** *Averaging results pertaining to the implementation of hybrid feedback via PWM control*, Dynamics and Control of Switched Electronic Systems, Springer, submitted.

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Contents

Part I

Background

Chapter 1

Introduction

Mathematical models of dynamical systems arising in engineering and science are often too complex to be analyzed directly as their exact closed-form solutions are possible only in very special cases. In general, the dynamic behavior and properties of the model need to be analyzed via various approximate methods, such as numerical solution methods [4, 57] and asymptotic methods [73, 119, 132]. These methods provide systematic procedures for approximating the solutions of the model and, hence, for analysis of the behavior of complex dynamical systems.

Asymptotic methods apply to systems modeled as

$$\dot{x} = f(t, x, \varepsilon) \ , \tag{1.1}$$

where ε is a positive "small" parameter. Asymptotic methods can be used to compute the solution $x(t, \varepsilon)$ of (1.1) via an approximate simpler system whose solution is denoted as $\bar{x}(t, \varepsilon)$. The approximating solution is such that the error $x(t, \varepsilon) - \bar{x}(t, \varepsilon)$ is small in some sense when $|\varepsilon|$ is small. Asymptotic methods can also be used to conclude stability properties of (1.1) from stability of the approximating system.

Among the numerous asymptotic methods, the averaging and the singular perturbation methods play a significant role as they apply to cases when a separation of time scales can be identified in the model; in other words, the variables in the model can be classified into "slow" and "fast". Such situations are very common in practice and these methods are ubiquitous in a range of engineering applications. Averaging applies to a class of time-varying systems of the form

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right) \ , \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter. Hence, the time variations of right hand side of the model are faster than the variations of the states of the system. The vector field is often assumed to be periodic with $f(t+T, x, 0) = f(t, x, 0)$ for some $T > 0$. On the other hand, the singular perturbation technique applies to models of the form:

$$\begin{aligned} \dot{x} &= f(t, x, z, u, \varepsilon) \\ \varepsilon \dot{z} &= g(t, x, z, u, \varepsilon) \ , \end{aligned} \tag{1.3}$$

where $u = u(t)$ is the control vector and the states x and z can be classified into slow and fast respectively.

Averaging is ubiquitous in analysis of several important classes of engineered systems. For instance, in power electronic systems, pulse width modulation (PWM) is prevalent and it can often be shown that the response of the system is equivalent to the average effect of the pulse train; hence, averaging techniques can be used in their analysis [83]. Certain control strategies are also amenable to analysis via the averaging method, such as the vibrational control [164], network control [147], adaptive control [5] and extremum seeking control [149, 150].

There are numerous examples of singularly perturbed systems that arise in engineering practice and especially in control engineering. For instance, in electromechanical systems such as electrical motors, the electrical variables (e.g. voltages and currents) change more rapidly in time than the mechanical variables (e.g. velocities and positions) and singular perturbation methods can be used for their analysis [76]. Indeed, in this case the electrical and mechanical variables can be respectively classified as “fast” and “slow”. Other examples of singularly perturbed systems are chemical reactors with slow and fast reactions [101]. Singular perturbations are also useful in situations when the time scale separation is enforced through the use of classes of controllers u, such as the high gain controllers [76, 138], backstepping controllers [79, 140, 159] or feedforwarding controllers [122]. Further applications can be found in fields of atmosphere or ocean science, finance, chemical engineering and so on [52, 75, 107, 119, 163].

This thesis concentrates on developing asymptotic methods - in particular, averaging and singular perturbations - for several classes of models that were not previously considered in the literature; the emphasis is on the development of tools that are useful in analysis and design of control systems. Novel technologies, new controller design techniques and classes of controllers necessitate the use of such new models for which asymptotic methods need to be developed.

This chapter is organized as follows. In Section 1.1, we introduce and motivate the classes models that we consider in the present dissertation. Background on averaging and singular perturbations methods are respectively presented in Sections 1.2 and 1.3. The main contributions of the thesis are summarized and its outline is presented in the last section.

1.1 Novel model classes: beyond the disturbance-free continuous-time systems

Averaging and singular perturbation methods have been typically addressed for continuous-time systems without disturbances, by considering models of the form (1.2) and (1.3) respectively. A good summary of these classical results can be found in [74] and references cited therein. However, understanding of robustness to disturbances is essential in control theory and asymptotic methods for continuous-time systems with exogenous disturbances were recently developed.

Indeed, results for averaging were developed in [110, 153] for continuous-time systems with disturbances w:

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x, w\right) , \tag{1.4}$$

and for singular perturbations were proposed in [37, 151] for continuous-time systems of the form:

$$\begin{aligned} \dot{x} &= f(t, x, z, w, \varepsilon) \\ \varepsilon\dot{z} &= g(t, x, z, w, \varepsilon) \ . \end{aligned}$$

The main purpose of this thesis is to develop averaging and singular perturbation asymptotic tools for stability and robust stability analysis of several classes of discrete-time, switched and hybrid models that were recently considered in the literature. We next introduce these classes of models and provide motivation for their investigation. More details on prior research work on averaging and singular perturbation methods are presented in the next two sections.

First, we consider averaging for a family of parameterized discrete-time models with disturbances of the form:

$$\frac{\Delta x}{\tau} = F_\tau \left(\frac{k\tau}{\varepsilon}, x(k\tau), w(k\tau) \right) , \tag{1.5}$$

where $\Delta x := x((k+1)\tau) - x(k\tau)$, τ is the sampling period and $\varepsilon > 0$ is a small parameter. These models arise when an approximate discrete-time model of a sampled-data plant is used to design a discrete-time controller [112]. The sampling period τ can be adjusted to reduce the mismatch between the exact and approximate discrete-time models of the plant.

Sampled-data systems are prevalent in the control engineering practice and currently attract a lot of attention in the literature, see [35, 36, 67]. The presence of a sampler in the closed loop makes the sampled-data system time-varying even if the plant and controller are time invariant. This complicates the analysis of sampled-data systems, especially when the plant is nonlinear [35, 36, 67].

A framework for controller design for nonlinear sampled-data plants via their approximate discrete-time models was proposed and developed in [109, 112, 113]. Parameterized discrete-time systems of the form (1.5) naturally arise within this design framework and their stability properties need to be ensured in order for the results in [109, 112, 113] to apply. Hence, the averaging results for (1.5) that we prove in Chapter 2 can be used together with the results in [109, 112] to design controllers for time-varying sampled-data plants, as illustrated by an example in Chapter 2. To the best of our knowledge this class of models has not been considered before in the context of averaging.

We also consider averaging for nonlinear and linear switched systems of the form:

$$\dot{x} = f_{\rho(\frac{t}{\varepsilon})}(x, w) , \tag{1.6}$$

where x is the state and w is the external disturbance; ε is a small positive parameter; there are N subsystems indexed by set $i \in S \triangleq \{1, 2, \cdots, N\}$ and $\rho : \mathbb{R}_+ \to S$ is a switching law.

Stability of switched systems is often based on slow switching assumption and many references use the notion of the "dwell time" to prove stability of the switched system [65, 88, 89, 104, 105, 161, 166]. We, on the other hand, consider rapidly switching systems with disturbances of the form (1.6) and use averaging techniques to investigate their properties. Switching systems of this form have been used in applications in areas such as power electronics [83], network control systems [127, 147], adaptive control systems [5, 41], synchronization of chaotic oscillators [95], control of multiple autonomous agents [126] and so on.

Recently, a new modeling framework was proposed for hybrid systems as outlined in [55]. Stability analysis for hybrid feedback control systems continues to attract attention since hybrid feedbacks enhance the capabilities of nonlinear feedback control even for a continuous-time plant. There are many circumstances where hybrid feedbacks play an important role, such as for systems with quantized signals [87], systems that do not admit control-Lyapunov functions [54, 129, 134], locomotion robotic systems [128, 133], autonomous vehicles systems [99], juggling systems [31, 137], motor drive control systems [90] and robot path following systems [64].

We consider in this thesis two classes of parameterized hybrid models that are presented next. The models we consider fit within the modeling framework proposed in [55]. We first consider averaging for the following class of hybrid systems:

$$\left.\begin{array}{l}\dot{x} = f_\varepsilon(x, w, \tau)\\ \dot{\tau} = \frac{1}{\varepsilon}\end{array}\right\} \qquad ((x, w), \tau) \in C \times \mathbb{R}_{\geq 0}$$
$$\left.\begin{array}{l}x^+ \in G(x, w)\\ \tau^+ \in H(x, w, \tau)\end{array}\right\} \qquad ((x, w), \tau) \in D \times \mathbb{R}_{\geq 0}\ , \tag{1.7}$$

where x is the state; w is the hybrid input signal; $\varepsilon > 0$ is a parameter; f_ε is the flow mapping and G is a set valued mapping that respectively reflect continuous-time and discrete-time dynamics of the state x; C and D are constraint sets that allow x to flow and jump respectively; H is the jump mapping for the timer τ.

The hybrid model (1.7) arises in hybrid feedback control systems that are driven by pulse-width-modulation (PWM). PWM is a technique in which the width of a train of voltage (or current) pulses is adjusted (modulated) by rapidly turning the switch between the supply and load on and off [152]. This technique is ubiquitous in electrical power systems that are an indispensable technology and prevalent in many branches of technology, manufacturing, transportation, and so on. The key elements of electrical power systems, electronic power converters, have a very high efficiency and power density and can operate at very high frequencies due to efficient power semi-conductors that allow for high frequency switching with minimal losses. Indeed, this allows for an efficient implementation of various PWM techniques [71, 97].

The PWM technique is used extensively in power electronics and finds wide applications in industry [40, 71, 83, 118, 146, 148]. The net effect of the modulated voltage pulse train on a load can be shown to be equal to the average voltage of the pulse train; while this observation can be proved in many situations, rigorous averaging techniques for general pulse width modulated systems are still not fully developed. The existing averaging results for PWM systems, e.g. [78, 83, 100], are tailored for the case when systems are controlled with a continuous feedback controller. Note that there are situations when certain closed-loop performance specifications can not be achieved for power converter systems with any continuous feedback controller whereas they are achievable with a hybrid controller, see [24, 32, 93, 98]. This observation provides a partial motivation for developing averaging techniques for hybrid systems in Chapter 4.

Finally, we consider a class of singularly perturbed hybrid systems:

$$\left.\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \dot{z} &= \tfrac{1}{\varepsilon}\psi(x,z,\varepsilon) \end{aligned}\right\} \qquad (x,z) \in C \times \Psi \tag{1.8}$$
$$(x,z)^+ \in G(x,z) \qquad (x,z) \in D \times \Psi \ ,$$

where x and z are the states; f and ψ are continuous functions; G is a set-valued mapping; C and D are constraint sets that allow for x to flow or jump respectively. The set Ψ is assumed to be compact as we wish to deal with compact attractors for the fast state z and without any assumption on the set-valued map G; if (1.8) admits solutions with a purely discrete-time domain then a jump rule like $z^+ = z$ will not allow z to converge to a compact set unless it is constrained to a compact set a priori. The small parameter $\varepsilon > 0$ ensures that the flow dynamics of z are much faster than x. Hybrid systems (1.8) can be used to model the dynamics of a hybrid feedback control system with fast but continuous actuators.

In the next two sections we provide the background on averaging and singular perturbations. We start from the classical results that consider disturbance-free continuous-time systems, and then discuss the results that are more closely related to models that we introduced in this section. We do not present a comprehensive overview of the literature and more details can be found in the cited references.

1.2 Averaging

Averaging was developed into a rigorous approximation theory during the second half of the 18^{th} century with its origins in astronomy and physics [23]. As mentioned in [132], the original work of obtaining approximate solutions for differential equations via averaging was first developed and used extensively by Lagrange [80] and Laplace in his study of the Sun-Jupiter-Saturn configuration [82].

The averaging method has attracted research attention ever since and found applications in areas such as power electronics [83], network control systems [127, 147], adaptive control systems [5, 41], synchronization of chaotic oscillators [95], control of multiple autonomous agents [126] and so on. Systems that are amenable to averaging exhibit a time-scale separation between the time variations of the state of the system and the time variations of the derivative of that state [23]. For such systems, the effect of fast oscillatory dynamics is averaged in an appropriate sense and then the average behavior of the system is captured by an appropriately defined time-invariant average system. It is then shown that the time-invariant average system approximates the actual time varying system in an appropriate sense. Consequently, the standard methods for time-invariant systems can be employed to consider properties for the actual time-varying system based on this average model.

Suppose we are given a time-varying system (1.2), where $f(., x)$ is assumed to be periodic of period $T > 0$. The classical averaging method shows that the following time-invariant system

$$\dot{x} = f_{av}(x) \tag{1.9}$$

can be used to analyze the properties of (1.2), where

$$f_{av}(x) := \frac{1}{T}\int_0^T f(\tau, x)d\tau \ .$$

The basic problem in the averaging method is to determine in what sense the behavior of the time-invariant average system (1.9) approximates the behavior of the actual time-varying system (1.2).

The classical averaging method, see [132], shows that the solutions of system (1.2) can be approximated by the solutions of its time-invariant average system on compact time intervals, with the error of order of $O(\varepsilon)$ on the time scale of order $O(1/\varepsilon)$. The Krylov-Bogolyubov-Mitropolsky method relaxes periodic assumption for vector fields f in (1.2) to almost-periodic [23].

Note that it is not necessary for f to be periodic or almost periodic; indeed, general averaging methods can be applied in this case [74, Chapter 10]. With the assumption of existence of a well-defined average, closeness between solutions of the time-varying system and solutions of its average on intervals of the order $O(1/\varepsilon)$ can be studied. The order of approximation error is determined by an order function $\delta(\varepsilon)$, which depends on how well the average system (1.9) approximates the actual system (1.2).

Systems for which the dynamics simultaneously depend on a "fast" and a "slow" time modeled as $\dot{x} = f_p(t/\varepsilon, t, x, \varepsilon)$ often arise in practice. In such cases, only the fast variations in the system are averaged whereas the slow variations are retained in the approximate model; this leads to the notion of "partial averages" considered in [16, 66, 132]. Despite the fact that the partial average is a time-varying system, partial averages may still greatly simplify the analysis and they find applications in various areas, such as celestial mechanics [132]. We use the notion of partial averages to study dynamical properties of switched linear systems in Chapter 3.

Averaging can also be used to conclude about the stability properties of the actual system (1.2) via the stability properties of its average system (1.9). First, we discuss the classical results that assume local asymptotic/exponential stability properties of the average system and then summarize results that use global asymptotic stability of the average system. In fact, it can be shown that stability of the average system typically implies a weaker stability property for the actual system. To be more precise, the solutions of the actual time-varying system are ultimately bounded with the ultimate bound that becomes arbitrarily small, if the parameter ε is sufficiently reduced. This topological nature of the convergence property of solutions is called practical stability.

Practical stability was used in the early work on stability analysis for continuous-time systems by using the averaging method, where the vector field f in (1.2) is assumed to be periodic or almost periodic. For instance, Hale [62] proved that for nonlinear systems (1.2), there exists a periodic solution in a neighborhood of the equilibrium point that is uniformly asymptotically stable, when the linearization of the averaged system (1.9) is exponentially stable.

Without the assumption of periodicity or almost periodicity for vector fields, the averaging results in [3, 74, 121] show that exponential stability of the averaged system implies practical exponential stability for the actual system. These results are based on the Lyapunov approach, where the time-invariant nonlinear average system is linearized at its equilibrium and exponential stability of the approximated linear model guarantees the existence of a Lyapunov function, which can be used in analysis of stability properties for the actual system.

Note that the results in [3, 62, 74, 121] focus on local stability of dynamical systems. In contrast, sufficient conditions are established in [2, 120] that can be used to conclude uniform semi-global practical asymptotic stability for the original time-varying system in case when the system is homogeneous with a positive order. The concept of semi-global asymptotic stability pertains to the case when one can prove that, by tuning the parameter ε, the domain of attraction can be arbitrarily enlarged [33].

Relaxing the homogeneous assumption, Teel et al. showed in [103, 156] that, under appropriate technical conditions, if the origin of the average system is globally asymptotically stable, then it is semi-globally practically asymptotically stable for the actual time-varying system.

The averaging results given above do not consider influence of input signals. Indeed, for continuous-time systems with exogenous disturbances, see (1.4), an appropriate averaging stability theory was derived in [110, 151, 153, 155], where the notions of strong and weak averages introduced in [110] play an important role. We summarize these concepts next as they are directly related to results presented in this thesis. A function $f_{wa}(x, w)$ is said to be a weak average of $s \mapsto f(s, x, w)$ in (1.4) if

$$\left| f_{wa}(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w) ds \right| \leq \beta_{av}(\max\{|x|, |w|, 1\}, T) \ ,$$

holds for arbitrary $T > 0$, where the function β_{av} is of class $\mathcal{KL}$ [1]. On the other hand, we have a strong average $f_{sa}(x, w)$ for $s :\mapsto f(s, x, w)$ if

$$\left| \frac{1}{T} \int_t^{t+T} [f_{sa}(x, w(s)) - f(s, x, w(s))] ds \right| \leq \beta_{av}(\max\{|x|, ||w||_\infty, 1\}, T),$$

for all $w \in \mathcal{L}_\infty$ [2]. Consequently, we can define the weak average $\dot{x} = f_{wa}(x, w)$ and the strong average $\dot{x} = f_{sa}(x, w)$ for system (1.4). Note that it is not necessary for f to be periodic or almost periodic to employ the strong or the weak average definition. Also note that the input signal w is a vector in the definition of weak average whereas it is a function in the strong average definition, which implies that weak averages pertain to slow-varying input signals and strong averages are applicable for both fast and slow varying signals.

With the strong and weak average definitions, robustness properties to dis-

[1] The definition of $\mathcal{KL}$ function is given at the beginning of Part II.
[2] For a measurable function $w(\cdot)$, it is called $w \in \mathcal{L}_\infty$ if $||w||_\infty < \infty$, where $||w||_\infty := ess \ \sup_{t \geq 0} |w(t)|$.

turbances for continuous-time systems (1.4) are considered in [110,153,155], where the notions of input-to-state stability (ISS) and derivative ISS (DISS) are used. ISS and DISS are important tools in analysis of robustness to disturbances for dynamical systems but they are only two of the many other possible robust stability properties for dynamical systems [6–8, 49, 63, 70, 96, 139]. The concept of ISS introduced by Sontag [143,144] provides a way to characterize the asymptotic behavior of nonlinear systems in the presence of perturbations. ISS reflects the relationships between states and inputs: bounded inputs imply bounded states. DISS is a generalization of ISS and it is useful in some important situations, where the infinity norm of the input signal and the infinity norm of its derivatives are rather small, e.g. in the output regulation context. For instance, one can get tighter estimation for the steady-state tracking due to time-varying and smooth reference signals in the output regulation context [8, 114].

It was shown in [110] that the actual time-varying system is semi-globally practically ISS (SGP-ISS) if its strong average is ISS, and it is semi-globally practically DISS (SGP-DISS) if its weak average is ISS. In [153, 155], one can get that solutions of system (1.4) can be made arbitrarily close to solutions of its strong or weak average systems on the finite time interval if ε can be reduced sufficiently small. As a generalization of results in [110], a unified framework for studying robustness properties to slowly-varying parameters, rapidly-varying signals and generalized singular perturbations is given in [151]. We can also get from [110] that strong averages exist for a smaller class of systems but using them one can state stronger stability results. On the other hand, weak averages exist for a larger class of systems but using them one can state weaker stability results. Nevertheless, weak averages are found useful in cases when disturbances are bounded and have bounded derivatives and such situation arises when one deals with cascaded systems.

We next summarize averaging results that are related to parameterized discrete-time systems, switched systems and hybrid systems discussed in Section 1.1.

For parameterized discrete-time systems (1.5), we are not aware of any averaging results. Hence, we next summarize averaging results on non-parameterized discrete-time systems that are most closely related to our results in Chapter 2. As mentioned in [84], the reference [61] is among the first to discuss stability properties of discrete-time systems with periodic vector fields. Early Russian citations that include [19] and [102] extended the work of [61] to non-periodic systems. Averaging of discrete systems has been also developed for applications in adaptive identification and control [15, 21, 22, 141]. In particular, it is shown in [15, 21, 141] that if the linearization of the averaged system is exponentially stable, then there exists a unique solution of the original discrete-time system in a neighborhood of the equilibrium point of the averaged system. Moreover, if the equilibrium points of both systems are identical, then the original system is locally exponentially stable.

Most averaging results for discrete-time systems focus on local stability properties of the actual system and assume that no exogenous signals exist. Additionally, these results can be applied for analysis of sampled-data systems only under the assumption that the exact discrete-time model of the plant is known, which is often not justified in sampled-data nonlinear systems. On the other hand, we consider non-local ISS properties for parameterized discrete-time systems in Chapter 2, that naturally arise when an approximate discrete-time model of a sampled-data nonlinear system is used for its stability analysis or controller design, see [109, 112]. Indeed, the results of this chapter can be used together with [109,112] to design controllers achieving ISS for nonlinear sampled-data systems for which the exact discrete-time model can not be analytically computed and we have to use an approximate discrete-time model for controller design and stability analysis.

In Chapter 3, robust stability analysis of fast switching systems is considered. Averaging methods play an important role in this context and find applications in areas such as power electronic systems, digital control systems, synchronization of chaotic oscillators, control of multiple autonomous agents and so on [20, 48, 124–126, 147]. Nevertheless, the averaging theory for switched systems is still underdeveloped.

Recent averaging results for systems with fast switching behavior show that under appropriate switching signals, the switched systems whose subsystems are not necessarily stable may still exhibit a stable behavior if the average system induced by the switching system is stable [127,131]. For instance, exponential stability of a class of linear switched systems was investigated under the assumption that its average system is exponentially stable [127]. In [131], the same authors use the averaging method to analyze the finite $\mathcal{L}_2$ gain property of rapidly switching linear systems of the form

$$\begin{aligned} \dot{x} &= A_{\rho(\frac{t}{\varepsilon})}x + B_{\rho(\frac{t}{\varepsilon})}w \\ y &= C_{\rho(\frac{t}{\varepsilon})}x \ , \end{aligned} \tag{1.10}$$

where x, u and y are states, input signals and outputs respectively; there are N time-invariant subsystems characterized by the matrices (C_i, A_i, B_i) with $i \in S := \{1, 2, \cdots, N\}$; $\rho : \mathbb{R}_+ \to S$ is a switching law.

It is shown in [131] that if the input matrix B_ρ does not switch, the $\mathcal{L}_2$ gain of the actual time-varying switched system is bounded by the $\mathcal{L}_2$ gain of its average

$$\begin{aligned} \dot{x} &= \bar{A}x + \bar{B}w \\ y &= \bar{C}x \end{aligned} \tag{1.11}$$

when the switching rate is increased, where $\bar{A} := \frac{1}{T}\int_0^T A_{\rho(s)}ds$ and $\bar{B}$ and $\bar{C}$ are defined similarly. In addition, it was illustrated via an example that if the input matrix B_ρ switches, then the $\mathcal{L}_2$ gain of the actual switched system may not be bounded by the $\mathcal{L}_2$ gain of its average when the switching rate increases. We consider averaging for general nonlinear switched systems and also give stronger conclusions for linear switched systems than [131]. The details of our contributions are given in Section 1.4.

In Chapter 4, we consider averaging of hybrid dynamical systems (1.7). As we already pointed out, hybrid systems are more general than pure continuous-time or discrete-time systems. They represent a relatively new area of research including a variety of challenging problems that may be approached at various levels of detail and sophistication [9, 165]. Consequently, most averaging results on hybrid dynamics are valid only for special classes of systems.

For instance, systems with dither signals are a special class of hybrid systems, for which stability analysis via the averaging method is given in [68, 69, 149, 150]. The dither signal has the effect of averaging the nonlinearity of Lipschitz continuous feedback systems [167]; it also provides excitation for parameter identification to find an extremum value of a known nonlinear mapping and leads to many practical implementations [1, 86, 123, 149, 150, 168]. Iannelli et al indicated how averaging can be used to infer rigorous practical stability of a class of nonlinear dither systems by analyzing the continuous-time average systems [68, 69, 167].

Note that for dither systems, a continuous-time or a non-hybrid average system is used to approximate the actual hybrid system. However, in some situations, it is more appropriate to approximate the time-varying hybrid system by a time-invariant hybrid system. For example, the hybrid systems (1.7) used to model the hybrid feedback control systems that are actuated through pulse-width modulation (PWM). PWM is a paradigm in which the actuator can only apply pulses of constant amplitude and modulated width [152]. In this case, it is desirable to prove that the pulse-width modulated implementation produces a closed-loop behavior that is similar to the behavior generated by implementing the hybrid feedback directly [154].

The results in [154] can handle such PWM hybrid feedback control examples. It is shown that asymptotic stability of time-varying hybrid systems can be concluded from asymptotic stability of its hybrid time-invariant average system. We extend the averaging results given in [154] to hybrid systems with disturbances in Chapter 4. Adapting the notions of strong and weak averages from [110], robustness properties to inputs signals are considered for the time-varying hybrid system based on ISS of its strong or weak hybrid average system. The problem is studied under the hybrid framework that combines the results in [25, 26, 29, 30, 53, 56], where many important results for continuous-time systems are carried over to the hybrid setting, so that we can use it as analysis tools to tackle challenging problems related to hybrid dynamics.

We next summarize pertinent results on singular perturbations that are relevant to results that we present in Chapter 5.

1.3 Singular perturbations

Singular perturbation techniques are used to study dynamical systems having a natural separation of time scales where all states can be grouped into slow and fast time scales. Such systems arise in all areas of science and engineering [76, 107]. These techniques are also used to analyze control problems for plants not in standard singular perturbation form. For example, the singular perturbation technique was applied to the high-gain feedback control, optimal control and stochastic control [76, 77, 138]. Many references on the applications of singular perturbation techniques in system control are provided in [106].

The basic intuitive ideas of singular perturbation theory can be found in Prandtl's study in 1904 on fluid dynamical boundary layers and early works by Laplace, Kirchhoff and others, see references in [116]. Almost half a century later, the benchmark works of Tikhonov [157] and Levinson [85] appeared.

The celebrated Levinson-Tikonov approach applies to continuous-time systems of the form (1.3). Assuming the existence of isolated real roots $z = h(t, x)$ for the algebraic equation $0 = g(t, x, z, 0)$, we can obtain auxiliary systems from (1.3): the slow (reduced) system

$$\frac{dx}{dt} = f(t, x, h(t, x), 0) \ , \tag{1.12}$$

and a family of fast (boundary layer) systems

$$\frac{dy}{d\tau} = g(t_0, x_0, y + h(t_0, x_0), 0) \ , \tag{1.13}$$

where the change of variables $y = z - h(t, x)$ is performed when the variables $t = t_0 + \varepsilon\tau$ and $x = x(t_0 + \varepsilon\tau, \varepsilon)$ are "frozen" by letting $\varepsilon = 0$ at $t = t_0$ and $x = x_0$. Then, one can relate the dynamical properties of the perturbed system (1.3) to properties of the auxiliary systems (1.12) and (1.13).

Assume that vector fields are Lipschitz continuous; the solutions of the boundary layer system (1.13) exponentially converge to a stable equilibrium manifold $h(t, x)$; and the reduced system (1.12) has a unique solution $\bar{x}(t)$ for initial condition $x(t_0)$, for some positive real number $t_1 > t_0$ and for all $t \in [t_0, t_1]$. The classical singular perturbation technique, see [74], shows that the solutions $x(t)$ and $z(t)$ of the actual system (1.3) can be approximated by $\bar{x}$, the solution y of the boundary layer system (1.13) and the quasi-steady state $\bar{z} := h(t, \bar{x}(t))$ of fast states for a sufficiently small $\varepsilon > 0$:

$$\begin{aligned} &x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon) \\ &z(t, \varepsilon) - h(t, \bar{x}(t)) - y\left(\frac{t}{\varepsilon}\right) = O(\varepsilon) \quad \forall\, t \in [t_0, t_1] \ . \end{aligned}$$

Moreover, exponential stability of the original system (1.3) can be guaranteed with exponential stability of the reduced and boundary layer systems (1.12) [74].

The singular perturbation technique has attained a high level of maturity in the theory of continuous-time and discrete-time control systems described by ordinary differential and difference equations respectively [60,60,91,92,106]. Analogs of the classical singular perturbation theory were also established for differential inclusions on finite time intervals [44, 160] and on infinite time intervals [162] with the assumptions that the boundary layer system converges to a Lipschitz set-valued map and that the reduced system is globally asymptotically stable.

Another direction for development of the singular perturbation theory is to relax the assumption of the classical singular perturbation theory that trajectories of the boundary layer system converge to an equilibrium manifold. Instead, it is assumed that the trajectories converge to a set on which one can average their steady state behavior to obtain the slow system. For example, the trajectories of the boundary layer may converge to a family of limit cycles parameterized by the slow state variables. The steady-state behavior then can be used to average the derivative of slow state variables. This idea can be found in the optimal control results in [50, 51], the work of Grammel in [58, 59], the work of Artstein in [10–12] where the averaging is done using invariant measures and the reduced system is typically a differential inclusion. More recently, we can find this idea in a unified framework for studying robustness to slowly-varying parameters, rapidly-varying signals and generalized singular perturbations in [151].

Results considering singularly perturbed hybrid systems are limited and most of them consider some special class of hybrid systems [14, 39, 47, 72, 135, 136, 145]. For instance, stability of singularly perturbed hybrid feedback control systems is considered in [135, 136]. In particular, it is showed in [136] that hybrid control systems are robust to filtered measurements, a class of singular perturbation, and the continuous-time implementation of the control signal. The stability result in [135] applies to hybrid control systems that are singularly perturbed by fast, continuous actuators. This singular perturbation result justifies hybrid control design based on a simplified plant model that ignores stable, fast actuator dynamics. The analysis implies that if a hybrid control system has a compact set that is globally asymptotically stable when the actuator dynamics are omitted, then the same compact set is semi-globally practically asymptotically stable for the actual hybrid system under perturbations by actuators. In Chapter 5, we consider the class of singularly perturbed hybrid systems (1.8) and use the averaging method to study slow dynamics and relax the assumption that solutions of boundary layer systems converge to an equilibrium manifold. Our results in Chapter 5 are also compared with the results of [135] via examples.

1.4 Contribution and outline

The dissertation is organized as follows. We present our research work in Part II. Some useful definitions and mathematical notation are first listed at the beginning of Part II. After that, Chapters 2-5 are included. Robustness properties to disturbances via averaging are considered for a family of parameterized discrete-time systems in Chapter 2, for switched systems in Chapter 3 and for a class of time-varying hybrid systems in Chapter 4. Asymptotic stability for singularly perturbed hybrid systems is analyzed in Chapter 5. We summarize the thesis and propose some topics for further research in Chapter 6 that is the only chapter of Part III. We next discuss contributions for our research work in Chapters 2-5.

In Chapter 2, we consider non-local ISS properties for parameterized discrete-time systems that naturally arise when an approximate discrete-time model of a sampled-data nonlinear system is used for its stability analysis or controller design. We show that under appropriate conditions, ISS of strong (or weak) average of the family of discrete-time systems implies SGP-ISS (or SGP-ISS like) properties for the actual family of systems. We also present general results on closeness of solutions of the actual system with solutions of its weak or strong average that only require the average system to be forward complete. Our results can be used together with [109, 112] to design controllers achieving ISS for nonlinear sampled-data systems for which the exact discrete-time model can not be analytically computed and we have to use an approximate discrete-time model for controller design and stability analysis.

In Chapter 3, we present averaging results on robustness analysis for both nonlinear and linear switched systems. We show that the notions of strong, weak, and partial strong average play an important role in the context of switched systems. A direct application of results in [110] yields conditions under which solutions of the strong/weak average can approximate well solutions of the actual switched system on finite time intervals, and ISS of the strong/weak average implies SGP-ISS/SGP-DISS of the switched system. Although these results follow directly from [110], to the best of our knowledge they were not known in the switched systems literature. Recent results in [131] that consider averaging of linear switched systems with disturbances (1.8) show that $\mathcal{L}_2$ gain of the actual switched system (1.8) is bounded by the $\mathcal{L}_2$ gain of its average (1.11). We provide stronger conclusions with which exponential ISS of the strong and the partial strong average system with linear gain imply exponential ISS with linear gain of the actual system. Similarly, exponential ISS of the weak average guarantees an appropriate exponential derivative ISS (DISS) property for the actual system. Moreover, using the Lyapunov method, we show that the estimates of the linear ISS gain of the actual system and its average converge to each other as the switching rate is increased.

In Chapter 4, we study the behavior of a class of hybrid dynamical systems with disturbances through its hybrid average system. Our results generalize the averaging results for hybrid systems in [154] to deal with exogenous disturbances. Closeness of solutions between the time-varying system and its weak or strong average on compact time domains is given under the assumption of forward completeness for the average system. We also show that ISS of the strong/weak average implies SGP-ISS/SGP-DISS of the actual hybrid system. Through a power converter example, we show that our results for hybrid systems can also be used as an analysis tool in the framework of designing a hybrid feedback control for PWM control systems.

Chapter 5 considers asymptotic stability of a class of singularly perturbed hybrid dynamical systems (1.8) using both the singular perturbation technique and the averaging method. The continuous-time boundary layer dynamics produce solutions that are assumed to generate a well-defined average vector field for the slow dynamics. This average, the projection of the jump map in the direction of the slow states, and flow and jump sets from the original dynamics define the reduced, or average, hybrid dynamical system. Assumptions about the average system lead to conclusions about the original, higher-dimensional system. For example, forward pre-completeness for the average system leads to a result on closeness of solutions between the original and average system on compact time domains. In addition, global asymptotic stability for the average system implies semiglobal, practical asymptotic stability for the original system. We give examples to illustrate the averaging concept and to relate it to classical singular perturbation results as well as to other singular perturbation results that have appeared recently for hybrid systems.

In classical singular perturbation theory, see [16, 74, 158], it is assumed that the boundary layer system has a globally asymptotically stable equilibrium manifold and the vector fields are Lipschitz continuous. In contrast, a compact set replaces such equilibrium manifold and no Lipschitz continuity condition is needed for vector fields in our results, which greatly weaken that fundamental assumption. Moreover, we illustrate through an example that our results can provide sharper conclusions than the recent stability analysis results on singularly perturbed hybrid systems in [135, 136].

Part II

Research Work

Notations

$|\cdot|$ refers to the Euclidean norm, $\mathbb{R}_{\geq 0} := [0, +\infty)$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \cdots\}$.

$\mathbb{B}$ is the closed unit ball in an Euclidean space, the dimension of which should be clear from the context.

$M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ refers to a set-valued map such that every value of the argument $x \in \mathbb{R}^n$ is mapped into a set $M(x) \subset \mathbb{R}^n$.

Given a set S, its closed convex hull denoted by $\overline{con} S$ is the smallest closed convex set that contains S. Given set $\mathcal{A} \subset \mathbb{R}^n$ and a $x \in \mathbb{R}^n$, define $|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |x - y|$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero, continuous and nondecreasing.

A function $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{GG}$ if it is zero at zero, continuous and nondecreasing in both arguments.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing.

A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is non-increasing and converging to zero as its argument grows unbounded.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is of class-$\mathcal{K}$ in its first argument and class of $\mathcal{L}$ in its second argument.

A class-$\mathcal{KL}$ function $\beta(\cdot, s)$ is called exponential if $\beta(r, s) = Kr \exp(-\lambda s)$ for some $K > 0$, $\lambda > 0$.

Chapter 2

Averaging of parameterized discrete-time Systems

2.1 Introduction

Sampled-data nonlinear systems often arise in engineering practice since digitally controlled continuous processes are prevalent and nonlinear plant dynamics are common in a range of applications [35, 36, 67]. Tools for analysis and controller design for sampled-data nonlinear systems are still underdeveloped. Indeed, the most common approach to controller design is emulation of continuous controllers where the controller is first designed ignoring sampling and then discretized for digital implementation. An alternative approach is to discretize the plant model and then design the controller in discrete-time. While this approach is common for linear plants, it is not applicable to most nonlinear plants since one can not find analytically the exact discrete-time model of the system as this requires an analytical solution of a set of nonlinear differential equations that model the plant dynamics.

On the other hand, an approximate discrete-time plant model, such as the approximate model obtained by using the Euler method, is readily available but it was shown in [109, 111–113] that controller design based on approximate discrete-time models needs to be carried out very carefully. Indeed, it was shown in [112, Example 3] that a family of controllers may stabilize a family of approximate discrete-time models for all sampling periods $\tau > 0$ but at the same time destabilize the family of exact discrete-time models for all sampling periods $\tau > 0$. Certain consistency conditions in [112] are needed to guarantee that a set is semi-globally practically stable in the sampling period τ for the family of exact discrete-time models if this set is uniformly globally asymptotically stable for the family of approximate discrete-time models. At the same time, semi-global practical stability of sets for the exact discrete-time models implies under weak conditions the same property for the actual sampled-data system, see [113]. Similar results in [109] presented conditions to guarantee semi-global practical input-to-state stability (SGP-ISS) of sets for exact discrete-time models implies the same property for the actual sampled-data system.

Indeed, the results in [109, 111–113] proposed a controller design framework for sampled-data nonlinear systems that is based on families of approximate discrete-time plant models parameterized by the sampling period. One typically needs to verify uniform stability properties of a family of approximate parameterized discrete-time models for stabilization of sampled-data nonlinear systems under this framework. This motivates the averaging results provided in the present chapter that can be used to conclude SGP-ISS of parameterized families of discrete time systems and to design controllers for nonlinear sampled-data systems together with the results in [109]. More details are given in Section 2.2.

Recall that most averaging results for discrete-time systems focus on local exponential stability for non-parameterized discrete-time systems, such as [15, 141]. Such results are useful in situations when the exact discrete-time model of the sampled-data system is known. We are not aware of discrete-time averaging results for systems with disturbances, which is the main focus of the present chapter. Moreover, our results can be used together with [109] to analyze ISS of sampled-data nonlinear systems for which we can not compute the exact discrete-time model, and we need to use an approximate model for stability analysis or controller design.

We adapt the strong and weak average definitions introduced in [110] for continuous-time systems to consider robustness to disturbances for discrete-time systems. We present conditions under which ISS of the strong average implies SGP-ISS of the family of time-varying parameterized discrete-time systems. We also prove similar results based on ISS of the weak average where we conclude an ISS like property that requires derivatives of disturbances to be bounded.

This chapter is organized as follows. Section 2.2 lists the preliminary results and Section 2.3 presents the parameterized discrete-time systems and average definitions under the discrete-time setting. The main results and an application example are given in Section 2.4 and Section 2.5 respectively. The last section contains some conclusions. Proofs of the main results are provided in the Appendix A.

2.2 Preliminaries

Consider a nonlinear sampled-data plant with disturbances

$$\dot{x} = f(t, x, u, w) \ , \tag{2.1}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ and $u \in \mathbb{R}^p$ are respectively the state, exogenous disturbance and control input. For a given function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, use the notation $w_f[k] := \{w(t) : t \in [k\tau, (k+1)\tau]\}$ with $k \in \mathbb{N}$ and the sampling period $\tau > 0$, and $w(k)$ is the value of the function $w(\cdot)$ at $t = k\tau$, $k \in \mathbb{N}$. The control is taken to be a piecewise constant signal with $u(t) = u(k\tau)$ for $t \in [k\tau, (k+1)\tau)$.

If we want to carry out a controller design in discrete-time then we need to compute the (exact) discrete-time model of the plant:

$$\begin{aligned} x(k+1) &= x(k) + \int_{k\tau}^{(k+1)\tau} f(s, x(s), u(k), w(s))ds \\ &:= F_\tau^e(k\tau, x(k), u(k), w_f[k]) \ , \end{aligned} \tag{2.2}$$

which is obtained by integrating (2.1) over one sampling interval $[k\tau, (k+1)\tau]$ from the initial time $k\tau$ and the initial state $x(k) := x(k\tau)$ with a constant control input $u(k) := u(k\tau)$ and given disturbance inputs $w_f[k]$. Note that (2.1) is a functional difference equation as it is dependent on $w_f[k]$. Since (2.1) is nonlinear, it is typically not possible to analytically compute the exact discrete-time model (2.2) for controller design.

Instead, we can use approximate discrete-time models for controller design. Different approximate discrete-time models can be obtained using different methods. For instance, with the assumption that the disturbances $w(\cdot)$ are constant during sampling intervals, $w(t) = w(k)$, $\forall\ t \in [k\tau, (k+1)\tau]$ and using a classical Runge-Kutta numerical integration scheme (e.g., Euler), one gets the approximate models that can be formed as:

$$x(k+1) = F_\tau^a(k\tau, x(k), u(k), w(k)) \ . \tag{2.3}$$

On the other hand, using the numerical integration schemes for systems with measurable disturbances in [46], we obtain another family of approximate discrete-time models:

$$x(k+1) = F_\tau^a(k\tau, x(k), u(k), w_f[k]) \ . \tag{2.4}$$

Assume that the sampling period τ is a design parameter which can be arbitrarily assigned. Consider a family of dynamical feedback controllers:

$$\begin{aligned} z(k+1) &= G_\tau(x(k), z(k)) \\ u(k) &= u_\tau(x(k), z(k)) \ , \end{aligned} \tag{2.5}$$

where $z \in \mathbb{R}^{n_z}$ and $G : \mathbb{R}^n \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$. Let

$$\begin{aligned} \bar{x} &:= \begin{bmatrix} x \\ z \end{bmatrix} \\ \mathcal{F}_\tau^i(k\tau, \bar{x}(k), \cdot) &:= \begin{pmatrix} F_\tau^i(k\tau, x(k), u_\tau(x(k), z(k)), \cdot) \\ G_\tau(x(k), z(k)) \end{pmatrix} \ . \end{aligned} \tag{2.6}$$

The superscript i is either e or a, where e stands for *exact* model and a stands for *approximate* model. This superscript is omitted if we refer to a general model. The third argument of $\mathcal{F}_\tau^i(k\tau, \bar{x}, \cdot)$ (fourth argument of F_τ^i) is either a vector $w(k)$ or a piece of function $w_f[k]$.

Then, a natural question is if we design a family of dynamical feedback controllers (2.5) such that a set is stable in an appropriate sense for the following family of approximate discrete-time closed-loop models

$$\bar{x}(k+1) = \mathcal{F}_\tau^a(k\tau, \bar{x}(k), w(k)) \tag{2.7}$$

or

$$\bar{x}(k+1) = \mathcal{F}_\tau^a(k\tau, \bar{x}(k), w_f[k]), \tag{2.8}$$

would it be also stable (maybe in some weaker sense) for the family of exact discrete-time models

$$\bar{x}(k+1) = \mathcal{F}_\tau^e(k\tau, \bar{x}(k), w_f[k]) \tag{2.9}$$

using the same family of controllers (2.5).

In fact, it was shown in [109] that this is not always true. To give the details, we denote the norm $||w||_\infty := ess\ \sup_{t\geq 0}|w(t)|$, write $w \in \mathcal{L}_\infty$ if there exists $r > 0$ such that $||w||_\infty \leq r$ and define Lyapunov-semiglobal-ISS for the family of approximate closed-loop models (2.7) or (2.8).

Definition 2.2.1. *The family of parameterized discrete-time systems (2.8) is Lyapunov-semiglobally-ISS if there exists functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\tilde{\gamma} \in \mathcal{K}$, and for any strictly real numbers $\Delta_1, \Delta_2, \delta_1, \delta_2$ there exist strictly positive real numbers τ^* and L such that for all $\tau \in (0, \tau^*)$ there exists a function $V_\tau : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $\bar{x} \in \mathbb{R}^{n+n_z}$ with $|\bar{x}| \leq \Delta_1$ and for all $w \in \mathcal{L}_\infty$ with $||w||_\infty \leq \Delta_2$ the following holds:*

$$\alpha_1(|\bar{x}|) \leq V_\tau(k\tau, \bar{x}) \leq \alpha_2(|\bar{x}|)$$

$$\frac{1}{\tau}\left[V_\tau(\mathcal{F}_\tau(k\tau, \bar{x}, w_f)) - V_\tau(k\tau, \bar{x})\right] \leq -\alpha_3(|\bar{x}|) + \tilde{\gamma}(||w_f||_\infty) + \delta_1 \quad \forall\ k \geq 0, \tag{2.10}$$

and, moreover, for all x_1, x_2, z with $|(x_1, z)^T|$, $|(x_2, z)^T| \in [\delta_2, \Delta_1]$, $\tau \in (0, \tau^*)$ and $k \geq 0$, we have $|V_\tau(k\tau, x_1, z) - V_\tau(k\tau, x_2, z)| \leq L|x_1 - x_2|$.

Remark 2.2.2. *For the family of parameterized discrete-time systems (2.7), the condition (2.10) is replaced by: for all $\tau \in (0, \tau^*)$, all $\bar{x} \in \mathbb{R}^{n+n_z}$ with $|\bar{x}| \leq \Delta_1$ and all $w \in \mathbb{R}^m$ with $|w| \leq \Delta_2$ we have:*

$$\frac{1}{\tau}\left[V_\tau(\mathcal{F}_\tau(k\tau, \bar{x}, w)) - V_\tau(k\tau, \bar{x})\right] \leq -\alpha_3(|\bar{x}|) + \tilde{\gamma}(|w|) + \delta_1.$$

With the above definition, the following conditions presented in [109] are required to guarantee that ISS properties of the approximate models (2.7) or (2.8) implies the similar property to the exact approximate models (2.9). [1]

1. **The family F^a_τ is one-step weakly consistent with F^e_τ**: for any given strictly positive real numbers $\Delta_x, \Delta_u, \Delta_w, \Delta_{\dot{w}}$, there exists a function $\alpha \in \mathcal{K}_\infty$ and $\tau^* > 0$ such that for all $\tau \in (0, \tau^*)$, all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ with $|x| \leq \Delta_x$ and $|u| \leq \Delta_u$ and function $w(\cdot)$ that are continuously differentiable and satisfy $||w_f||_\infty \leq \Delta_w$ and $||\dot{w}||_\infty \leq \Delta_{\dot{w}}$ we have that $|F^a_\tau - F^e_\tau| \leq \tau\alpha(\tau)$;

[1] Results in [109] are given for time invariant systems but these results can easily be extended to cover time-varying systems, see [81].

2. The family of control laws u_τ is locally uniformly bounded: for any $\Delta_{\bar{x}} > 0$ there exist strictly positive numbers τ^* and Δ_u such that for all $\tau \in (0, \tau^*)$ and all $|\bar{x}| \leq \Delta_{\bar{x}}$ we have $|u_\tau(\bar{x})| \leq \Delta_u$.

3. The family of approximate closed-loop models (2.7) or (2.8) is Lyapunov-semiglobally-ISS.

Note that the above conditions pertain to both approximate models (2.7) and (2.8). As a matter of fact, if we only consider the approximate discrete-time models (2.8), we may only need the following Conditions 4-6, of which the one-step strong consistency between F^a_τ and F^e_τ condition is independent to the derivative of disturbances.

4. **The family F^a_τ is one-step strongly consistent with F^e_τ**: for any given strictly positive real numbers $\Delta_x, \Delta_u, \Delta_w$, there exists a function $\alpha \in \mathcal{K}_\infty$ and $\tau^* > 0$ such that for all $\tau \in (0, \tau^*)$, all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $w \in \mathcal{L}_\infty$ with $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ and $||w_f||_\infty \leq \Delta_w$ such that $|F^a_\tau - F^e_\tau| \leq \tau\alpha(\tau)$;

5. The family of control laws u_τ is locally uniformly bounded: for any $\Delta_{\bar{x}} > 0$ there exist strictly positive numbers τ^* and Δ_u such that for all $\tau \in (0, \tau^*)$ and all $|\bar{x}| \leq \Delta_{\bar{x}}$ we have $|u_\tau(\bar{x})| \leq \Delta_u$.

6. The family of approximate closed-loop models (2.8) is Lyapunov-semiglobally-ISS.

The one-step strong or weak consistency condition is adapted from the numerical analysis literature and it holds for most commonly used approximations, such as Runge-Kutta methods. The second condition is easily checked once the control law (2.5) is obtained. The last condition is typically the hardest to check and it needs to be done on a case-by-case basis. This necessitates the development of various stability analysis tools for parameterized families of discrete-time systems (2.3) or (2.4) under the control law (2.5) that are useful in different situations. The main purpose of this chapter is to develop several such stability analysis tools that are based on the averaging theory.

Indeed, we define weak and strong averages to consider robustness to the disturbances with different properties for discrete-time models for families of discrete time models (2.7) or (2.8). The weak average definition pertains to slow varying disturbances that is consistent to the class of approximate models ((2.7), where w is assumed to be constant during the sampling interval, and we can apply Conditions 1-3 to ensure stability of the exact models. The strong average definition is applicable for both fast and slow varying signals and consequently pertains to both approximate models (2.7) and (2.8). Then, if we consider exact models through (2.8) using strong average, we only need to verify Conditions 4-6.

2.3 Parameterized discrete-time systems

We use sampled versions of a given continuous-time function in this chapter. Given a function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and a positive sampling period $\tau > 0$, we define its sampled version as $w_\tau := \{w(k\tau) : k \in \mathbb{N}\}$ but we omit the subscript τ for notational simplicity. Then, we define its infinity norm as $|w|_\infty := \max_{k\geq 0} |w(k\tau)|$.

Consider a family of time varying discrete-time systems parameterized by the sampling time interval $\tau > 0$:

$$\frac{\Delta x}{\Delta k} = F_\tau\left(k\tau, x, w\right) \qquad \Delta k = \tau \ , \tag{2.11}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the input signal, $\Delta x := x(k\tau+\Delta k) - x(k\tau)$, $k \geq 0$.

We also investigate the family of parameterized discrete-time systems that depends on a small parameter $\varepsilon > 0$:

$$\frac{\Delta x}{\Delta k} = F_\tau\left(\frac{k\tau}{\varepsilon}, x, w\right) \qquad \Delta k = \tau \ , \tag{2.12}$$

where the small parameter ε is used to imply that the time-varying terms change faster than states and hence they can be averaged. We require the following assumptions on local Lipschitz continuity and boundedness of $F_\tau(k\tau, x, w)$.

Assumption 2.3.1. *The family of parameterized functions $F_\tau(k\tau, x, w)$ is locally Lipschitz continuous in (x, w) uniformly in $k\tau$, $F_\tau(k\tau, 0, 0)$ is bounded.*

Next we adapt weak and strong average definitions that are introduced in [110] for continuous-time systems to families of discrete-time systems (2.12) so that we can obtain stability results that are fully consistent with [109, 112]. In particular, we want to check Condition 3 in the results outlined above.

Definition 2.3.2 (weak average)**.** *A function F_τ^{wa} is said to be the weak average of F_τ if there exists $\beta_{wa} \in \mathcal{KL}$ and $T^* > 0$ such that for all $T > T^*$, there exists*

$\tau^* = \tau^*(T)$, such that $\forall\ \tau \in (0, \tau^*)$ and $N\tau \geq T$, the following holds for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

$$\left| F_\tau^{wa}(x, w) - \frac{1}{N\tau} \sum_{s=k}^{k+N} F_\tau(s\tau, x, w)\Delta s \right| \leq \beta_{wa}(\max\{|x|, |w|, 1\}, N\tau) \ . \qquad (2.13)$$

The weak average of the parameterized family of discrete-time systems (2.12) is then defined as

$$\frac{\Delta y}{\Delta k} = F_\tau^{wa}(y, w) \qquad \Delta k = \tau \ . \qquad (2.14)$$

□

Let $\mathcal{L}_\mathcal{W}$ be a given subset of input signals $w : \text{dom } w \to \mathbb{R}^m$. We have the following strong average definition.

Definition 2.3.3 (strong average). *A function F_τ^{sa} is said to be the strong average of F_τ if there exists $\beta_{sa} \in \mathcal{KL}$ and and $T^* > 0$ such that for all $T > T^*$, there exists $\tau^* = \tau^*(T)$, such that $\forall\ \tau \in (0, \tau^*)$ and $N\tau \geq T$, the following holds for all $x \in \mathbb{R}^n$, $w \in \mathcal{L}_\mathcal{W}$*

$$\left| \frac{1}{N\tau} \sum_{s=k}^{k+N} \{F_\tau^{sa}(x, w(s\tau)) - F_\tau(s\tau, x, w(s\tau))\}\Delta s \right| \leq \beta_{sa}(\max\{|x|, |w|_\infty, 1\}, N\tau). \qquad (2.15)$$

The strong average of the parameterized family of discrete-time systems (2.12) is then defined as

$$\frac{\Delta y}{\Delta k} = F_\tau^{sa}(y, w) \qquad \Delta k = \tau \ . \qquad (2.16)$$

□

The above weak and strong average definitions are adapted from [110] where continuous-time systems are considered. The following remark using weak average as an example to illustrate that the average definitions for parameterized discrete-time systems that depend on the parameter τ are consistent to the definitions for continuous-time systems in [110].

