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Closure Operations on Subgroups

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ABSTRACT

During the past five years, a number of mathematicians have conducted research involving closure operations on the ideals of commutative rings. The most accessible paper on this is written by Neil Epstein [2], entitled "A Guide to Closure Operations in Commutative Algebra". This paper compiles much of the research done on the topic, and gives the reader an overview of closure operations on ideals, and includes examples, methods for constructions, and various special properties that arise from these operations as they pertain to ideals. However, very little research—if any—has been done on closures of subgroups.

This thesis aims to give a comprehensive overview of closure operations on subgroups, much in the spirit of Epstein's paper. We begin with an introduction to the notion of closure operations on subgroups, as well as providing examples to familiarize the reader with the idea of a "closed" subgroup. We then deviate from following Epstein's work to investigate ways homomorphisms preserve this notion of closures across groups, ultimately arriving at a closed-subgroup equivalent of the First Isomorphism Theorem. We also provide constructions for these operations, as well as include a chapter involving the dual notion of interior operations and how they can interact with closure operations.

In the interest of keeping the thesis self-contained, included is a preliminary chapter containing material from group theory that will resurface during the chap-

ters on closure and interior operations. This section largely focuses on examples to prepare the reader for the material that lies ahead, and provides theory only when appropriate.

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CONTENTS

1

PRELIMINARIES - GROUP THEORY

Since this paper focuses on closure operations on subgroups, it is important to familiarize the reader with the notion of algebraic groups and their properties. This chapter aims to get the reader up to speed by providing the necessary definitions, properties, and examples of group theory. All the material presented here gets used somewhere else in this paper, either in preparation for an important result or as it pertains to the information in subsequent chapters.

Most of the propositions, theorems, and lemmas presented in this chapter are found in [5] and [1]. If something was stated and proved in either of those texts, the proof will not be included here; the proofs provided are for the statements that are given as exercises, or are not found in either source.

1.1 GROUPS AND SUBGROUPS

Naturally, we begin with defining one of the most basic (and, for our purposes, most important) algebraic structures: the group.

Definition 1.1.1. A *group* is a set G together with a binary operation $\cdot : G \times G \to G$, satisfying the following three conditions:

(i) *Associativity*: For any $a, b, c \in G$ we have that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(ii) *Existence of identity element*: There exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$; and

(iii) *Existence of inverses*: For any $a \in G$ there is an element $a^{-1} \in G$ such that $a^{-1} \cdot a = a \cdot a^{-1} = e$.

The $\cdot$ in the above definition is referred to as *group multiplication*, and we often say that G is a *group under* (or *with respect to*) the group multiplication. Oftentimes the group is denoted $(G, \cdot)$, or just G when the binary operation is understood.

Let's look at a few examples (and a couple of non-examples) of groups:

Example 1.1.1. *An example of a group.* Let $G = \mathbb{R}\backslash\{0\}$ and let $\cdot = \times$ (usual scalar multiplication). It is easy to check that this satisfies the three properties given in Definition 1.1.1.

Example 1.1.2. *Another example of a group.* Consider $\mathbb{Z}_9 = \{\overline{0}, \overline{1}, \cdots, \overline{8}\}$ be the integers modulo 9 (we say two integers a, b are *congruent modulo n* if the quantity $a - b$ is divisible by n).

Let $G = U(\mathbb{Z}_9) := \{\overline{m} \mid (m, 9) = 1\} = \{\overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{7}, \overline{8}\}$, and let $\cdot = \times$, where $\overline{a} \times \overline{b} = \overline{a \times b}$. Then $(U(\mathbb{Z}_9), \times)$ is a group:

(i) *Associativity*. Take $\overline{a}, \overline{b}, \overline{c} \in U(\mathbb{Z}_9)$. Then

$$\overline{a} \times (\overline{b} \times \overline{c}) = \overline{a} \times (\overline{bc}) = \overline{abc} = (\overline{ab}) \times \overline{c} = (\overline{a} \times \overline{b}) \times \overline{c}.$$

(ii) *Existence of identity element*. $\overline{1}$ serves as the identity element, as for all $\overline{a} \in U(\mathbb{Z}_9)$ we have that

$$\overline{a} \times \overline{1} = \overline{a \times 1} = \overline{a} = \overline{1 \times a} = \overline{1} \times \overline{a}.$$

(iii) *Existence of inverses*. This can be found by calculating $\overline{a} \times \overline{b}$ for each $\overline{a}, \overline{b} \in U(\mathbb{Z}_9)$.

For example, $(\overline{2})^{-1} = \overline{5}$, since $\overline{2} \times \overline{5} = \overline{5} \times \overline{2} = \overline{10} = \overline{1}$.

Similarly, $(\overline{4})^{-1} = \overline{7}$, and so on.

So, $(U(\mathbb{Z}_9), \times)$ is a group.

For any integer n, the group $(U(\mathbb{Z}_n), \times)$ is known as the *multiplicative group* (or the *primitive residue classes*) *of integers modulo n*.

By going through the work in part (iii) of Example 1.1.2, we notice the following about elements and their inverses, which holds for any group:

Proposition 1.1.1. *Let $(G, \cdot)$ be a group. Then the inverse elements are unique; that is, for each $a \in G$ there exists exactly one $a^{-1} \in G$ satisfying property (iii) in Definition 1.1.1.*

Example 1.1.3. *A non-example of a group.* Let $G = \mathbb{N}$ and let $\cdot$ be scalar multiplication. We can see that associativity holds. Also, $1 \in G$, so that $1 \cdot n = n \cdot 1 = n$ for all $n \in G$. However, for all $n \in G$ with $n \neq 1$ there is no $m \in G$ such that $n \cdot m = m \cdot n = 1$ (the multiplicative inverse of n is $\frac{1}{n}$, which regretfully is not an element of $\mathbb{N}$). Hence, $\mathbb{N}$ is not a group under scalar multiplication.

Oftentimes, when talking about a group G, we write the group multiplication as ab instead of $a \cdot b$ when it is understood what the group multiplication is.

The following definition seems excessive; however, it is important to be aware that each group falls into one of two distinct categories:

Definition 1.1.2. A group G is *Abelian* (or *commutative*) if, for all $a, b \in G$, we have that $ab = ba$. Otherwise, we say G is *non-Abelian* (or *non-commutative*).

Throughout this text, groups will be referred to as "Abelian" rather than "commutative". This preference is due largely in part to the fact that *rings* satisfying such a property are always referred to as commutative; there is no such thing as an "Abelian ring". However, in the context of groups the two terms are completely interchangeable.

Example 1.1.4. *An example of an Abelian group*. Let G be as in Example 1.1.1. Then G is an Abelian group, since for all $a, b \in G$ we have that $a \cdot b = b \cdot a$.

Example 1.1.5. *A non-Abelian group*. Let $G = GL_2(\mathbb{Z})$ be the set of all 2×2 invertible matrices with coefficients in $\mathbb{Z}$ with respect to matrix multiplication.

Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Then $ab = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, but $ba = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq ab$.

So, $GL_2(\mathbb{Z})$ is not Abelian.

Definition 1.1.3. Let G be group under some binary operation $\cdot$, and let H a nonempty subset of G. We say H is a *subgroup of* G if H itself forms a group under the group multiplication, and we denote this as $H < G$.

Example 1.1.6. Let $G = \mathbb{Z}$ under $+$. Then $H = 2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ forms a subgroup of $\mathbb{Z}$:

Show associativity: This holds because associativity holds in G.

Show existence of identity element: 0 is the identity element of G.

$0 = 2 \cdot 0$, so 0 is indeed an element of H.

Show existence of inverses: Take $2n \in H$.

Then $-2n = 2(-n) \in H$ as well, so H contains the inverses of all its elements.

Thus $H < G$.

An important property of subgroups is the following proposition. The proof is omitted; however it can be found in [1] (pg. 47).

Proposition 1.1.2. *(The Subgroup Criterion). Let G be a group. Then a nonempty subset H of G is a subgroup of G iff for any* $a, b \in H$ *we also have that* $ab^{-1} \in H$.

It is also possible to "create" subgroups of a group G out of pre-existing ones. The following proposition illustrates one such way to go about doing this.

Proposition 1.1.3. *Let* $H, K < G$*. Then* $H \cap K < G$.

Note, however, that even if $H, K < G$, it is not necessarily the case that $H \cup K < G$:

Example 1.1.7. Let $G = \mathbb{Z}_6$ under addition (see Example 1.1.2). Let $H = \{\overline{0}, \overline{2}, \overline{4}\}$ and $K = \{\overline{0}, \overline{3}\}$. H and K are indeed subgroups of G.

Then $H \cup K = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$. But notice that $a, b \in H \cup K$ does not guarantee that $a + b \in H \cup K$; indeed, take $\overline{2}$ and $\overline{3}$. Then $\overline{2} + \overline{3} = \overline{2+3} = \overline{5} \notin H \cup K$.

Remark. When we have a group G and a subset $H \subseteq G$, and if it is the case that $a, b \in H$ does not imply that $ab \in H$ as in Example 1.1.7 (though it may be that all other properties of a subgroup are satisfied), then H cannot be a subgroup of G and we say that H is *not closed with respect to the group operation* (that H is not a subgroup follows from Definition 1.1.1).

It should be noted that a group G always has two subgroups—the subgroup $\{e\}$ and the group G itself—so it is meaningful to discuss subgroups in a general sense, as opposed to with respect to certain group structures.

In other words, any group G always has no fewer than two subgroups.

Definition 1.1.4. A subgroup H of G is called *trivial* if $H = \{e\}$. A subgroup H of G is *proper* if $H \neq G$ and $H \neq \{e\}$.

When discussing properties of subgroups, it is more interesting to look at the behavior of the proper subgroups. However, these properties also hold for the subgroups $\{e\}$ and G.

There are some subgroups have special properties with respect to the group itself. One such property is listed here:

Definition 1.1.5. Let $H < G$. We say H is *normal* in G if for all $g \in G$ we have that $gHg^{-1} = H$, and we denote this $H \triangleleft G$.

Note that any subgroup of an Abelian group is normal: let H be a subgroup of an Abelian group G. Then, for any $g \in G$, we have that $ghg^{-1} = g(hg^{-1}) = g(g^{-1}h) = (gg^{-1})h = h$ for all $h \in H$.

Another way to "create" subgroups involve using the elements of the group themselves:

Definition 1.1.6. Let a be an element in a group G. We define the *cyclic subgroup generated by a*, denoted $\langle a \rangle$, by

$$\langle a \rangle = \{a^n \mid n \in \mathbb{N}\}.$$

Similarly, for any set of elements $a_1, a_2, \ldots, a_k \in G$, we can define the *subgroup generated by $a_1, a_2, \ldots, a_k$* by

$$\langle a_1, a_2, \ldots, a_k \rangle = \{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k} \mid n_1, \ldots, n_k \in \mathbb{Z}\}.$$

A group G is said to be *finitely generated* if it is generated by finitely many elements $a_1, a_2, \ldots, a_k$ of G. We say, in this case, that G has k *generators*, and the set $\{a_1, a_2, \ldots, a_k\}$ is called the *generating set* of G.

