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An Application of Dynamic Economic Systems to the Gold Market

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B.S., Economics, University of Wyoming, 1999
M.S., Finance, University of Wyoming, 2002
M.A., Economics, University of New Mexico, 2006

DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy

Economics

The University of New Mexico
Albuquerque, New Mexico

December, 2009

DEDICATION

For my wife …

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ABSTRACT

The formulation of economic time series problems has a long tradition of progressing towards better estimation procedures of economic variables over time. This tradition, however has sometimes left a void in our understanding as we smooth and de-trend data to remove bias and correlation in attempts to correct for econometric problems over time. For the purposes of forecasting, the practitioner is often left with a choice of either a naive time series model or a static regression model with no effect across time. However, the treatment of such data can be difficult and often model fit can be relatively low. That does not need to necessarily be the case. Using dynamical systems methodology that has been recently developed in the fields of Physics and Biology and that is beginning to be used in Economics, we develop improved methods for estimation through a better characterization of the functional form of an econonomic variable over time, that does not have the constraints of linearity or independence that we often convey on time series data.

This dissertation will demonstrate the usefulness of dynamic systems methodology in regression analysis. We will find that dynamic systems analysis allows an economic variable changing over time to be split into random and deterministic components in order to better understand the root cause of why an economic variable is changing over time. Dynamic systems methodology will then be used to develop an equation that explains the behavior of the economic variable over time for the purposes of simulating possible changes to the economic system in the future. The equation of the dynamic system will also be used to perform a supply and demand analysis on an industry.

We study the dynamic system of the gold industry, an industry with a diverse and rich economic history. Gold has been valued by societies for hundreds of years due to its many uses: store of wealth, commodity, industrial metal, art. Therefore the price and production amounts of gold have been recorded by numerous countries for the last century. Additionally gold markets have sustained may changes which have been well documented. Because of the availability and diverse nature of data relating to the price of gold, the gold industry was used as a case study to demonstrate the usefulness and methodological differences of dynamic systems.

Dynamic systems methodology has undergone dramatic changes in other fields of study. This paper will "make a case" for the use of dynamic system methodology in economics in order to gain a more thorough understanding of how and why economic systems behave the way they do over time.

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Chapter 1 - Introduction

Overview

In general, some economists study the appropriate representation of economic variables over time and how changes in variables over time affect economic equilibrium. This has lead to a rich history in econometric time series analysis as well as the evolution of theoretical models to account for changes in economic understanding. Increases in computational speed and accuracy have lead to many new econometric tests and the development of new functional forms to represent economic data. This being said, there is still much work to be done in economic modeling. As a discipline, there is still a prevalence toward simplifying assumptions (linear systems, stationarity, etc.) to estimate and replicate dynamic economic systems. Before making simplfying assumptions we should first seek to understand the behavior of the data we are analyzing. In understanding the behavior or "character" of the data we are studying first, appropriate econometric assumptions can be made. Dynamic systems methodology offers a global approach to first understanding the character of economic data before making any modeling assumptions. Simplifying assumptions, without fully understanding the nature of the dynamic system, has left us with a gap in our knowledge of economic phenomena. The gap caused by simplification, leads to difficulties in analyzing market structure issues from an industrial organization standpoint.

The following research employs dyanamic systems to relax the constraints implicit with the normal simplifying assumptions of time series analysis. Reducing the simplifying assumptions in regression analysis allows the data to tell the story. Changes in our understanding of functional form can affect results of market structure models. This paper will demonstrate that economists need to understand dynamic systems in their general form with more rigor. A better understanding of dynamic systems will lead to avoiding many specification problems, frequently the cause of difficulties in understanding an industy or firm behavior.

The use of dynamic systems methodology is common in the study of physics. Dynamic systems are used to describe motion of an object over time, such as the swing of a pendulum or planetary orbits. In biology dynamic systems are used to assess changes in populations and the movement of diseases in a population. For instance chicken pox displays random behavior, suddenly expanding in one population and then dying off. Whereas the spread of measles flu displays deterministic behavior, moving from one geographic area to another in a more predictable pattern (Stone 1996).

This paper will show how to use dynamic system methodology to characterize an industry's economic system. The proper characterization of a dynamic system will allow for a better understanding of how economic variables behave. Tests will show how to describe the behavior of a dynamic system and how that information can be used to modify the choices we make about the functional form of economic variables.

To begin, we must describe what dynamic systems are and how they work. We will identify the various classifications of different dynamic systems and demonstrate problems with some conventional tests, such as autocorrelation. This paper will then explore tests to appropriately determine what type of dynamic system is present. Finally, the use of a case study of gold prices to give practical application to the methods developed will be presented.

The historical chronology of our understanding of dynamic systems and time series analysis helps to understand where we are currently. We will begin with a brief review of the history of dynamic systems and time series analysis in economics.

History of Time Series Analysis

The history of time series analysis in economics is as diverse as the many disciplines which have contributed to our understanding, identification and classification of probabilistic phenomena over time.

Pearson, Gauss and many others began by looking at discrete probabilities through flipping coins, which they used to develop the normal distribution (Pearson 1897). However, there was a disconnect between the discrete probability and the probability of an event occurring in the future. Once they realized that discrete probabilities didn't accuratley explain the chance of some events occuring over time, many started to study random processes including Yule, Pearson and others. Pearson was the first to use the term "random walk" (Pearson 1905) which was used to describe the behavior of a system in which the chance of an event occuring over time was not correlated with the previous events of that particular variable. Bachelier saw the "random walk" process as a stochastic difference equation of the form $y_t - y_{t-1} = \varepsilon$. (Bachelier 1900)

Using a deck of cards, many early econometricians produced a random series. They would take out the higher order cards from the deck, (Jacks, Queens and Kings) and designate the 10 remaing cards per suite to have a value of zero to ten. Each color represented positive or negative values. They would then draw a card, record the value, replace it in the deck and reshuffle (Yule 1921). These records of numbers are how early statiticians created a random series over time, which evolved into early time series analysis.

With time series analysis starting to pick up prevalence in the discipline of economics, more economists looked to natural processes to explain economic phenomena. As Stanley Jevons said "Time is the great independent variable of all change that which itself flows on uninterruptedly, and brings the variety which we call motion and life". (Jevons 1877) In fact the “father” of neoclassical economics himself, Jevons was convinced that economic downturns were correlated with high rates of sunspot activity (Jevons 1862). Although this theory eventually proved to be wrong, Jevons’ line of inquiry started the development of the autoregressive processes (AR).

The autoregressive process uses past data to predict future data. The early research on autoregressive processes was primarily conducted by Yule, in which he defined the sunspot data to be an AR(2) process or $y_t = f(y_{t-1}, y_{t-2}) + \varepsilon$ (Yule 1921). Yule then began to look for stationarity in time series and was instrumental in starting the development of using oscillators for time series analysis (Yule 1926). He classified time series data by four categories: random, conjunct, disjunct and oscillation.

A random series has no serial correlation, meaning the two events are not correlated over time. If events are purely random, then any past events will in no way effect what will happen in the future. In other words, what happens today does not influence tomorrow.

An example of a random series is consecutive rolls of a die, the outcome of each roll is independent of the previous roll.

A conjunct series has serial correlations which are all positive. These types of series tend to confer on one another in directionality. For instance, if a variable has risen in the previous period it is more likely to rise again. We see such behavior in stock prices movements.

A disjunct series has serial correlations which are all negative. This type of series also confers itself in directionality as the conjunct series, but in the opposite direction. So if a variable falls today it is more likely to fall tomorrow.

Finally, what Yule believed to be the most prevelant and often the least common type of random series used in economics, the oscillation series. The oscillation series has serial correlations which switch sign. In this case we see data that is constantly going up and down from period to period.

The identification of different catagories of the behavior of data over time was really the beginning of using economic time series data as a dynamic system. During this era, a lot of time was being spent studying random processes and how they follow a normal distribution. Slutsky (1927) saw these natural processes as a moving summation and started looking at time series problems through moving averages. The moving average is calculated by successively calculating an average of a defined interval that is shifted over time. Finally, it was Wold who put it all together to develop the random stationary process. (Wold 1938) The random stationary process is one in which the oscillation is random and the average remains at a constant level over time.

More recently (1940-1950), other economists expressed the need to understand dynamical systems such as Samuelson: “we should still need a theory of the path by which a given market approaches its equilibrium position, not for sake of the theory alone, but for the information that such knowledge throws upon the direction of displacement of the new equilibrium position as well” (Samuelson 1943). However, even Samuelson himself was conflicted over the appropriateness of using the understanding of dynamical systems from other disciplines, such as physics or biology and questioned their relevance. As a discipline, we have incorporated some of those concepts, such as logistic functions and Brownian Motion. We use the concepts of dynamic systems in natural resource economics for population changes as well as in explaining the movement of prices over time.

In 1965, Adelman noted that long cycles did exist in economic data. (Adelman 1965) Long cycles are oscillatory in nature but occur over large time intervals. In 1966 Granger noted that most economic phenomena exhibit low level frequency components (Granger 1966) and that lack of inclusion can lead to problematic modeling procedures. This was soon followed by Mandelbrot & Van Ness in 1968 who reclassified Brownian Motion into a more general form to allow for the inclusion of long memory processes. They stated that “empirical studies of random chance phenomena often suggest, on the contrary, a strong interdependence between distant samples” and additionally that: “It is known that economic time series “typically” exhibit cycles of all orders of magnitude, the slowest cycles having periods of duration comparable to the total sample size” (Mandelbrot/Van Ness,1968).

After this flurry of activity on memory processes, the next major piece of research in time series analysis was Box & Jenkins in 1976 with the formal derivation ARIMA(p,d,q) modeling. That is to say, short memory autoregressive processes (p) and longer memory non recursive moving averages (q) were included together into a modeling framework with an integrating factor (d). This set up a template that could capture some memory processes, at least short memory. However, long memory Brownian Motion was not included. Brownian Motion was not included due to the fact that the integrating factor only performs an exponential smoothing of the data, which does not work well with cyclical processes. This is because the integrating factor in the ARIMA modelling structure is forced to be an integer.

A few years later, Granger and Joyeux (1980) found that these low level frequencies may exist in ARMA models as long term memory and should be included. In 1981 Hoskings derived a method to include long memory processes, in what was considered a Fractional

ARIMA or FARIMA model. The FARIMA model allowed the integrating factor of the ARIMA model to vary more than just an integer value.

After this period, there came interest in chaotic processes as they are deterministic and a system of this type would have long memory processes among its attributes. Brock did most of the work, relating to macroeconomic phenomena. (Brock 1995)

As of late, more research has gone into dynamic systems and their properties for the purposes of simulation or to recover primitive functions of a dynamic system in motion. That is to say, that instead of decomposing a times series into additive components and determining an error term with Brownian Motion, more attention is being paid to other methods such as non-linear systems. Of particular interest, is whether or not a system is linear or non-linear and if it exhibits long memory or chaotic behavior (Frank/Stengos 1988, Brock 1988).

Aside from the economics literature, many other disciplines, such as physics and biology, have been working hard on dynamic system problems. Many books and articles have been produced on the topic. (Hilborn 2000, Williams 1997) May started chaotic research in biological processes (May 1973,1976,1996) through the use of attractor plots. Attractor plots are scatter plots of time series data over different time intervals. The attractor plot demonstrates whether a variable converges or diverges to a particular value over time. The attractor plot has been an important first step to visually understanding a dynamic system.

Many “classic” physics problems demonstrate chaotic behavior, such as the “double pendulum”. The double pendulum example describes the motion of two pendulums swung from the same axis will exhibit a chaotic type of behavior. Planetary orbits can behave in a chaotic way as well. That is to say, the attractor plot of a planet’s orbit over time can look random, but it is not.

At our current point in the history and evolution of time series analysis and dynamic systems, computing power has finally caught up with theory. Many dynamic systems do not have closed form solutions and must be solved numerically. The additional computing power allows us to be able to estimate and use many of these concepts in our analysis of economic problems.

The Gold Industry

For the first time, we utilize dynamic systems to evaluate the gold industry. Gold has a rich and diverse economic history which makes for an interesting study. Gold has been valued by societies for hundreds of years due to its many uses: store of wealth, commodity, industrial metal, art. Therefore the price and production amounts of gold have been recorded by numerous countries for the last century. Additionally gold markets have sustained may changes which have been well documented. Because of the availability and diverse nature of data relating to the price of gold, the gold industry is used as a case study to demonstrate the usefulness and methodological differences of dynamic systems.

Using dyanamic systems methodology we characterize the movements in the price of gold over time. We start by calculating the long run dependence of gold prices. The long run dependence is a measure of how important the history of the data is to its current price. To estimate the amount of dependence we will use the Hurst Exponent, which gives the level of long run dependence of data. We will discuss the various measures of the Hurst Exponent and derive them for the price of gold.

Secondly we determine how much of the price of gold is random and how much is deterministic. We do this by first defining what is random and deterministic. Then we will measure for determinism via the Lyapunov Exponent. We will find that the market price of gold has both a deterministic (intra market) and random (inter marekt) component.

After separating the intra and inter market components of the gold price, we test via regression for industry structure affects on the price of gold. We will also be able to develop a characteristic equation for the price of gold that incorporates the deterministic and random components. We will use this characteristic equation to simulate the effects of future events on the gold industry as well as to develop a linear supply and demand model for the U.S. gold industry. We will find that utilization of dynamic systems allows for an improved understanding to changes in the gold industry over time. It will be shown that increases in the number of firms in the gold industry make the market price of gold more subject to external events outside of the industry and thus more volatile and that a reduction in the number of firms in the industry will narrow the range of volatility in the market price of gold. From the supply and demand analysis we will discover that the majority of the changes in the market price of gold come from the demand curve. We will also learn how the intra and inter market prices affect the equilibrium price of gold. All of the tested relationships are only possible through an understanding of dynamic systems.

As our understanding of dynamic systems has evolved, so must our application of these principles to our discipline. Let us begin with what a dynamic system is and how they work.

CHAPTER 2 - DYNAMIC SYSTEMS

Dynamic systems include any series of data that propagates through time. In this chapter we will characterize the components of dynamic systems. First we will begin with describing the behavior of dynamic systems. Next we will characterize the two broad categories of dynamic systems, deterministic and random. We will then show a special deterministic system, a chaotic one. Following that we will discuss how memory (correlation of a variable over time) works and is measured in a dynamic system. We will then look at the Lyapunov Exponent test to determine if a dynamic system is deterministic. Finally we will discuss assumption differences in dynamic systems as well as what a general functional form without a linearity assumption would be for a dynamic system.

Behavior of Dynamic Systems

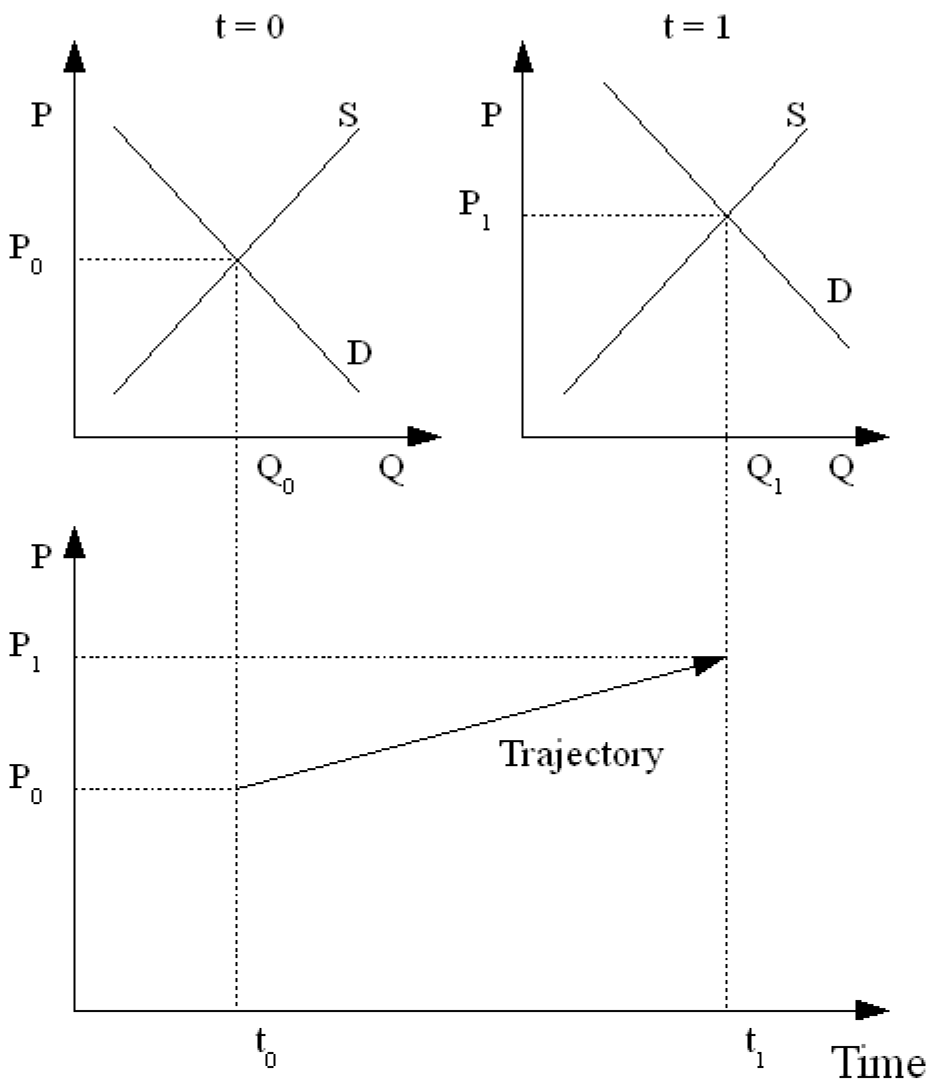
Dynamic systems are a functional form that explain the position of an object in space over time. That is to say, dynamic systems can identify the position of an object in space-time. Dynamic systems have been used extensively in the physical sciences to explain bodies in motion, as well as in the biological sciences for population growth. In economics, many market systems and economic models are dynamic in nature. We use dynamic systems to understand rational and adaptive expectations. Dynamic functions are used to determine rates of change in populations and sustainable yields. Dynamic functions are also used in optimal control theory. With all of our use of dynamic systems, we have done little in the discipline to understand how these systems behave and the impact of our assumptions. We will start with the development of what dynamic systems are and how they behave.

"A dynamical system is a rule for time evolution on a state space." (Meiss 2007) State space is the set of all possible states of the dynamic system and each state space is a unique coordinate point within the fixed set of the system. A dynamical system consists of a state space, and the coordinates of the system, at any instant, are described by the rule or functional form of the system. In economics, many of our dimensions of variables are dynamic in nature, such as interest rates, prices and quantities. A further definition may clarify: "Mathematically, a dynamical system is described by an initial value problem. The implication is that there is a notion of time and that a state at one time evolves to a state or possibly a collection of states at a later time. Thus states can be ordered by time, and time can be thought of as a single quantity." (Meiss 2007) Dynamic systems are deterministic because they have a functional form which identifies the state space and the evolution of the states over time completely.

For the economist, a deterministic dynamic system has a very profound meaning and effect for understanding an economic system. Determining the functional form that produces a system, is critical to identifying its behavior. If a system behaves in a dynamically deterministic fashion, then that system is always in equilibrium at every evolution on the state space. For instance, a pendulum has an equation that defines its motion completely. In order for the pendulum to swing back and forth, it is always dependent on where it was previously, as well as where it is going, in order for it to reverse direction. At every instance in time of the pedulum's arc, the pendulum must be in equilibrium or it would not be fully identified by the equation which governs its motion. The example of the pendulum shows that a process that is deterministic has to be in equilibrium always or the process could not be deterministic. The state space of the dynamic system is the evolution of the variable in the phase space across time. The variables that affect the pendulum's motion, that can be measured at any instance, are in the "phase space" of the dynamic system.

State space is the combination of the phase space and time. In supply and demand models, price and quantity make up the phase space. The phase space of the supply and demand model shows a static equilibrium point at an instant in time. The state space of the supply and demand model, is the description of the evolution of the equilibriums over time, which includes the phase space.

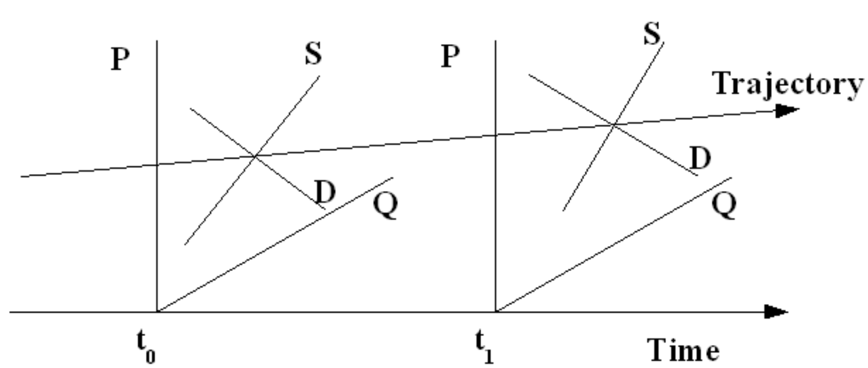
Figure 1 – Evolution of Static Equilibriums



We classically define a static equilibrium in price-quantity space, which is our phase space (Figure 1). As we have changes in supply and demand from time t=0 to time t=1 we arrive at a new static equilibrium, due to the demand curve shifting between t=0 to t=1. The resulting plot in the lower part of the figure, results from connecting the equilibria in the phase space over time. When we plot a variable across time, we lose a dimension. In this case we do not see the quantity dimension, although we know it is necessary for the formation of the equilibrium price over time. Often, we only look at either the trajectory of a price change over time or the static phase space without time. In either case, we may lose information about the variables that caused the supply or demand curves to shift.

In reality, the two graphs put together give the complete dynamic system in state space. The state space describes the evolution of the equilibrium point over time (Figure 2).

Figure 2 - Complete Dynamic System of Supply and Demand



Dynamical systems are deterministic, if there is a unique point to point evolution (trajectory) of the state space. Dynamic systems can be random, if there is a probability associated with the evolution of the movement between state space. For example, Figure 2 would represent a deterministic system if a function could be identified that would exactly explain the position of the price, quantity demanded and quantity supplied of any instance in time past, present and future. If no function could explain the position exactly, then there may be randomness in the system. However, the presence of randomness doesn't preclude the function from being dynamic in nature. The indentification of an appropriate function that describes the movement of all the variables in the state space, is important to our understanding of how economic variables evolve. If a system is deterministic, that suggests that the system is always in a static equilibrium in the phase space at any given point in time along the trajectory. That conclusion is very different than a random dynamic system where the exact trajectory is unknown. Understanding whether or not a dynamic system is deterministic or random is important to describing what type of behavior we might expect. Such as a system that drives toward a long run equilibrium or a system that will never reach a static long run equilibrium.

Certain tests can be used to demonstrate how deterministic or non-deterministic a system is. Before these are presented, let us better define both a deterministic and non deterministic system through example.

Deterministic Dynamic Systems

A deterministic dynamic system can be defined and better understood using the example of a simple, one dimensional model which propagates over time. For example the example that follows will utilize the logistic function to demonstrate the one dimensional model.

The logistic function is a non-linear dynamic system that describes the behavior of one variable in the past, present and the future. The logistic function is dependent on the function's previous value, time and the sensitivity of the growth rate. This function is frequently used by economists to model the supply or availability of various renewable resources, as well as population growth rates (Conrad 2002). The logistic function in discrete time takes the form:

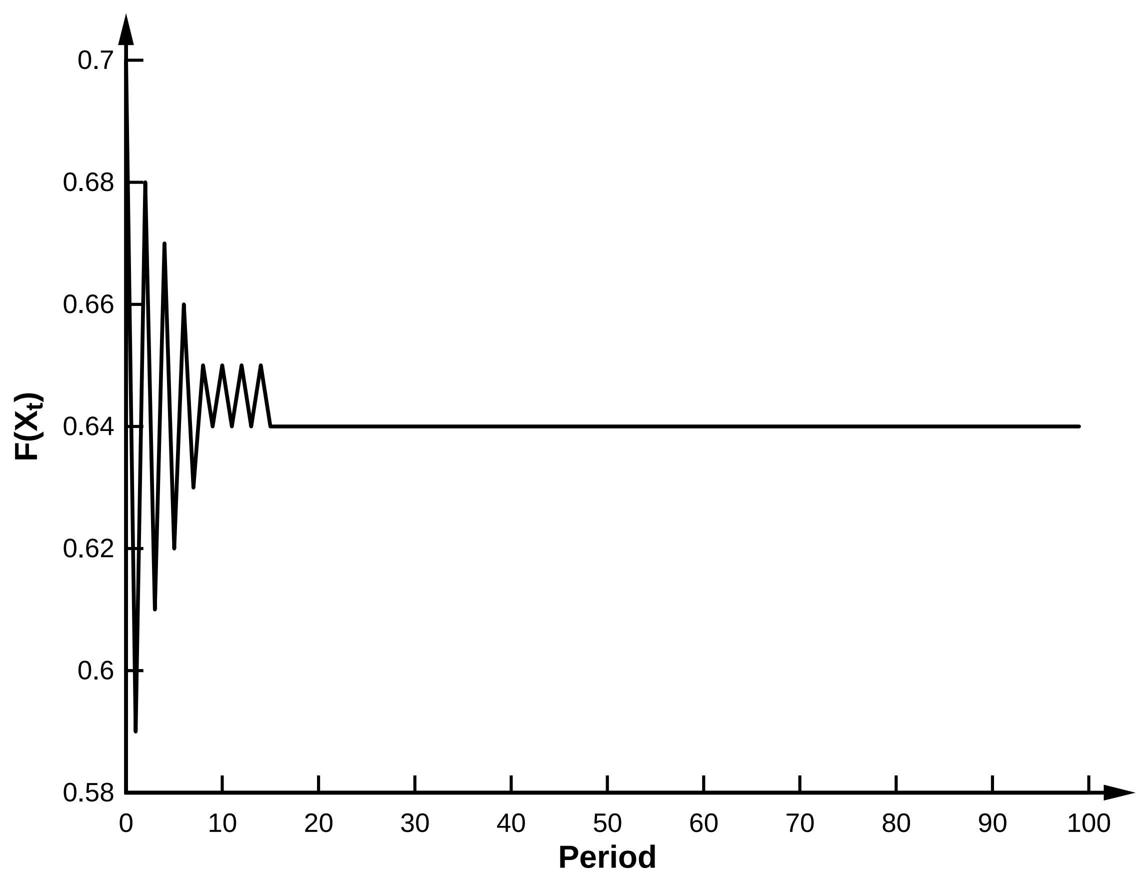
$$X_{t+1} = \alpha X_t(1 - X_t)$$

The coefficient of sensitivity "alpha", can theoretically take on any value from 0 to 4. Although this value seems arbitrary, all other values for the sensitivity coefficient cause the logistic function to become undefined. The value of X (growth rate) can take on a value between 0 and 1. The variable alpha causes the motion of the variable X over time. The state space for the logistic function will include all possible values for alpha, because the state space defines the entire range of possibilities of all trajectories. As alpha changes so does the evolution of X through the state space and all different iterations of

X through space-time are predetermined given a specific alpha. Since the logistic equation is a non-linear dynamic equation, the value of X in the future is dependent on its past by functional definition: $X_{t+1} = \alpha X_t(1 - X_t)$.

Consider the trajectry of X through the state space when the value of the alpha is 2.8. For consistency across examples, a starting value for X of $1/2$ will be utilized. In Figure 3 we can see that the value of X oscillates to a fixed point after aproximatley 15 time periods and converges to the value 0.64.

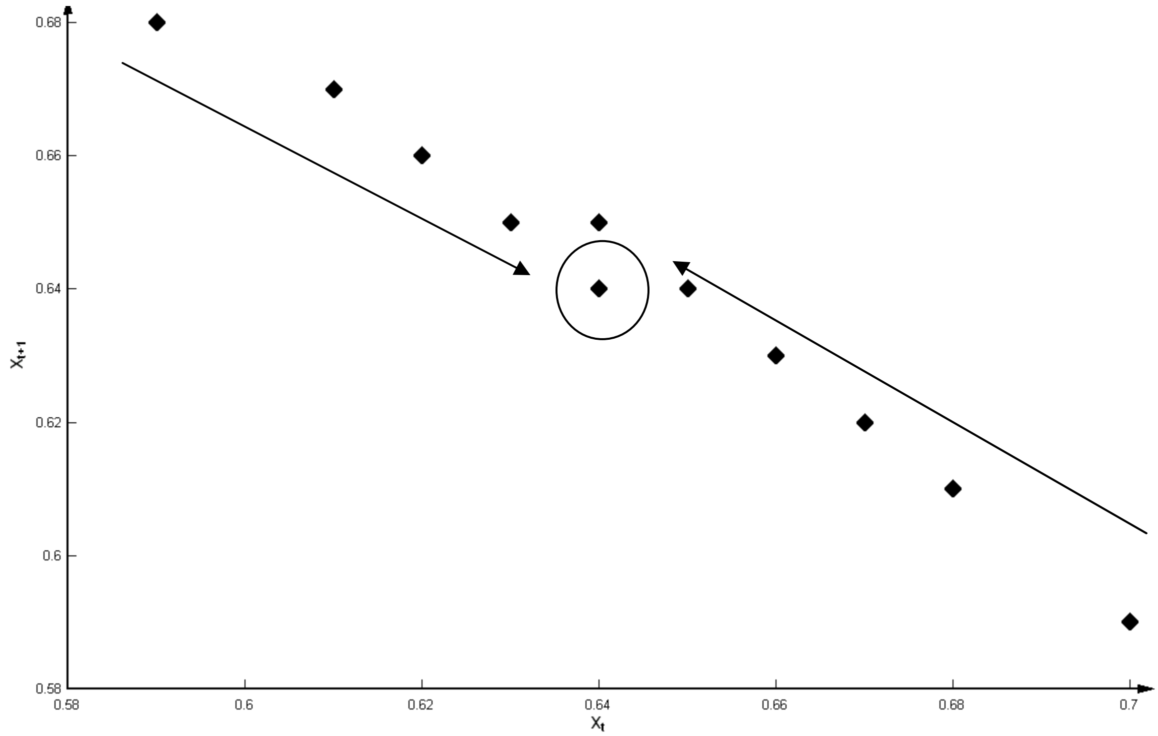
Figure 3 - Logistic Function α=2.8



The logistic function with value $\alpha = 2.8$ above, is an example of a dynamic system that is attenuating to a constant level. The evolution of the state space converges to one point in the set of all possible values in the state space. Another tool for understanding the convergence of a series to a particular value over time can be used, the attractor plot (May 1973).

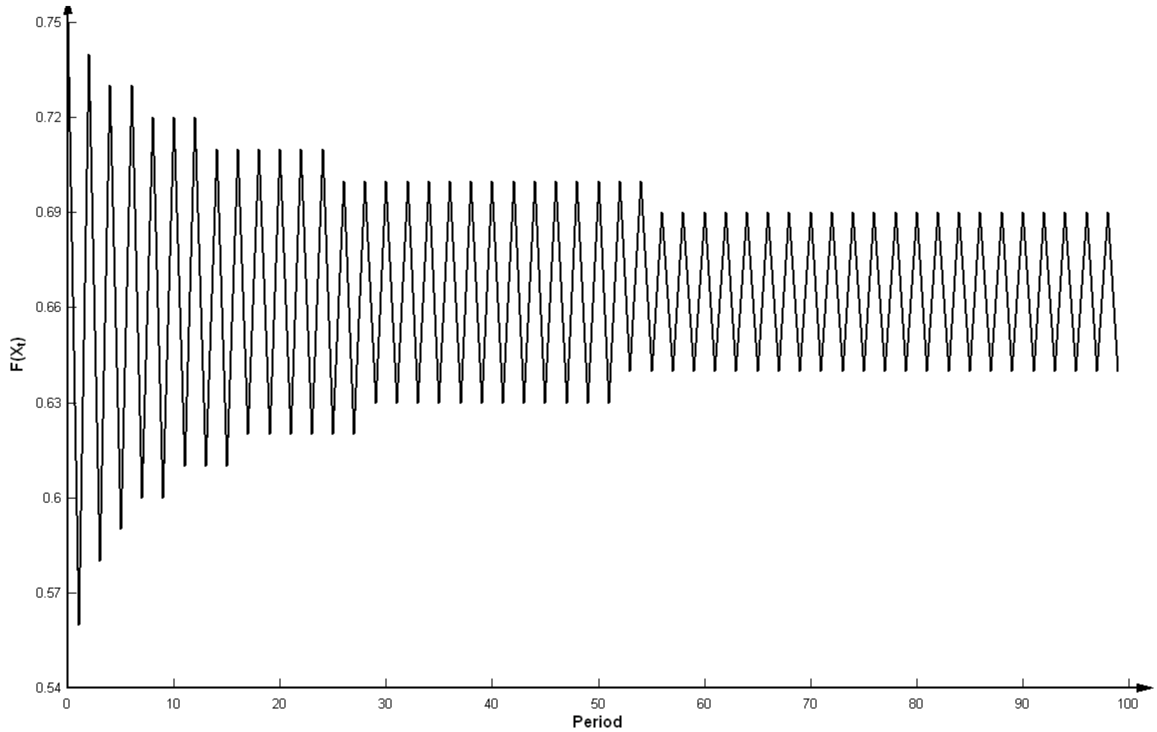
The attractor plot is one time step plotted against another time step so that the system is viewed in a time independent fashion. Based on the attractor plot (Figure 4) of the same logistic function we can see that the value of X is converging to a value of 0.64. We classify a deterministic system of this type, as stable and in this case has a "long run" equilibrium of a single value over time. Systems that converge to a single value are intuitively simple. Most of the use of dynamic systems in economics has stopped as this point.