Remark 2.3.4. *In [110], a locally Lipschitz continuous function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called the weak average for the function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ if there exist $\tilde{\beta}_{av} \in \mathcal{KL}$ and $T^* > 0$ such that for all $T \geq T^*$ and $t \geq 0$, the following holds:*

$$\left| f_{wa}(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w) ds \right| \leq \tilde{\beta}_{wa}(\max(|x|, |w|, 1), T) \; . \tag{2.17}$$

Let $t := k\tau$ and $\tilde{T} := N\tau$ for arbitrary $k \geq 0$, $N\tau > 0$ and fixed τ. We consider the inequality (2.13) in the weak average definition for parameterized discrete-time systems and get that

$$\begin{aligned} & \left| F_\tau^{wa}(x, w) - \frac{1}{N\tau} \sum_{s=k}^{k+N} F_\tau(s\tau, x, w) \Delta s \right| \\ & \leq \left| F_\tau^{wa}(x, w) - \frac{1}{\tilde{T}} \int_t^{t+\tilde{T}} F_\tau(h, x, w) dh \right| \\ & \quad + \left| \frac{1}{\tilde{T}} \int_t^{t+\tilde{T}} F_\tau(h, x, w) dh - \frac{1}{N\tau} \sum_{s=k}^{k+N} F_\tau(s\tau, x, w) \Delta s \right| \; , \\ & \leq \tilde{\beta}_{wa} \left(\max\{|x|, |w|, 1\}, \tilde{T} \right) + \frac{1}{\tilde{T}} \left| \int_t^{t+\tilde{T}} F_\tau(h, x, w) dh - \sum_{s=k}^{k+N} F_\tau(s\tau, x, w) \tau \right| \; , \end{aligned} \tag{2.18}$$

where $\tilde{\beta}_{wa}(\cdot)$ comes from (2.17). The second term of (2.18) denotes the error between the sum of $F_\tau(s\tau, x, w)$ from $s = k$ to $s = k + N$ and the integral of $F_\tau(h, x, w)$ for $h \in [k\tau, (k + N)\tau]$. To construct a $\beta_{wa} \in \mathcal{KL}$ to bound (2.18), it is required that the parameter τ is sufficiently small such that the second term of (2.18) can be bounded with a function $\gamma(|x|, |w|)$ of class-$\mathcal{GG}$. Note that $\frac{1}{\tilde{T}} \leq \frac{2}{\tilde{T}+1}$ for any $\tilde{T} \geq 1$. Let

$$\beta_{wa}(\max\{|x|, |w|, 1\}, N\tau) := \tilde{\beta}_{wa} \left(\max\{|x|, |w|, 1\}, \tilde{T} \right) + \frac{2}{\tilde{T} + 1} \gamma(|x|, |w|) \; .$$

Then, for any $N\tau \geq 1$, we have that

$$\left| F_\tau^{wa}(x,w) - \frac{1}{N\tau} \sum_{s=k}^{k+N} F_\tau(s\tau,x,w)\Delta s \right| \le \beta_{wa}(\max\{|x|,|w|,1\}, N\tau) \ ,$$

which gives the Definition 2.3.2 and illustrates that the weak average definition for parameterized discrete-time systems depends on τ. For general periodic systems, the sampling period τ is independent of T, but $\tau^* = \tau^*(T)$ is used for the aim of generalization. For instance, it is used to exclude the case when the sampling interval τ coincides with the integer multiple of period for periodic function F_τ. In this case, the average of F_τ is a constant that can not provide useful information about F_τ. □

Note that the main difference between the weak and strong averages is that in the definition of weak average the disturbance is kept constant in (2.13) whereas in the definition of strong average the inequality (2.15) needs to hold for all disturbances $w \in \mathcal{L}_\mathcal{W}$. In case when $w \equiv 0$ both definitions of average coincide.

Strong averages exist for a smaller class of systems but using them we can state stronger stability results. On the other hand, weak averages exist for a larger class of systems but using them we can state weaker stability results. Nevertheless, weak averages are found useful in cases when disturbances are bounded and have bounded derivatives and such situation arises when one deals with ISS of cascaded systems. Hence, the notions of weak and strong averages are useful in different situations and so we investigate both.

Remark 2.3.5. *A complete characterization of strong averages for continuous-time periodic systems was given in [110]. It can be shown in a similar manner to [110] that any $F_\tau(s\tau,x,w)$ that is periodic in $s\tau$ has a strong average if and only if the function F_τ has the structure as follows:*

$$F_\tau(k\tau,x,w) = F_\tau^1(k\tau,x) + F_\tau^2(x,w) \ , \tag{2.19}$$

and there exists the average $F_{av}(x)$ for $F_\tau^1(k\tau,x)$ according to either of our definitions (they coincide since F_τ^1 does not depend on the disturbance). Then, $F_\tau^{sa}(x,w) := F_{av}(x) + F_\tau^2(x,w)$ satisfies our definition of the strong average for F_τ.

The following example shows that for some systems, the weak average may exist whereas the strong average does not.

Example 2.3.6. *Consider the system*

$$\frac{\Delta x}{\Delta k} = -0.5x^3 + \cos\left(\frac{k\tau}{\varepsilon}\right) x^3 w \tag{2.20}$$

where $x, w \in \mathbb{R}$*. The weak average of* $-0.5x^3 + \cos(k\tau)x^3 w$ *is*

$$\frac{\Delta y}{\Delta k} = -0.5y^3 \ .$$

Indeed, setting $\tilde{s} = s\tau$ *and* $T := N\tau$ *we can write for sufficiently small* τ *that*

$$\begin{aligned} \left| \frac{1}{N\tau} \sum_{s=k}^{k+N} \cos(s\tau)x^3 w \cdot \tau \right| &= \left| \frac{1}{N\tau} \sum_{\tilde{s}=k\tau}^{k\tau+N\tau} \cos(\tilde{s})x^3 w \cdot \Delta\tilde{s} \right| \ , \\ &\leq \left| \frac{x^3 w}{T} \int_{k\tau}^{k\tau+T} \cos(\tilde{s}) d\tilde{s} \right| \ , \\ &\leq \frac{|x^3 w|\pi}{T} \leq \frac{2(\max\{|x|, |w|, 1\})^4 \pi}{T+1} \ , \end{aligned} \tag{2.21}$$

where the last inequality holds when $T \geq 1$ *and we can let* $\beta_{wa}(s,t) := \frac{2\pi s^4}{t+1}$.

Now, we will show that there does not exist strong average for system (2.20). Pick an arbitrary $\bar{x} \neq 0$ *and note that, for any given function* $F_\tau^{sa}(x,w)$*, we have two possibilities*

a. *either* $F_\tau^{sa}(\bar{x}, w) + 0.5\bar{x}^3 = 0$*,* $\forall\ w$*, or*
b. $\exists\ \bar{w}$ *such that* $F_\tau^{sa}(\bar{x}, \bar{w}) + 0.5\bar{x}^3 \neq 0$.

Suppose that $F_\tau^{sa}(x,w)$ *is the strong average for* $-0.5x^3 + \cos(k\tau)x^3 w$ *and case* ***a*** *holds. Let* $w(k\tau) = \cos(k\tau)$*,* $\tilde{s} = s\tau$*, and* $N_C\tau := C\pi$ *for* $C \in \mathbb{N}$*, similarly like (2.21), we have*

$$\begin{aligned} \left| \frac{1}{N_C\tau} \sum_{s=k}^{k+N_C} \bar{x}^3 \cos^2(s\tau) \cdot \tau \right| &= \left| \frac{1}{N_C\tau} \sum_{\tilde{s}=k\tau}^{k\tau+N_C\tau} \bar{x}^3 \cos^2(\tilde{s}) \Delta\tilde{s} \right| \ , \\ &\leq \left| \frac{1}{C\pi} \int_{k\tau}^{k\tau+C\pi} \bar{x}^3 \cos^2(\tilde{s}) d\tilde{s} \right| \ , \\ &= \frac{1}{2}|\bar{x}^3| > 0 \quad \forall\ C > 0 \ , \end{aligned}$$

*which does not converge to zero as C approaches infinity ($N_C\tau \to \infty$). Suppose now that $F_\tau^{sa}(x,w)$ is the strong average for $-0.5x^3 + \cos(k\tau)x^3 w$ and case **b** holds. Pick $w(k\tau) = \bar{w}$, set $N_C\tau := 2C\pi$, one gets*

$$\begin{aligned}
&\left|\frac{1}{N_C\tau}\sum_{s=k}^{k+N_C}(F_\tau^{sa}(\bar{x},\bar{w}) + 0.5\bar{x}^3 - \bar{x}^3\bar{w}\cos(s\tau))\cdot\tau\right| , \\
&= \left|\frac{1}{N_C\tau}\sum_{\tilde{s}=k\tau}^{k\tau+N_C\tau}(F_\tau^{sa}(\bar{x},\bar{w}) + 0.5\bar{x}^3 - \bar{x}^3\bar{w}\cos(\tilde{s}))\Delta\tilde{s}\right| , \\
&\le \left|\frac{1}{2C\pi}\int_{k\tau}^{k\tau+2C\pi}(F_\tau^{sa}(\bar{x},\bar{w}) + 0.5\bar{x}^3 - \bar{x}^3\bar{w}\cos(\tilde{s}))d\tilde{s}\right| , \\
&= |0.5\bar{x}^3 + F_\tau^{sa}(\bar{x},\bar{w})| > 0 \quad \forall\, C > 0 .
\end{aligned}$$

The left hand side in the above expression is larger than zero for all $C \in \mathbb{N}$ and it does not converge to zero as C approaches infinity ($N_C\tau \to \infty$). Hence, there does not exist a strong average for system (2.20). □

2.4 Main results

We first show that solutions of families of strong or weak averages can be arbitrarily close to solutions of the family of actual parameterized discrete-time systems on compact time intervals. Instead of requiring stability of strong/weak average systems, an appropriate concept of forward completeness that is weaker than stability, see Def. 2.4.1 below, is assumed.

Definition 2.4.1. *Let $\mathcal{L}_\mathcal{W}$ be a set of locally bounded functions, the system*

$$\frac{\Delta y}{\Delta k} = F_\tau(y,w) \qquad y(k_0\tau) = y_0 \;\; \Delta k = \tau \tag{2.22}$$

is said to be $\mathcal{L}_\mathcal{W}$-forward complete if for each $r > 0$ and $T > 0$ there exists $R \ge r$ and $\tau^* > 0$ such that, for all $\tau \in (0,\tau^*)$, $|y_0| \le r$ and $w \in \mathcal{L}_\mathcal{W}$, the solutions of (2.22) are contained in a closed ball of radius R for all $(k-k_0)\tau \in [0,T]$. □

To present results for strong and weak averages respectively, that pertain to input signals with different properties, we need some definitions to classify classes of disturbances.

Definition 2.4.2. *Let $\mathcal{L}_\mathcal{W}$ be a set of locally bounded functions, the set $\mathcal{L}_\mathcal{W}$ is* *equi-bounded if there exists a strictly positive real number* r *such that, for all* $w \in \mathcal{L}_{\mathcal{W}}$, $|w|_\infty \leq r$. □

Definition 2.4.3. *Let* $\mathcal{L}_{\mathcal{W}}$ *be a set of locally bounded functions, the set* $\mathcal{L}_{\mathcal{W}}$ *is equi-uniformly Lipschitz if there exists a strictly positive real number* ν *and* $\tau^* > 0$ *such that, for all* $\tau \in (0, \tau^*)$, $w \in \mathcal{L}_{\mathcal{W}}$, $\left|\frac{\Delta w}{\Delta k}\right|_\infty \leq \nu < \infty$. □

Note that sampling a bounded continuous-time function $w(\cdot)$ at any sampling period yields its sampled version $w_\tau = w(k\tau)$ that is still bounded. From Def. 2.4.3, we know that if $\dot{w}(\cdot)$ is bounded, then its sampled version $w_\tau = w(k\tau)$ will be equi-uniformly Lipschitz continuous.

Now, we are ready for the following theorems that give conditions under which the solution of the family of systems (2.12) are close to the solutions of its weak average (2.14) or strong average (2.16) on compact time intervals. The proofs are presented in Appendices A.1 and A.2.

Theorem 2.4.4 (Closeness to weak average)**.** *Suppose that Assumption 2.3.1 holds for the family of discrete-time systems (2.12), the set* $\mathcal{L}_{\mathcal{W}}$ *is equi-bounded and equi-uniformly Lipschitz, there exists a locally Lipschitz continuous function* $F_\tau^{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *that is weak average of* $F_\tau : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and the family weak average systems (2.14) are* $\mathcal{L}_{\mathcal{W}}$*-forward complete. Then, for each triple* (r, δ, T) *of strictly positive real numbers there exists a triple of* $(\tau^*, \varepsilon^*, \mu)$ *of strictly positive numbers such that, for each* $\tau \in (0, \tau^*)$, $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$, $|y_0| \leq r$, $w \in \mathcal{L}_{\mathcal{W}}$ *and for each* x_0 *such that* $|x_0 - y_0| \leq \mu$, *each solution* $x(k\tau, k_0, x_0, w)$ *of the family of systems (2.12) and the solution* $y((k-k_0)\tau, y_0, w)$ *of the weak average satisfy*

$$|x(k\tau, k_0\tau, x_0, w) - y((k-k_0)\tau, y_0, w)| \leq \delta, \quad \forall\ k :\ (k-k_0)\tau \in [0, T]\ . \qquad \square$$

Theorem 2.4.5 (Closeness to strong average)**.** *Suppose that Assumption 2.3.1 holds for the family of discrete-time systems (2.12), the set* $\mathcal{L}_{\mathcal{W}}$ *is equi-bounded, there exists a locally Lipschitz continuous function* $F_\tau^{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *that is strong average of* $F_\tau : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and the family strong average systems (2.16) are* $\mathcal{L}_{\mathcal{W}}$*-forward complete. Then, for each triple* (r, δ, T) *of the strictly positive real numbers there exists a triple* $(\tau^*, \varepsilon^*, \mu)$ *of the strictly positive numbers such that for each* $\tau \in (0, \tau^*)$, $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$, $|y_0| \leq r$, $w \in \mathcal{L}_{\mathcal{W}}$ *and for each* x_0 *such that* $|x_0 - y_0| \leq \mu$, *each solution* $x(k\tau, k_0, x_0, w)$ *of the family of systems (2.12) and the solution* $y((k-k_0)\tau, y_0, w)$ *of the strong average satisfies*

$$|x(k\tau, k_0\tau, x_0, w) - y((k-k_0)\tau, y_0, w)| \leq \delta, \quad \forall\, k:\ (k-k_0)\tau \in [0,T]\ .$$

□

Before we come to the main results of this chapter, a preliminary lemma is given and proved in Appendix A.3 first. For a given disturbance set $\mathcal{L}_{\mathcal{W}}$, we show that the family of discrete-time systems are semi-globally practically ISS on the set $\mathcal{L}_{\mathcal{W}}$ on compact time intervals, if and only if they are semi-globally practically ISS on the set $\mathcal{L}_{\mathcal{W}}$. Precise definitions are first given and followed by Lemma 2.4.8.

Definition 2.4.6. *The parameterized family of discrete-time systems (2.12) is said to be semiglobally practically ISS on the set $\mathcal{L}_{\mathcal{W}}$ on compact time intervals, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ such that for each triple of (r, δ, T) with $r > \delta \geq 0$ and $T > 0$, there exist positive real numbers τ^* and ε^* such that for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$, each $w \in \mathcal{L}_{\mathcal{W}}$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|w|_\infty \leq r$ and $|x(k_0\tau)| \leq r$, we have*

$$|x(k\tau)| \leq \max\{\beta(|x(k_0\tau)|, (k-k_0)\tau), \gamma(|w|_\infty)\} + \delta, \quad \forall\ (k-k_0)\tau \in [0,T]\ .$$

□

Note that above semiglobal practical ISS for system (2.12) is defined on compact time intervals. The following definition is used to show that such a property holds for infinite time intervals. We also include semi-global practical asymptotic stability for the parameterized family of discrete-time systems (2.12) in the following definition for the disturbance-free case.

Definition 2.4.7. *The parameterized family of discrete-time systems (2.12) is said to be semiglobally practically ISS on the set $\mathcal{L}_{\mathcal{W}}$, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ such that for each pair of (δ, r) with $r > \delta \geq 0$, there exist positive real numbers τ^* and ε^* such that for each $\tau \in (0, \tau^*)$, $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$, each $w \in \mathcal{L}_{\mathcal{W}}$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|w|_\infty \leq r$ and $|x(k_0\tau)| \leq r$, we have*

$$|x(k\tau)| \leq \max\{\beta(|x(k_0\tau)|, (k-k_0)\tau), \gamma(|w|_\infty)\} + \delta, \quad \forall\ (k-k_0)\tau \geq 0\ .$$

Moreover, when $w(\cdot) \equiv 0$, the parameterized family of discrete-time systems (2.12) is said to be semiglobally practically asymptotically stable if there exists $\beta \in \mathcal{KL}$ such that for each pair of (δ, r) with $r > \delta \geq 0$, there exist positive real numbers τ^* and ε^* such that for each $\tau \in (0, \tau^*)$, $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \leq r$, we have

$$|x(k\tau)| \leq \beta(|x(k_0\tau)|, (k - k_0)\tau) + \delta, \quad \forall\ (k - k_0)\tau \geq 0\ .$$

□

Lemma 2.4.8. *The parameterized family of discrete-time systems (2.12) is semi-globally practically ISS on the set $\mathcal{L}_{\mathcal{W}}$ on compact time intervals if and only if it is semi-globally practically ISS on the set $\mathcal{L}_{\mathcal{W}}$.* □

Next, with the above Lemma 2.4.8 and the results on closeness of solutions on compact time intervals in Theorem 2.4.4/2.4.5, we assume that the family of strong or weak averages is ISS and show the ISS properties for the actual parameterized discrete-time systems. We next give the definition of ISS and global asymptotic stability for time-invariant parameterized discrete-time systems.

Definition 2.4.9. *The parameterized family of discrete-time systems $\frac{\Delta y}{\Delta k} = F_\tau(y, w)$ with $\Delta k = \tau$ is said to be globally ISS on the set $\mathcal{L}_{\mathcal{W}}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ such that for all $w \in \mathcal{L}_{\mathcal{W}}$ each solution of the system satisfies*

$$|y(k\tau)| \leq \max\{\beta(|y(0)|, k\tau), \gamma(|w|_\infty)\}, \quad \forall\ k\tau \geq 0\ .$$

Moreover, the parameterized family of discrete-time systems $\frac{\Delta y}{\Delta k} = F_\tau(y)$ with $\Delta k = \tau$ is said to be globally asymptotically stable if there exist $\beta \in \mathcal{KL}$ such that each solution of the system satisfies

$$|y(k\tau)| \leq \beta(|y(0)|, k\tau), \quad \forall\ k\tau \geq 0\ .$$

□

Now, we come to the results that provide the conditions to guarantee that ISS of weak/strong average systems implies SGP-DISS/SGP-ISS for the actual family of parameterized discrete-time systems (2.12). The proof of Theorem 2.4.10 is provided in Appendix A.4 and the proof of Theorem 2.4.11 is omitted as it is identical to the proof of Theorem 2.4.10.

Theorem 2.4.10. *Suppose that Assumption 2.3.1 holds for the family of discrete-time systems (2.12), the set* $\mathcal{L}_{\mathcal{W}}$ *is equi-bounded and equi-uniformly Lipschitz, there exists a locally Lipschitz continuous function* $F_\tau^{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *that is weak average of* $F_\tau : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and the family of weak average systems (2.14) is globally ISS on the set* $\mathcal{L}_{\mathcal{W}}$*. Then, the family of discrete-time systems (2.12) is semi-globally practically ISS on the set* $\mathcal{L}_{\mathcal{W}}$*.* □

Theorem 2.4.11. *Suppose that Assumption 2.3.1 holds for the family of discrete-time systems (2.12), the set* $\mathcal{L}_{\mathcal{W}}$ *is equi-bounded, there exists a locally Lipschitz continuous function* $F_\tau^{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *that is strong average of* $F_\tau : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and the family of strong averages (2.16) is globally ISS on the set* $\mathcal{L}_{\mathcal{W}}$*. Then, the family of discrete-time systems (2.12) is semi-globally practically ISS on the set* $\mathcal{L}_{\mathcal{W}}$*.* □

We emphasize that the conclusion of Theorem 2.4.10 that exploits weak averages holds only for sets of disturbances $\mathcal{L}_{\mathcal{W}}$ that are equi-bounded and equi-uniformly Lipschitz. On the other hand, the conclusion of Theorem 2.4.11 that involves strong averages holds on larger sets of disturbances that are equi-bounded.

The results of Theorem 2.4.10 and 2.4.11 can be directly applied to a disturbance free case:

$$\frac{\Delta x}{\Delta k} = F_\tau\left(\frac{k\tau}{\varepsilon}, x\right) \qquad \Delta k = \tau \ , \tag{2.23}$$

and obtain the following corollary. Note that we can use the average for F_τ according to either of our definition of strong and weak average, as they coincide in the disturbance-free case.

Corollary 2.4.12. *Suppose the parameterized family of discrete-time systems (2.23) has a family of average systems* $\frac{\Delta y}{\Delta k} = F_\tau^{av}(y)$ *where* $\Delta k = \tau$*, if the family of average systems is globally asymptotically stable, then the family of discrete-time systems (2.23) is semi-globally practically asymptotically stable.* □

2.5 An application example

To illustrate applicability of the results of this chapter, we address stabilization for the single-degree-freedom oscillator system with a periodically time-varying mass, which is an important model that arises in the application of biomechanics, robotics, conveyor systems, fluid structure interaction problems and many other situations [117].

Consider the nonlinear model for Duffing oscillator with a periodically time-varying mass [115]:

$$y'' + kM(t)y + \gamma M(t)y^3 = u(t) \ ,$$

where $y(t)$ is the displacement of the center mass measured from its rest, $u(t)$ is the input, $k > 0$ and $\gamma \neq 0$ are stiffness coefficients of linear and cubic elastic restoring forces respectively. $M(t)$ is the total mass of the oscillator that is periodic in $\tilde{T}$ and satisfies [2]

$$M(t) = \begin{cases} m & t \in [n\tilde{T}, n\tilde{T} + c) \\ 0 & t \in [n\tilde{T} + c, (n+1)\tilde{T}) \end{cases}$$

where $n = 0, 1, \cdots$, m and c are positive constants.

To illustrate our results, u is implemented via a digital controller. Then, with $x_1 = y$ and $x_2 = y'$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kM\left(\frac{t}{\varepsilon}\right)x_1 - \gamma M\left(\frac{t}{\varepsilon}\right)x_1^3 + u(t) \ , \end{aligned} \tag{2.24}$$

where $u(t) = u(k\tau) := u(k)$, $\forall\ t \in [k\tau, (k+1)\tau), k \in \mathbb{N}$, $\tau > 0$ is the sampling interval and the small parameter $\varepsilon > 0$ is used to imply that $M(\cdot)$ is fast switching.

We use the Euler approximation and get the family of approximate models parameterized by $\Delta k = \tau$:

$$\begin{aligned} \frac{\Delta x_1(k)}{\Delta k} &= x_2(k) \\ \frac{\Delta x_2(k)}{\Delta k} &= -kM\left(\frac{k\tau}{\varepsilon}\right)x_1(k) - \gamma M\left(\frac{k\tau}{\varepsilon}\right)x_1(k)^3 + u(k) \ . \end{aligned} \tag{2.25}$$

Let $M_0 := \frac{cm}{\tilde{T}}$. Noting that the definitions of the strong and the weak average coincide without disturbances, the average of the family of discrete time systems (2.25) is

[2] The expression of total mass $M(t)$ for the oscillator comes from [115].

$$\begin{aligned} \frac{\Delta x_1(k)}{\Delta k} &= x_2(k) \\ \frac{\Delta x_2(k)}{\Delta k} &= -kM_0 x_1(k) - \gamma M_0 x_1^3(k) + u(k) \ . \end{aligned} \tag{2.26}$$

Indeed, setting $\tilde{s} = s\tau$ and $T := N\tau$ we have for sufficiently small τ that

$$\begin{aligned} & \left| \frac{1}{N\tau} \sum_{s=k}^{k+N} (kM(s\tau)x_1 + \gamma M(s\tau)x_1^3 - kM_0 x_1 - \gamma M_0 x_1^3)\tau \right| \\ &= \left| \frac{1}{N\tau} \sum_{\tilde{s}=k\tau}^{k\tau+N\tau} (kM(\tilde{s})x_1 + \gamma M(\tilde{s})x_1^3 - kM_0 x_1 - \gamma M_0 x_1^3)\Delta\tilde{s} \right| \ , \\ &\leq \left| \frac{kx_1 + \gamma x_1^3}{T} \int_{k\tau}^{k\tau+T} (M(\tilde{s}) - M_0) d\tilde{s} \right| \ , \\ &\leq \frac{2cm(k+\gamma)(\max\{|x|, 1\})^3}{T+1} \ , \end{aligned} \tag{2.27}$$

where the last inequality holds when $T \geq 1$ and we can let $\beta_{sa}(s,t) = \beta_{wa}(s,t) := \frac{2cm(k+\gamma)s^3}{t+1}$ in this case.

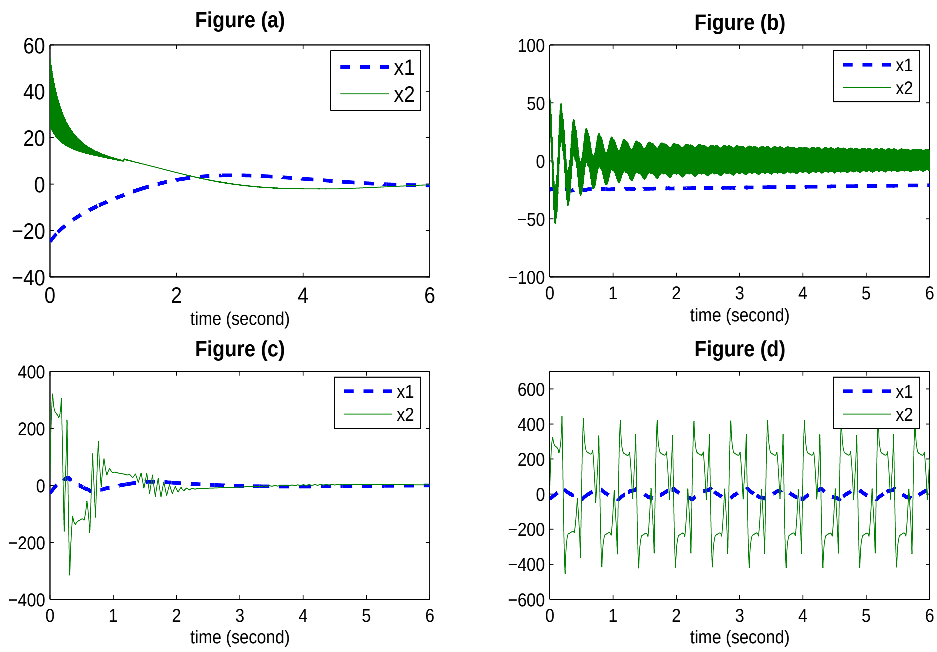

Figure 2.1: (a) $\tau = 0.005$ $\varepsilon = 0.005$ (b) $\tau = 0.1$ $\varepsilon = 0.005$ (c) $\tau = 0.01$ $\varepsilon = 0.09$ (d) $\tau = 0.01$ $\varepsilon = 0.097$

It is straight forward that under the controller $u(k) = \gamma M_0 x_1^3 - 2\sqrt{kM_0}x_2$, the closed loop of the average system is a linear system whose eigenvalues are within the unit circle and then it is globally exponentially stable. From the result in Corollary 2.4.12, we conclude that the family of discrete time systems (2.25) is semi-globally practically stable. Recall the sufficient conditions of [109] mentioned in Section 2.2 that guarantee the same controller $u(k)$ for the approximate model can stabilize the exact discrete-time model. With the fact that the control law is bounded on compact sets uniformly in small τ, we conclude semi-global practical stability of the sampled data system (2.24) with this controller $u(k)$.

To simulate the sampled data system (2.24), we assume that $\tilde{T} = 1s$ and have the trajectory of the solutions x_1 and x_2 under different values of parameter ε and sampling time interval τ in Fig. 2.1. The plots (a) and (c) of Fig. 2.1 show that when ε and τ are sufficiently small, the trajectories of the sampled-data system (2.24) converge to the equilibrium. On the other hand, when sampling time intervals τ or parameters ε are chosen rather large, see plots (b) and (d) in Fig. 2.1, trajectories of the system fluctuate in a rather large interval.

2.6 Conclusions

ISS of parameterized families of discrete-time systems was investigated via the averaging method in this chapter. The main results are useful when an approximate discrete-time model of a sampled-data system is used for controller design. We showed that under appropriate conditions, ISS of strong (or weak) average of the family of discrete-time systems implies SGP-ISS (or SGP-ISS like) properties for the actual family of systems. We showed via an example that the results can be used with [109] to design controllers via approximate discrete-time models that achieve ISS of sampled-data nonlinear systems. Moreover, we presented general results on closeness of solutions of the actual system with solutions of its weak or strong average that only require the average system to be forward complete.

Chapter 3

Averaging of Fast Switching Systems

3.1 Introduction

Averaging is an important analysis tool for rapidly switched systems that arise in a range of engineering applications, such as power electronics [83], network control systems [127, 147], adaptive control systems [5, 41], synchronization of chaotic oscillators [95], control of multiple autonomous agents [126] and so on. However, the averaging theory for fast switching systems is still underdeveloped. In the present chapter, we use the averaging technique to consider ISS stability for both nonlinear and linear switched systems. We directly apply the results in [110] to the averaging analysis of general switched nonlinear systems and derive stronger conclusions for linear switched systems.

Note that the averaging results for continuous-time systems in [110] do not deal exclusively with switched systems but they are general enough to include nonlinear switched systems as a special case. Indeed, we show in Section 3.3 that the notions of weak and strong averages that were pioneered in [110] play a significant role in the context of rapidly switched systems with disturbances. We provide conditions under which ISS of the strong average implies SGP-ISS of the nonlinear switched system. Using the notion of weak average, we can conclude a SGP-DISS property that requires also derivatives of disturbances to be bounded.

Moreover, the results in [110] are too weak whenever the ISS disturbance gain is linear and the decay of transients is exponential. Such a situation often arises in linear switched systems and, hence, there is a strong motivation for sharpening the results in [110], see Section 3.4. With the notions of strong and weak averages given in [110] and partial strong average that we propose here, we derive results for the case when the average system is ISS with an exponential $\mathcal{KL}$ estimate and a linear gain. We show that exponential ISS of the strong average implies exponential ISS with linear gain for the actual linear switched system for sufficiently high switching rates. We also show that if the weak average is exponentially ISS with linear gain, then the actual linear switched system satisfies an exponential DISS property.

In addition, using partial averaging [121], we present stronger conclusions for the case when there does not exist a strong average. The partial strong average is used to show that its exponential ISS implies exponential ISS properties for the actual system when switching is fast enough. Moreover, based on the Lyapunov method, we show that an estimate of the linear ISS gain of the actual system converges to the estimated ISS gain of its average as the switching rate is increased.

These new results provide novel insights on robustness in the context of linear switched systems, and we believe that these average notions will play an important role in future developments of averaging methodology for switched systems with disturbances.

Chapter 3 is organized as follows. We give definitions on average and ISS-like properties for continuous-time systems in Section 3.2. Sections 3.3 and 3.4 contain the main results on averaging of the general nonlinear switched system and the linear switched system, respectively. Conclusions are given in the last section and the proofs of the main results are presented in the Appendix B.

3.2 Definitions

Weak, strong and partial strong average definitions together with some ISS-like properties for continuous-time systems are given in this section. Strong and weak average definitions are useful to consider both nonlinear switched systems in Section 3.3 and linear switched systems in Section 3.4. The partial strong average definition is used to obtain stronger conclusions if there does not exist a strong average for a linear switched system and only weak conclusions can be obtained with the weak average definition.

For nonlinear continuous-time systems

$$\dot{x} = f(t, x, w) ,$$

where $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ are states and input signals respectively. Given a measurable function $w(\cdot)$, let $||w||_\infty := ess\ \sup_{t\geq 0} |w(t)|$ and denote $w \in \mathcal{L}_\infty$ if $||w||_\infty < \infty$. Then, we have the following weak and strong averages (introduced in [110]).

Definition 3.2.1 (Weak average). *A function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a weak average of $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ if there exist $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq 0$, $T \geq T^*$, $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, the following holds:*

$$\left| f_{wa}(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w) ds \right| \leq \beta_{av}(\max\{|x|, |w|, 1\}, T) .$$

□

Definition 3.2.2 (Strong average). *A function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a strong average of $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ if there exist $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq 0$, $T \geq T^*$, $x \in \mathbb{R}^n$ and $w \in \mathcal{L}_\infty$, the following holds:*

$$\left| \frac{1}{T} \int_t^{t+T} [f_{sa}(x, w(s)) - f(s, x, w(s))] ds \right| \leq \beta_{av}(\max\{|x|, ||w||_\infty, 1\}, T) .$$

□

For continuous-time systems with $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ of the form

$$\dot{x} = f_p\left(\frac{t}{\varepsilon}, t, x, w\right) ,$$

where the small parameter $\varepsilon > 0$ is used to show that some time-varying terms of f_p change faster than others time-varying terms, and for which we can define the partial strong average.

Definition 3.2.3 (Partial strong average). *A function $f_{psa} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a partial strong average of $f_p : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ if there exist $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq 0$, $T \geq T^*$, $x \in \mathbb{R}^n$ and $w \in \mathcal{L}_\infty$, the following holds:*

$$\left| \frac{1}{T} \int_t^{t+T} \left[f_{psa}(t, x, w(s)) - f_p\left(\frac{s}{\varepsilon}, t, x, w(s)\right)\right] ds \right| \leq \beta_{av}(\max\{|x|, ||w||_\infty, 1\}, T) .$$

□

We consider robustness to different classes of disturbances for switched systems in the present chapter. Consequently, we need definitions on robustness stability to disturbances for continuous-time systems, such as input-to-state stability

(ISS), derivative ISS, semi-global practical ISS (SGP-ISS), semi-global practical DISS (SGP-DISS) and exponential ISS. We also need definitions for subsets of input signals with specified properties, such as equi-essential boundedness and local equi-uniform Lipschitz continuity. Some of these properties have been defined in last chapter but in the discrete-time setting.

Equi-essential boundedness and local equi-uniform Lipschitz continuity for a subset of disturbances $w \in \mathcal{L}_{\mathcal{W}} \subset \mathcal{L}_\infty$ are first given and followed by definitions on robustness stability to disturbances.

Definition 3.2.4 (Equi-essential boundedness). *Given a subset $\mathcal{L}_{\mathcal{W}}$ of signals $w \in \mathcal{L}_\infty$, the set $\mathcal{L}_{\mathcal{W}}$ is called equi-essentially bounded if there exists a strictly positive real number Ω such that, for all $w \in \mathcal{L}_{\mathcal{W}}$, $||w||_\infty \leq \Omega$.* □

Definition 3.2.5 (Local equi-uniform Lipschitz continuity). *Given a subset $\mathcal{L}_{\mathcal{W}}$ of signals $w \in \mathcal{L}_\infty$, the set $\mathcal{L}_{\mathcal{W}}$ is called locally equi-uniformly Lipschitz continuous if there exists $L > 0$ such that, for all $w \in \mathcal{L}_{\mathcal{W}}$ and $(t, s) \geq 0$*

$$|w(s) - w(t)| \leq L|t - s|.$$

□

Note that $\mathcal{L}_{\mathcal{W}}$ is locally equi-uniformly Lipschitz continuous if each $w \in \mathcal{L}_{\mathcal{W}}$ is absolutely continuous and there exists a strictly positive real number Ω_1 such that, for all $w \in \mathcal{L}_{\mathcal{W}}$, $||\dot{w}||_\infty \leq \Omega_1$.

Definition 3.2.6 (ISS). *The system $\dot{x} = f(x, w)$ is said to be input-to-state stable with respect to (β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if each solution of the system satisfies*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(||w||_\infty) \ , \qquad \forall\ t \geq 0 \ .$$

□

Definition 3.2.7 (DISS). *The system $\dot{x} = f(x, w)$ is said to be derivative input-to-state stable with respect to $(\beta, \gamma, \gamma_1)$ with $\beta \in \mathcal{KL}$ and $\gamma, \gamma_1 \in \mathcal{G}$ if each solution of the system satisfies*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(||w||_\infty) + \gamma_1(||\dot{w}||_\infty) \ , \qquad \forall\ t \geq t_0 \geq 0 \ .$$

□

Definition 3.2.8 (SGP-ISS)**.** *The system $\dot{x} = f(t, x, w, \varepsilon)$ is said to be semi-globally practically input-to-state stable with respect to (β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if for any strictly positive real numbers r, Ω and ν, there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each solution of the system with $|x(t_0)| \leq r$ and $||w||_\infty \leq \Omega$ satisfies*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(||w||_\infty) + \nu \ , \quad \forall \ t \geq t_0 \geq 0.$$

□

The concept of derivative input-to-state stability (DISS) in Def. 3.2.7 was proposed in [8]. In contrast, the following definition give a stronger semi-global practical version of DISS, where $\gamma_1(s) \equiv 0$ in Def. 3.2.7.

Definition 3.2.9 (SGP-DISS)**.** *The system $\dot{x} = f(t, x, w, \varepsilon)$ is said to be semi-globally practically derivative input-to-state stable with respect to (β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if for any strictly positive real numbers r, Ω, Ω_1 and ν, there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each solution of the system with $|x(t_0)| \leq r$, $||w||_\infty \leq \Omega$ and $||\dot{w}||_\infty \leq \Omega_1$ satisfies*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(||w||_\infty) + \nu \ , \quad \forall \ t \geq t_0 \geq 0.$$

□

Definition 3.2.10 (Exponential ISS)**.** *The system $\dot{x} = f(t, x, w)$ is said to be exponentially input-to-state stable with linear gain $\gamma > 0$ if there exist positive constants K, λ such that each solution of the system satisfies [74, Section 4.9]*

$$|x(t)| \leq K \exp(-\lambda(t - t_0))|x(t_0)| + \gamma||w||_\infty \ , \quad \forall \ t \geq t_0 \geq 0 \ .$$

□

In the following section, we directly apply the averaging results in [110] and [153] to analyze ISS of rapidly switching nonlinear systems. Although the results presented in this section follow directly from [110] and [153], we present them here because to the best of our knowledge they appear to be new in the context of switched systems. Indeed, we are not aware of any averaging results for general nonlinear switched systems that investigate ISS. Moreover, Section 3.3 serves as a motivation for the use of the notions of strong and weak averages that we believe are going to be useful in a range of other averaging problems for switched systems with disturbances.

3.3 Nonlinear switched systems

Consider fast switching nonlinear systems of the form:

$$\dot{x} = f_{\rho(\frac{t}{\varepsilon})}(x, w) \ , \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the external disturbance and $\varepsilon > 0$ is a small parameter; there are N subsystems indexed by set $i \in S \triangleq \{1, 2, \cdots, N\}$, where $\rho : \mathbb{R}_+ \to S$ is a switching law. The switched system (3.1) shows a fast switching behavior when ε is sufficiently small. Note that for any fixed and arbitrarily small ε, the switching signal ρ guarantees that we do not have Zeno solutions. We need the following assumption on continuity and local boundedness of the mapping f of system (3.1).

Assumption 3.3.1. *$f_{\rho(t)}(x, w)$ is locally Lipschitz in x, w, uniformly in ρ, and there exists $c \geq 0$ such that $|f_{\rho(t)}(0, 0)| \leq c$ for all $t \geq 0$.* □

Applying the definitions of weak and strong averages in Def. 3.2.1 and 3.2.2 to consider nonlinear switched system (3.1), we get that its strong average is

$$\dot{x} = f_{sa}(x, w) \ , \tag{3.2}$$

and the weak average is

$$\dot{x} = f_{wa}(x, w) \ . \tag{3.3}$$

Directly applying the results in [110] and [153] to the switched system (3.1), we have the results on closeness of solutions between actual switched system (3.1) with solutions of its weak average (3.2) or strong average (3.3), see Subsection 3.3.1. We also present conditions under which ISS of the strong or weak average imply appropriate semi-global practical ISS properties of the actual system in Subsection 3.3.2.

3.3.1 Closeness of solutions

Here, we present the results on how the strong or weak average can approximate the actual switched system (3.1) based on forward completeness of the average system. The definition of $\mathcal{L}_{\mathcal{W}}$-forward completeness for a continuous-time system is first given.

Definition 3.3.2 ($\mathcal{L}_{\mathcal{W}}$-forward completeness)**.** *Given a subset $\mathcal{L}_{\mathcal{W}}$ of signals $w \in \mathcal{L}_\infty$ and a continuous function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the system $\dot{x} = f(x, w)$ is said to be $\mathcal{L}_{\mathcal{W}}$-forward complete if for each $r > 0$ and $T > 0$ there exists $R \geq r$ such that, for all $|x_0| \leq r$ and $w \in \mathcal{L}_{\mathcal{W}}$, the solutions of the system exist and are contained in a closed ball of radius R for all $t \in [0, T]$.* □

Without a stability assumption for the average system, the results of closeness of solutions in the following lemmas show that the strong/weak average system approximates well the actual switched system for some classes of disturbances.

Theorem 3.3.3 (Closeness to strong average)**.** *Suppose that Assumption 3.3.1 holds for the system (3.1), there exists a locally Lipschitz continuous function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ that is strong average of $t \mapsto f_{\rho(t)}(x, w)$, the set $\mathcal{L}_{\mathcal{W}}$ is equi-bounded and the strong average (3.2) of system (3.1) is $\mathcal{L}_{\mathcal{W}}$-forward complete. Then, for each triple of strictly positive real numbers (r, δ, T), there exists a couple of (ε^*, μ) of strictly positive numbers such that, for each $\varepsilon \in (0, \varepsilon^*)$, for all $t_0 \geq 0$, $|y_0| \leq r$, x_0 such that $|x_0 - y_0| \leq \mu$ and $w \in \mathcal{L}_{\mathcal{W}}$, each solution $x_\varepsilon(t, t_0, x_0, w)$ of the system (3.1) and the solution $y(t - t_0, y_0, w)$ of its strong average (3.2) satisfy*

$$|x_\varepsilon(t, t_0, x_0, w) - y(t - t_0, y_0, w)| \leq \delta \quad \forall\, t \in [t_0, t_0 + T].$$

□

Theorem 3.3.4 (Closeness to weak average)**.** *Suppose that Assumption 3.3.1 holds for the system (3.1), there exists a locally Lipschitz continuous function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ that is weak average of $t \mapsto f_{\rho(t)}(x, w)$, the set $\mathcal{L}_{\mathcal{W}}$ is equi-bounded and locally equi-uniformly continuous and the weak average (3.3) of system (3.1) is $\mathcal{L}_{\mathcal{W}}$-forward complete. Then, for each triple of strictly positive real numbers (r, δ, T), there exists a couple of (ε^*, μ) of strictly positive numbers such that, for all $\varepsilon \in (0, \varepsilon^*)$, for all $t_0 \geq 0$, $|y_0| \leq r$, x_0 such that $|x_0 - y_0| \leq \mu$ and $w \in \mathcal{L}_{\mathcal{W}}$, each solution $x_\varepsilon(t, t_0, x_0, w)$ of the system (3.1) and the solution $y(t - t_0, y_0, w)$ of its weak average (3.3) satisfy*

$$|x_\varepsilon(t, t_0, x_0, w) - y(t - t_0, y_0, w)| \leq \delta, \quad \forall\, t \in [t_0, t_0 + T].$$

□

Note that the result on closeness of solutions holds for equi-essentially bounded disturbances for strong averages, see Theorem 3.3.3, while the same conclusion holds for equi-essentially bounded and locally equi-uniformly Lipschitz continuous disturbances for weak averages in Theorem 3.3.4. The following example is used to illustrate this.

Example 3.3.5. *Consider the switched system*

$$\dot{x} = f_{\rho(\frac{t}{\varepsilon})}(x) + B_{\rho(\frac{t}{\varepsilon})}xw \tag{3.4}$$

where $x,\ w \in \mathbb{R}$*. The switching law* $\rho(\frac{t}{\varepsilon})$ *selects elements of the set* $S = \{1, 2\}$ *according to the policy*

$$\rho\left(\frac{t}{\varepsilon}\right) = \begin{cases} 1 & \frac{t}{\varepsilon} \in \left[n\pi, \frac{(2n+1)\pi}{2}\right) \\ 2 & \frac{t}{\varepsilon} \in \left[\frac{(2n+1)\pi}{2}, (n+1)\pi\right) \end{cases} \tag{3.5}$$

where $n \in \mathbb{N}_{\geq 0}$*. Let* $f_1(x) = f_2(x) = 0.1x$*,* $B_1 = 1$ *and* $B_2 = -1$*. Then, the weak average of system (3.4) is*

$$\dot{y} = 0.1y\ , \tag{3.6}$$

since the weak average definition in Def. 3.2.1 holds with letting $\beta_{av}(s,t) := \frac{\pi s^2}{t+1}$ *and the fact that*

$$\begin{aligned} \left| f_{wa}(x,w) - \frac{1}{T}\int_t^{t+T} f_{\rho(s)}(x,w)ds \right| &\leq \frac{|xw|}{T}\left|\int_t^{t+T} B_{\rho(s)}ds\right| \\ &\leq \frac{|xw|\pi}{2T} \\ &\leq \frac{(\max\{|x|,|w|,1\})^2\pi}{1+T}\ , \end{aligned}$$

where the last inequality holds for $T \geq 1$*. Moreover, using the same technique as in Example 2.3.6, one can show that there does not exist a strong average for system (3.4).*

We next consider two input signals: $w_1(t) = \sin(t)$ *and*

$$w_2(t) = \begin{cases} 1 & when \quad \rho(t) = 1 \\ -1 & when \quad \rho(t) = 2\ . \end{cases} \tag{3.7}$$

We have the following simulations that give the errors between the solution x *of the actual system (3.4) with the solution* y *of its weak average system (3.6) under* *different input signals.*

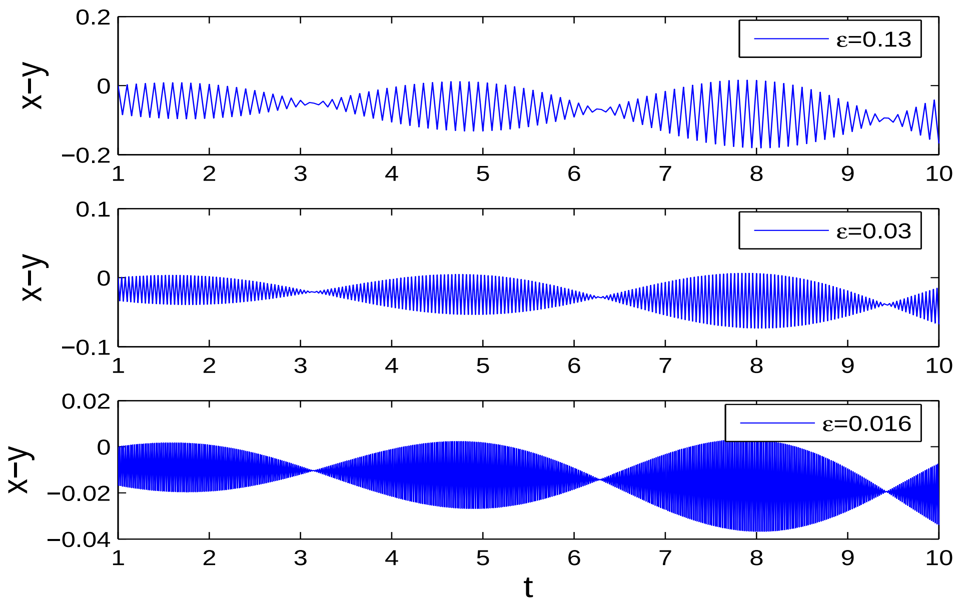

Figure 3.1: The error between solution x of system (3.4) with solution y of its weak average system (3.6) for the input $w_1 = \sin(t)$.

Since forward-completeness and Lipschitz properties hold naturally for linear systems and the weak average system (3.6) is linear and unstable, we know that conditions of Theorem 3.3.4 are satisfied for the sinusoidal signal w_1. This is consistent to the simulations results in Fig. 3.1, where the error between solutions of the actual switched system and solutions of its weak average system decreases when ε is reduced.

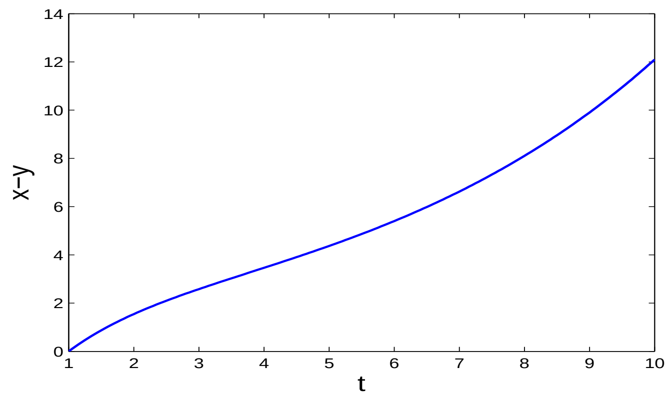

Figure 3.2: The error between solution x of system (3.4) with solution y of its weak average system (3.6) for the input w_2 in (3.7).

On the other hand, note that the actual system (3.4) with the given disturbance (3.7) that has unbounded derivatives evolves like

$$\dot{x} = 0.1x + x = 1.1x \ ,$$

for which the dynamics of state x is independent of the parameter ε. Then, the error $x - y$ can not be reduced by regulating ε, see Fig. 3.2, and the closeness of solutions between system (3.4) and its weak average system do not hold. □

3.3.2 ISS analysis

We considered the case when average systems are forward-complete and presented results on closeness of solutions in last subsection. In the present subsection, we assume that the strong or weak average is ISS and conclude semi-global practical ISS properties for the actual switched system. Using the definitions of ISS, SGP-ISS and SGP-DISS in Definitions 3.2.6, 3.2.8 and 3.2.9, we give the following results.

Theorem 3.3.6. *Suppose that Assumption 3.3.1 holds for the system (3.1), there exists a locally Lipschitz continuous function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ that is strong average of $t \mapsto f_{\rho(t)}(x, w)$, the set $\mathcal{L}_{\mathcal{W}}$ is equi-bounded and the strong average (3.2) of the system (3.1) is ISS with respect to (β, γ), then the system (3.1) is SGP-ISS with respect to (β, γ).* □

Theorem 3.3.7. *Suppose that Assumption 3.3.1 holds for the system (3.1), there exists a locally Lipschitz continuous function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ that is weak average of $t \mapsto f_{\rho(t)}(x, w)$, the set $\mathcal{L}_{\mathcal{W}}$ is equi-bounded and locally equi-uniformly continuous and the weak average (3.3) of the system (3.1) is ISS with respect to (β, γ), then the system (3.1) is SGP-DISS with respect to (β, γ).* □

We have stated that ISS of the strong average system implies the actual switched system is SGP-ISS. On the other hand, it is impossible to prove a similar result for weak averages without the assumption that the disturbances are absolutely continuous with w, $\dot{w} \in \mathcal{L}_\infty$. The following example shows that the weak average system maybe ISS, but the actual switched system is not ISS.

Example 3.3.8. *Consider the switched system (3.4) in Example 3.3.5 with the switched signal ρ given in (3.5). Let $f_1(x) = f_2(x) = -0.5x$, $B_1 = 1$ and $B_2 = -1$. Using the results in Example 3.3.5, we have that there does not exist a strong average for system (3.4) and its weak average is*

$$\dot{y} = -0.5y \ . \tag{3.8}$$

Note that the weak average (3.8) of system (3.4) is disturbance free and it is uniformly globally asymptotically stable. In other words, it is ISS with with zero disturbance gain. Now we consider a bounded disturbance for the actual system and we show that its solution grows unbounded, i.e. the actual system is not SGP-ISS.

Considering the input signal $w_2(t)$ *in (3.7), the actual system (3.4) with this disturbance evolves like*

$$\dot{x} = -0.5x + x = 0.5x,$$

which is independent of the perturbation $\varepsilon > 0$ *and has solutions that grow unbounded from any non-zero initial condition. Note that the actual system (3.4) is not SGP-ISS. This is due to the fact that the disturbance is discontinuous.* □

The weak average is quite useful for analysis of stability properties of several classes of time-varying interconnected systems, where the input of one subsystem is the output of another subsystem. In particular, an important motivation for use of weak average is the analysis of ISS of time-varying switched cascaded systems. Consider the system

$$\begin{aligned} \dot{\xi} &= f_{\rho_1(\frac{t}{\varepsilon})}(\xi, \eta) \\ \dot{\eta} &= f_{\rho_2(\frac{t}{\varepsilon})}(\eta) \end{aligned}$$

where $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$. Using Theorem 3.3.7, we have the following corollary.