A quick little property is the following:

Proposition 1.1.4. *Let a_1, a_2 be elements in a group G. Then $\langle a_1 \rangle < \langle a_1, a_2 \rangle$ (likewise, $\langle a_2 \rangle < \langle a_1, a_2 \rangle$).*

We can continue this via induction to obtain that any subgroup H generated by $n \in \mathbb{N}$ elements is contained in a subgroup generated by the generating set of H and appending an additional $m \in \mathbb{N}$ elements. In other words, it is always the case that

$$\langle a_1, a_2, \ldots, a_n \rangle < \langle a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m} \rangle,$$

with the a_{n+i} are distinct from the a_j, ($i = 1, \ldots m$ and $j = 1, \ldots, n$).

This topic will be utilized more in-depth in Section 1.2.

Definition 1.1.7. The *order* of a group G is the cardinality of G as a set; i.e. the order of G is $|G|$. We say G is *finite* if $|G|$ is finite, and *infinite* otherwise.

Similarly, the order of a subgroup H of G is just $|H|$.

Definition 1.1.8. $HK = \{hk \mid h \in H, k \in K\}$.

It is clear that $H \subseteq HK$ and $K \subseteq HK$. Moreover, when it is the case that HK forms a subgroups of G, then $H < HK$ and $K < HK$ (since each element $h \in H$ can be written as $he \in HK$; a similar argument holds for $K < HK$). Furthermore, if $K < H$, then $HK = H$ (see Proposition A.1.4).

1.2 NOETHERIAN GROUPS

The following proposition is found in [5], pg. 74 (presented there as Corollary 1.7); consequently its proof will be omitted.

Proposition 1.2.1. *Let G be a finitely generated Abelian group, with n generators. Then every subgroup $H < G$ is finitely generated by no more than n elements.*

Note here that "finitely generated Abelian group" means exactly what it sounds like: G is both finitely generated and Abelian. Sometimes a group will be defined as being *finitely generated Abelian* (e.g. "Suppose a group G is finitely generated Abelian, and..."); this just means that G is a finitely generated Abelian group.

Definition 1.2.1. A group G is *Noetherian* (or sometimes called *slender*) if it satisfies the ascending chain condition; that is, any ascending chain $\{e\} < H_1 < H_2 < \cdots$ of subgroups of G eventually terminates after finitely many iterations. In other words, if $\{e\} < H_1 < H_2 < \cdots < H_m < \cdots$ is a chain of subgroups of G, then $\exists$ $n \in \mathbb{N}$ such that $\{e\} < H_1 < H_2 < \cdots < H_{n-1} < H_n$ and $H_n = H_{n+k}$ $\forall$ $k \geq 1$.

Note that if G is any finite group, then the ascending chain condition (oftentimes abbreviated ACC) will always hold (since G has finitely many subgroups). So, any finite group G is Noetherian.

In fact,

Proposition 1.2.2. *Let G be a finitely generated Abelian group. Then G is Noetherian.*

It should be noted that the "finitely generated" assumption is not enough, and the premise that our group is Abelian is crucial; the following example illustrates this.

Example 1.2.1. *An example of a non-Noetherian finitely generated group.*

Let $\mathbb{F}_2 = \langle a, b\rangle$ be the *free group on two generators*, meaning every element of $\mathbb{F}_2$ is of the form $w = a^{n_1}b^{m_1}a^{n_2}b^{m_2}\cdots a^{n_k}b^{m_k}$, and there is no torsion among the elements a^i and b^j.

$\mathbb{F}_2$ is an infinite non-Abelian group: to see that $\mathbb{F}_2$ is non-Abelian, note that for $w_1 = ab$ and $w_2 = a^2b$, $w_1w_2 = aba^2b \neq a^2bab = w_2w_1$; to see that it is infinite, notice that $\langle a\rangle < \mathbb{F}_2$ and $\langle a\rangle$ is infinite, and hence so is $\mathbb{F}_2$.

Now consider the commutator subgroup $[\mathbb{F}_2, \mathbb{F}_2]$ of $\mathbb{F}_2$ (see Section 1.5.3). Note that $[\mathbb{F}_2, \mathbb{F}_2]$ is not finitely generated; this is immediate from the following claim:

Claim. $[\mathbb{F}_2, \mathbb{F}_2] = \langle [a^m, b^n] \mid m, n \in \mathbb{Z} \rangle$.

(The proof of this claim is given in Appendix A, Proposition A.1.6).

And so the generating set S of $[\mathbb{F}_2, \mathbb{F}_2]$ is countably infinite.

Take $x_1 = [a^{n_1}, b^{m_1}] \in S$. We can see that $\langle x_1 \rangle < [\mathbb{F}_2, \mathbb{F}_2]$.

Now take another element $x_2 = [a^{n_2}, b^{m_2}] \in S \backslash \{\langle x_1 \rangle\}$. Then we have that $\langle x_1, x_2 \rangle < [\mathbb{F}_2, \mathbb{F}_2]$, and subsequently $\langle x_1 \rangle < \langle x_1, x_2 \rangle$.

We can continue picking out these subgroups of $[\mathbb{F}_2, \mathbb{F}_2]$ to form the chain of subgroups

$$\{e\} < \langle x_1 \rangle < \langle x_1, x_2 \rangle < \langle x_1, x_2, x_3 \rangle < \cdots < \langle x_1, x_2, \ldots, x_k \rangle < \cdots$$

This chain does not satisfy the ACC by construction; we are able to find such a chain of subgroups by exploiting the fact that $\mathbb{F}_2$ contains a subgroup with infinitely many generators (despite having finitely many generators itself). So, $\mathbb{F}_2$ is a non-Noetherian, but finitely generated, group.

Similarly, it is not enough to assume that our group is only Abelian.

Example 1.2.2. *A non-Noetherian Abelian (but infinitely generated) group.*

Let $G = \mathbb{Q}$ under addition. It is clear that $\mathbb{Q}$ is Abelian, but not finitely generated; indeed, suppose $S = \{\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_k}\}$ is the (finite) generating set of $\mathbb{Q}$. But then $\frac{1}{a_1 a_2 \ldots a_k + 1} \in \mathbb{Q}$ is not in $\langle S \rangle$, so $\mathbb{Q}$ is not finitely generated.

Take $\frac{1}{2^k} \in \mathbb{Q}$ for each $k \in \mathbb{N}$. Then the following chain does not satisfy the ACC:

$$\{e\} < \left\langle \frac{1}{2} \right\rangle < \left\langle \frac{1}{4} \right\rangle < \left\langle \frac{1}{8} \right\rangle < \cdots < \left\langle \frac{1}{2^k} \right\rangle < \cdots$$

So $\mathbb{Q}$ has a non-ending chain of subgroups, and hence is not Noetherian.

Since it is the case that "finitely generated Abelian" $\not\Leftrightarrow$ "Noetherian", it would be good practice to provide examples of groups that are Noetherian, but do not satisfy the former property.

A simple example is as follows:

Example 1.2.3. *An example of a Noetherian non-Abelian group.* Any finite non-Abelian group will do, but we give an explicit example here:

Let G be the symmetric group on three letters

$$S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}.$$

Then this group is clearly Noetherian.

Since we have a foolproof condition for determining whether a group G is Noetherian, it's only fair that we have one to find when G is not Noetherian. The next proposition does just that.

Proposition 1.2.3. *Let G be an infinitely generated group. Then under no circumstance is G Noetherian.*

At last, these examples allow us to establish a definitive criterion for Noetherian groups:

Theorem 1.2.4. *Let G be a group. Then G is Noetherian iff every subgroup of G is finitely generated.*

The proof of this theorem is extremely straightforward: if G is Noetherian but contains an infinitely generated subgroup K, then any chain containing K terminates after finitely many iterations. In particular, any chain ending with K has finitely many subgroups in it, and so K must have been finitely generated (see Example 1.2.1). The converse is precisely Proposition 1.2.2.

1.3 GROUP HOMOMORPHISMS

As interesting as it is to talk about groups and their various subgroups and properties, it is also important to talk about mappings from one group to another. However, not any map from groups to other groups will get the job right; the particular mapping that works is known as the *homomorphism*, and needs to be defined accordingly.

Definition 1.3.1. Let G and G' be two groups, and $\varphi : G \to G'$. Then φ is a *homomorphism* if $\forall\ g, h \in G$ we have that $\varphi(gh) = \varphi(g)\varphi(h)$. In other words, φ is a homomorphism if it preserves group structure.

Oftentimes, the term *group homomorphism* will be used; this is to indicate that both the image and preimage of φ are groups (as opposed to specifying beforehand).

Naturally, homomorphisms are functions. So, we say that a homomorphism φ is a *monomorphism* if φ is also one-to-one; an *epimorphism* if φ is onto; and an *isomorphism* if φ is a bijection. If φ maps a group to itself, then φ is an *endomorphism*, and an endomorphism that is also an isomorphism is called an *automorphism*.

If there is an isomorphism between two groups H and G, then we say that G and H are *isomorphic* and we denote this as $G \simeq H$ (read "G is isomorphic to H").

We now proceed to see how subgroups interact with homomorphisms.

Proposition 1.3.1. *Let G, G' be groups and $\varphi : G \mapsto G'$ a homomorphism. If $H < G$ then $\varphi(H) < G'$. Furthermore, if φ is a monomorphism, then $\varphi(G) \simeq G$.*

The first statement follows from the fact that a homomorphism φ maps the identity element of G to the identity element of G'. Indeed, if e is the identity element of G and e' the identity element of G', then for $g \in G$ we have that

$$\varphi(h) = \varphi(eh) = \varphi(e)\varphi(h)$$

because φ is a homomorphism. From here we see that $\varphi(e) = e'$.

The isomorphism in the second statement is immediate once the injective property is imposed on φ, as $G \mapsto \varphi(G)$ is already onto.

Definition 1.3.2. Let $\varphi : G \mapsto G'$ be a group homomorphism. Then the *kernel of* φ is the set $\ker \varphi = \{g \in G \mid \varphi(g) = e \in G'\}$.

It is clear from the discussion following the previous proposition that $\ker \varphi$ is nonempty, as $e \in \ker \varphi$. In fact, we have that

Proposition 1.3.2. *For any group homomorphism $\varphi : G \mapsto G'$, ker φ is a subgroup of G.*

The following proposition is really just a list of properties of all homomorphisms. The proofs are omitted as they are very straightforward to see.

Proposition 1.3.3. *The following are facts about homomorphisms:*

(i) for any $g \in G$, $\varphi(g^n) = (\varphi(g))^n \ \forall \ n \in \mathbb{Z}$;

(ii) φ is a monomorphism $\iff$ ker $\varphi = \{e\}$; and

(iii) If φ, ψ are homo-/mono-/endo-/isomorphisms, then so are $\varphi \circ \psi$ and $\psi \circ \varphi$.

We end this section with a couple of examples:

Example 1.3.1. *An explicit example of a homomorphism.* Let $G = \mathbb{R}$ under scalar addition and $G' = \mathbb{R}^+$ under scalar multiplication. Define $\varphi : G \to G'$ by $\varphi(x) =$

e^x. Then φ is a homomorphism: take $x, y \in \mathbb{R}$. Then $\varphi(x+y) = e^{x+y} = e^x e^y = \varphi(x)\varphi(y)$.