Figure 4 - Attractor Plot of X_{t+1} vs. X_t for Logistic Function α=2.8



As a contrast to the "long run" stable equilibrium, let us look again at the same logistic function, but this time change the value of alpha to 3. In this case (Figure 5) the series is starting to converge to a particular point in the state space, but will never reach a single point.

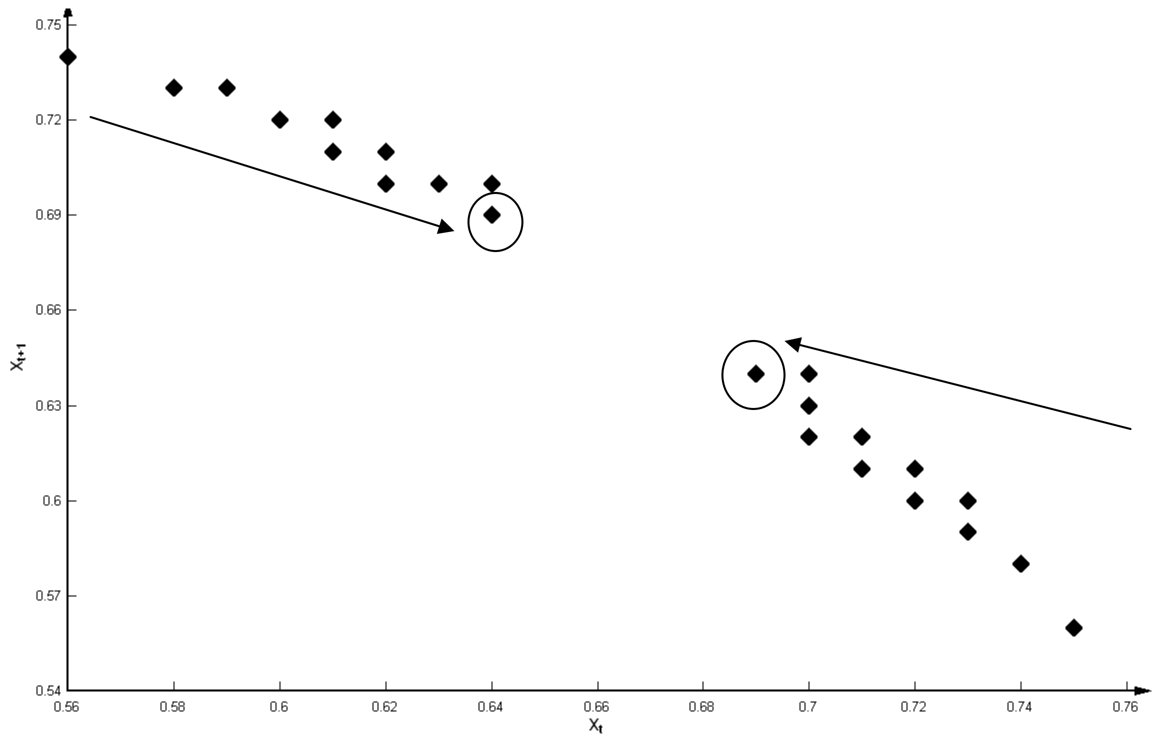
Figure 5 - Logistic Function α=3



In Figure 5 the system finds two equilibriums between the two points of approximatley 0.64 and 0.69. With no changes to the logistic equation, the system will continue to alternate between these two equilibrums indefinatley.

The above logistic function is also a stable "long run" equilibrium. We can also see this behavior in the attractor plot of the series (Figure 6) where the equilibrium does not stabilize to one point, but instead oscillates between two equilibriums of 0.64 and 0.69.

Figure 6 - Attractor Plot of X_{t+1} vs. X_t for Logistic Function α=3



In both of the two cases presented, the dynamic system is deterministic, but a small change in the initial value of alpha from 2.8 to 3 caused the equilibrium of the system to change from one stable value to a stable oscillation between two values. It is important to note that a small change in a deterministic dynamic system can have a profound effect on the resulting type of "long run" equilibrium.

We have seen examples of dynamic systems that are deterministic, however not all dynamic systems are deterministic, let us now focus our attention on a dynamic system that has some random component to it. As discussed, variables which move through time are dynamic in nature. There are two types of dynamic systems. Those that are completley defined, as demonstrated earlier, and those dynamic systems which have an element of randomness, also known as a stochastic process.

Random Dynamic Systems

Dynamic systems contain variables which move through time. Occasionally the variables of some dynamic systems are random. A random (stochastic) dynamic system is subject to the effects of noise (randomness). In economics, we usually consider only a specific type of noise "white noise". Noise is a random variable and can fluctuate with or without a regular pattern over time. Noise is common in many models, consider an AR(1) process:

$$X_{t+1} = \Theta_1 X_t + \varepsilon_t$$

where ε_t is a random process

The random dynamic system is still a dynamic system as the AR(1) equation completley describes how X_t evolves over time through state space. In the case of the AR(1) equation, the value of the variable in the next period is a function of the value of the variable in the previous period plus randomness (noise). Typically noise refers to the generating of fluctuations due to a large number of variables interacting in the system, considered to be a problem of omitted variables. Sometimes however, noise in a dynamic system is due to the variables in the system being probabilistic in nature and arises due to the confluence of these probabilistic variables interacting. (Chatfield 2004)

Noise can have a pattern, just as in a deterministic system, and can be one of three types. Noise can be persistent, anti persistent or completely random. All noise (regardless of type) falls under the general classification of Fractal Brownian Motion (FBM). As economists, we typically use the subset of FBM that is completly random or Regular Brownian Motion (RBM). FBM and RBM are related as:

$$RBM \subset FBM,$$

that is Random Brownian Motion is a subset of Fractal Brownian Motion. By definition FBM falls into three catagories: persistent, anti-persistent and random. Persistent and anti persistent randomness is serially correllated over time. What makes RBM special, is that there is no serial correlation of the noise between time periods. Persistent FBM is a situation in which the noise is positivley correllated over time. In persistent FBM, if the previous value of the noise is moving up, there is a higher probability of the present value moving up as well. Stock prices tend to exhibit this behavior, where the price of a stock has movements that directionally go the same way for a while such as: “up”, “up”, “up”, “down”, ”up”, “up”, etc. Anti persistent noise has serial correllations that are negative. In the anti persistent case, there is a higher likelihood that the series will alternate from “up” to “down” with more periodicity.

To better understand the full complement of Brownian Motions let us define randomness formally as was done by Mandelbrot and van Ness (1968). In their definition t designates time and ω is all the values of a random function where ω belongs to sample space Ω. Therefore, Regular Brownian Motion (completley random) has a mean of zero and constant variance between any two points such as $\left|t_2 - t_1\right|$. Thus an RBM process in completely random because $\mathbf{B}(t_2, \omega) - \mathbf{B}(t_1, \omega)$ are independent of one another. A RBM process is stationary because there is no serial correlation (dependence) between time periods.

For the two other noise cases, we need to add a parameter to capture the serial correlation. This parameter is called the Hurst Exponent (H), thus Fractional Brownian Motion is $\mathbf{B}_H(t, \omega)$ and to look for stationarity we now have:

$$\mathbf{B}_H(t_2, \omega) - \mathbf{B}_H(t_1, \omega)$$
$$= \frac{1}{\Gamma(H + \frac{1}{2})}\left\{\int_{-\infty}^{t_2} [(t_2 - s)^{H-\frac{1}{2}}]dB(s, \omega) - \int_{-\infty}^{t_1} [(t_1 - s)^{H-\frac{1}{2}}]dB(s, \omega)\right\}$$

The gamma function is: $\Gamma(\alpha) := \int_0^\infty X^{\alpha-1}e^{-X}dx$ and is used to ensure the Hurst Exponent (H) takes on a positive value. The range of the Hurst Exponent is: 0< H<1 and B is an FBM stochastic process.

With this formulation FBM falls into the three basic categories: anti persistent ($0 < H < \frac{1}{2}$), persistent ($\frac{1}{2} < H < 1$) and neutrally persistent (RBM) ($H = \frac{1}{2}$). The formal derivation of FBM means that stochastic processes can have a memory structure (Granger,1980) and that RBM really is a subset of FBM and is indeed a special stochastic process. In other words some random processes are correlated over time and the existence of correlation over time does not preclude a series from being random.

Further proof of serial correlation can be seen in the autocovariance of an incremental random process Z where $Z = [Z_k: k = 0,1,\dots]$. Thus for any time series data, Z is the random process. Whether Z is correlated or not can be defined as FBM by: $Z_t = \mathbf{B}_H(t_2, \omega) - \mathbf{B}_H(t_1, \omega)$. As such the corresponding autocovariance function between t_2 and t_1 is of the form:

$$E \frac{(\mathbf{B}_H(t_2, \omega)\mathbf{B}_H(t_1, \omega))}{(\mathbf{B}_H(t_2, \omega))}$$

The autocorrelation is the covariance divided by the variance. In another form, we could also define an h that is any fixed increment between observations so that the autocovariance is equivalent to :

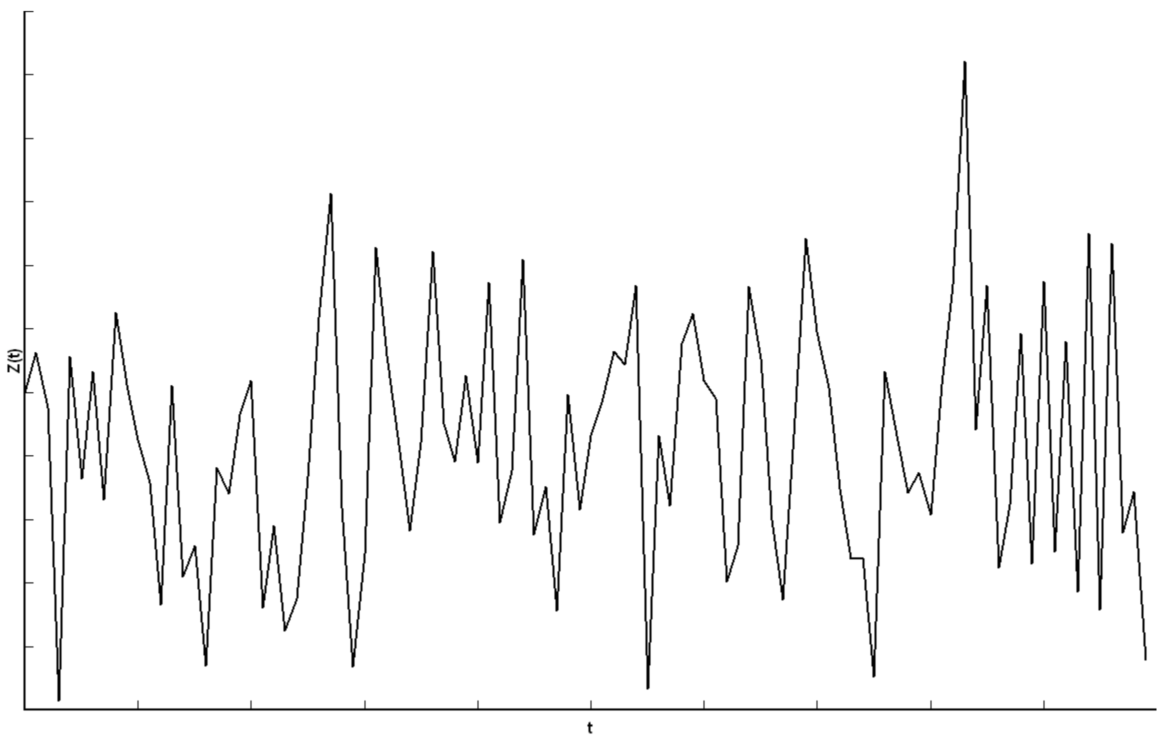
$$r(n) = \sigma^2 \frac{h^{2H}}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}), n = 1,2,\dots, h > 0$$

The general formulation of the ACF function between any two time steps is:

$$\gamma(t) = \frac{[t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}]}{2t_2^{2H}}$$

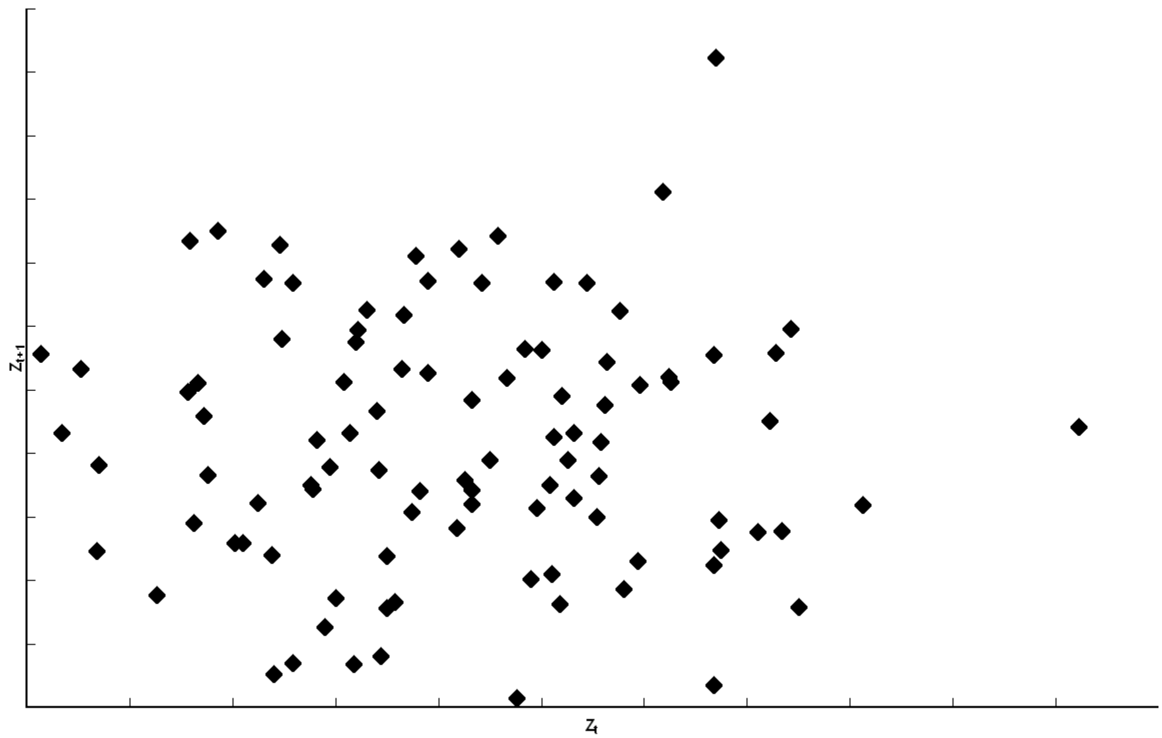
If $H = \frac{1}{2}$ then there is no serial correlation between time steps and we have a series that is random in the common definition of the term. RBM is a true subset of FBM. In fact, RBM is the special case of FBM when the Hurst Exponent is ½ . When the Hurst Exponent is ½ the covariances, or serial correlations in the random process go to zero in the numerator of the ACF function. All exponents in the ACF function become one. A graphic representation of an RBM process is in Figure 7 below.

Figure 7 - Regular Brownian Motion B_H = 1/2



Although visual inspection of any stocastic prosess is difficult, the movement of Z(t) in Figure 7 has no correlation between periods. To reiterate, RBM is special because there is no serial correlation of noise between time periods. Another way to visually inspect the process, is to look at an attractor plot of the RBM process as we did before for the deterministic system (Figure 8).

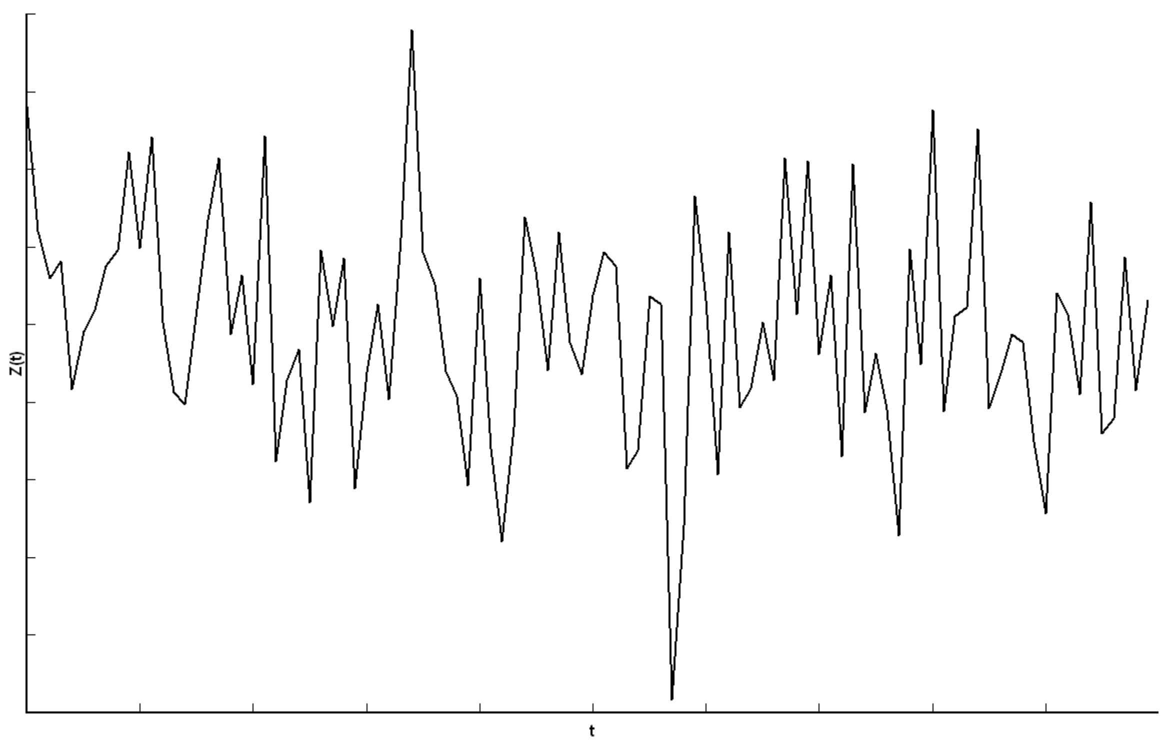
Figure 8 - Attractor Plot of RBM



In Figure 8 there is no discernable pattern or convergence of the trajectory to a particular value, which supports the lack of serial correlation between time periods, demonstrating that RBM is different. The clustering or lack of clustering in the attractor plot is determined by the Hurst Exponent.

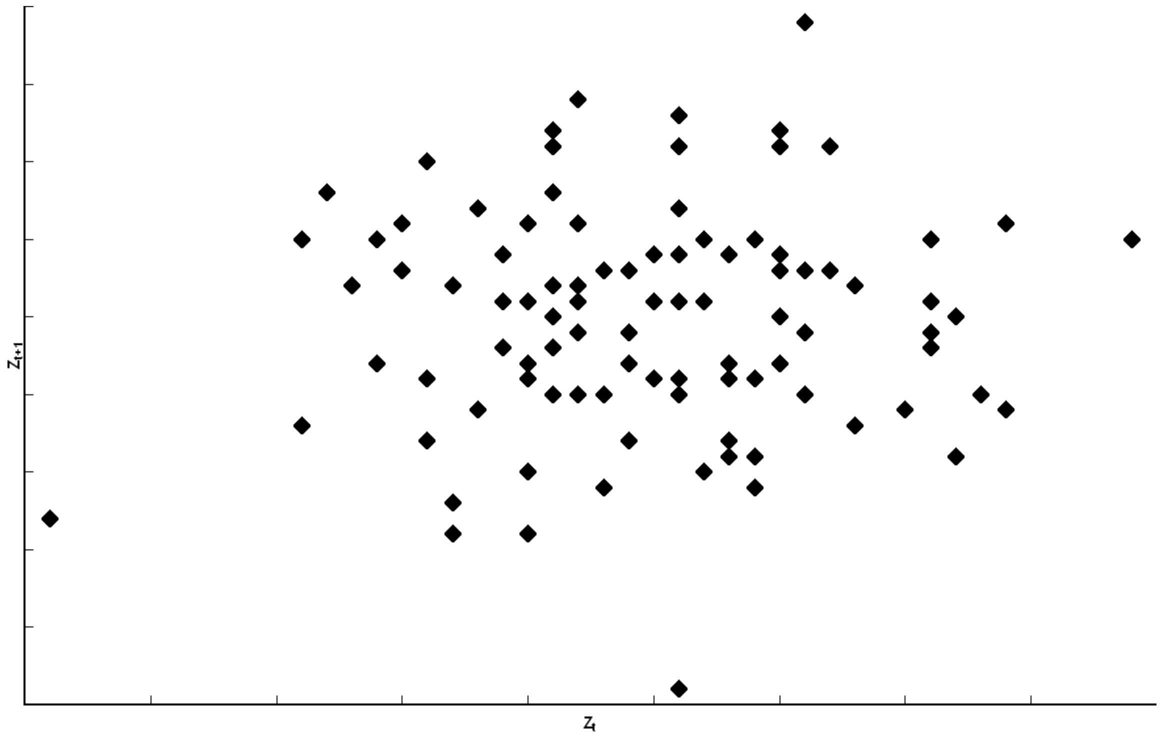
Again, it is worth noting, that random and deterministic behavior can be difficult to discriminate. Later we will discuss tests to identify the difference between random and deterministic behavior. In the case of persistent Fractal Brownian Motion, the terms are positively correlated to one another (Mandelbrot,1971) and in this case we should expect to see more of a "pattern". Using the general functional form for an FBM process and assigning a value of H = 0.9 for the Hurst Exponent, the persistent FBM in Figure 9 is generated. (code to generate FBM in Appendix)

Figure 9 - Persistent FBM B_H=.9



A visual inspection of the random process does, not clearly demonstrate if the process is correlated over time. However an attractor plot does provide a hint of correlation with a visual inspection (Figure 10).

Figure 10 - Attractor Plot of Persistent FBM



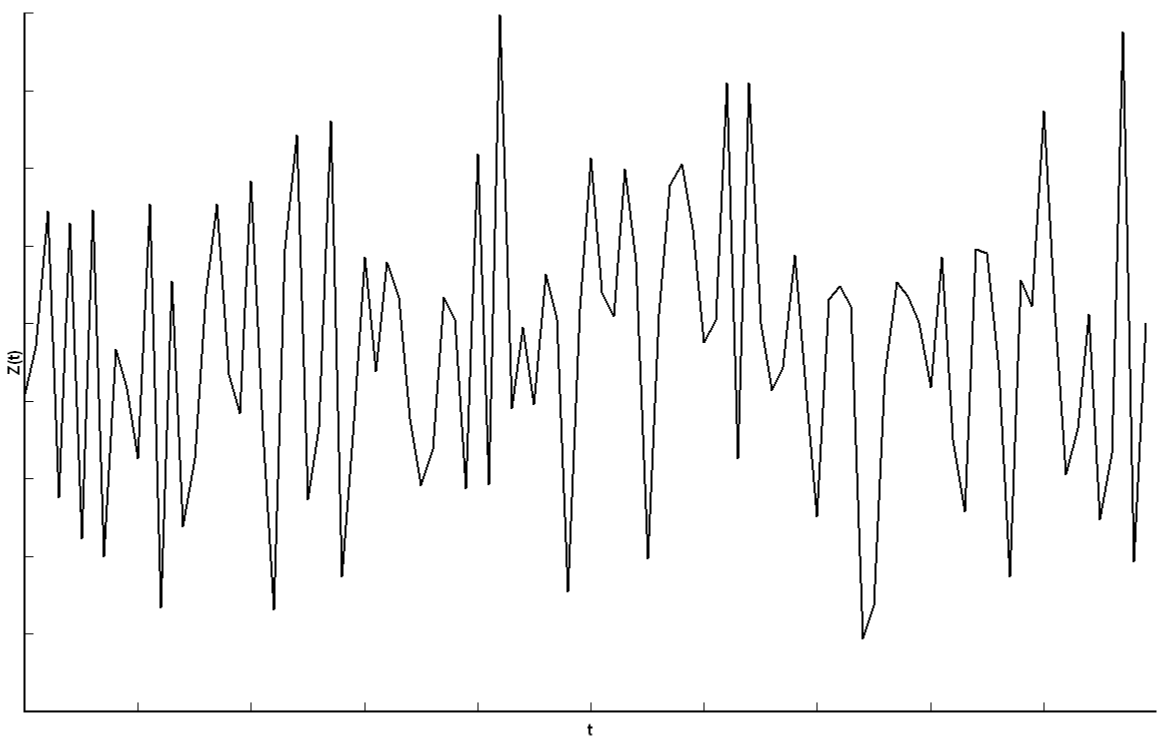
The difference between the persistent FBM attractor plot and the RBM attractor plot is the clustering of the Z values. Notice in Figure 10, that the random values are clustered together "tighter" than that of the RBM (Figure 8).

To reiterate, visual comparison of an RBM plot (Figure 7) and a persistent FBM plot (Figure 9) reveals little discernable differences. Close visual comparison of an RBM attractor plot (Figure 8) and a persistent FBM attractor plot (Figure 10) does begin to reveal the correlation between time periods. The attractor plots of these two different types of Brownian Motion demonstrate that not just one type of randomness exists.

Let us now review a third type of Brownian Motion, the antipersistent FBM. Again a visual inspection of a plot of anti-persistent FBM (Figure 11) does not obviously demonstrate a correlation over time.

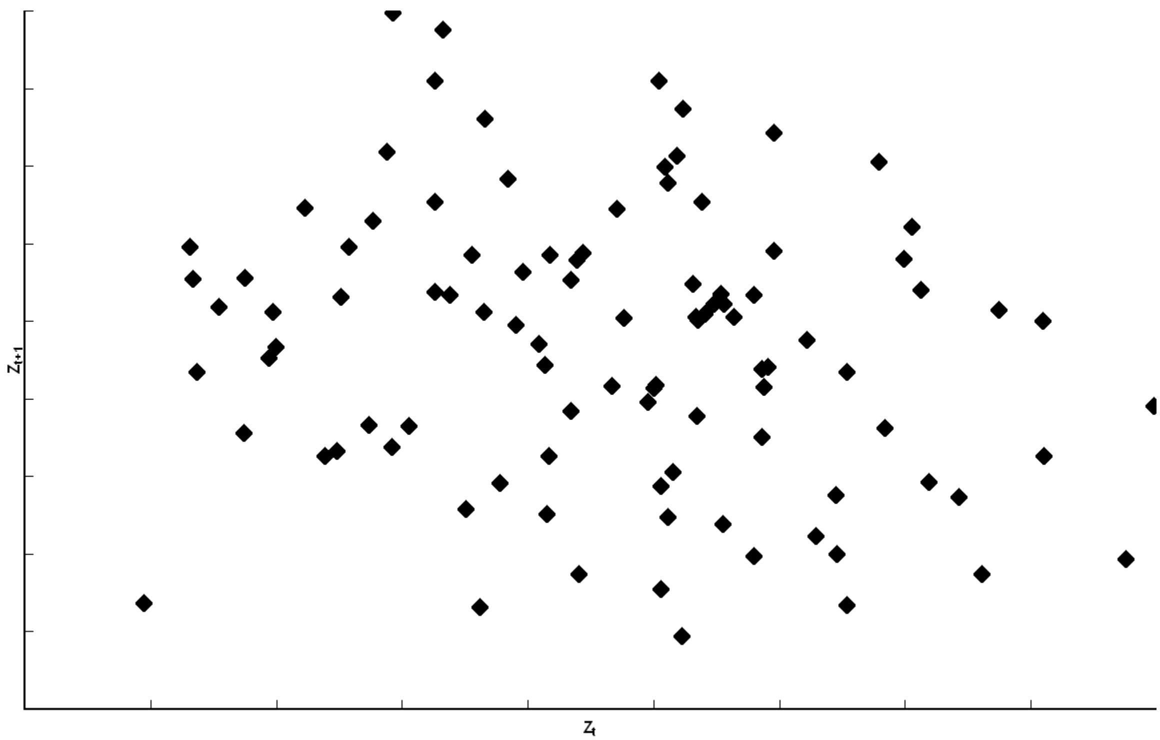
In the case of Anti persistent FBM, the terms are negatively correlated over time. To produce anti persistent FBM, we utilize the general form of the FBM equation and in this case use a value of H = 0.3 for the Hurst Exponent.

Figure 11 - Anti Persistent FBM B_H=.3



In this example we may see more “cycling” behavior, however the attractor plot will illuminate the difference in a more pronounced way (Figure 12).

Figure 12 - Attractor Plot of Anti Persistent FBM



In the case of anti persistent FBM the attrator plot (Figure 12) has values that are more "spread out" than those in the previous two attrator plots. Again demonstrating that there is correlation over time, but it is difficult to discern visually.

Regardless of the type of plot used, visual inspection alone does not adequately demonstrate correlation over time. As such, we need to test for the Hurst Exponent, which defines how a random process is correlated over time. We will discuss the tests a in a forth coming section.

Regardless of the noise type, it evolves, or moves through state space, the set of the state space changes according to the probabilistic properties of the noise variable itself. Due to noise in a random dynamic system, we cannot map by means of the attractor plot the complete evolution of the variable through space time. Instead a compendium (many "states of the world") of state space trajectories exist after each demarcation of time in the state space. Specifically, noise which is random, has infinite state spaces at each time demarcation and is only constrained by the probabilistic properties of the noise. This is not true of a deterministic dynamic system, which has a well defined dimension. (Longtin 2007) As will be discussed in the next section determining the size of the state space is the basis of methods used to distinguish random dynamic systems from deterministic systems.

The correlation of randomness over time is given as the integral of the autocorrelation function over all times in the state space. The problem with the measurement and detection of randomness, is that randomness typically occurs in conjunction with a variable of interest in the functional form. This is true of the AR(1) process, as with other random dynamic systems. Both processess occuring together can cause misidentification of the dynamic system as being random and not deterministic.

To estimate a dynamic system, it is common to use a linear approximation. A linear assumption can lead to errors in identifying the functional for of the system. For example given the function:

$$\frac{dx}{dt} = f(x, \sigma)$$

where: $\sigma = Noise$

The measurement of the observational noise over time would be given by:

$$Measurement(t) = F(x(t) + \sigma(t))$$

In this case the act of measurement is affected by the noise, but the variable of interest x(t) is not. A linear type of treatment can make the detection of deterministic systems difficult and cause mis-specification of a dynamic system. (Longtin 2007) Before we talk about how to detect the difference between the two, we need to discuss a special type of deterministic system that can mimic a random system.

Chaotic Dynamic Systems

In the previous section we argued, by example, that visual comparisons of a plot or attractor plot of an RBM, persistent FBM and of an anti-persistent FBM process do not clearly demonstrate correlation between time periods. However, we will see that in chaotic systems, attractor plots do provide a clear visual correlation over time.

A chaotic system is a condition in which the system appears to be random, but is in fact deterministic. The existence of chaotic systems further compounds the problem of specifying dynamic systems.

Refer back to the deterministic logistic function from the previous section, we can see chaos as well. As the alpha in the logistic function changes, so too does the equilibrium. A chaotic system, is one in which the alpha becomes "sensitive" enough to cause the logistic function to oscillate in a fashion that appears random. For the logistic function, this is true for any alpha with a value of greater than 3.57. Figure 13 is one such system. As before, the initial value of X is 0.5, but now the alpha is 3.95.