Corollary 3.3.9. *Suppose that* $f_{\rho_1(t)}(\xi, \eta)$ *satisfies Assumption 3.3.1, and for each* $r > 0$ *there exist* $R > 0$ *and* $\varepsilon^* > 0$ *such that* $|\eta| \le r$ *and* $\varepsilon \in (0, \varepsilon^*)$ *implies* $|f_{\rho_2(\frac{t}{\varepsilon})}(\eta)| \le R$. *If the weak average of the* ξ*-subsystem exists and is ISS with respect to* η *and if the* η*-subsystem is uniformly semi-globally practically asymptotically stable in* ε*, then the system (3.9) is uniformly semi-globally practically asymptotically stable in* ε*.* □

Note that to get the results for semi-global practical asymptotic stability of the cascade, the ξ-subsystem does not have to be uniformly semi-globally ISS with respect to η and instead we can use the semi-global practical DISS property that was concluded in Theorem 3.3.7. Note that this is weaker than the classical cascaded result, see [74, Lemma 4.7], in which ISS is used for the top system to conclude stability of the cascade. This observation was already made in [110].

Example 3.3.10. *Consider the switched cascaded system:*

$$\begin{aligned} \dot{\xi} &= -0.5\xi + B_{\rho(\frac{t}{\varepsilon})}\xi\eta \\ \dot{\eta} &= -(1 - 0.5B_{\rho(\frac{t}{\varepsilon})})\eta \ , \end{aligned} \tag{3.9}$$

where $B_{\rho(t)}$ comes from Example 3.3.8. The weak average of the ξ-subsystem (when η is regarded as the input) was shown in Example 3.3.8 to be ISS with zero gain. Note also that the η-subsystem is uniformly semi-globally practically asymptotically stable and hence from Corollary 3.3.9 we conclude that the cascade system (3.9) is uniformly semi-globally practically asymptotically stable in ε. This is despite the fact that the ξ-subsystem is not semi-globally practically ISS as shown in Example 3.3.8. □

3.4 Linear switched systems

We consider linear fast switching systems and present stronger results than those in the last section. We consider the linear switched system of the form:

$$\dot{x} = A_{\rho_1(\frac{t}{\varepsilon})}x + B_{\rho_2(\frac{t}{\varepsilon})}w \tag{3.10}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the input, (A_i, B_i) is a family of constant matrices that is parameterized by some index $i \in S \triangleq \{1, 2, \cdots, N\}$, $\rho_1, \rho_2 : \mathbb{R}_{\geq 0} \to S$ are piecewise constant functions of time, called switching signals. Suppose that for any fixed and arbitrary small $\nu > 0$, the interval between consecutive switching times is not smaller than ν for switching signals $\rho_1(t), \rho_2(t)$.

Using the following definition of average for switched matrices, together with the definitions of weak, strong and partial strong averages for continuous-time systems in Definitions 3.2.1-3.2.3, we can generate several type of averages for the linear switched system (3.10).

Definition 3.4.1 (Average for switched matrices)**.** *A constant matrix A_{av} is said to be an average of $A_{\rho(t)}$ if there there exist a class-$\mathcal{L}$ function σ and positive real numbers k and T^*, such that for all $t \geq 0$ and $T \geq T^*$, the following holds:*

$$\left| A_{av} - \frac{1}{T}\int_t^{t+T} A_{\rho(s)}ds \right| \leq k\sigma(T). \tag{3.11}$$

□

Note that the average of switched matrices does not necessarily imply that $\rho(\cdot) : \mathbb{R}_{\geq 0} \to S$ is periodic. On the other hand, suppose that $\rho(\cdot)$ is periodic of period T. It can be shown that the average matrix defined as $A_{av} = \frac{1}{T}\int_0^T A_{\rho(t)}dt$ satisfies Def. 3.4.1 for some k and $\sigma \in \mathcal{L}$. Let T_i be the length of time during one period for which $\rho(t) = i$. Then, it is not hard to see that

$$A_{av} = \frac{1}{T}\sum_{i=1}^{N} A_i T_i = \sum_{i=1}^{N} \lambda_i A_i \ ,$$

where by definition $\lambda_i = T_i/T$ and $\sum_{i=1}^{N} \lambda_i = 1$.

Using the Def. 3.4.1, averages for switched matrices, we will concentrate on three types of average systems for linear switched systems (3.10). In particular, it can be shown that if A_{av} and B_{av} are respectively averages of $A_{\rho_1(t)}$ and $B_{\rho_2(t)}$ in (3.10) under Def. 3.4.1, the system

$$\dot{x} = A_{av}x + B_{av}w \tag{3.12}$$

satisfies the weak average definition in Def. 3.2.1. On the other hand, if we have that $B_i = B$ for all $i \in \{1, 2, \cdots, N\}$, or in other words the matrix $B_{\rho_2(t)}$ in (3.10) satisfies $B_{\rho_2(t)} \equiv B$ for all $t \geq 0$, then the system

$$\dot{x} = A_{av}x + Bw \tag{3.13}$$

satisfies the definition of strong average in Def. 3.2.2. Finally, we will also consider the system

$$\dot{x} = A_{av}x + B_{\rho_2(\frac{t}{\varepsilon})}w \ . \tag{3.14}$$

It is not hard to show that this system is a partial strong average of system (3.10) under Def. 3.2.3.

Note that the definition of partial strong average is novel in the context of switched systems. Using this definition we can conclude exponential ISS properties for the actual linear switched system in cases when a strong average does not exist and weak averages would give too weak conclusions.

In order to state the results on robust stability for linear switched systems, we need the definition of exponential-ISS with linear disturbance gain in Def. 3.2.10. We show that the estimated linear ISS gains of the actual system and its average converge to each other as the switching rate is increased using Lyapunov method.

Definition 3.4.2. *For the linear system $\dot{y} = Ay + Bw$, if there exist $\gamma_a > 0$ and a symmetric positive definite constant matrix P such that for all y and w there exist positive real numbers c_1, c_2, and the quadratic Lyapunov function $V = y^T P y$ satisfying*

$$\begin{aligned} c_1|y|^2 \le V(y) &\le c_2|y|^2, \\ \frac{dV}{dy}(y)\,(Ay + Bw) &\le -|y|^2 + \gamma_a|w|^2 \quad \forall\ y, w\ , \end{aligned}$$

then the systems $\dot{y} = Ay + Bw$ is exponentially ISS. □

Definition 3.4.3. *For a time-varying linear system $\dot{y} = Ay + B_{\rho_2(\frac{t}{\varepsilon})}w$, if there exist positive real numbers ε^*, γ_a, c_1, c_2, and a continuously differentiable function $V(t, y)$ such that for all $\varepsilon \in (0, \varepsilon^*)$ and $t \ge 0$*

$$\begin{aligned} c_1|y|^2 &\le V(t,y) \le c_2|y|^2, \\ \frac{\partial V}{\partial t}(t,y) + \frac{\partial V}{\partial y}(t,y)\left(Ay + B_{\rho_2(\frac{t}{\varepsilon})}w\right) &\le -|y|^2 + \gamma_a|w|^2 \quad \forall\ y, w \end{aligned}$$

holds, then the systems $\dot{y} = Ay + B_{\rho_2(\frac{t}{\varepsilon})}w$ is exponentially ISS. □

Remark 3.4.4. *Let $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ be the minimum and the maximum eigenvalue of a matrix respectively. For a given matrix P, let $c_1 := \lambda_{min}(P)$ and $c_2 := \lambda_{max}(P)$. Then, exponentially ISS in Definitions 3.4.2 and 3.4.3 ensures that for the system $\dot{y} = Ay + Bw$ or $\dot{y} = Ay + B_{\rho_2(\frac{t}{\varepsilon})}w$, there exist $\gamma_a > 0$ and a quadratic Lyapunov function $V = y^T P y$ such that [3]:*

$$|x(t)| \le K\exp(-\lambda(t - t_0))|x_0| + \gamma||w||_\infty \quad \forall\ t \ge t_0 \ge 0\ ,$$

where parameters $K = \sqrt{\frac{c_2}{c_1}}$, $\lambda = \frac{1}{2c_2}$ and $\gamma = \sqrt{\frac{c_2\gamma_a}{c_1}}$. This is consistent to Def. 3.2.10 on exponential ISS for continuous-time systems. On the other hand, Def. 3.2.10 also guarantees the existence of the constant matrix P in Def. 3.4.2 that can be calculated through the Lyapunov matrix equation $A_{av}^T P + PA_{av} = -I$. □

The following results show that the actual linear switched system (3.10) is exponentially DISS if its weak average is exponential-ISS. In addition, if its strong or partial strong average system is exponential-ISS, then system (3.10) is exponentially ISS. The proofs of Theorems 3.4.5 and 3.4.7 are given in Appendices B.1 and B.2 respectively. The proof of Theorem 3.4.8 is omitted as it is nearly identical to the proof of Theorem 3.4.5.

Theorem 3.4.5. *Suppose that the weak average (3.12) of system (3.10) exists and is exponentially ISS. Then, for any $\delta > 0$ there exists $\varepsilon^* > 0$ and for all $\varepsilon \in (0, \varepsilon^*)$, $w, \dot{w} \in \mathcal{L}_\infty$ and $x_0 := x(t_0) \in \mathbb{R}^n$, the solution of system (3.10) satisfies:*

$$|x(t)| \leq (K+\delta)\exp(-(\lambda-\delta)(t-t_0))|x_0| + (\gamma+\delta)||w||_\infty + \delta||\dot{w}||_\infty \quad \forall \ t \geq t_0 \geq 0 \ , \tag{3.15}$$

where positive constants K, λ, γ come from Remark 3.4.4. Thus, the system (3.10) is exponentially derivative input-to-state stable (DISS) uniformly in ε. □

Related results were presented in [131] where $\mathcal{L}_2$ gain stability was considered for rapidly switching linear systems (1.10) that can be formed as (3.10) with letting $\rho_1(\frac{t}{\varepsilon}) = \rho_2(\frac{t}{\varepsilon}) := \rho(\frac{t}{\varepsilon})$. In [131], it is showed via an example that if the input matrix B_ρ switches, then the $\mathcal{L}_2$ gain of the actual switched system may not be bounded by the $\mathcal{L}_2$ gain of its average when the switching rate increases. Comparing with their results, we show in Theorem 3.4.5 that an estimate of ISS gain of the actual system also can be recovered by its weak average (note that δ in (3.15) can be arbitrarily small when ε is sufficiently small) if we restrict the derivatives of disturbances to be uniformly bounded. The following example illustrates this.

Example 3.4.6. *Consider the linear switched system (3.10) with $\rho_1(\frac{t}{\varepsilon}) = \rho_2(\frac{t}{\varepsilon}) := \rho(\frac{t}{\varepsilon})$. Let switching signals be the same as (3.5) in Example 3.3.8. Let $A_1 = A_2 = -0.5$, $B_1 = 1$ and $B_2 = -1$.*

As the input matrix B_ρ switches, we do not have a strong average for the switched system (3.10). Its weak average (3.8) given in Example 3.3.8 is disturbance free and it is uniformly globally exponentially stable. In other words, it is exponentially ISS with zero disturbance gain. Now we consider a bounded disturbance for the actual system and we show that its linear gain is not zero. For some $c > 0$, consider the following input:

$$w_1(t) = \begin{cases} c & when \quad \rho(t) = 1 \\ -c & when \quad \rho(t) = 2 \ . \end{cases}$$

Then, the actual system with the given disturbance evolves like

$$\dot{x} = -0.5x + c$$

and its solution is $x(t) = \exp(-0.5t)x_0 + 2c(1 - \exp(-0.5t))$. *For any* $x_0 > 0$, *we have that* $x(t) > 2c$ *and* $x(t) \to 2c$ *when* $t \to \infty$. *As* $||w||_\infty = c$ *we get the linear gain* γ *of the actual system satisfies* $\gamma \geq 2$, *which is much lager than the ISS gain of its weak average that is equal to zero. This is consistent to the results in [131].*

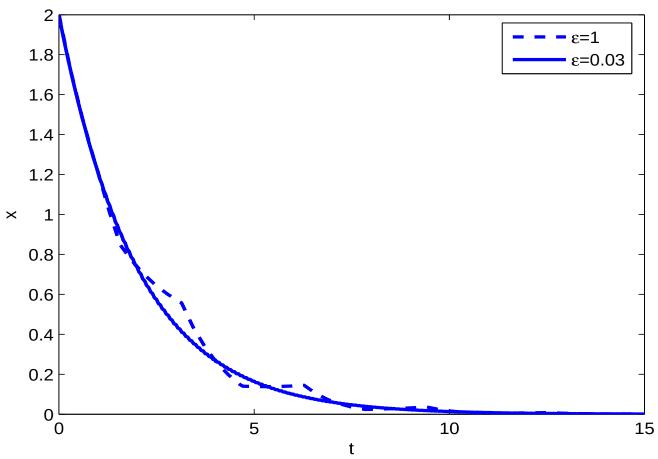

Figure 3.3: Dynamics of the actual switched system with disturbances $w_2(t) = \sin(t)$.

On the other hand, for some input signals with bounded derivatives, such as a sinusoidal signal $w_2(t) = \sin(t)$, *Theorem 3.4.5 shows that the actual systems is exponentially ISS with the estimated ISS gain converges to the estimated ISS gain of its weak average system. Although we may not get the analytical solution of the actual system but the simulation results in Fig. 3.3 shows that its solutions converge to the origin and then its ISS gain is zero. Hence, the results of Theorem 3.4.5 can be applied only when* $\dot{w} \in \mathcal{L}_\infty$ *is satisfied and conclusions we obtain are sharper than [131] under these conditions.* □

In the case when the input matrix B_ρ does not switch, [131] shows that the $\mathcal{L}_2$ gain of the actual time varying switched system is bounded by the $\mathcal{L}_2$ gain of its average as the switching rate is increased. We consider a different stability property (ISS as apposed to $\mathcal{L}_2$ stability), and we get a stronger conclusion that the estimated linear ISS gain can be recovered for linear switched systems that allow for strong averages, see Theorem 3.4.7.

Theorem 3.4.7. *Suppose that the strong average (3.13) of system (3.10) exists and is exponentially ISS. Then, for any $\delta > 0$ there exists $\varepsilon^* > 0$ and for all $\varepsilon \in (0, \varepsilon^*)$, $w \in \mathcal{L}_\infty$ and $x_0 := x(t_0) \in \mathbb{R}^n$, the solution of system (3.10) satisfies:*

$$|x(t)| \leq (K + \delta)\exp(-(\lambda - \delta)(t - t_0))|x_0| + (\gamma + \delta)||w||_\infty \qquad \forall t \geq t_0 \geq 0 \ ,$$

where positive constants K, λ, γ come from Remark 3.4.4. Thus, the system (3.10) is exponentially ISS uniformly in ε. □

When there does not exist a strong average for the linear switched system (3.10) and the conclusions on exponential-DISS for this switched system in Theorem 3.4.5 is too weak, we give the following result to obtain exponential ISS for such system.

Theorem 3.4.8. *Suppose that the partial strong average (3.14) of system (3.10) exists and is exponentially ISS. Then, for any $\delta > 0$ there exists $\varepsilon^* > 0$ and for all $\varepsilon \in (0, \varepsilon^*)$, $w \in \mathcal{L}_\infty$ and $x_0 \in \mathbb{R}^n$, the solution of system (3.10) satisfies:*

$$|x(t)| \leq (K + \delta)\exp(-(\lambda - \delta)(t - t_0))|x_0| + (\gamma + \delta)||w||_\infty \qquad \forall t \geq t_0 \geq 0 \ .$$

where positive constants K, λ, γ come from Remark 3.4.4. Thus, the system (3.10) is exponentially ISS uniformly in ε. □

In a similar fashion like Corollary 3.3.9, we can state a corollary that can be derived from Theorem 3.4.5 for the following cascaded linear switched system, for which the ξ-subsystem does not have to be uniformly exponentially ISS with respect to η to get the exponential stability of the cascade and instead we can use the exponential DISS property that was concluded in Theorem 3.4.5.

Consider a cascaded linear switched system

$$\begin{aligned} \dot{\xi} &= A_{\rho_1(\frac{t}{\varepsilon})}\xi + B_{\rho_2(\frac{t}{\varepsilon})}\eta \ , \\ \dot{\eta} &= A_{\rho_3(\frac{t}{\varepsilon})}\eta \ , \end{aligned} \tag{3.16}$$

where $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$.

Corollary 3.4.9. *Suppose that the weak average of ξ-subsystem exists and is exponentially ISS with respect to η, and the average of η-subsystem is exponentially stable, then there exists a $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the system (3.16) is exponentially stable uniformly in ε.*

Note that (3.10) is comprised of subsystems $\dot{x} = A_i x + B_i w$ that are not necessarily ISS, but the average system induced by subsystems and switching signals is ISS. Our main results show that stability of average systems implies the whole switched system is stable when the switching rate is high enough, and the following is a simple example that illustrates this fact.

Example 3.4.10. *Consider the linear switched system (3.10) with $\rho_1(\frac{t}{\varepsilon}) = \rho_2(\frac{t}{\varepsilon}) := \rho(\frac{t}{\varepsilon})$. Let the switching signal $\rho(\cdot)$ be the same as (3.5) in Example 3.3.8. Let $B_1 = B_2 = 0$,*

$$f_1(x) = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} x \ , \qquad f_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} x \ . \tag{3.17}$$

From the average definition (strong and weak average coincide in this case), the average system

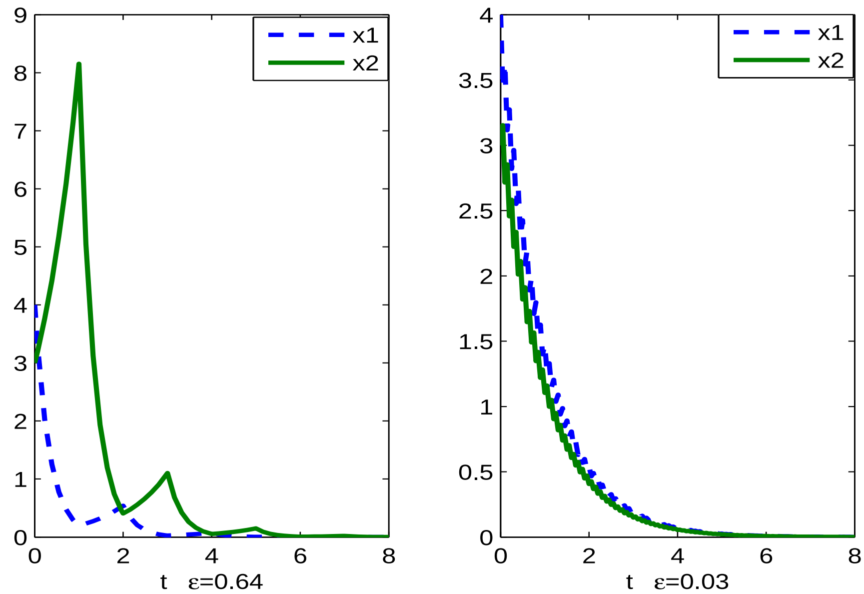

Figure 3.4: The trajectories of states x_1 and x_2 for system (3.10).

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x$$

induced by two unstable subsystems and the periodical switching signal (3.5) is an exponentially stable. Our results in Theorem 3.4.5 or 3.4.7 show that the actual system (3.10) is stable when ε is small enough. The simulation results in Fig. 3.4 illustrate this conclusion. □

From classical stability analysis theory for switched systems, e.g. [89], we know that slow switching assumption can guarantee that the whole switched system is stable if all subsystems are stable. However, we may not get the same conclusion for rapidly switching systems as the average system might not be ISS even if all subsystems are ISS. The following example is used to show this in zero disturbance case.

Example 3.4.11. *Consider the linear switched system (3.10) with $\rho_1(\frac{t}{\varepsilon}) = \rho_2(\frac{t}{\varepsilon}) := \rho(\frac{t}{\varepsilon})$. Let the switching signal be the same as (3.5) in Example 3.3.8, $B_1 = B_2 = 0$ and*

$$f_1(x) = \begin{bmatrix} -1 & -4 \\ 0 & -1 \end{bmatrix} x \ , \quad f_2(x) = \begin{bmatrix} -1 & 0 \\ -4 & -1 \end{bmatrix} x \ . \tag{3.18}$$

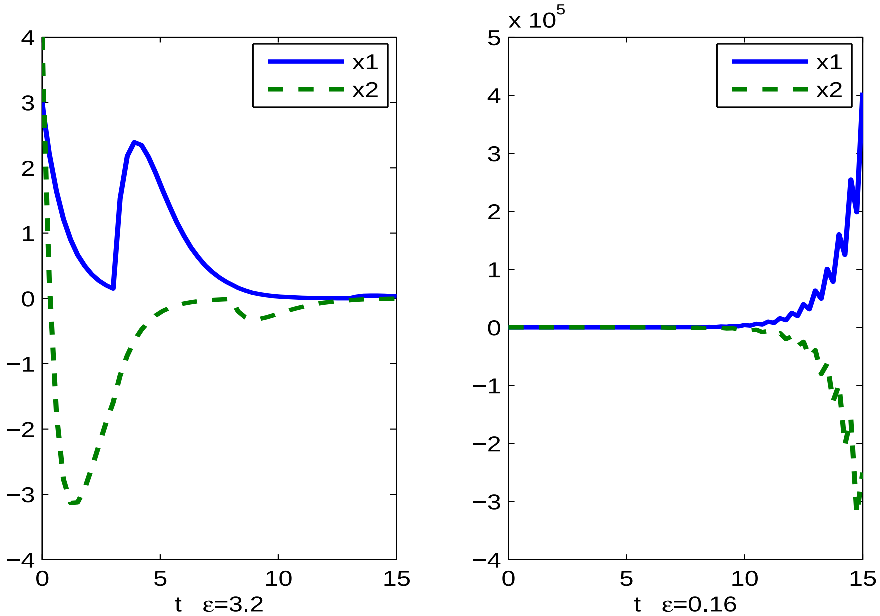

Figure 3.5: The trajectories of states x_1 and x_2 for system (3.10).

The average system

$$\dot{x} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} x$$

induced from both exponentially stable subsystems is unstable with real part of one eigenvalues ($\lambda_1 = 1$ and $\lambda_2 = -3$) being positive.

We can see in Fig. 3.5 that stability of the switched system can be kept when switching is slow enough, $\varepsilon = 3.2$ (the period of the switched signal is 10), and this property is lost when $\varepsilon = 0.16$ (the period of the switched signal is 0.5). □

3.5 Conclusions

ISS of fast switching systems with disturbances via the averaging method was investigated in this chapter. We directly applied the results in [110] to general switched nonlinear system with disturbances and presented conditions under which ISS of the strong average implies semi-global practical ISS of the actual switched system. We also showed a semi-global practical DISS property using the notion of weak average and requiring derivatives of disturbances to be bounded. Results were also derived for the strong or the weak average system to be ISS with an exponential $\mathcal{KL}$ estimate and a linear gain. We proved that exponential ISS of the strong and the partial strong average system implies exponential ISS for the actual linear switched system with their estimated linear gains converging to each other as the parameter is reduced. Moreover, exponential ISS of the weak average guarantees an appropriate DISS property for the actual system. We emphasize that one contribution of this chapter is a systematic use of strong, partial strong and weak averages for switched systems with disturbances that we believe will be very useful in a range of other averaging questions for switched systems.

Chapter 4

Input-to-State Stability of a Class of Hybrid Systems

4.1 Introduction

In this chapter, we consider a class of hybrid models via averaging. This hybrid model arises in hybrid feedback control systems that are driven by pulse-width-modulated (PWM) actuators. For such systems, it is desirable to prove that the PWM implementation of a given hybrid control law produces a closed-loop behavior that is similar to the behavior that would be achieved by implementing the hybrid feedback directly. In this case, a simpler hybrid system is used to approximate the actual hybrid system. See Section 1.1 for further motivation and related references.

Our results fit in the framework [25, 26, 29, 30, 53, 56] considering hybrid systems of the form

$$\begin{aligned} \dot{\xi} &\in \mathcal{F}(\xi) \qquad && \xi \in \mathcal{C} \\ \xi^+ &\in \mathcal{G}(\xi) \qquad && \xi \in \mathcal{D}\ , \end{aligned} \tag{4.1}$$

where the states agree with a set-valued flow mapping $\mathcal{F}$ with states constrained in the flow set $\mathcal{C}$ and a set-valued jump mapping $\mathcal{G}$ with states constrained in the jump set $\mathcal{D}$. This hybrid model captures a wide variety of dynamic phenomena including systems with logic-based state components that take values in a discrete set, as well as timers, counters, and other components. One can find how to cast hybrid automata, switched systems, as well as sampled-data and networked control systems, into such form [55]. Using this modeling framework, one can generalize most classical results on continuous-time systems to the hybrid setting.

Strong and weak average definitions are used to approximate the time-varying hybrid systems by time-invariant hybrid systems. We consider closeness of solutions of the strong average system and solutions of the actual hybrid system disturbed by bounded input signals on compact time domains assuming that the strong average system is forward pre-complete. Same conclusion can be obtained based on forward pre-completeness of the weak average system for bounded input signals with bounded derivatives.

We also show that ISS of the strong average implies semi-global practical ISS (SGP-ISS) of the actual system. In a similar fashion, ISS of the weak average implies semi-global practical derivative ISS (SGP-DISS) of the actual system. Using a PWM hybrid feedback control example, we illustrate how to apply our averaging results for hybrid systems to design a controller based on the simpler hybrid average system to stabilize the actual PWM hybrid feedback system.

Averaging of hybrid systems attracts attention since the averaging method can be used to simplify analysis of time-varying hybrid dynamics and hybrid dynamics naturally arise in a range of engineering applications, for example power electronics [34, 146], robotic manufacturing [142], automotive engine controlling [17], air traffic management systems [94], chemical process [9, 13, 45] and so on. However, most averaging results for hybrid systems focus on approximating a special class of hybrid system by a non-hybrid average system. For instance, one can find that a continuous-time system is the average for the dither system in [68,69]. These results are not applicable for the case we consider in this chapter.

Stability properties of a class of time-varying hybrid dynamical systems without disturbances are considered in [154] based on asymptotic stability of its hybrid average system. This chapter extends the results in [154] to hybrid dynamical systems with exogenous disturbances and gives sufficient conditions such that ISS of the strong or the weak average system imply ISS-like properties for the actual hybrid system.

The chapter is organized as follows. Some useful definitions are given in Section 4.2. Section 4.3 presents the definitions of strong and weak averages for hybrid systems. Section 4.4 contains the main results and Section 4.5 gives an application example in PWM hybrid feedback control to illustrate how to apply the main results. Conclusions are provided in the last section.

4.2 Useful definitions under hybrid frameworks

We present some useful definitions in this section for the hybrid framework provided in [25, 26, 29, 30, 53, 55, 56]. For this purpose, consider a hybrid system with $\xi \in \mathbb{R}^n$ and $w \in \mathcal{W} \subset \mathbb{R}^m$ of the form

$$H \quad \begin{array}{ll} \dot{\xi} = F(\xi, w) & (\xi, w) \in C \\ \xi^+ \in G(\xi, w) & (\xi, w) \in D\ . \end{array} \tag{4.2}$$

To define solution for hybrid system in (4.1) and solution pairs for hybrid system H in (4.2), we need the definitions on properties of set-valued mappings, such as outer semi-continuity and local boundedness.

Definition 4.2.1 (Outer semi-continuity)**.** *A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous at $x \in \mathbb{R}^n$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$ such that $y_i \to y$ we have $y \in M(x)$, and M is outer semi-continuous (OSC) if it is outer semi-continuous at each $x \in \mathbb{R}^n$.*

Definition 4.2.2 (Local boundedness)**.** *A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded if for any compact set $A \subset \mathbb{R}^n$ there exists $r > 0$ such that $M(A) := \bigcup_{x \in A} M(x) \subset r\mathbb{B}$; if M is OSC and locally bounded, then $M(A)$ is compact for any compact set A.*

Solutions for hybrid systems, under the framework in [25, 26, 29, 30, 53, 55, 56], are defined on hybrid time domains. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a compact hybrid time domain if $S = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. The set S is a hybrid time domain if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain. A hybrid time domain example from [55] is given in Fig. 4.1.

A function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is locally absolutely continuous if its derivative is defined almost everywhere and we have $x(t) - x(t_0) = \int_{t_0}^{t} \dot{x}(s) ds$ for all $t \geq t_0 \geq 0$, for which the precise definition see [108]. Now, we can define solutions for hybrid system in (4.1), also see [55], and solution pairs for hybrid system H in (4.2) [28].

Definition 4.2.3. *A hybrid signal is a function defined on a hybrid time domain. $w :$ dom $w \to \mathcal{W}$ is called a hybrid input if $w(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid signal $\xi :$ dom $\xi \mapsto \mathbb{R}^n$ is called a hybrid arc if $\xi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $\xi :$ dom $\xi \mapsto \mathbb{R}^n$ is a solution to the hybrid system (4.1) if $\xi(0, 0) \in \mathcal{C} \cup \mathcal{D}$ and:*

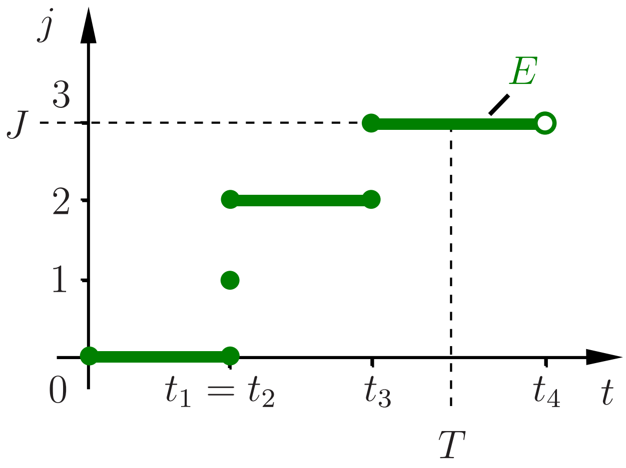

Figure 4.1: A hybrid time domain example.

1. *for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \mathrm{dom}\ \xi$, $\xi(t, j) \in \mathcal{C}$ and $\dot{\xi}(t, j) \in \mathcal{F}(\xi(t, j))$;*

2. *for all $(t, j) \in \mathrm{dom}\ \xi$ such that $(t, j+1) \in \mathrm{dom}\ \xi$, $\xi(t, j) \in \mathcal{D}$ and $\xi(t, j+1) \in \mathcal{G}(\xi(t, j))$.*

A hybrid arc ξ : $\mathrm{dom}\ \xi \mapsto \mathbb{R}^n$ and a hybrid input w : $\mathrm{dom}\ w \mapsto \mathcal{W}$ form a solution pair to system H in (4.2) if $\mathrm{dom}\ \xi = \mathrm{dom}\ w$, $(\xi(0, 0), w(0, 0)) \in C \cup D$ and

1. *for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \mathrm{dom}\ \xi$, $(\xi(t, j), w(t, j)) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j), w(t, j))$;*

2. *for all $(t, j) \in \mathrm{dom}\ \xi$ such that $(t, j + 1) \in \mathrm{dom}\ \xi$, $(\xi(t, j), w(t, j)) \in D$ and $\xi(t, j + 1) \in G(\xi(t, j), w(t, j))$.*

A solution or a solution pair is maximal if it cannot be extended. □

An example of a trajectory for a hybrid system, obtained from [55], is given in Fig. 4.2; its hybrid time domain dom x is given in Fig. 4.1.

Let $C_\Omega := \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B}$ such that $(\xi, w) \in C)\}$ and $F_\Omega(\xi) := \{v \in \mathbb{R}^n : v = F(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B}$ and $(\xi, w) \in C\}$. In order to exploit recent results in the literature on robustness for hybrid systems, we make the following assumptions.

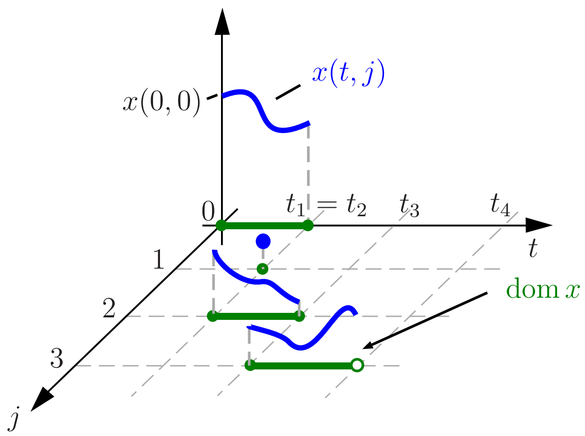

Figure 4.2: A solution of a hybrid system.

Assumption 4.2.4. *The sets $C \subset \mathbb{R}^n \times \mathbb{R}^m$, $D \subset \mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{W} \subset \mathbb{R}^m$ are closed; $F : C \to \mathbb{R}^n$ is continuous, for each $\Omega \geq 0$ and $\xi \in C_\Omega$, the set $F_\Omega(\xi)$ is convex; $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded, and for each $(\xi, w) \in D$, $G(\xi, w)$ is nonempty.* □

Assumption 4.2.4 lists some basic regularity conditions that combine what is typically assumed in continuous-time and in discrete-time systems. On the other hand, the convexity condition in Assumption 4.2.4 is used to guarantee robustness to disturbances for hybrid systems and the results in this chapter are based on robustness properties of the hybrid system. Example 4.4.10 that shows up later illustrates necessities of the convexity condition.

Given any hybrid signal $w : \text{dom } w \mapsto \mathcal{W}$, let $\Gamma(w)$ denote the set of $(t, j) \in \text{dom } w$ such that $(t, j+1) \in \text{dom } w$, and define

$$|w| := \max \left\{ \underset{(t,j)\in \text{dom } w \setminus \Gamma(w)}{ess\ \sup} |w(t, j)|, \sup_{(t,j)\in\Gamma(w)} |w(t, j)| \right\}. \tag{4.3}$$

For a hybrid arc w, $\dot{w}(t, j)$ is well defined for almost all t such that $(t, j) \in \text{dom } w$. Noting the fact that $\dot{w}$ can be defined arbitrarily on the set $\{t : (t, j) \in \text{dom } w\}$ of nonzero (Lebesgue) measure, the following definition for $|\dot{w}|$ is well defined and is consistent with the definition in (4.3):

$$|\dot{w}| := \underset{(t,j)\in \text{dom } w}{ess\ \sup} |\dot{w}(t, j)| . \tag{4.4}$$

Let $\mathcal{L}_{\mathcal{W}}$ be a given subset of hybrid signals $w : \text{dom } w \to \mathcal{W}$. The definitions of equi-essential boundedness and local equi-uniform Lipschitz continuity for a set of hybrid signals follows from Definitions 3.2.4 and 3.2.5, respectively. Note that a sufficient condition for $\mathcal{L}_{\mathcal{W}}$ to be locally equi-uniformly Lipschitz continuous is that there exists a strictly positive real number Ω_1 such that, for each $w \in \mathcal{L}_{\mathcal{W}}$, $w(\cdot, j)$ is locally absolutely continuous for each j and for all $(t, j) \in \text{dom } w$ such that $|\dot{w}| \leq \Omega_1$.

We consider closeness between solutions of hybrid dynamical systems to solutions of its average systems on compact time domains as one of the main results. These results are stated under the assumption that the average hybrid system is forward pre-complete. The definitions of forward pre-completeness and closeness of hybrid signals are given here.

Definition 4.2.5. *(Forward completeness) A hybrid solution pair is said to be forward complete if its domain is unbounded. A hybrid solution pair is said to be forward pre-complete if its domain is compact or unbounded. System H is said to be forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$ if all maximal solution pairs (ξ, w) with $\xi(0,0) \in K_0$ and w with $|w| \leq \Omega$ are forward pre-complete.* □

For pure continuous-time systems, forward completeness guarantees the existence of solutions with no finite-time escape occurring. In contrast, forward pre-completeness of hybrid systems, where the prefix "pre-" is used since it is not a requirement that maximal solutions to be complete, shows that solutions are contained in a compact set in compact time domains. See more details in [55] on considering properties of hybrid systems without insisting on completeness of solutions. With forward pre-completeness of hybrid systems, we consider closeness of solutions of the actual hybrid system with solutions of its approximation.

Note that closeness between hybrid signals may not hold uniformly in t, see an example in Fig. 4.3. Two hybrid solutions that start from initial point x_0 and $x_0 + \delta$ are plotted in Fig. 4.3 respectively, where $x_0 := x(0, 0)$ and $\delta > 0$ is a small number. The plot in the left corresponds to the solution components on hybrid time domains and the right plot illustrates the same solution components parameterized by t. The trajectories of these two solutions are not close in the uniform distance, in particular, at nearby jump times. On the other hand, the neighborhoods around its pieces shows closeness of their graphs which motivates the use of graph distance for describing closeness of hybrid solutions, see the following concept of (T, ρ)-closeness for hybrid signals [55].

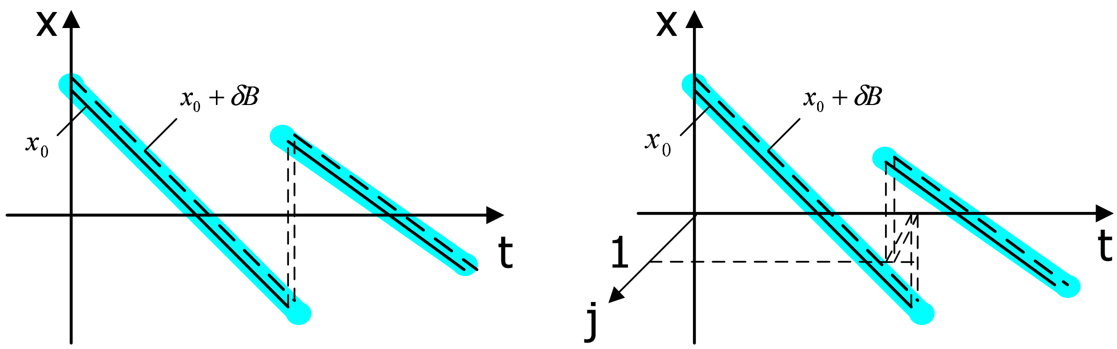

Figure 4.3: Closeness of hybrid signals.

Definition 4.2.6 (Closeness of hybrid signals)**.** *Two hybrid signals* $\xi_1 : \text{dom}\ \xi_1 \mapsto \mathbb{R}^n$ *and* $\xi_2 : \text{dom}\ \xi_2 \mapsto \mathbb{R}^n$ *are said to be* (T, ρ)*-close if:*

1. *(a) for each* $(t, j) \in \text{dom}\ \xi_1$ *with* $t + j \leq T$ *there exists* s *such that* $(s, j) \in \text{dom}\ \xi_2$*, with* $|t - s| \leq \rho$ *and* $|\xi_1(t, j) - \xi_2(s, j)| \leq \rho$*,*

2. *(b) for each* $(t, j) \in \text{dom}\ \xi_2$ *with* $t + j \leq T$ *there exists* s *such that* $(s, j) \in \text{dom}\ \xi_1$*, with* $|t - s| \leq \rho$ *and* $|\xi_2(t, j) - \xi_1(s, j)| \leq \rho$*.*

□

We also study robust stability properties of hybrid systems with the definitions of ISS, SGP-ISS and SGP-DISS that are given below. Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, a function $\chi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a proper indicator function for $\mathcal{A}$ on $\mathbb{R}^n$ if χ is continuous, $\chi(x) = 0$ if and only if $x \in \mathcal{A}$, and $\chi(x) \to \infty$ when $|x| \to \infty$. To study stability concepts with respect to a certain measure instead of a vector norm, in the following we let $\mathcal{A} \subset \mathbb{R}^n$ be nonempty and compact and let $\chi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a proper indicator for $\mathcal{A}$.

The definition of input-to-state stability (ISS) for hybrid system H (4.2) given in [28] is first recalled.

Definition 4.2.7 (ISS)**.** *System* H *in (4.2) is called ISS with respect to* (χ, β, γ) *with* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{G}$ *if for all* $\xi(0, 0) = \xi_0 \in \mathbb{R}^n$ *and* $w \in \mathbb{R}^m$ *each solution pair* (ξ, w) *satisfies*

$$\chi(\xi(t, j)) \leq \max\{\beta(\chi(\xi_0), t + j), \gamma(|w|)\}\ , \quad \forall (t, j) \in \text{dom}\ \xi\ . \tag{4.5}$$

□

Note that the existence of a class-$\mathcal{KL}$ function β in Definition 4.2.7 for the compact set $\mathcal{A}$ is equivalent to stability and pre-attractivity of the set $\mathcal{A}$ as defined in [30]. For more details see [56, Theorem 6.5].

Semi-global practical-ISS and semi-global practical derivative-ISS are defined for a hybrid system H_μ with data $(F_\mu, G_\mu, C_\mu, D_\mu)$ that depends on a small parameter $\mu > 0$ with $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$:

$$H_\mu \quad \begin{array}{ll} \dot{x} \in F_\mu(x,w) & x \in C_\mu \\ x^+ \in G_\mu(x,w) & x \in D_\mu \ . \end{array} \tag{4.6}$$

Definition 4.2.8 (SGP-ISS)**.** *System H_μ in (4.6) is called semi-globally practically ISS with respect to (χ, β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if, for each compact set $K_0 \subset \mathbb{R}^n$ and any positive real numbers Ω and ν there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*]$, each solution pair (x, w) with $x_0 := x(0,0) \in K_0$ and $|w| \leq \Omega$ satisfies*

$$\chi(x(t,j)) \leq \max\{\beta(\chi(x_0), t+j), \gamma(|w|)\} + \nu \ , \quad \forall (t,j) \in \text{dom } x \ .$$

□

Definition 4.2.9 (SGP-DISS)**.** *System H_μ in (4.6) is called semi-globally practically derivative ISS with respect to (χ, β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if, for each compact set $K_0 \subset \mathbb{R}^n$ and each triple of positive real numbers (Ω, Ω_1, ν), there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*]$, each solution pair (x, w), where $w(\cdot, j)$ is locally absolutely continuous, with $x_0 := x(0,0) \in K_0$, $|w| \leq \Omega$ and $|\dot{w}| \leq \Omega_1$ satisfies:*

$$\chi(x(t,j)) \leq \max\{\beta(\chi(x_0), t+j), \gamma(|w|)\} + \nu \ , \quad \forall (t,j) \in \text{dom } x \ .$$

□

4.3 Strong and weak averages

In this section, a class of time-varying hybrid systems is presented and definitions of weak and strong average for such systems are given. In addition, we introduce functions that are used in a coordinate transformation that facilitates establishing the averaging results for the given class of systems. Basic requirements of these functions are established in several lemmas.

Consider a class of time-varying hybrid systems H_ϵ that depends on a small parameter $\varepsilon > 0$

$$H_\epsilon \qquad \left.\begin{array}{l} \dot{x} = f_\varepsilon(x,w,\tau) \\ \dot{\tau} = \frac{1}{\varepsilon} \end{array}\right\} \quad ((x,w),\tau) \in C \times \mathbb{R}_{\geq 0} \qquad \left.\begin{array}{l} x^+ \in G(x,w) \\ \tau^+ \in H(x,w,\tau) \end{array}\right\} \quad ((x,w),\tau) \in D \times \mathbb{R}_{\geq 0} \ , \tag{4.7}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}_{\geq 0}$.

Assumption 4.3.1. *Sets (C, D) and the jump mapping $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ satisfy Assumption 4.2.4; $\tau \mapsto f_\varepsilon(x,w,\tau)$ is measurable for each $(x,w) \in C$; and for each $\delta > 0$ and compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exist $M(K) > 0$ and $\varepsilon^*(K,\delta) > 0$ such that*

$$\begin{aligned} |f_0(x,w,\tau)| &\leq M \quad \forall((x,w),\tau) \in (C \cap K) \times \mathbb{R}_{\geq 0} \ , \\ |f_\varepsilon(x,w,\tau) - f_0(x,w,\tau)| &\leq \frac{\delta}{3} \quad \forall((x,w),\tau,\varepsilon) \in (C \cap K) \times \mathbb{R}_{\geq 0} \times (0,\varepsilon^*] \ . \end{aligned} \tag{4.8}$$

□

Note that only local boundedness but no continuity condition is needed for the flow mapping f_0. For instance, we show that it holds for PWM hybrid feedback control systems in Section 4.5, for which a differential equation with the discontinuous right-hand side is used to describe flow dynamics. Conditions for (G, C, D) in the above assumption are also mild and they guarantee that the sets of solutions of H have good sequential compactness properties [55].

We next define weak and strong averages that are taken from [110] for the flow mapping $f_0 : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ on the flow set C. Comparing with average definitions in the previous chapters, we require bounds of the average error to hold only for $x \in C$. For simplicity, the following average definitions are defined on the time domain t instead of hybrid time domain (t, j).

Definition 4.3.2. *(Weak Average) For a function $f_0 : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the function $f_{wa} : C \to \mathbb{R}^n$ is said to be a weak average on C if for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists a class-$\mathcal{L}$ function σ_K such that, for all $((x,w),\tau,T) \in (C \cap K) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$:*

$$\left| f_{wa}(x,w) - \frac{1}{T} \int_\tau^{\tau+T} f_0(x,w,s)ds \right| \leq \sigma_K(T) \ .$$

□

We next define strong average for a subset of input signals. As the strong average is defined on the time domain t, we need the notation for sets of input signals defined on t. For any functions $\tilde{w} \in \mathcal{L}_\infty : \mathbb{R}_{\geq 0} \to \mathcal{W}$, let $\tilde{\mathcal{L}}_\mathcal{W}$ be a subset of input signals. We have the following strong average definition.

Definition 4.3.3. *(Strong Average) For a function $f_0 : C_1 \times \mathcal{W} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the function $f_{sa} : C_1 \times \mathcal{W} \to \mathbb{R}^n$ is said to be a strong average on $C_1 \times \mathcal{W}$ if for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists a class-$\mathcal{L}$ function σ_K such that, for all $\tilde{w} \in \tilde{\mathcal{L}}_\mathcal{W}$ with $((x, \tilde{w}(s)), \tau, T) \in ((C_1 \times \mathcal{W}) \cap K) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ for all $s \geq 0$, the following holds:*

$$\left| \frac{1}{T} \int_\tau^{\tau+T} [f_{sa}(x, \tilde{w}(s)) - f_0(x, \tilde{w}(s), s)] ds \right| \leq \sigma_K(T) \ .$$

□

Letting f_{wa} come from Definition 4.3.2 and (G, C, D) from (4.7), the weak average H_{wa} of system H_ϵ is

$$H_{wa} \qquad \begin{array}{ll} \dot{\xi} = f_{wa}(\xi, w) & \qquad (\xi, w) \in C \\ \xi^+ \in G(\xi, w) & \qquad (\xi, w) \in D \ . \end{array} \tag{4.9}$$

Similarly, for the case where $C = C_1 \times \mathcal{W}$, the strong average H_{sa} of system H_ϵ is

$$H_{sa} \qquad \begin{array}{ll} \dot{\xi} = f_{sa}(\xi, w) & \qquad (\xi, w) \in C \\ \xi^+ \in G(\xi, w) & \qquad (\xi, w) \in D \ , \end{array} \tag{4.10}$$

where f_{sa} comes from Definition 4.3.3.

To study closeness of solutions between system H_ϵ and its weak or strong average, we define the functions η_{wa} and η_{sa} used in a coordinate transformation to facilitate the averaging results for weak average and strong average case respectively. Similar techniques have been used to average continuous-time systems in [74, Chapter 10]. Let f_{wa} and f_{sa} come from the definitions of weak average and strong average. For each $((x, w), \tau, \mu) \in C \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$, let

$$\eta_{wa}(x, w, \tau, \tau_0, \mu) := \int_{\tau_0}^{\tau} \exp(\mu(s - \tau))[f_0(x, w, s) - f_{wa}(x, w)] ds \ . \tag{4.11}$$

Let $0 \leq \tau_0 \leq \tau_1$ and $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ be given. For each $\tau \in [\tau_0, \tau_1]$ and $(x, \mu) \in C_1 \times \mathbb{R}_{\geq 0}$, let

$$\eta_{sa}(x, \tilde{w}, \tau, \tau_0, \mu) := \int_{\tau_0}^{\tau} \exp(\mu(s - \tau))[f_0(x, \tilde{w}(s), s) - f_{sa}(x, \tilde{w}(s))]ds \; . \qquad (4.12)$$

We assume that, when $\mu = 0$, η_{sa} and η_{wa} are locally Lipschitz, uniformly in τ and τ_0, as stated below in Assumptions 4.3.4 and 4.3.5. These assumptions may hold even when f is not periodic in τ nor continuous in (x, w). The pulse-width modulated hybrid control example in Section 4.5 illustrates this situation. Let $\bar{N} := \{1, \cdots, n\}$. For each $i \in \bar{N}$, η^i_{sa} represents the ith component of η_{sa}, and similarly for η^i_{wa}.

Assumption 4.3.4. *For a function f_0 defined on $C \times \mathbb{R}_{\geq 0}$, f_{wa} is a continuous function that is a weak average of f_0 on C and, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $L(K)$ such that, for all $i \in \bar{N}$, $((x_1, w_1), \tau_a), ((x_2, w_2), \tau_b) \in (C \cap K) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$:*

$$\begin{aligned} &\left|\eta^i_{wa}(x_1, w_1, \tau_a, \tau_0, 0) - \eta^i_{wa}(x_2, w_2, \tau_b, \tau_0, 0)\right| \\ &\qquad \leq L(|x_1 - x_2| + |w_1 - w_2| + |\tau_a - \tau_b|) \; . \end{aligned}$$

□

Assumption 4.3.5. *For a function f_0 defined on $C_1 \times \mathcal{W} \times \mathbb{R}_{\geq 0}$, where $C \subset C_1 \times \mathcal{W}$, f_{sa} is a continuous function that is a strong average of f_0 on $C_1 \times \mathcal{W}$ and, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $L(K)$ such that, for all $i \in \bar{N}$, $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x_1, \tilde{w}(s)), \tau_a), ((x_2, \tilde{w}(s)), \tau_b) \in ((C_1 \times \mathcal{W}) \cap K) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$:*

$$\left|\eta^i_{sa}(x_1, \tilde{w}, \tau_a, \tau_0, 0) - \eta^i_{sa}(x_2, \tilde{w}, \tau_b, \tau_0, 0)\right| \leq L(|x_1 - x_2| + |\tau_a - \tau_b|) \; .$$

□

4.4 Main results

In this section, we present the results on closeness of solutions for hybrid system H_ϵ and its averages systems. We also state results on semiglobal practical ISS (DISS) properties of H_ϵ assuming ISS of its strong average (weak average). For different classes of input signals, we first show that solutions of system H_ϵ and solutions of its strong or weak average system are (T, ρ)-close (Def. 4.2.6) using forward pre-completeness (Def. 4.2.5) of the average system. The proofs of Theorems 4.4.1 and 4.4.2 are given in Appendix C.1.

Theorem 4.4.1. *(Weak average) Suppose that the set $\mathcal{L}_\mathcal{W}$ is equi-essentially bounded and locally equi-uniformly Lipschitz continuous, system H_ϵ in (4.7) satisfies Assumptions 4.3.1 and 4.3.4, and its weak average system H_{wa} satisfies Assumptions 4.2.4 and is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $T \geq 0$ and $\rho > 0$, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and $w \in \mathcal{L}_\mathcal{W}$, for each solution pair (x, w) to H_ϵ with $x(0,0) \in K_0$ there exists some solution pair (ξ, w_1) to H_{wa} with $\xi(0,0) \in K_0$ and $|w_1| \leq |w|$ such that x and ξ are (T, ρ)-close.* □

If, there exists a strong average for system H_ϵ, we can obtain a stronger conclusion in the following theorem, for which the results on closeness of solutions of the actual hybrid system H_ϵ and solutions of its strong average hold without requiring the derivative of input signals to be bounded.