Although it now seems clear thanks to the previous example, it should be noted that it is of utmost importance that a homomorphism obeys the binary operations on G and G'.

Example 1.3.2. *An abstract example of a homomorphism.* Let G be a group, and let $\varphi : G \to G$ be defined by $\varphi(g) = g^2$. Then φ is a homomorphism (more precisely, φ is an endomorphism) $\iff$ G is Abelian.

To see this, first suppose φ is a homomorphism. By definition of φ, $\varphi(ab) = (ab)^2 = abab$. On the other hand, since φ is a homomorphism, $\varphi(ab) = \varphi(a)\varphi(b) = a^2b^2$. Then $abab = a^2b^2$ for any $a, b \in G$ and hence G is Abelian.

Similarly, suppose G is Abelian. Then the computations above give us the equality $\varphi(ab) = \varphi(a)\varphi(b)$, so that φ is a homomorphism.

1.4 NORMAL SUBGROUPS CONTINUED

1.4.1 *Order and Cosets*

Definition 1.4.1. The *order* of an element $a \in G$ is the smallest positive integer $n \in \mathbb{Z}$ such that $a^n = e$ and this will be denoted by $|a|$.

Note that $|a| = |\langle a \rangle|$.

Definition 1.4.2. Let $H < G$, and $a \in G$. Then the *right coset* of H in G is the set $aH = \{ah \mid h \in H\}$. Similarly, the *left coset* of H in G is $Ha = \{ha \mid h \in H\}$.

Note that if G is not Abelian, we may not have that $aH = Ha$ for some subgroup H.

Example 1.4.1. (See [5], pg. 40). Let $G = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$, and let H be the subgroup generated by (1 2). Take $a = (1\ 3)$. Then $aH = \{(1\,3), (1\,2\,3)\}$, but $Ha = \{(1\,3), (1\,3\,2)\}$.

In fact, no left coset of H is also a right coset. As another example, take $b = (2\,3)$. Then $bH = \{(2\,3), (1\ 3\ 2)\}$ while $Hb = \{(2\,3), (1\,2\,3)\}$.

Note. Here, G is known as the *symmetric group on* 3 *letters*, and is denoted S_3. S_n is defined similarly for any natural number n.

This example illustrates two points: left and right cosets are *subsets* of G, not *subgroups*; and, more importantly, left and right cosets appear to be disjoint; from Example 1.4.1, we see that $aH \neq bH$ and $Ha \neq Hb$. Indeed,

Proposition 1.4.1. *Let $H < G$. Then any two left (right) cosets of H are either disjoint or equal.*

Proof. This is proven in [1], pg. 80 (proof of Proposition 4). ☐

Another important fact that we need is

Proposition 1.4.2. *For a subgroup $H < G$, then the number of left cosets of H in G is equal to the number of right cosets.*

Although this intuitively seems that it would be true in the case where G is finite, it is not necessarily obvious if G is infinite. The proof is provided in the Appendix as Proposition A.1.9 for the general case.

The final result given in this subsection is

Proposition 1.4.3. *Let H, K be subgroups of a group G. Then HK is a subgroup of G $\iff HK = KH$.*

1.4.2 *Characteristic Subgroups*

Along with normal subgroups, another special class of subgroups is known as the *characteristic subgroups*. Don't let the shortness of this section fool you: these subgroups will play an important role in Chapter 2, when we investigate the interaction between closure operations and homomorphisms. So, let's first introduce ourselves to this notion of characteristic subgroups.

Definition 1.4.3. Let $H < G$. We say H is *characteristic in* G (or that H is a *characteristic subgroup of* G), and denote this "H char G" as in [1], if for all automorphisms ψ of G we have that $H = \psi(H)$.

Remark. [5] presents Definition 1.4.3 a bit differently: $H < G$ is said to be *characteristic* if for all automorphisms φ of G we have that $\varphi(H) < H$. Notice that these two definitions yield the same result at the end of the day, and so we will use them interchangeably when it is most convenient to do so.

Also, note that any group G has at least two characteristic subgroups: the trivial subgroup and the group itself. So, it is meaningful to use the notion of characteristic subgroups in a general sense.

1.4.3 *Normal Subgroups (Revisited) and Quotient Groups*

Recall that a subgroup N of G is *normal* if $gNg^{-1} = N$ for every $g \in G$ (Def. 1.1.5). We can now characterize this notion of a normal subgroup in terms of cosets of N in G.

Theorem 1.4.4. *Let $N < G$. Then the following are equivalent:*

(i) *$gN = Ng$ for all $g \in G$;*

(ii) *For all* $g \in G$, $gNg^{-1} \subset N$;

(iii) $gNg^{-1} = N$.

So, this theorem allows us to say that a subgroup N of G is *normal* in G if every right coset of N is also a left coset.

Theorem 1.4.5. *Let* $N \triangleleft G$ *and let* G/N *denote the set of all (left) cosets of* N *in* G*. Then* G/N *is a group under the binary operation given by* $(aN)(bN) = abN$ *and* $|G/N| = [G:N]$.

When N is as in Theorem 1.4.5, we say that the group G/N is called the *quotient* (*factor*) *group* of G by N. The notation G/N is read as "G mod (modulo) N".

We will now present a result that enhances that of Proposition 1.3.2:

Proposition 1.4.6. *Let* $\varphi : G \to G'$ *be a group homomorphism. Then* $\ker \varphi \triangleleft G$.

Lemma 1.4.7. *Let* $K, N < G$ *with* $N \triangleleft G$*. Then*

(i) $(N \cap K) \triangleleft K$;

(ii) $N \triangleleft (N \vee K)$;

(iii) $NK = N \vee K = KN$; *and*

(iv) *If* $K \triangleleft G$ *and* $K \cap N = \langle e \rangle$, *then* $nk = kn$ *for all* $k \in K$, $n \in N$.

One final result about normal subgroups:

Proposition 1.4.8. *Let* $\varphi : G \to G'$ *be an isomorphism. If* $N \triangleleft G$, *then* $\varphi(N) \triangleleft G'$.

1.5 SOME IMPORTANT SUBGROUPS OF G

We conclude this chapter with a look at a few specific subgroups created by elements of a group G. These are important enough to warrant their own section;

they come up again in subsequent chapters, so getting the necessary basics out of the way here is essential. The proofs of the propositions, on the other hand, are a burden and can be found in section A.1.

1.5.1 *The Normalizer of a Subgroup*

Definition 1.5.1. Let G be a group, and take $H < G$. We define the *normalizer* of H in G, denoted $N_G(H)$, by

$$N_G(H) = \{g \in G \mid \; gHg^{-1} = H\}.$$

Note that if it is the case that $N_G(H) = G$, then H is, by definition, normal in G. Also note that $N_G(H)$ is never empty, as $H \subseteq N_G(H)$.

Proposition 1.5.1. *$N_G(H)$ is a subgroup of G.*

Proposition 1.5.2. *(Properties of the Normalizer). The following are properties of $N_G(H)$, where H is taken to be any subgroup of G:*

(i) $H < N_G(H)$;

(ii) $H \lhd N_G(H)$; *and*

(iii) $H \subseteq K$ *and* $H \lhd K \implies K \subseteq N_G(H)$.

1.5.2 *The Join of Two (or More) Subgroups*

Definition 1.5.2. Let $K, H < G$. We define the *join* of H and K, denoted as $H \vee K$, by

$$H \vee K = \langle H \cup K \rangle.$$

In other words, $H \vee K = \langle H \cup K \rangle$ is generated by the elements in H and K.

An element x of $H \vee K$ is of the form $x = h_1^{r_1} k_1^{s_1} h_2^{r_2} k_2^{s_2} \cdots h_m^{r_m} k_m^{s_m}$, where $h_i \in H$, the $k_i \in K$, and r_i, $s_i \in \mathbb{Z}$ for all $i = 1, \ldots, m$.

We may, for all intents and purposes, consider the join to be the way we can take "unions" of subgroups and get another subgroup. What makes this definition successful in creating a subgroup, and the union a relative failure, is the fact that the join is *generated* by the elements of both H and K, and not merely just containing them.

Proposition 1.5.3. *Take $H, K \subseteq G$. Then $H \vee K = \langle H \cup K \rangle < G$.*

Of course, we can generalize the definition of the join to include any number of subgroups of G; consequently, the join of an arbitrary number of subgroups of G is again a subgroup of G.

Another fact of the join is

Proposition 1.5.4. $H \vee K = \bigcap \{ J < G \mid H < J \text{ and } K < J \}$.

1.5.3 *The Commutator Subgroup*

Definition 1.5.3. Let G be a group. The *commutator subgroup of* G, denoted $[G, G]$, is

$$[G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle.$$

Here, the quantity $aba^{-1}b^{-1}$ is known as a *commutator* of G, and is written $[a, b]$.

Proposition 1.5.5. *$[G, G]$ is in fact a subgroup of G.*

1.5.4 *The Stabilizer of a Group Action*

Before we get into the subgroup given in this section (and the next), we present a preliminary definition:

Definition 1.5.4. A *group action* of a group G on a set $S \subseteq G$ is a map from $G \times S \to S$ that satisfies the following:

(i) $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s \ \forall \ g_1, g_2 \in G$ and $\forall \ s \in S$; and

(ii) $e \cdot s = s \ \forall \ s \in S$.

Oftentimes, we just say that G *acts on* S.

In most cases, we can write the group action as gs, rather than $g \cdot s$, when there is no confusion regarding the binary operation bestowed upon the group.

Definition 1.5.5. Let G be a group that acts on a set S. We define the *stabilizer of* G *at* s to be the set

$$G_s := \{g \in G \mid gs = s\}$$

(sometimes G_s is called the *subgroup fixing* s).

True to its parenthetical name, we have that

Proposition 1.5.6. $G_s < G$.

A similar definition and result is obtained by looking at the stabilizers of subgroups of G at s (Example 3.1.5 takes advantage of this notion).

1.5.5 *The Normal Closure & Normal Core of a Subgroup*

Definition 1.5.6. Let G act on S via right conjugation (i.e. $g \mapsto g^{-1}sg$). The *normal* (or *conjugate*) *closure of S of G*, denoted $\langle S^G \rangle$, is generated by the set

$$S^G = \{g^{-1}sg \mid g \in G, s \in S\}.$$

As you may have come to expect,

Proposition 1.5.7. *The following are true:*

(i) $\langle S^G \rangle < G$. *In fact,* $\langle S^G \rangle \triangleleft G$*; and*

(ii) Suppose $N \triangleleft G$. *Then* $N = \langle N^G \rangle$.

Another fact about the normal closure is that it is the smallest normal subgroup containing S. This is seen by the following:

Proposition 1.5.8. *Let* $S \subseteq G$. *Then* $\langle S^G \rangle = \bigcap H_i$, *where* $S \subseteq H_i$ *and* $H_i \triangleleft G \ \forall\ i$.

A dual notion to the normal closure is that of the *normal core*:

Definition 1.5.7. The *normal core of a subgroup H of G* with respect to a subset S of G is defined to be

$$Core_S(H) = \bigcap_{s \in S} s^{-1}Hs.$$

As expected,

Proposition 1.5.9. *Take* $S = G$ *in the previous definition. Then the following hold for the normal core:*

(i) $Core_G(H) \triangleleft G$;

(ii) If $N \triangleleft G$, *then* $N = Core_G(N)$*; and*

(iii) $Core_G(N)$ *is the largest normal subgroup of* G *contained in* N.