Figure 13 - Logistic Function α=3.95

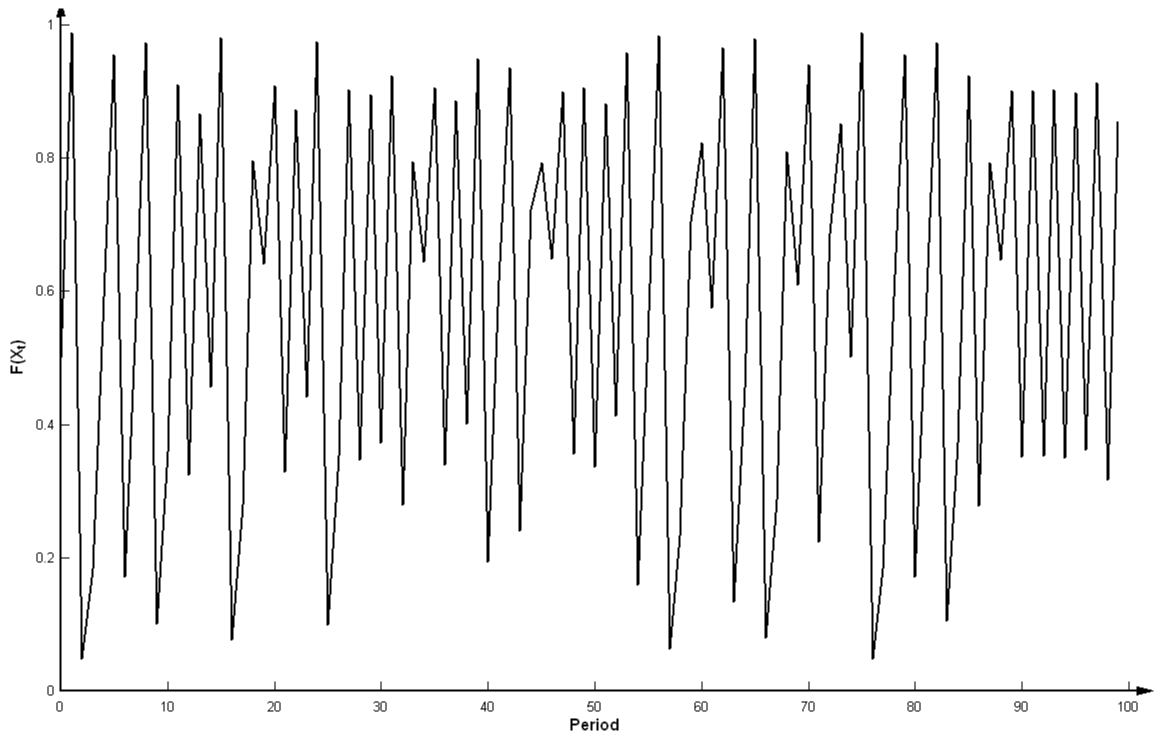
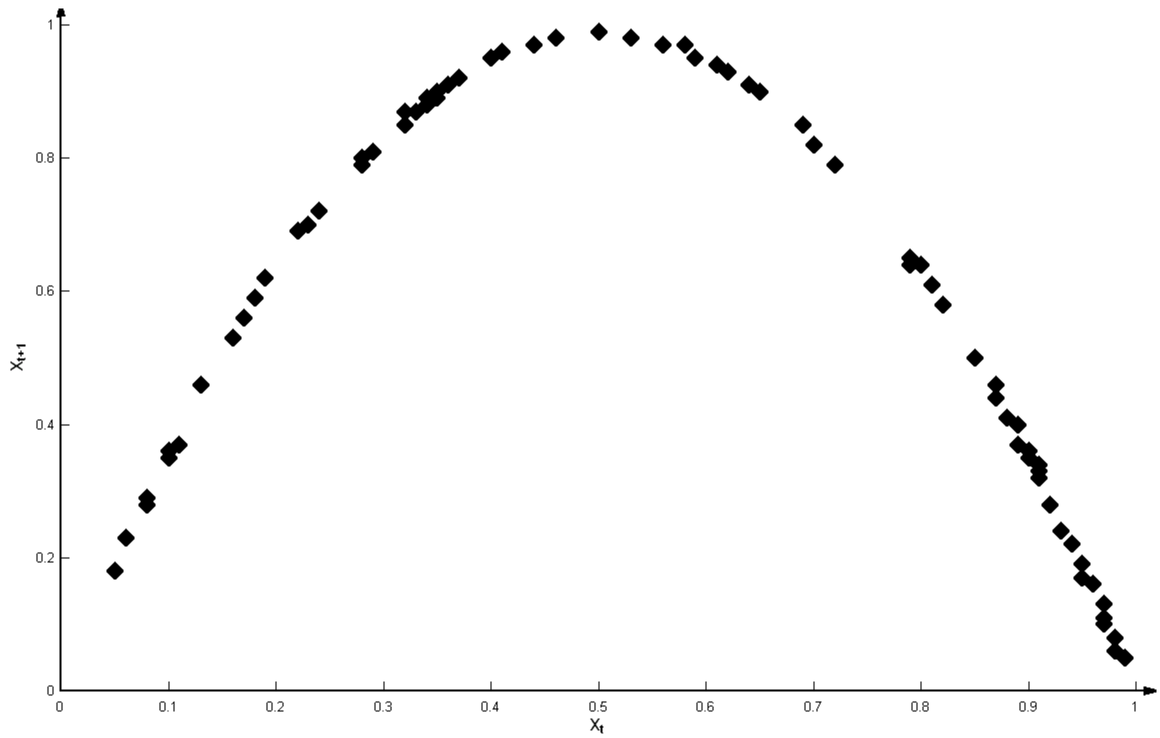


Figure 13 appears to have behavior that is random. Visual inspection of the plot (Figure 13) shows no obvious correlation. It would be easy to come to the conclusion that this plot is random. Discerning the difference between the random processess and the deterministic one is difficult because they share similar properties such as constancy of mean (0.56) and variance (0.09) throughout either system. In the case of this specific function, we know that it is not random and we can verify this by looking at the attractor plot when the alpha is 3.95 (Figure 14).

Figure 14 - Attractor Plot of Logistic Function α=3.95



The attractor plot of the above deterministic logistic function does not drive to a particular value or values. Instead the equilibrium in the system is the parabola shown. All values on the parabola are equilibria. In a chaotic system, the equilibrium becomes the locus of points described by the parabola. The behavior of a deterministic system can sometimes change from one equilibrium to many, simply by changing the alpha coefficient. For example, in Figure 14 changing the alpha from 2.8 to 3.95 would cause the system to become chaotic. Once a system is chaotic it is very sensitive to changes. This sensitivity is what causes some systems to exhibit very volatile behavior, which happens to mimic a random system.

To find out whether or not a system is deterministic or random, we must determine how much "memory" (how past events affect future events) a dynamic system has. We will do this by estimating the Hurst Exponent that was presented earlier.

Dynamic Systems and Long Memory Processes

Dynamic systems, whether they be random or deterministic, exhibit to a certain degree a behavior called “long run dependency” or “Long Memory Processes” (Granger 1966). In long run dependency events far back in time affect the evolution of the series through its state space today. Because of long memory processes, any changes in a dynamic system can affect the evolution of the trajectory through state space for very long periods, sometimes for years to come.

Common measures of this phenomena are measured linearly, through the autocovariance and autocorrelation functions, and dimensionally through the use of the Hurst Exponent. (Hurst,1951)

Many dynamical systems do not posess constant variance and stationarity. The autocovariance function implies that the covariance between two time segments of an object’s state space, the covariance, only arises as a function of the absolute distance between the two points in time. The autocorrelation function is similar, as it is the autocovariance function normalized by the variance. This results in the autocorrelation coefficient. Due to their assumptions of constant variance and stationarity, autocovariance and autocorrelation functions are not the best methods to use for testing dynamic systems. This does not mean that the measures of autocovariance and autocorrelation are of no consequence. They can help as a basic start to analyzing a dynamic system.

To get a better idea of the degree of long run dependency in a system a better measure is needed and comes in the form of the Hurst Exponent. The Hurst Exponent not only looks at autocovariance and autocorrelation, but also how much the past influences the future.

The Hurst Exponent was originally developed by Harold Hurst in 1951 for use in hydrology to determine optimal dam sizing for the Nile river. Hurst wanted to know how much a previous years rainfall affected the height of the Nile river. The measure he developed gave him insight into how long a rainfall would cause an increase in the height of the Nile. The Hurst Exponent is a measurement that is non-deterministic in nature and measures what is observed. Currently, there are five methods for estimation of the Hurst Exponent (H). In no particular order they are: re-scaled range, autocorrelation, absolute moment method, aggregated variance method and periodogram method. The original method developed by Hurst was the re-scaled range method.

We will begin the explanation of tests for the Hurst Exponent with the original re scaled range test.

Re-Scaled Range Test

In the re-scaled range test the Hurst Exponent is related to dimensional space (D) of the system by the equation:

$$D = 1 - H$$

The Hurst Exponent itself is bounded from 0 to 1. The scale between 0 and 1 describes the behavior of the series. For example $H = \frac{1}{2}$ Random Brownian Motion $H > \frac{1}{2}$ indicates a persistent effect of previous data on current data. For example if we have a high data point it is likely to be followed by a high data point again. $H < \frac{1}{2}$ indicates anti-persistent behavior, meaning a high value is likely to be followed by a low value. H = 0 indicates some other type of noise such as pink or white noise.

One way the Hurst Exponent can be estimated is through the use of a re scaled range analysis. To perform this type of analysis one starts with the amount of data you have. For example let us assume we have 100 observations x(1), x(2), ..., x(100). We first start by removing any trend by subtracting the mean (m) from each observation and develop the series x'(1), x'(2), ...x'(100) where x'(t) = x(t) – m.

Next, a set of partial sums are formed where x''(1) = x'(1), x''(2) = x'(1) + x'(2) etc. until x''(n) = x'(1) + x'(2) + ... + x'(n). Since this series is a sum of a mean-zero variable, the series will be positive if the majority of variables is positive x'(n) and vice versa if negative. Next, the range R is defined as R = max x'' - min x''. Finally, the range is scaled by the standard deviation (s) of the series to get the re-scaled range (RR) or RR = R / s.

Feller (1951) has proven that if the re-scaled range is independent (no serial correlation) and has finite variance it follows that $RR = kn^{\frac{1}{2}}$ where k is a constant and n is the number of observations. To test this, a regression is run in the form of log(RR) = a + b log(n) over many ranges of the observations where "a" is a constant and "b" is the slope parameter that should correspond to $\frac{1}{2}$. Hurst found that this did not hold and that $RR = kn^H$ where H is the Hurst Exponent. Tested by running the regression log(RR) = a + H log(n) over various ranges, this method can be tedious to perform. For example, if you had 1024 observations you would need to run this analysis over the entire range and then again for the first and last 512 observations. Then again on all four 256 observations sets and so on verifying that the Hurst Exponent was the same over all. This makes the re scaled range method computationally cumbersome which limits its use. Another way to calculate the Hurst Exponent is through the fractal dimension by estimating D using FARIMA, although there is some debate over the correct value of D to use.

The use of FARIMA in estimating the Hurst Exponent from a re-scaled range perspective comes from the relationship with the dimensional space given earlier. FARIMA allows the "d" parameter in the FARIMA(p,d,q) model to be estimated. We will see an example in the case study section of this paper. The next Hurst Exponent estimation method to discuss is the Autocorrelation method.

The Autocorrelation Method for Hurst Estimation

To estimate a Hurst Exponent using the Autocorrelation Method, one needs to calculate a sufficient number of lags to perform the analysis. In the case of this analysis, the Hurst Exponent is related to the Autocorrelation Function (ACF) via the slope coefficient of the estimate of the log of the ACF versus the log of the frequency. To perform this test one should calculate the ACF of a series until the ACF is negative and use all of the positive values as a data series. A regression run on the natural log of the ACF values versus the natural log of the lags of the ACF values is used to estimate the Hurst Exponent. The Hurst Exponent is related to the slope coefficient via:

$$H = 1 + \frac{\alpha}{2}$$

Where α = slope of regression. Here again it is important to have a sufficient amount of points. However unlike the re scaled range method, the range of the data does not have to be a power of 2. The ACF method is easier to calculate than the re scaled range method.

Absolute Moment Method for Hurst Estimation

To estimate the Hurst Exponent with the absolute moment method, one starts estimation by dividing a series of length n into shorter segments of length m and then averaging the series over each m length segment.

$$X^m(k) := \frac{1}{m}\sum\nolimits_{i=(k-1)m+1}^{km} X_i, k = 1,2,\ldots,\frac{n}{m}$$

To get the absolute moment (AM) of the series:

$$AM_n^m = \frac{1}{N/m}\sum\nolimits_{k=1}^{N/m} |X^m(k) - \bar{X}|^n$$

This method is generally used for n=1. If n=2 or larger it reduces to the aggregated variance method. In a log/log plot of the absolute moments versus m the slope(α) of the linear fit is related to the Hurst Exponent as:

$$\alpha = n(H-1) \text{ or } \alpha = H-1 \text{ if n=1}$$

$$\text{so } H = \alpha + 1$$

If there is no long run dependence, then the Hurst Exponent again will be 0.5.

<u>Aggregated Variance Method for Hurst Estimation</u>

To estimate the Hurst Exponent with the aggregated variance method one starts as in the absolute moment method, by dividing a series of length n into shorter segments of length m and then averaging the series over each m length segment.

$$X^m(k) := \frac{1}{m}\sum_{i=(k-1)m+1}^{km} X_i, k = 1,2,\ldots,\frac{n}{m}$$

Then the sample variance is calculated for each m length segment and the log of the variance is plotted against the log of m, as done in previous methods. Once again, the slope (α) of the linear regression of the log/log plot is related to the Hurst Exponent.

$$H = \frac{\alpha}{2} + 1$$

As before, if H = 0.5 then the series has no long range dependence.

<u>Periodogram Method for Hurst Estimation</u>

To estimate the Hurst Exponent with the Periodogram method, one estimates the slope of the log of the Periodogram (I) versus the log of the Frequency over the entire domain from 0 to π. The Hurst Exponent is:

$$H = \frac{1 - \alpha}{2}$$

Where α is the slope of the regression. To perform the periodogram analysis one needs to use the Fourier equation (Wei 2006) to estimate the Fourier coefficients a_k and b_k:

$$Z_t = \sum_{k=0}^{\frac{n}{2}} (a_k \cos(w_k t) + b_k \sin(w_k t))$$

where:

Z_t=Series

$w_t = \frac{2\pi k}{n}$=Frequency

$$a_k = \frac{1}{n}\sum_{t=1}^{n} Z_t \cos(w_k t), k = 0 \; and \; k = \frac{n}{2} \, if \; n \; is \; even$$

$$a_k = \frac{2}{n}\sum_{t=1}^{n} Z_t \cos(w_k t)\,, k = 1,2,\ldots,\frac{n-1}{2} \, if\ n\ is\ odd$$

and

$$b_k = \frac{2}{n}\sum_{t=1}^{n} Z_t \sin\,(w_k t), \qquad k = 1,2,\ldots,\frac{n-1}{2}$$

To calculate the periodogram the Fourier coefficients are used to calculate the periodogram (I) where (Wei 2006):

$$I(w_k) = na_0^2, k = 0$$

$$I(w_k) = \frac{n}{2}(a_k^2 + b_k^2), k = 1,2,\ldots,\frac{n-1}{2}$$

$$I(w_k) = na_{\frac{n}{2}}^2, k = \frac{n}{2} when\ n\ is\ even$$

Once the periodogram coefficients are calculated, the regression of log(I) versus log(Frequency) is used to estimate the Hurst Exponent. The estimate of the slope of the regression is used in the calculation of the Hurst Exponent.

Regardless of the estimation method that is applied to estimate the Hurst Exponent, the Hurst Exponent measures the correlation of data over time. The Hurst Exponent is helpful in characterizing the dynamic system. If H = 0.5 then there is no memory in the system and the system is completley random (RBM). If H is not 0.5 then the system may be deterministic (FBM) or some combination of the two. Keeping the concept of "memory" of a system in mind let us look at the problems of the traditional time series approach to classifying a system.

Confusion in Testing for Types of Dynamic Systems

Previously, it was demonstrated how the Hurst Exponent is helpful in characterizing a dynamic system. However, with our current methods of time series classification, we can have difficulty in determining an appropriate model specification. For example, let us visually inspect two graphs A and B (Figures 15 and 16) containing different simulated time series data each over 100 periods and try to determine which series has a random component and which one does not.

Figure 15 - Graph A

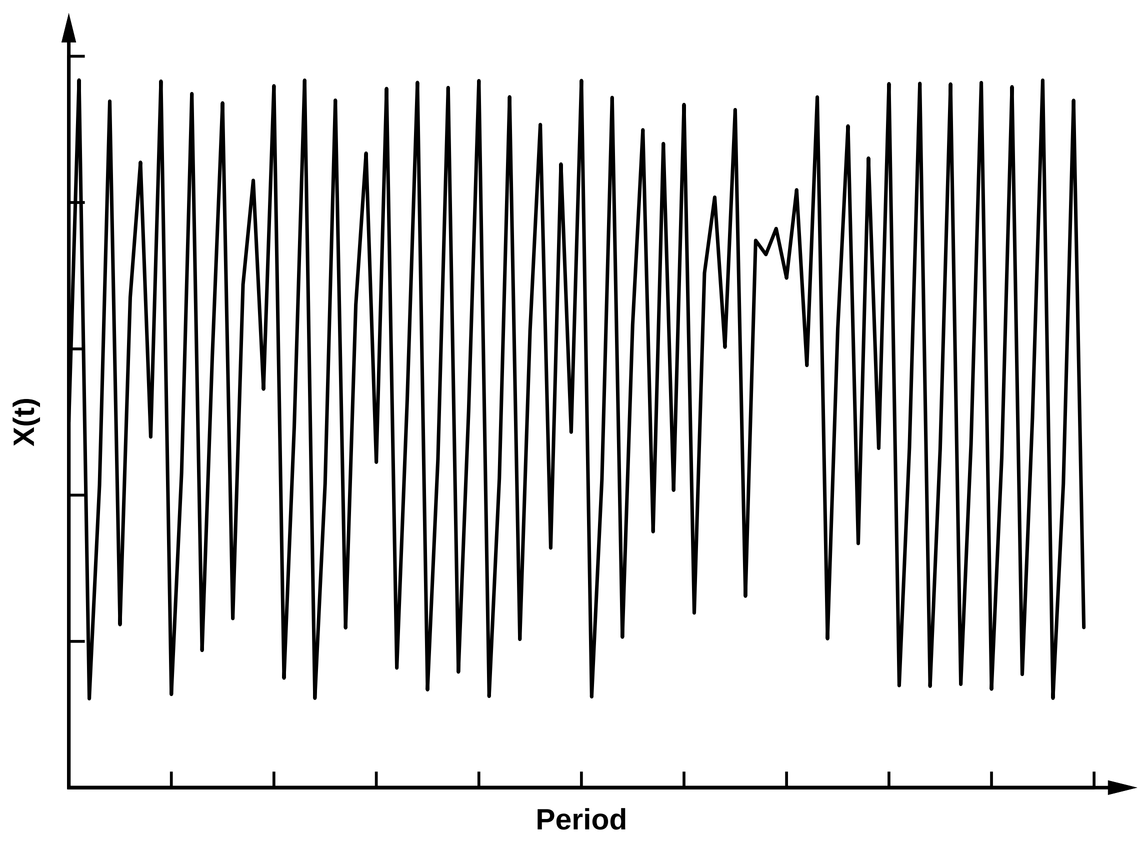
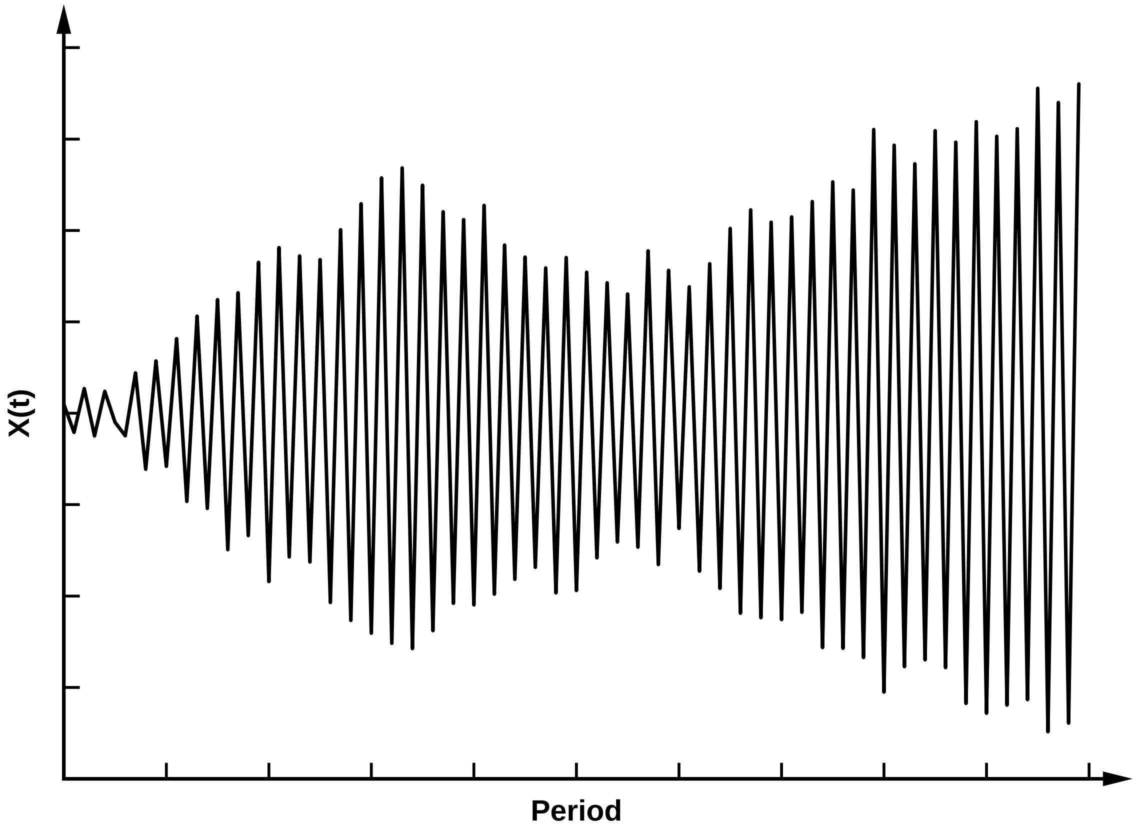


Figure 16 - Graph B



If you think the graph of the data in Figure 16 is not random than you would be incorrect. The series in Figure 16 does contain a random component whereas the data in Figure 15 is not random at all. This is an interesting problem for economic analysis. Just so we are working from the same information, let us look at the functions that produced both data sets.

Graph A (Figure 15): $X_{t+1} = \alpha X_t(1 - X_t)$

Graph B (Figure 16): $X_{t+1} = -X_t + \varepsilon$

$$where:$$

$$\alpha = sensitivity$$

$$\varepsilon = random\ normal\ error\ term$$

As will be discussed in a moment, the first equation is a logistic and is completely deterministic. Whereas the second function (Figure 16), is a standard autoregressive function with a random process.

To detect whether or not a series is random, it is conventional to begin with a traditional time series testing method such as the autocorrelation function (ACF). Recall that the ACF is the autocovariance between to time steps, divided by the variance. In Figures 17 an 18 below we see the ACF plots for both functions.

Figure 17 - ACF Plot of Graph A Data

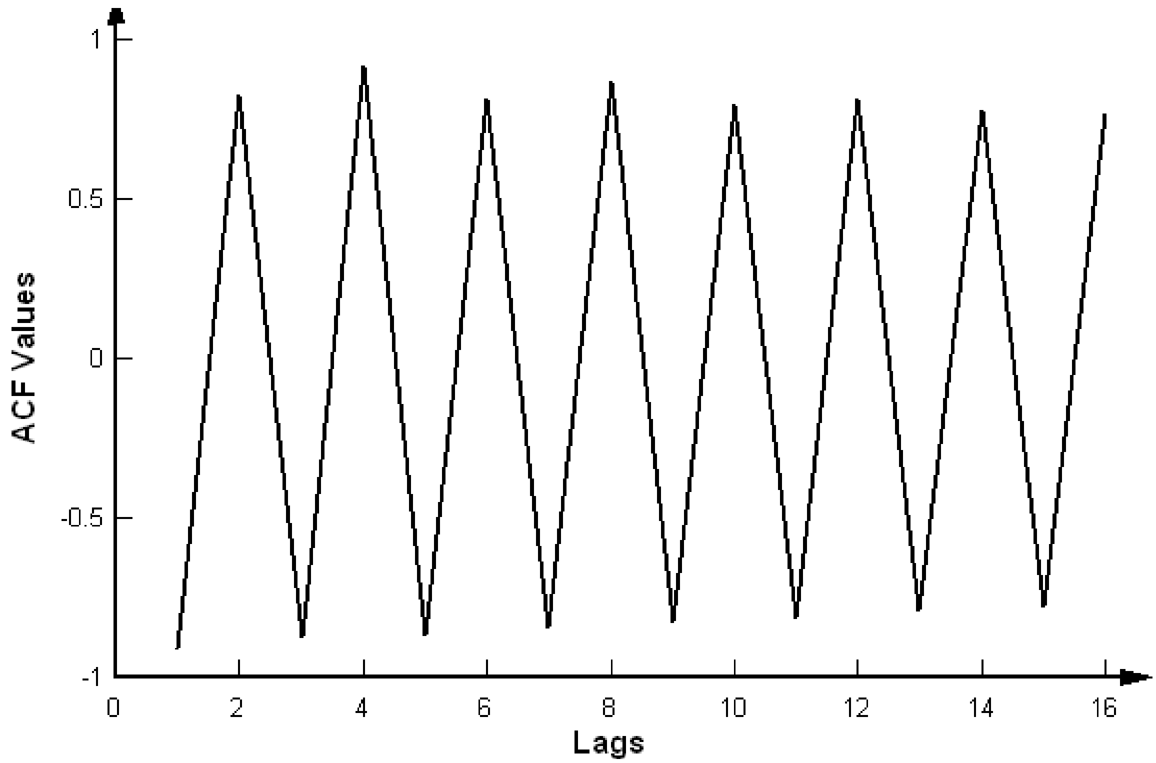
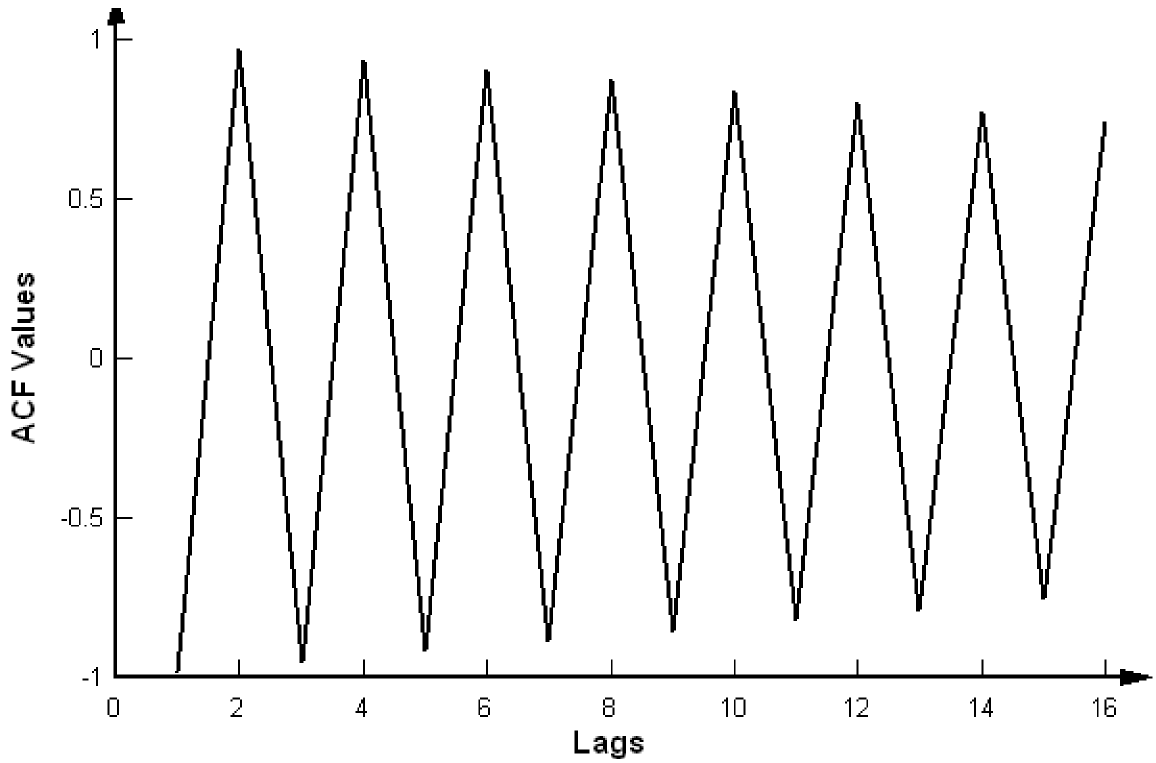


Figure 18 - ACF Plot of Graph B Data



Notice that in both cases, each equation shows some cycling behavior and both have similar magnitudes of ACF values. In a standard time series approach, we would be tempted to use an autoregressive (AR) model. Indeed if we did, we would find that the data for Figure 18 could be reduced nicely using an AR(1) model and that the ACF of the residuals would be stationary. However the data for Figure 17 would not reduce and would need additional components. Using a traditional ARMA approach an ARMA(1,1) model would fit the data in Figure 17 well and the residuals would be considered stationary. Note that in both cases, there would not be a unit root problem as both final models would be outside the unit circle.

At this point in time, the reader may question why not use ARMA or ARIMA models if we can get a reasonable forecast? The answer: in the case of data from Figure 17 the sensitivity of the coefficient α is important to describing how the system behaves. When using a linearly additive estimation technique such as ARMA to estimate, the sensitivity of the coefficient looses information due to the linear measurement. This causes a misidentification of the functional form of the system.

So let us now start our discussion of what a dynamic system is and how it works, by taking a step back and defining a dynamic system more generally and more precisely and how it applies to economics.

Identifying a Dynamic System as Deterministic

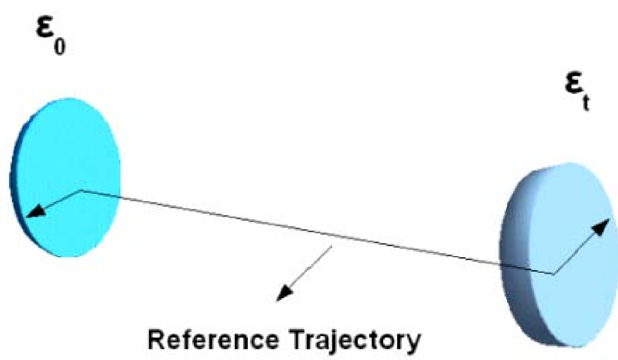
Although it can be rather difficult to detect, there are ways to test a system for deterministic and chaotic behavior. The most common test is that of the Lyapunov Exponent. The Lyapunov Exponent is defined as:

$$\lambda = \frac{1}{t}\ln(\frac{\varepsilon_t}{\varepsilon_0})$$

Where ε at any time step is defined as the difference between the actual series and the reference trajectory. The reference trajectory is generated numerically by creating a series that is close to the actural series. In estimation of the Lyapunov Exponent the distance (seperation) of the reference trajectory is continually increased. Lyapunov Exponents are calculated over a range of different trajectories to evaluate a series of data.

To perform this test, one needs to measure the divergence of the trajectory away from a reference trajectory at various time steps. If the system is deterministic or chaotic that implies a predetermined path will exist, so the actual trajectory and the reference trajectory would be close to one another. As we can see in Figure 19 below, we have data that falls within a point on the hyper sphere at ε_0 and at another point in time t. The data should be dimensionally close, if the series is deterministic.

Figure 19 - Reference Trajectory



If a series is chaotic, then it is by definition deterministic. Therefore the actual series, must lie in the same dimension as the reference trajectory, if the Lyapunov Exponent is positive. One note when testing for chaos, is that it may not exist on all n-dimensions. As such, testing amongst various dimensions is necessary, since not all dimensions form a contact manifold (intersection) with one another.

The Lyapunov Exponent characterizes the rate at which close trajectories separate. Because rates of separation can differ depending on an objects orientation, there are many Lyapunov Exponents. The number of Lyapunov Exponents depends on how the object may separate. Thus it is common to look for the largest Lyapunov Exponent, as this is the maximum amount of divergence.

Again if a series is random the state space will have infinite dimension. Also, the Lyapunov Exponent can give us an idea of the behavior of the series. For example, if a system is conservative, that is to say does not dissipate, then the sum of all the Lyapunov Exponents will equal zero. If a system dissipates, then the sum of the Lyapunov Exponents is negative. The sum is positive if the series gains momentum.

Using the Lyapunov Exponent helps us to learn if a dynamic system has deterministic behavior. Testing the Lyapunov Exponent against multiple dimensions is necessary. Using the largets Lyapunov Exponent produces the largest amount of divergence. The largest Lyapunov Exponent tells us how many dimensions the system has and therefore whether or not a series is deterministic (limited dimensions) or random (infinite dimension). Now let us look at differences between testing for linear and non-linear dynamic systems.

Testing Dynamical Systems

A dynamical system may or may not lie in a restricted dimensional space (some are deterministic and some are random) however, all move through space time. Since dynamical systems may or may not be linear, there is a different approach to testing for their existence than standard linear tests. In other words, different assumptions must be used since a dynamic system may or may not be linear. The table below shows the differences between the two major types of dynamical models, linear and non linear.