Theorem 4.4.2. *(Strong average) Suppose that the set $\mathcal{L}_\mathcal{W}$ is equi-essentially bounded, system H_ϵ in (4.7) satisfies Assumptions 4.3.1 and 4.3.5, and its strong average system H_{sa} satisfies Assumptions 4.2.4 and is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $T \geq 0$ and $\rho > 0$, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and $w \in \mathcal{L}_\mathcal{W}$, for each solution pair (x, w) to H_ϵ with $x(0,0) \in K_0$ there exists some solution pair (ξ, w_1) to H_{sa} with $\xi(0,0) \in K_0$ and $|w_1| \leq |w|$ such that x and ξ are (T, ρ)-close.* □

For the 0-input case, the result on closeness between solutions of the hybrid system H_ϵ and solutions of its average system can be directly derived from Theorem 4.4.1 or 4.4.2. In this case, assumptions and definitions for system H_ϵ with disturbances can be directly used with letting $w = 0 \subset \mathbb{R}^m$ and dom $w :=$ dom ξ. Note that the weak and the strong averages for system H_ϵ coincide for zero input signals. Let H_{av} denote this average system. Applying the results in Theorem 4.4.1 or 4.4.2, we can directly get the following corollary that is identical to [154, Theorem 1].

Corollary 4.4.3. *Suppose that system H_ϵ, where $w = 0 \subset \mathbb{R}^m$, in (4.7) satisfies Assumptions 4.3.1 and 4.3.4, and its average system H_{av} satisfies Assumption 4.2.4 and is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $T \geq 0$ and $\rho > 0$, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, each solution x to H_ϵ with $x(0,0) \in K_0$ there exists some solution ξ to H_{av} with $\xi(0,0) \in K_0$ such that x and ξ are (T, ρ)-close.* □

The results on closeness of solutions in Theorems 4.4.1 and 4.4.2 hold for arbitrarily large compact hybrid time domains if we assume forward pre-completeness of the average system. We conclude that the strong/weak average system approximate well solutions of the original system on compact time domains, where disturbances are required to be Lipschitz and bounded for weak averages whereas only bounded for the strong averages.

We next consider stability properties of system H_ϵ if we assume ISS of its weak/strong average. In Theorem 4.4.4, we show that ISS (Def. 4.2.7) of the weak average system implies semi-global practical derivative ISS (Def. 4.2.9) of the original system H_ϵ. When the strong average system is ISS, we conclude semi-global practical ISS (Def. 4.2.8) for system H_ϵ in Theorem 4.4.5. The proofs are given in Appendix C.2.

Theorem 4.4.4. *Suppose that the set $\mathcal{L}_{\mathcal{W}}$ is equi-essentially bounded and locally equi-uniformly Lipschitz continuous, system H_ϵ in (4.7) satisfies Assumptions 4.3.1 and 4.3.4 and its weak average system H_{wa} satisfies Assumptions 4.2.4 and is ISS with respect to (χ, β, γ). Then, system H_ϵ is SGP-DISS with respect to (χ, β, γ).* □

The above results show that if the weak average system is ISS with functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ for the class of bounded disturbances that have bounded derivatives, the actual system is SGP-DISS with the same β and γ. Furthermore, we can obtain the same conclusion for system H_ϵ with bounded disturbances if its strong average system is ISS, see Theorem 4.4.5.

Theorem 4.4.5. *Suppose that the set $\mathcal{L}_{\mathcal{W}}$ is equi-essentially bounded, system H_ϵ in (4.7) satisfies Assumptions 4.3.1 and 4.3.5 and its strong average system H_{sa} satisfies Assumption 4.2.4 and is ISS with respect to (χ, β, γ). Then, system H_ϵ is SGP-ISS with respect to (χ, β, γ).* □

Note that Theorem 4.4.4 pertains to bounded input signals with bounded derivatives. In the following example, we can see that for bounded disturbances that do not have bounded derivatives, ISS of the weak average system does not guarantee robustness to disturbances for the original system.

Example 4.4.6. *Consider a hybrid system of the form*

$$\left.\begin{array}{l}\dot{x} = f(x, w, \tau)\\ \dot{\tau} = \frac{1}{\varepsilon}\end{array}\right\} \qquad (x, w, \tau) \in C \times \mathbb{R} \times \mathbb{R}_{\geq 0}$$
$$\left.\begin{array}{l}x^+ = g(x)\\ \tau^+ = 0\end{array}\right\} \qquad (x, w, \tau) \in D \times \mathbb{R} \times \mathbb{R}_{\geq 0}\ , \tag{4.13}$$

where constraint sets $C := \mathbb{R}_{\geq 0}$*,* $D := \mathbb{R}_{\leq 0}$*, and*

$$\begin{aligned} f(x, w, \tau) &:= -kx^3 + \cos(\tau)x^3 w \\ g(x) &:= -x \end{aligned} \tag{4.14}$$

for the parameter $k \in (0, 0.5)$*. The flow dynamics of hybrid system (4.13) agree with the continuous-time system* $\dot{x} = -kx^3 + \cos\left(\frac{t}{\varepsilon}\right)x^3 w$ *for* $x \in C$ *considered in [110, Example 1]. It is showed in [110, Example 1] that there does not exist a strong average for the function* $f : C \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *in (4.14) and its weak average is* $f_{wa}(x) = -kx^3$*. Then, from (4.9), we have that the weak average of system (4.13) is*

$$\begin{aligned} \dot{y} &= f_{wa}(y) & y &\in C \\ y^+ &= g(y) & y &\in D\ , \end{aligned} \tag{4.15}$$

where $g : \mathbb{R} \to \mathbb{R}$*, sets* C *and* D *come from (4.13).*

Let $\mathcal{A} := \{0\}$*. Consider a Lyapunov function candidate* $V(y) = \frac{1}{2}y^2$*. Note that this* $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *satisfies*

$$\begin{aligned} u_C(y) &< 0 & \forall\, y &\in C \setminus \mathcal{A} \\ u_D(y) &= 0 & \forall\, y &\in D\ , \end{aligned} \tag{4.16}$$

where $u_C(y) := \langle \nabla V(y), f_{wa}(y)\rangle = -ky^4$ *and* $u_D(y) := V(g(y)) - V(y)$*. For any* $\mu > 0$*, noting that the set* $V^{-1}(\mu) \cap \{y \in \mathbb{R} | u_C(y) = u_D(y) = 0\}$ *is empty and using the LaSalle's Principle for hybrid systems [55, Theorem 23], we can get that* $\mathcal{A}$ *is globally asymptotically stable. In other words, the weak average system is ISS with zero disturbance gain. We next show that the actual system (4.13) is* *not SGP-ISS. In fact, the original system exhibits finite time escapes under the action of bounded signals.*

Consider a bounded continuous input signal $w(\tau) = \cos(\tau)$ *that can be rewritten as* $w_\varepsilon(t) = \cos\left(\frac{t}{\varepsilon}\right)$ *on the* t *time domain. Note that* $|w_\varepsilon| = 1$ *for any* ε *but* $|\dot{w}_\varepsilon| = \frac{1}{\varepsilon}$ *that becomes arbitrarily large when* ε *is sufficiently small. Thus, the signal* w *is not locally equi-uniformly Lipschitz continuous, see Def. 3.2.5. Recall that*

$$\int_t^T \cos^2(s) = 0.5T + 0.25(\sin(2t + 2T) - sin(2t)) \ ,$$

By direct integration of $\dot{x} = -kx^3 + \cos\left(\frac{t}{\varepsilon}\right) x^3 w$ *with the input signal* $w_\varepsilon(t) = \cos\left(\frac{t}{\varepsilon}\right)$*, we have*

$$\int_{x(t_0)}^{x(t)} \frac{dx}{x^3} = \int_{t_0}^{t} \left(\cos^2\left(\frac{s}{\varepsilon}\right)\right) ds$$

and

$$x^2(t) = \frac{x^2(t_0)}{1 - 2\psi(\varepsilon, t, t_0)x^2(t_0)} \ , \tag{4.17}$$

where

$$\psi(\varepsilon, t, t_0) = (0.5 - k)(t - t_0) + 0.25\varepsilon \left(\sin\left(\frac{2t}{\varepsilon}\right) - \sin\left(\frac{2t_0}{\varepsilon}\right)\right) \ .$$

Fix $t_0 \geq 0$, $\varepsilon > 0$ *and let* $x(t_0) := x(t_0, 0) = 1$*. Considering (4.17), we know that there exists some* $t_1 \geq t_0$ *such that* $\psi(\varepsilon, t_1, t_0) = 0.5$ *since* $(0.5 - k) \in (0, 0.5)$*. Moreover, we have that* $(t_1, 0) \in \operatorname{dom} x$ *for the solution* x *of actual hybrid system (4.13) as the solution* $x(t, 0)$ *with the initial condition* $x(t_0, 0) = 1$ *will stay in the set* C $(x(t, 0) \in \mathbb{R}_{\geq 0})$ *and keep flowing when* $t_0 \leq t \leq t_1$*. Then, there are finite escape times for such a maximal solution* x *and the actual hybrid system* (4.13) *is not semi-globally practically ISS.* □

We consider robustness to disturbances for hybrid systems H_ϵ in Theorems 4.4.4 and 4.4.5. Our results generalize the asymptotic stability analysis results of a class of hybrid systems for the disturbance-free case in [154]. We can directly apply Theorem 4.4.4 or 4.4.5 to consider hybrid system H_ϵ when $w \equiv 0$ and get a corollary that gives the same conclusion as [154, Theorem 4]. Let $\mathcal{A} \subset \mathbb{R}^n$ be nonempty and compact and $\chi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a proper indicator for $\mathcal{A}$. The precise definitions on global asymptotic stability of system H and semi-global practical asymptotic stability of system H_μ when $w \equiv 0$ are given first. After that, the corollary is presented.

Definition 4.4.7 (GAS)**.** *System H in (4.2), when $w \equiv 0$, is called globally asymptotically stable with respect to (χ, β) with $\beta \in \mathcal{KL}$ if for all $\xi(0,0) = \xi_0 \in \mathbb{R}^n$ each solution ξ satisfies*

$$\chi(\xi(t,j)) \leq \beta(\chi(\xi_0), t+j)\} \qquad \forall\ (t,j) \in \text{dom}\ \xi\ . \tag{4.18}$$

□

Definition 4.4.8 (SGP-AS)**.** *System H_μ in (4.6), when $w \equiv 0$, is called semi-globally practically asymptotically stable with respect to (χ, β) with $\beta \in \mathcal{KL}$ if for each compact set $K_0 \subset \mathbb{R}^n$ and any positive real number ν there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*]$, each solution x with $x_0 := x(0,0) \in K_0$ satisfies*

$$\chi(x(t,j)) \leq \beta(\chi(x_0), t+j) + \nu \qquad \forall\ (t,j) \in \text{dom}\ x\ .$$

□

Corollary 4.4.9. *Suppose that system H_ϵ in (4.7) with $w \equiv 0$ satisfies Assumptions 4.3.1 and 4.3.4 and its average system H_{av} satisfies Assumption 4.2.4 and is GAS with respect to (χ, β). Then, system H_ϵ is SGP-AS with respect to (χ, β).*

□

The above results are based on the property that hybrid systems are robust to perturbations. In particular, we need that forward pre-completeness or ISS of hybrid system H in (4.2) imply similar properties for its inflated system H_δ:

$$H_\delta \qquad \begin{array}{ll} \dot{\bar{x}} \in F_\delta(\bar{x}, w) & (\bar{x}, w) \in C_\delta \\ \bar{x}^+ \in G_\delta(\bar{x}, w) & (\bar{x}, w) \in D_\delta\ . \end{array} \tag{4.19}$$

For a parameter $\delta > 0$, the data $(F_\delta, G_\delta, C_\delta, D_\delta)$ are defined as

$$\begin{aligned} F_\delta(\bar{x}, w) &:= \overline{con} F((\bar{x} + \delta\mathbb{B}, w) \cap C) + \delta\mathbb{B} \\ G_\delta(\bar{x}, w) &:= G((\bar{x} + \delta\mathbb{B}, w) \cap D) + \delta\mathbb{B} \\ C_\delta &:= \{(\bar{x}, w) : (\bar{x} + \delta\mathbb{B}, w) \cap C \neq \emptyset\} \\ D_\delta &:= \{(\bar{x}, w) : (\bar{x} + \delta\mathbb{B}, w) \cap D \neq \emptyset\} \ . \end{aligned}$$

As part of the proof for the main results of this chapter, we prove some preliminary results in the Appendices C.1 and C.2 that show robustness properties to perturbations for hybrid system H in (4.2).

Without convexity condition for the flow mapping F in Assumption 4.2.4, robustness to perturbations for hybrid system H is not guaranteed. The following example, see also [27, Remark 3], is used to illustrate that forward completeness of a hybrid system without convexity assumption for its flow mapping F may not be preserved under a small perturbation.

Example 4.4.10. *Let $x = [x_1, x_2]^T \in \mathbb{R}^2$ and $w = [w_1, w_2]^T \in \mathcal{W} \subset \mathbb{R}^2$. Consider the system H (4.2) with the data*

$$\begin{aligned} F(x, w) &:= \begin{bmatrix} x_1^3 |w_1 - w_2| \\ w_1 - w_2 \end{bmatrix}, \\ G(x, w) &:= 0, \\ C &:= \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\} \times \mathcal{W}, \\ D &:= \mathbb{R}^2 \times \mathcal{W}. \end{aligned}$$

Note that for hybrid system H with above (F, G, C, D), all solution pairs starting from the set $(\mathbb{R}^2 \times \mathcal{W}) \setminus C$ are forced to jump to the origin by the mapping G. At the same time, the signal $w_1 - w_2$ will drive the solution that starts in the set C to leave this set and then jump to the origin whenever $w_1(\cdot, 0) - w_2(\cdot, 0) \neq 0$, or keep it staying at the initial position when $w_1(\cdot, 0) - w_2(\cdot, 0) = 0$. Thus, we know that the hybrid system H is forward pre-complete. While, for its inflated system H_δ, one can find $x(\cdot, 0)$ to flow in C_δ and blow up on the compact time domain. Indeed, noting the analytic solution of $\dot{x_1} = x_1^3 |w_1 - w_2|$:

$$x_1(t, 0) = \frac{1}{\sqrt{\frac{1}{x_1^2(0,0)} - 2\int_0^t |w_1(\tau, 0) - w_2(\tau, 0)| d\tau}} \ ,$$

we can see $x(\cdot, 0)$ *blow up in the* x_1 *coordinate for an appropriately chosen* $w(\cdot, 0)$ *where* (x, w) *is the maximal solution pair that starts in* $C_\delta \times \mathcal{W}$ *with* $C_\delta := \{x : (x + \delta\mathbb{B}) \cap C \neq \emptyset\}$ *for a positive real number* δ. □

4.5 PWM hybrid feedback control example

Pulse-width-modulated (PWM) control strategy is useful for systems controlled by on-off switches, which are commonly utilized to model switching power electronic systems and find wide application in industry [40,83,118,146,148]. In this section, we take PWM hybrid feedback control systems as an example to show how to apply the results presented above.

In particular, Subsection 4.5.1 illustrates how to model a hybrid feedback controlled PWM power converter as hybrid systems of the form (4.7). In Subsection 4.5.2, we consider strong and weak averages for the PWM hybrid feedback control system and apply the results given in Section 4.4 to analyze its ISS properties. Moreover, we revisit the power converter example to show that we can design a hybrid controller based on the simpler average model such that the actual converter system can be stabilized using the same controller.

4.5.1 Models

To show how to model a general PWM hybrid feedback control system as hybrid system H_ϵ, we first consider a single rate PWM boost power converter example, see Fig. 4.4. For this PWM power converter, the open-loop model considered in [78] and closed-loop model with a continuous feedback controller presented in [83, Section 4] are first given. We also present the closed-loop system when this power converter is controller by a general hybrid controller. After that, we consider a general continuous-time plant with hybrid feedbacks that are implemented via multi-rate PWM. We show that the class of hybrid systems (4.7) considered in this chapter include this general multi-rate PWM hybrid model as a special case.

Example 4.5.1. *Suppose that the boost converter in Fig. 4.4 operates in the continuous conduction mode [83]; in this case there are two configuration modes for the converter system corresponding to the on/off state of the switches. Namely, the mode* q_1 *corresponds to the switch* SW_1 *on and* SW_2 *off and the mode* q_2 *corresponds to* SW_1 *off and* SW_2 *on. A PWM produces a signal in form of a pulse train (e.g. that takes values only at* 0 *and* 1*) of a constant frequency but with*

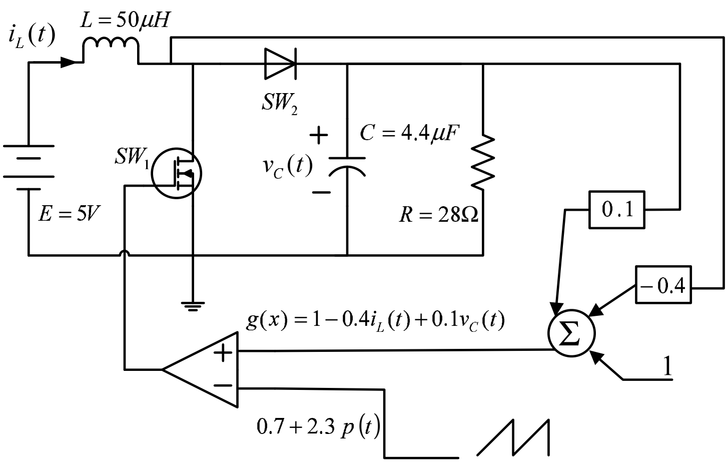

Figure 4.4: Continuous-time feedback control boost converter [83].

pulses that have a varying width; the pulse width is modulated using the measurements from the plant to determine the duty ratio, i.e., the ratio of time spent at 0 *to the time spent at* 1 *or* SW_1 *on to off for the converter system in Fig. 4.4.*

Let ξ_1 *denote the instantaneous value of the inductor current* i_L *and* $\xi_2 := v_C$ *be the capacitor voltage. Let* $\xi := (\xi_1, \xi_2)$*. Considering the circuit in Fig. 4.4, we have that dynamics of states* ξ *agree with* $\dot{\xi} = A_{q_i}\xi + B_{q_i}$ *on the* q_i *configuration for* $i = 1, 2$ *[83], where*

$$A_{q_1} := \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \quad A_{q_2} := \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \quad B_{q_1} = B_{q_2} := \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}.$$

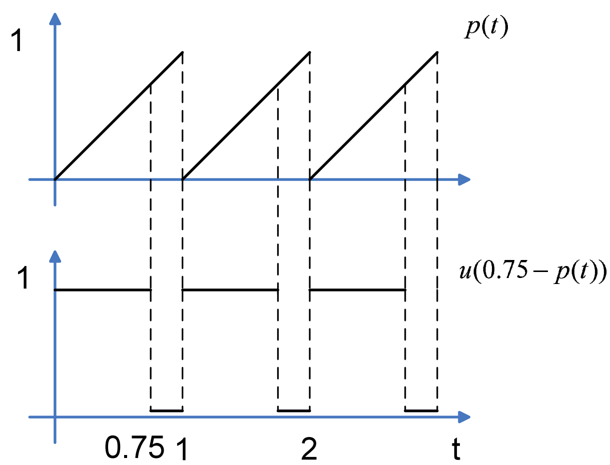

Figure 4.5: A typical triangle switching signal $p(t)$ in PWM control systems and $u(d - p(t))$ for $d = 0.75$.

Noting that the point of equilibrium of the converter system can be moved to the origin using a coordinate transformation, one can consider the stabilization of the origin for the closed-loop converter system. For the converter example in [83], see Fig. 4.4, note that the triangle switched signal is denoted by $0.7 + 2.3\ p(t)$, *where* $p(t)$ *is periodic in* t *satisfying* $p(t) = \frac{t}{T}$ *for* $t \in [0, T)$ *and* $T > 0$. *Then, we have the closed-loop model of the converter system [83]:*

$$\dot{\xi} = A_{q_2}\xi + B_{q_1} + (A_{q_1}\xi - A_{q_2}\xi)\, u(d(\xi) - p(\tau)), \tag{4.20}$$

where the duty ratio function $d(\xi) := \frac{g(\xi)-0.7}{3-0.7}$, *with* $g(\xi) = 1 - 0.4\xi_1 + 0.1\xi_2$ *is scaled using the minimum and the maximum values of the triangle signal so that* $d(\xi)$ *takes values in* $[0, 1]$; $u : \mathbb{R} \to [0, 1]$ *is the unit step function with* $u(s) = 1$ *for* $s \geq 0$ *and* $u(s) = 0$ *for* $s < 0$. *Fig. 4.5 is an example of* $u(d - p(t))$ *for* $d = 0.75$ *and* $T = 1$.

The open-loop model for this converter system is given in [78]:

$$\dot{\xi} = A_{q_2}\xi + B_{q_1} + (A_{q_1}\xi - A_{q_2}\xi)\, u(d - p(\tau)), \tag{4.21}$$

where $d \in [0, 1]$ *is the duty cycle for the open-loop PWM operation.*

Note that there are situations when certain closed-loop performance specifications can not be achieved with any linear feedback controller whereas they are achievable with a hybrid controller, see [18]. For instance, a switched controller designed via Lyapunov approach in [24] is employed to control a power converter system in [93] and it was shown to provide better performance on transient and steady dynamics than continuous PID controllers. More details can be found in the survey of hybrid control techniques for power converter systems [32, 98]. This observation provides a partial motivation for developing averaging techniques for hybrid systems that can be used to analyze a class of hybrid PWM systems in general and power converter systems in particular.

We next consider the same converter but instead of the continuous controller $d(\xi)$ *in Fig. 4.4 we want to apply a hybrid feedback controller, see Fig. 4.6. Suppose that hybrid controller* $h : \bar{C} \times \bar{D} \to [0, 1]$ *that was designed to satisfy given performance specifications is given as:*

$$\begin{aligned} \dot{\eta} &= R(\eta, \xi) & (\xi, \eta) \in \bar{C} \\ \eta^+ &\in S(\eta, \xi) & (\xi, \eta) \in \bar{D} \\ h &:= h(\eta, \xi) \ , \end{aligned} \tag{4.22}$$

where $\eta \in \mathbb{R}^n$*;* $\bar{C}$*,* $\bar{D}$ *are the constraint sets that allow flows and jumps for* η*;* $R : \bar{C} \to \mathbb{R}^n$ *is a flow mapping and* $S : \bar{D} \rightrightarrows \mathbb{R}^n$ *is a set-valued mapping. Note that states* η *may include physical variables together with logic variables or operation modes that are used to describe the hybrid feedback control law.*

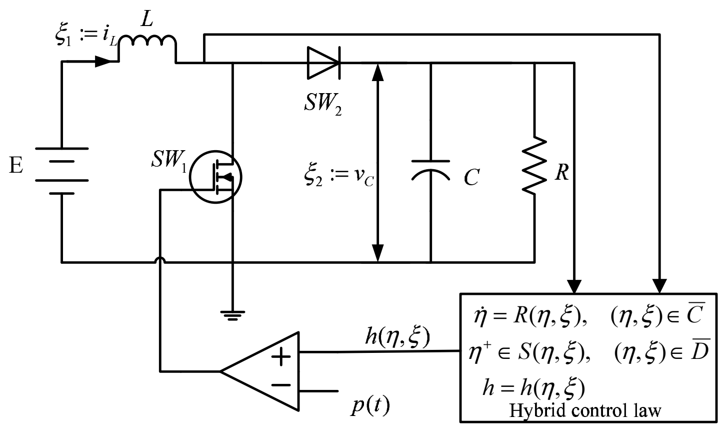

Figure 4.6: Hybrid feedback control boost converter

Applying this $h(\eta, \xi)$ *to the open-loop converter system (4.21), we have that the closed-loop model of the converter system in Fig. 4.6 is:*

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} A_{q_2}\xi + B_{q_1} \\ R(\eta, \xi) \end{bmatrix} \\ &\quad + \begin{bmatrix} A_{q_1}\xi - A_{q_2}\xi \\ 0 \end{bmatrix} \cdot u(h(\eta, \xi) - p(\tau)) \qquad (\xi, \eta) \in \bar{C} \\ \begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} &\in \begin{bmatrix} \xi \\ S(\eta, \xi) \end{bmatrix} \qquad (\xi, \eta) \in \bar{D} \ . \end{aligned} \tag{4.23}$$

Averaging results from [83] can be used to analyze the model (4.20) but not the model (4.23). In this chapter we presented results that can be used to analyze models of the form (4.23). With these analysis results, one can design hybrid controllers $h(\eta, \xi)$ *based on the simpler average model to obtain similar properties of the actual PWM closed-loop system under the same* h*.*

Note that the average model of open-loop converter system (4.21) is given in [78]:

$$\dot{\xi} = A_{q_2}\xi + B_{q_1} + d\,(A_{q_1} - A_{q_2})\,\xi, \tag{4.24}$$

where the duty cycle $d \in [0,1]$ can be taken as a control signal. Suppose a controller $h : \bar{C} \cup \bar{D} \to [0,1]$ is designed to stabilize the open-loop average system (4.24) by letting $d := h$. Then, our results in this chapter can be used to analyze the stability properties of the PWM converter system (4.23) through stability of the closed-loop of system (4.24) using the same controller h. □

We assume that the controller h and the triangle signal p in Example 4.5.1 satisfy $h(\eta,\xi) : \bar{C} \times \bar{D} \to [0,1]$ and $p : \mathbb{R}_{\geq 0} \to [0,1]$ respectively. The following remark shows that we may consider h and p only with their images in $[0,1]$ without loss of generality.

Remark 4.5.2. *Suppose that we need a controller $\tilde{U} = \tilde{U}(\xi,\eta)$ that takes values in $[a,b]$ to stabilize the plant and achieve appropriate performance. To implement this controller via PWM we need to get an average ranging from a to b using a step function $\tilde{u}(\cdot)$ that satisfies $\tilde{u}(s) = a$ for $s < 0$ and $\tilde{u}(s) = b$ for $s \geq 0$. Suppose also that we want to use a triangle signal $\hat{p}(\cdot) = c + kp(\cdot) \in [c, c+k]$, with $k > 0$ to implement this controller, where $p(\cdot)$ is the triangle wave defined earlier. Then, we need a duty cycle function $\hat{U}(\xi,\eta)$, generated from $\tilde{U}(\xi,\eta)$, but taking values in $[c, c+k]$. In particular, we take $\hat{U}(\xi,\eta) := c + k\frac{\tilde{U}(\xi,\eta)-a}{b-a}$. The PWM control that we need to implement is then*

$$\tilde{u}(\hat{U}(\xi,\eta) - \hat{p}(\tau)) \ ,$$

which can be written in other ways and it is also equal to

$$a + (b-a)u(h(\xi,\eta) - p(\tau)) \ ,$$

where

$$\begin{aligned} h(\xi,\eta) &:= \frac{\tilde{U}(\xi,\eta) - a}{b-a}, \\ u(s) &:= \frac{\tilde{u}(s) - a}{b-a}, \\ p(\tau) &:= \frac{\hat{p}(\tau) - c}{k} \ . \end{aligned}$$

□

Note that the averaging results of this chapter pertain to a more general class of PWM systems that is presented next. We consider a general continuous-time plant with disturbances controlled by hybrid feedbacks that are implemented via multi-rate PWM. Consider a continuous-time plant with states $\xi \in \mathbb{R}^n$, disturbances $w \in \mathcal{W} \subset \mathbb{R}^m$ and outputs $y \in \mathbb{R}^l$:

$$\begin{aligned} \dot{\xi} &= O(\xi, w) + \sum_{i}^{k} P_i(\xi, w) h_i \ , \\ y &= Q(\xi, w) \ . \end{aligned} \tag{4.25}$$

For this continuous-time plant, the hybrid feedback controllers h_i are given through the following auxiliary hybrid system with states $\eta \in \mathbb{R}^h$:

$$\begin{aligned} &\dot{\eta} = R(\eta, y) && (\eta, y) \in C_1 \\ &\eta^+ \in S(\eta, y) && (\eta, y) \in D_1 \\ &h_i = h_i(\eta, y) \ , \end{aligned}$$

where $C_1,\ D_1 \in \mathbb{R}^h \times \mathbb{R}^l$ are the constraint sets that allow flows and jumps for η; $S : \mathbb{R}^h \rightrightarrows \mathbb{R}^h$ is a set-valued mapping that is outer semi-continuous, locally bounded and for each $(\eta, y) \in D_1$, $S(\eta, y)$ is nonempty; functions $O : \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$ and $R : C_1 \to \mathbb{R}^h$ are continuous while $P_i : \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$, $Q : \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^l$ and $h_i : \mathbb{R}^h \times \mathbb{R}^l \to [0, 1]$ are locally Lipschitz.

Let

$$\begin{aligned} C &:= C_1 \times \mathcal{W}, \quad D := D_1 \times \mathcal{W}, \quad h_i(x, w) := h_i(\eta, Q(\xi, w)), \\ x &:= \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \psi_i(x, w) := \begin{bmatrix} P_i(\xi, w) \\ 0 \end{bmatrix}, \\ \mathcal{F}_0(x, w) &:= \begin{bmatrix} O(\xi, w) \\ R(\eta, Q(\xi, w)) \end{bmatrix}, \\ \mathcal{G}(x, w) &:= \begin{bmatrix} \xi \\ S(\eta, Q(\xi, w)) \end{bmatrix}. \end{aligned}$$

In the case when feedback controllers h_i are implemented by multi-rate PWM, the closed-loop of system (4.25) becomes:

$$\left.\begin{aligned}\dot{x} &= \mathcal{F}(x,w,\tau)\\ \dot{\tau} &= \tfrac{1}{\varepsilon}\end{aligned}\right\} \qquad ((x,w),\tau) \in C \times \mathbb{R}_{\geq 0}$$

$$\left.\begin{aligned}x^{+} &\in \mathcal{G}(x,w)\\ \tau^{+} &= \tau\end{aligned}\right\} \qquad ((x,w),\tau) \in D \times \mathbb{R}_{\geq 0}\ , \tag{4.26}$$

where $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the jump mapping, C and $D \subset \mathbb{R}^n \times \mathbb{R}^m$ are given sets that allow for flow and jump for the designed hybrid feedback controller and

$$\mathcal{F}(x,w,\tau) := \mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)u(h_i(x,w) - p_i(\tau))\ . \tag{4.27}$$

The second term of $\mathcal{F}$ in (4.27) is used to model a multi-rate implementation of a PWM hybrid controller. As $p_i(\tau)$ are the only time-varying terms, the small parameter $\varepsilon > 0$ in (4.26) is used to guarantee that the switching signals p_i change fast compared with state ξ and so the effect of p_i can be averaged.

4.5.2 Averaging analysis

We next consider the PWM hybrid feedback control system with disturbances in (4.26) to illustrate how our results can be applied so that the ISS properties of the actual closed loop of system (4.26) can be studied through its time-invariant average system.

First, we show that there exists a weak average for function $\mathcal{F} : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ in (4.27) on the set C and Assumption 4.3.4 holds for the PWM control system in (4.26). Given $T > \max\{T_1, \cdots, T_n\}$, let $k_i = k_i(T) \in \mathbb{Z}_{\geq 0}$ and $\tilde{T}_i \in [0, T_i)$ satisfying $T = k_i T_i + \tilde{T}_i$. Note that $k_i(T) \to \infty$ when the given T approaches infinity. For all $(x,w) \in C$, we get

$$\begin{aligned}\frac{1}{T}\int_{\tau}^{\tau+T} &\left\{\mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)u(h_i(x,w) - p_i(s))\right\} ds\\ &= \mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)\frac{1}{T}\left\{\int_{\tau}^{\tau+k_iT_i} u(h_i(x,w) - p_i(s))ds\right.\\ &\qquad\qquad \left.+ \int_{\tau+k_iT_i}^{\tau+k_iT_i+\tilde{T}_i} u(h_i(x,w) - p_i(s))ds\right\}\end{aligned}$$

$$= \mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)\left(\frac{k_iT_i}{k_iT_i+\tilde{T}_i}h_i(x,w) + \frac{v_i(x,w,\tilde{T}_i)}{k_iT_i+\tilde{T}_i}\right),$$

where $v_i(x,w,\tilde{T}_i) := \int_\tau^{\tau+\tilde{T}_i} u(h_i(x,w)-p_i(s))ds$ that satisfies $|v_i(x,w,\tilde{T}_i)| \le \tilde{T}_i$.

Let

$$f_{wa}(x,w) := \mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)h_i(x,w) \ .$$

Note that for any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $r > 0$ such that, for all $(x,w) \in C \cap K$:

$$\left| f_{wa}(x,w) - \frac{1}{T}\int_\tau^{\tau+T}\left\{\mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)u(h_i(x,w)-p_i(s))\right\}ds \right|$$
$$\le \frac{1}{T}\sum_{i=1}^{m}|g_i(x,w)v_i(x,w,\tilde{T}_i)| \le \frac{r}{T+1} := \sigma_K(T) \ ,$$

which shows that f_{wa} agrees with Definition 4.3.2. Let $\mathcal{G}, C, D$ come from (4.26). Then, the hybrid system H_{wa}

$$\begin{array}{ll} \dot{x} = f_{wa}(x,w) & (x,w) \in C \\ x^+ \in \mathcal{G}(x,w) & (x,w) \in D \ , \end{array} \tag{4.28}$$

with same $\mathcal{G}, C, D$ in (4.26), is the weak average for the PWM closed-loop control system.

Next we verify Assumption 4.3.4. Considering the definition of η_{wa} in (4.11), it follows for each $\tau \in (0, \min_i\{T_i\})$ and $\tau_0 \in [0,\tau]$ that

$$\begin{aligned} &\eta_{wa}(x,w,\tau,\tau_0,0) \\ &= \int_{\tau_0}^{\tau}\left(\mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)u_i(h_i(x,w)-p_i(s)) - f_{wa}(x,w)\right)ds \ , \\ &= \sum_{i=1}^{m} g_i(x,w)\int_{\tau_0}^{\tau}[u_i(h_i(x,w)-p_i(s)) - h_i(x,w)]ds \ , \\ &= \sum_{i=1}^{m} g_i(x,w)(\min\{(\tau-\tau_0), h_i(x,w)\} - (\tau-\tau_0)h_i(x,w)) \ , \end{aligned} \tag{4.29}$$

which is bounded for any (x, w) in a compact set and locally Lipschitz as functions g_i and h_i are locally Lipschitz. Then, Assumption 4.3.4 holds for the function $\eta_{wa}(x, w, \tau, \tau_0, 0)$.

Recall that for any function $\mathcal{F}(t, x, w)$ that is periodic in t, the necessary and sufficient condition for existence of strong averages is that function $\mathcal{F}$ has the following structure [110]:

$$\mathcal{F}(t, x, w) = \mathcal{F}_1(t, x) + \mathcal{F}_2(x, w) \ .$$

We can get that for PWM control system (4.26) there exists a strong average if $g_i(x, w)$ and $h_i(x, w)$ are independent of w, i.e., $g_i(x, w) := g_i(x)$ and $h_i(x, w) := h_i(x)$. In this case, following the calculations used to establish the weak average, we get that $f_{sa}(x, w) := f_0(x, w) + \sum_{i=1}^{m} g_i(x)h_i(x)$ on the set C, at least when C has the form $C = C_1 \times \mathcal{W}$, and the strong average of system (4.26) is

$$\begin{array}{ll} \dot{x} = f_{sa}(x, w) & (x, w) \in C \\ x^+ \in \mathcal{G}(x, w) & (x, w) \in D \ . \end{array} \tag{4.30}$$

Using the definition of η_{sa} in (4.12), we have

$$\begin{aligned} &\eta_{sa}(x, w, \tau, \tau_0, 0) \\ &\quad = \int_{\tau_0}^{\tau} \left(f_0(x, w) + \sum_{i=1}^{m} g_i(x)u_i(h_i(x) - p_i(s)) - f_{sa}(x, w) \right) ds \ , \\ &\quad = \sum_{i=1}^{m} g_i(x) \int_{\tau_0}^{\tau} (u_i(h_i(x) - p_i(s)) - h_i(x))ds \ . \end{aligned}$$

Noting (4.29), it follows that Assumption 4.3.5 holds for the function $\eta_{sa}(x, w, \tau, \tau_0, 0)$.

The above analysis shows that Assumptions 4.3.4 and 4.3.5 hold with functions h_i and g_i being locally Lipschitz. Note that only local boundedness but no continuity condition is required for the flow mapping of the actual hybrid systems in Assumption 4.3.1. This condition holds for PWM hybrid feedback control systems (4.26) and then we can get that the results in Section 4.4 can be applied under some mild regular conditions. The following corollaries come directly from Theorems 4.4.4-4.4.5. Using these corollaries we can analyze robust stability of the time-varying PWM control system (4.26) based on its weak or strong average system.

Corollary 4.5.3. *Suppose that the set $\mathcal{L}_{\mathcal{W}}$ is equi-essentially bounded and locally* *equi-uniformly Lipschitz continuous, the PWM hybrid control system in (4.26) satisfies Assumptions 4.3.1 and its weak average system H_{wa} satisfies Assumption 4.2.4 and is ISS with respect to (χ, β, γ). Then, the PWM hybrid control system in (4.26) is SGP-DISS with respect to (χ, β, γ).* □

Corollary 4.5.4. *Suppose that the set $\mathcal{L}_{\mathcal{W}}$ is equi-essentially bounded, the PWM hybrid control system in (4.26) satisfies Assumptions 4.3.1 and its strong average system H_{sa} satisfies Assumption 4.2.4 and is ISS with respect to (χ, β, γ). Then, the PWM hybrid control system in (4.26) is SGP-ISS with respect to (χ, β, γ).* □

Recall that the power converter model in Example 4.5.1 is disturbances free. We can apply the results obtained in Corollary 4.4.9 to revisit Example 4.5.1.

Example 4.5.5. *Note that the single rate PWM power converter system (4.23) in Example 4.5.1 is a special case of PWM hybrid feedback control systems in (4.26). We can directly apply the results on strong (4.30) or weak average (4.28) of general hybrid feedback control systems (they coincide for the zero-input case) to get the average for the closed-loop converter system (4.23):*

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} A_{q_2}\xi + B_{q_1} \\ R(\eta,\xi) \end{bmatrix} + \begin{bmatrix} A_{q_1}\xi - A_{q_2}\xi \\ 0 \end{bmatrix} h(\eta,\xi) & (\xi,\eta) \in \bar{C} \\ \begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} &\in \begin{bmatrix} \xi \\ S(\eta,\xi) \end{bmatrix} & (\xi,\eta) \in \bar{D}\ . \end{aligned} \tag{4.31}$$

For the PWM hybrid feedback power converter system (4.23) and its average system (4.31), Assumptions 4.3.4 and 4.3.5 hold since the local boundedness and Lipschitz condition are naturally satisfied for linear systems since the auxiliary system generating hybrid feedback controllers for linear plant is usually linear.

Then, if we design hybrid feedbacks such that the closed-loop average model (4.31) for the power converter is globally asymptotically stable, Corollary 4.4.9 shows that the actual converter system (4.23) under the same hybrid controller is SGP-AS. □

4.6 Conclusions

We considered ISS properties for a class of time-varying hybrid dynamic systems via the averaging method. Using the notions of strong and weak average, the time-varying hybrid system is approximated by a time-invariant hybrid system. We showed that solutions of the actual time-varying hybrid system and solutions of its weak or strong average can be made arbitrarily close on compact time domains by reducing the parameter ε if the average system is forward pre-complete. Our main results also showed that ISS of the strong (weak) average implies SGP-ISS (SGP-DISS) of the actual system. A PWM hybrid feedback control power converter example was used to illustrate our results.

Chapter 5

Averaging in Singularly Perturbed Hybrid Systems

5.1 Introduction

We consider a class of singularly perturbed hybrid systems in this chapter using both the singular perturbation technique and the averaging method. Such a class of hybrid systems can be used to model the dynamics of a hybrid feedback control system with fast but continuous actuators. We analyze the singularly perturbed hybrid system via a reduced-order hybrid system, which is defined by the average vector field for the slow dynamics that is generated by solutions of continuous-time boundary layer dynamics, the projection of the jump map in the direction of the slow states, and flow and jump sets from the original dynamics.

Averaging method is used in [16] together with the singular perturbation technique to consider continuous-time systems when the boundary layer system is time-varying and possesses a time-varying integral manifold on which the derivative of slow state variables can be averaged. Such results can be applied to adaptive control systems [130] and extremum seeking control systems [149]. The averaging method is also helpful in considering the singular perturbation problem when the boundary layer system is not time varying. Instead of insisting that trajectories of the boundary layer system converge to an equilibrium manifold, as in the classical singular perturbation theory, a set is used to replace the equilibrium manifold. For instance, trajectories of the boundary layer system might be assumed to converge to a family of limit cycles parameterized by slow state variables, which then can be used to average the derivative of slow state variables [151].

Combining averaging and singular perturbation techniques, we first consider closeness between solutions of the average system and solutions of the slow dynamics for the actual perturbed hybrid system on compact time domains based on forward pre-completeness of the average system. We also consider stability properties of the singularly perturbed hybrid system with the assumption that its average system is asymptotically stable. We show that a compact set is semi-globally practically asymptotically stable for the actual hybrid system if it is globally asymptotically stable for the average system. Compared to hybrid singular perturbation results in [135], our averaging models in some cases are better approximations for the original system and using them we can draw stronger stability conclusions for the original system. An example is used to illustrate this claim.

The chapter is organized as follows. We provide some useful preliminary results in Section 5.2. Section 5.3 introduces a class of singularly perturbed hybrid systems and the average definition, and illustrates the existence of averages through several examples. The intuitional idea on relating the actual perturbed system with its average system through the state augmentation and the coordinate transformation is given in Section 5.4. Section 5.5 is the main results and Section 5.6 contains the conclusions.

5.2 Preliminaries

We provide some preliminary results that consider robustness to small perturbations for hybrid systems in this section. Consider a hybrid system with states $\xi \in \mathbb{R}^n$ of the form

$$H \quad \begin{array}{ll} \xi' \in F(\xi) & \xi \in C \\ \xi^+ \in G(\xi) & \xi \in D \ , \end{array} \tag{5.1}$$

and list some mild conditions in Assumption 5.2.1 that are assumed to hold for the system H.

Assumption 5.2.1. *The sets $C, D \subset \mathbb{R}^n$ are closed; $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded and for each $\xi \in C$, $F(\xi)$ is nonempty and convex; $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded, for each $\xi \in D$, $G(\xi)$ is nonempty.* □

We consider closeness between solutions of the reduced hybrid average system and solutions of the slow dynamics of the original singularly perturbed hybrid system based on forward pre-completeness of the average system. Recall Def. 4.2.5, we have that a hybrid solution is said to be forward pre-complete if its domain is compact or unbounded. System H is said to be forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ if all maximal solutions ξ with $\xi(0,0) \in K_0$ are forward pre-complete.

We need the following definition on (T, J, ρ)-closeness for hybrid signals, which has the same implication as (T, ρ)-closeness for hybrid signals in Def. 4.2.6 but we use two positive real numbers T and J instead of only T to define the compact time domain. This (T, J, ρ)-closeness concept is useful in proving the main results of this chapter.

Definition 5.2.2 ((T, J, ρ)-closeness)**.** *Two hybrid signals* $\xi_1 : \text{ dom } \xi_1 \mapsto \mathbb{R}^n$ *and* $\xi_2 : \text{ dom } \xi_2 \mapsto \mathbb{R}^n$ *are said to be* (T, J, ρ)*-close if:*

1. *for each* $(t, j) \in \text{dom } \xi_1$ *with* $t \le T$ *and* $j \le J$ *there exists* s *such that* $(s, j) \in \text{dom } \xi_2$*, with* $|t - s| \le \rho$ *and* $|\xi_1(t, j) - \xi_2(s, j)| \le \rho$*,*

2. *for each* $(t, j) \in \text{dom } \xi_2$ *with* $t \le T$ *and* $j \le J$ *there exists* s *such that* $(s, j) \in \text{dom } \xi_1$*, with* $|t - s| \le \rho$ *and* $|\xi_2(t, j) - \xi_1(s, j)| \le \rho$*.*

□

We also consider the stability properties of the perturbed system under the assumption that a compact set is globally asymptotically stable for its average system. Let a compact set $\mathcal{A} \subset \mathbb{R}^n$ be given. Recall the definitions on global asymptotic stability (GAS) of the set $\mathcal{A}$ in Def. 4.4.7 for hybrid system H in (5.1), and semi-global practical asymptotic stability (SGP-AS) of the set $\mathcal{A}$ in Def. 4.4.8 for hybrid system H_μ in (4.6) when $w \equiv 0$.

Next, we present the preliminary results in Proposition 5.2.3 on properties of system H based on its forward pre-completeness and Lemmas 5.2.4-5.2.5 on robustness to small perturbations for system H. For this purpose, we consider a hybrid system H_δ inflated from the system H in (5.1):

$$H_\delta \quad \begin{array}{ll} \bar{x}' \in F_\delta(\bar{x}) & \bar{x} \in C_\delta \\ \bar{x}^+ \in G_\delta(\bar{x}) & \bar{x} \in D_\delta \ , \end{array} \tag{5.2}$$

where $\bar{x} \in \mathbb{R}^n$, and for a parameter $\delta > 0$, the data $\{F_\delta, G_\delta, C_\delta, D_\delta\}$ are defined as

$$\begin{aligned} F_\delta(\bar{x}) &:= \overline{con} F((\bar{x} + \delta\mathbb{B}) \cap C) + \delta\mathbb{B} , && (5.3) \\ G_\delta(\bar{x}) &:= G((\bar{x} + \delta\mathbb{B}) \cap D) + \delta\mathbb{B} , && (5.4) \\ C_\delta &:= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap C \neq \emptyset\} , \\ D_\delta &:= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap D \neq \emptyset\} . \end{aligned}$$

Proposition 5.2.3, also see [56, Corollary 4.7], discusses the compactness of the reachable set for the system H based on its forward pre-completeness. For a given compact set $K_0 \subset \mathbb{R}^n$, let $S(K_0)$ denote the set of maximal solutions ξ to system H in (5.1) with $\xi(0,0) \in K_0$. Then, we have the results precisely given below.

Proposition 5.2.3. *Suppose that system H in (5.1) satisfies Assumption 5.2.1 and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $T, J \geq 0$ the reachable set*

$$R(K_0, T, J) := \{\xi(\tau, j) : \xi \in S(K_0), \tau \leq T, j \leq J\} \tag{5.5}$$

is compact. □

The following Lemma 5.2.4, also as [56, Corollary 5.5], is about the closeness of solutions between the system H with the system H_δ inflated from H by a small parameter $\delta > 0$.

Lemma 5.2.4. *Suppose that the system H in (5.1) satisfies Assumption 5.2.1, and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J there exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ and any solution $\bar{x}$ of the inflated system H_δ formed as (5.2) with $\bar{x}(0,0) \in K_0 + \delta\mathbb{B}$ there exists a solution ξ to the system H with $\xi(0,0) \in K_0$ such that $\bar{x}$ and ξ are (T, J, ρ)-close.* □

When system H is globally asymptotically stable, Lemma 5.2.5 shows that its inflated system H_δ is SGP-AS, also see [56, Theorem 6.6].

Lemma 5.2.5. *Suppose that system H in (5.1) satisfies Assumption 5.2.1, and the compact set $\mathcal{A}$ is globally asymptotically stable with respect to $\beta \in \mathcal{KL}$ for system H. Then, the compact set $\mathcal{A}$ is SGP-AS for the system H_δ.* □

5.3 Singularly perturbed hybrid systems

We present a class of singularly perturbed hybrid systems in this section, for which the analysis is developed based two time scales, (τ, j) and (t, j) with $\tau = \varepsilon t$, with the notations $x' = \frac{dx}{d\tau}$, $\dot{x} = \frac{dx}{dt}$. The average definition is given. Existence of the average is discussed, for which we give some conclusion in Lemma 5.3.5 and illustrate this through several examples.

Consider a class of singularly perturbed hybrid systems with the time variables (τ, j):

$$H_\varepsilon \qquad \left.\begin{array}{l} x' = f(x, z, \varepsilon) \\ z' = \frac{1}{\varepsilon}\psi(x, z, \varepsilon) \end{array}\right\} \qquad (x, z) \in C \times \Psi \tag{5.6}$$
$$\qquad (x, z)^+ \in \mathcal{G}(x, z) \qquad\qquad (x, z) \in D \times \Psi \ ,$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $C, D \subset \mathbb{R}^n$, $\Psi \subset \mathbb{R}^m$, $f : C \times \Psi \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\psi : C \times \Psi \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$, and $\varepsilon > 0$ is a small parameter that ensures the flow dynamics of z are much faster than x.

Let $f_0(x, z) := f(x, z, 0)$ and $\psi_0(x, z) := \psi(x, z, 0)$. We assume that system H_ε satisfies the following conditions.

Assumption 5.3.1. *The sets C and D are closed and the set Ψ is compact. $\mathcal{G}$ is outer semi-continuous and locally bounded, and for each $(x, z) \in D \times \Psi$, $\mathcal{G}(x, z)$ is nonempty. $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \mathbb{R}^m$ are continuous, and for each $\delta > 0$ and compact $K \subset \mathbb{R}^n$ there exists $\varepsilon^* := \varepsilon^*(K, \delta) > 0$ such that*

$$\left.\begin{array}{l} |f(x, z, \varepsilon) - f_0(x, z)| \leq \delta \\ |\psi(x, z, \varepsilon) - \psi_0(x, z)| \leq \delta \end{array}\right\} \ \forall \ ((x, z), \varepsilon) \in ((C \cap K) \times \Psi) \times (0, \varepsilon^*] \ . \tag{5.7}$$

□

The set Ψ is assumed to be compact as we wish to deal with compact attractors for the fast state z and without any assumption on the set-valued map $\mathcal{G}$; if (5.6) admits solutions with a purely discrete-time domain then a jump rule like $z^+ = z$ will not allow z to converge to a compact set unless it is constrained to a compact set a priori.

To facilitate the definition of the boundary layer system (see (5.9) below), the system H_ε is also expressed with the time variables (t, j) with $t := \tau/\varepsilon$:

$$H_\varepsilon \qquad \left.\begin{aligned} \dot{x} &= \varepsilon f(x,z,\varepsilon) \\ \dot{z} &= \psi(x,z,\varepsilon) \end{aligned}\right\} \qquad (x,z) \in C \times \Psi \tag{5.8}$$
$$(x,z)^+ \in \mathcal{G}(x,z) \qquad (x,z) \in D \times \Psi \ .$$

Then, we can define the boundary layer system of the system H_ε as

$$H_{bl} \qquad \left.\begin{aligned} \dot{x}_{bl} &= 0 \\ \dot{z}_{bl} &= \psi_0(x_{bl}, z_{bl}) \end{aligned}\right\} \qquad (x_{bl}, z_{bl}) \in C \times \Psi \ , \tag{5.9}$$

which is obtained by ignoring the jump mapping and setting $\varepsilon = 0$ in (5.8). Note that the continuous-time boundary layer model only includes the flow dynamics of the original system H_ε as we consider the case when the dynamics of singular perturbations are continuous-time varying.

We next give the average definition for flow dynamics of system H_ε based on functions $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \mathbb{R}^m$ that denote slow and fast flow dynamics respectively. The average for hybrid system H_ε is given after that.

Definition 5.3.2. *For functions $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \mathbb{R}^m$, the set-valued mapping $F_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be an average of f_0 with respect to ψ_0 on $C \times \Psi$ if for each compact set $K \subset \mathbb{R}^n$ there exists a class-$\mathcal{L}$ function σ_K such that, for each $L > 0$, $x \in C \cap K$ and each function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $\dot{z}_{bl} = \psi_0(x, z_{bl})$ there exists a measurable function $f_{z_{bl}} : [0, L] \to \mathbb{R}^n$ such that $f_{z_{bl}}(s) \in F_{av}(x)$ for all $s \in [0, L]$ and the following holds:*

$$\left| \frac{1}{L} \int_0^L [f_0(x, z_{bl}(s)) - f_{z_{bl}}(s)] ds \right| \leq \sigma_K(L) \ . \tag{5.10}$$

□

For the singularly perturbed system H_ε modeled in (5.6) or (5.8) with a well-defined average, we have its average system $H_{av} := \{F, G, C, D\}$ formed as (5.1) with

$$F(x) := F_{av}(x) \qquad \forall \ x \in C \ , \tag{5.11}$$

$$G(x) := \{v_1 \in \mathbb{R}^n : (v_1, v_2) \in \mathcal{G}(x,z), \ (z, v_2) \in \Psi \times \mathbb{R}^m\} \ , \tag{5.12}$$

where F_{av} comes from the definition of the average and G is the projection of $\mathcal{G}(x,z)$ in the x direction. The flow mapping of the average system, according to the average definition, is constructed by solutions of the boundary layer system so that fast dynamics of the original system are averaged into slow dynamics. As mentioned before, we consider the case when singular perturbations are continuously time-varying, and then the jump mapping of the average system is consistent to the jump dynamics for the slow state x of the original system.

Before analyzing the singularly perturbed system H_ε through its average system H_{av}, we assume that a well-defined average is admitted by H_ε.