2

CLOSURE OPERATIONS

This chapter is the main focus of this paper. The material from sections 1, 4, and 5 stems from the research done by Epstein (and others), as it applies to groups, with properties and examples being modified accordingly.

The topics covered in sections 2 and 3, however, are an investigation of the nature in which closure operations and homomorphisms interact with one another. This investigation is not done in vain—homomorphisms play such an essential role in the theory of groups and, as we shall see, even the First Isomorphism Theorem has a closed-subgroup analogue.

2.1 CLOSURE OPERATIONS

We, of course, begin with the foundational knowledge of closure operations. The majority of the information in this section is a direct translation of the material found in Section 2 of [2].

Definition 2.1.1. Let G be a group. A *closure operation cl* on a set of subgroups S of G is a set map $cl : S \to S$, $H \mapsto H^{cl}$, satisfying the following three properties:

(i) (*Extension*). $H \subseteq H^{cl}$ for all $H \in S$;

(ii) (*Idempotence*). $H^{cl} = (H^{cl})^{cl}$ for all $H \in S$; and

(iii) (*Order-Preserving*). If $K \subseteq H$, where $H, K \in S$, then $K^{cl} \subseteq H^{cl}$.

If S is the set of all subgroups of G, we say cl is a *closure operation on* G.

A subgroup $H \in S$ is *cl-closed* if $H = H^{cl}$. We say G is *fully closed under cl* if every subgroup of G is cl-closed.

To familiarize ourselves with this notion, let's look at several examples of closure operations on a group G. These examples will come back in subsequent sections and chapters.

Example 2.1.1. (*Identity*). $id : S \to S$, where $H \mapsto H$, is a closure operation. In fact, any subgroup H of a group G is cl-closed under this closure operation.

Example 2.1.2. (*Trivial closure*). Define the *trivial closure* $cl_G : S \to S$ by $H \mapsto G$.

Proposition 2.1.1. *The trivial closure is a closure operation*

Proof. Show extension: $H \subseteq H^{cl_G} = G$ always.

So extension holds.

Show idempotence: We can see that $G = H^{cl_G}$.

Then $(H^{cl_G})^{cl_G} = G^{cl_G} = G$, so indeed $H^{cl_G} = (H^{cl_G})^{cl_G}$.

So idempotence holds.

Show order-preserving: Let $K \subseteq H$.

Then $K^{cl_G} = G \subseteq G = H^{cl_G}$.

So, the trivial closure is in fact a closure operation. □

Remark. The idea of the trivial closure cl_G can be used to define other "trivial closure"-like operations on specific sets of subgroups of G. To do so, let S be a collection of subgroups of G such that $\forall\ H \in S\ \exists\ M < G$ such that $H < M$. We can then define cl_M, which we will call the *trivial closure restricted to* M, to be such that $H^{cl_M} = M\ \forall\ H \in S$. This idea is nothing novel, since this is essentially more of a ”localized” trivial closure (this notion of trivial closures restricted to subgroups will come into play in Section 2.5).

Example 2.1.3. (*Join*). Fix some subgroup $K \in S$. Then $H \mapsto H \vee K$, called the *join of H with respect to K*, is a closure operation:

Proof. Show extension: $H \subseteq H \vee K$,

since $e \in K \implies he = h \in H \vee K$ for all $h \in H$.

So extension holds.

Show idempotence: Again, we need only show

$(H \vee K) \vee K \subseteq H \vee K$.

Take $x \in H \vee K$.

Then there exists $a_1, a_2, \ldots, a_k \in (H \vee K) \cup K$ and

$r_1, r_2, \ldots, r_k \in \mathbb{Z}$ such that $x = a_1^{r_1} \cdots a_k^{r_k}$.

If $a_i \in K$, there is nothing to show.

If $a_i \in H \cup K$, then there are $b_{i_1}, \ldots, b_{i_m} \in H \cup K$

and $s_{i_1}, \ldots, s_{i_m} \in \mathbb{Z}$ such that $a_i = b_{i_1}^{s_{i_1}} \cdots b_{i_m}^{s_{i_m}}$

Thus x is a product of powers of a_i and b_{i_j}, with

$a_i, b_{i_j} \in H \cup K$ for all i, j.

Thus $x \in H \vee K$.

So idempotence holds.

Show order-preserving: Take $J \subseteq H$. Show $J \vee K \subseteq H \vee K$:

$x \in J \vee K \implies$ there exist $a_i \in J \cup K$,

$i = 1, \ldots, n$, such that $x = a_1^{r_1} \cdots a_n^{r_n}$ for $r_1, \ldots, r_n \in \mathbb{Z}$.

So $a_i \in J \cup K \implies a_i \in J$ or $a_i \in K$.

$a_i \in K$ tells us nothing, but $a_i \in J \implies a_i \in H$ (since $J \subseteq H$).

So $x = a_1^{r_1} \cdots a_n^{r_n} \in \langle H \cup K \rangle = H \vee K$.

So the join with respect to a subgroup K is a closure operation. ∎

When talking generally about this closure operation, we will say that G is *closed with respect to the join*. In this case, it will be understood that we are taking the join with respect to a fixed $K < G$.

Example 2.1.4. (*Normal closure*). Take $H < G$. Then $H \mapsto \langle H^G \rangle$, called the *normal closure of H in G*, is a closure operation:

Note. By Proposition 1.5.7 $\langle H^G \rangle < G$.

Proof. *Show extension*: $H \subseteq \langle H^G \rangle$, since $h = e^{-1}he \in H^G \subseteq \langle H^G \rangle$

for all $h \in H$.

So extension holds.

Show idempotence: This follows immediately by applying

Proposition 1.5.7(ii) to Proposition 1.5.7(i).

So idempotence holds.

Show order-preserving: Let $J \subseteq H$, and take $j \in \langle J^G \rangle$.

Then $j = (g_1^{-1} j_1 g_1) \cdots (g_n^{-1} j_n g_n)$, with $g_i \in G$ and $j_i \in J$

for all i.

$J \subseteq H \implies j_i \in H$ for all $i = 1, \ldots, n$.

Therefore $j \in \langle H^G \rangle$.

So the normal closure of H in G is a closure operation. ∎

One may be tempted to not bother with the normal closure of H in G, and instead would look only at the (arguably "simpler") normalizer of H in G (see Definition 1.5.1). To our dismay, however, the normalizer does not determine a closure operation as illustrated in Example 2.3.2.

Example 2.1.5. (*Radical*). Take a subgroup H of a Noetherian Abelian group G (hence H is finitely generated), and define the *radical of H*, which we will denote $\text{rad}(H)$, by

$$\text{rad}(H) := \{g \in G \mid \exists\, n > 0 \text{ s.t. } g^n \in H\}.$$

(This definition is just a direct translation of the radical of an ideal).

Then $\text{rad}(H)$ defines a closure operation:

Proof. To see $\text{rad}(H) < G$, consult Proposition A.2.1.

Show extension: This is clear.

So extension holds.

Show idempotence: Only need $\text{rad}(\text{rad}(H)) \subseteq \text{rad}(H)$.

Take $h \in \text{rad}(\text{rad}(H))$.

Then $\exists\ n > 0$ such that $h^n \in \text{rad}(H)$.

$\implies \exists\ m > 0$ such that $(h^n)^m = h^{mn} \in H$.

$\implies h \in \text{rad}(H)$ since $mn > 0$.

So idempotence holds.

Show order-preservation: This is also clear.

If $J \subseteq H$, then $j \in \text{rad}(J) \implies j^n \in J \subseteq H \implies j \in \text{rad}(H)$.

So the radical defines a closure operation on Abelian groups. ☐

The radical can be used for arbitrary Abelian groups G. The focus on finitely generated Abelian groups here allows us to construct the radical in two different ways; see Example 2.5.5 in Section 2.5.

Following the example set by [2], we have the following properties of cl:

Proposition 2.1.2. *Let G be a group, and cl a closure operation on G. Let $H < G$ and $\{H_i\}_{i \in I}$ a collection of subgroups of G. Then*

(i) If each H_i is cl-closed, so is $\bigcap_i H_i$;

(ii) $\bigcap_i H_i^{cl}$ is cl-closed; and

(iii) H^{cl} is the intersection of all cl-closed subgroups that contain H.

Proof. Show (i): By extension, we know $\bigcap_i H_i \subseteq (\bigcap_i H_i)^{cl}$.

Show $\supseteq$: Note that $\bigcap_i H_i \subseteq H_j\ \forall\ j$.

Order-preservation $\implies (\bigcap_i H_i)^{cl} \subseteq H_j^{cl} = H_j$ since H_j cl-closed.

This holds $\forall\, j \implies (\bigcap_i H_i)^{cl} \subseteq \bigcap_i H_i$.

Show (ii): Again, by extension we have $\bigcap_i H_i^{cl} \subseteq (\bigcap_i H_i^{cl})^{cl}$.

Show $\supseteq$: Know $\bigcap_i H_i^{cl} \subseteq H_j^{cl}\ \forall\, j$.

$\implies (\bigcap_i H_i^{cl})^{cl} \subseteq (H_j^{cl})^{cl} = H_j^{cl}$ by idempotence.

True $\forall\, j \implies (\bigcap_i H_i^{cl})^{cl} \subseteq \bigcap_i H_i^{cl}$.

Show (iii): Let K be a cl-closed subgroup containing H.

Then by order-preservation and idempotence, $H^{cl} \subseteq K$.

This holds for all such $K \implies H^{cl}$ is contained in such an intersection.

But then H^{cl} is also one of these such K, so equality follows. □

2.2 CLOSURE OPERATIONS AND HOMOMORPHISMS

Let $\varphi : G \to G'$ be a group homomorphism. We already know by definition that φ preserves group structure. This then begs the question: when does φ also preserve closures? In other words, what conditions must φ (and perhaps even the closure itself) satisfy in order to send closed subgroups of G to closed subgroups of G'? This is the question we seek to answer in this section.

In the interest of keeping things simple (and probably reasonable), we only consider the case in which groups are endowed with the same closure operation. With this in mind, we open with a definition and an example.

Definition 2.2.1. Let $\varphi : G \to G'$ be a group homomorphism, and cl a closure operation on both G and G'. We say that *φ preserves cl on S*, where S is some set of subgroups of G, if $\forall\, H \in S$ we have that $\varphi(H^{cl})$ is cl-closed.

If $\varphi(H^{cl})$ is cl-closed $\forall\ H < G$, we will say that φ *preserves cl on* G, or simply just that φ *preserves cl*.

Example 2.2.1. *Homomorphisms preserve the identity closure.* Again, let $\varphi : G \to G'$ be a group homomorphism, and let cl be the identity closure on G and G'.

Then φ preserves this closure.

Proof. This is extremely straightforward; indeed, since $H < G$, we
have that $\varphi(H) < G'$.

Then $H^{cl} = H \mapsto \varphi(H) = (\varphi(H))^{cl}$ since the identity closure
is a closure operation on both groups.