Table 1 - Linear vs. Non-Linear Systems

Linear	**Non-Linear**
Constant Mean	May or May Not Have Constant Mean
Invertible	Not Invertible
Variable is Independent and Identically Distributed	Variable is Not Independent and Identically Distributed
Series Has Infinite Dimension	Series Lies in Restricted Space
Series is Additively Separable	Series is Not Additively Separable

To test whether a system is dynamical, and to what degree, one needs to investigate the state space as well as the long run dependence with the methodologies previously discussed (Hurst Exponent, Lyapunov Exponent, ACF). Let us not forget that seeing oscillations in a series, may not mean the system is non linear. Conversley seeing a pattern that is flat, does not mean a system is linear. Again, to see this notion you can refer to the logistic function discussed in Figures 3, 5 and 13.

When using dynamic systems we have to be careful with more traditional techniques. When a series is differenced to make it stationary using standard time series techniques, we have demonstrated that the measurmentof sensitivity is lost. Also, we lose the signal that the process is generating naturally as it travels through space-time.

Dynamical systems are not simply identified by an observation at a given point in time. The dimensions that we view observationally are only one part of the objects cause of trajectory at a given point. As we will see when testing dynamical systems, there are other dimensions at work “behind the scenes” that produce the observation. Recall when we looked at the supply and demand model over time, we were able to view price changes over time. But the dimension we lost was quantity.

Since we do not see all dimensions in a dynamic system, it is important to estimate them through the use of manifolds. A manifold is an abstract space in which every point has a neighborhood that resembles coordinate space. The dimension is the minimum number of coordinates needed to specify every point within the manifold. Thus dimensional space is important in understanding dynamical systems. A line, or a circle has a manifold of one and a plane would have a manifold of two. So in defining a dynamical system, there needs to be enough manifolds used so that we get a picture that resembles coordinate space. The number of dimensions used is analogous to creation of a space that houses all points of that effect, such as mass or acceleration. The use of manifolds and dimensional space, describes how to define the general functional form of a dynamic equation. Using this information helps to determine if a dynamic system is linear or not, which is important in assigning a correct functional form to economic data.

A General Functional Form for Dynamic Systems

In general, a manifold is defined such that the dimensions within it, define the behavior of the series. For example, given any one point we define the point by its dimensional coordinates.

$$P_1 := (P_{1x}, P_{1y}, P_{1z})$$

There may be more dimensions than the example lists. Each one of these dimensional coordinates contains a vector of possibilities. In a general form more familiar to economists, we would use a Hamiltonian to define these dimensions. For example a common motion problem would be defined as:

$$\mathrm{Z} := (p, q, t)$$

Where p is momentum, q is the generalized coordinates and t is time. In this case the value of the series at any instant is:

$$\dot{p} = \frac{-\partial \mathrm{Z}}{\partial q}$$

$$\dot{q} = \frac{\partial \mathrm{Z}}{\partial p}$$

Where p = momentum vector and q = generalized coordinates vector. In this case, the vector of time goes away. In terms of dimensions, we find an odd number because time is always its own dimension. However, what is usually lacking in dynamical models are the other dimensions of behavior. For example randomness, mass (stock), acceleration (extraction), historical dependence, etc. So a general form of a dynamical economic system would look as follows:

$$p_t := (m_t, a_t, r_t, d_t, dp_t)$$

where:

$$P_t = observtion, m_t = mass, a_t = acceleration, r_t = random,\ d_t = deterministic, dp_t = dependence\ (memory)$$

The Hamiltonian in this case would be structured as before but p is instead given in the defining equality above.

$$Z := Z(p, q, t)$$

Of course by redefining the Hamiltonian in this way, it makes integration difficult. That is why manifolds need to be used on each variable.

We have described the behavior of dynamic systems and their various catagories (deterministic and random), as well as derived tests for determing if a dynamic system has memory or is deterministic. We have described the difference in assumptions between linear and non-linear dynamic systems and what a general functional form of a dynamic system looks like. Using the methodology presented so far we will separate the deterministic and random components of economic variables which will allow us to characterize a dynamic economic system. We can then study the results of the characterization and the impact the components of a dynamic system has on economic variables.

In the following case study we will see that the gold industry is an example of a dynamic system with both random and deterministic components. Using the Hurst Exponent, Lyapunov Exponent and autocorrelation tests, we will be able to separate the deterministic from the random. This information will allow us to characterize the behavior of gold prices based on intra and inter market events. An equation will be formulated based on the dynamic behavior of gold prices. The characteristic equation of gold prices will allow for simulation of market events and the construction of a supply and demand curve for the current US gold market.

CHAPTER 3 – THE CASE OF GOLD

There are many examples of dynamic systems in economics. The changes in prices of commodities are examples of dynamic systems that interest economists. Of particular interest to many individuals, is the change in the price of gold over time. Consumers and economists have an interest in gold for many reasons: gold's history of regulation, various uses (products, investment) and recent volatile history. Characterizing the dynamic system of gold prices will allow for the measurement of how market structure changes affect the price of gold. We will find in our case study that industry concentration causes the price of gold to become less volatile. Furthermore characterizing an appropriate function for the evolution of gold prices over time, will allow the reconstruction of supply and demand curves for the US gold industry.

To study the dynamic system of gold prices, we will start by testing for long run dependence and deterministic behavior. We will use all of the various methods for estimating long run dependence with the Hurst Exponent given in the previous chapter.

We will then test for deterministic behavior by estimating Lyapunov Exponents for gold price, again using the methods described previously, to find out how deterministic or random gold prices are.

Once we have determined the amount of long run dependence and the deterministic portion of the gold price, we will separate the two. Seperation of the deterministic portion of the gold price from the random portion will be conducted through the use of a space-time regression, utilizing the estimates of the Hurst and Lyapunov Exponents.

After the deterministic and random portions of the price of gold are separated we will study how sensitive the gold industry is to external and internal events. An equation for gold prices, based on deterministic and random components will be developed. This equation will then be used to simulate possible affects from future events. Using the dynamical equation for gold prices, we will see how external or internal effects on the gold industry can affect gold prices.

Finally, we will use the dynamic equation for gold prices to develop a supply and demand curve for US production and consumption of gold. Before we begin let us get an idea of the gold industry, past and present.

A Brief History of the Gold Industry

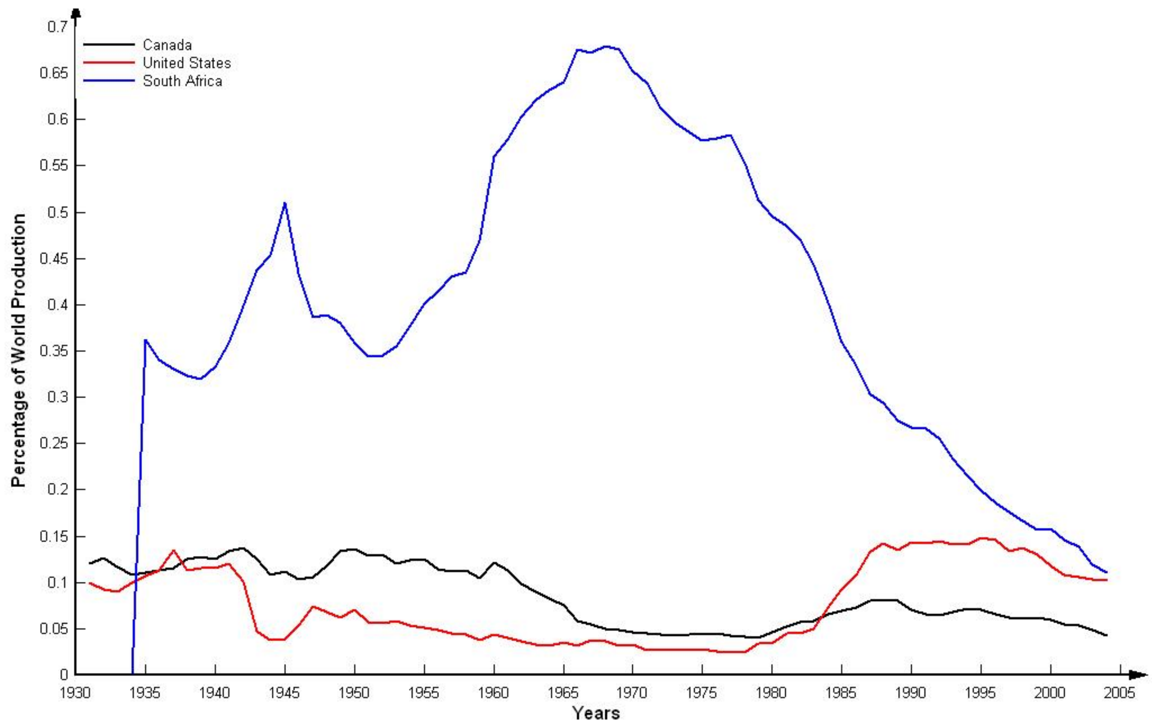
The history of gold is multifaceted. In ancient civilizations gold was used for jewelry and ceremonial purposes (NMA 2008). Gold began to be used instead of silver for coinage in many societies over the centuries (NMA 2008). In more recent times, throughout the last century, governments have used gold as a monetary standard and have controlled the price up until the ending of the gold standards in the 1970s (NMA 2008). After the deregulation of gold, the industry expanded with many new new mining firms. The expansion of the gold industry was short lived due to mergers and consolidation over the last decade or so. A peak in merger activity in 2001 resulted in 40.9 billion dollars worth of mergers in the gold industry (Ericsson 2001-02). From 1990 to 2001 the Herfindahl-Hirschman Index for the 10 largest firms increased from 395 to 457 or 62 points (Ericsson 2001-02). The control over the production of gold has declined since deregulation, to historic lows (Ericsson 1994). To get an unbiased historical perspective of the gold industry, gold production data by country was gathered from the "Minerals Yearbook" (USGS) from 1931 to 2006. The results by country and by decade are in Table 2.

Table 2 - Percentage of World Gold Production by Country (by percent)

	1930s	1940s	1950s	1960s	1970s	1980s	1990s	2000s
Canada	11.82	12.00	12.10	6.93	4.50	5.85	6.99	5.45
Mexico	2.41	1.79	1.07	0.42	0.40	0.50	0.73	1.12
United States	10.71	7.19	5.22	2.90	2.90	6.39	14.16	11.50
Bolivia	0.05	0.05	0.06	0.13	0.08	0.13	0.44	0.42
Brazil	0.60	0.64	0.49	0.26	0.51	4.07	3.02	1.79
Chile	0.64	0.73	0.37	0.11	0.25	1.08	1.71	1.68
Colombia	1.19	1.61	1.09	0.63	0.60	1.59	1.16	1.36
Ecuador	0.23	0.25	0.07	0.03	0.02	0.17	0.50	0.16
Peru	0.45	0.60	0.41	0.21	0.23	0.42	2.23	6.75
Venezuela	0.33	0.24	0.13	0.05	0.04	0.12	0.39	0.37
Finland	0.01	0.03	0.06	0.04	0.06	0.09	0.12	0.20
France	0.24	0.15	0.12	0.09	0.13	0.15	0.18	0.08
Sweden	0.56	0.39	0.25	0.18	0.16	0.22	0.26	0.19
India	1.11	0.66	0.57	0.26	0.24	0.14	0.10	0.15
Japan	1.94	0.67	0.61	0.48	0.43	0.35	0.38	0.33
Philippines	1.61	0.80	1.13	0.82	1.36	1.94	1.23	1.39
South Africa	18.65	40.81	39.28	49.81	61.81	43.59	23.67	14.20
Australia	2.07	3.10	2.98	1.75	1.45	4.38	11.58	10.75
New Zealand	0.26	0.44	0.12	0.03	0.02	0.07	0.41	0.38

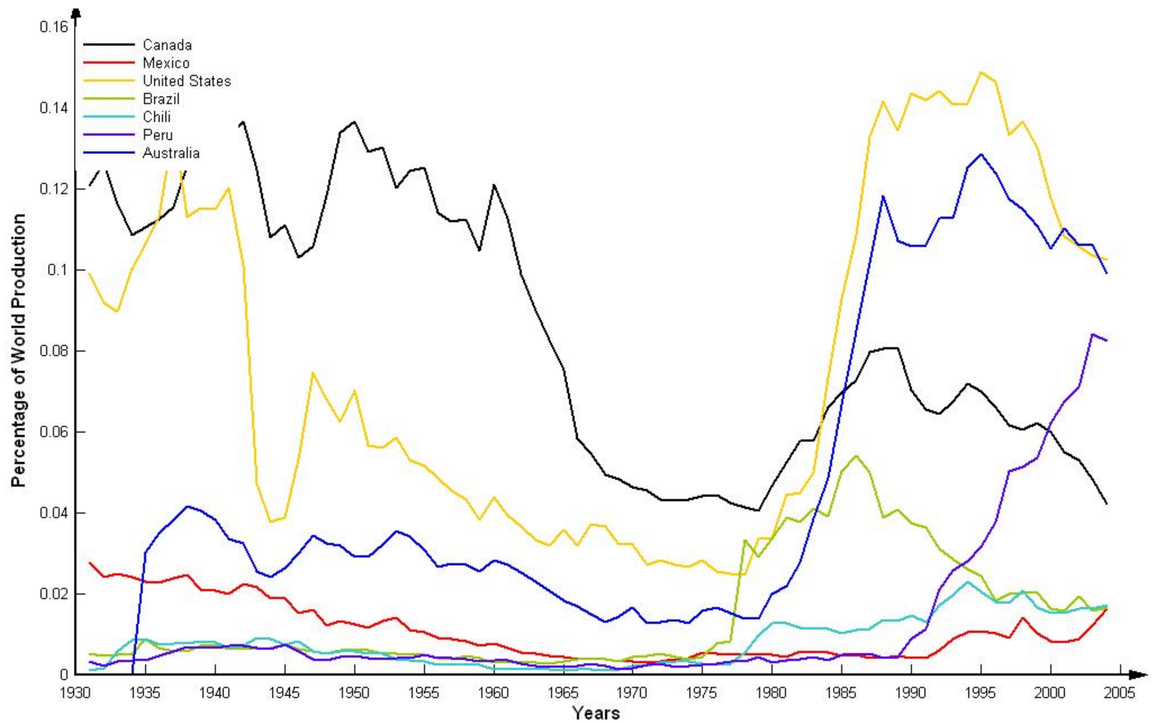
South Africa represented the largest portion of production in the market for gold over a number of decades. Figure 20 shows the comparison of three of the largest gold producing nations: South Africa, United States and Canada.

Figure 20 - Percentage of World Gold Production by Country (1931-2006)



Prior to industry deregulation, South Africa was the largest producer of gold. After deregulation, countries such as the United States and Australia increased production significantly and South Africa decreased production significantly. For a more detailed perspective of the rest of the gold producing countries South Africa was removed due to scaling issues (Figure 21).

Figure 21 - Percentage of World Gold Production by Country (1931-2006)



Currently no country represents more than 15% of the production in the entire industry. As such, market structure is vital to understanding the changes in the price of gold.

To understand the affect of market structure on the price of gold, we must first characterize the system of gold prices to find out if gold is deterministic, random or a combination of both. We will begin by estimating long run dependence in the price of gold.

Testing for Long Run Dependence in Gold Prices

For the purposes of characterizing the price of gold, nominal price data was collected. The data used are nominal monthly average gold prices from January 1968 to February of 2008 (Figure 22).

Figure 22 - Average Monthly Gold Price

As we have previously demonstrated, a visual inspection of the plot does not adequatley convey if gold prices are deterministic, random, have long run dependence, etc. Therefore, we will perform various tests of the Hurst Exponent to determine if long run dependence exists in the price of gold (see Appendix for computer code). Calculating the re-scaled range estimate for long run dependence, we found the following results (Figure 23).

The value of the Hurst Exponent estimated is equal to 0.9788 and suggests that a persistent long run dependency exists. To verify the validity of the estimate, we will continue with, the ACF Hurst Estimation technique (Figure 24).

Figure 23 – Re-Scaled Range Analysis of Monthly Gold Prices

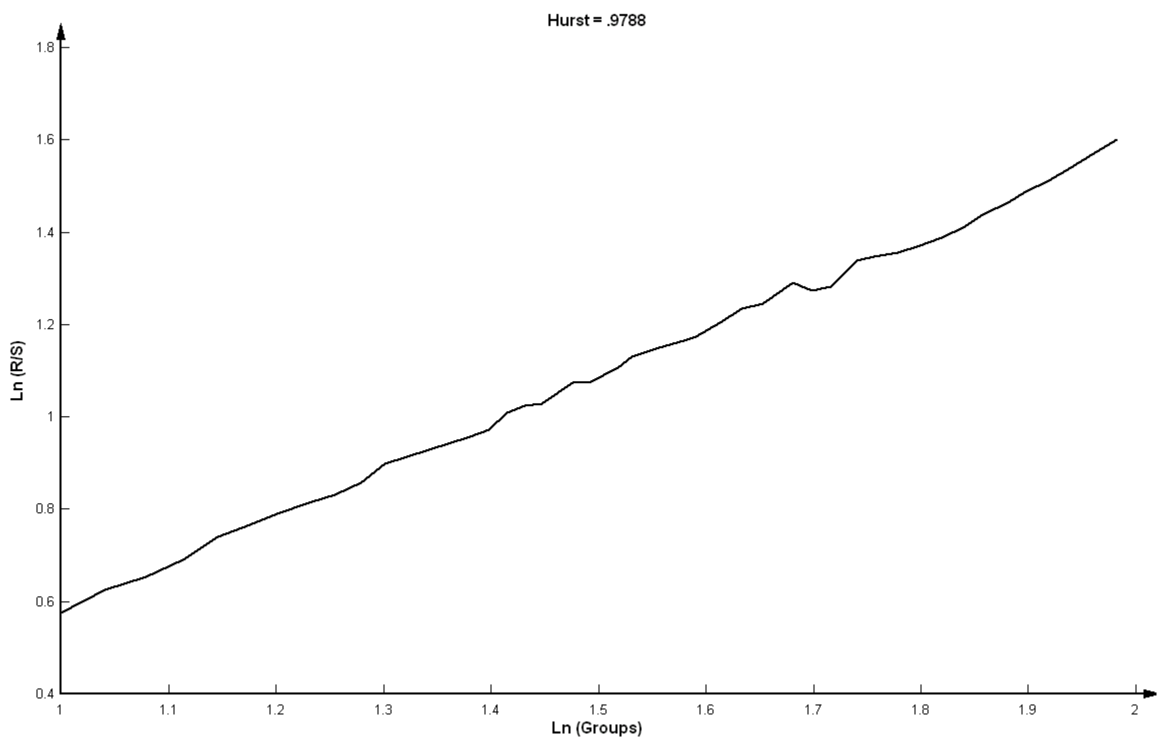
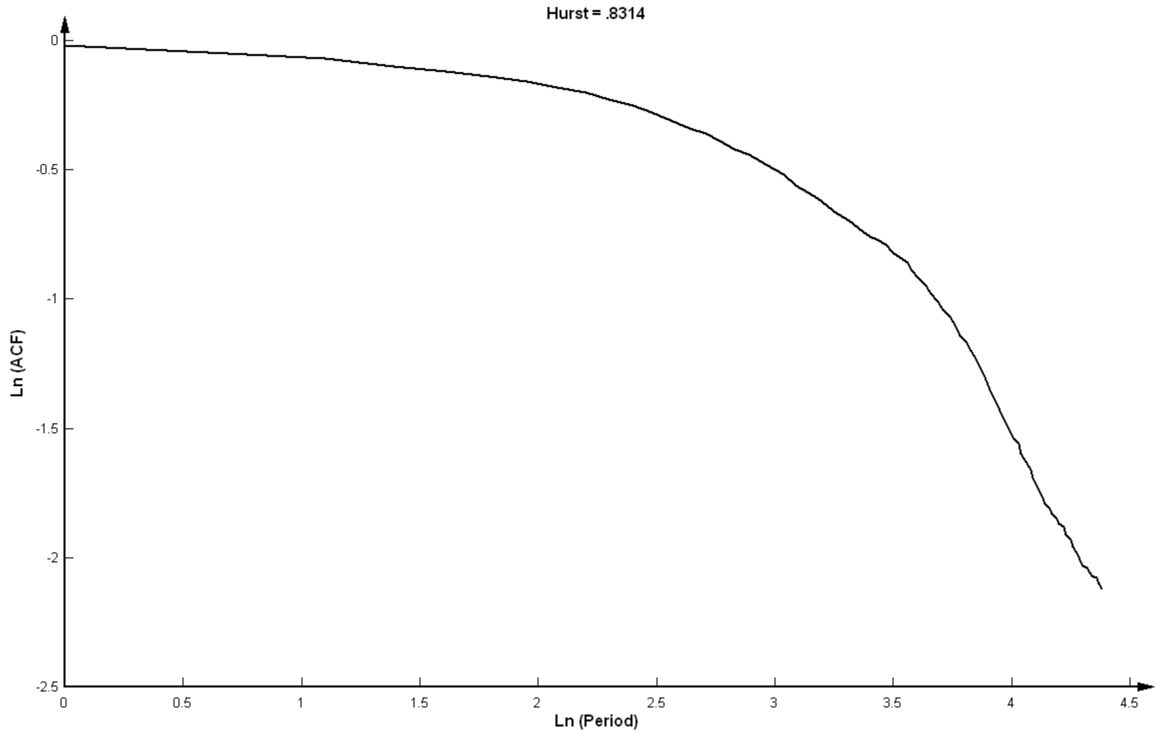
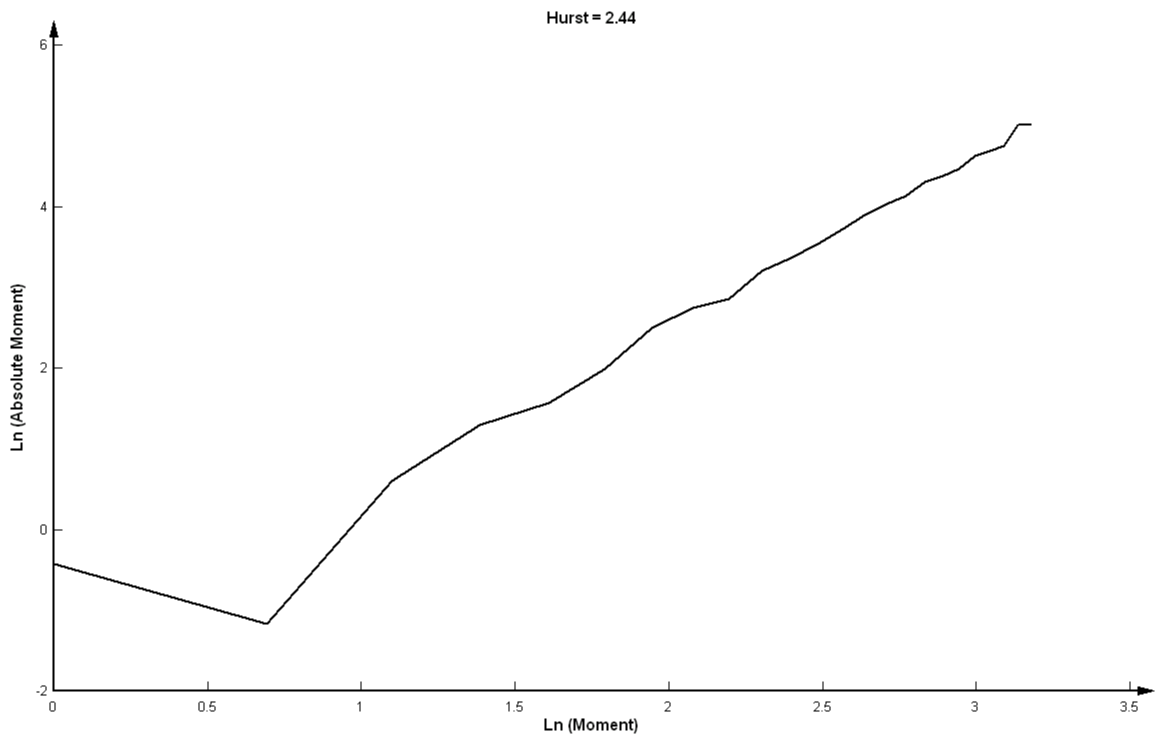


Figure 24 - ACF Analysis of Monthly Gold Prices



In the case of the ACF method H = 0.8314 which again suggests persistence in the series over time. Continuing our estimation, the Absolute Moments Method estimate is 2.44 (Figure 25).

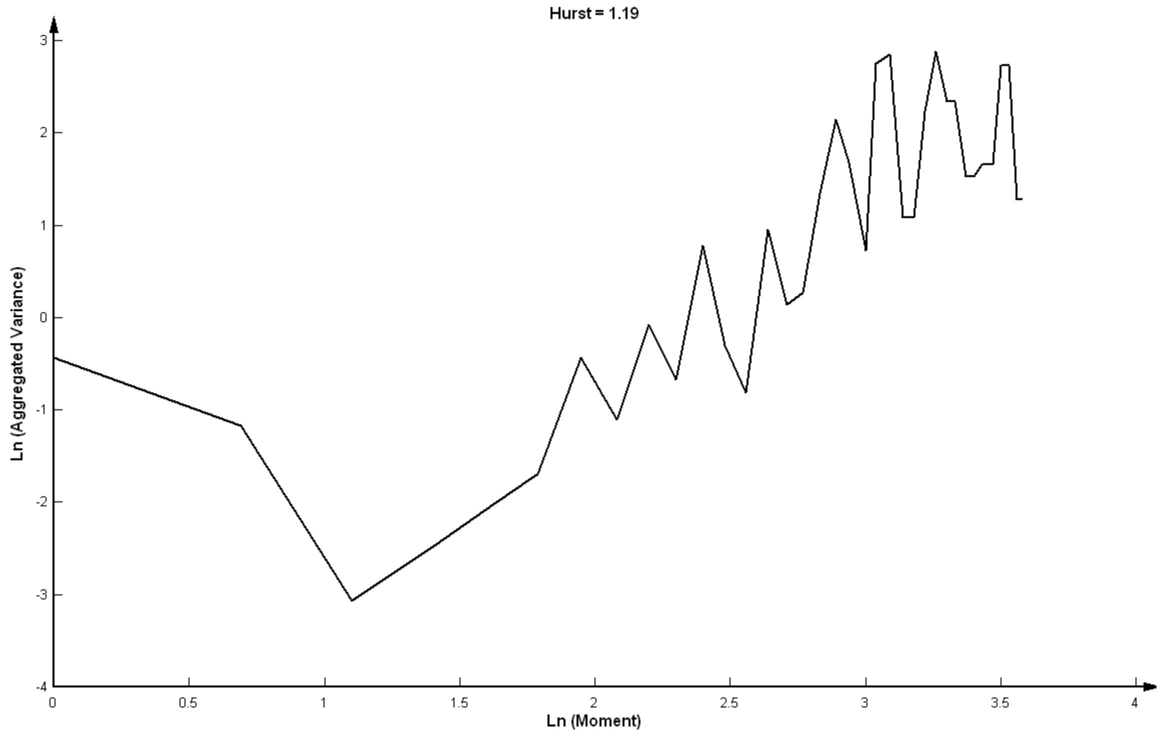
Figure 25 - AMM Analysis of Monthly Gold Prices



Since H = 2.44 this again suggests long run dependence in the data.

The Aggregated Variance Method (Figure 26) has an estimate of 1.19 for the Hurst Exponent.

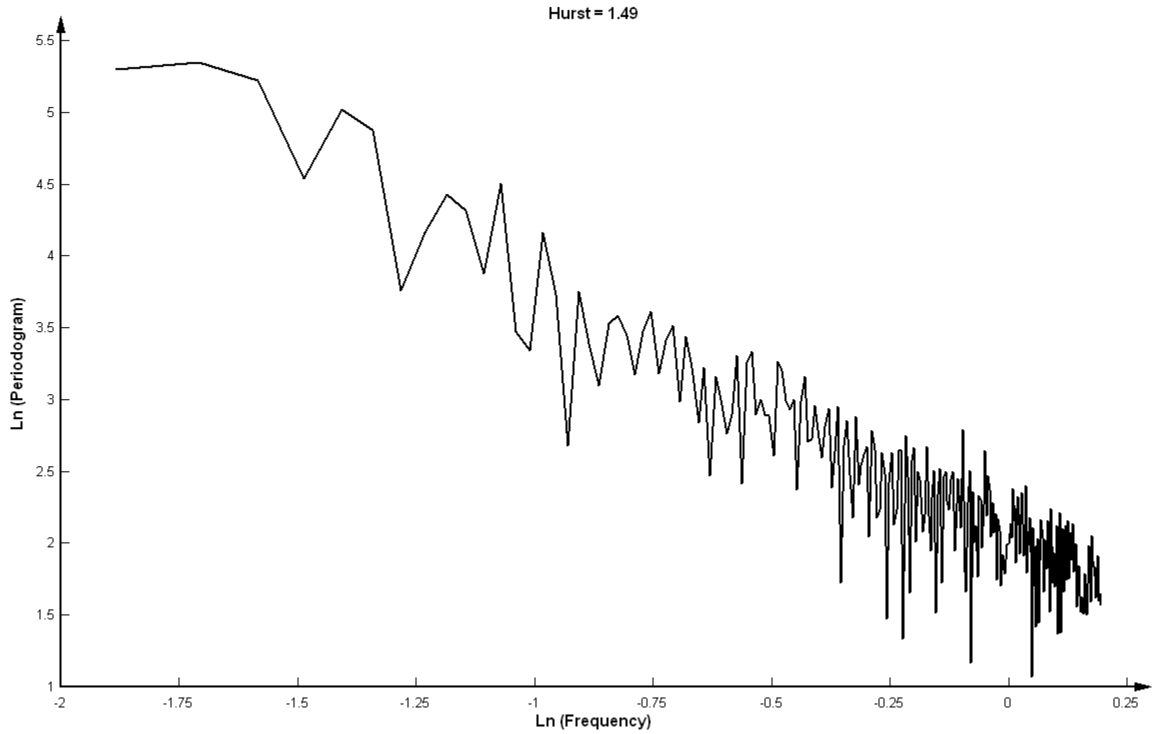
Figure 26 - AVM Analysis of Monthly Gold Prices



In this case H=1.19 again suggesting persistence.

With the Periodogram Method (Figure 27) we get a Hurst Estimate of 1.49, again suggesting persistence.

Figure 27 - Periodogram Analysis of Monthly Gold Prices



Finally, one more way to derive the Hurst exponent for gold prices is to use the FARIMA model. The differencing parameter, or exponent on the autoregressive and moving average terms, is allowed to vary in order to estimate the dependency on dimensional space and the amount of the dependency is given by the Hurst Exponent. The results of an FARIMA(1,d,1) model, that is to say, 1 Auto Regressive and 1 Moving Average component allowing the fractional differencing (d) component to be estimated to determine the level of dependency, Hurst Exponent is 1-d (Table 3).

Table 3 - Estimation of Hurst Exponent with FARIMA

Coefficients	Estimate	Std. Error	z Value Pr(>| z|)
d	4.583e-05	0.000	Inf<2e-16***
AR(1)	9.995e-01	0.000	Inf<2e-16***
MA(1)	-2.980e-01	7.075e-03	-42.12<2e-16***
Significance: 0'***' Log likelihood: -2115			

The Hurst Exponent is 1-d (1-4.583e-05) = 0.999, thus, the Hurst Exponent = 0.999. The fact that the Hurst Exponent is greater than ½ shows that the dependency on the past is rather large. The derived Hurst Exponent shows that there is dependence between prices over very long intervals of time in gold prices.

A quick review of the results of our Hurst Exponent tests (Table 4) shows the prominence of the result that some long memory process exists in the price of gold because all tests, and the average of all tests, are greater than 1/2. The variations in the results of the different Hurst Exponent tests are due to the different methods employed to estimate the long run dependence.