Assumption 5.3.3. *The function $f_0 : C \times \Psi \to \mathbb{R}^n$ admits an outer semi-continuous, locally bounded and convex set-valued average mapping $F_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with respect to $\psi_0 : C \times \Psi \to \mathbb{R}^m$ on the set $C \times \Psi$.* □

We provide some sufficient conditions such that Assumption 5.3.3 holds in Lemma 5.3.5. We show that the existence of the average can be guaranteed with the conditions listed in Assumption 5.3.4 and Lemma 5.3.5. The idea of Lemma 5.3.5 is implicit in the results of [151] and the proof is provided in the Appendix D.2.

Assumption 5.3.4. *For a given compact set $\Omega \subset C \times \Psi$, there exist an outer semi-continuous, locally bounded and convex set-valued mapping $F_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a class-$\mathcal{L}$ function σ_Ω such that, for each $L > 0$, $(x, z_{bl}(0)) \in \Omega$ and function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $\dot{z}_{bl} = \psi_0(x, z_{bl})$ there exists a measurable function $f_{z_{bl}} : [0, L] \to \mathbb{R}^n$ such that $f_{z_{bl}}(s) \in F_{av}(x)$ for all $s \in [0, L]$ and the following holds:*

$$\left| \frac{1}{L} \int_0^L [f_0(x, z_{bl}(s)) - f_{z_{bl}}(s)] ds \right| \leq \sigma_\Omega(L) \ .$$

□

Lemma 5.3.5. *Suppose that the singularly perturbed system H_ε in (5.6) satisfies Assumptions 5.3.1. Assumption 5.3.3 holds if for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $\Omega \subset (C \cap K) \times \Psi$ such that Assumption 5.3.4 holds and Ω is globally asymptotically stable for the boundary layer system in (5.9) with C replaced with $C \cap K$.* □

We next present a few examples to show some situations when Assumption 5.3.3 holds. For instance, it holds in the case when the boundary layer system H_{bl} has a globally asymptotically stable quasi-steady state equilibrium manifold

$h : C \to \Psi$, which is an essential assumption for classical singular perturbation theory. The existence of the average $F_{av} : C \rightrightarrows \mathbb{R}^n$ of the $f_0 : C \times \Psi \to \mathbb{R}^n$ with respect to $\psi_0 : C \times \Psi \to \mathbb{R}^m$ for the perturbed system H_ε in (5.8) is considered as an example.

Example 5.3.6. *To show the existence of the average of f_0 with respect to ψ_0 when the boundary layer system globally asymptotically converges to an equilibrium manifold, we assume that the following conditions hold. The functions f_0 and ψ_0 are assumed to be continuous. Moreover, there exists a continuous function $h : C \to \Psi$ such that, for each compact set $K \subset \mathbb{R}^n$ and $x \in C \cap K$, the compact set $\Omega := \{(x, z_{bl}) : x \in C \cap K, z_{bl} = h(x)\}$ is globally asymptotically stable for the boundary layer system H_{bl} formed in (5.9) with C replaced by $C \cap K$. Then, we show that the function $x \mapsto f_0(x, h(x))$ is the average of f_0 with respect to ψ_0.*

Based on Lemma 5.3.5, we just need to show that Assumption 5.3.4 holds. From global asymptotic stability of Ω, in particular, its forward invariance, we have that if $(x, z_{bl}(0)) \in \Omega$ then $z_{bl}(s) = h(x)$ for all $s \geq 0$. It is then immediate that Assumption 5.3.4 holds for $f_{av}(x) = f_0(x, h(x))$ using $\sigma_\Omega(L) \equiv 0$.

The following example shows existence of the average of f_0 with respect to ψ_0 if the dynamics of the boundary layer system follow an oscillator that converges to a stable limit cycle. Moreover, we revisit this example in Section 5.5 to illustrate that we can get stronger results than [135] by using averaging in the context of singularly perturbed hybrid systems.

Example 5.3.7. *Consider a singularly perturbed system with $x \in \mathbb{R}$ and $z \in \mathbb{R}^2$:*

$$\left.\begin{aligned} \dot{x} &= \varepsilon f_0(x, z) \\ \dot{z} &= \psi_0(z) + \varepsilon \psi_1(x, z) \end{aligned}\right\} \qquad (x, z) \in C \times \Psi \ , \tag{5.13}$$

where $C := \{x : x \geq 0\}$, Ψ is a compact set satisfying $\mathbb{S}^1 \subset \Psi \subset \mathbb{R}^2 \setminus \{0\}$ with $\mathbb{S}^1$ being the unit circle, $\psi_1 : C \times \Psi \to \Psi$ is locally bounded, and

$$\begin{aligned} f_0(x, z) &:= -(0.5x + xz_1) \ , \\ \psi_0(z) &:= \begin{bmatrix} z_1 - z_2 - z_1\sqrt{z_1^2 + z_2^2} \\ z_1 + z_2 - z_2\sqrt{z_1^2 + z_2^2} \end{bmatrix} \ . \end{aligned} \tag{5.14}$$

It is convenient to consider the dynamics $\dot{z} = \psi_0(z)$ in polar coordinates with

$z_1 = \rho \sin\theta$ and $z_2 = \rho\cos\theta$: $\dot{\rho} = \rho(1-\rho)$ and $\dot{\theta} = 1$, which shows that the solution of $\dot{z} = \psi_0(z)$ is an oscillator that asymptotically converges to a limit cycle on the unit circle $\mathbb{S}^1 \subset \Psi$.

For each compact $K \subset \mathbb{R}$, let $\Omega := (C \cap K) \times \mathbb{S}^1$. Noting that Ω is globally asymptotically stable for the boundary layer system H_{bl} of (5.13) with C replaced by $(C\cap K)$, the solution z_1 of system H_{bl} with $(x, z(0)) \in \Omega$ satisfies $z_1(t) = \sin(t+\phi)$, where the parameter ϕ is determined by the initial condition of z. Then, for each $M > 0$, $L > 0$, $x \in [-M, M]$ and solution $z_1 : [0, L] \mapsto \Psi$ of system H_{bl} satisfying $(x, z(0)) \in \Omega$, it follows that,

$$\begin{aligned}\left|\frac{1}{L}\int_0^L [-(0.5x + xz_1(s)) + 0.5x]ds\right| &\leq \frac{|x|}{L}\left|\int_0^L \sin(s+\phi)ds\right| \\ &\leq \frac{2M}{L} := \sigma_\Omega(L)\ ,\end{aligned}$$

where the last inequality holds since the integration of a sinusoid function over the time less than one period is less than 2. From Lemma 5.3.5 and the fact that σ_Ω is of class-$\mathcal{L}$, we have that $F_{av}(x) := -0.5x$ is the average of f_0 with respect to ψ_0 in (5.13) on $C \times \Psi$. □

Note that global asymptotic stability of the boundary layer system in Example 5.3.6 or the condition that the boundary layer system converges to a stable limit cycle in Example 5.3.7 is not necessary for the existence of an average. To illustrate this, we revisit Example 5.3.7 and redefine $\psi_0 : \Psi \to \Psi$ such that the boundary layer system contains equilibria that are neither stable nor attractive but the average is equal to what would be obtained by restricting the initial conditions of the boundary layer system to these equilibria. Note that the average F_{av} agrees with continuous functions in the above two examples. On the other hand, it is a set-valued mapping in the following example.

Example 5.3.8. *Consider the singularly perturbed system (5.13) with replacing ψ_0 in (5.14) as*

$$\psi_0(z) := \begin{bmatrix} -\frac{z_1\left(\sqrt{z_1^2+z_2^2}-1\right)^3}{\sqrt{z_1^2+z_2^2}} + z_2(z_1+z_2-1)^2 + z_2\left(1-\sqrt{z_1^2+z_2^2}\right)^2 \\ -\frac{z_2\left(\sqrt{z_1^2+z_2^2}-1\right)^3}{\sqrt{z_1^2+z_2^2}} - z_1(z_1+z_2-1)^2 - z_1\left(1-\sqrt{z_1^2+z_2^2}\right)^2 \end{bmatrix}. \quad (5.15)$$

Like in Example 5.3.7, the set Ψ is a compact set satisfying $\mathbb{S}^1 \subset \Psi \subset \mathbb{R}^2 \setminus \{0\}$.

Noting the dynamics $\dot{z} = \psi_0(z)$ in polar coordinates:

$$\begin{aligned} \dot{\rho} &= -(\rho - 1)^3 \ , \\ \dot{\theta} &= (\rho \sin(\theta) + \rho \cos(\theta) - 1)^2 + (1 - \rho)^2 \ , \end{aligned} \tag{5.16}$$

we know that θ is unbounded when the solution of $\dot{z} = \psi_0(z)$ starts off the unit circle $\mathbb{S}^1$, since the first term of righthand of dynamics of θ is positive and second term is not integrable. Noting that $\dot{\theta} > 0$ and $\theta(t)$ is unbounded when t grows, we know that the equilibria $(0, 1)$ and $(1, 0)$ of the boundary layer system H_{bl} are neither stable nor attractive for any solution of system H_{bl} that starts in $\Psi \setminus \mathbb{S}^1$.

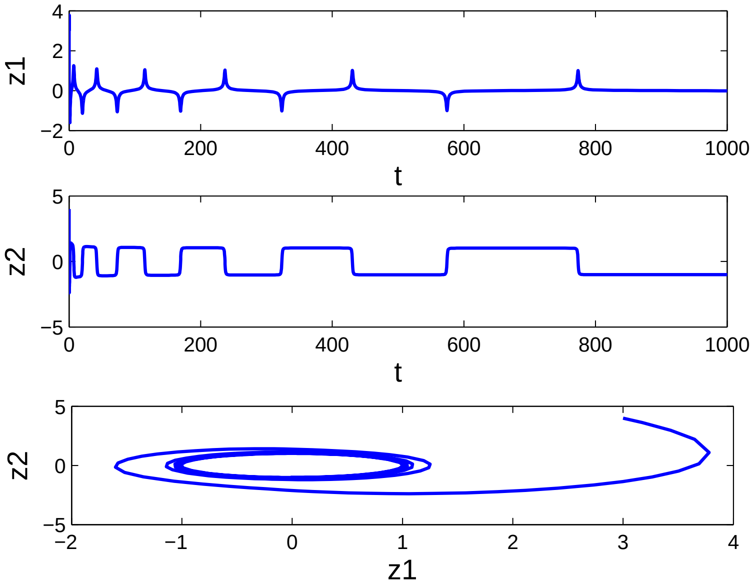

Figure 5.1: Trajectories of the solution z of $\dot{z} = \psi_0(z)$.

From Fig. 5.1, we can see that solutions of $\dot{z} = \psi_0(z)$ that start off the unit circle $\mathbb{S}^1$, tend toward $\mathbb{S}^1$ while rotating in the counterclockwise direction with motion that becomes arbitrarily slow at points arbitrarily close to equilibria $(0, 1)$ or $(1, 0)$. For each $M > 0$, let $K := [-M, M]$ and $\Omega := (C \cap K) \times \mathbb{S}^1$. It is clear that Ω is globally asymptotically stable for the boundary layer system with C replaced by $C \cap K$. We next consider if there exists a $\sigma_\Omega \in \mathcal{L}$ such that Assumption 5.3.4 holds to invoke Lemma 5.3.5.

Note that the boundary layer system (5.16) degenerates into $\dot{\theta} = \phi(\theta)$ when $z(0) \in \mathbb{S}^1$ with $\phi(\theta) := (\sin(\theta) + \cos(\theta) - 1)^2$. Moreover, each solution z_1 of system H_{bl} with $(x, z(0)) \in \Omega$ has the form of

$$z_1(t) = \sin(\theta(t, \theta(0))) \tag{5.17}$$

where $\theta(0)$ *is determined by the initial condition* $z(0)$*. As the function* $\phi(\theta)$ *is periodic in* θ *with the period* 2π*, we consider the solution* z_1 *for* $\theta(0) \in [0, 2\pi]$*.*

Let $\Theta := [0, \pi/2]$ *and* $\Omega_1 := [-M, M] \times \{\rho = 1, \theta \in \Theta\}$*. We first check if Assumption 5.3.4 holds for all* $(x, z(0)) \in \Omega_1 \subset \Omega$*. Note that* $\theta = 0$ *and* $\theta = \pi/2$ *are the equilibria of* $\dot{\theta} = \phi(\theta)$*, for which the solution* $\theta(t, \theta(0))$ *is forward invariant in the set* Θ*. Then, the solution* z_1 *in (5.17) picks value in* $[0, 1]$ *and the function* $f_0(x, z)$ *in (5.13) satisfies* $f_0(x, z(s)) \in [-0.5x, -1.5x]$ *for all solutions* (x, z) *of system* H_{bl} *with* $(x, z(0)) \in \Omega_1$*. Let* $F_{av}(x) := [-0.5, -1.5]x$*. Then, for each solution* z *of system* H_{bl} *of (5.13) starting on the set* Ω_1*, we can always find* $f_{z_{bl}}(s) \in F_{av}(x)$ *such that Assumption 5.3.4 holds for arbitrary class-*$\mathcal{L}$ *function* σ_Ω*.*

We next consider Assumption 5.3.4 when $(x, z(0)) \in \Omega \setminus \Omega_1$*. Note that* $\Omega \setminus \Omega_1 = [-M, M] \times \{\rho = 1, \theta \in (\pi/2, 2\pi)\}$*. For arbitrarily small* $\delta \in (0, 0.5]$ *and* $\theta(t, \theta(0))$ *satisfies* $\dot{\theta} = \phi(\theta)$*, let*

$$\begin{aligned} \vartheta &:= \arcsin(\delta) \in [0, \pi/6] \\ T(\delta) &:= \{t : \theta(t, \vartheta + \pi/2) = 2\pi - \vartheta\} , \end{aligned} \tag{5.18}$$

where $T(\delta)$ *is the time for function* $\theta(t, \theta(0))$ *going through that starts from* $\theta(0) > \pi/2$ *in the neighborhood of* $\pi/2$ *and reach some point that is less than but quite close to* 2π*.*

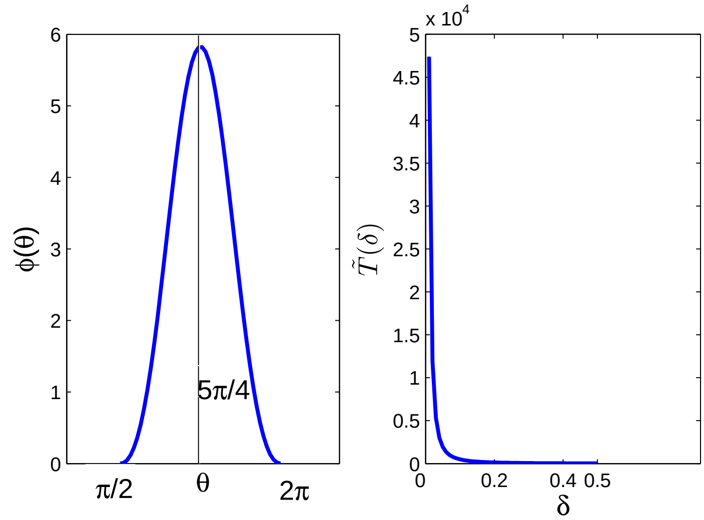

Figure 5.2: Functions $\phi(\theta)$ and $\tilde{T}(\delta)$.

Noting $\phi : (\pi/2, 2\pi) \to [0, (\sqrt{2}+1)^2]$ *is symmetric about* $\theta = 5\pi/4$ *and is* *strictly increasing on* $[\pi/2 + \vartheta, 5\pi/4]$*, see Fig. 5.2, we can get that* $\dot{\theta} \geq \phi(\vartheta)$ *for* $\theta \in [\pi/2 + \vartheta, 5\pi/4]$*. Since* $\phi(\vartheta) = 2(1-\delta)(1-\sqrt{1-\delta^2})$*, we have* $T(\delta) \leq \frac{3\pi/4 - \arcsin(\delta)}{(1-\delta)(1-\sqrt{1-\delta^2})} := \tilde{T}(\delta)$*. Noting that* $\tilde{T} : (0, 0.5] \to \mathbb{R}_{>0}$ *is continuous, strictly decreasing, bounded away from zero and* $\lim_{\delta\to 0} \tilde{T}(\delta) = \infty$*, there exists a* $\alpha \in \mathcal{K}_\infty$ *such that* $\alpha(1/\delta) = \tilde{T}(\delta)$ *for* $\delta \in (0, 0.5]$*.*

For each solution of $\dot{\theta} = \phi(\theta)$ *with* $\theta(0) \in (\pi/2, 2\pi)$*, let* $T_1 := \{t \geq 0 : \theta(t) = \pi/2 + \vartheta\}$ *and* $T_1 := 0$ *if* $\theta(t) \geq \pi/2 + \vartheta$ *for all* $t \geq 0$*, let* $T_2 \geq T_1 := \{t \geq 0 : \theta(t) = 2\pi - \vartheta\}$ *and* $T_2 := T_1$ *if* $\theta(t) \geq 2\pi - \vartheta$ *for all* $t \geq 0$*. Noting that* $T_2 - T_1 \leq T_\delta$ *in (5.18), we have for each* $L > 0$*:*

$$\begin{aligned} \frac{1}{L}\int_0^L |\sin(\theta(s))| ds &\leq \frac{1}{L}\int_0^{T_1} 1 ds + \int_{T_1}^{T_2} 1 ds + \int_{T_2}^{L} \delta ds \\ &\leq \frac{T_1}{L} + \frac{\tilde{T}(\delta)}{L} + \delta = \frac{T_1}{L} + \frac{\alpha(1/\delta)}{L} + \delta \ . \end{aligned} \tag{5.19}$$

Noting that (5.19) holds for arbitrary $\delta \in (0, 1/2]$*, it holds for*

$$\delta = \min\left\{0.5, \ \frac{1}{\alpha^{-1}\left(\sqrt{L}\right)}\right\} \ . \tag{5.20}$$

Let $c := \frac{T_1}{L}$ *and note that* $c \in [0,1]$ *from the definition of* L *and* T_1 *in (5.19). Then, we have for each* $M > 0$*,* $L > 0$*,* $x \in [-M, M]$ *and solution* $z_1 : [0, L] \mapsto \Psi$ *of system* H_{bl} *satisfying* $(x, z(0)) \in (\Omega \setminus \Omega_1)$*, the function* $f_{z_{bl}}(s) = -(0.5 + c)x \in F_{av}(x)$ *satisfies:*

$$\begin{aligned} &\left|\frac{1}{L}\int_0^L [-(0.5x + xz_1(s)) - f_{z_{bl}}(s)] ds\right| \\ &= \left|\frac{1}{L}\int_0^L [-(0.5x + xz_1(s)) + (0.5 - c)x] ds\right| \\ &\leq |x| \left(\frac{1}{L}\int_0^L |\sin(\theta(s))| ds - c\right) \\ &\leq M\left(\frac{\max\left\{\alpha(2), \sqrt{L}\right\}}{L} + \min\left(0.5, \frac{1}{\alpha^{-1}\left(\sqrt{L}\right)}\right)\right) \\ &:= \sigma_\Omega(L) \ . \end{aligned}$$

The function σ_Ω *defined by the inequalities above is of class-*$\mathcal{L}$*, which together* *with Lemma 5.3.5 shows that the set-valued mapping* $F_{av}(x) := [-1.5, -0.5]x$ *is average of system (5.13) when* ψ_0 *in (5.14) is replaced with (5.15).* □

To illustrate how to get the jump mapping G of the averaged system from $\mathcal{G}$ of the actual hybrid system, a simple example is given.

Example 5.3.9. *Consider the hybrid system* H_ε *with the data* $(f_0, \psi_0, \mathcal{G}, C, D, \Psi)$ *formed as (5.8) with* C, Ψ, f_0 *given in (5.13), and for some* $\gamma > 0$ *with* $\mathcal{G}, D$ *being defined as*

$$\mathcal{G}(x,z) := \begin{bmatrix} -\gamma x + z_1^2 \\ g(x,z) \end{bmatrix} \quad D := \{x : x \leq 0\} \tag{5.21}$$

where $g(x,z)$ *is an arbitrary function. Noting the definition of* G *in (5.12) and the average of* f_0 *with respect to* ψ_0 *on* $C \times \Psi$ *from Examples 5.3.7 and 5.3.8, we get the average of the hybrid system* H_ε *with* ψ_0 *in (5.14) is*

$$\begin{array}{rcll} \xi' & = & -0.5\xi & \xi \in C \\ \xi^+ & \in & -\gamma\xi + [c_3, c_4] & \xi \in D \ , \end{array} \tag{5.22}$$

and for ψ_0 *in (5.15) is*

$$\begin{array}{rcll} \xi' & \in & [-1.5, -0.5]\xi & \xi \in C \\ \xi^+ & \in & -\gamma\xi + [c_3, c_4] & \xi \in D \ , \end{array} \tag{5.23}$$

where the positive real numbers $c_3 := \min_{z\in\Psi}\{z_1^2\}$ *and* $c_4 := \max_{z\in\Psi}\{z_1^2\}$. □

5.4 Coordinate transformation

To employ a coordinate transformation, a continuous function that reflects accumulating errors between the actual system and its average is usually constructed to facilitate averaging techniques, see early works in [22] for adaptive control systems, general averaging theory for continuous-time systems [74] and for hybrid dynamical systems [154]. In the present section, we provide some intuition on how to construct a function η that facilitates the averaging method and use coordinate transformations to relate the perturbed hybrid system H_ε with its average system H_{av}. The property of η is given in Lemma 5.4.1 that is essential to prove the main results presented in Section 5.5.

The function η used in the coordinate transformation is constructed by augmenting the actual time-varying hybrid system H_ε. Let the set valued mapping F_{av} come from Definition 5.3.2. For a $\mu \geq 0$, let

$$F_{av}^{\mu}(x) := \overline{con} F_{av}((x + \mu\mathbb{B}) \cap C) + \mu\mathbb{B} \ .$$

We intersect the sets C, D with K and augment the perturbed system H_ε in (5.8) with $\mu \geq 0$ and the state $\eta \in \mathbb{R}^n$ to obtain the following system:

$$H_K \quad \begin{array}{l} \left.\begin{array}{l} x' = f(x, z, \varepsilon) \\ z' = \frac{1}{\varepsilon}\psi(x, z, \varepsilon) \\ \eta' \in \frac{1}{\varepsilon}[f(x, z, \varepsilon) - F_{av}^{\mu}(x) - \mu\eta] \end{array}\right\} \quad \forall\ (x, z, \eta) \in (C \cap K) \times \Psi \times \mathbb{R}^n \\ \left.\begin{array}{l} (x, z)^+ \in \mathcal{G}(x, z) \\ \eta^+ = 0 \end{array}\right\} \quad \forall\ (x, z, \eta) \in (D \cap K) \times \Psi \times \mathbb{R}^n \ . \end{array} \tag{5.24}$$

For each solution (x, z, η) of H_K, letting $x = \bar{x} + \varepsilon\eta$ and noting $\eta^+ = 0$, it follows that

$$\begin{array}{rcl} \bar{x}' & \in & F_{av}^{\mu}(x) + \mu\eta \ , \\ \bar{x}^+ & = & x^+ \ . \end{array}$$

Considering the definition of G in (5.12), we have

$$\begin{array}{ll} \bar{x}' \in F_{av}^{\mu}(\bar{x} + \varepsilon\eta) + \mu\eta & (\bar{x} + \mu\eta) \in C \\ \bar{x}^+ \in G(\bar{x} + \varepsilon\eta) & (\bar{x} + \mu\eta) \in D \ . \end{array} \tag{5.25}$$

Note that all solutions x of the system H_K are constrained to the compact set $K \cup G(K \cap D)$, with the compactness coming from the semi-continuity of the jump mapping G. From (5.25), we know that if μ, ε can be chosen sufficiently small so that there exists a solution η of H_K such that μ, $\mu|\eta(t, j)|$ and $\varepsilon|\eta(t, j)|$ are bounded by any given $\delta/2 > 0$, noting that $F_{av}^{\delta/2}(x + \delta/2) \in F_{av}^{\delta}(x)$ from (5.3) and definition of G_δ in (5.4), then we get a system that is inflated from the average system of H_K by δ:

$$\begin{aligned} &\bar{x}' \in F^{\delta}_{av}(\bar{x}) && \bar{x} \in C_{\delta} \\ &\bar{x}^{+} \in G(\bar{x} + \delta\mathbb{B}) \subset G_{\delta}(\bar{x}) && \bar{x} \in D_{\delta}\ . \end{aligned} \tag{5.26}$$

The requirements on $\mu|\eta|$ and $\varepsilon|\eta|$ are established in the following results. The proof of Lemma 5.4.1 is given in the Appendix D.1 while Corollary 5.4.2 follows from Lemma 5.4.1 and the discussion above.

Lemma 5.4.1. *Suppose that Assumption 5.3.1 holds for the singularly perturbed system H_ε in (5.8). Then, for any $\nu > 0$ and compact set $K \subset \mathbb{R}^n$ there exists $(\mu, \varepsilon^*) > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, each solution (x, z) of the system H_K in (5.24) there exists a solution η of H_K with $\eta\,(0,0) = 0$ that satisfies*

$$\mu|\eta(t,j)| \le \nu, \qquad \forall\ (t,j) \in \text{dom}\ (x,z). \tag{5.27}$$

□

Corollary 5.4.2. *Suppose that Assumption 5.3.1 holds for the singularly perturbed system H_ε in (5.8). Then, for any $\delta > 0$ and any compact set $K \subset \mathbb{R}^n$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and each solution (x, z) of the system H_K in (5.24) there exists a solution η of H_K with $\eta\,(0,0) = 0$ such that $\varepsilon|\eta(t,j)| \le \delta/2$ for all $(t,j) \in \text{dom}\ (x,z)$ and $\bar{x} := x - \varepsilon\eta$ is the solution of the system H_δ in (5.2) inflated from the average system H_{av} of H_ε.* □

Using Lemma 5.2.4 and forward pre-completeness of the average system H_{av} of the perturbed system H_ε, we can consider closeness between solutions of H_ε and solutions of system H_{av}. Similarly, with Lemma 5.2.5, we can consider the stability properties of H_ε assuming global asymptotic stability of its average system H_{av}. Note that the compact set K that is used to define the augmented system H_K can be constructed using the forward pre-completeness and global asymptotic stability of H_{av} respectively.

5.5 Main results

We first present results on closeness of the slow solutions x of the singularly perturbed system H_ε to the solutions of its average system H_{av} on compact time domains in Theorem 5.5.1, under the assumption that the system H_{av} is forward pre-complete from a given compact set. We also consider stability properties of the actual hybrid system H_ε based on global asymptotic stability of its average system H_{av}. After that, we consider a continuous-time plant with a hybrid controller implemented through a fast actuator as an example to illustrate the main results of the present chapter.

Theorem 5.5.1. *Suppose that the singularly perturbed system H_ε in (5.6) satisfies Assumptions 5.3.1 and 5.3.3 and that its average system H_{av} defined in (5.1), (5.11) and (5.12) is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J, there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$ and each solution x to system H_ε with $x(0,0) \in K_0$, there exists some solution ξ to system H_{av} with $\xi(0,0) \in K_0$ such that x and ξ are (T, J, ρ)-close.* □

Proof. Let the compact set $K_0 \subset \mathbb{R}^n$, $T, J > 0$ and $\rho \in (0,1)$ be given. From Lemma 5.2.4, the set K_0 with T, J and $\frac{\rho}{2}$ generate a $\delta^* \in (0, \rho/2)$ such that, for all $\delta \in (0, \delta^*]$ and for any solution $\bar{x}$ with $\bar{x}(0,0) \in K_0 + \delta\mathbb{B}$ of system H_δ in (5.2) inflated from the average system H_{av} defined in (5.1), (5.11) and (5.12), there exists a solution ξ with $\xi(0,0) \in K_0$ of system H_{av} such that ξ and $\bar{x}$ are $\left(T, J, \frac{\rho}{2}\right)$-close. Consider a $\delta \in (0, \delta^*]$.

Let $S_{av}(K_0)$ denote the set of maximal solutions to the average system H_{av} in (5.1), (5.11) and (5.12) with $\xi(0,0) \in K_0$. Define

$$\begin{aligned} R_{av}(K_0, T, J) &:= \{\xi(\tau, j) : \ \xi \in S_{av}(K_0), \ \tau \le T, \ j \le J\} \\ K &:= (R_{av}(K_0, T, J) + \mathbb{B}) \cup G((R_{av}(K_0, T, J) + \mathbb{B}) \cap D) \ , \end{aligned} \tag{5.28}$$

where K is compact from Proposition 5.2.3 and the outer semi-continuity and local boundedness of the jump map G.

Let $\varepsilon_1^* > 0$ and $\mu > 0$ be generated by Lemma 5.4.1 with the compact set K and δ. Let $\varepsilon^* := \min\{\varepsilon_1^*, \mu\}$ and consider an $\varepsilon \in (0, \varepsilon^*]$. Then, Corollary 5.4.2 shows that, for each solution (x, z, η) of system H_K with $(x(0,0), z(0,0)) \in K_0 \times \Psi$ and $\eta(0,0) = 0$, $\bar{x}(\tau, j) = x(\tau, j) - \varepsilon\eta(\tau, j)$ is a solution of the system H_δ in (5.26) inflated from H_{av}. From the fact that $\delta < \rho/2$, x is $\frac{\rho}{2}$ close to $\bar{x}$ and then it is (T, J, ρ)-close to ξ, the latter being a solution of the average system starting in K_0.

Now, consider a solution $\tilde{x}$ of the system H_ε in (5.6) with $(\tilde{x}(0,0), \tilde{z}(0,0)) \in K_0 \times \Psi$. According to the discussion above, if $\tilde{x}(\tau, j) \in K$ for all $(\tau, j) \in \text{dom}\,(\tilde{x}, \tilde{z})$ with $\tau \le T$ and $j \le J$, then there exists a solution ξ of the average system H_{av} such that $\tilde{x}$ is also (T, J, ρ)-close to ξ. Otherwise, suppose that there exists

$(\tau, j) \in$ dom $(\tilde{x}, \tilde{z})$ such that $\tilde{x}(s, i) \in K$ for all $(s, i) \in$ dom $(\tilde{x}, \tilde{z})$ satisfying $s \leq \tau$, $i \leq j$ and either

Case 1 $(\tau, j+1) \in$ dom $(\tilde{x}, \tilde{z})$ and $\tilde{x}(\tau, j+1) \notin K$ or else,

Case 2 there exist a monotonically decreasing sequence $r_i > 0$ satisfying $\lim_{i\to\infty} r_i = \tau$, $(r_i, j) \in$ dom $(\tilde{x}, \tilde{z})$ and $\tilde{x}(r_i, j) \notin K$ for each i.

Note that the solution $\tilde{x}(s, i)$ must agree with a solution of system H_K up to time (τ, j). Because of the relationship stated in Corollary 5.4.2, between H_K and the average system, $\tilde{x}$ must satisfy $\tilde{x}(\tau, j) \in R_{av}(K_0, T, J) + \rho\mathbb{B}$. Then, for Case 1, using the definition of K above, $\tilde{x}(\tau, j+1) \in G((R_{av}(K_0, T, J) + \rho\mathbb{B}) \cap D) \subset K$, which is a contradiction since $\rho < 1$, while, for Case 2, there exists $\tilde{\rho} \in (\rho, 1)$ such that, for large i, $\tilde{x}(r_i, j) \in R_{av}(K_0, T, J) + \tilde{\rho}\mathbb{B} \subset K$, which is a contradiction. This establishes the result. □

Next, in Theorem 5.5.2, the stability properties of system H_ε are considered under a global asymptotic stability assumption on the average system H_{av}.

Theorem 5.5.2. *Suppose that the singularly perturbed system H_ε in (5.6) satisfies Assumptions 5.3.1 and 5.3.3 and the compact set $\mathcal{A}$ is globally asymptotically stable for its average system H_{av} defined in (5.1), (5.11) and (5.12) with respect to $\beta \in \mathcal{KL}$. Then, the compact set $\mathcal{A} \times \Psi$ is SGP-AS for system H_ε with respect to β.* □

Proof. Let $\nu \in (0, 1)$ and the compact set $K_0 \subset \mathbb{R}^n$ be given. Let $\beta \in \mathcal{KL}$, the compact set $\mathcal{A} \subset \mathbb{R}^n$ and the proper indicator function $\chi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for the set $\mathcal{A}$ come from the definition of global asymptotic stability for the average system H_{av}. Using Lemma 5.2.5, let $\frac{\nu}{3}$ and the compact set K_0 generate a $\delta > 0$ such that each solution $\bar{x}$ of system H_δ inflated from H_{av} with $\bar{x}(0, 0) \in K_0 + \delta\mathbb{B}$ satisfies

$$\chi(\bar{x}(\tau, j)) \leq \beta(\chi(\bar{x}(0, 0)), \tau + j) + \frac{\nu}{3} \quad \forall\ (\tau, j) \in \text{dom}\ \bar{x}\ . \tag{5.29}$$

Define

$$\begin{aligned} K_1 &:= \left\{ x \in \mathbb{R}^n : \chi(x) \leq \beta\left(\max_{\bar{x}\in K_0} \chi(\bar{x}), 0\right) + 1 \right\} \\ K &:= K_1 \cup G(K_1 \cap D)\ . \end{aligned} \tag{5.30}$$

The set K is compact because of continuity of the β, compactness of the set $\mathcal{A}$ and outer semi-continuity of the set mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

With the fact that for all $m \geq 0$, $\beta(m,s)$ converges to zero as $s \geq 0$ approaches infinity, let $\varepsilon_1^* > 0$ be such that, for all $x \in K$ and $\bar{x} \in K + \varepsilon_1^*\mathbb{B}$ satisfying $|x - \bar{x}| \leq \varepsilon_1^*$, the following holds:

$$\begin{aligned} \chi(x) &\leq \chi(\bar{x}) + \frac{\nu}{3} \\ \beta(\chi(\bar{x}), s) &\leq \beta(\chi(x), s) + \frac{\nu}{3}, \quad \forall\, s \in \mathbb{R}_{\geq 0}\,. \end{aligned} \tag{5.31}$$

System H_K defined in (5.24) is introduced. For each solution (x, z) of the system H_L, let Lemma 5.4.1 with the δ and the set K generate $\varepsilon_2^* > 0$, $\mu > 0$ and the solution η of system H_K. Let $\varepsilon^* := \min\{\varepsilon_1^*, \varepsilon_2^*, \mu\}$ and consider a $\varepsilon \in (0, \varepsilon^*]$. Then, Corollary 5.4.2 shows that for each $\varepsilon \in (0, \varepsilon_2^*]$ and solution (x, z, η) of system H_K with $(x(0,0), z(0,0)) \in K_0 \times \Psi$ and $\eta(0,0) = 0$, $\bar{x} := x - \varepsilon\eta$ is the solution to the inflated system H_δ, and then (5.29) holds.

Using (5.31), for all solutions (x, z) to system H_K with $(x(0,0), z(0,0)) \in K_0 \times \Psi$ and $(\tau, j) \in \text{dom}\,(x, z)$, we have

$$\begin{aligned} \chi(x(\tau, j)) &\leq \chi(\bar{x}(\tau, j)) + \frac{\nu}{3} \\ &\leq \beta(\chi(\bar{x}(0,0)), \tau + j) + \frac{2\nu}{3} \\ &\leq \beta(\chi(x(0,0)), \tau + j) + \nu\,. \end{aligned} \tag{5.32}$$

In particular, since $\nu < 1$, each solution to system H_K starting in K_0 remains in the compact set

$$K_\nu := \left\{ x \in \mathbb{R}^n : \ \chi(x) \leq \beta\left(\max_{\bar{x} \in K_0} \chi(\bar{x}), 0 \right) + \nu \right\}.$$

With $\nu < 1$, K_ν is contained in K defined in (5.30). Considering the solutions $(\tilde{x}, \tilde{z})$ of the perturbed system H_ε in (5.6) with $(\tilde{x}(0,0), \tilde{z}(0,0)) \in (C \cap K) \times \Psi$, we show that for the solution $\tilde{x}$ such that $\tilde{x}(s, i) \in K$ up to $s \leq \tau$, $i \leq j$ and Cases 1-2 in the proof of Theorem 5.5.1 are assumed to occur. As $\tilde{x}$ must agree with a solution of H_K up to time (τ, j). It then satisfies (5.32). Noting the definition of K in (5.30), neither of Cases 1-2 can occur, which proves the result. □

In the following example, we consider a continuous-time plant with a hybrid controller implemented through a fast actuator and with additive disturbances that are fast but have zero average. In this case, the averaging approach to singular perturbations, as studied in the current chapter, allows treating both disturbances within one framework to assess stability properties of the closed-loop system.

Example 5.5.3. *Consider a continuous-time system with states $\xi \in \mathbb{R}^n$, disturbances $w \in \mathbb{R}^m$ and parameter $\varepsilon > 0$:*

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)u + \ell(\xi)Qw \\ \varepsilon\dot{w} &= Sw \ , \end{aligned} \tag{5.33}$$

where functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times l}$ and $\ell : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous; $u \in \mathbb{R}^l$ is the control vector, Q is a constant matrix, the matrix S is stable such that $\dot{w} = Sw$ generates sinusoids or exponentially decaying sinusoids.

Let $\mathbb{Z}_1$ be a finite subset of the integers and $\kappa : \mathbb{R}^n \times \mathbb{Z}_1 \to \mathbb{R}^l$ be continuous. We consider the case that the control signals u are generated by fast actuators through $\epsilon\dot{\zeta} = A\zeta + B\kappa(\xi, q)$ and $u = C\zeta$ with a Hurwitz matrix A and matrices C and B of appropriate dimensions satisfying $-CA^{-1}B = 1$. Then, the overall dynamical system is

$$\left.\begin{aligned} &\dot{\xi} = f(\xi) + g(\xi)C\zeta + \ell(\xi)Qw \\ &\dot{q} = 0 \\ &\epsilon\dot{w} = Sw \\ &\epsilon\dot{\zeta} = A\zeta + B\kappa(\xi, q) \end{aligned}\right\} ((\xi, q), (w, \zeta)) \in \bar{C} \times \Psi$$
$$\left.\begin{aligned} &\xi^+ = \xi \\ &q^+ \in G(\xi, q) \\ &w^+ = w \\ &\zeta^+ = \zeta \end{aligned}\right\} ((\xi, q), (w, \zeta)) \in \bar{D} \times \Psi \ , \tag{5.34}$$

where $\zeta \in \mathbb{R}^l$, $\bar{C}$, $\bar{D}$ are closed subsets of $\mathbb{R}^n \times \mathbb{Z}_1$, Ψ is compact, G is outer semicontinuous, locally bounded and non-empty on $\bar{D}$.

Considering the results of Examples 5.3.6 and 5.3.8 and the fact that the sum of sinusoids has zero mean [74, Execise 10.12], we get that the average of system (5.34) is:

$$\left.\begin{aligned}\dot{\xi} &= f(\xi) + g(\xi)\kappa(\xi, q) \\ \dot{q} &= 0\end{aligned}\right\} \qquad (\xi, q) \in \bar{C}$$
$$\left.\begin{aligned}\xi^{+} &= \xi \\ q^{+} &\in G(\xi, q)\end{aligned}\right\} \qquad (\xi, q) \in \bar{D} \;. \tag{5.35}$$

Note that for the closed-loop original system (5.34), regularity conditions for $(G, \bar{C}, \bar{D})$ and continuity of the flow mapping required in Assumption 5.3.1 are satisfied. Then, we can directly apply Theorem 5.5.2 to conclude that if there exists a compact set $A_1 \subset \mathbb{R}^n$ such that $\mathcal{A} := A_1 \times \mathbb{Z}_1$ is globally asymptotically stable for the average system (5.35), then set $\mathcal{A}$ is semi-globally practically asymptotically stable the original system (5.34). □

In classical singular perturbation theory, say [16,74,158], the boundary layer system H_{bl} is assumed to have a globally asymptotically stable equilibrium manifold. Such an assumption is formulated as follows.

Assumption 5.5.4. *For the boundary layer system H_{bl} in (5.9), the function $h : C \to \Psi$ is continuous and for each compact set $K \subset \mathbb{R}^n$, the compact set*

$$\mathcal{M}_K := \{(x, z_{bl}) : x \in C \cap K, z_{bl} = h(x)\}$$

is globally asymptotically stable with respect to $\beta \in \mathcal{KL}$.

As shown in Example 5.3.6, Assumption 5.5.4 is sufficient to guarantee Assumption 5.3.3. On the other hand, as shown in Example 5.3.8, Assumption 5.5.4 is not necessary to guarantee Assumption 5.3.3. From Example 5.3.6, we know that the function $x \mapsto f_{av}(x) := f_0(x, h(x))$ is the average of f_0 with respect to ψ_0 for the system H_ε based on Assumption 5.5.4. Then, the average system $H_{av} := \{F, G, C, D\}$ of the perturbed system H_ε is formed as (5.1) with same G in (5.12) and

$$F(x) := f_0(x, h(x)), \quad \forall\, x \in C \;. \tag{5.36}$$

The following two corollaries follow directly from our main results. Note that these results are more general than [16,74,151,158] since the assumption of Lipschitz continuity for the functions f_0 and ψ_0 in [16,74,151,158] are not needed in the current chapter.

Corollary 5.5.5. *Suppose that the singularly perturbed system H_ε in (5.6) satisfies Assumptions 5.3.1 and 5.5.4 and its average system H_{av} defined in (5.1), (5.36) and (5.12) is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$ and each solution x to system H_ε with $x(0,0) \in K_0$ there exists some solution ξ to system H_{av} with $\xi(0,0) \in K_0$ such that x and ξ are (T, J, ρ)-close.* □

Corollary 5.5.6. *Suppose that the singularly perturbed system H_ε in (5.6) satisfies Assumption 5.3.1 and 5.5.4 and the compact set $\mathcal{A}$ is globally asymptotically stable for its average system H_{av} defined in (5.1), (5.36) and (5.12) with respect to $\beta \in \mathcal{KL}$. Then, the compact set $\mathcal{A} \times \Psi$ is SGP-AS for system H_ε with respect to β.* □

We next compare our results with [135], which considers a class of hybrid control systems singularly perturbed by fast but continuous actuators, where a reduced system that omits the actuator dynamics is used in analysis of stability properties of the actual system. To extend the classical singular perturbation theory to the hybrid setting, the equilibrium manifold in Assumption 5.5.4 is replaced by a set-valued mapping $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ in [135]. The closed-loop of the hybrid control system considered in [135] is formed as

$$\begin{aligned} \text{diag}\,(I_n, \varepsilon I_m)\, y' &\in F_1(y) \qquad y \in C \times \Psi \\ y^+ &\in \mathcal{G}(y) \qquad y \in D \times \Psi\,, \end{aligned} \tag{5.37}$$

where $y := (x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, I_n and I_m respectively denote the $n \times n$ and $m \times m$ identity matrices, $F_1 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$. In [135], the reduced system $H_r := \{F, G, C, D\}$ of the perturbed system (5.37) is defined as (5.1) with the set-valued mapping $\mathcal{H}$ and

$$\begin{aligned} F(x) &:= \overline{con}\{v_1 \in \mathbb{R}^n : (v_1, v_2) \in F_1(x, z), z \in \mathcal{H}(x), v_2 \in \mathbb{R}^m\}\,, \\ G(x) &:= \{v_1 \in \mathbb{R}^n : (v_1, v_2) \in \mathcal{G}(x, z), (z, v_2) \in \Psi \times \Psi\}\,. \end{aligned} \tag{5.38}$$

Note that G defined in (5.38) is a projection of $\mathcal{G}$ to the subspace of the slow state x, which is same as the definition of (5.12) for the average system, except that the image v_2 of jump mapping for fast states z is constrained to the compact set Ψ in (5.38) and v_2 is allowed to pick values in $\mathbb{R}^m$ for our results. Comparing with [135], the results of this chapter give sharper results in some cases and we revisit Examples 5.3.7 and 5.3.8 with the result of Example 5.3.9 to illustrate this.

Example 5.5.7. *Consider the hybrid system $H_\varepsilon := (f, \psi, \mathcal{G}, C, D, \Psi)$ formed as (5.8) with with f, C, Ψ given in (5.13), ψ in (5.14) or (5.15), and $\mathcal{G}, D$ defined in (5.21). From the definition of the reduced system in (5.38) given in [135], noting the fact that the boundary layer system of system (5.13) converges to a stable limit cycle on $\mathbb{S}^1$ and letting c_3 and c_4 come from Example 5.3.9, the reduced system of [135] is*

$$\begin{aligned} \xi' &\in -0.5\xi + |\xi|\mathbb{B} && \xi \in C \\ \xi^+ &\in -\gamma\xi + [c_3, c_4] && \xi \in D \ . \end{aligned} \tag{5.39}$$

Note that there are solutions for the reduced system (5.39) that exponentially grow unbounded.

From the average definition, we get that the average system of system H_ε is formed as (5.22) or (5.23) in Example 5.3.9. Recalling that $C := \{\xi : \xi \geq 0\}$ in (5.13), the jump mapping makes all solutions starting from the set D go back to the flow set C and we know that the flow dynamics globally exponentially converge to the origin from (5.22). Then, we can conclude stability of system H_ε via stability of its average system (5.22) using Theorem 5.5.2, but we cannot draw this stability conclusion from the reduced system (5.39). □

5.6 Conclusions

We analyzed the properties of a class of hybrid dynamical systems using singular perturbations theory and the averaging method. We showed that if there exists a well defined average for the actual perturbed hybrid system, the slow solutions of the actual system are arbitrarily close to the solution of the average system that approximates the slow dynamics of the actual system on compact time domains for arbitrarily small values of the singular perturbation parameter. We also showed that global asymptotic stability of a compact set for the average system implies that the set is semi-globally practically asymptotically stable for the actual perturbed system. Through several examples, we showed that the condition to guarantee the existence of the average is not stringent and our results are more general than the classical singular perturbation theory, where the asymptotic stability of the boundary layer system or local Lipschitz continuity of the vector fields are assumed.

Part III

Future Work

Chapter 6

Summary and future work

We first summarize main results of this dissertation and emphasize their contributions. In the second section, we propose some topics for further research.

6.1 Summary of the thesis

Averaging and singular perturbation techniques were utilized to consider several classes of dynamical systems in this thesis. We extended the classical averaging theory to consider new addressed classes of systems: the parameterized discrete-time systems, switched systems and hybrid systems. For input signals with different properties, we studied the trajectories and analyzed the robustness to disturbances of these classes of dynamical systems using the concepts of strong and weak averages. We also combined the averaging method and the singular perturbation technique for analysis of asymptotic stability for a class of hybrid dynamical systems. Our contributions in Chapters 2-5 are discussed below in more detail.

In Chapter 2, for the family of parameterized discrete-time systems with bounded disturbances, we showed that if the strong average system is forward complete, its solutions can be made arbitrarily close to solutions of the actual system on compact time intervals. The same conclusion holds if the weak average system is forward complete and the actual system is affected by bounded disturbances that also have bounded derivatives. We also showed that ISS of the strong averages implies semi-global practical ISS (SGP-ISS) of the actual family of parameterized discrete-time systems, and ISS of the weak averages implies semi-global practical derivative ISS (SGP-DISS) of the actual systems instead. For the case of no disturbance, our results show that strong and weak averages coincide and if such a average system is globally asymptotically stable then the actual system is semi-globally practically asymptotically stable. In other words, the bounds of the solutions of the actual system, have an additive offset that becomes arbitrarily small and the estimate of the domain of attraction can be arbitrarily enlarged by tuning certain parameter of the actual system. To the best of our knowledge, this result is weaker than the prior results on averaging of discrete-time systems where exponential stability is typically assumed.

Together with the results in [109], the averaging results for families of parameterized discrete-time systems given in Chapter 2 can be used to design controllers achieving ISS for nonlinear sampled-data systems, for which the exact discrete-time model can not be analytically computed and we have to use an approximate discrete-time model for controller design and stability analysis. For approximate discrete-time models that are time-varying, it is still challenging to design a controller based on this model. Through an example of a Duffing oscillator, we showed that one can design a controller based on its time-invariant average systems to achieve ISS for the closed-loop system such that the actual approximate model of the sample-data system is SGP-ISS. Then, the results in [109] imply that the original sampled-data system has similar stability properties.

In Chapter 3, both nonlinear and linear switched systems are considered. For nonlinear switched systems, we presented conditions under which ISS of the strong average implies SGP-ISS of the actual switched system. We also showed a SGP-DISS property using the notion of weak average and requiring derivatives of disturbances to be bounded. For the linear switched system, we adapted strong and weak average definitions from [110] and introduced a partial strong average to consider its robust stability when the average system is ISS with an exponential $\mathcal{KL}$ estimate and a linear gain.

We proved that exponential ISS of the strong and the partial strong average system implies exponential ISS for the actual linear switched system with their estimated linear gains converging to each other as the parameter is reduced. Moreover, exponential ISS of the weak average guarantees an appropriate DISS property for the actual system. Our results provide stronger conclusions than the prior results on the same topic [131] as we show that the estimated linear gain of the linear switched system converges to the estimated linear gain of the strong or partially strong average. Moreover, an estimate of ISS gain of the actual system also can be recovered by its weak average when switching rate is large enough if we restrict the derivatives of disturbances to be uniformly bounded.

One contribution of these results is a systematic use of strong, partial strong and weak averages for switched systems with disturbances that we believe will be very useful in a range of other averaging questions for switched systems. The results we obtained in Chapter 3 can be applied to analyze global stability or robustness to disturbances for various applications [20, 124, 126, 147].

In Chapter 4, we consider ISS properties for a class of hybrid systems using the averaging method that had not been addressed before. Using the notions of strong and weak average, a time-varying hybrid system is approximated by a time-invariant hybrid system. For hybrid systems with bounded input signals, we presented results on closeness of solutions between the actual system and solutions of its strong average with the assumption that this strong average system is forward complete. We obtained this result with the weak average system assuming its forward completeness for bounded input signals that have bounded derivatives. We also showed that ISS of the strong average implies SGP-ISS of the actual system. In a similar fashion, ISS of the weak average implies SGP-DISS of the actual system.

The results in Chapter 4 can be used as an analysis tool for closed-loop of pulse-width-modulated (PWM) hybrid feedback control systems. Indeed, the averaging results for hybrid systems in Chapter 4 shown to be useful when designing hybrid feedbacks that are implemented via PWM. For the hybrid feedback controller that is applied directly to a continuous-time plant and the closed-loop system is stable, our results show that the same controller implemented via PWM can also stabilize the actual system. A simple power converter example is used to illustrate how to apply the averaging results for hybrid systems for controllers design.

In Chapter 5, we considered a class of hybrid systems singularly perturbed by fast but continuous actuators. Combining both the averaging and the singular perturbation techniques, we studied the properties of this class of hybrid systems based on its reduced hybrid system, which is defined by the average vector field for the slow dynamics that is generated by solutions of continuous-time boundary layer dynamics, the projection of the jump map in the direction of the slow states, and flow and jump sets from the original dynamics. With this reduced hybrid model, we presented the results that for each solution of the slow dynamics of the actual singularly perturbed system, there exits a solution of its average system such that they are arbitrarily close for small enough values of the singular perturbation parameter. Moreover, we also showed that the global asymptotic stability of a compact set for the average system implies that the set is semi-globally practically asymptotically stable for the actual perturbed system.

The results in Chapter 5 are more general than classical singular perturbation theory as they weaken the typical assumptions that require the existence of an asymptotically stable manifold for the boundary layer system and local Lipschitz continuity of the vector fields. We showed that the existence of asymptotically stable manifold of the boundary layer system implies the existence of the average for the actual system, but using several examples we showed that this is not necessary. Indeed, we require a weaker assumption for the boundary layer system, for which the solutions asymptotically converge to a compact set. Through examples, we also showed that our results give sharper conclusions than other singular perturbation results for hybrid systems that have appeared recently in the literature.

6.2 Future work

There are many possible future directions for this research. The following list highlights but some of the possibilities for future studies and research topics.

With the assumption of existence of a well defined average for parameterized discrete-time systems, switched systems and hybrid systems, we considered closeness between solutions of the original time-varying system and solutions of its average on compact time intervals. The order of approximation error $O(\delta)$ is determined by an order function $\delta(\varepsilon)$ that depends on properties of the actual system and the input signals. Note that higher order approximations are useful in some cases. We can apply the results in [132, Chapter 3] on second order approximations for nonlinear systems to work on more precise approximations to improve closeness of solutions results in Chapters 2-4.

We presented results on global ISS properties for the three classes of systems mentioned above. In some situations, local or regional stability is of interest. For instance, prior averaging results for discrete-time systems focus on local exponential stability [15, 21, 141]. For this case, it is useful to draw conclusions on local robust stability for the actual systems based on local exponential ISS of their average systems. On the other hand, when the approximation of the actual system has a stronger stability property, such as uniform global exponential stability, further work can be done to find precise conditions that the original system is also globally exponentially stable. For systems with disturbances, one can study its global exponential ISS.