Hence φ preserves the identity closure. □

It turns out that $\varphi : G \to G'$ needs more than just being a homomorphism; indeed, the following is a non-example of φ preserving a closure operation on both G and G':

Example 2.2.2. *Non-epimorphisms do not preserve the trivial closure.* Consider $\mathbb{Z}$ and $\mathbb{Q}$ under addition, and let $\varphi : \mathbb{Z} \to \mathbb{Q}$ be defined by $\varphi(z) = z$. It is clear that φ is a homomorphism.

Take $H < \mathbb{Z}$ (any subgroup will do). Then $H^{cl} = \mathbb{Z}$.

But then $\varphi(H^{cl}) = \varphi(\mathbb{Z}) = \mathbb{Z} \neq \mathbb{Q}$; that is, $\varphi(H^{cl}) \neq \mathbb{Q}$ for any subgroup H.

So homomorphisms that are not epimorphisms do not preserve the trivial closure.

In light of this example, we can now insist that our homomorphisms be onto. This update in the criteria does not affect the first example; it is easy to see that the identity closure is preserved under epimorphisms.

This also agrees with the following proposition, found in [4]:

Proposition 2.2.1. *Let $f : G \to H$ be a nonzero homomorphism mapping a group G to a simple group H. Then f is an epimorphism $\iff$ $\langle K^{f(G)} \rangle = f(G)$ for all nontrivial subgroups K of $f(G)$.*

Proposition A.2.2 provides a proof of this.

At the end of the day, it would seem perfectly reasonable to think that *isomorphisms* are the ones to preserve closure operations—they do, after all, preserve nearly everything else about a group's structure. However, it is not the case that an isomorphism $\varphi : G \to G'$ will preserve any closure operation $cl\ \forall\ H < G$; a counterexample will be given later on in this section.

Before we get too ahead of ourselves, let's be reminded that we are searching for criteria in which φ *will* preserve cl. To simplify matters further, we will first look at how automorphisms (isomorphisms between G and itself) interact with closure operations. Once we get an idea what's going on there, we will then turn our attention to isomorphisms between two groups G and G'.

We continue along with our examples:

Proposition 2.2.2. *Let $\psi : G \to G$ be an automorphism. Then ψ preserves the normal closure for all $H < G$.*

Proof. We want to see that $\psi(\langle H^G \rangle)$ is closed in G.

By extension, $\psi(\langle H^G \rangle) \subseteq (\psi(\langle H^G \rangle))^{cl} = \langle (\psi(\langle H^G \rangle))^G \rangle$.

We just need the other containment.

Take $x \in \langle (\psi(\langle H^G \rangle))^G \rangle$.

Then $x = (g_1^{-1} p_1 g_1)(g_2^{-1} p_2 g_2) \cdots (g_n^{-1} p_n g_n)$, where $p_i \in \psi(\langle H^G \rangle)$ $\forall\ i = 1, \ldots, n$.

ψ automorphism $\implies$ $p_i = \psi(g_{i_1}^{-1} h_{i_1} g_{i_1} \cdots g_{i_m}^{-1} h_{i_m} g_{i_m})$
$= {g'_{i_1}}^{-1} \psi(h_{i_1}) g'_{i_1} \cdots {g'_{i_m}}^{-1} \psi(h_{i_m}) g'_{i_m}$,

where m is taken to be the largest index of the lengths

of the p_i.

So $x = g_1^{-1}(g_{1_1}'^{-1}\psi(h_{1_1})g_{1_1}' \cdots g_{1_m}'^{-1}\psi(h_{1_m})g_{1_m}')g_1 \cdots$
$g_n^{-1}(g_{n_1}'^{-1}\psi(h_{n_1})g_{n_1}' \cdots g_{n_m}'^{-1}\psi(h_{n_m})g_{n_m}')g_n.$

Note that $g_1^{-1}(g_{1_1}'^{-1}\psi(h_{1_1})g_{1_1}'g_{1_2}'^{-1}\psi(h_{1_2})g_{1_2}' \cdots g_{1_m}'^{-1}\psi(h_{1_m})g_{1_m}')g_1$

$$= g_1^{-1}g_{1_1}'^{-1}\psi(h_{1_1})g_{1_1}'(g_1g_1^{-1})g_{1_2}'^{-1}\psi(h_{1_2})g_{1_2}'(g_1g_1^{-1}) \cdots \cdot (g_1g_1^{-1})g_{1_m}'^{-1}\psi(h_{1_m})g_{1_m}'g_1$$
$$= (g_{1_1}'g_1)^{-1}\psi(h_{1_1})(g_{1_1}'g_1)(g_{1_2}'g_1)^{-1}\psi(h_{1_2})(g_{1_2}'g_1) \cdots \cdot (g_{1_m}'g_1)^{-1}\psi(h_{1_m})(g_{1_m}'g_1)$$
$$\in \psi(\langle H^G \rangle).$$

This holds $\forall\ i$, so $x \in \psi(\langle H^G \rangle)$.

Thus ψ preserves the normal closure. □

We also have the following:

Proposition 2.2.3. *Let ψ be an automorphism of G. Then*

(i) ψ preserves the trivial closure $H \mapsto G$;

(ii) ψ preserves the identity closure $H \mapsto H$; and

(iii) If G is Noetherian and Abelian, then ψ preserves the radical closure.

The proofs are extremely straightforward and will be omitted.

The only closure from Section 2.1 that has not been dealt with in this setting is the join with respect to a fixed subgroup K. A slight modification is needed (Example 2.2.3 omits this modification to illustrate its necessity), and it that the subgroups in question be *characteristic* (see Definition 1.4.3).

From here on out we will assume *cl* is not the identity closure, since that closure operation trivially(!) satisfies all the properties we will show.

Proposition 2.2.4. *Let $H < G$, and take some automorphism ψ of G. Let $K < G$ be fixed. Then ψ preserves the join with respect to K iff K is characteristic in G.*

Proof. $(\Rightarrow)$: Suppose ψ preserves the join with respect to K.

Then $\psi(H \vee K) = (\psi(H \vee K))^{cl}$.

Note that $\psi(H \vee K) = \psi(H) \vee \psi(K)$ (see Proposition A.2.3).

Then $\psi(H \vee K) = (\psi(H \vee K))^{cl}$

$\implies \psi(H) \vee \psi(K) = (\psi(H) \vee \psi(K)) \vee K$

$\implies K = \psi(K)$.

So K char G.

$(\Leftarrow)$: Now suppose K char G.

Know $\psi(H \vee K) = \psi(H) \vee \psi(K)$.

So $\psi(H \vee K) = \psi(H) \vee K$ since K char G.

Also $(\psi(H \vee K))^{cl} = (\psi(H) \vee \psi(K)) \vee K = \psi(H) \vee K$.

Thus ψ preserves the join with respect to K. □

With this example in mind, we can now show the following:

Theorem 2.2.5. *Let G be a group, C the set of all characteristic subgroups of G, and $cl : C \to C$ a closure operation. Then an automorphism $\psi : G \to G$ preserves cl.*

Proof. ψ preserves cl if $\psi(H^{cl})$ is cl-closed; i.e. if $\psi(H^{cl}) = (\psi(H^{cl}))^{cl}$.

Since $cl : C \to C$, H^{cl} char G.

Therefore $\psi(H^{cl}) = H^{cl}$.

Then $(\psi(H^{cl}))^{cl} = (H^{cl})^{cl} = H^{cl} = \psi(H^{cl})$. □

An immediate consequence is the following:

Corollary. *Let $\psi : G \to G$ be an automorphism, and cl a closure operation on C (as in Theorem 2.2.5). Then $\psi(H^{cl}) = (\psi(H))^{cl}\ \forall\ H \in C$.*

Proof. Since H char G, $\psi(H) = H$.

$\implies (\psi(H))^{cl} = H^{cl}$.

On the other hand, H^{cl} char G, so $\psi(H^{cl}) = H^{cl}$.

Therefore $(\psi(H))^{cl} = \psi(H^{cl})$. □

So in essence, if H char G, then cl and ψ "commute" for any automorphism ψ of G.

Unfortunately, it is not the case that automorphisms preserve all closure operations; Proposition 2.2.4 gives us the counterexample to such a claim, which we will elaborate upon further here.

Example 2.2.3. Let $G = \mathbb{R}^+$ under scalar multiplication, and take the automorphism $\psi(x) = x^2$ (see Example 1.3.2). Let $K = \mathbb{Q}^+$ be our fixed subgroup, and take $H = \langle\sqrt{2}\rangle$.

Let's suppose that ψ preserves the join with respect to $\mathbb{Q}^+$, so $\psi(\langle\sqrt{2}\rangle \vee \mathbb{Q}^+) = \psi(\langle\sqrt{2}\rangle) \vee \mathbb{Q}^+$. Take $x = (\sqrt{2})^4(\frac{1}{3}) \in \psi(\langle\sqrt{2}\rangle) \vee \mathbb{Q}^+$. Since ψ is an automorphism, there is an $x' \in \langle\sqrt{2}\rangle \vee \mathbb{Q}^+$ such that $\psi(x') = x$. So, $\psi(x') = (x')^2 = (\sqrt{2})^4(\frac{1}{3})$.

But this means that $x' = (\sqrt{2})^2(\frac{1}{3})^{\frac{1}{2}} \notin \langle\sqrt{2}\rangle \vee \mathbb{Q}^+$, so ψ does not preserve the join with respect to $\mathbb{Q}^+$.

To summarize: in showing that an automorphism ψ preserves a closure operation cl, the key is that under certain circumstances ψ "commutes" with cl; without this, we do not have that ψ sends closed subgroups to closed subgroups. We were only able to show this for characteristic subgroups; sadly, we could not show this for an arbitrary collection of subgroups S of G.

So, if we want to show that an *isomorphism* φ preserves a closure operation cl, it needs to be the case that φ and cl "commute" in the same way the automorphism ψ does; i.e. that $\varphi(H^{cl}) = (\varphi(H))^{cl}$ for some collection of subgroups $\{H_i\}_{i \in I}$ of G. Fortunately for us, in this case we do not need to restrict ourselves to certain classes of subgroups—with the right choice of cl, we can have that φ and cl "commute" for any $H < G$!

To get an idea of which closure operations satisfy this "commutative" property, we will turn to several examples. Note that in each of these, $\varphi : G \to G'$ denotes an isomorphism.

Example 2.2.4. (*Identity*). Suppose G, G' are endowed with the identity closure, so $H^{cl} = H$ for $H < G$. Then:

$\varphi(H^{cl}) = \varphi(H)$; and

$(\varphi(H))^{cl} = \varphi(H) = \varphi(H^{cl})$.

So the identity closure satisfies this property.

Example 2.2.5. (*Trivial closure*). Now consider G, G' with the trivial closure, so $H \mapsto G$, $H' \mapsto G'$ $\forall$ H, H'. Then:

$\varphi(H^{cl}) = \varphi(G) = G'$ (since φ isomorphism); and

$(\varphi(H))^{cl} = G' = \varphi(H^{cl})$.

So the trivial closure satisfies this property.