Table 4 - Summary of Hurst Exponents for Monthly Gold Price

RS Method	0.98
ACF Method	0.83
AMM Method	2.44
AVM Method	1.19
Periodogram Method	1.49
FARIMA Method	0.99
Average Hurst Exponent for All Methods	1.32

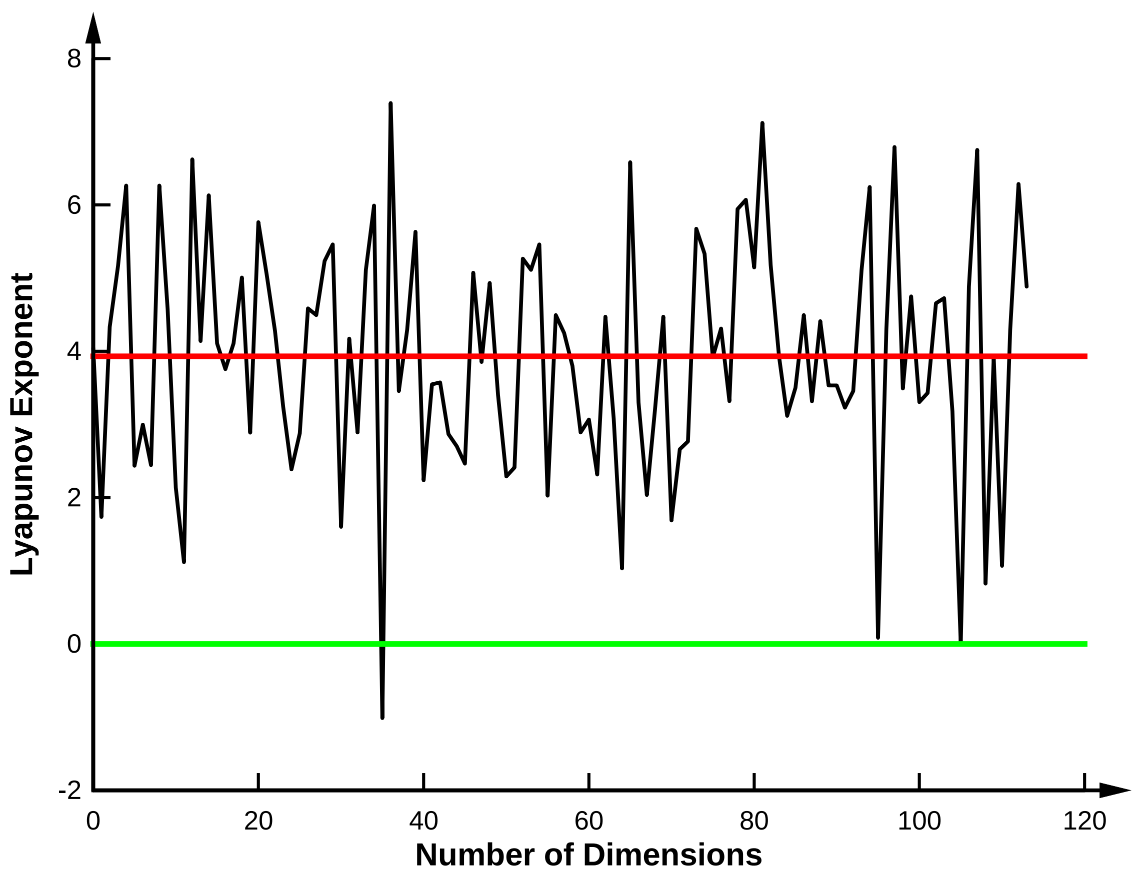
Due to the variety of estimating techniques for the Hurst Exponent results of those techniques will vary. In our case the AMM, AVM and Periodogram methods produced results that were outside of the theoretical limit of the Hurst Exponent. This is the result of difficulties in estimation of the Hurst Exponent. As of this moment there is no consensus as to which test is preferred given a particular set of conditions. This is why we performed all tests for the Hurst Exponent. All tests show a preponderance of of the Hurst Exponent having a value greater than 1/2. Testing for the Hurst Exponent has determined the existence of long run dependence in gold prices. We have shown that the price of gold has a large persistent memory over time. However, we need to classify this dependence. For instance, what is the functional form? To classify the dependence we must perform a check on whether the system is random, deterministic or some combination of both.

The Lyapunov Exponent will identify if the dynamic system of gold prices has a deterministic or random component. Once the component is identified we will then separate the deterministic portion of the data from the random portion.

Estimating the Lyapunov Exponent for Gold Prices

To test for the dimensional space, a Lyapunov Exponent was calculated as seen in Figure 28. The calculation determined that there are a finite number of dimensions, suggesting that part of the appropriate functional form, is multiplicative in nature. This is suggested because there is a portion of the system of gold prices that is not IID (Idependently Idenentically Distributed) and therefore not linear.

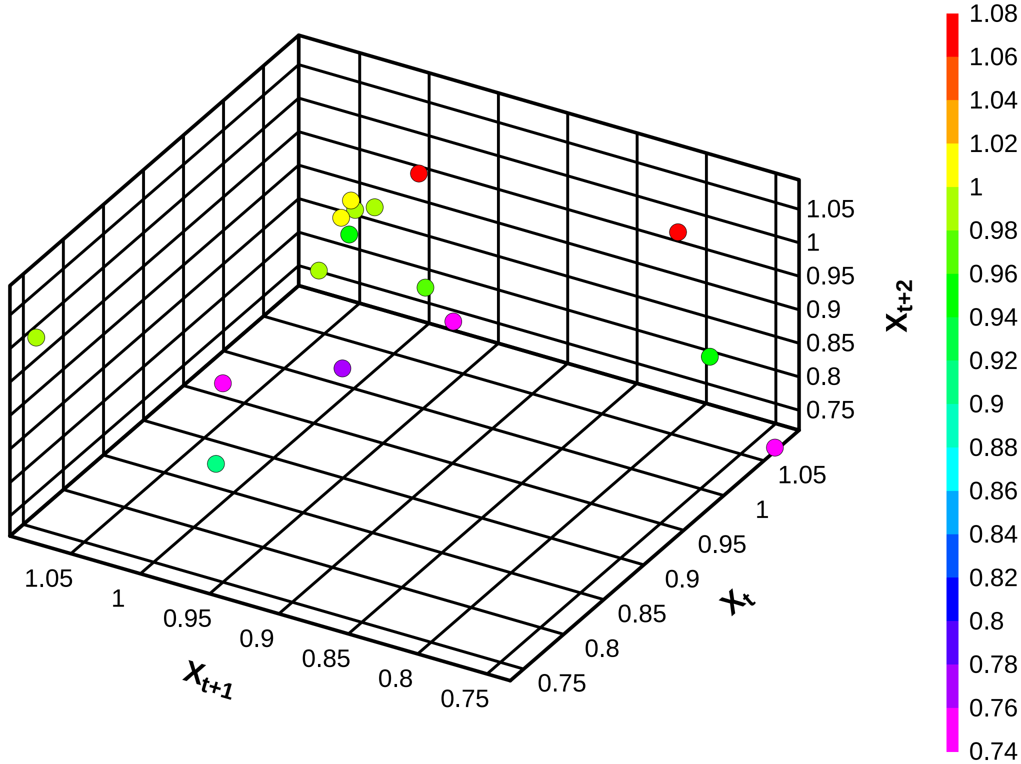
Figure 28 - Lyapunov Exponent Estimation of Monthly Gold Prices



The red line represents the average Lyapunov Exponent over the entire series of gold data, which has a value of 3.93. This demonstrates the dimensions are finite in number due to the Lyapunov Exponent being positive. As such, a portion of the system is deterministic.

To confirm the deterministic result found, a polynomial equation was used to create the reference trajectory. Then the Lyapunov Exponents were calculate by running regressions on ln(et) vs ln(e0) over different ranges of the data. The results were of lower magnitude, due to the polynomial reference trajectory being not as accurate as the programmed computer procedure. But the polynomial reference line still indicates some level of chaotic behavior. Figure 29 shows an attractor plot of the Lyapunov Exponents calculated with the polynomial methed. All Lyapunov Exponents are positive, indicationg deterministic behavior. Also the attractor plot shows that the Lyapunov Exponents are bounded to a close area of values.

Figure 29- Attractor Plot of Lyapunov Exponents



The range of Lyapunov Exponents using the polynomial method was:

Table 5 - Range of Lyapunov Exponents from Polynomial Method

Maximum Lyapunov Exponent	1.08
Minimum Lyapunov Exponent	0.52
Average Lyapunov Exponent	0.93

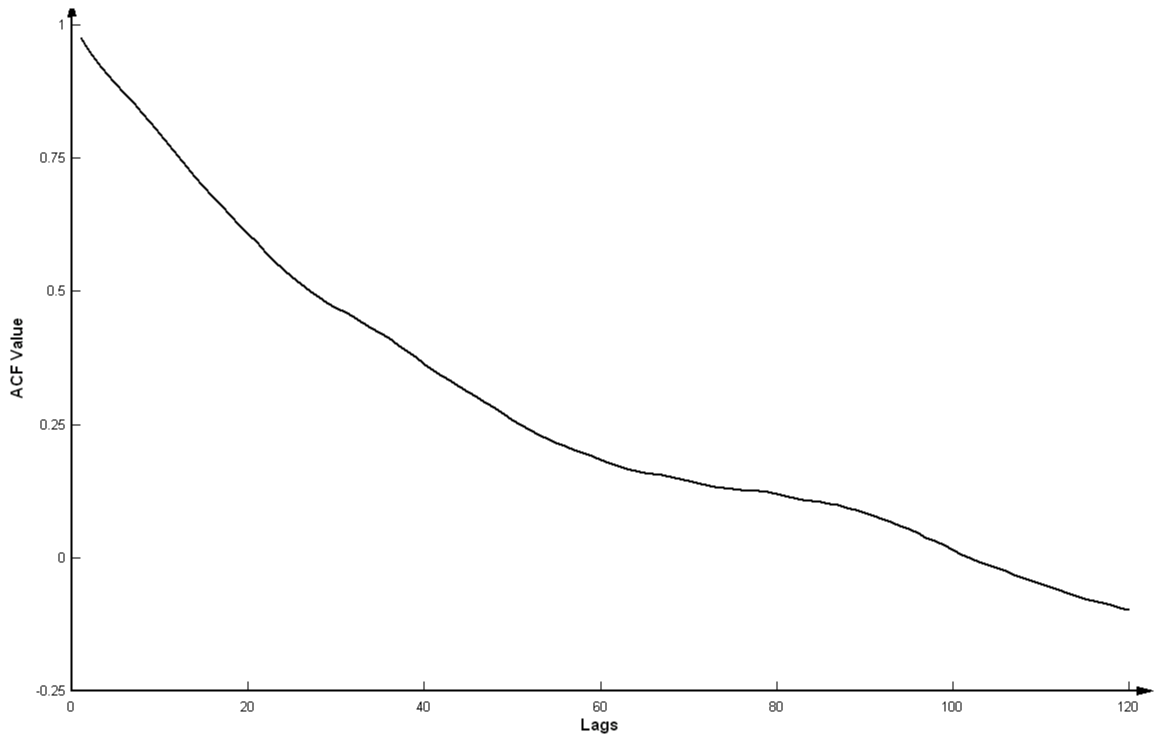
Given the results from both methods, we conclude that because all the Lyapunov Exponents are positive, then there is some deterministic chaotic behavior in gold prices.

With the existence of both random and deterministic components verified, the next step is to separate the two. To separate random IID and deterministic components of the price of gold, a space-time regression (Deutsch & Pfeifer 1981, Gooijer & Anderson 1985) must be used.

Seperating the Deterministic and the Random Components of Gold Price

Using the Lyapunov Exponent we have demonstrated that a portion of the gold price is deterministic in nature. To separate the random and deterministic, we need to check for linear dependency by utilizing the autocorrelation function. The ACF is a measure of the strength of the relationship of the prices to themselves over time. If the series exhibits a high degree of autocorrelation, that means that the series is very dependent on its previous values over time using the linear IID assumption. When the ACF is zero, there is no relationship between the current price and the price that came before it. We want to separate the data at this point, because the covariance across time in the data is zero when the ACF function has a value of zero. At the point where the ACF is zero, the affects to the system are random. In the case of gold, Figure 30 shows that the lag required to make the ACF coefficient equal to zero is 103 months. Figure 30 shows that the linear serial correlation is large for a long period in time. At the point where the ACF value is zero (103 months) there is no more linear serial correlation. This indicates that the point of seperation between the deterministic and random components in the gold price begins at 103 months.

Figure 30 - ACF Plot of Monthly Gold Data



This is confirmation of the long memory we are seeing in the process and will be the starting point to sperate the deterministic and random components of the signal. We can use the IID assumption in a linear model when there is no serial correlation. To further test the memory, and determine if gold prices are random or not, we need to check the dimensional space. If a series is random, then it will exhibit infinite dimensional space. Whereas a deterministic function will have a finite set of dimensions. The Lyapunov Exponent will give us an indication of the correct functional form for the signal.

The space-time regression routine separates the aggregate signal (nominal gold price) by its serial correlation and dimensional space. The starting of the estimation parameters is at the ACF lag of 103 months and the Lyapunov Exponent result for the dimensional space requirement from our Lyapunov Exponent estimation.

The Hurst, ACF and Lyapunov Exponent all indicate long run dependency and chaotic behavior in gold prices. We now turn to separating the data in order to isolate the component of the price that exhibits long run dependency versus the component that does not. To do this a Space-Time separation procedure was performed using the information gathered from the previous three tests. The results of this procedure produce a data series for the exogenous, or inter market effects, that does not have any long run dependency. In other words it is stationary.

To verify, an FARIMA(0,d,0) model was performed on the transformed data to test for long run dependency. A Hurst exponent of 0.5 is measured indicating the transformed data is now stationary. The long run dependent data is formed by subtracting the actual data from the random data. The descriptive statistics are in Table 6.

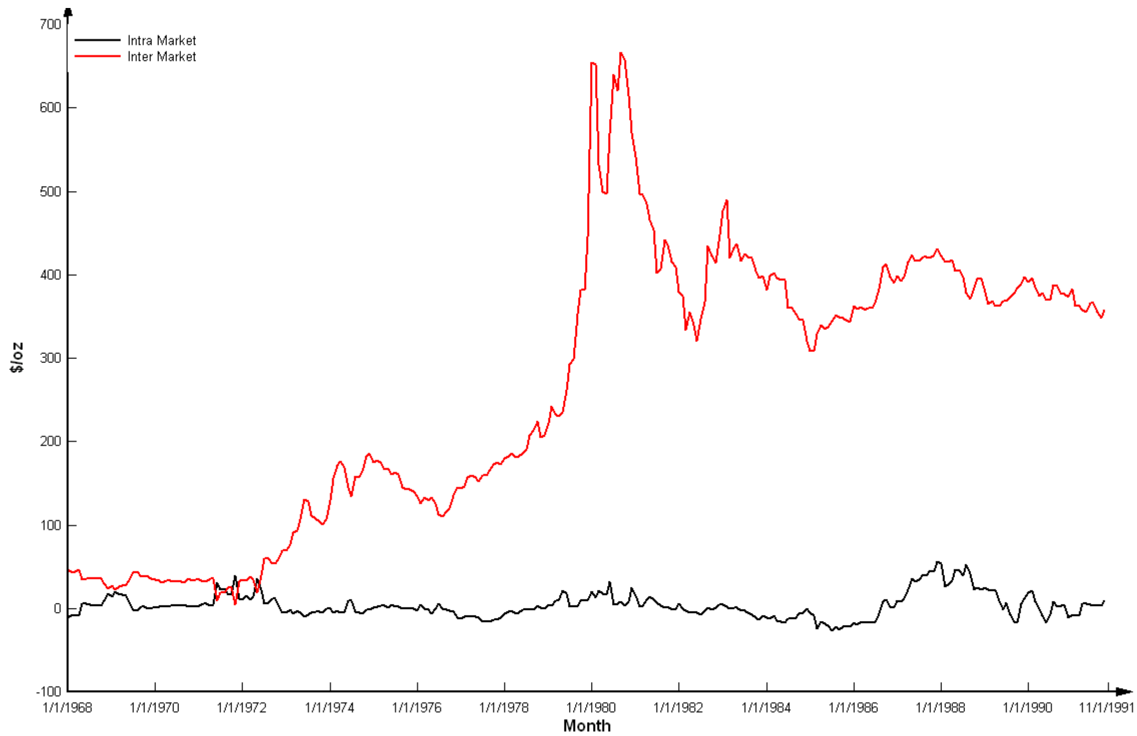
Table 6 - Descriptive Statistics of both Gold Price Signals

Intra Market Descriptive Statistics						
Minimum	1st Quartile	Median	Mean	Std. Dev	3rd Quartile	Maximum
-27.04	0.04	1.745	2.64	3.09	3.07	55.92
Inter Market Descriptive Statistics						
Minimum	1st Quartile	Median	Mean	Std. Dev	3rd Quartile	Maximum
3.97	182.00	347.00	311.70	164.30	394.70	920.60

We have now separated the long run dependent portion on the price (endogenous or intra market) and the random portion (exogenous or inter market), as shown in Table 6.

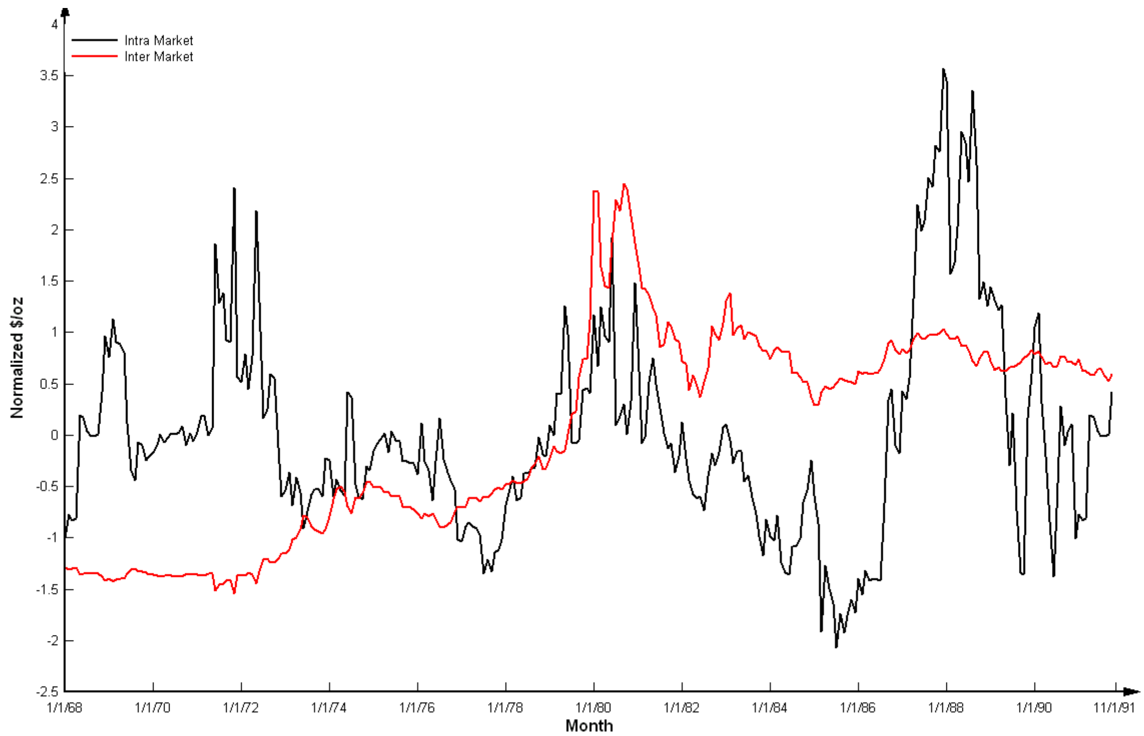
Figure 31 shows the relative impacts of the two components. The endogenous component accounts for less than 20% of the total price - indicationg gold prices are highly susceptible to external or inter market factors.

Figure 31 - Nominal Monthly Gold Price Impacts



To illustrate the movement of the intra and inter market portions of price, both signals were normalized (Figure 32). The intra and inter market factors do not move in relation to one another. When we look at the intra market portion of the price we can see the series cycle in what appears to be a chaotic manner (Figure 32). This is because the intra market portion is deterministic so the behavior is not random. Recall the plot of the chaotic logistic equation as an example of what chaotic behavior looks like. We will test for the existence of chaotic behavior to determine if the intra market price of gold moves up and down with a frequency similar to the pattern of the chaotic logistic function shown previously.

Figure 32 - Normalized Monthly Gold Price Impacts



The functional form for the deterministic part of the series needs to be determined. For continuity and ease we utilize the logistic functional form that was discussed previously.

To verify that the logistic function fits the intra market signal we will test its validity via regression. To use the logistic function, the data has been scaled into the range of 0 to 1 and then an estimate of the value of the variable of sensitivity alpha is calculated using the method of non-linear least squares. The results are shown in Table 7.

Table 7 - Estimation of Logistic Function

Coefficient	Estimate	Std. Error	T Value	Pr(>| t|)
Alpha	3.639	0.202	18.05	2e-16***
Significance: 0'***'				
Residual standard error: 0.321 on 274 degrees of freedom				

The estimate of the alpha term in the logistic is significant at more than 1%. The value of 3.639 is inside of the theoretical limit of the logistic function and is a value that suggests chaotic behavior. The non-linear least squares method indicates that the logistic function is a statistically appropriate function to describe the deterministic component of gold prices. This will allow us to utilize the logistic function to characterize the deterministic component in gold prices. The knowledge that the logistic function is possibly chaotic will prove useful later on. Understanding that the gold industry is sensitive to industry changes will help in the understanding of policy changes that can effect the gold industry.

With the price of gold separated into deterministic and random components, we can now perform regressions on each series. Next, we will use our separated gold price signals, as well as world production data, to understand how price and quantity interact in the gold industry.

The Gold Industry and Gold Prices

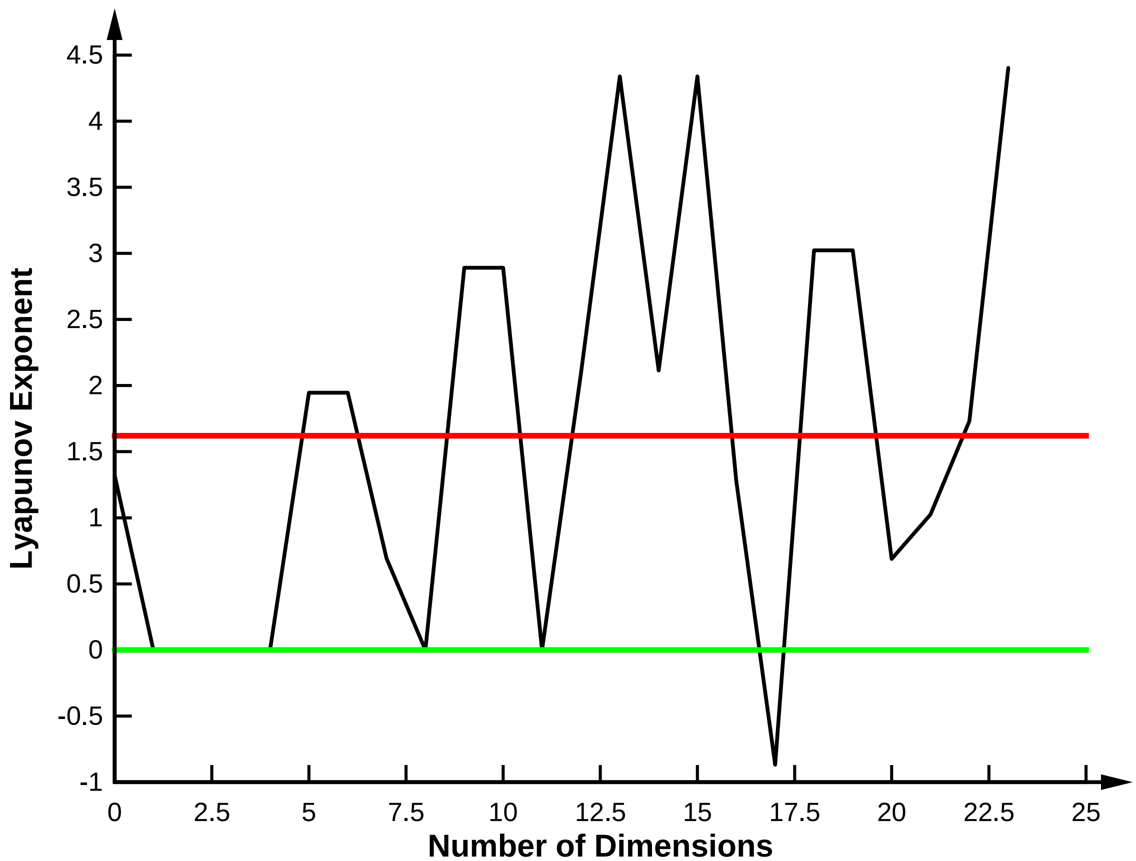
Given that gold prices exhibit long memory and chaotic behavior, this will allow us to develop a model that does not impose the normal restrictions of linear additivity. Thus, the data itself determines the model. Table 8 presents nominal annual descriptive statistics for the gold industry. For the last century the average price of gold is $136.33 per ounce, while the average production in the U.S. has been 120.25 tons and average U.S. consumption has been 150.03 tons per year. The intra and inter market signals were developed using the previously discussed methodology. The average price of the intra market signal is $1.66 per ounce, whereas the average price of the inter market signal is $137.99 per ounce. Notice the vast difference in range between the two signals, the intra market prices ranged from $-10.75 to $8.21 and the inter market prices ranged from $19 to $615.78.

Table 8 - Descriptive Statistics of the Annual Gold Market

Variable	Mean	Median	Max	Min	Std
Price($)	136.33	35.03	612.56	17.06	163.36
US Production (tons)	120.25	71.8	366	29.7	98.39
Consumption (tons)	150.03	120	667	12.7	102.02
World Production (tons)	1260.84	1170	2600	481	615.16
Intra Market Signal ($)	-1.66	-3.22	8.21	-10.75	3.09
Inter Market Signal ($)	137.99	37.94	615.78	19	164.3

Since the production data is annual, the price of gold was split on an annual basis using the same methodology as before. Again, evidence of chaotic behavior exists as we can see from the Lyapunov Exponents of the price of gold which are predominatley positive as well as in finite number(Figure 33).

Figure 33 - Yearly Gold Price Lyapunov Exponents

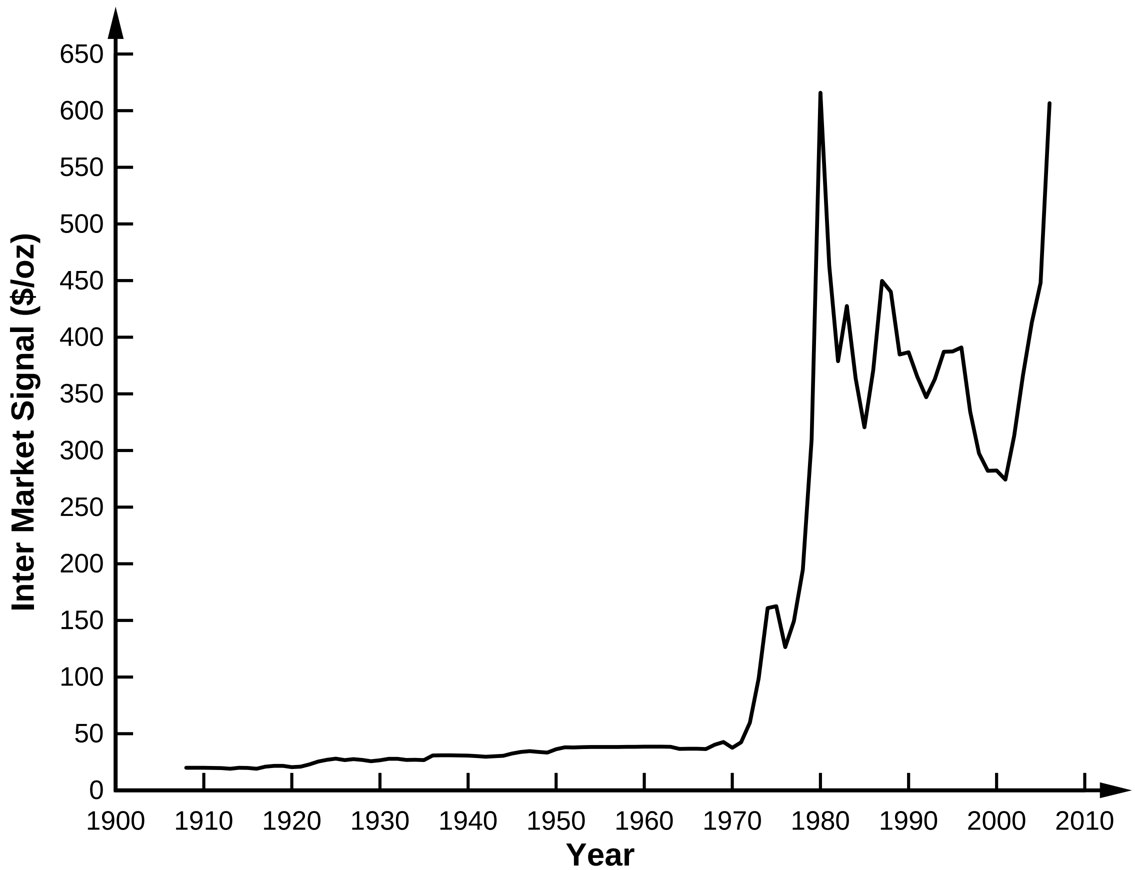


Evidence that a long memory process exists is also supported by the ACF estimate of the Hurst Exponent which is 0.723 for an annual measurement. The value of 0.723 for the Hurst Exponent means that there is some positive persistent behavior (memory) in annual gold prices. The persistent behavior detected by the Hurst Exponent means that the current market price of gold is dependent on previous prices.

Using the same methodology as used in the previous section to separate the monthly gold price, the intra market and inter market annual signals can also be separated. These signals take the form seen in Figures 34 and 35. The descriptive statistics are in the previous table (Table 8). Figures 34 and 35 show the yearly intra and inter market signals.

Figure 34 - Intra Market Gold Price Signals (1908-2006)

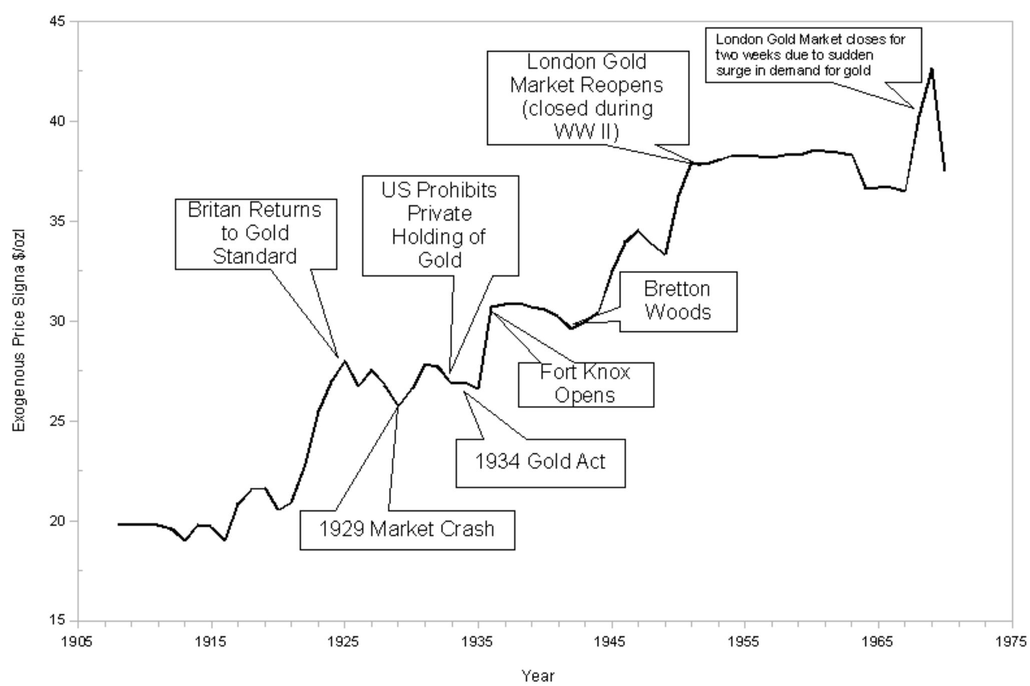
Figure 35 - Inter Market Gold Price Signals (1908-2006)



As with the monthly gold data, we can see that the intra market effect on price is low compared to the inter market effect. This means that currently gold prices are extremely sensitive to, and dominated by inter market (exogenous) factors. In comparing both price signals the complete deregulation of the gold industry in the 1970s caused the price of gold to be more susceptible to exogenous factors. This corresponds to the increase in the inter market signal in Figure 35. Comparing the two signals allows us to also infer that as the number of gold mining firms has increased, the intra market signal has less of an impact on the market price. In other words, as the gold industry's concentration decreases, the price is more subject to external factors. As the industry consolidates, the intra market signal becomes more significant and the market price of gold is less affected by inter market effects.

To appreciate how separation of price signals can be beneficial, let us look at the inter market signal and significant events that happened in the gold market from 1908-1970. Remember, that since the intra market effect has been removed, we are actually looking at the true magnitude of the inter market effect. The inter market signal here is the same as in Figure 35, the scale is changed to display the occurance of the events clearly.

Figure 36 - Exogenous Events vs. Inter Market Price



From the graph it can be seen that this pure Fractal Brownian Motion signal clearly displays the exogenous events. The key here, is that looking only at this particular part of the price signal, we can now tell what the correctly scaled impact from an inter market event truly was. It is important to note that this type of analysis can be used on any frequency of observation, in this case the data was yearly. Note that you cannot disaggregate or aggregate signals, in other words, we would not be able to take this yearly inter market signal and calculate for a monthly time frame or vice versa.