We analyzed robustness to disturbances using the concept of ISS for switched systems and stated that our results give sharper conclusions than [131], where the finite $\mathcal{L}_2$ gain property of rapidly switching linear systems was considered using the averaging method. One can go further with our averaging result in Chapter 3 that employ strong, partial strong and weak averages to consider the linear switched model in [131] and give stronger results on its finite $\mathcal{L}_2$ gain property.

Note that a partial strong average notion is introduced in Chapter 3 to provide stronger results when there does not exist a strong average for linear switched systems and using weak averages only obtain too weak conclusions. This concept of partial averaging can also be applied to consider, e.g., stability for nonlinear switched systems

$$\dot{x} = f_{\rho\left(\frac{t}{\varepsilon}\right)}(x) + g_{\rho(t)}(x) \ , \tag{6.1}$$

and hybrid systems with flow dynamics of the form:

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right) + g(t, x) \ . \tag{6.2}$$

One can average the first term $f_{\rho(\cdot)}(x)$ in (6.1) and $f(\cdot, x)$ in (6.2) respectively to get a simpler but still time-varying approximation for the actual system. Then, the results in Chapters 3 and 4 can be generalized to analyze robustness to disturbances for such classes of systems using the partial averaging notions.

Note that our averaging results for hybrid systems in Chapter 4 consider the case when flow dynamics of the hybrid systems agree with differential equations but without assuming that the vector fields are continuous. Hence, our results are applicable for systems with dither signals, for which the dynamics are denoted by a differential equation with discontinuous righthand side. We can revisit the models with dither signals in [68, 69, 149, 150] for stability analysis and work on new results for such systems with external disturbances. One can also consider extremum seeking control problem [1, 86, 123, 168] in the case when plants are affected by disturbances using the results in Chapter 4 that pertain to discontinuity of dynamics for its closed-loop system.

The averaging results for hybrid system in Chapter 4 can also be generalized to consider the class of hybrid systems:

$$
\begin{aligned}
\dot{x} &\in F\left(\frac{t}{\varepsilon}, x\right) & x &\in C \\
x^+ &\in G(x) & x &\in D \ ,
\end{aligned}
$$

where the state x agrees with differential inclusions instead of differential equations on the flow constraint set C. The above hybrid systems can take the models in [42,43] that present averaging results for differential inclusions as a special case.

Our averaging results for hybrid systems apply to the case when hybrid models only have time varying flow dynamics that can be averaged. The jump dynamics of the original hybrid system is the same as in its average system. On the other hand, for hybrid systems with almost periodical changes in jump dynamics, one can develop a hybrid average system that includes averaging behavior for both flow and jump dynamics to approximate the actual system. Then, stability analysis based on such a hybrid average model can recover the averaging results not only for pure continuous-time systems but also for pure discrete-time systems, e.g., the results in [141].

For the class of singularly perturbed hybrid systems we considered in Chapter 5, where hybrid control systems are assumed implemented by continuous and fast actuators, the vector filed that denotes slow flow dynamics is assumed to be continuous. The continuity assumption on the slow vector field can be relaxed to an assumption that small perturbations to the solutions of the boundary layer system lead to small changes in the integral that defines the average vector field. These generalizations are useful for recovering the averaging results of [154]. Also, one can consider set-valued boundary layer dynamics. These generalizations are useful for recovering the singular perturbation results in [135].

As one more step, one can generalize the results in Chapter 5 to analyze asymptotic stability via averaging for a class of hybrid systems

$$
\left.
\begin{aligned}
\dot{x} &\in \varepsilon F(x, z, \varepsilon) \\
\dot{z} &\in \Psi(x, z, \varepsilon)
\end{aligned}
\right\} \quad (x, z) \in C
$$
$$
(x, z)^+ \in G(x, z) \qquad (x, z) \in D \ ,
$$

where the flow mapping F, Ψ and jump mapping G are set-valued. The above hybrid system can include the models in [44, 160, 162] as a special case, where the singular perturbation results for differential inclusions are provided. We can also consider robustness to disturbances for the slow states of such class of systems assuming appropriate ISS properties for that reduced order system and the boundary layer system.

Moreover, we concluded in Chapter 5 that the slow dynamics of the singularly perturbed hybrid systems admit a semi-globally asymptotically stable set if a well defined average system exists and this set is globally asymptotically stable for its average system. Instead of asymptotic stability, we can study exponential stability properties for the singularly perturbed hybrid systems based on exponential stability of its reduced average hybrid system.

Appendix A

To prove Theorems 2.4.4 and 2.4.5, we first present a lemma that considers closeness of points of some functions on finite time intervals.

Lemma A.0.1. *Let the set of functions* $\tilde{w}_i(k\tau)(i = 1, \cdots, n)$*,* $\widetilde{\mathcal{W}}$*, be equi-bounded and equi-uniformly Lipschitz, then for any* $\tilde{\delta} > 0$ *and all* $\tilde{w} \in \widetilde{\mathcal{W}}$ *there exists* $\rho^* > 0$ *such that, for each* $\rho \in (0, \rho^*]$ *and* $\tau < \rho$*, the following holds:*

$$|\tilde{w}_i(k\tau) - \tilde{w}_i(k_0\tau)| \leq \tilde{\delta} \quad \forall\ (k - k_0)\tau \in [0, \rho]\ .$$

□

Proof of Lemma A.0.1

Given arbitrary $\tilde{\delta} > 0$. Let r and ν come from the definitions of equi-boundedness and equi-uniform Lipschitzness. Let

$$\rho^* := \frac{1}{\nu} \ln \left(1 + \frac{\tilde{\delta}}{r}\right)\ .$$

Consider $\rho \in (0, \rho^*]$, and let

$$e_k := |\tilde{w}_i(k\tau) - \tilde{w}_i(k_0\tau)|, \quad \forall (k - k_0)\tau \in [0, \rho]\ .$$

As $\widetilde{\mathcal{W}}$ is equi-bounded and equi-uniformly Lipschitz, for each $\tilde{w}_i \in \widetilde{\mathcal{W}}$,

$$|\tilde{w}_i((k + 1)\tau) - \tilde{w}_i(k\tau)| \leq \nu\tau|\tilde{w}(k\tau)|\ .$$

Assume for the purpose of induction that

$$e_m \leq (\exp(\nu m\tau) - 1)\, |\tilde{w}_i(k_0\tau)| \quad m\tau \in [0, \rho]\ . \tag{A.1}$$

This is trivially true for $m = 0$. Noting that for all $m\tau \in [0, \rho]$,

$$\begin{aligned} e_{m+1} &= |\tilde{w}_i((m+1)\tau) - \tilde{w}_i(k_0\tau)| \\ &\leq |\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + |\tilde{w}_i((m+1)\tau) - \tilde{w}_i(m\tau)| \\ &\leq |\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + \nu\tau|\tilde{w}_i(m\tau)| \\ &\leq |\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + \nu\tau(|\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + |\tilde{w}_i(k_0\tau)|) \\ &= (1+\nu\tau)e_m + \nu\tau|\tilde{w}_i(k_0\tau)| \\ &\leq (1+\nu\tau)(\exp(\nu m\tau) - 1)|\tilde{w}_i(k_0\tau)| + \nu\tau|\tilde{w}_i(k_0\tau)| \\ &= \{(1+\nu\tau)\exp(\nu m\tau) - 1\}|\tilde{w}_i(k_0\tau)| \\ &\leq (\exp(\nu(m+1)\tau) - 1)|\tilde{w}_i(k_0\tau)| \ , \end{aligned}$$

one gets that the inductive hypothesis (A.1) holds. With the fact that $|\tilde{w}(k_0\tau)| \leq r$ for all $\tilde{w}(k_0\tau) \in \tilde{\mathcal{W}}$, $m\tau \leq \rho$ and noting the definition of ρ, it follows for each $\tau < \rho$ that (A.1) satisfies

$$|e_m| \leq \tilde{\delta} \quad \forall m\tau \in [0, \rho] \ ,$$

and this gives the conclusion. □

A.1 Proof of Theorem 2.4.4

Part 1. Definition of τ^*, ε^* and μ:

Given arbitrary $\delta, T > 0$. Without loss of generality, assume $\delta < 1$. Let r and ν come from the definitions of equi-boundedness and equi-uniform Lipschitzness of $\mathcal{W}$. Let $R \geq r$ and τ_1^* comes from $\mathcal{W}$-forward completeness of the weak average. Then, from the definition of weak average and Lipschitz condition of $F_\tau(k\tau, x, w)$, it follows that for all w satisfying $|w| \leq r$, there exists $L > 0$ such that, for all $\tau \in (0, \tau_1^*)$, $|y| \leq R+1$ and $|x| \leq R+1$:

$$|F_\tau^{wa}(x, w) - F_\tau^{wa}(y, w)| \leq L|x - y| \ ,$$

and there exists a finite positive number B such that

$$B := \max_{k\tau \geq 0, |x| \leq R+1, |y| \leq R+1, |w| \leq r} \{|F_\tau(k\tau, x, w)|, |F_\tau^{wa}(y, w)|\} \ . \tag{A.2}$$

Then, let

$$\mu := \frac{\delta}{2\exp\frac{2LT+L^2T}{2}} \ , \tag{A.3}$$

and in preparation for defining ε^*, let

$$G\left(\frac{k\tau}{\varepsilon}, \tilde{w}\right) := \tilde{w}_1^T \left\{F_\tau\left(\frac{k\tau}{\varepsilon}, \tilde{w}_2, \tilde{w}_3\right) - F_\tau^{wa}(\tilde{w}_2, \tilde{w}_3)\right\} \tag{A.4}$$

with $\tilde{w}_i$ being components of appropriate dimension of a vector $\tilde{w}$. Let $\widetilde{\mathcal{W}}$ be the set of functions

$$\tilde{w}(k\tau) := \begin{bmatrix} \tilde{w}_1(k\tau) \\ \tilde{w}_2(k\tau) \\ \tilde{w}_3(k\tau) \end{bmatrix},$$

that is equi-bounded and equi-uniformly Lipschitz. Let τ_2^* come from the definition of equi-uniform Lipschitzness of $\mathcal{W}$, and $\rho > 0$ be such that, for all $\tilde{w} \in \widetilde{\mathcal{W}}$, $k_i\tau \geq 0$, $(k - k_i)\tau \in [0, \rho]$ and $\tau \in (0, \tau_2^*)$:

$$\left|G\left(\frac{k\tau}{\varepsilon}, \tilde{w}(k\tau)\right) - G\left(\frac{k\tau}{\varepsilon}, \tilde{w}(k_i\tau)\right)\right| \leq \frac{\delta^2}{8T\exp(2LT + L^2T)} \ . \tag{A.5}$$

Such ρ exists since G is Lipschitz uniformly in $\tilde{w}$, and from Lemma A.0.1, $\tilde{w}(k\tau)$ and $\tilde{w}(k_i\tau)$ can be arbitrarily close for each $\tau \in (0, \tau_2^*)$ and all $(k - k_i)\tau \in [0, \rho]$, if ρ is sufficiently small. Moreover, the quantity being bounded in (A.5) is zero when $k\tau = k_i\tau$. Then, the left hand side of (A.5) can be made arbitrarily small by choosing small enough ρ.

Let $\beta_{wa} \in \mathcal{KL}$ and $T^* > 0$ and τ_3^* come from the definition of weak average. Let $\tilde{T} > T^*$, $\tau_3^* = \tau_3^*(\tilde{T})$ be such that for all $\tau \in (0, \tau_3^*)$ and $\tilde{N}\tau \geq \tilde{T}$:

$$\beta_{wa}(\max\{(R+1), r\}, \tilde{N}\tau) \leq \frac{\delta^2}{8T(1+3B)\exp(2LT + L^2T)} \ . \tag{A.6}$$

Consider $\tau \in (0, \tau^*)$ with $\tau^* := \min\{\tau_1^*, \tau_2^*, \tau_3^*\}$. Without loss of generality for the fast sampling system, assume $\tau < 1$ and let

$$\varepsilon^* := \min\left\{\frac{\rho}{\tilde{N}\tau}, \frac{\delta^2}{16B\tilde{N}\tau(1+3B)\exp(2LT + L^2T)}\right\}. \tag{A.7}$$

Part 2. Error of solutions:

For any fixed $\tau \in (0, \tau^*)$. Consider $k_0\tau \geq 0$, $y_0 \in \mathbb{R}^n$ with $|y_0| \leq r$, $\varepsilon \in (0, \varepsilon^*)$, $x_0 \in \mathbb{R}^n$ such that $|x_0 - y_0| \leq \mu$. For $(k - k_0)\tau \in [0, T]$ and $w \in \mathcal{W}$, let

$$E(k\tau) := x(k\tau, k_0\tau, x_0, w) - y((k - k_0)\tau, y_0, w) \ ,$$

and note that $|E(k_0\tau)| \leq \mu \leq \frac{\delta}{2} < 1$. If $|E(k\tau)| < 1$ for all $(k - k_0)\tau \in [0, T]$, then define $\bar{k}\tau = k_0\tau + T$. Otherwise, define

$$\bar{k}\tau := \max_{s \in [0,T]} \{s : |E(k\tau)| < 1 \quad \forall k\tau \in [0, s]\} \ .$$

Note that $\bar{k}\tau > k_0\tau$, $E(\cdot)$ and $x(\cdot, k_0\tau, x_0, w)$ are defined on $[k_0\tau, \bar{k}\tau]$. Let $\tilde{w}(k\tau) \in \widetilde{\mathcal{W}}$ be such that, for all $k\tau \in [k_0\tau, \bar{k}\tau]$,

$$\begin{bmatrix} \tilde{w}_1(k\tau) \\ \tilde{w}_2(k\tau) \\ \tilde{w}_3(k\tau) \end{bmatrix} = \begin{bmatrix} E^T(k\tau) + \tau\phi^T(k\tau) + \frac{\tau}{2}\psi^T(k\tau) \\ x(k\tau, k_0, x_0, w) \\ w(k\tau) \end{bmatrix}, \tag{A.8}$$

with

$$\begin{aligned} \psi(k\tau) &:= F_\tau\left(\frac{k\tau}{\varepsilon}, x, w\right) - F_\tau^{wa}(x, w) \ , \\ \phi(k\tau) &:= F_\tau^{wa}(x, w) - F_\tau^{wa}(y, w) \ . \end{aligned} \tag{A.9}$$

Such a $\tilde{w}(k\tau) \in \widetilde{\mathcal{W}}$ exists since $\tilde{w}_3 \in \mathcal{W}$, and for all $k\tau \in [k_0\tau, \bar{k}\tau]$, $|E(k\tau)| < 1$, and from (A.2), we know $|\phi| \leq 2B$ and $|\psi| \leq 2B$. Then, for each $\tau \in (0, \tau^*)$ and all $k\tau \in [k_0\tau, \bar{k}\tau]$, it follows that $|\tilde{w}_1|_\infty \leq (1 + 3B)$, and $\left|\frac{\Delta\tilde{w}_1}{\Delta k}\right|_\infty \leq 2B + 3L(B + \nu)$. Moreover, since $|y((k - k_0)\tau, y_0, w)| \leq R$ for all $k\tau \in [0, T]$, it follows that $|x(k\tau, k_0, x_0, w)| \leq R + 1$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$ and $\left|\frac{\Delta\tilde{w}_2}{\Delta k}\right|_\infty \leq B$ from (A.2).

Using the simplified notation x and y to replace $x(k\tau, k_0\tau, x_0, w)$ and $y((k - k_0)\tau, y_0, w)$, and define the difference of $E(k\tau)$ as

$$
\begin{aligned}
H(k\tau) &:= \frac{\Delta E(k\tau)}{\Delta k} \\
&= \frac{E(k\tau+\Delta k)-E(k\tau)}{\Delta k} \\
&= \frac{x(k\tau+\Delta k)-x(k\tau)}{\Delta k} - \frac{y(k\tau+\Delta k)-y(k\tau)}{\Delta k} \\
&= F_\tau\left(\frac{k\tau}{\varepsilon}, x, w\right) - F_\tau^{wa}(y,w)\ ,
\end{aligned} \tag{A.10}
$$

where the last equality comes from (2.12) and (2.14). Comparing (A.10) with (A.9), one gets $H(k\tau) = \psi(k\tau) + \phi(k\tau)$. Moreover, for all $(k-k_0)\tau \in [0, \bar{k}\tau]$, consider a scalar-valued function $V(k\tau) := \frac{1}{2}E^T(k\tau)E(k\tau)$, we have

$$
\begin{aligned}
\frac{\Delta V(k\tau)}{\Delta k} &= \frac{1}{2}\frac{E^T(k\tau+\Delta k)E(k\tau+\Delta k)-E^T(k\tau)E(k\tau)}{\Delta k} \\
&= \frac{1}{2}\frac{(E(k\tau+\Delta k)+E(k\tau))^T(E(k\tau+\Delta k)-E(k\tau))}{\Delta k} \\
&= \frac{1}{2}\left(2E(k\tau)+\Delta k\frac{\Delta E(k\tau)}{\Delta k}\right)^T\frac{\Delta E(k\tau)}{\Delta k} \\
&= E^T(k\tau)H(k\tau) + \frac{1}{2}\Delta k H^T(k\tau)H(k\tau)\ .
\end{aligned} \tag{A.11}
$$

Substituting $H(k\tau)$ with the expression of $\phi(k\tau)$ and $\psi(k\tau)$ in (B.8), noting (A.8) and the definition of $G(\cdot)$ in (A.4), and using the following inequality from the Lipschitz condition of weak average

$$
|\phi(k\tau)| \le L|E(k\tau)|\ , \tag{A.12}
$$

it follows for $(k-k_0)\tau \in [0, \bar{k}\tau]$ that:

$$
\begin{aligned}
\frac{\Delta V(k\tau)}{\Delta k} &= E^T(k\tau)(\psi+\phi) + \frac{1}{2}(\psi+\phi)^T(\psi+\phi)\tau \\
&= E^T(k\tau)\phi + \frac{\tau}{2}\phi^T\phi + E^T(k\tau)\psi + \tau\phi^T\psi + \frac{\tau}{2}\psi^T\psi \\
&\le V(k\tau)(2L+L^2\tau) + E^T(k\tau)\psi + \tau\phi^T\psi + \frac{\tau}{2}\psi^T\psi \\
&= V(k\tau)(2L+L^2\tau) + (E^T(k\tau) + \tau\phi^T + \frac{\tau}{2}\psi^T)\psi \\
&= V(k\tau)(2L+L^2\tau) + G\left(\frac{k\tau}{\varepsilon}, \tilde{w}\right)\ .
\end{aligned}
$$

By standard comparison theorems in [74], there exists $W(k\tau)$ with $W(k_0\tau) = \frac{1}{2}\mu^2$

such that $V(k\tau) \le W(k\tau)$ and satisfy the equation

$$W((k+1)\tau) = (2L\tau + L^2\tau^2 + 1)W(k\tau) + G\left(\frac{k\tau}{\varepsilon}, \tilde{w}\right)\Delta k \ .$$

With the fact that $N\tau \le T$ and the definition of μ in (A.3), one knows $V(k_0\tau) \le \frac{1}{2}\mu^2 = \frac{\delta^2}{8\exp(2LT+L^2T)}$. Then, with the inequality

$$\{1 + (2L\tau + L^2\tau^2)\}^N \le \exp(2LN\tau + L^2N\tau^2) \ ,$$

it follows from the definition of τ that:

$$\begin{aligned}
& V(k\tau) \\
& \le (2L\tau + L^2\tau^2 + 1)^{k-k_0}V(k_0\tau) + \sum_{s=k_0}^{k-1}(2L\tau + L^2\tau^2 + 1)^{k-1-s}G\left(\tfrac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \\
& \le \exp(2LN\tau + L^2N\tau^2)V(k_0\tau) + \exp(2LN\tau + L^2N\tau^2)\sum_{s=k_0}^{k-1}G\left(\tfrac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \\
& \le \exp(2LT + L^2T\tau)V(k_0\tau) + \exp(2LT + L^2T\tau)\sum_{s=k_0}^{k-1}G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \\
& \le \frac{\delta^2}{8} + \exp(2LT + L^2T)\sum_{s=k_0}^{k-1}G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \ .
\end{aligned}$$

Fix $k\tau \in [k_0\tau, \bar{k}\tau]$ and set m to be the largest nonnegative integer such that $m \le \frac{(k-k_0-1)\tau}{\varepsilon\tilde{N}\tau}$ with $\varepsilon\tilde{N}\tau \le \rho$, where ρ is a positive real number such that inequality (A.5) holds. Letting $(k_i - k_0)\tau = i\varepsilon\tilde{N}\tau$ for $i = 0, 1, \cdots, m$, one gets

$$\begin{aligned}
V(k\tau) \ \le\ & \frac{\delta^2}{8} + \exp(2LT + L^2T)\sum_{s=k_m}^{k-1}G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \\
& + \exp(2LT + L^2T)\sum_{i=0}^{m-1}\sum_{s=k_i}^{k_{i+1}}G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \ . \qquad (A.13)
\end{aligned}$$

From the definition of m, we have $(k_{i+1}-k_i)\tau = \varepsilon\tilde{N}\tau$ and $(k-k_m-1)\tau \le \varepsilon\tilde{N}\tau$ for $k\tau \in [k_0\tau, \bar{k}\tau]$. Noting (A.2), (A.4), the definition of ε and $|E(k\tau)| < 1$ holds for all $k\tau \in [k_0\tau, \bar{k}\tau]$, one gets

$$\left|G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\right| \leq 2B|\tilde{w}_1(k\tau)| \leq 2B(1+3B) \ , \tag{A.14}$$

and

$$\begin{aligned} &\exp(2LT + L^2T)\sum_{s=k_m}^{k-1} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right)\Delta s \\ &\quad \leq \varepsilon\tilde{N}\tau 2B(1+3B)\exp(2LT+L^2T) \leq \frac{\delta^2}{8} \ . \end{aligned} \tag{A.15}$$

Define the function $k\tau = \varepsilon\zeta\tau$, using which we map the countable set $\{k_0\tau, k_1\tau, k_2\tau, \cdots\}$ into a set $\{\zeta_0\tau, \zeta_1\tau, \zeta_2\tau, \cdots\}$ with $k_0\tau = \varepsilon\zeta_0\tau$, $k_i\tau = k_0\tau + i\Delta k$ and $\zeta_i\tau = \zeta_0\tau + i\Delta\zeta$, for $i = 1, 2, \cdots$. Under the mapping, the corresponding family of systems (2.12) could be written as

$$\frac{\Delta x}{\Delta\zeta} = \varepsilon F_\tau(\zeta\tau, x, w) \qquad \Delta\zeta = \frac{\tau}{\varepsilon} \ , \tag{A.16}$$

with the fixed initial time $k_0\tau = \varepsilon\zeta_0\tau$. Then, if (2.12) satisfies

$$|x(k\tau)| \leq \beta(|x_0|, (k-k_0)\tau) \ ,$$

where $x_0 = x(k_0\tau)$, then the family of systems (A.16) satisfies

$$|x(\zeta\tau)| \leq \beta(|x_0|, \varepsilon(\zeta-\zeta_0)\tau)$$

with $x_0 = x(\varepsilon\zeta_0\tau)$.

Then, the countable set $\{k_0\tau, k_1\tau, k_2\tau, \cdots\}$ is mapped by the function $k\tau = \varepsilon\zeta\tau$ into a set $\{\zeta_0\tau, \zeta_1\tau, \zeta_2\tau, \cdots\}$, with $k_0\tau = \varepsilon\zeta_0\tau$, $k_i\tau = k_0\tau + i\varepsilon\tilde{N}\tau$ and $\zeta_i\tau = \zeta_0\tau + i\tilde{N}\tau$, for $i = 1, 2, \cdots$. From (A.6), the following holds:

$$
\begin{aligned}
&\left|\sum_{s=k_i}^{k_{i+1}} G(\frac{s\tau}{\varepsilon}, \tilde{w}(k_i\tau))\Delta s\right| \\
&\leq |\tilde{w}_1^T(k_i\tau)| \cdot \left|\varepsilon\tilde{N}\tau F_\tau^{wa}(\tilde{w}_2, \tilde{w}_3) - \sum_{k=k_i}^{k_{i+1}} F_\tau(\frac{s\tau}{\varepsilon}, \tilde{w}_2, \tilde{w}_3)\Delta s\right| \\
&= (1+3B) \cdot \left|\varepsilon\tilde{N}\tau F_\tau^{wa}(\tilde{w}_2, \tilde{w}_3) - \varepsilon \sum_{\zeta=\zeta_i}^{\zeta_{i+1}} F_\tau(\zeta\tau, \tilde{w}_2, \tilde{w}_3)\Delta\zeta\right| \\
&\leq \varepsilon\tilde{N}\tau(1+3B) \cdot \left|F_\tau^{wa}(x,w) - \frac{1}{\tilde{N}\tau}\sum_{\zeta=\zeta_i}^{\zeta_i+\tilde{N}} F_\tau(\zeta\tau, x, w)\Delta\zeta\right| \\
&\leq \varepsilon\tilde{N}\tau(1+3B)\beta_{wa}(\max\{(R+1), r\}, \tilde{N}\tau) \\
&\leq \varepsilon\tilde{N}\tau\frac{\delta^2}{8T\exp(2LT+L^2T)}\ .
\end{aligned} \tag{A.17}
$$

Substituting (A.15) into (A.13), noting the fact $m\varepsilon\tilde{N}\tau \leq T$ and combining with the inequalities (A.17), (A.5), we get

$$
\begin{aligned}
V(k\tau) &\leq \frac{\delta^2}{4} + \exp(2LT+L^2T) \cdot \sum_{i=0}^{m-1}\sum_{s=k_i}^{k_{i+1}} \Big\{G\left(\frac{s\tau}{\varepsilon}, x(k_i\tau), w(k_i\tau)\right) \\
&\quad + \left|G\left(\frac{s\tau}{\varepsilon}, x(s\tau), w(s\tau)\right) - G\left(\frac{s\tau}{\varepsilon}, x(k_i\tau), w(k_i\tau)\right)\right|\Big\}\Delta s \\
&\leq \frac{\delta^2}{4} + \exp(2LT+L^2T)m\varepsilon\tilde{N}\tau \cdot \frac{2\delta^2}{8T\exp(2LT+L^2T)} \\
&\leq \frac{\delta^2}{2}\ .
\end{aligned}
$$

As $V(k\tau) \leq \frac{\delta^2}{2}$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$ and $V(k\tau) = \frac{1}{2}E^T(k\tau)E(k\tau)$, one knows that $|E(k\tau)| \leq \delta < 1$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$. From the definition of $\bar{k}$, it follows that $(\bar{k}-k_0)\tau \leq T$ so that $|E(k\tau)| \leq \delta$ for all $(k-k_0)\tau \in [0,T]$. This establishes the result. □

A.2 Proof of Theorem 2.4.5

The proof of Theorem 2.4.5 follows exactly the same steps as the proof of Theorem 2.4.4 with following changes. With the strong average definition, instead of (A.17) we use

$$
\begin{aligned}
&\left|\sum_{s=k_i}^{k_{i+1}} G(\frac{s\tau}{\varepsilon}, \tilde{w}_1, \tilde{w}_2, \tilde{w}(s\tau))\Delta s\right| \\
&\leq \varepsilon\tilde{N}\tau|\tilde{w}_1^T|\frac{1}{\tilde{N}\tau}\cdot\left|\sum_{\zeta=\zeta_i}^{\zeta_{i+1}}\{F(\zeta\tau, \tilde{w}_2, \tilde{w}_3(\varepsilon\zeta\tau)) - F_{sa}(\tilde{w}_2, \tilde{w}_3(\varepsilon\zeta\tau))\}\Delta\zeta\right| \\
&\leq \varepsilon\tilde{N}\tau(1+3B)\beta_{sa}(\max\{(R+1), r\}, \tilde{N}\tau) \\
&\leq \varepsilon\tilde{N}\tau\frac{\delta^2}{8T\exp(2LT+L^2T)}\ . \qquad (A.18)
\end{aligned}
$$

Define $\tilde{w}(k\tau)$ as (A.8). Similarly like (A.5), we can show that for each $\tau \in (0, \tau^*)$ there exists sufficiently small $\rho > 0$ such that, for all $k_0\tau \geq 0$ and $(k-k_i)\tau \in [0, \rho]$:

$$
\left|G\left(\frac{k\tau}{\varepsilon}, \tilde{w}(k\tau)\right) - G\left(\frac{k\tau}{\varepsilon}, \tilde{w}_1(k_i\tau), \tilde{w}_2(k_i\tau), \tilde{w}_3(k\tau)\right)\right| \leq \frac{\delta^2}{8T\exp(2LT+L^2T)}\ . \qquad (A.19)
$$

Note that we need the closeness of points of $\tilde{w}_1(\cdot)$ and $\tilde{w}_2(\cdot)$ on finite time intervals with the equi-boundedness of $\tilde{w}_3 \in \mathcal{W}$ to show the existence of such ρ. From the result in Lemma A.0.1, the equi-boundedness and the equi-uniform Lipschitzness of $\tilde{w}_1(\cdot)$ and $\tilde{w}_2(\cdot)$ presented in the proof of Theorem 2.4.4, $\tilde{w}_1(k\tau)$ with $\tilde{w}_1(k_i\tau)$ and $\tilde{w}_2(k\tau)$ with $\tilde{w}_2(k_i\tau)$ can be arbitrarily close for all $(k-k_i)\tau \in [0, \rho]$ when ρ is sufficiently small. That is, there exists $\rho > 0$ such that (B.11) holds as G is Lipschitz uniformly in $\tilde{w}$, and for $k\tau = k_i\tau$ the quantity being bounded in (B.11) is zero and its left hand side can be made arbitrarily small by choosing ρ sufficiently small.

Using the inequalities (A.15), (A.18), (B.11) and the fact $m\varepsilon\tilde{N}\tau \leq T$, one knows that for all $k\tau \in [k_0\tau, \bar{k}\tau]$ the following holds:

$$
\begin{aligned}
V(k\tau) \;\leq\; & \frac{\delta^2}{4} + \exp(2LT + L^2T) \cdot \sum_{i=0}^{m-1} \sum_{s=k_i}^{k_{i+1}} \Big\{ G\left(\frac{s\tau}{\varepsilon}, \tilde{w}_1(k_i\tau), \tilde{w}_2(k_i\tau), \tilde{w}_3(s\tau)\right) \\
& + \left| G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) - G\left(\frac{s\tau}{\varepsilon}, \tilde{w}_1(k_i\tau), \tilde{w}_2(k_i\tau), \tilde{w}_3(s\tau)\right)\right| \Big\} \Delta s \\
\leq\; & \frac{\delta^2}{4} + \exp(2LT + L^2T) m\varepsilon\tilde{N}\tau \\
& \cdot \left\{ \frac{\delta^2}{8T\exp(2LT + L^2T)} + \frac{\delta^2}{8T\exp(2LT + L^2T)} \right\} \\
\leq\; & \frac{\delta^2}{2} \,.
\end{aligned}
$$

This establishes the result in the same way as Theorem 2.4.4. □

A.3 Proof of Lemma 2.4.8

A trajectory approach that also taken in [151] is utilized to prove the following preliminary result. The sufficiency is straightforward. For considering the necessity, note $(k - k_0)\tau \in [0, T]$, take arbitrary $(\frac{\delta}{2}, r)$, and let $T > 0$ be large enough such that for all $\beta(\max\{r, \gamma(r) + \delta\}, s\tau) \leq \frac{\delta}{2}$, $\forall s\tau \in [T, \infty)$. From this, estimate the trajectory of the $x(k\tau)$ step by step and finish the proof.

A For all $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \leq \max\{r, \gamma(r) + \delta\}$, $w \in \mathcal{L}_{\mathcal{W}}$ with $|w|_\infty \leq r$ and $s\tau \in [T, \infty)$

$$
\begin{aligned}
\max\{\beta(|x(k_0\tau)|, s\tau), &\gamma(|w|_\infty)\} + \frac{\delta}{2} \\
&\leq \max\{\beta(\max\{r, \gamma(r) + \delta\}, s\tau), \gamma(|w|_\infty)\} + \frac{\delta}{2} \\
&\leq \max\left\{\frac{\delta}{2}, \gamma(|w|_\infty)\right\} + \frac{\delta}{2} \leq \gamma(|w|_\infty) + \delta \qquad \text{(A.20)}
\end{aligned}
$$

B From the assumption that the family of systems (2.12) is semi-globally ISS on finite time intervals, for the particular values $2T$, $\frac{\delta}{2}$, $\max\{r, \gamma(r) + \delta\} > 0$, we get a $\tau^* > 0$ and a $\varepsilon^* > 0$ such that for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \leq \max\{r, \gamma(r) + \delta\}$, $|x(k\tau)| \leq \max\{\beta(|x(k_0\tau)|, (k - k_0)\tau), \gamma(|w|_\infty)\} + \frac{\delta}{2}$, $\forall (k - k_0)\tau \in [0, 2T]$. Together with (A.20) it follows that $|x(k\tau)| \leq \gamma(|w|_\infty) + \delta$, $\forall (k - k_0)\tau \in [T, 2T]$, and in particular one gets that $|x(T)| \leq \gamma(r) + \delta$.

C With initial value $\bar{x}(\bar{k}_0\tau) = x(T)$, repeated application of **A** and **B**, for $|\bar{x}(\bar{k}_0\tau)| \leq \max\{r, \gamma(r)+\delta\}$ and $s\tau \in [\bar{k}_0\tau, \bar{k}\tau_0+T]$, we have $\max\{\beta(|\bar{x}(\bar{k}_0\tau)|, s\tau), \gamma(|w|_\infty)\} + \frac{\delta}{2} \leq \gamma(|w|_\infty) + \delta$, and $|\bar{x}(k\tau)| \leq \max\{\beta(|\bar{x}(\bar{k}_0\tau)|, s\tau), \gamma(|w|_\infty)\} + \frac{\delta}{2}$, $\forall(k - \bar{k}_0)\tau \in [0, 2T]$. It follows that $|x(k\tau)| \leq \gamma(|w|_\infty) + \delta$, $\forall(k - k_0)\tau \in [T, 3T]$ and repeating the process yields that, for all $k_0\tau \geq 0$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \leq \max\{r, \gamma(r) + \delta\}$, $|x(k\tau)| \leq \gamma(|w|_\infty) + \delta$ hold $\forall(k - k_0)\tau \in [T, \infty)$. □

A.4 Proof of Theorem 2.4.10

From Lemma 2.4.8, it is just necessary to show that the family of systems (2.12) is semiglobally practically ISS on finite time interval on the set $\mathcal{L}_\mathcal{W}$. Taking arbitrary triple (r, δ, T), let $\tilde{\delta} > 0$ and $T > 0$ satisfy

$$\max_{d\in[0,r],(k-k_0)\tau\in[0,T]} \Big[\beta(d + \tilde{\delta}, (k - k_0)\tau) - \beta(d, (k - k_0)\tau)\Big] + \tilde{\delta} \leq \delta \ . \qquad \text{(A.21)}$$

Using the result of Theorem 2.4.4, for some sufficiently small numbers $\tau^* > 0$ and $\varepsilon^* > 0$, for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$, there exits $\tilde{\delta} \geq 0$ and $(k - k_0)\tau \in [0, T]$, such that the solution $x(k\tau, k_0\tau, x_0, w)$ of the family of systems (2.12) and the solution of the family of weak average systems satisfy

$$|x(k\tau, k_0\tau, x_0, w) - y((k - k_0)\tau, y_0, w)| \leq \tilde{\delta} \ . \qquad \text{(A.22)}$$

Using the simplified notation $x(k\tau)$ and $y(k\tau)$ to replace $x(k\tau, k_0\tau, x_0, w)$ and $y((k - k_0)\tau, y_0, w)$, the global ISS of the family of weak average systems on the set $\mathcal{L}_\mathcal{W}$ guarantee that for any $y(k_0\tau) \in \mathbb{R}^n$ and $w \in \mathcal{L}_\mathcal{W}$, we have

$$|y(k\tau)| \leq \max\{\beta(|y(k_0\tau)|, (k - k_0)\tau), \gamma(|w|_\infty)\} \ \ \forall(k - k_0)\tau \geq 0 \ . \qquad \text{(A.23)}$$

Note that for any $y(k\tau)$ and $x(k\tau)$ satisfy the inequality (A.22), $|x(k_0\tau)-y(k_0\tau)| \leq \tilde{\delta}$ holds. Using (A.21), (A.22) and (A.23), one gets for all $(k - k_0)\tau \in [0, T]$,

$$
\begin{aligned}
|x(k\tau)| &\leq |y(k\tau)| + |x(k\tau) - y(k\tau)| \\
&\leq \max\{\beta(|y(k_0\tau)|, (k-k_0)\tau), \gamma(|w|_\infty)\} + \tilde{\delta} \\
&\leq \max\{\beta(|x(k_0\tau)| + \tilde{\delta}, (k-k_0)\tau), \gamma(|w|_\infty)\} + \tilde{\delta} \\
&\leq \max\{\beta(|x(k_0\tau)|, (k-k_0)\tau), \gamma(|w|_\infty)\} + \delta \ .
\end{aligned}
$$

The result then follows by applying Theorem 2.4.8. □

Appendix B

The proof of Theorems 3.4.5 starts from the following technical lemma.

Lemma B.0.1. *Suppose that the weak average of system (3.10) exists and satisfies Assumption 3.4.2 with Lyapunov function* V *and positive constants* c_1, c_2 *and* γ_a*. Then, for any* $\tilde{\delta} \in (0,1)$ *there exists* $\tilde{\tau}^* > 0$ *such that, for each* $\tau \in (0, \tilde{\tau}^*)$ *there exist* $\varepsilon^* > 0$ *and an increasing sequence of times* $t_i (i \in \mathbb{N}) : t_{i+1} - t_i \leq \tau$ *with* $t_i \to \infty$ *as* $i \to \infty$*, and then for all* $t_i \geq t_0$*,* $\varepsilon \in (0, \varepsilon^*)$*,* $w, \dot{w} \in \mathcal{L}_\infty$ *and* $x_0 := x(t_0) \in \mathbb{R}^n$*, the solution of system (3.10) satisfies:*

$$\frac{V(x(t_{i+1})) - V(x(t_i))}{\tau} \leq -\left(1 - \tilde{\delta}\right)|x(t_i)|^2 + \left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tilde{\delta}||\dot{w}||_\infty^2 . \quad \text{(B.1)}$$

□

Lemma B.0.2. *Suppose that the strong average of system (3.10) exists and satisfies Assumption 3.4.2 with Lyapunov function* V *and positive constants* c_1, c_2 *and* γ_a*. Then, for any* $\tilde{\delta} \in (0,1)$ *there exists* $\tilde{\tau}^* > 0$ *such that, for each* $\tau \in (0, \tilde{\tau}^*)$ *there exist* $\varepsilon^* > 0$ *and an increasing sequence of times* $t_i (i \in \mathbb{N}) : t_{i+1} - t_i \leq \tau$ *with* $t_i \to \infty$ *as* $i \to \infty$*, and then for all* $t_i \geq t_0$*,* $\varepsilon \in (0, \varepsilon^*)$*,* $w \in \mathcal{L}_\infty$ *and* $x_0 := x(t_0) \in \mathbb{R}^n$*, the solution of system (3.10) satisfies:*

$$\frac{V(x(t_{i+1})) - V(x(t_i))}{\tau} \leq -\left(1 - \tilde{\delta}\right)|x(t_i)|^2 + \left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 . \quad \text{(B.2)}$$

□

Proof of Lemma B.0.1

Let a positive real number $\tilde{\delta} < 1$ be given. Let the quadratic Lyapunov function V and positive constants c_1, c_2 and γ_a come from Assumption 1. Let $k_1, k_2, T^* > 0$ and $\sigma_1, \sigma_2 \in \mathcal{L}$ come from the definition of average for matrices A_{ρ_1} and B_{ρ_2}. In preparation for defining ε^*, let $\tilde{T}_1 \geq T^*$ and $\tilde{T}_2 \geq T^*$ satisfying

$$\begin{aligned}\sigma_1(\tilde{T}_1) &\leq \frac{\tilde{\delta}}{8k_1c_2}\\ \sigma_2(\tilde{T}_2) &\leq \frac{\tilde{\delta}}{8k_2c_2}\ ,\end{aligned}$$

and define $\tilde{T} = \max\{\tilde{T}_1, \tilde{T}_2\}$. Let τ_1 satisfies $\tau_1 \leq \frac{\tilde{\delta}}{8c_2a_m}$ and a strictly positive real number $a_m = \max_{t\geq t_0}\{|A_{\rho 1(t)}|, |B_{\rho 2(t)}|\}$ be given. Let $\tau_2 > 0$ be such that for any $\tilde{\delta}$, $t_i \geq t_0$ and positive constant $\tilde{k}$ such that

$$s \in [t_i, t_i + \tau_2] \Rightarrow \tilde{k}s\exp(2a_m s) \leq \tilde{\delta}\ .$$

Then define $\tilde{\tau}^* := \min\{\tau_1, \tau_2\}$. For all $\tau \in (0, \tilde{\tau}^*)$, let $\varepsilon^* := \left\{\frac{\tau}{\tilde{T}}\right\}$, and $t_i = t_0 + i\varepsilon\tilde{T}$ for all $\varepsilon \in (0, \varepsilon^*)$ and $i \in \mathbb{N}$. From the definition of ε^*, we have $t_{i+1} - t_i = \varepsilon\tilde{T} \leq \tau$. Applying the Lyapunov candidate function V in Assumption 3.4.2 to system (3.10) for all $t \in [t_i, t_{i+1}]$, it follows that

$$\begin{aligned}&\frac{\partial V}{\partial x}(x(t,t_i))\{A_{\rho 1(\frac{t}{\varepsilon})}x(t,t_i) + B_{\rho 2(\frac{t}{\varepsilon})}w(t)\} \qquad \text{(B.3)}\\ &\quad= \frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(t)\} - \frac{\partial V}{\partial x}(x(t_i))\{B_{av}w(t) - B_{av}w(t_i)\}\\ &\quad-\frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(t_i)\} + \frac{\partial V}{\partial x}(x(t_i))\{A_{\rho 1(\frac{t}{\varepsilon})}x(t_i) + B_{\rho 2(\frac{t}{\varepsilon})}w(t_i)\}\\ &\quad+\frac{\partial V}{\partial x}(x(t,t_i))\{A_{\rho 1(\frac{t}{\varepsilon})}x(t,t_i) + B_{\rho 2(\frac{t}{\varepsilon})}w(t)\}\\ &\quad-\frac{\partial V}{\partial x}(x(t_i))\{A_{\rho 1(\frac{t}{\varepsilon})}x(t_i) + B_{\rho 2(\frac{t}{\varepsilon})}w(t_i)\}\ .\end{aligned}$$

Integrating both sides of the inequality along the solution of $x(t, t_i)$ over the interval $[t_i, t_{i+1}]$ and with the fact $\left|\frac{\partial V}{\partial x}(x(t_i))\right| \leq 2c_2|x(t_i)|$, we get

$$
\begin{aligned}
&\frac{V(x(t_{i+1})) - V(x(t_i))}{\varepsilon\tilde{T}} \\
&\leq \underbrace{\frac{1}{\varepsilon\tilde{T}}\int_{t_i}^{t_{i+1}} \frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(s)\}ds}_{1} \\
&\quad + \underbrace{\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}} B_{av}(w(s) - w(t_i))ds\right|}_{2} \\
&\quad + \underbrace{\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left\{\left|\int_{t_i}^{t_{i+1}} \{A_{av} - A_{\rho 1(\frac{s}{\varepsilon})}\}x(t_i)ds\right| + \left|\int_{t_i}^{t_{i+1}} \{B_{av} - B_{\rho 2(\frac{s}{\varepsilon})}\}w(t_i)ds\right|\right\}}_{3} \\
&\quad + \underbrace{\frac{1}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}} \left\{\frac{\partial V}{\partial x}(x(s))\left(A_{\rho 1(\frac{s}{\varepsilon})}x(s) + B_{\rho 2(\frac{s}{\varepsilon})}w(s)\right)\right.\right.}_{4} \\
&\quad \underbrace{\left.\left. - \frac{\partial V}{\partial x}(x(t_i))\left(A_{\rho 1(\frac{s}{\varepsilon})}x(t_i) + B_{\rho 2(\frac{s}{\varepsilon})}w(t_i)\right)\right\} ds\right|}_{4} \; .
\end{aligned}
$$

We now turn to bounding each of the terms on the right-hand side: 1. From Assumption 1, it follows that the term 1 is bounded by

$$
\begin{aligned}
&\frac{1}{\varepsilon\tilde{T}}\int_{t_i}^{t_{i+1}} \frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(s)\}ds \\
&\leq \frac{1}{\varepsilon\tilde{T}}\int_{t_i}^{t_{i+1}} \{-|x(t_i)|^2 + \gamma_a|w(s)|^2\}ds \\
&\leq -|x(t_i)|^2 + \gamma_a||w||_\infty^2 \; .
\end{aligned}
$$

2. With the definition of τ, we have for all $t \in [t_i, t_{i+1}]$ that term 2 is bounded by

$$
\begin{aligned}
\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}} B_{av}(w(s) - w(t_i))ds\right| &\leq 2c_2 a_m \tau |x(t_i)| \cdot ||\dot{w}||_\infty \\
&\leq \frac{\tilde{\delta}}{8}(|x(t_i)|^2 + ||\dot{w}||_\infty^2) \; . \qquad \text{(B.4)}
\end{aligned}
$$

3. Under two time scale behavior of $A_{\rho 1(\frac{s}{\varepsilon})}$ and $B_{\rho 2(\frac{s}{\varepsilon})}$, set $s = \varepsilon\nu$, and consider the average definition for matrices, we have that term 3 can be bounded by

$$
\begin{aligned}
&\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left\{\left|\int_{t_i}^{t_{i+1}}\{A_{av}-A_{\rho 1(\frac{s}{\varepsilon})}\}x(t_i)ds\right|+\left|\int_{t_i}^{t_{i+1}}\{B_{av}-B_{\rho 2(\frac{s}{\varepsilon})}\}w(t_i)ds\right|\right\}\\
\leq\ & 2c_2|x(t_i)|\left\{\left|A_{av}-\frac{1}{\tilde{T}}\int_{t_i}^{t_i+\tilde{T}}A_{\rho 1(\nu)}d\nu\right|\cdot|x(t_i)|\right.\\
&\left.+\left|B_{av}-\frac{1}{\tilde{T}}\int_{t_i}^{t_i+\tilde{T}}B_{\rho 2(\nu)}d\nu\right|\cdot||w||_\infty\right\}\\
\leq\ & 2c_2|x(t_i)|\{k_1\sigma_1(\tilde{T})|x(t_i)|+k_2\sigma_2(\tilde{T})||w||_\infty\}\\
\leq\ & \frac{1}{4}\{\tilde{\delta}|x(t_i)|^2+\tilde{\delta}|x(t_i)|\cdot||w||_\infty\}\leq\frac{3\tilde{\delta}}{8}|x(t_i)|^2+\frac{\tilde{\delta}}{8}||w||_\infty^2\ .
\end{aligned}
$$

Finally, term 4 is bounded by

$$
\begin{aligned}
&\left|\frac{\partial V}{\partial x}(x(t,t_i))\{A_{\rho 1(\frac{t}{\varepsilon})}x(t,t_i)+B_{\rho 2(\frac{t}{\varepsilon})}w(t)\}-\frac{\partial V}{\partial x}(x(t_i))\{A_{\rho 1(\frac{t}{\varepsilon})}x(t_i)+B_{\rho 2(\frac{t}{\varepsilon})}w(t_i)\}\right|\\
\leq\ & \left|\frac{\partial V}{\partial x}(x(t,t_i))\left\{A_{\rho 1(\frac{t}{\varepsilon})}(x(t,t_i)-x(t_i))+B_{\rho 2(\frac{t}{\varepsilon})}(w(t)-w(t_i))\right\}\right|\\
&+\left|\left\{\frac{\partial V}{\partial x}(x(t,t_i))-\frac{\partial V}{\partial x}(x(t_i))\right\}\{A_{\rho 1(\frac{t}{\varepsilon})}x(t_i)+B_{\rho 2(\frac{t}{\varepsilon})}w(t_i)\}\right|\\
\leq\ & 2c_2a_m\left\{|x(t,t_i)|\cdot(|x(t,t_i)-x(t_i)|+|w(t)-w(t_i)|)\right.\\
&\left.+|x(t,t_i)-x(t_i)|(|x(t_i)|+||w||_\infty)\right\}\ . \qquad \text{(B.5)}
\end{aligned}
$$

As for all $t\in[t_i,t_{i+1}]$ and $t_i\geq t_0$, the solution of system (3.10) satisfies $x(t,t_i)=x(t_i)+\int_{t_i}^{t}(A_{\rho_1(s)}x(s)+B_{\rho_2(s)}w(s))ds$. With the definition of a_m, it is obvious that for all $t\in[t_i,t_{i+1}]$

$$|x(t,t_i)|\leq\exp(a_m\tau)|x(t_i)|+(\exp(a_m\tau)-1)||w||_\infty. \qquad \text{(B.6)}$$

Noting $|w(t)-w(t_i)|\leq||\dot{w}||_\infty\tau$ and $|x(t,t_i)-x(t_i)|\leq a_m\tau(|x(t,t_i)|+||w||_\infty)$, it follows that the inequality (B.5) is bounded by $2\tau\exp(2a_m\tau)c_2\{5a_m^2(|x(t_i)|^2+||w||_\infty^2)+a_m||\dot{w}||_\infty^2\}$. Let $\bar{a}=\min\{a_m,1\}$ and $\tilde{k}:=\frac{1}{4c_2\bar{a}^2}$. Note that

$$\tilde{k}>\frac{\max\{|x(t_i)|^2,||w||_\infty^2,||\dot{w}||_\infty^2\}}{4c_2\{5a_m^2(|x(t_i)|^2+||w||_\infty^2)+a_m||\dot{w}||_\infty^2\}}$$

and the definition of τ_2, it follows that term 4 is bounded by $\frac{\tilde{\delta}}{2}(|x(t_i)|^2+||w||_\infty^2+$

$||\dot{w}||_\infty^2$) for all $\tau \in (0, \tilde{\tau}^*)$. Combining the upper bound of four terms in (B.4), we complete the proof:

$$\begin{aligned}\frac{V(x(t_{i+1})) - V(x(t_i))}{\tau} &\leq -\left(1-\tilde{\delta}\right)|x(t_i)|^2 + \left(\gamma_a + \frac{5\tilde{\delta}}{8}\right)||w||_\infty^2 + \frac{5\tilde{\delta}}{8}||\dot{w}||_\infty^2 \\ &\leq -\left(1-\tilde{\delta}\right)|x(t_i)|^2 + \left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tilde{\delta}||\dot{w}||_\infty^2 .\end{aligned}$$

□

Proof of Lemma B.0.2

Given $\tilde{\delta} < 1$, define $\tilde{T}_1$, τ_2 and a_m same as the proof of Lemma B.0.1. Let $\tilde{T} := \tilde{T}_1$, $\tau^* := \tau_2$, $t_i = t_0 + i\varepsilon\tilde{T}$ for all $\varepsilon \in (0, \varepsilon^*)$ and $i \in \mathbb{N}$ with ε^* defined in (A.7). Then, the majority of the proof of Lemma B.0.2 is same as the proof of Lemma B.0.1, except equations (B.3) and (B.4) are replaced by

$$\begin{aligned}&\frac{\partial V}{\partial x}(x(t,t_i))\{A_{\rho_1(\frac{t}{\varepsilon})}x(t,t_i) + Bw(t)\} = \frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + Bw(t)\} \\ &\quad -\frac{\partial V}{\partial x}(x(t_i))\{A_{av} - A_{\rho_1(\frac{t}{\varepsilon})}\}x(t_i) + \left\{\frac{\partial V}{\partial x}(x(t,t_i)) - \frac{\partial V}{\partial x}(x(t_i))\right\}Bw(t) \\ &\quad +\frac{\partial V}{\partial x}(x(t,t_i))A_{\rho_1(\frac{t}{\varepsilon})}x(t,t_i) - \frac{\partial V}{\partial x}(x(t_i))A_{\rho_1(\frac{t}{\varepsilon})}x(t_i) \end{aligned} \tag{B.7}$$

and

$$\begin{aligned}\frac{V(x(t_{i+1}) - x(t_i))}{\varepsilon\tilde{T}} &\leq \underbrace{\frac{1}{\varepsilon\tilde{T}}\int_{t_i}^{t_{i+1}}\frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + Bw(s)\}ds}_{1} \\ &\quad + \underbrace{\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}}\{A_{av} - A_{\rho_1(\frac{s}{\varepsilon})}\}x(t_i)ds\right|}_{2} \\ &\quad + \underbrace{\frac{1}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}}\left(\frac{\partial V}{\partial x}(x(s)) - \frac{\partial V}{\partial x}(x(t_i))\right)Bw(s)ds\right|}_{3} \\ &\quad + \underbrace{\frac{1}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}}\left\{\frac{\partial V}{\partial x}(x(s))A_{\rho_1(\frac{s}{\varepsilon})}x(s) - \frac{\partial V}{\partial x}(x(t_i))A_{\rho_1(\frac{s}{\varepsilon})}x(t_i)\right\}ds\right|}_{4} .\end{aligned} \tag{B.8}$$