Example 2.2.6. (*Join*). Fix $K < G$ and $\varphi(K) < G'$. Then the join of H (resp. H') with respect to K (resp. $\varphi(K)$) also has that property:

$\varphi(H^{cl}) = \varphi(H \vee K) = \varphi(H) \vee \varphi(K)$; and

$(\varphi(H))^{cl} = \varphi(H) \vee \varphi(K) = \varphi(H^{cl})$.

So in this instance, the join with respect to K (and $\varphi(K)$) satisfies this property.

Example 2.2.7. (*Normal closure*).

$\varphi(H^{cl}) = \varphi(\langle H^G \rangle) = \langle \varphi(H)^{G'} \rangle$; and

$(\varphi(H))^{cl} = \langle \varphi(H)^{G'} \rangle = \varphi(H^{cl})$.

So the normal closure satisfies this property.

Note. To see that $\varphi(\langle H^G \rangle) = \langle \varphi(H)^{G'} \rangle$:

$h \in \langle H^G \rangle \implies h = \Pi_{i=1}^n g_i^{-1} h_i g_i$, where $g_i \in G$ and $h_i \in H$.

Thus $\varphi(h) \in \varphi(\langle H^G \rangle)$, so $\varphi(h) = \Pi_{i=1}^n \varphi(g_i)^{-1} \varphi(h_i) \varphi(g_i)$

$\in \langle \varphi(H)^{G'} \rangle$.

So $\varphi(\langle H^G \rangle) \subseteq \langle \varphi(H)^{G'} \rangle$.

The reverse containment can be seen by tracing the above set of steps backwards.

Example 2.2.8. (*Radical*). For this example, we assume G and G' are both Noetherian Abelian groups. We let cl be the radical of H, so $H \mapsto \text{rad}(H)$. Then:

$\varphi(H^{cl}) = \varphi(\text{rad}(H))$; and

$(\varphi(H))^{cl} = \text{rad}(\varphi(H)) = \varphi(H^{cl})$.

So for Noetherian Abelian groups, the radical satisfies this property.

Note. Recall $\text{rad}(H) = \{g \in G \mid g^n \in H \text{ for some } n > 0\}$.

Then $\varphi(\text{rad}(H)) = \varphi(\{g \in G \mid g^n \in H\})$

$= \{\varphi(g) \in G' \mid (\varphi(g))^n \in \varphi(H)\}$ (since φ isomorphism)

$= \text{rad}(\varphi(H))$.

Remark. The only case in which the desired relationship between cl and φ isn't automatically achieved is in Example 2.2.6; if, for example, we fixed a subgroup $K < G$ and a subgroup $K' < G'$ with $K' \neq \varphi(K)$, then the result will fail. Of all the examples we have provided, this is the only one that—much like in the case with automorphisms—will cause problems.

In light of these examples, we present the following definition:

Definition 2.2.2. Let $G \simeq G'$, and cl a closure operation on both G and G'. We say that cl is *obedient* (or that cl is an *obedient closure*) if $\forall\ H < G$ we have that $\varphi(H^{cl}) = (\varphi(H))^{cl}$. Otherwise, cl is said to be *naughty*.

So in the most general sense the trivial, identity, radical (when it's defined), and normal closures are all obedient; meanwhile, the join is naughty unless we take cl on G' to be the join with respect to $\varphi(K)$, (where cl on G is the join with respect to $K < G$).

With this, we finally have our result we seek!

Theorem 2.2.6. *Let $\varphi : G \to G'$ be an isomorphism, and cl an obedient closure operation on both G and G'. Then φ preserves cl.*

Proof. Let cl be an obedient closure.

Then $\varphi(H^{cl}) = (\varphi(H))^{cl}\ \forall\ H < G$.

Then $(\varphi(H^{cl}))^{cl} = ((\varphi(H))^{cl})^{cl} = (\varphi(H))^{cl} = \varphi(H^{cl})$.

Therefore φ preserves cl. ∎

The next section is something of an application of Theorem 2.2.6.

2.3 A "FIRST ISOMORPHISM THEOREM" FOR CLOSED SUBGROUPS

Since we have just established a criterion in which an isomorphism φ preserves a closure operation cl, it is natural to ask: what criteria is needed in order to find an analogous First Isomorphism-like theorem for closed subgroups?

In a perfect world, the First Isomorphism Theorem would translate over into something like this:

If $\varphi : G \to G'$ is a group homomorphism and cl a closure operation on G and G', then $\forall\, H < G$, $H^{cl}/ker(\varphi|_{H^{cl}}) \simeq$ [some closed subgroup in G'].

Of course, we went through the effort of finding out when φ sends closed subgroups to closed subgroups, so we can find an explicit closed subgroup *depending on* $\varphi(H)$ in G', rather than just some arbitrary closed subgroup. But first, some notation:

Notation. For a homomorphism φ of G and subgroup $H < G$, we will denote the restriction of φ to H by φ_H.

With that out of the way, we can just go ahead and state our theorem:

Theorem 2.3.1. *(A First Isomorphism Theorem for Closed Subgroups). Let $\varphi : G \to G'$ be a group homomorphism and cl a closure operation on both G and G' such that $\varphi(H^{cl}) = (\varphi(H))^{cl}\ \forall\, H < G$. Then $H^{cl}/ker(\varphi_{H^{cl}}) \simeq (\varphi(H))^{cl}$.*

Note. Example 2.2.1 provides a closure operation *cl* that satisfies this "obedience" property for homomorphisms, so the premises of the theorem are meaningful.

The proof of Theorem 2.3.1 is surprisingly similar to that of the First Isomorphism Theorem for groups. We shall go through the process here anyway.

Proof. Let $K := \ker(\varphi_{H^{cl}})$.

Define the map $\pi : H^{cl}/K \to \varphi(H^{cl})$ by $\pi(hK) = \varphi(h)$.

Show π well-defined: Suppose $hK = gK$.

Then $\pi(hK) = \varphi(h)$ and $\pi(gK) = \varphi(g)$.

$hK = gK \implies h^{-1}g \in K$, so $\varphi(h^{-1}g) = e' \in G'$.

Thus $\varphi(h)^{-1}\varphi(g) = e'$, so $\varphi(h) = \varphi(g)$.

Therefore $\pi(hK) = \pi(gK)$.

So π is well-defined.

Show π homomorphism: Take $hK, gK \in H^{cl}/K$.

Then $\pi(hK \cdot gK) = \pi(hgK) = \varphi(hg) = \varphi(h)\varphi(g)$
$= \pi(hK)\pi(gK)$.

So π is a homomorphism.

Show π injective: Suppose $\pi(hK) = \pi(gK)$.

Then $\varphi(h) = \varphi(g)$.

$\implies \varphi(h)^{-1}\varphi(g) = \varphi(h^{-1}g) = e'$.

$\implies h^{-1}g \in K$, so $hK = gK$.

So π is injective.

Show π onto: This is immediate by construction:

$$\pi(H^{cl}/K) = \{\varphi(h) \mid \pi(hK) = \varphi(h) \; \forall \; h \in H^{cl}\} = \varphi(H^{cl}).$$

So π is onto.

Thus π is an isomorphism, so $H^{cl}/K \simeq \varphi(H^{cl})$.

Since $\varphi(H^{cl}) = (\varphi(H))^{cl}$, we have that $H^{cl}/K \simeq (\varphi(H))^{cl}$. ∎

An immediate consequence is

Corollary. *If φ were instead taken to be an isomorphism in Theorem 2.3.1, then $H^{cl} \simeq (\varphi(H))^{cl}$.*

2.4 NOT-QUITE CLOSURE OPERATIONS

In this section, we will briefly go over what is known as a *preclosure* operation. This was alluded to in Section 2.1 when discussing the normal closure; indeed, we shall go over the normalizer of a subgroup and explicitly show that it does not satisfy the requirements to be a closure operation. As usual, we start with the definition, but then we will launch right in to the example.

Definition 2.4.1. Let G be a group. A *preclosure operation* $^*cl$ on a set of subgroups S of G is a set map $^*cl : S \to S$, $H \mapsto H^{^*cl}$, satisfying the following two properties:

(i) (*Extension*). $H \subseteq H^{*cl}$ for all $H \in S$;

(ii) (*Order-Preserving*). If $K \subseteq H$, where $H, K \in S$, then $K^{*cl} \subseteq H^{*cl}$.

If S is the set of all subgroups of G, we say $^*cl$ is a *preclosure operation on G*.

If a preclosure operation $^*cl$ also satisfies that $(H^{*cl})^{*cl} = H^{*cl}$ (idempotence) for all $H < G$, then $^*cl$ is a *closure operation* on G.

Example 2.4.1. (*Normalizer*). Let $^*cl : S \to S$, where $H \mapsto N_G(H)$ (see Definition 1.5.1). Then the normalizer is a preclosure operation.

Proof. Show extension: This is clear by Proposition 1.5.2 (i).

Show order-preserving: Suppose $K \subseteq H$, and take $g \in N_G(K)$.

Then $gKg^{-1} = K \subseteq H \subseteq gHg^{-1}$.

But then there exist k_1, k_2 such that $gk_1g^{-1} = k_2 \in H$.

So $g \in N_G(H)$.

Hence $H \mapsto N_G(H)$ is a preclosure operation. ◻

It is not immediately clear that the normalizer is only a preclosure operation, and not a full-blown closure operation. Indeed, although any closure operation is a preclosure, the converse does not necessarily hold. The following example illustrates this fact.

Example 2.4.2. (*A Dihedral group*). Recall the Dihedral group of order 8, denoted here by $D_4 = \{e, r, r^2, r^3, f_{1,3}, f_{2,4}, f_x, f_y\}$, where r is the clockwise rotation of a square with vertices $1, 2, 3, 4$ (hence r^n represents n successive clockwise rotations), f_x and f_y are flips about the x- and y-axes, respectively, and $f_{1,3}$ and $f_{2,4}$ are flips along diagonals, exchanging vertex 2 with 4 and 1 with 3, respectively (see Figure 1).

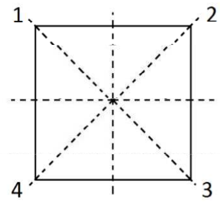

Figure 1.: The Dihedral group D_4 acts on this square, where the dotted lines indicate the axes of the flips.

Consider the subgroup $H = \{e, f_{1,3}\}$. Taking the normalizer $N_{D_4}(H)$ as our preclosure operation, we find that $N_{D_4}(H) = \{e, r^2, f_{1,3}, f_{2,4}\}$. (In fact, for all other elements $g \in D_4 \backslash H$ we get that $gHg^{-1} = \{e, f_{2,4}\}$.)

But then, $r \in N_{D_4}(N_{D_4}(H))$, since $r(N_{D_4}(H))r^{-1} = N_{D_4}(H)$, so we have that $N_{D_4}(N_{D_4}(H)) \neq N_{D_4}(H)$.

So, while we have that extension and order-preservation will always occur with the normalizer, the normalizer of this particular subgroup H of D_4 does not satisfy the idempotence condition of a closure operation and hence the normalizer is not always a closure operation.