To test that the events listed in Figure 36 had a significant impact on the FBM signal a trend stationary AR(1) regression was performed (Table 9). Due to the amount of data points available the following equation was used for the test in Table 9 to keep adequate degrees of freedom. One dummy variable was used to test for the significance of all events, the impact of each individual event was not tested.

$$P_t = \propto_0 + \ \beta_1 \, Trend + \ \beta_2 \, AR(1) + \ \beta_3 \, Events + \ \varepsilon_t$$

Table 9 - Estimation of Exogenous Events on Inter Market Price

Dependent variable: Inter Market Gold Price
Number of observations: 98

Variable	Coefficient	St. Error	t-statistic	Sign.
Constant	-16.072	10.301	-1.560	[0.1220]
Trend	0.562	0.265	2.124	[0.0363]
AR(1)	0.929	0.048	19.328	[0.0000]
Events	25.88	13.922	1.859	[0.0661]

R^2adj. = 92.70%
R^2 = 92.93%
DW = 1.6369
S.E. = 44.4921
Residual sum of squares: 186077.134
Maximum loglikelihood: -508.954
AIC = 10.489
F(3,94) = 411.7796 [0.0000]
Normality: $\chi^2(2)$ = 1287.958 [0.0000]
Heteroskedasticity: $\chi^2(1)$ = 10.519 [0.0012]
Functional form: $\chi^2(1)$ = 12.0982 [0.0005]
AR(1) in the error: $\chi^2(1)$ = 1.9386 [0.1638]

The events had a statistically significant affect on the inter market price signal (Table 9). The coefficient estimate of the external events shows a $25.88 impact on the inter market price signal per event. Further investigation of the individual events in Figure 36 was done by using a dummy variable for each event, given the following equation.

$$P_t = \propto_0 + \ \beta_1 \, Trend + \ \beta_2 \, AR(1) + \ \beta_3 \, Event\ 1 + \ \beta_4 Event\ 2 + \ \beta_5 Event\ 3 + \ \beta_6 Event\ 4 + \ \beta_7 Event\ 5 + \ \beta_8 Event\ 6 + \ \beta_9 Event\ 7 + \ \beta_{10} Event\ 8 + \ \varepsilon_t$$

The results of the regression are given in Table 10. The results of the estimated coefficients are not very robust the estimated coefficients have the appropriate sign we would expect from visual inspection of Figure 36 with the exception of Event 7.

Table 10 - Estimation of Individual Events on Inter Market Price

Dependent variable: Inter Market Gold Price				
Number of observations: 98				
Variable	Coefficient	St. Error	t-statistic	Sign.
Constant	-13.314	11.248	-1.184	[0.2398]
Trend	0.628	0.286	2.198	[0.0306]
AR(1)	0.909	0.051	17.837	[0.0000]
Britan Returns to Gold Standard (Event 1)	1.042	1.043	0.964	[0.3377]
1929 Market Crash (Event 2)	-0.964	1.043	-0.859	[0.3927]
U.S. Prohibits Private Holding of Gold (Event 3)	-0.765	1.044	-6.279	[0.0000]
1934 Gold Act (Event 4)	-0.753	1.042	-6.891	[0.0000]
Fort Knox Opens (Event 5)	1.121	1.044	2.665	[0.0092]
Bretton Woods (Event 6)	-0.996	1.042	-0.108	[0.9145]
London Gold Market Reopens (Event 7)	-0.988	1.042	-0.302	[0.7630]
London Gold Market Closes (Event 8)	1.003	1.042	0.079	[0.9373]
R^2adj. = 91.86%				
R^2 = 92.70%				
DW = 1.6573				
S.E. = 46.988				
Residual sum of squares: 192084.071				
Maximum loglikelihood: -510.511				
AIC = 10.663				
F(10,87) = 110.487 [0.0000]				
Normality: $\chi^2(2)$ = 2109.259 [0.0000]				
Heteroskedasticity: $\chi^2(1)$ = 7.775 [0.0053]				
Functional form: $\chi^2(1)$ = 16.1233 [0.0001]				
AR(1) in the error: $\chi^2(1)$ = 1.6677 [0.1966]				

To discern what type of randomness is occuring we need to test the series for its Hurst Exponent. Please note, we previously used the Hurst Exponent to test for long run memory over the entire signal. This time we are using the Hurst Exponent to test for correlation of FBM to determine the type of randomness. The ACF method of Hurst Exponent estimation derives a value of 0.726.

The result of the ACF test shows that the series is persistent FBM. Knowing this, allows us not to falsely conclude that the part of the series that is random is not RBM, what we see is FBM. This is important in a dynamical system because we now know that the inter market impacts are not of the RBM type and that the randomness is indeed serially correlated over time. This demonstrates a significant interpretation change of the randomness. It means that there is a more likely chance of movements conferring on each other up to a switch point. For example, we are more likely to see many upward movements before a downward movement. The persistence is an important consideration in understanding the systems behavior.

To further investigate the separation and dimensional space of the variables, let us look at three dimensional attractor plots of the two signals (Figures 37 and 38).

Figure 37 - Attractor Plot of Intra Market Gold Prices

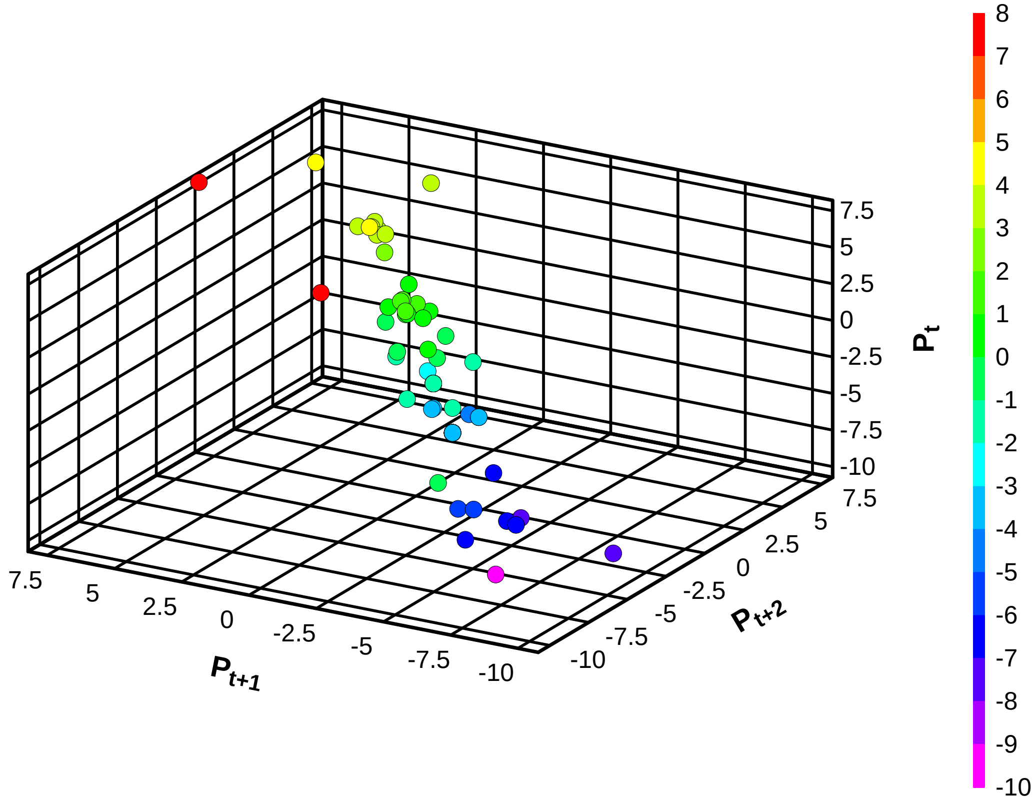
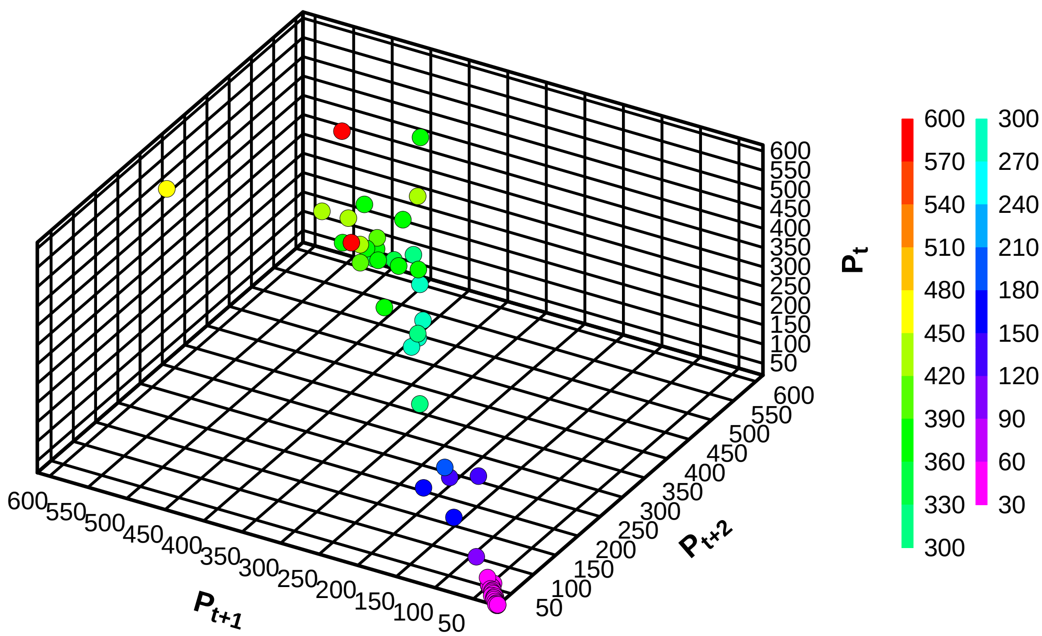


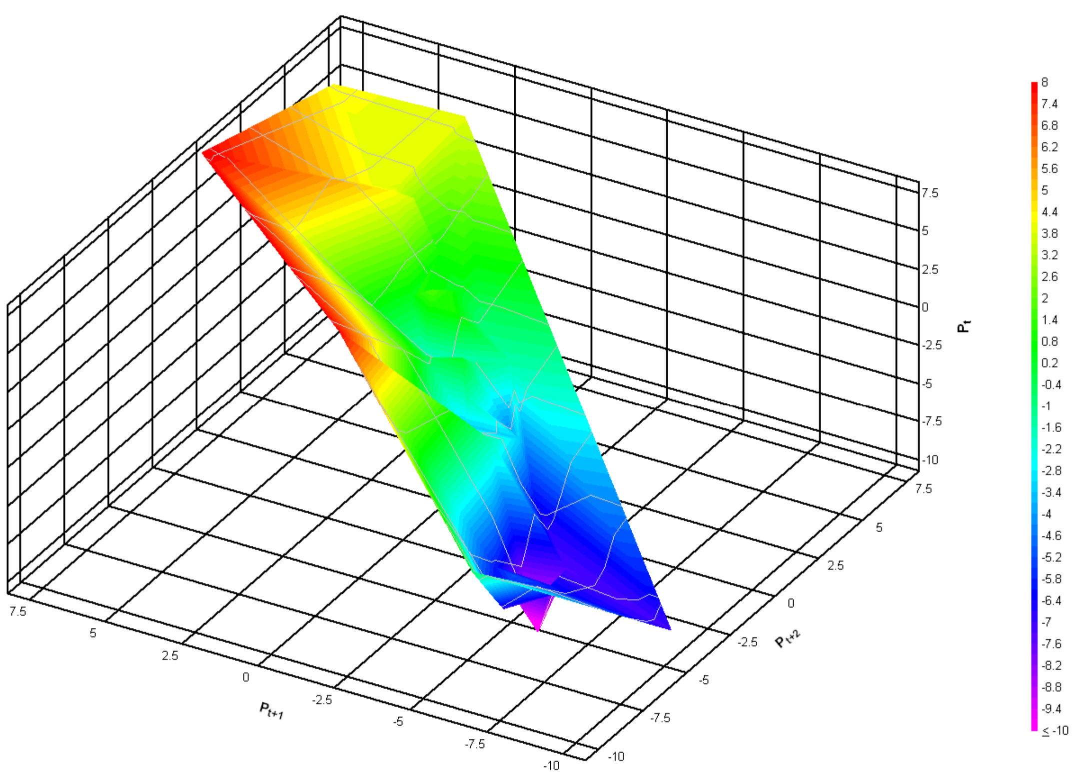
Figure 38 - Attractor Plot of Yearly Inter Market Gold Prices



The attractor plots of the intra market signal are consistent with deterministic behavior, because they are dimensionally close. We can see in Figure 37 that all of the intra market prices remain in a resticted dimensional space. In the case of the inter market prices, we can see that they are persistent FBM because the points remain in relativley close dimensional space. Both price signals have some persistance (memory) in each price signal. We can see that the level of divergence is much less than the inter market. These results demonstrate that the random component has a larger impact on the overall price of gold.

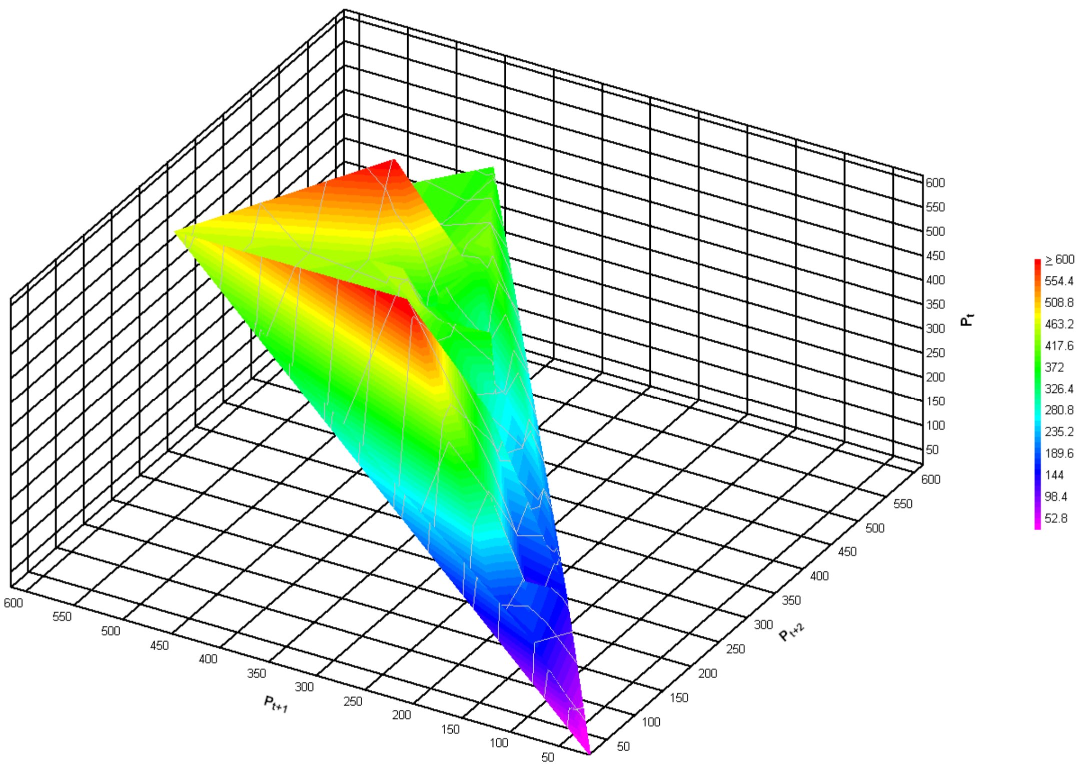
We have specifically used the Lyapunov Exponent to separate the series by nearby trajectories, thus demonstrating the cycling behavior on the intra market prices. Looking at surface plots of the same attractors, we can see the limited dimensional space and grouping of prices (Figure 39 and 40). The surface plots allow for an interpolation of all between the points from each scatter plot. The highest values are in red for each plot and the lowest values are in purple. The surface plots show an interesting topography to each price signal. For example in Figure 39 there is a low spot at approximately $P_t = -10$, $P_{t+1} = -6$, $P_{t+2} = -5$. This combination of these three prices over time shows the persistance in the data, where one low value causes the value to stay low in the future.

Figure 39 - Surface Plot of Yearly Intra Market Gold Prices



In Figure 40 we can see the high peaks in red also demonstrate the persistance in the inter market price, where a high value causes high values to persist into the future.

Figure 40 - Surface Plot of Yearly Inter Market Gold Prices



Both the surface plots of the intra and inter market prices show groupings of the prices (peaks and valleys) in a more pronounced manner than the scatter plots. The surface plots confer the result that there are a finite number of dimensions in both price signals and that persistance exists in both price signals.

We can now look at other impacts on gold prices, other than the two signals by themselves. Examination of regressions on the two signals of different explanatory variables will show how production can affect the price of gold. In both the intra and inter market values, some autoregressive components can be seen in the plots of the autocorrelation function (Figure 41 and 42).

Figure 41 - ACF of Intra Market Price Signal

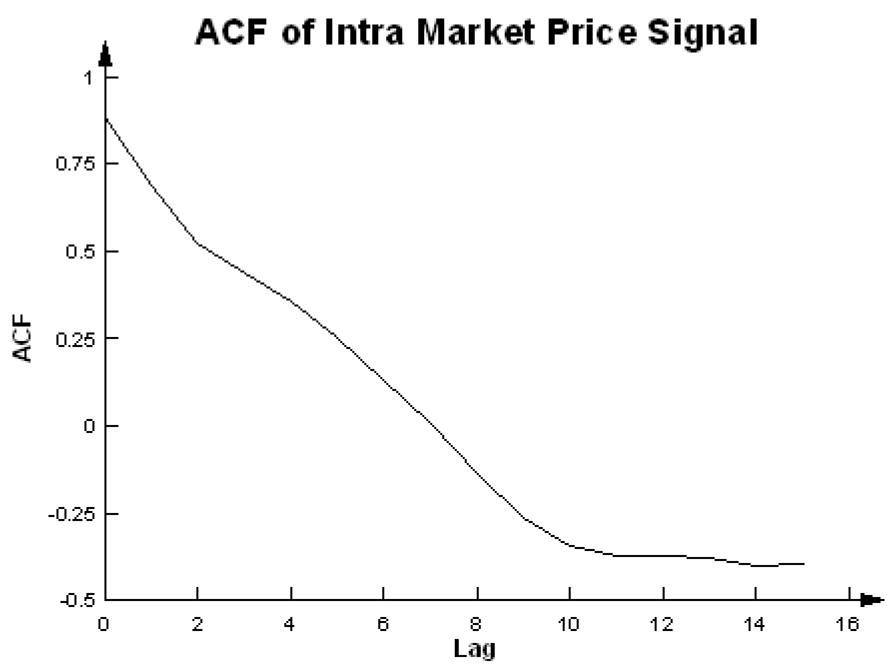
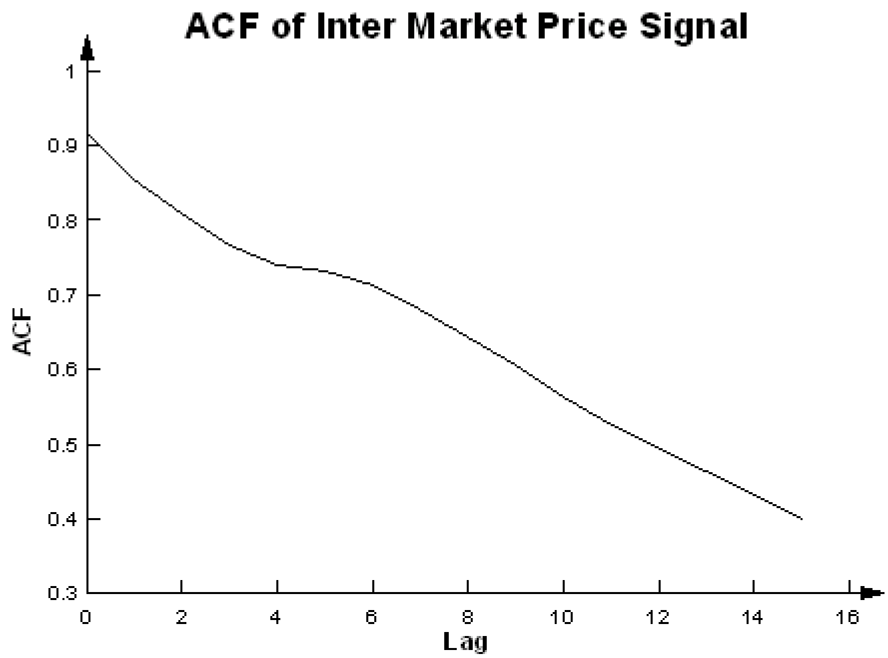


Figure 42 - ACF of Inter Market Price Signal



Both ACF plots suggest serial correlation of both signals in a linear sense due to the high ACF values over many lags. The slowly decaying ACF values versus the lags show that autoregressive components will be necessary in both regressions to remove linear serial correlation.

Using the percentage of world production by country, we can run regressions which have that results confer the description of the two signals. The first regression run is on the inter market signal, identifying how production by country affects the inter market price signal. Four of the largest gold producing countries were chosen (Canada, U.S., Mexico and South Africa). The following regression equation was used.

$$Inter\ Market\ Price_t = \alpha_0 + \beta_1\ Canada_t + \beta_2\ Mexico_t + \beta_3\ South\ Africa_t + \beta_4\ U.S._t + \beta_5\ AR(1) + \varepsilon_t$$

The results of the inter market price signal, are listed in Table 11 (the signal was normalized to improve estimation).

Table 11 - Estimation of Industry Structure on Inter Market Price

Dependent Variable: Normalized Inter Market Signal
Number of Observations: 74

Variable	Coefficient	St. Error	T-Statistic	Sign.
Constant	0.68	0.28	2.41	[0.0187]
Canada	-2.96	1.6	-1.85	[0.0689]
Mexico	-8.69	6.89	-1.26	[0.2115]
South Africa	-0.6	0.37	-1.62	[0.1088]
United States	-0.78	1.44	-0.54	[0.5890]
AR(1)	0.87	0.06	14.82	[0.0000]

Adjusted R^2 = 91.46%
R^2 = 92.05%
DW = 1.8487
S.E = 0.2927
Residual Sum of Squares: 5.8257
Maximum Loglikelihood: -10.9554
AIC = 0.4853
F(5,68) = 157.3769 [0.0000]
Normality: $\chi^2(2)$ = 1495.556 [0.0000]
Heteroskedasticity: $\chi^2(1)$ = 3.7302 [0.0534]
Functional Form: $\chi^2(1)$ = 16.1094 [0.0001]
AR(1) in the error: $\chi^2(1)$ = 0.4761 [0.4902]

The signs of the coefficients are negative as economic theory would predict. As the industry consolidates, the inter market signal weakens. Meaning that as competition goes down, the gold industry is less sensitive to inter market events. Conversely , as competition increases, market share is diluted, and external events have a greater impact on the overall price of gold. As one country produces a larger share of the world production, the result is that the inter market signal is reduced.

Conversely if we look at the intra market price versus the United States production (Table 12), we see that the effect is the opposite from that of the inter market. The following regression equation was used to estimate the coefficients.

$$Intra\ Market\ Price_t = \alpha_0 + \beta_1\, U.S._t + \beta_2\, AR(1) + \beta_3\, AR(2) + \varepsilon_t$$

Although the results are not as robust as the inter market price, the estimated coefficient suggests that as the percentage of world production by the United States grows, the intra market price increases. If the Intra market price is increasing the inter market price would be declining, resulting in the market price of gold becomes less subject to inter market events such as those seen in Figure 36 (wars, stock market crash, etc.). .

Table 12 - Estimation of U.S Production on Intra Market Price

Dependent Variable: Normalized Intra Market Signal
Number of Observations: 74

Variable	Coefficient	St. Error	T-Statistic	Sign.
Constant	-0.03	0.11	-0.26	[0.7982]
United States	0.52	1.25	0.42	[0.6733]
AR(1)	1.25	0.11	11.93	[0.0000]
AR(2)	-0.46	0.1	-4.37	[0.0000]

Adjusted R^2 = 79.84%
R^2 = 80.67%
DW = 1.7111
S.E = 0.448
Residual Sum of Squares: 14.047
Maximum Loglikelihood: -43.52
AIC = 1.311
F(3,70) = 97.3597 [0.0000]
Normality: $\chi^2(2)$ = 145.0713 [0.0000]
Heteroskedasticity: $\chi^2(1)$ = 0.2308 [0.6309]
Functional Form: $\chi^2(1)$ = 0.6767 [0.4107]
AR(1) in the error: $\chi^2(1)$ = 0.418 [0.5179]

The separation of the price into its two components helps us to quantify the different effects of market structure. The separate regressions allow for a more detailed account of what is occuring in the gold industry. Without separating the signals, the sign change in US production would not be obvious, due to the greater magnitude of the inter market signal. This is clearly demonstrated when we look at regressions of the same independent variables versus the entire price signal (Table 12). The following equation for estimation was used.

$$Gold\ Price_t = \alpha_0 + \beta_1\ Canada_t + \beta_2\ Mexico_t + \beta_3\ South\ Africa_t + \beta_4\ U.S._t + \beta_5\ AR(1) + \varepsilon_t$$

Please note that the U.S. coefficient is negative as it was in the inter market regression. However the estimated coefficient is now larger in magnitude from -0.78 to -0.82. While the regression results are not robust enough to be statistically different, the differences in the estimates suggests that the separation of the market price into the two signal offers a different perspective on the data. The only conclusion which can be drawn, when not separating the signals, is that as gold production increases the price of gold goes down. This conclusion is more generalized and does not show as much detail as to how the market structure of the gold industry affects the price of gold.

Table 13 - Estimation of Market Structure of Entire Gold Price

Dependent Variable: Normalized Gold Price
Number of Observations: 74

Variable	Coefficient	St. Error	T-Statistic	Sign.
Constant	0.67	0.28	2.47	[0.0159]
Canada	-2.99	1.61	-1.85	[0.0680]
Mexico	-8.76	6.89	-1.27	[0.2082]
South Africa	-0.63	0.37	-1.69	[0.0950]
United States	-0.82	1.45	-0.57	[0.5724]
AR(1)	0.86	0.06	14.75	[0.0000]

Adjusted R^2 = 91.34%
R^2 = 91.93%
DW = 1.8620
S.E = 0.2941
Residual Sum of Squares: 5.8835
Maximum Loglikelihood: -11.3209
AIC = 0.4952
F(5,68) = 154.9573 [0.0000]
Normality: $\chi^2(2)$ = 1564.79 [0.0000]
Heteroskedasticity: $\chi^2(1)$ = 3.6514 [0.0560]
Functional Form: $\chi^2(1)$ = 16.8505 [0.0000]
AR(1) in the error: $\chi^2(1)$ = 0.387 [0.0619]

For example, the difference in the coefficient on US production (Table 11 and 13) is 0.04. This may seem insignificant, but an extra 0.04 tons of gold in the world could cause a large impact on price. In the case of looking for the intra market effect, we can see that because the inter market signal is so large, the estimate below is contrary to the results we would expect (Table 14). The following estimation equation was used.

$$Gold\ Price_t = \alpha_0 + \beta_1\, U.S._t + \beta_2\, AR(1) + \beta_3\, AR(2) + \varepsilon_t$$

Table 14 - Estimation of U.S Production on Entire Gold Price

Dependent Variable: Normalized Gold Price
Number of Observations: 74

Variable	Coefficient	St. Error	T-Statistic	Sign.
Constant	0.08	0.08	0.95	[0.3430]
United States	-0.59	0.99	-0.59	[0.5561]
AR(1)	1.04	0.12	8.71	[0.0000]
AR(2)	-0.07	0.12	-0.57	[0.5696]

Adjusted R^2 = 90.69%
R^2 = 91.07%
DW = 1.9562
S.E = 0.305
Residual Sum of Squares: 6.512
Maximum Loglikelihood: -15.078
AIC = 0.5426
F(3,70) = 237.9363 [0.0000]
Normality: $\chi^2(2)$ = 1080.687 [0.0000]
Heteroskedasticity: $\chi^2(1)$ = 6.718 [0.0095]
Functional Form: $\chi^2(1)$ = 34.4353 [0.0000]
AR(1) in the error: $\chi^2(1)$ = 3.761 [0.0525]

We would expect to see, from our previous analysis, that the coefficient for US production should be positive, however the magnitude of the inter market signal causes the coefficient to be negative. We cannot estimate the true effect of market structure using a regression without separating the signals. This is important for simulations, as we will see in the next section.

Simulation of Gold Industry Events

To analyze the affect of potential future events on the industry, we can simulate different scenarios and see the impact the change in market structure has on the market price. Given our previous analysis, we know that there is a deterministic and a random component to the market price of gold. The deterministic component is multiplicative in nature and the random component displays signs of persistent FBM. The following general equation for gold price will be used for simulation:

$$p_t = \beta\alpha p_{t-1}(1 - p_{t-1}) + \theta\mathbf{B}_H(t, \omega)$$

Where:

β = Scaling factor on the logistic function

θ = Scaling Factor of FBM

Simulation allows us to vary the magnitudes of the deterministic and random components. Simulation also provides the level of oscillation in the deterministic component and the level of memory in the random component. Both the deterministic and random components are produced on a zero to one scale, as such the scaling factor puts the prices into the range of current prices. Because the previous empirical tests of the FBM component showed persistence, a value of 0.9 was assigned for the Hurst Exponent. The Hurst Exponent could be changed to test for what would happen to structural changes in the type of randomness, if desired. However for the purposes of this paper, we will only evaluate changes in the deterministic and random components.

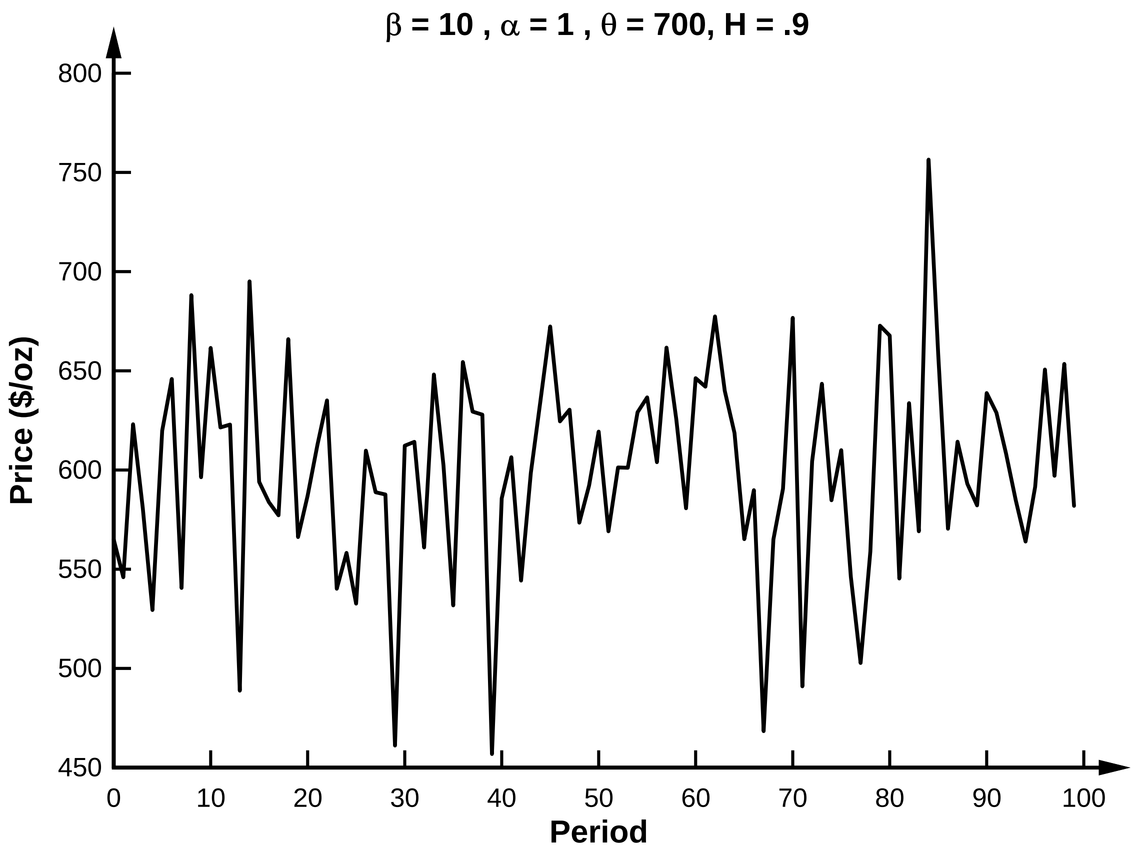
We simulate four scenarios. The first base case is consistent with a large number of firms in the industry and a large random component. The second case is consistent with a smaller number of firms in the industry and a large random component. The third case is consistent with a small number of firms in the industry and a small random component. The fourth case is consistent with a large number of firms in the industry and a small random component. The table below summarizes the four simulations.

Table 15 - Design of Simulations

Degree of Randomness		Number of Firms: Many	Number of Firms: Few
	High	Simulation 1	Simulation 2
	Low	Simulation 4	Simulation 3

Figure 43 shows a simulation of the gold price path under the scenario of a stationary deterministic component (α=1) and a large random component. This scenario describes a situation in which the gold industry is reaching perfect competition. Since α=1 the deterministic portion of the price signal remains constant. Thus this simulation result is only influenced by the random signal.

Figure 43 - Simulation of Gold Price (1)



This figure (43) demonstrates that the price of gold is subject to external events which are the only cause for these price changes. As market share per firm declines, the intra market signal will be reduced to a fixed value. This will leave the market price of gold highly subject to inter market events.