Similarly, term 1 is bounded by $-|x(t_i)|^2 + \gamma_a||w||_\infty^2$ from Assumption 1. Using the definition of average for matrices, and letting $s = \varepsilon\nu$, the upper bound of term 2 is

$$\begin{aligned}\frac{2c_2(t_i)|}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}}\{A_{av} - A_{\rho_1(\frac{s}{\varepsilon})}\}x(t_i)ds\right| &\leq 2c_2|x(t_i)|^2\left|A_{av} - \frac{1}{\tilde{T}}\int_{t_i}^{t_i+\tilde{T}} A_{\rho_1(\nu)}d\nu\right| \\ &\leq 2c_2k_1\sigma_1(\tilde{T})|x(t_i)|^2 \leq \frac{\tilde{\delta}}{4}|x(t_i)|^2 \ . \quad \text{(B.9)}\end{aligned}$$

Let $\tilde{k} := \frac{1}{12c_2a_m^2}$. With $\tilde{k} > \frac{\max\{|x(t_i)|^2,||w||_\infty^2\}}{4c_2a_m^2(4|x(t_i)|^2+3||w||_\infty^2)}$ and from the definition of $\tilde{\tau}^*$, we have that for all $t \in [t_i, t_{i+1}]$ and $t_i \geq t_0$, term 3 is bounded by

$$\begin{aligned}&\left|\frac{\partial V}{\partial x}(x(t,t_i)) - \frac{\partial V}{\partial x}(x(t_i))\right| \cdot |Bw(s)| \\ &\quad \leq 2c_2a_m|x(t,t_i) - x(t_i)| \cdot ||w||_\infty \\ &\quad \leq 2c_2a_m^2\tau(|x(t,t_i)| + ||w||_\infty) \cdot ||w||_\infty \\ &\quad \leq 2\tau\exp(a_m\tau)c_2a_m^2(|x(t_i)| + ||w||_\infty)||w||_\infty \\ &\quad \leq \frac{\tilde{\delta}}{4}(|x(t_i)|^2 + ||w||_\infty^2) \ , \qquad \text{(B.10)}\end{aligned}$$

and the upper bound of term 4 is

$$\begin{aligned}&\left|\frac{\partial V}{\partial x}(x(t,t_i))A_{\rho_1(\frac{t}{\varepsilon})}x(t,t_i) - \frac{\partial V}{\partial x}(x(t_i))A_{\rho_1(\frac{t}{\varepsilon})}x(t_i)\right| \\ &\leq \left|\frac{\partial V}{\partial x}(x(t,t_i))A_{\rho_1(\frac{t}{\varepsilon})}(x(t,t_i) - x(t_i))\right| \\ &\quad + \left|\frac{\partial V}{\partial x}(x(t,t_i)) - \frac{\partial V}{\partial x}(x(t_i))\right| \cdot |A_{\rho_1(\frac{t}{\varepsilon})}x(t_i)| \\ &\leq 2\tau\exp(2a_m\tau)c_2a_m^2(4|x(t_i)|^2 + 3||w||_\infty^2) \leq \frac{\tilde{\delta}}{2}(|x(t_i)|^2 + ||w||_\infty^2) \ . \quad \text{(B.11)}\end{aligned}$$

Combining the upper bounds of all terms into (B.8), this complete the proof. □

B.1 Proof of Theorem 3.4.5

Let the quadratic Lyapunov function V and positive constants c_1, c_2 and γ_a come from Assumption 3.4.2, K, λ and γ come from Remark 3.4.4. Given any $0 < \tilde{\delta} < 1$,

$\tilde{\tau}^*$ is then determined from Lemma B.0.1 by $\tilde{\delta}$. Let

$$a_m := \max_{t \geq t_0}\{|A_{\rho_1(t)}|, |B_{\rho_2(t)}|\} \ , \quad c := 1 - \tilde{\delta} > 0 \ . \tag{B.12}$$

Let $\tau_3 = \frac{c_2}{c}$ and $\tau^* = \min\{\tilde{\tau}^*, \tau_3\}$, then we have $0 < \left(1 - \frac{c\tau}{c_2}\right) < 1$ when $\tau \in (0, \tau^*)$. Let ε^* be determined by τ from Lemma B.0.1, and the proof is followed for all $\varepsilon \in (0, \varepsilon^*)$. For preparing the definition of δ, let

$$m := 1 - \frac{c\tau}{c_2} \ , \quad \mu := \sqrt{\frac{\tau}{c_1(1-m)}} \ . \tag{B.13}$$

Then, let

$$\begin{aligned}
\delta_1 &= (\exp(a_m\tau) - 1)(\gamma + K + 1) + \exp(a_m\tau)\left(\gamma\sqrt{\frac{\tilde{\delta}}{1-\tilde{\delta}}} + \mu\sqrt{\tilde{\delta}}\right) \ , \\
\delta_2 &= \frac{\tilde{\delta}}{2c_2}, \ \delta_3 = \left(\exp\left(2\tau\left(a_m + \lambda - \frac{\tilde{\delta}}{2c_2}\right)\right) - 1\right)K \ , \\
\delta &:= \max\{\delta_1, \delta_2, \delta_3, \tilde{\delta}\} \ .
\end{aligned} \tag{B.14}$$

With a_m defined above, for all $t \geq t_0$, solutions of system (3.10) satisfy $|x(t)| \leq \exp(a_m(t - t_0))|x_0| + (\exp(a_m(t - t_0)) - 1)||w||_\infty$, and then there exists a t_{i_0} such that $t_{i_0} - t_0 \leq \tau$ implies

$$|x(t_{i_0})| \leq \exp(a_m\tau)|x_0| + (\exp(a_m\tau) - 1)||w||_\infty \ . \tag{B.15}$$

From Lemma B.0.1, for all $\varepsilon \in (0, \varepsilon^*)$ and any t_{i_0} satisfying $t_{i_0} - t_0 \leq \tau$ and $t_{i_0+k} - t_{i_0+k-1} \leq \tau$, $\forall k \in \mathbb{N}$, we have

$$\frac{V(x(t_{i_0+1})) - V(x(t_{i_0}))}{\tau} \leq -c|x(t_{i_0})|^2 + \left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tilde{\delta}||\dot{w}||_\infty^2 \ . \tag{B.16}$$

Using $V(x) \leq c_2|x|^2$, it follows that

$$V(x(t_{i_0+1})) \leq \left(1 - \frac{c\tau}{c_2}\right)V(x(t_{i_0})) + \tau\left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tau\tilde{\delta}||\dot{w}||_\infty^2 \ .$$

With the definition of m, we have $0 < m < 1$ and

$$V(x(t_{i_0+1})) \leq mV(x(t_{i_0})) + \tau\left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tau\tilde{\delta}||\dot{w}||_\infty^2 \ .$$

By repeating this argument, and using $(1-z)^n \leq \exp(-nz)$, $\forall z \in (0,1)$, we have $\forall n \in \mathbb{N}$

$$\begin{aligned} & V(x(t_{i_0+n})) \\ \leq \ & m^n V(x(t_{i_0})) + \sum_{k=1}^{n} m^{k-1}\tau(\gamma_a + \tilde{\delta})||w||_\infty^2 + \sum_{k=1}^{n} m^{k-1}\tau\tilde{\delta}||\dot{w}||_\infty^2 \\ \leq \ & \exp\left(-\frac{c\tau n}{c_2}\right)V(x(t_{i_0})) + \frac{\tau(\gamma_a+\tilde{\delta})}{(1-m)}||w||_\infty^2 + \frac{\tau\tilde{\delta}}{(1-m)}||\dot{w}||_\infty^2 \ . \end{aligned}$$

From $c_1|x|^2 \leq V(x) \leq c_2|x|^2$, we know that

$$|x(t_{i_0+n})|^2 \leq \frac{c_2}{c_1}\exp\left(-\frac{c\tau n}{c_2}\right)|x(t_{i_0})|^2 + \frac{\tau(\gamma_a+\tilde{\delta})}{c_1(1-m)}||w||_\infty^2 + \frac{\tau\tilde{\delta}}{c_1(1-m)}||\dot{w}||_\infty^2 \ .$$

With the definition of μ and K, and noting $t_{i_0+n} - t_{i_0} \leq n\tau$, it follows that

$$\begin{aligned} |x(t_{i_0+n})| \leq \ & K\exp\left(-\frac{c}{2c_2}(t_{i_0+n} - t_{i_0})\right)|x(t_{i_0})| \\ & + \mu\left(\sqrt{\gamma_a} + \sqrt{\tilde{\delta}}\right)||w||_\infty + \mu\sqrt{\tilde{\delta}}||\dot{w}||_\infty \ . \end{aligned} \tag{B.17}$$

Letting $\bar{\lambda} = \frac{c}{2c_2}$, and by repeating the same argument, one knows that for every $j \in \mathbb{N}$

$$\begin{aligned} |x(t_{i_0+jn})| \leq \ & K\exp\left(-\bar{\lambda}(t_{i_0+jn} - t_{i_0})\right)|x(t_{i_0})| \\ & + \mu\left(\sqrt{\gamma_a} + \sqrt{\tilde{\delta}}\right)||w||_\infty + \mu\sqrt{\tilde{\delta}}||\dot{w}||_\infty \ . \end{aligned} \tag{B.18}$$

For every $t \geq t_{i_0}$, there exist $j \in \mathbb{N}$ such that $t \in [t_{i_0+jn}, t_{i_0+jn+1})$. While (B.18) gives the evolution of the flow at times t_{i_0+jn}, $\forall j \in \mathbb{N}$, but gives no information about the flow between the times t_{i_0+jn}. Considering (B.6), it follows that $|x(t, t_{i_0+jn})| \leq \exp(a_m\tau)|x(t_{i_0+jn})| + (\exp(a_m\tau) - 1)||w||_\infty$, $\forall t \in [t_{i_0+jn}, t_{i_0+jn+1})$.

Noting (B.17), it follows that for all $t \geq t_{i_0}$, the solution of system (3.10) satisfies

$$\begin{aligned} |x(t,t_{i_0})| \leq\ & \exp(a_m\tau)K\exp(-\bar{\lambda}(t_{i_0+jn}-t_{i_0}))|x(t_{i_0})| + (\exp(a_m\tau)-1)||w||_\infty \\ & +\mu\exp(a_m\tau)\sqrt{\tilde{\delta}}||\dot{w}||_\infty + \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty \\ \leq\ & \exp((a_m+\bar{\lambda})\tau)K\exp\left(-\bar{\lambda}(t-t_{i_0})\right)|x(t_{i_0})| + \mu\exp(a_m\tau)\sqrt{\tilde{\delta}}||\dot{w}||_\infty \\ & + (\exp(a_m\tau)-1)\,||w||_\infty + \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty\ . \end{aligned} \tag{B.19}$$

With $\gamma = \sqrt{\frac{c_2\gamma_a}{c_1}}$, the definitions of m and δ, we have

$$\begin{aligned} \mu\sqrt{\gamma_a} &= \sqrt{\frac{\tau\gamma_a}{c_1(1-m)}} \\ &= \sqrt{\frac{c_2\gamma_a}{c_1\left(1-\tilde{\delta}\right)}} = \gamma\cdot\sqrt{1+\frac{\tilde{\delta}}{1-\tilde{\delta}}} \end{aligned}$$

and

$$\begin{aligned} & \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right) + (\exp(a_m\tau)-1)(K+1) \\ \leq\ & \exp(a_m\tau)\left(\gamma+\gamma\sqrt{\frac{\tilde{\delta}}{1-\tilde{\delta}}}+\mu\sqrt{\tilde{\delta}}\right) + (\exp(a_m\tau)-1)(K+1) \\ \leq\ & \gamma+\delta\ . \end{aligned} \tag{B.20}$$

Considering the definition of δ, we know that

$$\exp(2(a_m+\bar{\lambda})\tau)K = \exp\left(2\left(a_m+\lambda-\frac{\tilde{\delta}}{2c_2}\right)\tau\right)K \leq K+\delta\ . \tag{B.21}$$

Then, combining $\bar{\lambda} = \frac{1-\tilde{\delta}}{2c_2} \geq \lambda - \delta$, (B.14), (B.15) and (C.26) into (B.23), the following completes the proof:

$$
\begin{aligned}
|x(t)| &\leq \exp(2(a_m+\bar{\lambda})\tau)K\exp\left(-\bar{\lambda}(t-t_0)\right)|x_0| \qquad \text{(B.22)}\\
&\quad +\mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty+\exp(a_m\tau)\mu\sqrt{\tilde{\delta}}||\dot{w}||_\infty\\
&\quad +\left(\exp(a_m\tau)-1\right)(K+1)||w||_\infty\\
&\leq (K+\delta)(\exp(-(\lambda-\delta)(t-t_0))|x_0|+(\gamma+\delta)||w||_\infty+\delta||\dot{w}||_\infty\ .
\end{aligned}
$$

B.2 Proof of Theorem 3.4.7

The proof of Theorem 3.4.7 is exactly same as the proof of Theorem 3.4.5 in Section B.1 with the following changes. Given any $0<\tilde{\delta}<1$, $\tilde{\tau}^*$ is then determined from Lemma B.0.2 by $\tilde{\delta}$. Let $a_m := \max_{t\geq t_0}\{|A_{\rho_1(t)}|, |B_{\rho_2(t)}|\}$ and $c := 1-\tilde{\delta} > 0$. Let $\tau_3 = \frac{c_2}{c}$ and $\tau^* = \min\{\tilde{\tau}^*, \tau_3\}$. Let ε^* be determined by τ from Lemma B.0.2, and the proof is followed for all $\varepsilon \in (0,\varepsilon^*)$. Define the $\delta > 0$ as (B.14).

Using the results from Lemma B.0.2, for all $\varepsilon \in (0,\varepsilon^*)$ and any t_{i_0} satisfying $t_{i_0} - t_0 \leq \tau$ and $t_{i_0+k} - t_{i_0+k-1} \leq \tau$, $\forall k \in \mathbb{N}$, the following inequality replacing (B.16) in Section B.1 holds.

$$
\frac{V(x(t_{i_0+1}))-V(x(t_{i_0}))}{\tau} \leq -c|x(t_{i_0})|^2+\left(\gamma_a+\tilde{\delta}\right)||w||_\infty^2\ .
$$

With the definition of m in (B.13), we have

$$
V(x(t_{i_0+1})) \leq mV(x(t_{i_0}))+\tau\left(\gamma_a+\tilde{\delta}\right)||w||_\infty^2\ .
$$

By repeating this argument, and using $(1-z)^n \leq \exp(-nz)$, $\forall z \in (0,1)$, we have $\forall n \in \mathbb{N}$

$$
\begin{aligned}
V(x(t_{i_0+n})) &\leq m^nV(x(t_{i_0}))+\sum_{k=1}^{n}m^{k-1}\tau(\gamma_a+\tilde{\delta})||w||_\infty^2\\
&\leq \exp\left(-\frac{c\tau n}{c_2}\right)V(x(t_{i_0}))+\frac{\tau(\gamma_a+\tilde{\delta})}{(1-m)}||w||_\infty^2\ .
\end{aligned}
$$

Then, (B.23) and (B.22) in the proof of Theorem 3.4.5 are replaced by

$$
\begin{aligned}
|x(t,t_{i_0})| \le{}& \exp((a_m+\bar{\lambda})\tau)K\exp\left(-\bar{\lambda}(t-t_{i_0})\right)|x(t_{i_0})| \\
&+ \left(\exp(a_m\tau)-1\right)||w||_\infty + \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty\,. \qquad \text{(B.23)}
\end{aligned}
$$

and

$$
\begin{aligned}
|x(t)| \le{}& \exp(2(a_m+\bar{\lambda})\tau)K\exp\left(-\bar{\lambda}(t-t_0)\right)|x_0| + \left(\exp(a_m\tau)-1\right)(K+1)||w||_\infty \\
&+ \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty \\
&\le (K+\delta)(\exp(-(\lambda-\delta)(t-t_0))|x_0| + (\gamma+\delta)||w||_\infty,
\end{aligned}
$$

which give the conclusion of Theorem 3.4.7.

Appendix C

C.1 Proofs of Theorems 4.4.1 and 4.4.2

We need some technique results that are presented and proved in Subsection C.1.1 to show Theorems 4.4.1 and 4.4.2.

C.1.1 Technique results

Lemma C.1.1. *For a function f_0 defined on $C \times \mathbb{R}_{\geq 0}$, suppose f_{wa} is a continuous function that is a weak average of f_0 on C. Then, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a function α_K of class-$\mathcal{G}$ such that for all $((x, w), \mu, \tau) \in (C \cap K) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$:*

$$\mu |\eta_{wa}(x, w, \tau, \tau_0, \mu)| \leq \alpha_K(\mu) \ .$$

□

Proof of Lemma C.1.1

The proof uses the same technical method as Lemma 1 in [154] and follows the calculation of [74, p. 415]. Let the compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ be given. From the definitions of the weak average, for each $((x, w), \tau, \mu) \in C \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$, the following holds:

$$\begin{aligned} &|\eta_{wa}(x, w, \tau + T, \tau_0, 0) - \eta_{wa}(x, w, \tau, \tau_0, 0)| \\ &= \left| \int_{\tau}^{\tau+T} [f_0(x, w, s) - f_{wa}(x, w(s))] ds \right| \leq T \sigma_K(T) \ . \end{aligned} \tag{C.1}$$

Integrating by parts in the definition of η_{wa}, we have:

$$
\begin{aligned}
&\eta_{wa}(x,w,\tau,\tau_0,\mu)\\
&= \left[\exp(\mu(s-\tau))\int_{\tau_0}^{s}(f_0(x,w,r)-f_{wa}(x,w))dr\right]_{\tau_0}^{\tau}\\
&\quad -\mu\int_{\tau_0}^{\tau}\exp(\mu(s-\tau))\int_{\tau_0}^{s}(f_0(x,w,r)-f_{wa}(x,w))drds,\\
&= \eta_{wa}(x,w,\tau,\tau_0,0)-\mu\int_{\tau_0}^{\tau}\exp(\mu(s-\tau))\eta_{wa}(x,w,s,\tau_0,0)ds\ . \qquad (C.2)
\end{aligned}
$$

Then, adding and subtracting $\mu\eta_{wa}(x,w,\tau,\tau_0,0)\int_{\tau_0}^{\tau}\exp(\mu(s-\tau))ds$ to the right hand side of (C.4), we obtain

$$
\begin{aligned}
&\eta_{wa}(x,w,\tau,\tau_0,\mu)\\
&= \exp(-\mu(\tau-\tau_0))\eta_{wa}(x,w,\tau,\tau_0,0)\\
&\quad +\mu\int_{\tau_0}^{\tau}\exp(-\mu(\tau-s))[\eta_{wa}(x,w,\tau,\tau_0,0)-\eta_{wa}(x,w,s,\tau_0,0)]ds\ .
\end{aligned}
$$

Let $\hat{\tau}:=\tau-\tau_0$. Using the fact $\eta_{wa}(x,w,\tau_0,\tau_0,0)=0$ and (C.1), it follows that

$$
\begin{aligned}
\mu|\eta_{wa}(x,w,\tau,\tau_0,\mu)| &\le \exp(-\mu(\tau-\tau_0))\mu(\tau-\tau_0)\sigma_K(\tau-\tau_0)\\
&\quad +\mu^2\int_{\tau_0}^{\tau}\exp(-\mu(\tau-s))(\tau-s)\sigma_K(\tau-s)ds\\
&= \exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau})+\mu^2\int_0^{\hat{\tau}}\exp(-\mu r)r\sigma_K(r)dr\\
&= \exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau})+\int_0^{\mu\hat{\tau}}\exp(-z)z\sigma_K\left(\frac{z}{\mu}\right)dz\ .
\end{aligned}
$$

There are two possibilities for $\mu\tilde{\tau}$: $\mu\tilde{\tau}\le\sqrt{\mu}$ and $\mu\tilde{\tau}\ge\sqrt{\mu}$. In the first case, we have

$$
\exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau})+\int_0^{\mu\hat{\tau}}\exp(-z)z\sigma_K\left(\frac{z}{\mu}\right)dz\le\sqrt{\mu}\sigma_K(0)+\frac{\mu}{2}\sigma_K(0)\ .
$$

For the second case, using $\eta\exp(-\eta)\le\exp(-1)$ for all $\eta\ge 0$ and $\int_0^{\infty}\exp(-z)zdz=1$, and then

$$\begin{aligned}
&\exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau}) + \int_0^{\mu\hat{\tau}} \exp(-z)z\sigma_K\left(\frac{z}{\mu}\right)dz \\
&\leq \exp(-1)\sigma_K\left(\frac{1}{\sqrt{\mu}}\right) + \sigma_K(0)\int_0^{\mu\hat{\tau}} zdz + \sigma_K\left(\frac{1}{\sqrt{\mu}}\right)\int_{\mu\hat{\tau}}^{\infty} z\exp(-z)dz \\
&\leq (\exp(-1)+1)\sigma_K\left(\frac{1}{\sqrt{\mu}}\right) + \frac{\mu}{2}\sigma_K(0) \ .
\end{aligned}$$

Then, let

$$\alpha_K(\mu) := \frac{\mu}{2}\sigma_K(0) + \max\{\sqrt{\mu}\sigma_K(0), \sigma_K\left(\frac{1}{\sqrt{\mu}}\right)(\exp(-1)+1)\} \ .$$

Since σ_K is of class-$\mathcal{L}$, it follows that α_K is of class-$\mathcal{G}$. □

Lemma C.1.2. *For a function f_0 defined on $C_1 \times \mathcal{W} \times \mathbb{R}_{\geq 0}$, where $C \subset C_1 \times \mathcal{W}$, suppose f_{sa} is a continuous function that is a strong average of f_0 on $C_1 \times \mathcal{W}$. Then, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a function α_K of class-$\mathcal{G}$ such that for all $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x, \tilde{w}(s)), \mu, \tau) \in ((C_1 \times \mathcal{W}) \cap K) \times \mathbb{R}_{\geq 0} \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$, the following holds:*

$$\mu|\eta_{sa}(x, \tilde{w}, \tau, \tau_0, \mu)| \leq \alpha_K(\mu) \ .$$

□

Proof of Lemma C.1.2:

Let the compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ be given. From the definitions of the strong average, for any $0 \leq \tau_0 \leq \tau_1$, $\tau, (\tau + T) \in [\tau_0, \tau_1]$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $(x, \tilde{w}(s)) \in (C_1 \times \mathcal{W}) \cap K$ for all $s \in [\tau_0, \tau_1]$, the following holds:

$$\begin{aligned}
&|\eta_{sa}(x, \tilde{w}, \tau + T, \tau_0, 0) - \eta_{sa}(x, \tilde{w}, \tau, \tau_0, 0)| \\
&= \left|\int_{\tau}^{\tau+T} [f_0(x, \tilde{w}(s), s) - f_{sa}(x, \tilde{w}(s))]ds\right| \leq T\sigma_K(T) \ . \qquad \text{(C.3)}
\end{aligned}$$

Integrating by parts in the definition of η_{sa}, we have:

$$\begin{aligned}
&\eta_{sa}(x,\tilde{w},\tau,\tau_0,\mu)\\
&= \left[\exp(\mu(s-\tau))\int_{\tau_0}^{s}(f_0(x,\tilde{w}(r),r)-f_{sa}(x,\tilde{w}(r)))dr\right]_{\tau_0}^{\tau}\\
&\quad -\mu\int_{\tau_0}^{\tau}\exp(\mu(s-\tau))\int_{\tau_0}^{s}(f_0(x,\tilde{w}(r),r)-f_{sa}(x,\tilde{w}(r)))drds,\\
&= \eta_{sa}(x,\tilde{w},\tau,\tau_0,0)-\mu\int_{\tau_0}^{\tau}\exp(\mu(s-\tau))\eta_{sa}(x,\tilde{w},s,\tau_0,0)ds\ . \qquad \text{(C.4)}
\end{aligned}$$

Then, adding and subtracting $\mu\eta_{sa}(x,\tilde{w},\tau,\tau_0,0)\int_{\tau_0}^{\tau}\exp(\mu(s-\tau))ds$ to the right hand side of (C.4), we obtain

$$\begin{aligned}
&\eta_{sa}(x,\tilde{w},\tau,\tau_0,\mu)\\
&= \exp(-\mu(\tau-\tau_0))\eta_{sa}(x,\tilde{w},\tau,\tau_0,0)\\
&\quad +\mu\int_{\tau_0}^{\tau}\exp(-\mu(\tau-s))[\eta_{sa}(x,\tilde{w},\tau,\tau_0,0)-\eta_{sa}(x,\tilde{w},s,\tau_0,0)]ds\ .
\end{aligned}$$

Then, the following steps are identical to the proof of Lemma C.1.1 with replacing (C.1) and (C.2) by (C.3) and (C.4) respectively. □

Claim C.1.3. *Under Assumption 4.3.4, for each compact set* $K\subset\mathbb{R}^n\times\mathbb{R}^m$ *there exists* $L(K)$ *such that, for each* $i\in\bar{N}$, $\mu>0$, $((x_1,w_1),\tau_a),((x_2,w_2),\tau_b)\in(C\cap K)\times\mathbb{R}_{\geq 0}$ *and* $\tau_0\in[0,\min\{\tau_a,\tau_b\}]$*:*

$$\left|\eta_{wa}^{i}(x_1,w_1,\tau_a,\tau_0,\mu)-\eta_{wa}^{i}(x_2,w_2,\tau_b,\tau_0,\mu)\right|\leq 2L(|x_1-x_2|+|w_1-w_2|+|\tau_a-\tau_b|)\ .$$

□

Proof of Claim C.1.3

Let the compact $K\subset\mathbb{R}^n\times\mathbb{R}^m$ be given. Integrate by parts in the definition of η_{wa} to get (C.2). Then, for all $i\in\bar{N}$, $((x_1,w_1),\tau_a),((x_2,w_2),\tau_b)\in(C\cap K)\times\mathbb{R}_{\geq 0}$ and $\tau_0\in[0,\min\{\tau_a,\tau_b\}]$, it follows from Assumption 4.3.4 that:

$$\begin{aligned}
&\left|\eta^i_{wa}(x_1, w, \tau_a, \tau_0, \mu) - \eta^i_{wa}(x_2, w, \tau_b, \tau_0, \mu)\right| \\
&\quad \leq \left|\eta^i_{wa}(x_1, w, \tau_a, \tau_0, 0) - \eta^i_{wa}(x_2, w, \tau_b, \tau_0, 0)\right| \\
&\qquad +\mu \int_{\tau_0}^{\tau_a} \exp(\mu(s-\tau_a))|\eta^i_{wa}(x_1, w, s, \tau_0, 0) - \eta^i_{wa}(x_2, w, s+\tau_b-\tau_a, \tau_0, 0)|ds \\
&\quad \leq L(|x_1 - x_2| + |\tau_a - \tau_b|)\left(1 + \mu \int_{\tau_0}^{\tau_a} \exp(\mu(s-\tau_a))ds\right) \\
&\quad \leq 2L(|x_1 - x_2| + |\tau_a - \tau_b|) \ , \qquad \text{(C.5)}
\end{aligned}$$

where the last inequality in (C.5) follows from the fact $\mu \int_{\tau_0}^{\tau_a} \exp(\mu(s-\tau_a))ds \leq 1$ for any $\mu, \tau_0, \tau_a \geq 0$. □

Claim C.1.4. *Under Assumption 4.3.5, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists $L(K)$ such that, for each $i \in \bar{N}$, $\mu > 0$, $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x_1, \tilde{w}(s)), \tau_a)$, $((x_2, \tilde{w}(s)), \tau_b) \in ((C_1 \times \mathcal{W}) \cap K) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$, the following holds:*

$$\left|\eta^i_{sa}(x_1, \tilde{w}, \tau_a, \tau_0, \mu) - \eta^i_{sa}(x_2, \tilde{w}, \tau_b, \tau_0, \mu)\right| \leq 2L(|x_1 - x_2| + |\tau_a - \tau_b|) \ .$$

□

Proof of Claim C.1.4

Integrate by parts in the definition of η_{sa} to get (C.4). The proof of Claim C.1.4 is identical to the proof of Claim C.1.3 with replacing (C.2) by (C.4). □

Claim C.1.5. *Let $J \subset \mathbb{R}^n$ be closed, $L > 0$, and $M > 0$. For a vector-valued function $f := (f_1, \cdots, f_n)$ where $f_i : J \to \mathbb{R}$ are real-valued functions, define*

$$\tilde{g}_i(x) := \sup_{z \in J}\{f_i(z) - L|x - z|\} \ .$$

Let

$$\mathrm{sat}(s) := \frac{Ms}{\max\{M, |s|\}} \ , \qquad \text{(C.6)}$$

and $g(x) := \mathrm{sat}(\tilde{g}(x))$ with $\tilde{g} := (\tilde{g}_1, \cdots, \tilde{g}_n)$. Then, the function g satisfies the following properties:

1. $|g(x)| \leq M$ *for all* $x \in \mathbb{R}^n$,

2. $|g(x) - g(y)| \leq \sqrt{n}L|x - y|$ *for all* $x, y \in \mathbb{R}^n$ *and*

3. *if, for all* $i \in \{1, \cdots, n\}$, $x, y \in J$, $|f(x)| \leq M$ *and* $|f_i(x) - f_i(y)| \leq L|x - y|$, *then* $g(x) = f(x)$ *for all* $x \in J$.

□

Proof of Claim C.1.5

Noting $g(x) = \text{sat}(\tilde{g}(x))$ and (C.6), it is straightforward that the first property is satisfied. Let $\bar{N} = \{1, \cdots, n\}$. Let $k \in \bar{N}$ satisfy $|\tilde{g}_k(x) - \tilde{g}_k(y)| = \max_{i \in \bar{N}} |\tilde{g}_i(x) - \tilde{g}_i(y)|$. Without loss of generality, assume $\tilde{g}_k(x) \geq \tilde{g}_k(y)$. Using the fact $|\text{sat}(\xi) - \text{sat}(\psi)| \leq |\xi - \psi|$ for all $\xi, \psi \in \mathbb{R}^n$, the extended function g satisfies

$$
\begin{aligned}
|g(x) - g(y)| &= |\text{sat}(\tilde{g}(x)) - \text{sat}(\tilde{g}(y))| \\
&\leq |\tilde{g}(x) - \tilde{g}(y)| \\
&= \left(\sum_{i=1}^{n} |\tilde{g}_i(x) - \tilde{g}_i(y)|^2 \right)^{\frac{1}{2}} \\
&\leq \left(n \cdot |\tilde{g}_k(x) - \tilde{g}_k(y)|^2 \right)^{\frac{1}{2}} \\
&\leq \left(n \cdot \sup_{a \in J} |L|a - x| - L|a - y||^2 \right)^{\frac{1}{2}} \\
&\leq \left(nL^2 \cdot \sup_{a \in J} |a - x - a + y|^2 \right)^{\frac{1}{2}} = \sqrt{n}L|x - y| \ ,
\end{aligned}
$$

for all $x, y \in \mathbb{R}^n$. Now, consider the third property. Let $M > 0$ be such that

$$
|f(x)| \leq M \quad \forall\, x \in J. \tag{C.7}
$$

Let $L > 0$ be such that

$$
|f_i(x) - f_i(y)| \leq L|x - y| \quad \forall\, x, y \in J, i \in \bar{N}. \tag{C.8}
$$

Using (C.8), for $x \in J$, we have

$$
\begin{aligned}
f_i(x) &\le \sup_{z\in J}\{f_i(z) - L|x-z|\} \\
&= \tilde{g}_i(x) = \sup_{z\in J}\{f_i(z) - f_i(x) + f_i(x) - L|x-z|\} \le f_i(x) \ .
\end{aligned}
$$

Noting the construction of g_i in (C.6), we have that $g_i(x) \ge \mathrm{sat}\,(f_i(x))$ with letting $z = x$ for all $x \in J$. Moreover, with we have for any $x \in J$:

$$
\begin{aligned}
g_i(x) &= \mathrm{sat}\left(\sup_{z\in J}\{f_i(z) - f_i(x) + f_i(x) - L|x-z|\}\right) \\
&\le \mathrm{sat}\,(f_i(x)) = f_i(x) \ ,
\end{aligned}
$$

which shows $\tilde{g}_i(x) = f_i(x)$ for all $x \in J$. Then, with (C.7) and the definition of g, it follows that $g(x) = \mathrm{sat}(f(x)) = f(x)$ when $x \in J$. □

Claim C.1.6. *The hybrid arc ξ is a solution to the hybrid inclusion*

$$
H_\Omega \quad \begin{array}{ll} \dot{\xi} \in F_\Omega(\xi) & \xi \in C_\Omega \\ \xi^+ \in G_\Omega(\xi) & \xi \in D_\Omega \end{array} \tag{C.9}
$$

that is extended from system H in (4.2) for some $\Omega \ge 0$ with the data $(F_\Omega, G_\Omega, C_\Omega, D_\Omega)$ being defined as:

$$
\begin{aligned}
F_\Omega(\xi) &:= \{v \in \mathbb{R}^n : v = F(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ and } (\xi, w) \in C\} \\
G_\Omega(\xi) &:= \{v \in \mathbb{R}^n : v \in G(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ and } (\xi, w) \in D\} \\
C_\Omega &:= \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ such that } (\xi, w) \in C)\} \\
D_\Omega &:= \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ such that } (\xi, w) \in D)\} \ ,
\end{aligned} \tag{C.10}
$$

if and only if there exists a hybrid input w_1 such that (ξ, w_1) is a solution pair to system H in (4.2) with $|w_1| \le \Omega$. □

Proof of Claim C.1.6

The proof is identical to [28, Claim 3.7].

Proposition C.1.7. *Suppose that system H in (4.2) satisfies Assumption 4.2.4,* *and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $\rho > 0$ and $T \geq 0$ there exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, each solution pair $(\bar{x}, w)$ of system H_δ in (4.19) with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ and $|w| \leq \Omega$ there exists a solution pair (ξ, w_1) to system H with $\xi(0,0) \in K_0$ and $|w_1| \leq |w|$ such that $\bar{x}$ and ξ are (T, ρ)-close.* $\square$

Proof of Proposition C.1.7

Let the compact set K_0 and $\Omega \geq 0$ be given. For some $\delta > 0$, let $(\bar{x}, w)$ be a solution pair to system H_δ with $\bar{x}(0,0) \in (K_0+\delta\mathbb{B})$ and w with $|w| \leq \Omega$. Let $\Omega_1 := |w| \in [0, \Omega]$. Consider a hybrid inclusion $H_{(\Omega_1,\delta)} := \{F_{(\Omega_1,\delta)}, G_{(\Omega_1,\delta)}, C_{(\Omega_1,\delta)}, D_{(\Omega_1,\delta)}\}$ formed as (C.9) with its data being constructed from the system $H_\delta := \{F_\delta, G_\delta, C_\delta, D_\delta\}$ in (4.19):

$$\begin{aligned} F_{(\Omega_1,\delta)}(\bar{x}) &:= \{v \in \mathbb{R}^n : v \in F_\delta(\bar{x}, w), w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ and } (\bar{x}, w) \in C_\delta\} \\ G_{(\Omega_1,\delta)}(\bar{x}) &:= \{v \in \mathbb{R}^n : v \in G_\delta(\bar{x}, w), w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ and } (\bar{x}, w) \in D_\delta\} \\ C_{(\Omega_1,\delta)} &:= \{\bar{x} : \exists w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ such that } (\bar{x}, w) \in C_\delta)\} \\ D_{(\Omega_1,\delta)} &:= \{\bar{x} : \exists w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ such that } (\bar{x}, w) \in D_\delta)\} \,. \end{aligned} \tag{C.11}$$

Note that the data $\{F_{(\Omega_1,\delta)}, G_{(\Omega_1,\delta)}, C_{(\Omega_1,\delta)}, D_{(\Omega_1,\delta)}\}$ in (C.11) satisfies

$$\begin{aligned} F_{(\Omega_1,\delta)}(\bar{x}) &= \overline{con} F_{\Omega_1}((\bar{x} + \delta\mathbb{B}) \cap C_{\Omega_1})) + \delta\mathbb{B} \\ G_{(\Omega_1,\delta)}(\bar{x}) &= G_{\Omega_1}((\bar{x} + \delta\mathbb{B}) \cap D_{\Omega_1}) + \delta\mathbb{B} \\ C_{(\Omega_1,\delta)} &= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap C_{\Omega_1} \neq \emptyset\} \\ D_{(\Omega_1,\delta)} &= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap D_{\Omega_1} \neq \emptyset\} \,, \end{aligned} \tag{C.12}$$

with $\{F_{\Omega_1}, G_{\Omega_1}, C_{\Omega_1}, D_{\Omega_1}\}$ defined as (C.10). From (C.12), it is straightforward that $H_{(\Omega_1,\delta)}$ is an inclusion inflated from $H_{\Omega_1} := \{F_{\Omega_1}, G_{\Omega_1}, C_{\Omega_1}, D_{\Omega_1}\}$.

Consider arbitrary $\rho > 0$ and $T \geq 0$. Note that forward pre-completeness of H_{Ω_1} on the set K_0 comes from Claim C.1.6 and the assumption that H is forward pre-complete from K_0. Noting that for each $\xi \in C_{\Omega_1}$, $F_{\Omega_1}(\xi)$ is convex in Assumption 4.2.4, we have $F_{\Omega_1}(\xi) = \overline{con}\ F_{\Omega_1}(\xi)$ for each $\xi \in C_{\Omega_1}$. Using the results of [56, Corollary 5.2] and [56, Theorem 5.4], we have that there exists a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ and for each solution $\bar{x}$ of $H_{(\Omega_1,\delta)}$ with $\bar{x}(0,0) \in K_0 + \delta\mathbb{B}$ there exists a solution ξ to H_{Ω_1} with $\xi(0,0) \in K_0$ such that $\bar{x}$

and ξ are (T, ρ)-close. Consider H_δ in (4.19) and $H_{(\Omega_1,\delta)}$ in (C.11) with $\delta \in (0, \delta^*]$. Note that for each solution pair $(\bar{x}, w)$ of system H_δ there exists a solution $\bar{x}$ to the inclusion $H_{(\Omega_1,\delta)}$. Considering any solution pair $(\bar{x}, w)$ of system H_δ with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ and w with $|w| = \Omega_1 \in [0, \Omega]$, noting the closeness of solutions $\bar{x}$ to $H_{(\Omega_1,\delta)}$ and ξ to H_{Ω_1} and applying Claim C.1.6 completes the proof. □

Proposition C.1.8. *Suppose that system H in (4.2) satisfies Assumption 4.2.4 and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $T \geq 0$ the reachable set*

$$R_T(K_0, \Omega) := \{\xi(t, j) : (\xi, w) \in S(K_0), t + j \leq T, |w| \leq \Omega\} \tag{C.13}$$

is compact, where $S(K_0)$ denotes the set of maximal solution pairs (ξ, w) to system H in (4.2) with $\xi(0,0) \in K_0$. □

Proof of Proposition C.1.8

The result follows using Claim C.1.6 and the result of [56, Corollary 4.7]. □

C.1.2 Proof of Theorem 4.4.1

Let $\Omega, \Omega_1 > 0$ come from the definitions of equi-essential boundedness and local equi-uniform Lipschitz continuity respectively. Let the compact set K_0 be given. Let $T \geq 0$ and $\rho > 0$ be given. Apply Proposition C.1.7 with the set K_0 and (T, ρ, Ω) to generate a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ and the system H_δ inflated from the weak average system H_{wa}, for each solution pair $(\bar{x}, w)$ to H_δ with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ there exists a solution pair (ξ, w_1) to system H_{wa} with $\xi(0,0) \in K_0$ and $|w_1| \leq |w|$ such that the solutions $\bar{x}$ and ξ are $(T, \frac{\rho}{2})$-close. Without loss of generality, assume that $\delta < 1$ and $\rho < 1$.

Let $S_{wa}(K_0)$ denote the set of maximum solution pairs to the weak average system H_{wa} with $\xi(0,0) \in K_0$ and define the set K as

$$\begin{aligned} R_T(K_0, \Omega) &:= \{\xi(t, j) : (\xi, w) \in S_{wa}(K_0), t + j \leq T, |w| \leq \Omega\}, \\ K_1 &:= R_T(K_0) + \mathbb{B}, \\ K &:= K_1 \cup G((K_1 \times \Omega\mathbb{B}) \cap D), \end{aligned} \tag{C.14}$$

where K_1 is compact from Proposition C.1.8. The set K is also compact as G is outer semi-continuous and locally bounded.

Set $\bar{K} := K \times \Omega\mathbb{B} \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $\eta_{wa}(x, w, \tau, \tau_0, \mu)$ be defined as (4.11). Let $\bar{K}$ generate $L\left(\bar{K}\right) \geq 1$ such that Assumption 4.3.4 holds for all $((x_1, w_1), \tau_a)$, $((x_2, w_2), \tau_b) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$ with $L := L\left(\bar{K}\right)$. Let $\bar{K}$ and Lemma C.1.1 generate $\alpha_{\bar{K}}$ and pick $\mu > 0$ such that $\alpha_{\bar{K}}(\mu) \leq \frac{\delta}{3}$. Then, for this μ, for all $((x, w), \tau) \in \left(C \cap \bar{K}\right) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$, we have $|\eta_{wa}(x, w, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}$.

Let $J := \left(C \cap \bar{K}\right) \times \mathbb{R}_{\geq 0}$. Claim C.1.5 gives us a new function $\tilde{\eta}_{wa}$ that defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. For the picked μ, the following properties are satisfied for $\tilde{\eta}_{wa}$.

1. for all $(x, w, \tau) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$:
$$|\tilde{\eta}_{wa}(x, w, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}, \tag{C.15}$$
2. $|\tilde{\eta}_{wa}(x_1, w_1, \tau_a, \tau_0, \mu) - \tilde{\eta}_{wa}(x_2, w_2, \tau_b, \tau_0, \mu)| \leq 2\sqrt{n}L(|x_1 - x_2| + |w_1 - w_2| + |\tau_a - \tau_b|)$ for each (x_1, w_1, τ_a), $(x_2, w_2, \tau_b) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$,
3. $\tilde{\eta}_{wa}(x, w, \tau, \tau_0, \mu) = \eta_{wa}(x, w, \tau, \tau_0, \mu)$ for all $((x, w), \tau) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$.

Let Assumption 4.3.1, δ and the set $\bar{K}$ generate $M\left(\bar{K}\right) \geq 1$ and ε_1^* such that the bounds (4.8) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$. Let $\varepsilon_2^* = \frac{\delta}{6\sqrt{n}L(M+1+\Omega_1)}$, $\varepsilon_3^* = \frac{3\rho\mu}{2\delta}$, $\varepsilon_4^* = 3\mu$ and $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$. Consider $\varepsilon \in (0, \varepsilon^*]$.

Let (x, w, τ) be a solution to the system

$$H_K \quad \left.\begin{array}{l} \dot{x} = f_\varepsilon(x, w, \tau) \\ \dot{\tau} = \frac{1}{\varepsilon} \end{array}\right\} \quad ((x, w), \tau) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}, \qquad \left.\begin{array}{l} x^+ \in G(x, w) \cap K \\ \tau^+ \in H(x, w, \tau) \end{array}\right\} \quad ((x, w), \tau) \in (D \cap \bar{K}) \times \mathbb{R}_{\geq 0}. \tag{C.16}$$

Note that the system H_K agrees with system H_ϵ but with G intersected with K and C, D intersected with $K \times \Omega\mathbb{B}$. By construction, for each $(t, j) \in \text{dom}\,(x, w, \tau)$, we have $(x(t, j), w(t, j)) \in \bar{K}$. With (C.15) and the definitions of ε, we have for all $(t, j) \in \text{dom}\,(x, w, \tau)$:

$$|\varepsilon\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)| \leq \frac{\varepsilon\delta}{3\mu} \leq \delta \tag{C.17}$$

holds for all $\tau_0 \in [0, \tau(t,j)]$. For each $(t,j) \in \text{dom}\ (x,w,\tau)$, define

$$\bar{x}(t,j) = x(t,j) - \varepsilon\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu), \tag{C.18}$$

with $\tau_0 := \tau(t_j, j)$ and $t_j := \min\{t : (t,j) \in \text{dom}\ (x,w,\tau)\}$. It follows that $\bar{x}$ is a hybrid arc. For each $(t,j) \in \text{dom}\ \bar{x}$ such that for all $(t,j+1) \in \text{dom}\ \bar{x}$,

$$(x(t,j),w(t,j)) = (\bar{x}(t,j) + \varepsilon\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu),w(t,j)) \in D \cap \bar{K},$$

which with (C.17) implies that $(\bar{x}(t,j),w(t,j)) \in D_\delta$ and

$$\begin{aligned}
\bar{x}(t,j+1) &= x(t,j+1) - \varepsilon\tilde{\eta}_{wa}(x(t,j+1),w(t,j+1),\tau(t,j+1),\tau_0,\mu) \\
&\in (G((x(t,j),w(t,j)) \cap D) + \delta\mathbb{B}) \cap K \\
&\subset G((x(t,j),w(t,j)) \cap D) + \delta\mathbb{B} \\
&= G((\bar{x}(t,j) + \varepsilon\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu),w(t,j)) \cap D) + \delta\mathbb{B} \\
&\subset G((\bar{x}(t,j) + \delta\mathbb{B},w(t,j)) \cap D) + \delta\mathbb{B} \\
&= G_\delta(\bar{x}(t,j),w(t,j)) \ .
\end{aligned}$$

Moreover, for each j such that the set $I_j := \{t : (t,j) \in \text{dom}\ \bar{x}\}$ has nonempty interior and for all $t \in I_j$,

$$(x(t,j),w(t,j)) = (\bar{x}(t,j) + \varepsilon\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu),w(t,j)) \in C \cap \bar{K}$$

implies that $(\bar{x}(t,j),w(t,j)) \in C_\delta$. Noting $\tilde{\eta}_{wa}$ is globally Lipschitz continuous, $\bar{x}(\cdot,j)$ is locally absolutely continuous and the set $\mathcal{L}_{\mathcal{W}}$ is locally equi-uniformly Lipschitz continuous, and for almost all $t \in I_j$ we have

$$
\begin{aligned}
&\dot{\bar{x}}(t,j)\\
&\in \;\; \dot{x}(t,j) - \varepsilon\frac{\partial\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)}{\partial x}\dot{x}(t,j)\\
&\quad -\varepsilon\frac{\partial\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)}{\partial w}\dot{w}(t,j) - \frac{\partial\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)}{\partial \tau}\\
&= \;\; f_\varepsilon(x(t,j),w(t,j),\tau(t,j)) - \varepsilon\frac{\partial\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)}{\partial x}\dot{x}(t,j)\\
&\quad -\varepsilon\frac{\partial\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)}{\partial w}\dot{w}(t,j) - f_0(x(t,j),w(t,j),\tau(t,j))\\
&\quad + f_{wa}(x(t,j),w(t,j)) - \mu\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu)\\
&\in \;\; f_{wa}(\bar{x}(t,j)+\varepsilon\tilde{\eta}_{wa}(x(t,j),w(t,j),\tau(t,j),\tau_0,\mu),w(t,j)) + \frac{\delta\mathbb{B}}{3}\\
&\quad +\varepsilon 2\sqrt{n}L(M+1+\Omega_1)\mathbb{B} + \alpha_K(\mu)\mathbb{B}\\
&\in \;\; F(\bar{x}(t,j)+\delta\mathbb{B},w(t,j)) + \delta\mathbb{B}\\
&\subset \;\; F_\delta(\bar{x}(t,j),w(t,j))\;, \qquad\qquad (\text{C.19})
\end{aligned}
$$

where

$$
\left[\frac{\partial\tilde{\eta}_{wa}(x,w,\tau,\tau_0,\mu)}{\partial x}, \frac{\partial\tilde{\eta}_{wa}(x,w,\tau,\tau_0,\mu)}{\partial w}, \frac{\partial\tilde{\eta}_{wa}(x,w,\tau,\tau_0,\mu)}{\partial \tau}\right]
$$

can be considered as generalized Jacobian of $\tilde{\eta}_{wa}$. The sequence of equalities and inclusions in (C.19) hold from the results in (Section 2.6, [38]) with Assumption 4.3.1, definitions of ε^* and μ. Then, it follows that $(\bar{x}, w)$ is the solution pair of system H_δ, and we can conclude that for each $(\bar{x}, w)$ there exists some solution pair (ξ, w_1) to system H_{wa} such that $\bar{x}$ and ξ are $(T, \frac{\rho}{2})$ close. Moreover, from the definition of $\bar{x}$ in (C.18) and definition of ε^*, we know that for the solution pair (x, w) to system H_K, x is $(T, \frac{\rho}{2})$ close to $\bar{x}$ and then it is (T, ρ)-close to ξ.

Next, consider solution pairs of system H_ϵ that start in K_0. Let $(\tilde{x}, w)$ be a solution pair to system H_ϵ with $\tilde{x}(0,0) \in K_0$ and $|w| \leq \Omega$. If $\tilde{x} \in K$ for all $(t,j) \in \text{dom } \tilde{x}$ with $t + j \leq T$, then for each solution pair $(\tilde{x}, w)$ of H_ϵ, there exists some solution pair (ξ, w_1) of system H_{wa} such that $\tilde{x}$ is also (T, ρ) close to ξ. Now, suppose that there exists $(t,j) \in \text{dom } \tilde{x}$ such that $\tilde{x}(s,i) \in K$ satisfying $s + i \leq t + j$ and either

1. $(t, j+1) \in \text{dom } \tilde{x}$ and $\tilde{x}(t, j+1) \notin K$ or else,

2. there exist a monotonically decreasing sequences r_i with the limit $\lim_{i\to\infty} r_i = t$ such that $(r_i, j) \in \text{dom } \tilde{x}$ and $\tilde{x}(r_i, j) \notin K$ for each i.

The solution pair $(\tilde{x}, w)$ must agree with a solution pair of system H_K up to time (t, j), and thus must satisfy $\tilde{x} \in R_T(K_0) + \rho\mathbb{B}$. If this follows, by the definition of K in (C.14) and $\rho < 1$, which implies that $R_T(K_0) + \rho\mathbb{B}$ is contained inside of K, that neither of these two case can occur. This establishes the result.

C.1.3 Proof of Theorem 4.4.2

The proof of Theorem 4.4.2 follows exactly the same steps in the proof of Theorem 4.4.1 with following changes. Let $\Omega \geq 0$ comes from the definition of equi-essential boundedness and δ be same as the proof of Theorem 4.4.1. Let K be defined as (C.14) for strong average system H_{sa}. Let the set $\bar{K} := K \times \Omega\mathbb{B}$ and δ generate $M\left(\bar{K}\right) \geq 1$ and ε_1^* such that bounds (4.8) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$.

Let $\eta_{sa}(x, \tilde{w}, \tau, \tau_0, \mu)$ be defined as (4.12). Let $\bar{K}$ generate $L\left(\bar{K}\right) \geq 1$ such that Assumption 4.3.5 holds with $L := L\left(\bar{K}\right)$. Let the set $\bar{K}$ and Lemma C.1.2 generate $\alpha_{\bar{K}}$ and pick $\mu > 0$ such that $\alpha_{\bar{K}}(\mu) \leq \frac{\delta}{3}$. Then, for this μ, for all $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x, \tilde{w}(s)), \tau) \in \left((C_1 \times \mathcal{W}) \cap \bar{K}\right) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$, we have $|\eta_{sa}(x, \tilde{w}, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}$.

Let $J := ((C_1 \times \mathcal{W}) \cap \bar{K}) \times \mathbb{R}_{\geq 0}$. Using the result in Claim C.1.5, we have the function $\tilde{\eta}_{sa}$ such that, for the picked μ and for all $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$, the following properties are satisfied:

1. for each $(x, \tilde{w}(s), \tau) \in \mathbb{R}^n \times \mathbb{R}^m \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$:
$$|\tilde{\eta}_{sa}(x, \tilde{w}, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu} \ , \tag{C.20}$$
2. $|\tilde{\eta}_{sa}(x_1, \tilde{w}, \tau_a, \tau_0, \mu) - \tilde{\eta}_{sa}(x_2, \tilde{w}, \tau_b, \tau_0, \mu)| \leq 2\sqrt{n}L(|x_1 - x_2| + |\tau_a - \tau_b|)$ for all $(x_1, \tilde{w}(s), \tau_a)$, $(x_2, \tilde{w}(s), \tau_b) \in \mathbb{R}^n \times \mathbb{R}^m \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$,
3. $\tilde{\eta}_{sa}(x, \tilde{w}, \tau, \tau_0, \mu) = \eta_{sa}(x, \tilde{w}, \tau, \tau_0, \mu)$ for all $((x, \tilde{w}(s)), \tau) \in ((C_1 \times \mathcal{W}) \cap \bar{K}) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$.