2.5 THE CONSTRUCTION OF CLOSURE OPERATIONS

Here we find group-theory analogues of the constructions given in Section 3.2 of [2]. He provides six constructions, most of which get used for constructing his examples of closure operations on ideals; likewise, we also give six constructions, which are near-carbon copies of the ones given in his paper with appropriate modifications. However, we only use three of these constructions to create our examples. Later on, in Chapter 3, a seventh construction will be introduced.

Construction 1. *Let $\varphi : G \to G'$ be a group homomorphism and d a closure operation on G'. For $H < G$ define $H^c := \varphi^{-1}((\varphi(H))^d)$. Then c is a closure operation on G.*

Remark. Construction 1 has nothing to do with preserving closure operations; we are merely *defining* the "pullback" of the closure of the image of a subgroup to be closed (cf. Proposition 2.2.4, Definition 2.2.2 and the remark preceding it, and Theorem 2.2.6).

Let's verify that this construction is indeed a closure operation. We will provide the verification for all closure operation constructions in this section.

Proof. Show extension: Take $h \in H$.

Then $\varphi(h) \in \varphi(H) \subseteq (\varphi(H))^d$ by extension of d.

Thus $h \in \varphi^{-1}((\varphi(H))^d)$.

So extension holds.

Show idempotence: Need only show $(H^c)^c \subseteq H^c$.

Take $h \in (H^c)^c$.

Then $h \in \varphi^{-1}((\varphi[\varphi^{-1}((\varphi(H))^d)])^d)$.

$\implies \varphi(h) \in (\varphi[\varphi^{-1}((\varphi(H))^d)])^d$

$= ((\varphi(H))^d)^d = (\varphi(H))^d$ by idempotence of d.

So idempotence holds.

Show order-preservation: $K \subseteq H \implies \varphi(K) \subseteq \varphi(H)$

$\implies (\varphi(K))^d \subseteq (\varphi(H))^d$ by order-preservation of d.

$\implies K^c = \varphi^{-1}((\varphi(K))^d) \subseteq \varphi^{-1}((\varphi(H))^d) = H^c$.

So this construction does indeed define a closure operation. ∎

Construction 2. *Let $\{cl_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closure operations on the subgroups of a group G. Then for any subgroup H of G, $H^{cl} := \bigcap_{\alpha \in A} H^{cl_\alpha}$ also gives a closure operation.*

Proof. *Show extension*: Since cl_α is a closure operation for all $\alpha \in A$,

then $H \subseteq H^{cl_\alpha}$ for any $H < G$.

Thus $H \subseteq \bigcap_{\alpha \in A} H^{cl_\alpha} = H^{cl}$.

So extension holds.

Show idempotence: As usual, we only need to see $(H^{cl})^{cl} \subseteq H^{cl}$.

$(H^{cl})^{cl} = \bigcap_{\alpha \in A}(H^{cl})^{cl_\alpha} = \bigcap_{\alpha \in A}(\bigcap_{\alpha \in A} H^{cl_\alpha})^{cl_\alpha}$,

so $h \in (H^{cl})^{cl} \implies h \in (H^{cl})^{cl_\alpha}$ for all α.

$H^{cl} \subseteq H^{cl_\alpha}$ for all α, so $h \in (H^{cl})^{cl_\alpha} \subseteq (H^{cl_\alpha})^{cl_\alpha} = H^{cl_\alpha}$

since cl_α is a closure operation.

This holds for all $\alpha \in A$, so $h \in H^{cl}$ and hence $(H^{cl})^{cl} \subseteq H^{cl}$.

So idempotence holds.

Show order-preserving: Take $K \subseteq H$.

By definition, $K^{cl} = \bigcap_\alpha K^{cl_\alpha}$, and $H^{cl} = \bigcap_\alpha H^{cl_\alpha}$.

$K^{cl_\alpha} \subseteq H^{cl_\alpha}$ for all α, so $K^{cl} = \bigcap_\alpha K^{cl_\alpha} \subseteq \bigcap_\alpha H^{cl_\alpha} = H^{cl}$.

So this construction defines a closure operation. □

Construction 3. *Let G be a Noetherian Abelian group, and let $\{cl_\alpha\}_{\alpha \in A}$ be a directed set of closure operations on the subgroups of G; that is, for all $\alpha_1, \alpha_2 \in A$ there is $\beta \in A$ such that $H^{cl_{\alpha_i}} \subseteq H^{cl_\beta}$ for $i = 1, 2$. Then $H^{cl} := \bigcup_{\alpha \in A} H^{cl_\alpha}$ gives us a closure operation.*

Proof. *Show extension*: $H \subseteq H^{cl_\alpha}$ for all α

$\implies H \subseteq \bigcup_{\alpha \in A} H^{cl_\alpha} = H^{cl}$.

Show idempotence: Need to see $(H^{cl})^{cl} \subseteq H^{cl}$.

$(H^{cl})^{cl} = \bigcup_{\alpha \in A}(H^{cl})^{cl_\alpha}$.

Since cl_α is a closure operation, H^{cl_α} is a subgroup for all

$\alpha \in A$ (see Definition 2.1.1—cl maps subgroups to

subgroups).

The directed-set property of $\{cl_\alpha\}_{\alpha \in A} \implies H^{cl} = H^{cl_i}$ for

some $i \in A$:.

For any $\alpha_1, \alpha_2 \in A$, $H^{cl_{\alpha_1}}, H^{cl_{\alpha_2}} \subseteq H^{cl_\beta}$, some $\beta \in A$.

For two other indices $\alpha_3, \alpha_4 \in A$, $\exists\ \gamma \in A$ such that

$H^{cl_{\alpha_3}}, H^{cl_{\alpha_4}} \subseteq H^{cl_\gamma}$.

Then $\exists\ \delta \in A$ such that $H^{cl_\beta}, H^{cl_\gamma} \subseteq H^{cl_\delta}$.

So we have a chain that goes something like

$\{e\} \subseteq H^{cl_{\alpha_1}} \subseteq H^{cl_\beta} \subseteq H^{cl_\delta} \subseteq \cdots$

since we can continue this process.

G satisfies ACC, so this chain terminates at, say, H^{cl_ω}.

Then $\bigcup_{\alpha \in A} H^{cl_\alpha} \subseteq H^{cl_\omega}$ since $H^{cl_\alpha} \in H^{cl_\omega}$ for all $\alpha \in A$.

Also, $\omega \in A \implies H^{cl_\omega} \subseteq \bigcup_{\alpha \in A} H^{cl_\alpha}$.

Thus $H^{cl_\omega} = \bigcup_{\alpha \in A} H^{cl_\alpha} = H^{cl}$.

So $H^{cl} = H^{cl_\omega}$ for some $\omega \in A$.

Note here that this means that H^{cl} is a subgroup.

So $(H^{cl})^{cl} = (H^{cl_i})^{cl} = (H^{cl_i})^{cl_j}$ for some $i, j \in A$.

Now take $cl_k \in \{cl_\alpha\}$ such that $H^{cl_i}, H^{cl_j} \subseteq H^{cl_k}$.

$\therefore (H^{cl})^{cl} = (H^{cl_i})^{cl_j} \subseteq (H^{cl_k})^{cl_k} = H^{cl_k} \subseteq H^{cl}$.

So idempotence holds.

Show order-preservation: Take $K \subseteq H$.

Then $K^{cl_\alpha} \subseteq H^{cl_\alpha}\ \forall\ \alpha \in A$

$\implies \bigcup_{\alpha \in A} K^{cl_\alpha} \subseteq \bigcup_{\alpha \in A} H^{cl_\alpha}$

$\implies K^{cl} \subseteq H^{cl}$.

So this construction defines a closure operation. ☐

Construction 4. *Let d be an operation on (the subgroups of) G that is a preclosure. Let S be the set of all closure operations on G defined by $c \in S \iff H^d \subseteq H^c\ \forall\ H < G$. Then the map $H \mapsto H^{d^\infty} := \bigcap_{c \in S} H^c$ is a closure operation (called the* idempotent hull *of d).*

Proof. This follows immediately using Construction 2. ☐

Construction 5. *Let c be a closure operation on finitely generated subgroups H of G, and define c_f by setting*

$$H^{c_f} := \bigvee\{K^c \mid K \text{ finitely generated and } K < H\}.$$

Then c_f is a closure operation.

Note. In [2], his analogous construction is done using unions of finitely generated ideals; in our case, it is not enough to take unions of closures of subgroups, as—per Example 1.1.7—the union of subgroups is not necessarily again a subgroup. Here, we must take the *join* of subgroups, as this guarantees we are constructing yet another subgroup.

Proof. *Show extension*: Take $h \in H$, and consider $\langle h \rangle < H$.

Then $H = \bigvee_{h \in H}\langle h \rangle$ (see Proposition A.2.4 for proof).

c is a closure operation, so we have that $\langle h \rangle \subseteq \langle h \rangle^c \subseteq H^{c_f}$.

Therefore $H \subseteq \bigvee\langle h \rangle \subseteq \bigvee\langle h \rangle^c \subseteq H^{c_f}$.

So extension holds.

Show idempotence: Need only show $(H^{c_f})^{c_f} \subseteq H^{c_f}$.

Take $h \in (H^{c_f})^{c_f}$.

Then $h = k_1^{r_1}k_2^{r_2}\cdots k_n^{r_n}$, where $k_i \in K_i^c$ (we know there are finitely many K_i since H finitely generated).

$K_i^c \subseteq H^{c_f}$ since $K_i^c \subseteq H^c$ for each i.

Therefore $h \in \bigvee K_i^c \subseteq \bigvee H^{c_f} = H^{c_f}$.

$h \in (H^{c_f})^{c_f}$ arbitrary, so we have that $(H^{c_f})^{c_f} \subseteq H^{c_f}$.

So idempotence holds.

Show order-preservation: Let $J \subseteq H$.

Note that since H is finitely generated, so is J.

Take $j \in J^{c_f}$.

Then $j \in \bigvee\{K^c \mid K < J$ finitely generated$\}$.

Since J finitely generated, then there are only finitely many

K^c in the join.

So $j = k_1^{r_1} k_2^{r_2} \cdots k_n^{r_n}$, where $k_i \in K_i^c$.

$K_i \subseteq J \subseteq H \implies K_i^c \subseteq H^c$ since c is a closure operation.

Then $j \in H^c$.

H finitely generated and $H < H \implies H^c \subseteq H^{c_f}$.

Therefore $j \in H^c \subseteq H^{c_f}$.

Thus $J^{c_f} \subseteq H^{c_f}$.

So this construction defines a closure operation. ☐

Construction 6. *Let S be a set, and H a subgroup of G. Then the map $H \mapsto H^{cl} := \{g \in G \mid gS \subseteq HS\}$ gives a closure operation.*

Proof. Show extension: This is clear, as $hS \subseteq HS$ for all $h \in H$,

hence $H \subseteq H^{cl}$.

So extension holds.

Show idempotence: Need to see $(H^{cl})^{cl} \subseteq H^{cl}$.

Take $h \in (H^{cl})^{cl}$.

Then $hS \subseteq (H^{cl})S$.

So there is $h' \in H^{cl}$ such that $hS \subseteq h'S$.

So $hS \subseteq h'S \subseteq HS$ since $h' \in H^{cl}$.

So idempotence holds.

Show order-preservation: Suppose $K \subseteq H$, and take $k \in K^{cl}$.