The simulation in Figure 44 has the same random component and scale as in Figure 45. The change in the simulation this time, is making the deterministic component oscillate (α=3.5). This could be a situation of consolidation of firms within the industry, with the random component still having a greater share of causation of gold prices. In this case, both the random and deterministic components confer on one another causing an increase in the volitility of gold prices. This simulation represents the current state of the gold industry. As the gold industry has been consolidating over the past decade the intra market signal is getting larger in magnitude and more chaotic, however the FBM still is much larger.

Figure 44 - Simulation of Gold Price (2)

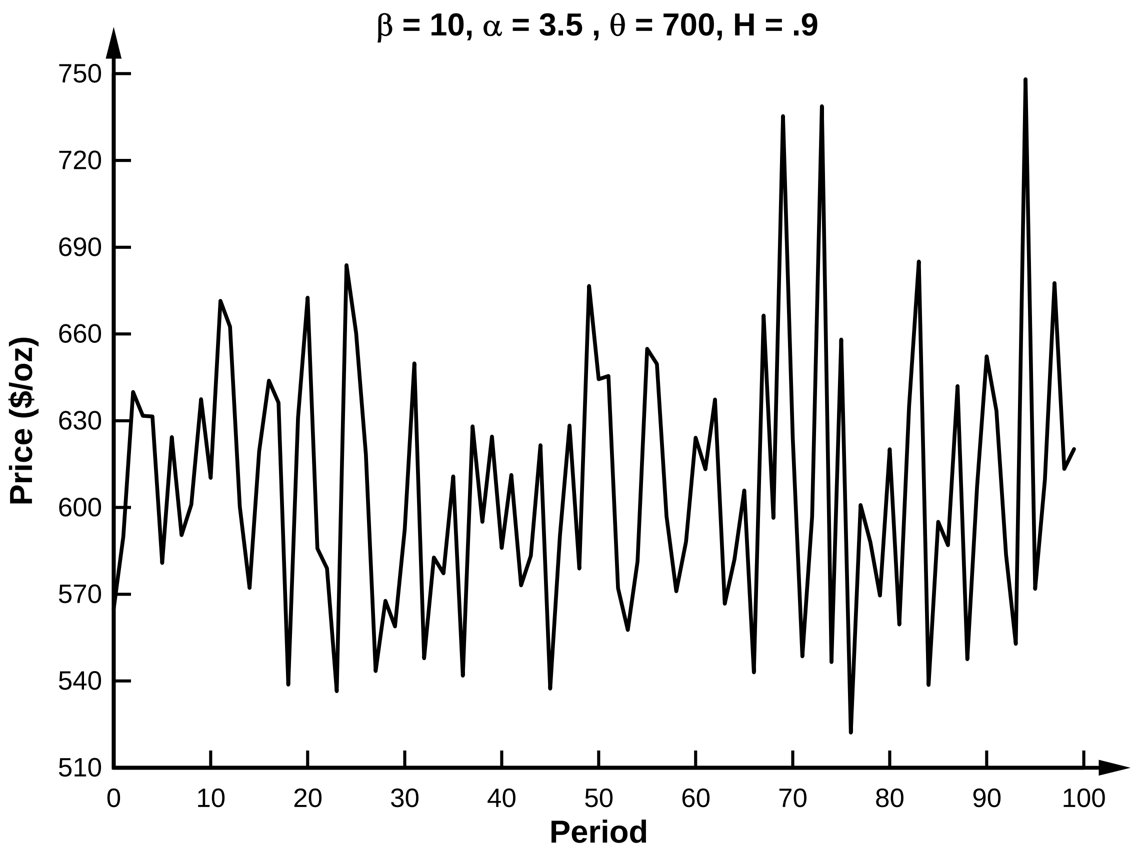
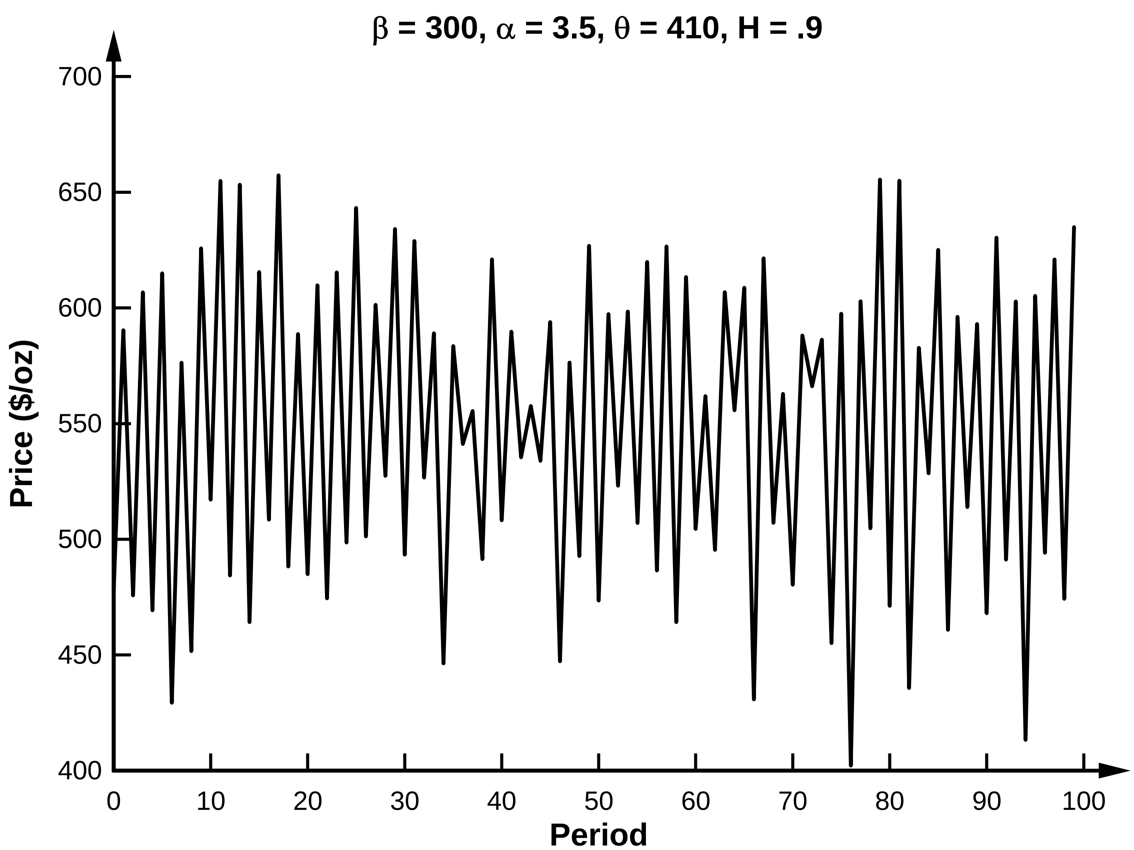


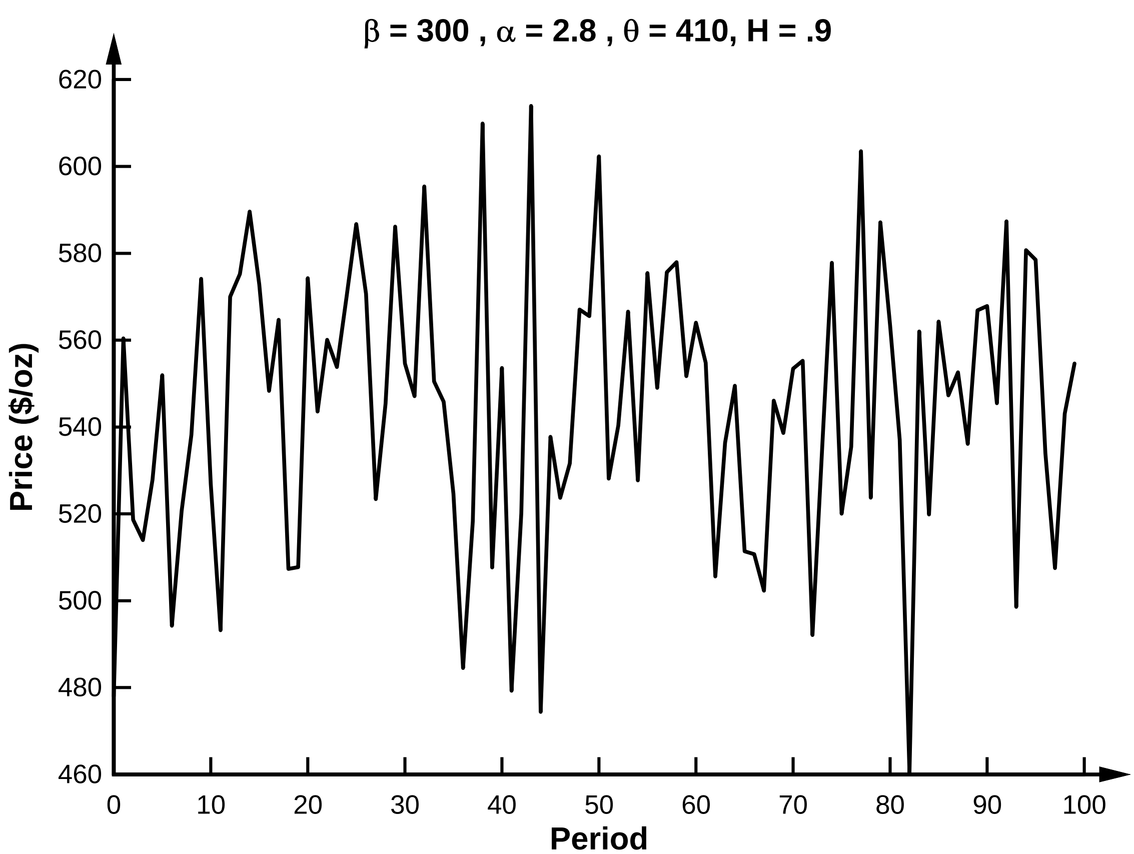
Figure 45 uses the same oscillation as in Figure 44, but this time the random component is of lower magnitude. This is a historic scenario where more regulation is causing the price of gold to be less subject to external events and there are fewer firms. In this case the oscillatory behavior of the deterministic signal, overrides the random component and more regular oscillations in the price of gold occur. This simulation describes a gold industry that has few firms and is stable to external events. Historically this scenario has occurred and describes the situation of gold being regulated for currencies.

Figure 45 - Simulation of Gold Price (3)



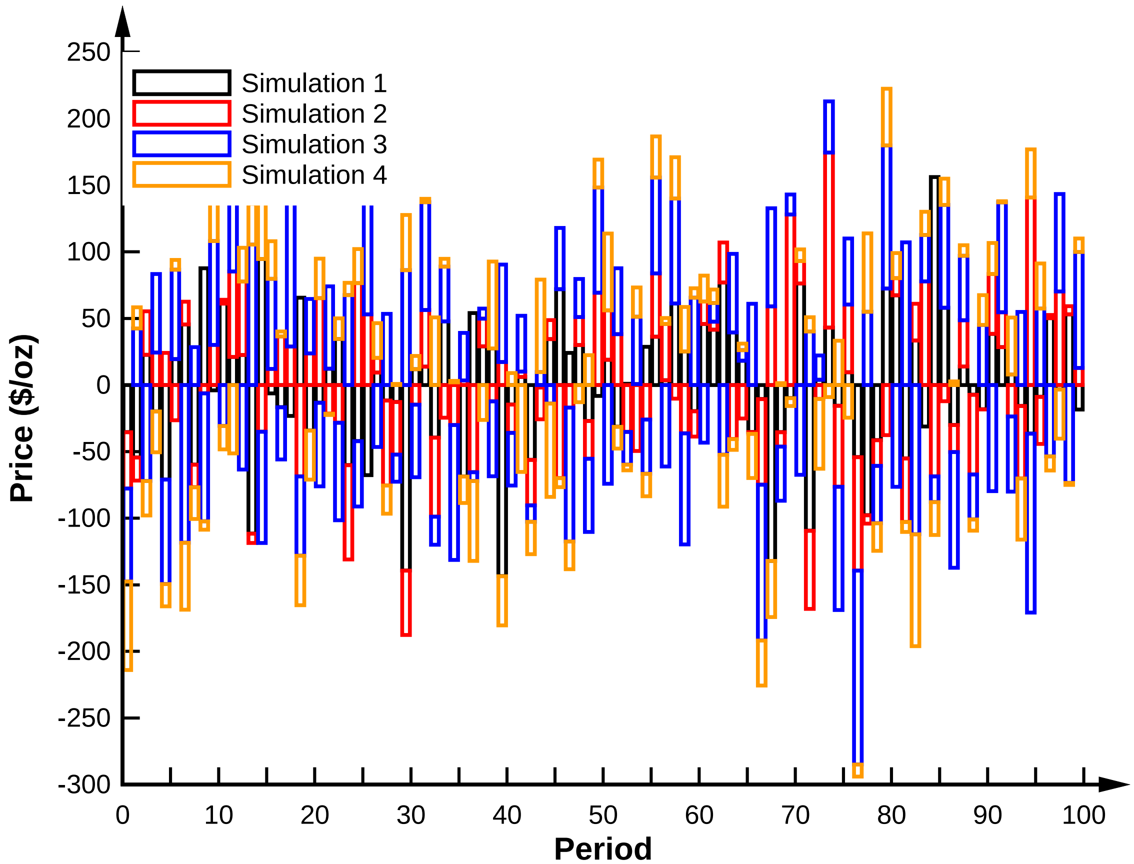
In Figure 46 the deterministic component attenuates to a "long run" value. Meaning that over time, only the random component causes changes in the price of gold. As in the previous scenario if regulation kept the effects from external events low, but in this case there is more competition in the industry. Gold prices would be less effected by intra industry changes and more by inter market events. What makes this simulation different from the one in Figure 44 is the magnitude of the FBM, which is lower causing the range on gold prices to be smaller.

Figure 46 - Simulation of Gold Price (4)



To recap, in Figure 43 the Brownian motion is the larger of the two effects and in Figure 44 the FBM is still the larger of the effects, but the deterministic signal is chaotic. In Figure 44 even though the magnitude of the deterministic effect is small, the amount of cycling (chaotic behavior) still has an impact on the price, however it is minimized by the much larger in magnitude inter market component. In Figures 45 and 46 the deterministic components are larger in magnitude than the random components. In Figure 46 the level of chaotic behavior declines, as there is a significantly larger decrease in cycling behavior. A comparison of the magnitudes of change between the four scenarios is given in the stacked bar graph (Figure 47). All four scenarios were centered around their mean for comparision. From Figure 47 we see that industry consolidation (scenarios 2 & 3) would lead to more chaotic behavior in prices and consequently more up and down changes in the price of gold. This is in contrast to scenarios 1 and 4 where the inter market price has more of an impact on the overall price. In that case we can see swings in the price of gold, but they occur with less regular frequency.

Figure 47 – Scenario Comparison



The importance of being able to simulate the system in accordance with its character, is that it can lead to a better understanding of how changes in industry structure can affect the market price. With simulation based on a more accurate functional form, we can better understand the impact of various changes. We have learned that changes in market structure affect the volatility in the price of gold because it changes the magnitudes of the deterministic and random effects. This type of simulation can help us determine how sensitive an economic system is to both intra and inter market changes.

In the case of the gold industry, currently the inter market signal is so much larger than the intra market signal that the exogenous market impacts override many of the deterministic behaviors in the price signal. If the current conditions of the gold industry continue, we can expect to see gold prices remaining volatile for some time to come.

From the simulation results, we conclude that the market structure of the gold industry will influence the price of gold. As competition in the gold industry decreases, the intra market price becomes chaotic, whether or not it has a large impact on the price is dependent on the size of the inter market component. As competition in the gold industry grows and more firms arise, the intra market price becomes less chaotic and more stable and the market price of gold is influenced more by exogenous events.

Using the information and results derived so far we can construct a linear model of the market for gold in the U.S. that includes the behavior of gold prices. Our results so far have shown that the price of gold has an intra and inter market component. We have tested both components and learned that the intra market component has chaotic behavior and that the inter market component is persistent FBM. We will use a similar functional form for the the movement of gold prices over time as was used in the simulations as this is the appropriate characterization of the movement in the market price of gold over time. With this information we will develop a supply and demand model that can give insight into gold industry events and their impact on the market price of gold.

Constructing the U.S Demand and Supply for Gold

Employing the functional form of both a deterministic component and a random component for gold prices (that has been developed, tested and simulated) a demand and supply curve can be constructed for the U.S. Gold industry, based off of the data from 1970 to 2008. Data prior to 1970 was excluded because of the influence of the regulation of gold prices, which caused a dampening of the motion of the variables. Again we use the logistic function with a random component for the price equation. Price over time is:

$$p_t = \alpha x_{t-1}(1 - x_{t-1})p_{t-1} + \mathbf{B}_H(t-1, \omega)$$

Where:

x = growth rate of price

α = sensitivity of deterministic component

$\mathbf{B}_H$ = FBM

To relate the price with quantity demanded and quantity supplied, linear equations will be used for ease and because the scatter plot of the variables suggest that a linear approximation to be appropriate (Figure 48). The relationship between price and quantity in the phase space is linearly defined as:

$$\text{Demand} \equiv \; p_t = \gamma - \delta q_t$$

$$\text{Supply} \equiv p_t = \iota + \kappa q_t$$

Replacing p_t in the supply and demand equations with the equation for gold price over time gives the following price and quantity relationships:

$$\text{Demand} \equiv q_t = \frac{\gamma - \alpha x_{t-1}(1-x_{t-1})p_{t-1} - \mathbf{B}_H(t-1,\omega)}{\delta} \tag{1}$$

$$\text{Supply} \equiv q_t = \frac{\alpha x_{t-1}(1-x_{t-1})p_{t-1} + \mathbf{B}_H(t-1,\omega) - \iota}{\kappa} \tag{2}$$

Since the logistic function is the growth rate, we can replace $\alpha x_{t-1}(1 - x_{t-1})$ with x_t and solve for equilibrium price and quantity relationships.

Where: $\upsilon = \gamma + \iota$

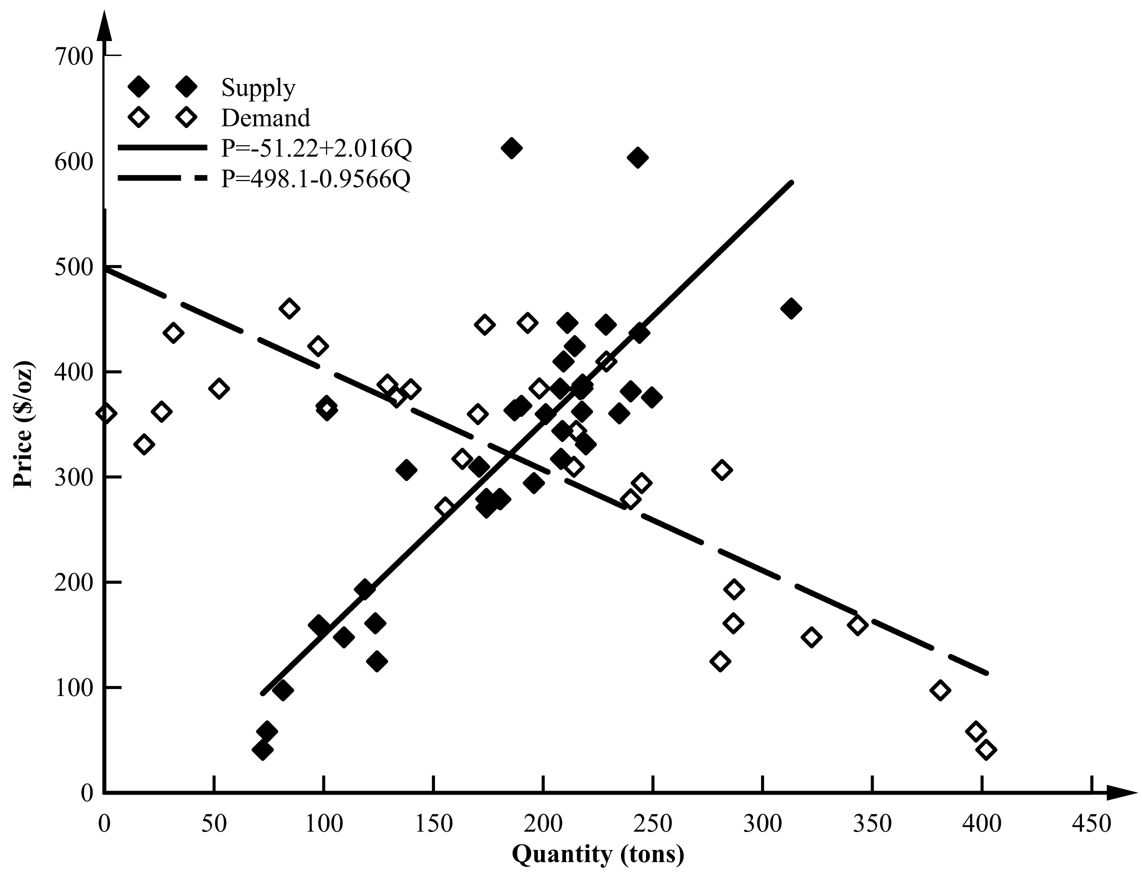
$$p_t^* = \frac{\upsilon - \mathbf{B}_H(t,\omega)}{x_{t+1}} \tag{3}$$

$$\text{Demand} \equiv q_t^* = \frac{\gamma x_{t+1} - \upsilon - \mathbf{B}_H(t,\omega)}{\delta x_{t+1}} \tag{4}$$

$$\text{Supply} \equiv q_t^* = \frac{\upsilon - \iota x_{t+1} - \mathbf{B}_H(t,\omega)}{\kappa x_{t+1}} \tag{5}$$

Some important items to note are that both equilibrium price and quantity are determined by both deterministic growth as well as randomness. Using equations 1 and 2, as well as the intra and inter market signals estimated earlier the points for the price/quantity relationship in the phase space can be plotted in Figure 48. To calculate the demand and supply curves of the U.S. gold industry through 2008 equations 3, 4 and 5 are used to create the lines of best fit through the points via the simplex method. To create the best fit $\gamma,\delta,\iota,\kappa$ and x are all interated where each p* and q* have the smallest mean squared error. The linear demand and supply curves are the fitted p*, q* combinations of best fit to the points from the demand and supply schedules (Figure 48).

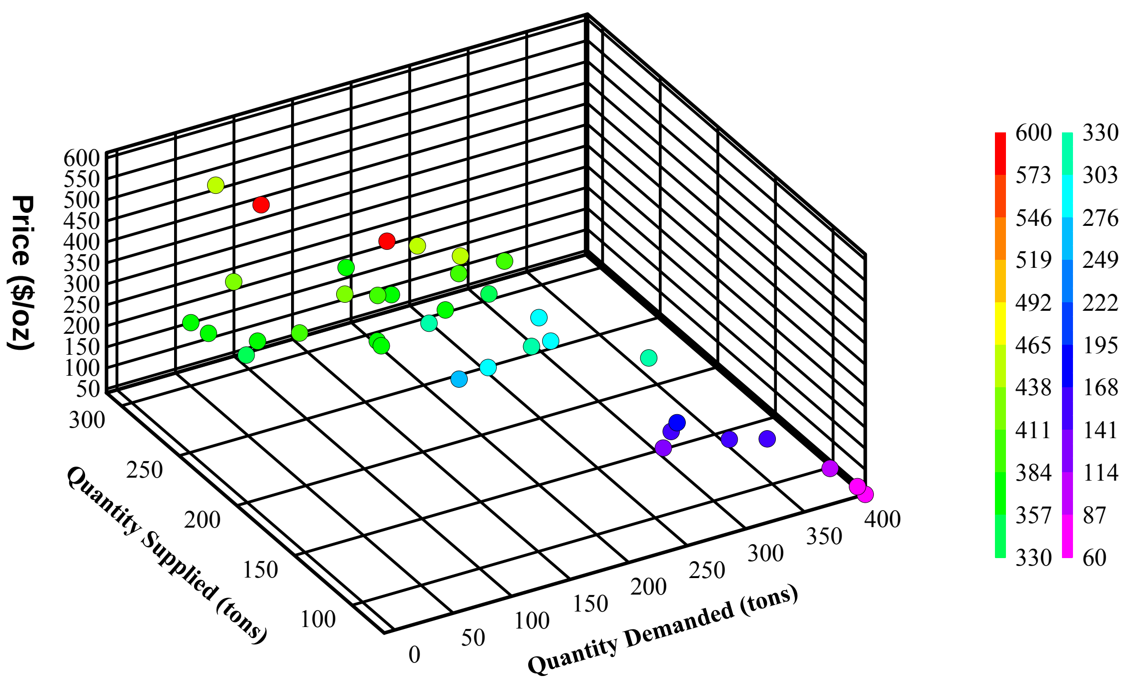
Figure 48 - U.S. Demand and Supply of Gold



As previously determined, the impact from the random component is rather large for the gold industry, therefore it is no surprise that there is some dispersion in the supply and demand points. Even with that dispersion, there is still a strong estimable linear relationship for both curves. From Figure 48, the equlbrium price from 1970 – 2006 is $321.32 and the equilibrium quantity is 184.76 tons per year in the U.S. From the characterization of the gold price, we know that deviations from the equilibrium are in large part caused by random exogenous events. This is due to the magnitude of the inter market price being much greater than that of the intra market price.

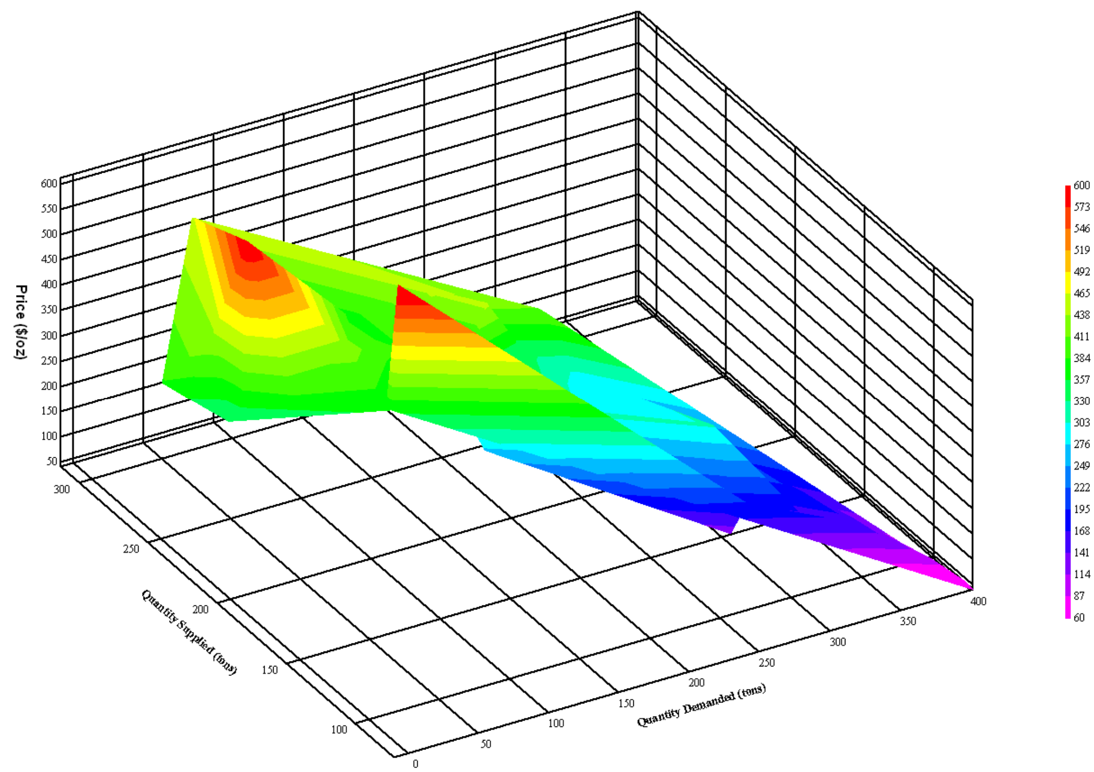
To investigate the equilibrium further, we can look at an attractor plot (Figure 49) of the price, quantity demanded and quantity supplied. Again we can see that randomness is causing dispersion in the equilibriums but that the dimensional space is still relatively close. Meaning that the equilibrium is still being influenced by deterministic behaviors.

Figure 49 - Attractor Plot of: Price and Quantity of Gold



Furthermore, a surface plot (Figure 50) of the three variables (price, quantity demanded, quantity supplied) gives an interpolation of all possible equilibrium points within the system. As economic theory would suggest, the quantity supplied is higher when the price is high and the quantity demanded is high when the price is low. The estimated supply and demand equations concur with economic theory.

Figure 50 - Surface Plot of: Price, Quantity Demanded and Quantity Supplied of Gold



We can also look at equilibrium growth rates from the deterministic component of the price for both the demand and supply equations in the equilibrium. The growth rates are derived by solving p* and q* for x. recall that the growth rates come from the deterministic (intra market) component of the gold price. Solving for the equilibrium growth rates gives the change in quantity demanded or supplied from the sensitivity of the intra market change in price.

$$\text{Equilibrium Quantity Demanded Growth Rate} \equiv x_{t+1}^{*} = \frac{\upsilon(1+\delta)+\mathbf{B}_H(t,\omega)(1-\delta)}{\gamma}$$

$$\text{Equilibrium Quantity Supplied Growth Rate} \equiv x_{t+1}^{*} = \frac{\upsilon(1-\kappa)-\mathbf{B}_H(t,\omega)(1+\kappa)}{\iota}$$

Based on the equations, the equilibrium growth rates are partially dependent on the random component as well as the constant and slope terms from the supply and demand equations. In the case of gold, because the random component is larger, a small change in the inter market effect can cause a large change in the growth (reduction) of the quantity demanded and quantity supplied. The magnitude of the effect of the random price component causes the elasticity of the curves. At the equilibrium over time from 1970 - 2008 the elasticity of demand and supply are:

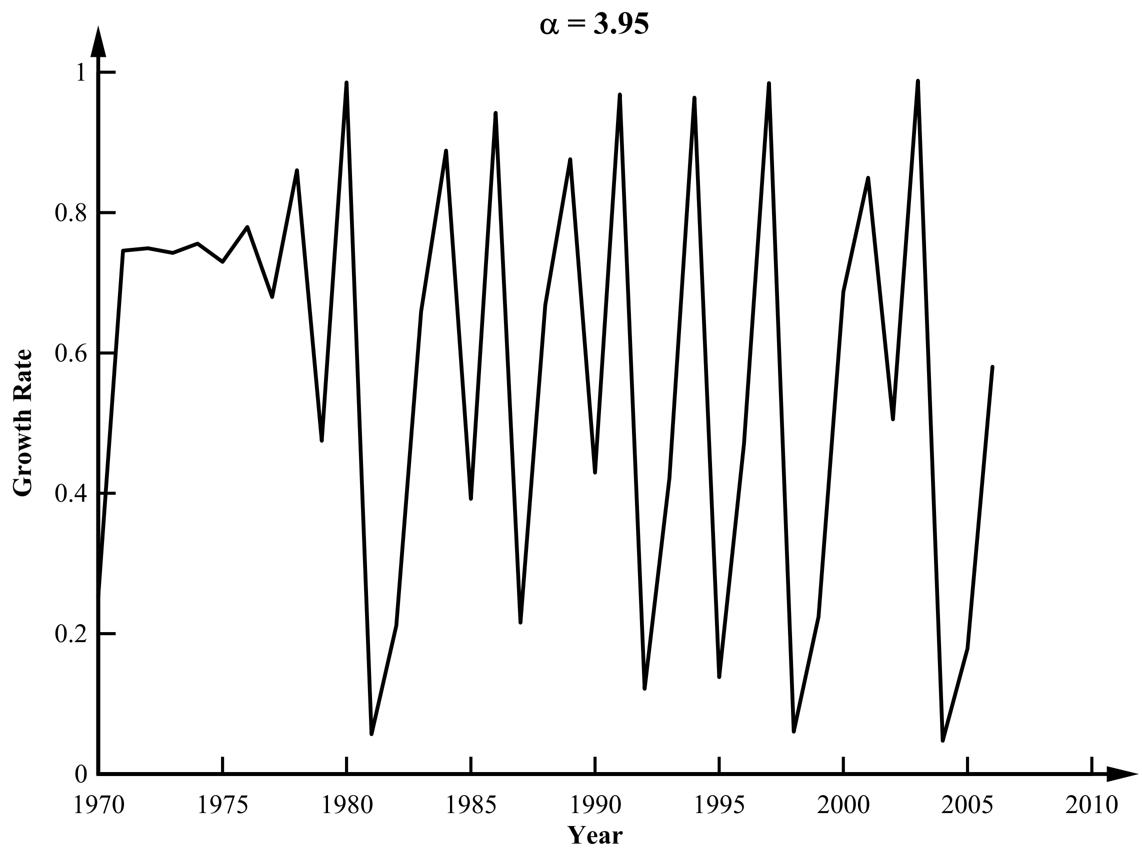
$$Elasticity\ of\ Demand \equiv \varepsilon_d = 1.82$$

$$Elasticity\ of\ Supply \equiv \varepsilon_s = 0.86$$

From an economic theory standpoint, this makes sense. Suppliers of gold are not as responsive to changes in price as demanders. This is because mining gold involves a lot of capital that is not easy to change in a short period of time.