Let Assumption 4.3.1, δ and the set $\bar{K}$ generate $M\left(\bar{K}\right) \geq 1$ and ε_1^* such that the bounds (4.8) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$. Let $\varepsilon_2^* = \frac{\delta}{6\sqrt{n}L(M+1)}$, $\varepsilon_3^* = \frac{3\rho\mu}{2\delta}$, $\varepsilon_4^* = 3\mu$ and $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$. Consider $\varepsilon \in (0, \varepsilon^*]$.

Letting (x, w, τ) be a solution to the system H_K in (C.16), it follows from the construction of H_K that $(x(t, j), w(t, j)) \in \bar{K}$ for all $(t, j) \in \text{dom}\ (x, w, \tau)$. Let $\tau_0 := \tau\left(t_j^0, j\right)$ and $\tau_1 := \tau\left(t_j^1, j\right)$ with $t_j^0 := \min\{t : (t, j) \in \text{dom}\ (x, w, \tau)\}$

and $t_j^1 := \max\{t : (t,j) \in \text{dom } (x,w,\tau)\}$. Let $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ be defined as $\tilde{w}(\tau(s,j)) := w(s,j)$ for each $s \in \{t : (t,j) \in \text{dom } (x,w,\tau)\}$. With (C.20) and the definition of ε, it follows that for all $(t,j) \in \text{dom } (x,w,\tau)$,

$$|\varepsilon\tilde{\eta}_{sa}(x(t,j), \tilde{w}, \tau(t,j), \tau_0, \mu)| \leq \frac{\varepsilon\delta}{3\mu} \leq \delta \ . \tag{C.21}$$

For each $(t,j) \in \text{dom } (x,w,\tau)$, define

$$\bar{x}(t,j) = x(t,j) - \varepsilon\tilde{\eta}_{sa}(x(t,j), \tilde{w}, \tau(t,j), \tau_0, \mu) \ .$$

For each $(t,j) \in \text{dom } \bar{x}$ such that for all $(t,j+1) \in \text{dom } \bar{x}$,

$$\begin{aligned}
&(x(t,j), w(t,j)) \\
&\qquad = (\bar{x}(t,j) + \varepsilon\tilde{\eta}_{sa}(x(t,j), \tilde{w}, \tau(t,j), \tau_0, \mu), w(t,j)) \in D \cap \bar{K}
\end{aligned}$$

with (C.21) implies that $(\bar{x}(t,j), w(t,j)) \in D_\delta$ and

$$\begin{aligned}
&\bar{x}(t,j+1) \\
&\quad = \quad x(t,j+1) - \varepsilon\tilde{\eta}_{sa}(x(t,j+1), \tilde{w}, \tau(t,j+1), \tau_0, \mu) \\
&\quad \in \quad (G((x(t,j), w(t,j)) \cap D) + \delta\mathbb{B}) \cap K \\
&\quad \subset \quad G((x(t,j), w(t,j)) \cap D) + \delta\mathbb{B} \\
&\quad = \quad G((\bar{x}(t,j) + \varepsilon\tilde{\eta}_{sa}(x(t,j), \tilde{w}, \tau(t,j), \tau_0, \mu), w(t,j)) \cap D) + \delta\mathbb{B} \\
&\quad \subset \quad G((\bar{x}(t,j) + \delta\mathbb{B}, w(t,j)) \cap D) + \delta\mathbb{B} \\
&\quad = \quad G_\delta(\bar{x}(t,j), w(t,j)) \ .
\end{aligned}$$

Moreover, for each j such that the set $I_j := \{t : (t,j) \in \text{dom } \bar{x}\}$ has nonempty interior and for all $t \in I_j$,

$$\begin{aligned}
&(x(t,j), w(t,j)) \\
&\qquad = (\bar{x}(t,j) + \varepsilon\tilde{\eta}_{sa}(x(t,j), \tilde{w}, \tau(t,j), \tau_0, \mu), w(t,j)) \in C \cap \bar{K}
\end{aligned}$$

implies that $(\bar{x}(t,j), w(t,j)) \in C_\delta$. Noting the definition of $\tilde{w}$, instead of (C.19), we have

$$
\begin{aligned}
\dot{\bar{x}}(t,j) \in\ & \dot{x}(t,j) - \varepsilon\frac{\partial\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu)}{\partial x}\dot{x}(t,j) \\
& - \frac{\partial\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu)}{\partial\tau} \\
=\ & f_\varepsilon(x(t,j),w(t,j),\tau(t,j)) - f_0(x(t,j),\tilde{w}(\tau(t,j)),\tau(t,j)) \\
& -\varepsilon\frac{\partial\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu)}{\partial x}\dot{x}(t,j) \\
& + f_{sa}(x(t,j),\tilde{w}(\tau(t,j)) - \mu\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu) \\
=\ & f_\varepsilon(x(t,j),w(t,j),\tau(t,j)) - f_0(x(t,j),w(t,j),\tau(t,j)) \\
& -\varepsilon\frac{\partial\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu)}{\partial x}\dot{x}(t,j) \\
& + f_{sa}(x(t,j),w(t,j)) - \mu\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu) \\
\in\ & f_{sa}(\bar{x}(t,j) + \varepsilon\tilde{\eta}_{sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu),w(t,j)) + \frac{\delta\mathbb{B}}{3} \\
& + \varepsilon 2\sqrt{n}L(M+1)\mathbb{B} + \alpha_K(\mu) \\
\in\ & F(\bar{x}(t,j) + \delta\mathbb{B}, w(t,j)) + \delta\mathbb{B} \\
\subset\ & F_\delta(\bar{x}(t,j),w(t,j))\ .
\end{aligned} \tag{C.22}
$$

Then, it follows that $(\bar{x}, w)$ is the solution pair to system H_δ, and we can conclude that for any solution pair $(\bar{x}, w)$ there exists some solution pair (ξ, w_1) to system H_{sa} such that $\bar{x}$ and ξ are $(T, \frac{\varrho}{2})$-close. Then, with the same steps in proof of Theorem 4.4.1, we can complete the proof.

C.2 Proofs of Theorems 4.4.4 and 4.4.5

C.2.1 Technique results

Proposition C.2.1. *Suppose that system H in (4.2) satisfies Assumption 4.2.4, and it is ISS with respect to (χ, β, γ). Then, for each compact set $K_0 \subset \mathbb{R}^n$ and each pair of $(\Omega, \nu) \geq 0$ there exists $\delta > 0$ such that for each solution pair $(\bar{x}, w)$ of system H_δ in (4.19) with $\bar{x}_0 := \bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ and $|w| \leq \Omega$, the following holds:*

$$
\chi(\bar{x}(t,j)) \leq \max\{\beta(\chi(\bar{x}_0), t+j), \gamma(|w|)\} + \nu \quad \forall (t,j) \in \text{dom } \bar{x}\ . \tag{C.23}
$$

□

Proof of Proposition C.2.1

The proof is based on the trajectory method used in [151]. Let the compact set K_0 be given. Let $\Omega, \nu \geq 0$ be arbitrary. Due to the compactness of K_0 and continuity of γ, there exist $m > \nu + \gamma(\Omega)$ such that $K_0 + \mathbb{B}$ is contained in a compact set $\{\bar{x} \in \mathbb{R}^n : \chi(\bar{x}) \leq m\}$. Pick large enough $T \geq 0$ so that $\beta(m, r) \leq \frac{\nu}{2}$ for $r \geq T$.

For the compact set K_0 and Ω, let K be the reachable set defined as (C.13) for system H for any $(t, j) \in \text{dom } \xi$ with $t + j \leq 2T$, which is compact from Proposition C.1.8 with forward pre-completeness of system H on K_0, thanks to the assumed ISS property. Let $M \geq 0$ be such that $\max_{\xi \in K_0} \chi(\xi) \leq M$. Using the continuity of χ and β, and the fact that $\beta(s, l)$ approaches zero as $l \geq 0$ tends to infinity, let $\rho_1^* > 0$ be small enough such that

$$\beta(s, l - \rho_1^*) - \beta(s, l) \leq \frac{\nu}{6} \quad \forall s \leq M, l \geq 0 \ . \tag{C.24}$$

Let ρ_2^* be sufficiently small such that, for all $\xi \in K$ and $\bar{x} \in (K + \rho_2^* \mathbb{B})$ satisfying $|\xi - \bar{x}| \leq \rho_2^*$, we have

$$\begin{aligned} \chi(\bar{x}) &\leq \chi(\xi) + \frac{\nu}{6} \\ \beta(\chi(\xi), l) &\leq \beta(\chi(\bar{x}), l) + \frac{\nu}{6}, \quad \forall\, l \geq 0 \ . \end{aligned} \tag{C.25}$$

Let $\rho = \min\{\rho_1^*, \rho_2^*\}$ and $\xi_0 := \xi(0, 0)$. Let Proposition C.1.7 with $(2T, \rho, \Omega)$ and the set K_0 generate a $\delta^* > 0$. Consider $\delta \in (0, \delta^*]$ and without loss of generality assume that $\delta < 1$. From Proposition C.1.7, we know that for each solution pair $(\bar{x}, w)$ of system H_δ with $\bar{x}_0 \in (K_0 + \delta\mathbb{B})$ there exists some solution pair (ξ, w_1) of system H with $\xi_0 \in K_0$ and $|w_1| \leq |w|$ such that $\bar{x}$ and ξ are $(2T, \rho)$-close. Then, with ISS property of H and the definitions of ρ^* in (C.24) and (C.25), we have for all $(t, j) \in \text{dom } \bar{x}$ with $0 \leq t + j \leq 2T$, all solution pairs $(\bar{x}, w)$ of system H_δ with $\bar{x}_0 \in (K_0 + \delta\mathbb{B})$ satisfy

$$\begin{aligned}
\chi(\bar{x}(t,j)) &\leq \chi(\xi(s,j)) + \frac{\nu}{6} \\
&\leq \max\{\beta(\chi(\xi_0), t+j-\rho), \gamma(|w_1|)\} + \frac{\nu}{6} \\
&\leq \max\{\beta(\chi(\xi_0), t+j), \gamma(|w_1|)\} + \frac{\nu}{3} \\
&\leq \max\{\beta(\chi(\bar{x}_0), t+j), \gamma(|w|)\} + \frac{\nu}{2} \\
&\leq \max\{\beta(m, t+j), \gamma(|w|)\} + \frac{\nu}{2} \ . \qquad \text{(C.26)}
\end{aligned}$$

In particular, from the choice of T, (C.26) shows that $\chi(\bar{x}(t,j)) \leq \max\left\{\frac{\nu}{2}, \gamma(|w|)\right\} + \frac{\nu}{2} \leq \gamma(|w|) + \nu$ for all $(t,j) \in \text{dom } \bar{x}$ with $T \leq t+j \leq 2T$.

Let $\bar{x}_T := \bar{x}(s,i)$ and inputs $\bar{w}(\cdot,\cdot) := w(s+\cdot, i+\cdot)$ for each (s,i) such that $(s,i) \in \text{dom } \bar{x}$ and $s+i = T$. For $(t,j) \in \text{dom } \bar{x}$ satisfying $2T \leq t+j \leq 3T$, using $m > \gamma(\Omega) + \nu$, (C.26) implies

$$\begin{aligned}
\chi(\bar{x}(t,j)) &\leq \max\{\beta(\chi(\bar{x}_T), t+j), \gamma(|\bar{w}|)\} + \frac{\nu}{2}, \\
&\leq \max\left\{\beta(\gamma(|w|) + \nu, t+j), \gamma(|w|)\right\} + \frac{\nu}{2} \\
&\leq \gamma(|w|) + \nu \ .
\end{aligned}$$

Using this fact recursively shows that $\chi(\bar{x}(t,j)) \leq \gamma(|w|) + \nu$ for all $(t,j) \in \text{dom } \bar{x}$ with $t+j \geq T$. This bound and (C.26) establish the bound in (C.23). □

C.2.2 Proof of Theorem 4.4.4

Let $\Omega, \Omega_1 > 0$ come from the definitions of equi-essential boundedness and local equi-uniform Lipschitz continuity respectively. Let functions (χ, β, γ) come from the definition of ISS in Def. 3.2.6 for system H_{wa}. Let the compact set $K_0 \subset \mathbb{R}^n$ be given, and define

$$\begin{aligned}
K_1 &:= \{x \in \mathbb{R}^n : \\
&\qquad \chi(x) \leq \max\left\{\beta\left(\max_{\bar{x}\in K_0} \chi(\bar{x}), 0\right), \gamma(\Omega)\right\} + 1\right\} \\
K &:= K_1 \cup G((K_1 \times \mathcal{W}) \cap D) \ . \qquad \text{(C.27)}
\end{aligned}$$

The set K is a compact because of continuity of the proper indicator χ and outer semi-continuity of the set mapping $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Let $\nu \in (0,1)$. From the Proposition C.2.1, the ν, Ω and the compact set K generate a $\delta > 0$ such that, each solution pair $(\bar{x}, w)$ of system H_δ inflated from H_{wa} with $\bar{x}_0 := \bar{x}(0,0) \in K + \delta\mathbb{B}$ satisfies

$$\chi(\bar{x}(t,j)) \leq \max\{\beta(\chi(\bar{x}_0), t+j), \gamma(|w|)\} + \frac{\nu}{3} \quad \forall (t,j) \in \text{dom } \bar{x} \ . \tag{C.28}$$

Without loss of generality, assume that $\delta < 1$. Let $\bar{K} := K \times \Omega\mathbb{B} \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $\bar{K}$ and Lemma C.1.1 generate $\alpha_{\bar{K}}$ and pick $\mu > 0$ such that $\alpha_{\bar{K}} \leq \frac{\delta}{3}$. Let $\bar{K}$, δ and Assumption 4.3.1 generate $M\left(\bar{K}\right) > 1$ and $\varepsilon_1^* > 0$ such that bounds (4.8) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$. Let Assumption 4.3.4 and the set $\bar{K}$ generate $L := L\left(\bar{K}\right) \geq 1$ so that Assumption 4.3.4 holds for all $((x_1, w_1), \tau_a)$, $((x_2, w_2), \tau_b) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}$. Let $\varepsilon_2^* = \frac{\delta}{6\sqrt{n}L(M+1+\Omega_1)}$, $\varepsilon_3^* = 3\mu$.

System H_K defined in (C.16) is introduced. With the continuity of the proper indicator χ and class-$\mathcal{KL}$ function β and the fact that $\beta(m,s)$ converges to zero as $s \geq 0$ approaches infinity for all $m \geq 0$, let $\varepsilon_4^* > 0$ be such that, for all $x \in K$ and $\bar{x} \in K + \varepsilon_4^* L\mathbb{B}$ satisfying $|x - \bar{x}| \leq \varepsilon_4^* L$, the following holds:

$$\begin{aligned} \chi(x) &\leq \chi(\bar{x}) + \frac{\nu}{3} \\ \beta(\chi(\bar{x}), s) &\leq \beta(\chi(x), s) + \frac{\nu}{3}, \quad \forall s \in \mathbb{R}_{\geq 0} \ . \end{aligned} \tag{C.29}$$

Letting $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$, for each $\varepsilon \in (0, \varepsilon^*]$, define $\bar{x}$ as (C.18) with the same construction method in the proof of Theorem 4.4.1. Then, we can show that $(\bar{x}(t,j), w(t,j))$ is a solution pair to the inflated system H_δ, and then (C.28) holds. Letting $x_0 := x(0,0)$ and using (C.29), for all solution pairs $(x, w) \in K$ to system H_K and $(t,j) \in \text{dom } x$, we have

$$\begin{aligned} \chi(x(t,j)) &\leq \chi(\bar{x}(t,j)) + \frac{\nu}{3}, \\ &\leq \max\{\beta(\chi(\bar{x}_0), t+j), \gamma(|w|)\} + \frac{2\nu}{3} \\ &\leq \max\{\beta(\chi(x_0), t+j), \gamma(|w|)\} + \nu \ . \end{aligned} \tag{C.30}$$

In particular, since $\nu < 1$, each solution pair to system H_K starting in K_0 remains in the compact set

$$K_\nu := \left\{ x \in \mathbb{R}^n : \ \chi(x) \leq \max \left\{ \beta \left(\max_{\bar{x} \in K_0} \chi(\bar{x}), 0 \right), \gamma(\Omega) \right\} + \nu \right\} .$$

With $\nu < 1$ and continuity of χ, K_ν is contained in K_1 defined in (C.27). Finally, using the same steps in the proof of Theorem 4.4.1, and the bound (C.30) on the solution pairs of system H_K to get conclusions about the solutions of system H_ϵ with $x_0 \in K_0$, and which establishes the result. □

C.2.3 Proof of Theorem 4.4.5

The proof is nearly identical to the proof of Theorem 4.4.4 except that $\mu > 0$ is generated from Lemma C.1.2 and the definition of ε_2^* should be replaced by $\varepsilon_2^* := \frac{\delta}{6\sqrt{n}L(M+1)}$, where $L := L\left(\bar{K}\right) \geq 1$ is generated by Assumption 4.3.5 and the set $\bar{K}$. □

Appendix D

D.1 Proof of Lemma 5.4.1

To prove Lemma 5.4.1, we need some technical results that are given first. Consider an arbitrary compact set $K \subset \mathbb{R}^n$. Let $S_{bl}(K)$ denote the set of maximal solutions $(x_{bl}, z_{bl}) : \mathrm{dom}\ (x_{bl}, z_{bl}) \mapsto C \times \Psi$ of the boundary layer system in (5.9) for $(x_{bl}, z_{bl}(0)) \in (C \cap K) \times \Psi$. We first define functions $\mathcal{Y}_{bl} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \mathbb{R}^m$, which are constructed by piece-wise concatenating the solutions (x_{bl}, z_{bl}) of the system H_{bl}.

Definition D.1.1. *For any $L > 0$ and $T \geq 0$, let $n := \left\lceil \frac{T}{L} \right\rceil$ and $\mathcal{F}(L, K, T)$ be a set of functions $\mathcal{Y}_{bl} : [0, T] \to C \times \Psi$ with $\mathcal{Y}_{bl} := (\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ such that for each integer $k \in \{0, \cdots, n\}$ there exists $y^k := \left(x^k_{bl}, z^k_{bl}\right) \in S_{bl}(K)$ with $L \in \mathrm{dom}\ \left(x^k_{bl}, z^k_{bl}\right)$:*

$$\mathcal{Y}_{bl}(s + kL) = y^k(s), \quad \forall\ s \in [0, L)\ s.t.\ (s + kL) \leq T\ . \tag{D.1}$$

□

Claim D.1.2. *Suppose that the set-valued mapping F_{av}, that is outer semi-continuous, locally bounded and convex, is an average of f_0 with respect to ψ_0 on $C \times \Psi$. Then, for each compact set $K \subset \mathbb{R}^n$, $L > 0$, $T \geq 0$ and all functions $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$, there exists a measurable function $f_{\mathcal{Z}_{bl}} : [0, T] \to \mathbb{R}^n$ such that $f_{\mathcal{Z}_{bl}}(s) \in F_{av}(\mathcal{X}_{bl}(s))$ for all $s \in [0, T]$, and the following holds for all $t \in [0, T]$:*

$$\left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{\mathcal{Z}_{bl}}(s)] ds \right| \leq t\sigma_K(L) + L\sigma_k(0)\ .$$

□

Proof of Claim D.1.2

Let a compact set $K \subset \mathbb{R}^n$, $T \geq 0$ and $L > 0$ be given and the $\mathcal{L}$-class function σ_K be generated by the set K from the definition of average of f_0 with respect to ψ_0. For each solution (x_{bl}, z_{bl}) of the boundary layer system, it follows that there exists a function $f_{z_{bl}} : [0, L] \to \mathbb{R}^n$ with $f_{z_{bl}}(s) \in F_{av}(x_{bl})$ for $s \in [0, L]$ from average definition such that:

$$\left| \int_0^L [f_0(x_{bl}, z_{bl}(s)) - f_{z_{bl}}(s)] ds \right| \leq L\sigma_K(L) \ . \tag{D.2}$$

For any $t \in [0, T]$ and given $L > 0$, let $n := \left\lfloor \frac{t}{L} \right\rfloor$ and then we have $t := nL + \tilde{t}$ with $0 \leq \tilde{t} < L$. For each solution (x_{bl}^k, z_{bl}^k) for $k = \{1, \cdots, n\}$ of the boundary layer system, let $f_{z_{bl}}^k$ be generated by the average definition such that (D.2) holds. Let

$$f_{\mathcal{Z}_{bl}}(s + kL) := f_{z_{bl}}^k(s) \ , \quad \forall \ s \in [0, L) \text{ s.t. } (s + kL) \leq T \ . \tag{D.3}$$

With the definition of the function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ in (D.1), we have

$$\begin{aligned} &\left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{\mathcal{Z}_{bl}}(s))] ds \right| \\ &\leq \ \left| \sum_{k=0}^{n-1} \int_{kL}^{(k+1)L} \left[f_0(x_{bl}^k, z_{bl}^k(s - kL)) - f_{z_{bl}^k}(s) \right] ds \right| \\ &\quad + \left| \int_{nL}^{nL+\tilde{t}} \left[f_0(x_{bl}^n, z_{bl}^n(s - nL)) - f_{z_{bl}^n}(s) \right] ds \right| \\ &\leq \ nL\sigma_K(L) + \tilde{t}\sigma_K(\tilde{t}) \leq t\sigma_K(L) + L\sigma_k(0), \quad \forall \ t \in [0, T] \ . \end{aligned}$$

□

Claim D.1.3. *Suppose that the set-valued mapping F_{av}, that is outer semi-continuous, locally bounded and convex, is an average of f_0 with respect to ψ_0 on $C \times \Psi$. Then, for each compact set $K \subset \mathbb{R}^n$ and $\nu > 0$ there exist $L, \mu^* > 0$ such that, for each $T \geq 0$, $\mu \in [0, \mu^*]$ and function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ there exists a measurable function $f_{\mathcal{Z}_{bl}} : [0, T] \to \mathbb{R}^n$ such that $f_{\mathcal{Z}_{bl}}(s) \in F_{av}(\mathcal{X}_{bl}(s))$ for all $s \in [0, T]$, and η satisfying*

$$\dot{\eta} = -\mu\eta + f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{\mathcal{Z}_{bl}}, \quad \eta(0) = 0,$$

the following holds:

$$\mu|\eta(t)| \leq \nu \ , \qquad \forall \ t \in [0, T] \ .$$

□

Proof of Claim D.1.3

Let a compact set $K \subset \mathbb{R}^n$, $T \geq 0$ and $\nu > 0$ be given. Let $L > 0$ be large enough such that $\sigma_K(L) \leq \frac{\nu}{2(\exp(-1)+1)}$, where σ_K is a $\mathcal{L}$-class function generated by the set K from the definition of average of f_0 with respect to ψ_0. Let $\mu^* = \frac{\nu}{4L\sigma_K(0)}$ and consider a $\mu \in [0, \mu^*]$. For each function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$, let $f_{\mathcal{Z}_{bl}}$ be generated by Claim D.1.2.

Let the function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be such that

$$\dot{\phi} = f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{\mathcal{Z}_{bl}} \ , \quad \phi(0) = 0 \ .$$

Then, we have that $\dot{\eta} - \dot{\phi} = -\mu(\eta - \phi) - \mu\phi$. Integrating this differential equation, we have

$$\eta(t) = \phi(t) - \int_0^t \exp(\mu(s-t))(\mu\phi(s))ds \ . \tag{D.4}$$

Adding and subtracting $\mu\phi(t)\int_0^t \exp(\mu(s-t))ds$ to the right hand side of (D.4), we obtain

$$\eta(t) = \exp(-\mu t)\phi(t) + \int_0^t \exp(-\mu(t-s))[\mu(\phi(t) - \phi(s))]ds \ .$$

For any $T \geq 0$, we have that

$$|\phi(t)| = \left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{\mathcal{Z}_{bl}}(s)]ds \right| \leq t\sigma_k(L) + L\sigma_K(0)$$

holds for all $t \in [0, T]$ from Claim D.1.2.

For any $s \in [0, T]$, letting

$$\tilde{\mathcal{X}}_{bl}(\cdot) := \mathcal{X}_{bl}(s+\cdot)$$
$$\tilde{\mathcal{Z}}_{bl}(\cdot) := \mathcal{Z}_{bl}(s+\cdot)$$
$$\tilde{f}_{\mathcal{Z}_{bl}}(\cdot) := f_{\mathcal{Z}_{bl}}(s+\cdot)\ ,$$

it follows that

$$\begin{aligned}
|\phi(t)-\phi(s)| &= \left|\int_s^t [f_0(\mathcal{X}_{bl}(r),\mathcal{Z}_{bl}(r)) - f_{\mathcal{Z}_{bl}}(r)]dr\right| \\
&= \left|\int_0^{t-s} [f_0(\mathcal{X}_{bl}(s+y),\mathcal{Z}_{bl}(s+y)) - f_{\mathcal{Z}_{bl}}(s+y)]dy\right| \\
&= \left|\int_0^{t-s} [f_0(\tilde{\mathcal{X}}_{bl}(y),\tilde{\mathcal{Z}}_{bl}(y)) - \tilde{f}_{\mathcal{Z}_{bl}}(y)]dy\right| \\
&\le (t-s)\sigma_k(L) + L\sigma_K(0)\ . \qquad \text{(D.5)}
\end{aligned}$$

Noting (D.5), the definitions of L and μ, and using the fact that $y\exp(-y) \le \exp(-1)$ for all $y \ge 0$ and $\int_0^\infty \exp(-y)y dy = 1$, it follows that

$$\begin{aligned}
\mu|\eta(t)| &\le \mu\exp(-\mu t)(t\sigma_K(L) + L\sigma_K(0)) \\
&\quad + \mu^2\int_0^t \exp(-\mu(t-s))[(t-s)\sigma_K(L) + L\sigma_K(0)]ds \\
&= \sigma_K(L)\mu t\exp(-\mu t) + \mu L\sigma_K(0)\exp(-\mu t) + \mu^2\int_0^t \exp(-\mu r)[r\sigma_K(L) + L\sigma_k(0)]dr \\
&\le \sigma_K(L)\exp(-1) + \sigma_K(L)\int_0^{\mu t}\exp(-y)y dy + \mu L\sigma_K(0)\int_0^{\mu t}\exp(-y)dy \\
&\le \sigma_K(L)\exp(-1) + \mu L\sigma_K(0) + \sigma_K(L)\int_0^\infty \exp(-y)y dy \\
&\quad + \mu L\sigma_K(0)\int_0^\infty \exp(-y)dy \\
&\le \sigma_K(L)(\exp(-1)+1) + 2\mu L\sigma_K(0) \le \nu, \quad \forall\ t\in[0,T]\ .
\end{aligned}$$

□

We also give the following claim that can be easily obtained by viewing u_2 as an input to an exponentially stable linear system with η being initialized as zero.

Claim D.1.4. *For any function η satisfying $\dot{\eta} = -\mu\eta + u_2$ with $\eta(0) = 0$, the following holds:*

$$\mu|\eta(t)| \leq ||u_2|| \ , \quad \forall\, t \geq 0 \ , \tag{D.6}$$

where $u_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $||u_2|| = \text{ess}_{t\geq 0}|u_2(t)|$. □

Proof of Claim D.1.4

Consider the differential equation $\dot{\eta} = -\mu\eta + u_2$ with the Lyapunov function $V(\eta) = \eta^T\eta$. Note that for all $\mu|\eta| > |u_2|$,

$$\dot{V} = 2\eta^T(-\mu\eta + u_2) < 0 \ ,$$

which implies that $|\eta(t)| \leq \frac{||u_2||}{\mu}$ for all $t \geq 0$ and completes the proof. □

Consider a system inflated from the boundary layer system by $\delta > 0$ and its flow set intersected with a compact set $K \subset \mathbb{R}^n$:

$$\left.\begin{array}{l} \dot{x} \in \delta\mathbb{B} \\ \dot{z} \in \overline{con}\ \psi_0\left(\left((x,z) + \delta B\right) \cap (C \times \Psi)\right) + \delta\mathbb{B} \end{array}\right\} , \ (x,z) \in (C_\delta \cap K) \times \Psi_\delta \ , \tag{D.7}$$

where

$$\begin{aligned} C_\delta &:= \{x : (x + \delta\mathbb{B} \cap C) \neq \emptyset\} \\ \Psi_\delta &:= \{z : (z + \delta\mathbb{B}) \cap \Psi \neq \emptyset\} \ . \end{aligned}$$

With the definition of H_{bl}^{δ}, we give the following claim.

Claim D.1.5. *For each triple of strictly positive real number (μ, L, ρ) and compact set $K \subset \mathbb{R}^n$ there exists $\delta > 0$ such that, for each each $\tilde{T} > 0$ and solution (x, z) of the system H_{bl}^{δ} in (D.7) with $\left[0, \tilde{T}\right] = \text{dom}\ (x, z)$ there exists a function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ where $T \geq \tilde{T} - \rho$ such that:*

$$\left.\begin{array}{l} |x(t) - \mathcal{X}_{bl}(t)| \leq \mu \\ |z(t) - \mathcal{Z}_{bl}(t)| \leq \mu \end{array}\right\} , \quad \forall\, t \in \left[0, \min\left\{T, \tilde{T}\right\}\right] \ .$$

□

Proof of Claim D.1.5

Let $\mu, L, \rho > 0$ and any compact set $K \subset \mathbb{R}^n$ be given. Let Assumption 5.3.1 and the compact set K generate a $M > 0$ such that $|\psi_0(x,z)| \leq M$ for all $(x,z) \in (C \cap K) \times \Psi$. Let

$$\mu_1 := \min\left\{\frac{\mu}{M+2}, \rho\right\} .$$

Let δ be generated by Lemma 5.2.4 with the set K and a triple of determined numbers $(L+1, 0, \mu_1)$. For each solution (x,z) of the system H_{bl}^δ in (D.7) and $\tilde{T} > 0$ such that $\left[0, \tilde{T}\right] = \text{dom }(x,z)$, consider any $T \geq \tilde{T} - \rho$.

For the given L, the determined $\tilde{T}$ and T, let

$$n := \left\lceil \frac{\min\left\{T, \tilde{T}\right\}}{L} \right\rceil .$$

For each $k \in \{1, \cdots, n-1\}$, each solution (x,z) of the system H_{bl}^δ and $l \in [0, L+1]$, Lemma 5.2.4 with $(L+1, 0, \mu_1)$ guarantees that there exists a solution $\left(x_{bl}^k, z_{bl}^k\right) \in S_{bl}(K)$ of the boundary layer system H_{bl} and $t \in \text{dom }\left(x_{bl}^k, z_{bl}^k\right)$ with $|t - l| \leq \mu_1$ such that:

$$\begin{aligned} |x(l+kL) - x_{bl}^k(t)| &\leq \mu_1 , \\ |z(l+kL) - z_{bl}^k(t)| &\leq \mu_1 . \end{aligned} \tag{D.8}$$

With the fact that $|\dot{x}| \leq 1$ and $|\dot{z}| \leq M+1$ for the system H_{bl}^δ, we have the solution (x,z) of H_{bl}^δ starting from the point $\left(x_{bl}^k(t), z_{bl}^k(t)\right)$ such that $|x(0) - x_{bl}^k(t)| \leq \mu_1$ and $|z(0) - z_{bl}^k(t)| \leq \mu_1$ satisfies

$$\left.\begin{aligned} |x(\tilde{t}) - x_{bl}^k(t)| &\leq 2\mu_1 \leq \mu \\ |z(\tilde{t}) - z_{bl}^k(t)| &\leq (M+2)\mu_1 \leq \mu \end{aligned}\right\}, \quad \forall \ \left|\tilde{t} - t\right| \leq \delta, \tag{D.9}$$

which with Lemma 5.2.4 implies that for each $k \in \{1, \cdots, n-1\}$ and solution (x,z) of H_{bl}^δ there exists some solution $\left(x_{bl}^k, z_{bl}^k\right)$ of H_{bl} such that

$$|x(t+kL) - x_{bl}^k(t)| \leq \mu$$
$$|z(t+kL) - z_{bl}^k(t)| \leq \mu$$

for all $t \in [0, L+1-\mu_1]$. Noting the fact that $\mu_1 \leq \rho$ from its definition, it follows that the conclusion

$$|x(t+kL) - x_{bl}^k(t)| \leq \mu$$
$$|z(t+kL) - z_{bl}^k(t)| \leq \mu$$

also holds for all $t \in [0, L+1-\rho]$. Similarly, when $k = n$, we get

$$|x(t+nL) - x_{bl}^k(t)| \leq \mu$$
$$|z(t+nL) - z_{bl}^k(t)| \leq \mu$$

for all $t + nL \leq \tilde{T} + 1 - \rho$ and then holds for $t + nL \leq \min\left\{T, \tilde{T}\right\}$. Noting the definition of the function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ in (D.1), the conclusion is obtained. □

Proof of Lemma 5.4.1

Let a compact set $K \subset \mathbb{R}^n$ and $\nu > 0$ be given. Let $\mu^*, L > 0$ be generated from Claim D.1.3 with $\frac{\nu}{3}$ and the set K. Let $M > 0$ be such that $\max\{|f_0(x,z)|, |F_{av}(x)|\} \leq M/2$ for all $(x,z) \in (C \cap K) \times \Psi$ from the continuity of f_0 and local boundedness of the set-valued mapping F_{av}.

Considering the continuity property of the function f_0 in Assumption 5.3.1, it follows that for any $\nu > 0$ there exists $\mu \in (0, \mu^*)$ such that

$$\max\{|x_1 - x_2|, |z_1 - z_2|\} \leq \mu \Rightarrow f_0(x_1, z_1) - f_0(x_2, z_2)| \leq \frac{\nu}{6}. \tag{D.10}$$

Let $\rho = \frac{\nu}{3(M+1)}$. Let Claim D.1.5, the compact set K with (μ, L, ρ) generate a $\delta_1 > 0$. Let $\delta := \min\{\delta_1, \nu/6, 1\}$ and Assumption 5.3.1 with K and δ generate $\varepsilon_1^* > 0$ such that the bounds (5.7) hold for all $\varepsilon \in (0, \varepsilon_1^*]$. Let $\varepsilon^* := \min\left\{\varepsilon_1^*, \frac{\delta}{M+1}\right\}$

and consider a $\varepsilon \in (0, \varepsilon^*]$.

For each solution (x, z) of the system H_K and $(t, j) \in \text{dom } (x, z)$, let $I_j := \{t : (t, j) \in \text{dom } (x, z)\}$, $t_j^0 := \min\{t, t \in I_j\}$ and $\tilde{T}_j := \max\{t, t \in I_j\}$. From the construction of the augmented system H_K in (5.24), for each $j \in \{(t, j) \in \text{dom } (x, z)\}$, the solution $\eta : \left[t_j^0, \tilde{T}_j\right] \mapsto \mathbb{R}^n$ of system H_K agrees with

$$\dot{\eta} \in -\mu\eta + f(x, z, \varepsilon) - F_{av}^{\mu}(x), \quad \eta\left(t_j^0\right) = 0. \tag{D.11}$$

For any $\tilde{T}_j \geq \rho$, let $T_j := \tilde{T}_j - \rho$. For each solution (x, z) of the system H_K, let the function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(K, L, T_j)$ come from Claim D.1.5 and the function $f_{\mathcal{Z}_{bl}} : [t_j^0, T_j] \to \mathbb{R}^n$ be generated by such $(\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ from Claim D.1.3. Noting the definition of ε and the fact that the solution (x, z) of the system H_K only flows for all $t \in \left[t_j^0, T_j\right]$, the solution (x, z) agrees with the system H_{bl}^{δ} defined in (D.7). Then, the solution of the differential equation

$$\dot{\eta} = -\mu\eta + u_1 + u_2, \quad \eta\left(t_j^0\right) = 0. \tag{D.12}$$

with

$$\begin{aligned} u_1 &:= f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{\mathcal{Z}_{bl}}(t), \\ u_2 &:= [f(x, z, \varepsilon) - f_0(x, z)] + [f_0(x, z) - f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl})], \end{aligned}$$

agrees with the differential inclusion in (D.11), which due to the fact that $f_{\mathcal{Z}_{bl}}(s) \in F_{av}(\mathcal{X}_{bl}(s)) \subset F_{av}(\mathcal{X}_{bl})$ from Claim D.1.3, $|\mathcal{X}_{bl}(s) - x(s)| \leq \mu$ for all $t \in [t_j^0, T_j]$ from Claim D.1.5, and then $F_{av}(\mathcal{X}_{bl}) \subset F_{av}(x + \mu\mathbb{B}) \subset F_{av}^{\mu}(x)$. Then, for each solution (x, z) of system H_K if we can find a solution η of (D.12) such that $\mu|\eta(t)| \leq \nu$, then the proof of the lemma is completed.

Consider the response of η separately under inputs u_1 and u_2 on the time interval $\left[t_j^0, T_j\right]$. For the effect of u_1, Claim D.1.3 shows that $\mu|\eta| \leq \nu/3$. Moreover, Claim D.1.5 with (D.10) shows that $|u_2(t)| \leq \nu/3$ for $t \in [0, T_j]$ and Claim D.1.4 guarantees that $\mu|\eta(t)| \leq \nu/3$ for all $t \in [0, T_j]$ under u_2. Driven by both u_1 and u_2, we know that $\mu|\eta(t)| \leq 2\nu/3$ for all $t \in [0, T_j]$ from the superposition principle.

Using the fact that $|u_1(t) + u_2(t)| \leq M + 1$ for all $t \geq 0$ from the definition of ε_1^* and considering $|\eta(t)|$ on $t \in \left[T_j, \tilde{T}_j\right]$ from an initial point $|\eta(T_j)| \leq \frac{2\nu}{3\mu}$, we can get

$$\mu|\eta(t)| \leq \frac{2\nu}{3} + (M+1)\rho \leq \nu \quad \forall\, t \in \left[T_j, \tilde{T}_j\right] \ . \tag{D.13}$$

Similarly, we can consider $|\eta(t)|$ when $\tilde{T}_j \leq \rho$ from $\eta\left(t_j^0\right) = 0$ and get

$$\mu|\eta(t)| \leq (M+1)\rho \leq \frac{\nu}{3} \quad \forall\, t \in \left[t_j^0, \tilde{T}_j\right] \ . \tag{D.14}$$

which completes the proof.

D.2 Proof of Lemma 5.3.5

We need two technical lemmas and the following notations to prove the conclusion of Lemma 5.3.5. For a compact set $\Omega \subset C \times \Psi$, let $S_{bl}(\Omega)$ denote the set of maximal solutions $(x_{bl}, z_{bl}) : \mathrm{dom}(x_{bl}, z_{bl}) \to C \times \Psi$ of the boundary layer system H_{bl} for $(x_{bl}, z_{bl}(0)) \in \Omega$. Let $\mathcal{F}(L, \Omega, T)$ be defined same as $\mathcal{F}(L, K, T)$ in Def. D.1.1 with replacing $S_{bl}(K)$ by $S_{bl}(\Omega)$.

Lemma D.2.1. *Suppose that Assumption 5.3.4 holds for a compact set $\Omega \subset C \times \Psi$. Then, for the set Ω there exists $\alpha_\Omega \in \mathcal{K}_\infty$ such that, for each $\nu \in (0,1]$ there exists $L > 0$ such that, for each $T \geq 0$ and each function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, \Omega, T)$ there exists a measurable function $f_{\mathcal{Z}_{bl}} : [0,T] \to \mathbb{R}^n$ such that $f_{\mathcal{Z}_{bl}}(s) \in F_{av}(\mathcal{X}_{bl}(s))$ for all $s \in [0,T]$, and the following holds for all $t \in (0,T]$:*

$$\frac{1}{t}\left|\int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{\mathcal{Z}_{bl}}(s)]ds\right| \leq \nu + \frac{\alpha_\Omega(1/\nu)}{t}.\square$$

Proof of Lemma D.2.1

Let $\sigma_\Omega \in \mathcal{L}$ come from Assumption 5.3.4. Let arbitrary $T \geq 0$ and $\nu \in (0,1]$ be given and let $L = \sigma_\Omega^{-1}(\nu)$. Let $\alpha_\Omega \in \mathcal{K}_\infty$ be such that

$$\sigma_\Omega^{-1}(1/s) \leq \frac{1}{\sigma_\Omega(0)}\alpha_\Omega(s) \qquad \forall s \in [1, \infty)\ , \tag{D.15}$$

the existence of α_Ω is due to $\sigma_\Omega^{-1}(\cdot)$ is non-decreasing and $\sigma_\Omega^{-1}(1/s)$ is bounded for

$s \in [1, \infty)$. Then, we have $L\sigma_\Omega(0) \le \alpha_\Omega(1/\nu)$. For each $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, \Omega, T)$, let $f_{\mathcal{Z}_{bl}} : [0, T] \to \mathbb{R}^n$ be generated by Claim D.1.2. Combining the fact that $L\sigma_\Omega(0) \le \alpha_\Omega(1/\nu)$ and the result of Claim D.1.2 completes the proof. □

Lemma D.2.2. *Suppose that functions $f_0 : C \times \Psi \to \mathbb{R}^n$ and $\psi_0 : C \times \Psi \to \Psi$ are continuous. Then, for each triple of strictly positive real number (μ, L, ρ) there exists $\delta > 0$ such that, for each compact set $\Omega \subset C \times \Psi$ and solution (x, z) of the boundary layer system (5.9) with $(x(0), z(0)) \in \Omega + \delta\mathbb{B}$ there exists a function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, \Omega, T)$ where $T \ge \tilde{T} - \rho$ and $\left[0, \tilde{T}\right] = \text{dom } (x, z)$ such that*

$$\left.\begin{array}{l} |x(t) - \mathcal{X}_{bl}(t)| \le \mu \\ |z(t) - \mathcal{Z}_{bl}(t)| \le \mu \end{array}\right\}, \qquad \forall\, t \in \left[0, \min\left\{T, \tilde{T}\right\}\right]. \quad \square$$

Proof of Lemma D.2.2

Let a compact set $\Omega \subset C \times \Psi$ be given. Let $\mu, L, \rho > 0$ be given. Let $\mu_1 := \min\{\mu, \rho\}$. Let $L + 1$ and μ_1 generate a $\delta_1 > 0$ by using continuous dependence of ψ_0 on initial conditions to guarantee that solutions of the boundary layer system H_{bl} satisfies

$$|z(t, z_1) - z(t, z_2)| \le \mu_1 \quad \forall\, t \in [0, L + 1 - \mu_1] \text{ and } |z_1 - z_2| \le \delta_1\ . \tag{D.16}$$

Let $\delta := \min\{\delta_1, \mu_1\}$. For each solution $(x, z) \in S_{bl}(\Omega)$ and $\tilde{T} > 0$ such that $\left[0, \tilde{T}\right] = \text{dom } (x, z)$, consider any $T \ge \tilde{T} - \rho$.

For the given L, the determined $\tilde{T}$ and T, let $n := \left\lceil \frac{\min\left\{T, \tilde{T}\right\}}{L} \right\rceil$. Noting the definition of δ_1, for each $k \in \{1, \cdots, n-1\}$ and solution (x, z) of system H_{bl} with $(x(0), z(0)) \in \Omega + \delta\mathbb{B}$, it follows that there exists a solution $\left(x^k_{bl}, z^k_{bl}\right) \in S_{bl}(\Omega)$ of system H_{bl} and $t \in \text{dom } \left(x^k_{bl}, z^k_{bl}\right)$ with $t \in [0, L + 1 - \mu_1]$:

$$\begin{array}{c} |x(t + kL) - x^k_{bl}(t)| \le \mu_1 \\ |z(t + kL) - z^k_{bl}(t)| \le \mu_1\ . \end{array}$$

When $k = n$, we get that

$$|x(t+nL) - x_{bl}^n(t)| \leq \mu_1$$
$$|z(t+nL) - z_{bl}^n(t)| \leq \mu_1 \ ,$$

for all $t + nL \leq \min\{T, \tilde{T}\}$. Considering the definition of function $(\mathcal{X}_{bl}, \mathcal{Y}_{bl}) \in \mathcal{F}(L, \Omega, T)$ and $\mu_1 \leq \mu$, we obtain the conclusion. □

Proof of Lemma 5.3.5

Let a compact set $K \subset \mathbb{R}^n$ be given. Let the compact set $\Omega \subset (C \cap K) \times \Psi$ satisfy Assumption 5.3.4 and be GAS for the boundary layer system H_{bl} with C replaced by $C \cap K$. Let $\beta \in \mathcal{KL}$ and $\sigma_\Omega \in \mathcal{L}$ come from Definition **??** and Assumption 5.3.4, respectively. Let the set K with Assumptions 5.3.1 and 5.3.4 generates a $M_K > 0$ such that $|f_0(x, z)| + |F_{av}(x)| \leq M_K$ for all $(x, z) \in (C \cap K) \times \Psi$. For each $\nu \in (0, 1]$, let $L > 0$ and $\alpha_\Omega \in \mathcal{K}_\infty$ be generated by the set Ω and $\nu/3$ from Lemma D.2.1.

Considering continuity of f_0, it follows that for any $\nu > 0$ there exists a $\mu_1 \in (0, 1)$ such that

$$\max\{|x_1 - x_2|, |z_1 - z_2|\} \leq \mu_1 \Rightarrow f_0(x_1, z_1) - f_0(x_2, z_2)| \leq \frac{\nu}{3} \ . \qquad \text{(D.17)}$$

Noting outer semi-continuity of the set-valued mapping F_{av}, for any $\nu > 0$ there exists a $\mu_2 > 0$ such that for all $\mu \in (0, \mu_2)$ and each measurable function $f_1 : [0, L] \to \mathbb{R}^n$ with $f_1(s) \in F_{av}(x(s))$ for $s \in [0, L]$, the function $f_1^\mu(s) \in F_{av}(x(s) + \mu\mathbb{B})$ for $s \in [0, L]$ satisfies

$$|f_1(s) - f_1^\mu(s)| \leq \frac{\nu}{3}, \quad \forall \ s \in [0, L]. \qquad \text{(D.18)}$$

Let $\mu := \min\{\mu_1, \mu_2\}$. Let $(\mu, L, 1)$ generates a $\delta > 0$ from Lemma D.2.2.

Noting that $L(\cdot)$ is non-increasing function from the fact that $L = \sigma_\Omega^{-1}(\nu/3)$ in the proof of Lemma D.2.1, it follows from the proof of Lemma D.2.2 that $\delta(\nu)$ increases when ν grows. From the continuity condition of ψ_0 in Assumption 5.3.1 and the set Ω being GAS for system H_{bl} with respect to β, it follows that there exists a continuous, strictly decreasing function $T_K : (0, 1] \to \mathbb{R}_{\geq 0}$ satisfies

$$\beta\left(\max_{(x,z) \in (C \cap K) \times \Psi} |(x, z)|_\Omega, T_K(\delta)\right) \leq \delta. \qquad \text{(D.19)}$$

Noting the properties of the function $T_K : (0,1] \to \mathbb{R}_{\geq 0}$, the fact that $\lim_{\delta\to 0} T_K(\delta) = \infty$ and $\delta(\cdot)$ is a continuous increasing function, there exists $\hat{\alpha}_K \in \mathcal{K}_\infty$ such that $T_K(\delta(\nu)) \leq \hat{\alpha}_K(1/\nu)$.

Noting the definition of $M_K > 0$ and $f_{z_{bl}}(s) \in F_{av}(x)$ for all $s \in [0, L]$, for each solution $(x, z) \in (C \cap K) \times \Psi$ of system H_{bl}, we have:

$$\begin{aligned}\frac{1}{L}\left|\int_0^{T_K(\delta(\nu))} [f_0(x, z(s)) - f_{z_{bl}}(s)]ds\right| &\leq \frac{M_K T_K(\delta(\nu))}{L} \\ &\leq \frac{M_K \hat{\alpha}_K(1/\nu)}{L}.\end{aligned} \tag{D.20}$$

From (D.19), we have that all solutions (x, z) of system H_{bl} satisfy $(x(t), z(t)) \in \Omega + \delta\mathbb{B}$ when $t \geq T_K(\delta)$. Let functions $(\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ come from Lemma D.2.2 and $f_{\mathcal{Z}_{bl}}$ be generated by $(\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ from Lemma D.2.1. Noting the definition of $f_{\mathcal{Z}_{bl}}$ in (D.3) and Lemma D.2.2, we have $f_{\mathcal{Z}_{bl}}(s) = f^{\mu}_{z_{bl}}(s) \in F_{av}(\mathcal{X}_{bl}(s))$ and $|f_{\mathcal{Z}_{bl}}(s) - f_{z_{bl}}(s)| \leq \frac{\nu}{3}$ holds for $s \in [0, L]$ from (D.18). Let

$$\begin{aligned}\tilde{\mathcal{X}}_{bl}(\cdot) &:= \mathcal{X}_{bl}(\cdot + T_K(\delta(\nu))) \\ \tilde{\mathcal{Z}}_{bl}(\cdot) &:= \mathcal{Z}_{bl}(\cdot + T_K(\delta(\nu))) \\ \tilde{f}_{\mathcal{Z}_{bl}}(\cdot) &:= f_{\mathcal{Z}_{bl}}(\cdot + T_K(\delta(\nu)))\ .\end{aligned}$$

Noting (D.17) and the results of Lemmas D.2.1 and D.2.2, it follows that

$$\begin{aligned}&\frac{1}{L}\left|\int_{T_K(\delta(\nu))}^{L} [f_0(x, z(s)) - f_{z_{bl}}(s)]ds\right| \\ &\leq \frac{1}{L}\left|\int_{T_K(\delta(\nu))}^{L-1} [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{\mathcal{Z}_{bl}}(s)]ds\right| + \frac{1}{L}\left|\int_{T_K(\delta(\nu))}^{L-1} [f_{\mathcal{Z}_{bl}}(s) - f_{z_{bl}}(s)]ds\right| \\ &\quad + \frac{1}{L}\int_{T_K(\delta(\nu))}^{L-1} |f_0(x, z(s)) - f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s))|ds + \frac{1}{L}\left|\int_{L-1}^{L} [f_0(x, z(s)) - f_{\mathcal{Z}_{bl}}(s)]ds\right|, \\ &\leq \frac{1}{L}\left|\int_0^{L-1-T_K(\delta(\nu))} [f_0\left(\tilde{\mathcal{X}}_{bl}(s), \tilde{\mathcal{Z}}_{bl}(s)\right) - \tilde{f}_{\mathcal{Z}_{bl}}(s)]ds\right| + \frac{2\nu}{3} + \frac{M_K}{L}, \\ &\leq \nu + \frac{\alpha_\Omega(3/\nu)}{L} + \frac{M_K}{L}.\end{aligned} \tag{D.21}$$

Combining (D.20) and (D.21), we have

$$\frac{1}{L}\left|\int_{T_K(\delta(\nu))}^{L}[f_0(x,z(s))-f_{z_{bl}}(s)]ds\right| \leq \nu + \frac{\alpha_K(1/\nu)}{L} + \frac{M_K}{L}, \tag{D.22}$$

where $\alpha_K(s) = \alpha_\Omega(2s) + M_K\hat{\alpha}_K(s)$ is of class-$\mathcal{K}_\infty$ with α_Ω and $\hat{\alpha}_K$ of class-$\mathcal{K}_\infty$. Noting that (D.22) holds for arbitrary $\nu \in (0,1]$, let $\nu = \min\left\{1, \frac{1}{\tilde{\alpha}_K\left(\sqrt{L}\right)}\right\}$ and substitute it in (D.22), we get for each solution $(x,z) \in (C \cap K) \times \Psi$ of system H_ε, there exists a $f_{z_{bl}}(s) \in F_{av}(x)$ such that

$$\begin{aligned}
&\frac{1}{L}\left|\int_0^L [f_0(x,z(s))-f_{z_{bl}}(s)]ds\right| \\
\leq\ & \min\left\{1, \frac{1}{\tilde{\alpha}_K\left(\sqrt{L}\right)}\right\} + \frac{\max\left\{\alpha_K(1), \sqrt{L}\right\}}{L} + \frac{M_K}{L}, \\
:=\ & \sigma_K(L),
\end{aligned}$$

where σ_K is of class-$\mathcal{L}$. Noting that the definition of average holds with this σ_K, we know that F_{av} is the average of f_0 with respect to ψ_0 on $C \times \Psi$ and which gives the conclusion.

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