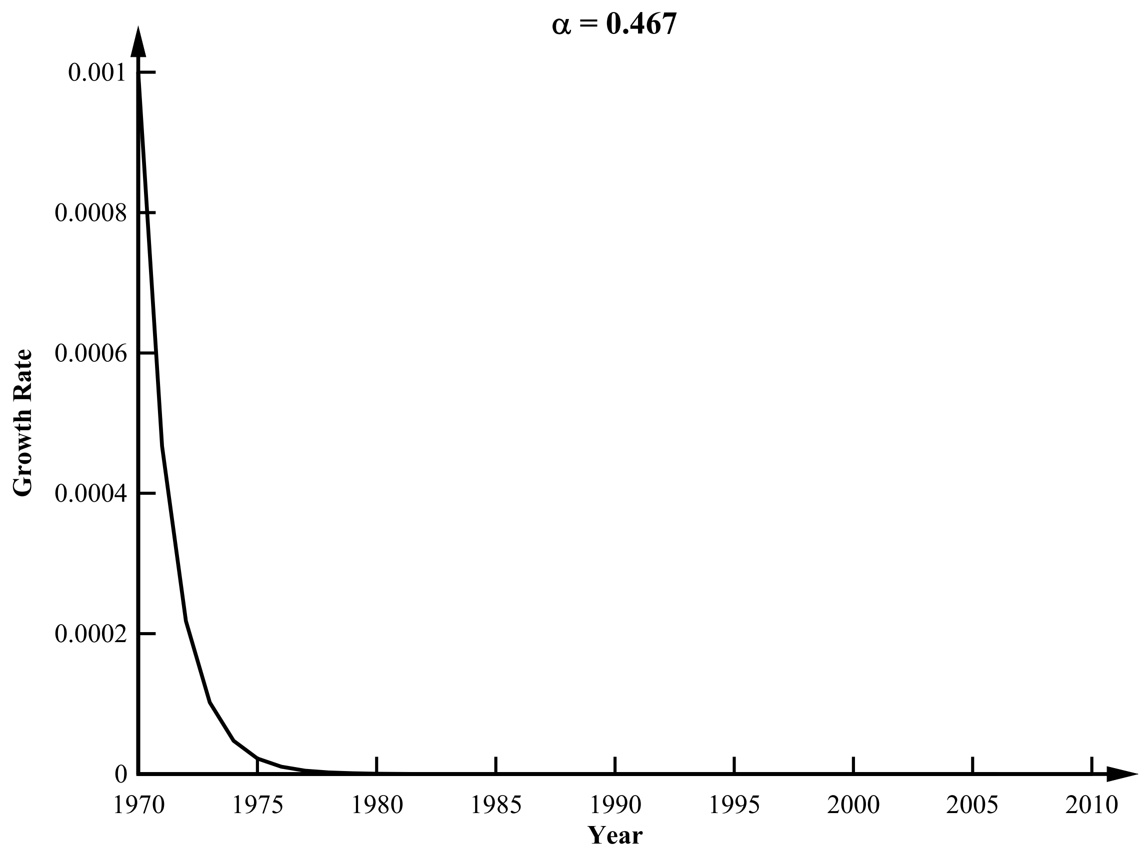
The elasticity measures give some understanding of the sensitivity of quantity demanded and quantity supplied to changes in price. However, the demand and supply growth rates show the volatility of the deterministic part of the price signal over time. The demand growth rate is much larger and more sensitive to random changes than the supply growth rate, as can be seen in Figures 51 and 52. Solving the equilibrium growth rates for the sensitivity coefficient in the logistic function for both demand and supply shows how sensitive or not both series are. The quantity demanded has an alpha of 3.95 whereas the quantity supplied has an alpha of 0.467. The elasticities of both curves are a rough indicator of this behavior, but it is through the use of the logistic equation that we can get a better picture of how volatile the two components are. In Figure 51 we see that the growth in demand for gold has fluctuated since the 1970's due to the chaotic behavior of the system. This means that the demand curve for gold is shifting more often than the supply curve. The changes in the market price of gold come largely from shifts in the demand curve, not the supply curve.

Figure 51 - Equilibrium Growth Rates for U.S. Gold Demand



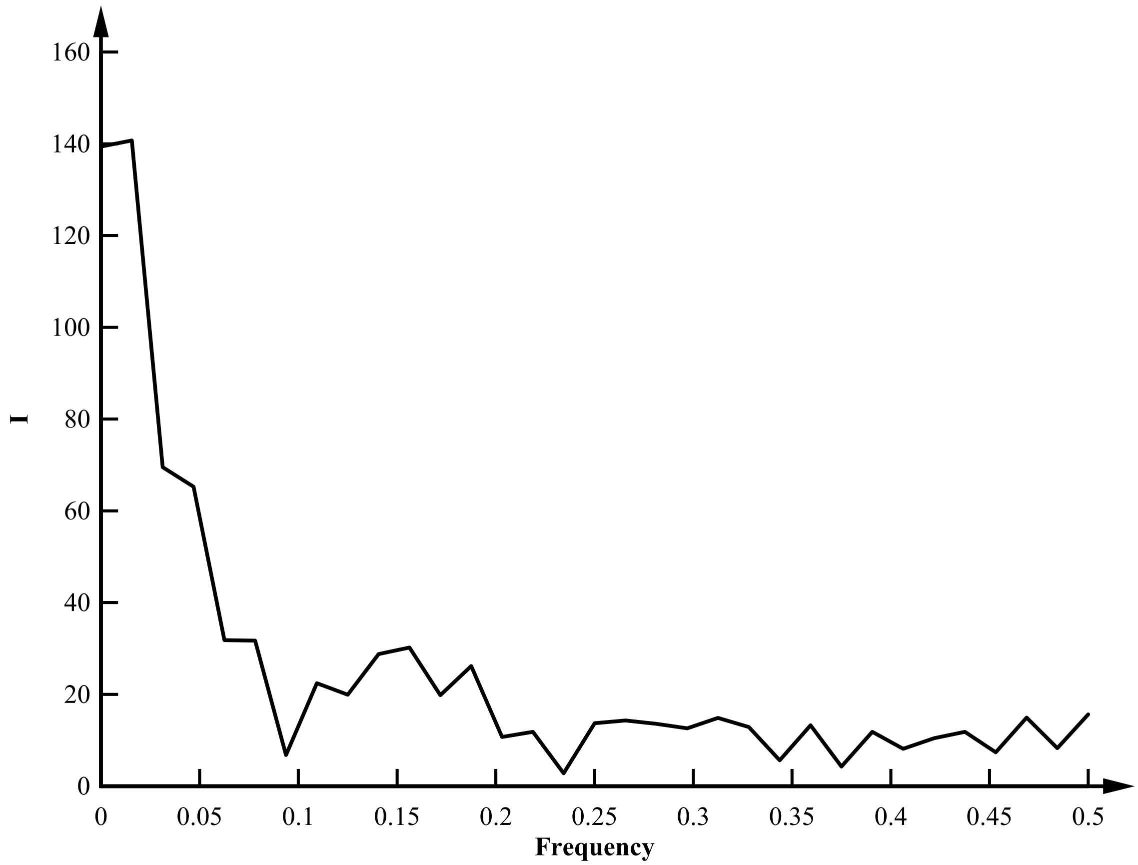
In Figure 52, the growth rate in supply has diminished to near zero since the 1970’s as more mines have opened. The growth rate shows that the deterministic component of price that affects the quantity supplied, has declined to a constant level. With no significant changes in the market structure of the gold industry we should expect that in the future, the growth rate in the supply of gold will continue to fall, internally in the industry.

Figure 52 - Equilibrium Growth Rates for U.S. Supply of Gold



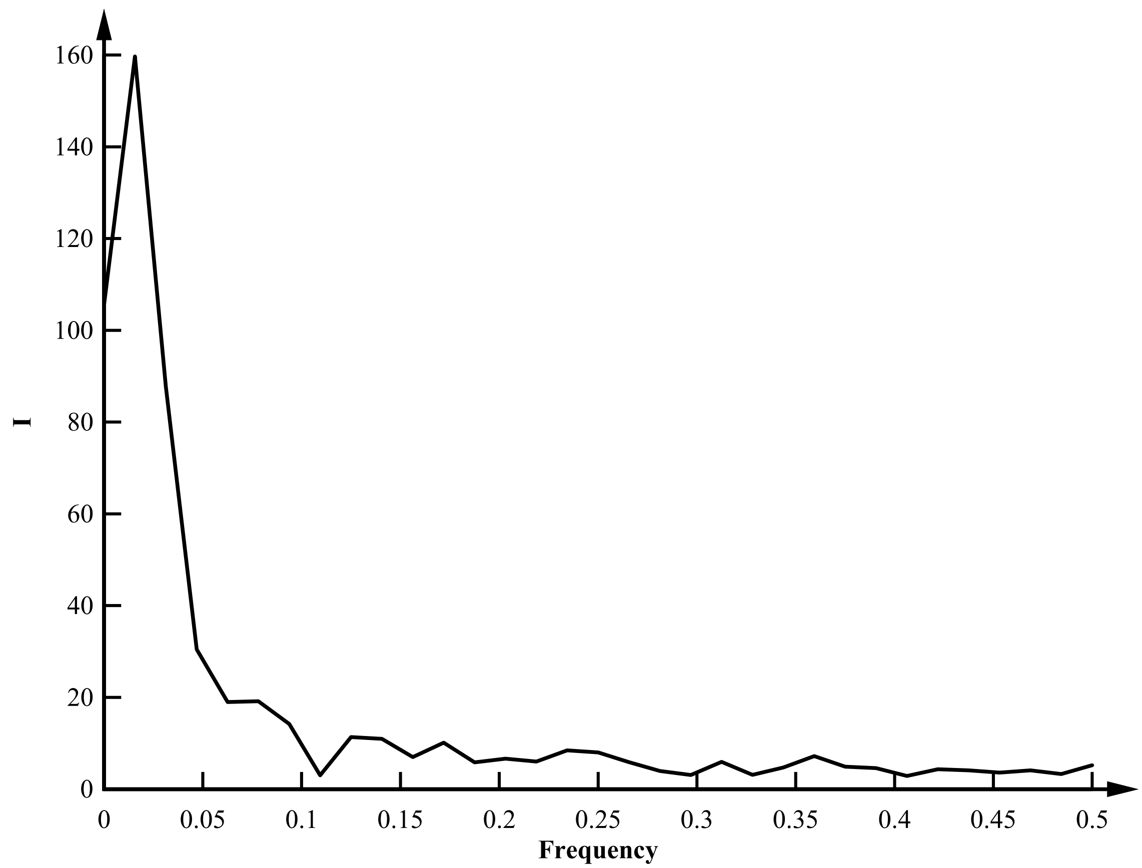
A spectral analysis of the demand (Figure 53) and supply (Figure 54) confirms both behaviors of the growth rates. The spectral analysis decomposes the growth rate into a series of composite wavelengths for the entire signal. In Figure 53 there are many cycles in the spectrum such as the long cycle at 0.02 as well as shorter cycles at 0.15 and 0.36. The spectrum for the growth rate in demand shows many oscillations. The oscillations in the spectrum confer with Figure 51.

Figure 53 - Spectral Analysis of U.S. Gold Demand



In Figure 54 the spectrum of supply growth also shows one long cycle at 0.02 which was seen at the same frequency as in Figure 53. This cycle has such a large time frame that it is irrelavant to our current analysis. After the 0.02 frequency, Figure 54 shows no cycles at all, instead the growth rate is decaying to a constant level.

Figure 54 - Spectral Analysis of U.S Supply of Gold

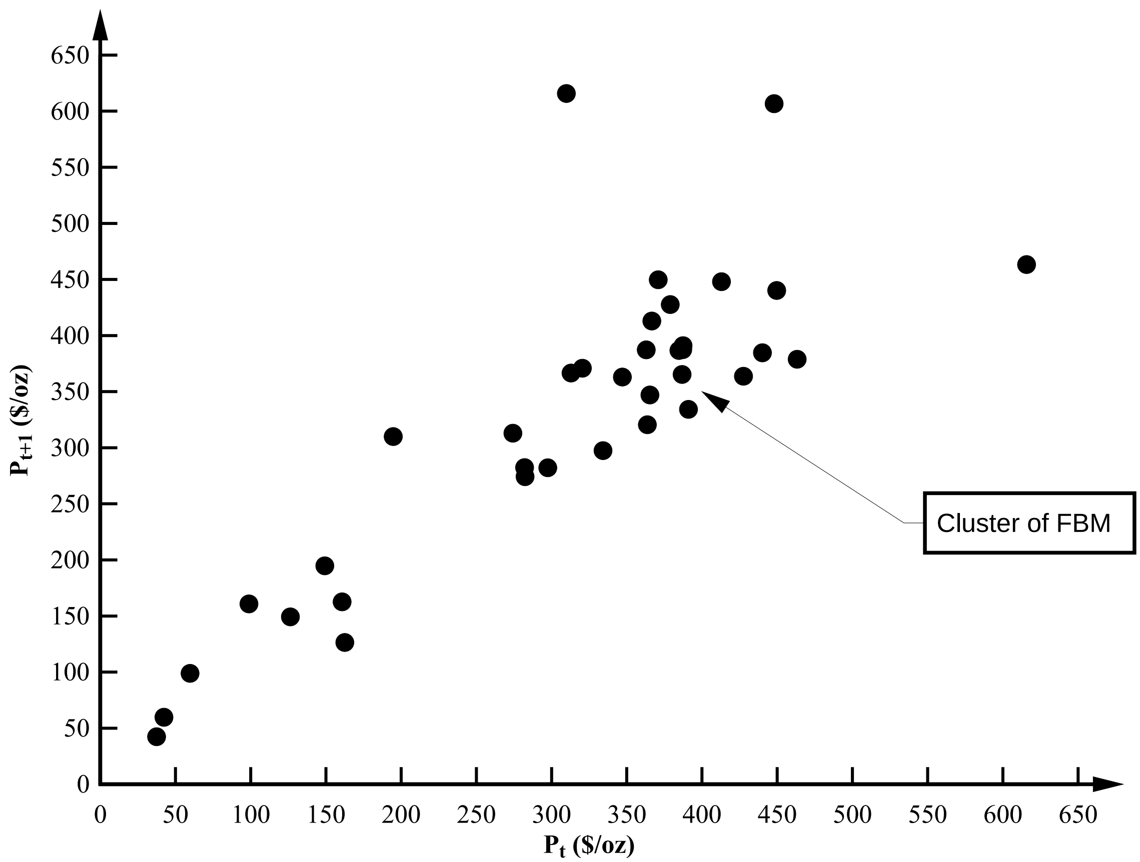


Lastly we need to look at the random component, which in the case of the gold industry is persistent FBM. Solving for an equilibrium amount of the randomness between the quantity demanded and quantity supplied yields the following equation.

$$B_H^*(t, \omega) = \frac{\gamma^2(1-\kappa) - \iota^2(1+\delta) - \gamma\iota(\kappa+\delta)}{\iota(1-\delta) + \gamma(1+\kappa)}$$

An attractor plot of the random component shows that the equilibrium value of the random component is roughly \$350 (Figure 55). Which means that most times, approximatley $350 of the market price of gold comes from inter market events. In 2008 this constituted roughly half of the market price of gold. The clustering of the FBM around $350 shows the magnitude that exogenous events have on the market price of gold. If inter market events lessen in the future they would cause the market price of gold to fall dramatically.

Figure 55 - Attractor Plot for U.S. Inter Market Gold Prices



With the demand and supply analysis complete we draw the following conclusions. Both the demand and supply for gold are sensitive to the inter market randomness that the industry faces. The magnitude of that sensitivity is partially determined by the elasticity of the supply and demand. Over time the demand curve changes more dramatically than the supply curve as seen by the growth rates. This means that changes in demand have a greater impact on changes in the total market price than changes in supply. In the case of gold industry specific regulations, there will not be much impact on the total market price because of the size of the inter market component compared to that of the intra market component. Instead, external changes that influence the system will have more of an impact on equilibrium. Also we know that the quantity demanded changes over time in a chaotic manner, this means that small changes to consumer demand will cause a large change in the market price. The quantity supplied is not chaotic therefore changes in industry structure will not have as significant and impact on the market price as the demand curve. Overall the U.S. market for gold is dominated by inter market events. The inter market events affect the rate of change in the demand curve the most. At this time the industry structure of the gold market has little to do with the market price which again comes predominatley from inter market events. Because the demand side of the gold market is sensitive to changes, it is easily swayed by exogenous events. This causes the market price of gold to be highly volatile and may be why we see the market price of gold change rapidly and often.

CHAPTER 4 – CONCLUSION

In economics, the study of time series analysis has overshadowed the field of dynamic systems.

Recent innovations in computational speed and accuracy have led to the development of many new functional forms and econometric tests. Even with our expanding knowledge, we still find simplified assumptions are being applied to dynamic systems, instead of a global process of first understanding the character of the data and then making assumptions based off of the data. The most common simplifying assumption is the linear model. Because the linear model is additive, behavior which is not additive in nature is lost during estimation. Many times, linear models that do not have a good fit, are said to be random due to the size of the error. Consequently, effects such as chaotic behavior are not addressed when using a linear model for estimation. This leads to a large gap in our understanding of the behavior of the dynamic system being studied.

Dynamic systems describe how a variable or variables change over time. Many economic systems are dynamic in nature, which requires proper characterization of their behavior. To do this we need to distinguish between two basic types of dynamic systems: random and deterministic.

A deterministic system is one in which the behavior of the variable is completely determined across time. A deterministic system can exhibit many behaviors such as: convergence of a system to a single value, convergence to a few values and no convergence at all. Deterministic systems are always in equilibrium. We used the example of a pendulum. In order for the pendulum to swing back and forth it is always dependent on where it was previously, as well as where it is going, in order for it to reverse directions. Because a deterministic system explains a variable over time it is always in equilibrium.

Deterministic dynamic systems that do not converge to a single value or values can give the impression of being random (chaotic). A chaotic system is a condition in which the system appears to be random, but is in fact deterministic. We used the logistic function to demonstrate all varieties of deterministic systems. When the alpha coefficient in the logistic function was low, we saw a system that attenuated to a single value. As the alpha coefficient of the logistic function increased, the behavior of the system showed more oscillations and eventually became chaotic. If a system becomes chaotic, then it is very volatile and sensitive to changes. To identify the equilibriums, and observe deterministic behavior, attractor plots were used. The attractor plots for the deterministic system show the equilibrium or equilibriums of the system.

A random dynamic system is problistic in nature and may or may not be correlated over time. In economics, we usually use RBM as the assumption of error in models or for data that is random. As we have learned, this assumption may not be the case as RBM is a subset of all types of FBM. Randomness can have persistence, anti-persistence or no persistence (RBM) over time. Allowing the possibility for randomness to be correlated over time, has an impact on how a dynamic system evolves. This is why we see random economic variables have some patterns, such as the price of a stock tends to have a series of up movements before a down movement or vice versa.

To characterize a dynamic system, one should first identify if it is deterministic, random or some combination of the two. A proper characterization will identify whether the function that describes the motion is linear or not. Based on this information, one should then develop a function that will more accurately explain the behavior of the dynamic system. Proper characterization can be found through the use of existing tests such as the Hurst Exponent, Lyapunov Exponent and autocorrelation.

The Hurst Exponent has two uses in analyzing a dynamic system. First, the Hurst Exponent tells us if there is persistence or long term dependency in the data. This lets us know how influential history is to the system. We discussed five different methods to determine the Hurst Exponent: re scaled range, autocorrelation, absolute moments method, aggregated variance, and periodogram methods. Long term dependency can be an indicator that the system may not be linear. The Hurst Exponent also tells us what type of randomness is occurring in a system.

The Lyapunov Exponent tells us if the dynamic system has some deterministic components. We can also utilize the Lyapunov Exponent to separate the random from the deterministic, which allows us to derive the intra and inter market pieces of the dynamic system.

Currently, we frequently over use the autocorrelation test based on the assumption that every dynamic system is separable in a linear fashion. A comparison of two different equations, one linear and one multiplicative, showed that the autocorrelations for both appeared to be nearly the same. If a linear time series technique such as ARMA is used, prediction results may be good, however there is still no understanding of how the dynamic system is evolving. Also, there is no measure as to how sensitive the dynamic system is to other economic variables. As demonstrated, large amounts of information are lost if conclusions are drawn solely based on the autocorrelation.

In this paper we have seen how dynamic systems can be used to improve our understanding of economic variables. We used a case study of the gold industry to explore how dynamic systems methods can be applied in economics.

Historically the gold industry has seen many changes. Gold was used in ancient times for religious purposes, and evolved to coinage and monetary standards as well as being a commodity and industrial metal. Current changes to the gold industry include price deregulation in the 1970's and the subsequent industry expansion. Recently the gold industry has seen concentration in the form of increased merger activity in the last decade.

To study the gold industry we first looked at gold production by country. We saw that as deregulation occurred the market for gold became more competitive and some countries lost large shares of the gold market. The price of gold increased significantly after the period of deregulation.

To understand where the changes in the price were coming from, the monthly price of gold was characterized. Performing all tests for the Hurst Exponent showed that long run dependence existed in gold prices. In other words, the price of gold today is dependent on what the price of gold was previously.

We also found there to be deterministic behavior in the price of gold via the Lyapunov Exponent. Knowing that there was some deterministic behavior, we had to identify when the series did not have any linear serial correlation through the ACF. With the two pieces of information about the gold price, the series was split into the deterministic and random components using a space-time regression. The deterministic component of price is the industry (intra) market and the random component of gold price is exogenous (inter) market effects.

We found that the intra market component of the price of gold has decreased in its magnitude of importance over time. The intra market price signal also showed that as competition in the gold industry increased, firms lost market power and the market price of gold became increasingly subject to exogenous factors. This lines up nicely with economic theory in that as an industry gets more competitive, firms become price takers and not price makers. Additionally, the intra market component was found to be chaotic. Meaning that the strategic behavior that is occurring in the gold industry causes the intra market price to be volatile, due to the sensitivity of firms competing in the gold industry. Over the same period in time, we determined the inter market component of gold price has increased in importance and magnitude. We classified the type of randomness in the gold industry to be persistent FBM.

To be able to compare the price signals to production, the same analysis had to be conducted with annual data. The results of the regressions on country production, versus the price signals, showed expected relationships. First, we were able to verify how the inter market signal and production were related. We found that as a country's market share increases, the inter market price falls, causing the intra market price to have a greater impact on the overall price of gold.

From a market structure standpoint, this makes sense. As an industry consolidates, we would expect to see more strategic behavior. The regression on the intra market price and production conferred this same result. Having the intra market signal separated out allows for the study of how individual firms can impact an industry.

In our analysis, we looked at the United States. We saw that as US production of gold increased, so did the intra market price. To show the usefulness of separating the two signals, we contrasted our regressions with a regression on the overall price of gold. We found that the overall regression was unable to discriminate between the deterministic and random effects. This comparison demonstrates that separating out the inter and intra market price provides a more thorough understanding of the behavior of the dynamic system.

Having found significance in both the intra and inter market price signals, we developed a dynamic equation for the changes in gold price over time based on both components. This equation was then used to simulate what would happen to gold prices over time based on changes to the dynamic system. We concluded that as the gold industry consolidates, the price of gold becomes less volatile and less subject to random events. We contrastingly observed that as the gold industry becomes more competitive, the price of gold becomes more volatile and more subject to external events. This is important to our understanding of policy on the gold industry. Regulations that change the industry's structure can cause the price of gold to become more or less volatile depending on whether or not they increase or decrease competition. Also because the current state of the gold industry is one in which the inter market component is larger than the intra market component we can conclude that industry specific regulations on price will have little impact on the overall market price. Using simulation on a dynamic system, after the system has been characterized, can allow for a better understanding of how policy decisions can impact the system. For example, we learned that if there were some policy change, such as anti-trust legislation, which forced the gold industry to become more diluted, the affect of the policy would make the market price of gold more volatile and more subject to changes from random events. The development of an appropriate dynamic equation, can allows us to simulate the consequences of a variety of changes to an economic system.

Finally, we used our dynamic gold price equation to develop supply and demand curves. We found that the US equilibrium price and quantity for gold per year over the last 30 years is $321.32 and 184.76 tons, respectively. We found that the demand curve for gold is elastic and the supply of gold is inelastic. We learned that the inter market signal currently accounts for approximately $350 of the equilibrium price. Since we had a logistic function for the price of gold, we were able to estimate growth rates for the quantity demanded and the quantity supplied. We discovered that the growth rate in the quantity demanded is chaotic and the growth rate in the quantity supplied is continually decreasing. This is important because it identifies the demand curve as the primary cause of volatility in the market price of gold. Because the growth in the quantity demanded is chaotic it is very sensitive to small changes. This is why consumer behavior in the gold markets causes such a large change in the price of gold. Overall we found that the main cause of changes in the market price of gold comes from the inter market price signal, predominately driven by consumers.

As a discipline, there is still a prevalence toward simplifying assumptions to estimate and replicate dynamic economic systems. The over simplification of assumptions leads to large gaps in our understanding and analysis of market structure issues. Overall we have seen how dynamic systems can be used to improve our understanding of economic variables. The use of dynamic systems can lead to the avoidance of many specification and interpretation difficulties. In the case of gold prices, we know that there is a larger portion of the current market price of gold that comes from inter market (random) than from intra market (deterministic) effects. Without a dynamic systems approach we would not be able to reach this conclusion, because we would not be able to characterize the movement in the price of gold over time appropriately. We would also not have been able to modify the supply and demand model to measure the deterministic growth rates of quantity demanded and quantity supplied. Without the growth rates we would not be able to determine the sensitivity of consumers and firms interacting in the gold industry.

Dynamic systems methodologies has a host of different uses in economics. As we have shown, dynamic systems can be used to characterize the evolution of a variable over time. The characterization of an economic variable can allow for a more complete understanding of economic policies as well. Knowing if a policy would make an economic system chaotic or not could be invaluable to understanding the full ramifications of policy.

As we increase our use of dynamic systems in economics, we will be better able to understand how policy decisions and other industry changes will affect industries and markets. It is through a better characterization of how an economic system propagates over time that will allow for a better understanding of the impact of changes to economic systems.

APPENDIX

Re Scaled Range Test for Hurst Exponent

```
% set default values;
clear;
format short;
% import file;
u = csvread(filename);
x = u;
x = x.';
N=length(x);
iRS=[];
RS=[];
for k=1:length(ii)
i=ii(k);
a=floor(N/i);
X=matrix(x(1:a*i),i,a);
ave=mean(X,'r');
mmat=[];
for k=1:i
mmat=[mmat;ave];
end
cumdev=X-mmat;
cumdev=cumsum(cumdev);
rm=max(cumdev,'r')-min(cumdev,'r');
sm=stdev(cumdev,'r');
sm=stdev(X,'r');
ind=find(sm);
if (ind<>[])
iRS=[iRS i];
RS=[RS mean(rm./sm)];
end
end
```

ACF Method of Hurst Exponent

```
% set default values;
clear;
format long;
setprintlimit(10000);
% import file;
u = csvread(filename);
x = u;
% get basic series info
n = length(x);
mx = mean(x);
k = floor(n/6); % calculate variances
var1 = zeros(n*k,2); for (i = 1:n);
var1(i,1) = x(i,1) - mx; var1(i,2) = var1(i,1)^2; end
%get sum of squares
svar1 = sum(var1);
% calc acov and acf
acov = zeros(k,1);
acf = zeros(k,1);
for (j = 1:k);
sumacov = 0;
for (m = 1:n);
sumacov = sumacov + var1(m+j,1) * var1(m,1);
end
acov(j,1) = sumacov;
end
for (j = 1:k);
acf(j,1) = acov(j,1) / svar1(1,2);
end
%create periods
for (i = 1:k);
period(i,1) = i;
end
% Get Logs
lacf = log(acf);
lperiod = log(period);
% Run regression
B = inv((lperiod' * lperiod)) * lperiod' * lacf;
%calc hurst
hurstacf = 1 + B/2
```

AMM Method of Hurst Exponent

```
% set default values;
clear;
format long;
setprintlimit(10000);
% import file;
u = csvread(filename);
x = u;
% get basic series info n = length(x);
mx = mean(x);
k = floor(n/6); % for sufficient amount of lags
nmoment = 24; % enter number of moments to get
for (t = 1:nmoment);
grp = floor(n/t); %set group size
%reset variables
avegrp = 0;
var2 = 0;
rng = 0;
rng1 = 0;
rng2 = 0;
%get average for groups
for (i = 1:grp:n);
rng(i,1) = i;
end
rng1 = nonzeros(rng); % start group
rng2 = rng1 -1; % stop group
tstep = length(rng1);
for (i = 1:tstep-1);
sumx = 0;
for (j = rng1(i,1):rng2(i+1,1));
sumx = sumx + x(j,1);
end
avegrp(i,1) = sumx/grp;
end
% calculate absolute moments
var2 = abs(avegrp - mx);
AM(t,1) = sum(var2)/grp;
moment(t,1) = t;
end
%create logs
lmoment = log(moment);
lAM = log(AM);
%Run regression
```

```
Bamm = inv((lmoment' * lmoment)) * lmoment' * lAM;
%calc hurst
hurstamm = Bamm + 1
```

AVM Method of Hurst Exponent

```
% set default values;
clear;
format long;
setprintlimit(10000);
% import file;
u = csvread(filename);
x = u;
% get basic series info
n = length(x);
mx = mean(x);
k = floor(n/6); % for sufficient amount of lags
%Aggregated Variance for Hurst
nmoment1 = 36; % enter number of moments to get
for (t = 1:nmoment1);
grp = floor(n/t); %set group size
%reset variables
avegrp = 0;
var2 = 0;
rng = 0;
rng1 = 0;
rng2 = 0;
%get average for groups
for (i = 1:grp:n);
rng(i,1) = i;
end
rng1 = nonzeros(rng); % start group
rng2 = rng1 -1; % stop group
tstep = length(rng1);
for (i = 1:tstep-1);
sumx = 0;
for (j = rng1(i,1):rng2(i+1,1));
sumx = sumx + x(j,1);
end
avegrp(i,1) = sumx/grp;
end
% calculate avm
var2 = (avegrp - mx); % need to fix variance method
AVM(t,1) = sum(var2)/grp;
```

```
moment1(t,1) = t;
end
%create logs
lmoment1 = log(moment1);
lAVM = log(AVM);
%Run regression
Bavm = inv((lmoment1' * lmoment1)) * lmoment1' * lAVM;
%calc hurst
hurstavm = Bavm/2 + 1
```

Periodogram Method of Hurst Exponent

```
% set default values;
clear;
format long;
setprintlimit(10000);
% import file;
u = csvread(freemat);
x = u;
% get basic series info
n = length(x);
mx = mean(x);
%add if statement
k = floor((n-1)/2); %odd series
% k = floor(n/2); %even series
%create spectral table
for (i = 1:k);
spectrum(i,1) = i; %calc k
spectrum(i,2) = (2*pi*i)/n; %calc freq
spectrum(i,3) = (2*pi)/spectrum(i,2); %calc period
end;
%find variances
for (i = 1:n);
variances(i,1) = x(i,1) - mx; %obs - mean
variances(i,2) = variances(i,1)^2; % get ss
end;
% get acfs acov
for (j = 1:k);
for (i = 1:k);
variances(i,3) = variances(i+j,1) * variances(i,1);
end;
spectrum(j,4) = sum(variances(:,3))/sum(variances(:,2)); %acf for given k lag
spectrum(j,5) = sum(variances(:,3))/n; %acov for k
end;
```

```
% get a   b fourier coeffs
for (j = 1:k);
coa = 0;
cob = 0;
for (i = 1:n);
coa = x(i,1)*cos(2*pi*j*i/n) + coa;
cob = x(i,1)*sin(2*pi*j*i/n) + cob;
end;
spectrum(j,6) = (2/n)*coa;
spectrum(j,7) = (2/n)*cob;
end;
%get periodogram I
for (i = 1:k);
spectrum(i,8) = (n/2)*(spectrum(i,6)^2 + spectrum(i,7)^2);
end;
%calculate Hurst
lni = log(spectrum(:,8));
lnf = log(spectrum(:,2));
p = polyfit(lnf,lni,1);
Hurst = (1 – p(1))/2
```

Generation of Fractal Brownian Motion

```
% set default values;
clear;
format long;
setprintlimit(10000);
H = .9;
B = 2;
Q = 4;
T = 100;
%calc first step
X(1,1) = randn;
xsub(1,1) = X(1,1);
% create cov vector
for (i=1:T);
r(i,1) = e^(-B^-i);
end
% create weight Vector
for (i=1:T);
W(i,1) = (H*(2*H -1)*(B^(1-H) - B^(-1+H)))/gamma(3-2*H) * B^(-2*(1-H)*i);
end
% generate fbm
for (j=2:T);
for (i=2:T);
xsub(i,1) = W(i,1)*(r(i,1)*xsub(i-1,1) + (1-r(i,1)^2)^.5*randn);
end
X(j,1) = sum(xsub);
end
```

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