Then $kS \subseteq KS \subseteq HS$, so $k \in H^{cl}$.

So this construction defines a closure operation. ☐

We now take the time to see how our known closure operations (see Section 2.1) are built from these constructions. As we have seen above, Constructions 1, 2, and 4 apply to arbitrary subgroups and so can be used in the most general sense; these will be the ones utilized below.

The approach we will take is to use only the closure operations we have previously constructed to make our example closures. For now, we can assume the trivial and identity closures are "given" to serve as a starting point, though we can certainly construct them using interior operations, as we shall see here:

Example 2.5.1. *Example 2.1.1 revisited.* The identity closure $H \mapsto H$ does not need any construction (though one could use Construction 7, introduced in Section 3.3, to do so).

Example 2.5.2. *Example 2.1.2 revisited.* The trivial closure $H \mapsto G$ also does not need any construction, but Construction 7 will do the trick.

Example 2.5.3. *Example 2.1.3 revisited.* Now take our closure to be the join with respect to a fixed subgroup K of G; i.e. $H^{cl} = H \vee K$.

By Proposition 1.5.4, $H \vee K = \bigcap \{J < G \mid H < J \text{ and } K < J\}$. Take cl_J to be the trivial closure operation cl restricted to J; that is, $H^{cl_J} = J$ for all $H < J$ (thus $K^{cl_J} = J$ as well).

Then, by Construction 2, $H \vee K = \bigcap_{J \supseteq (H \cup K)} H^{cl_J}$.

Example 2.5.4. *Example 2.1.4 revisited.* We now look at the normal closure $H \mapsto \langle H^G \rangle$.

By Proposition 1.5.8, $\langle H^G \rangle = \bigcap H_i$, where $H \subseteq H_i$ and $H_i \lhd G \; \forall \; i \in I$, I some indexing set. Let cl_i be the trivial closure restricted to H_i; that is, $H^{cl_i} = H_i$ for each i.

Then by Construction 2, $\langle H^G \rangle = \bigcap_{i \in I} H^{cl_i}$.

Example 2.5.5. *Example 2.1.5 revisited.* Consider now the radical closure $H \mapsto \text{rad}(H)$ on the subgroups of a Noetherian Abelian group G. We use two different constructions to create this closure.

Method 1: Construction 5. By assumption, H is finitely generated. Let $S = \{h_1, h_2, \ldots, h_n\}$ be the generating set of H.

For each $h_i \in S$, $\exists\ g_i \in G$ and $n_i > 0$ such that $g_i^{n_i} = h_i$ (note that it may just be the case that $n_i = 1$ and $g_i = h_i$; nevertheless there will always be such a $g_i \in G$ and $n_i > 0$). There may also be a number of g_i and corresponding n_i such that $g_i^{n_i} = h_i$ — we will take the g_i with the largest corresponding n_i (e.g. if $h_i = 64$, then $2^6 = 4^4 = 8^2 = 64$, we take $g_i = 2$) since that g_i also generates the other potential choices. In either case, we know that there are finitely many g_i (see Method 2).

Now let $\{g_1, g_2, \ldots, g_n\}$ be our set of g_i. For each $K < H$, define our closure operation c to be the trivial closure restricted to $\langle g_1, g_2, \ldots g_n \rangle$, so $K^c = \langle g_1, g_2, \ldots g_n \rangle$ $\forall\ K < H$.

Note that since H is finitely generated, so is each $K < H$ by Proposition 1.2.1. Thus, by Construction 5,

$$H^{c_f} = \bigvee \{K^c \mid K < H\} = \bigvee \langle g_1, g_2, \ldots g_n \rangle = \langle g_1, g_2, \ldots g_n \rangle.$$

Of course, $H^{c_f} = \text{rad}(H)$. To see this, note that by the way we defined H^{c_f}, $\text{rad}(H) \subseteq H^{c_f}$. To see the other inclusion, take $x \in H^{c_f}$, so $x = g_1^{r_1} g_2^{r_2} \cdots g_n^{r_n}$ for some powers r_i. Let m be the least common multiple of the n_i such that $g_i^{n_i} \in H$ for each i. Then $x^m = (g_1^{r_1} g_2^{r_2} \cdots g_n^{r_n})^m = g_1^{r_1 m} g_2^{r_2 m} \cdots g_n^{r_n m} = (g_1^m)^{r_1} (g_2^m)^{r_2} \cdots (g_n^m)^{r_n} \in H$ since now each of the $g_i^m \in H$.

This completes this construction of $\text{rad}(H)$.

Method 2: Construction 3. Let $S = \{h_1, h_2, \ldots, h_n\}$ be the generating set for H as before.

For each $h_i \in S$, let G_{h_i} be the set of all $g_{i_j} \in G$ such that $\exists\, n_{i_j} \geq 1$ with $g_{i_j}^{n_{i_j}} = h_i$ — since $H = \bigvee \langle h_i \rangle$, it is enough to list the elements in G_{h_i} from the largest such n_{i_j} to 1 since any other such g_{i_j} will have come from H.

Since $\langle h_i \rangle < \langle g_{i_j} \rangle\ \forall\, j$, then $\langle h_1, \ldots, h_i, \ldots, h_n \rangle < \langle h_1, \ldots, g_{i_j}, \ldots, h_n \rangle\ \forall\, j$. Likewise, for $n_{i_k} > n_{i_j}$ $\langle h_1, \ldots, g_{i_k}^{n_{i_k}}, \ldots, h_n \rangle < \langle h_1, \ldots, g_{i_j}^{n_{i_j}}, \ldots, h_n \rangle$. We can see from here that for each h_i there are finitely many elements in G_{h_i} since H is Noetherian. So, we take $g_i \in G_{h_i}$ with the largest corresponding n_{i_j}, just as in Method 1, since that element generates the other elements in G_{h_i}.

We define our closure operation by sending individual generators to their appropriate g_i, so we map $H = \langle h_1, \ldots, h_i, \ldots, h_n \rangle$ to $\langle h_1, \ldots, g_i, \ldots, h_n \rangle$ by cl_i, where $1 \leq i \leq n$. In a similar manner, $cl_{i,j}$ sends our subgroup H to $\langle h_1, \ldots, h_{i-1}, g_i, h_{i+1}, \ldots, h_{j-1}, g_j, h_{j+}$ i.e. the subscript i, j of cl denotes which generators h_k of H get sent to their corresponding elements g_k. We continue defining our closure operations inductively. We can then see that $cl_{1,2,\ldots,n}$ is the trivial closure restricted to $\langle g_1, \ldots, g_n \rangle$, and we will define cl_0 to be the identity closure.

This set of closure operations defines a directed set: indeed, any two such closures of H are always going to be contained in $\langle g_1, \ldots, g_n \rangle$ (though less obvious examples can be used).

Therefore, by Construction 3 (and Method 1) we can see that $\mathrm{rad}(H) = \bigcup H^{cl_{i_1,\ldots,i_k}} = H^{cl_{1,2,\ldots,n}} = \langle g_1, \ldots, g_n \rangle$.

This completes this construction of $\mathrm{rad}(H)$.

3

INTERIOR OPERATIONS AND THEIR RELATIONSHIP WITH CLOSURE OPERATIONS

In the same vein as Chapter 2, we can also discuss the notion of an *interior operation* on, or just an *interior* of, a subgroup $H < G$. The majority of this chapter will be devoted to the interaction between an interior operation and closure operations, taking an approach similar to that in analysis or topology. After all, what is "closed" without "open"?

3.1 INTERIOR OPERATIONS

Definition 3.1.1. Let G be a group. An *interior operation int* on a set of subgroups S of G is a set map $int : S \to S$, $H \mapsto H^{int}$, satisfying the following three properties:

(i) (*Inclusion*). $H^{int} \subseteq H$ for all $H \in S$;

(ii) (*Idempotence*). $H^{int} = (H^{int})^{int}$ for all $H \in S$; and

(iii) (*Order-Preserving*). If $K \subseteq H$, where $H, K \in S$, then $K^{int} \subseteq H^{int}$.

If S is the set of all subgroups of G, we say int is an *interior operation on G*.

A subgroup $H \in S$ is *int-open* if $H = H^{int}$.

Example 3.1.1. (*Identity*). $int : S \to S$, where $H^{int} = H$ is an interior operation.

Note. If *cl* and *int* are operations on G such that $\forall\ H < G\ H$ is both *cl*-closed and *int* open, then *cl* and *int* are the identity closure and interior, respectively. Note that we can, however, have non-identity closure and interior operations that "fix" a certain set of subgroups of G (see the remark following Example 3.3.2 in Section 3.3 for one such example).

Example 3.1.2. (*Trivial interior*). Define $int_t : S \to S$ by $H^{int_t} = \{e\}$. That the trivial interior is an interior operation is clear.

Example 3.1.3. (*Intersection*). Fix some subgroup K of G. (Note that $H \cap K$ is indeed a subgroup of G for all $H < G$).

Then $int : H \to H \cap K$, called the *intersection with respect to K*, is an interior operation.

Proposition 3.1.1. *The map $H \mapsto H \cap K$, $K < G$ fixed for all $H < G$, is an interior operation.*

Proof. Show inclusion: $H \cap K \subseteq H$ for all subgroups H.

Show idempotence: Need only see that $(H \cap K) \cap K \subseteq H \cap K$.

Take $h \in (H \cap K) \cap K$. Then $h \in H \cap K$ and $h \in K$.

So $(H \cap K) \cap K \subseteq H \cap K$.

So idempotence holds.

Show order-preserving: Let $J \subseteq H$, and take $h \in J \cap K$.

Then $h \in J$ and $h \in K$.

$J \subseteq H \implies h \in H$.

So $h \in H$ and $h \in K \implies h \in H \cap K$.

Therefore $J \cap K \subseteq H \cap K$.

So the intersection with respect to K is indeed an interior operation. ∎

Example 3.1.4. (*The kernel of a homomorphism*). Let $\varphi : G \to G$ be a homomorphism mapping G to itself (not necessarily an automorphism). For each $H < G$, define

$$ker_H\varphi = ker(\varphi|_H) = \{h \in H \mid \varphi(h) = e\}.$$

We will call $\ker_H\varphi$ the *kernel of φ with respect to H*.

Proposition 3.1.2. *The map $H \mapsto ker_H\varphi$ is an interior operation.*

Proof. Let $H^{int} = \ker_H\varphi$.

Show inclusion: Clearly $H^{int} \subseteq H$.

Show idempotence: Know $(H^{int})^{int} \subseteq H^{int}$ by (i).

Show $\supseteq$: $(H^{int})^{int} = \{h \in ker_{H^{int}}\varphi \mid \varphi(h) = e\}$

$= \{h \in H \mid \varphi(\varphi(h)) = e\}$.

Take $h \in H^{int}$.

Then $\varphi(h) = e \implies \varphi(\varphi(h)) = e \implies h \in (H^{int})^{int}$.

So idempotence holds.

Show order-preserving: Let $K \subseteq H$.

Then $k \in K^{int} \implies \varphi(k) = e \implies k \in H^{int}$ since $K \subseteq H$.

Therefore $K^{int} \subseteq H^{int}$.

So $\ker_H\varphi$ is indeed an interior operation. ☐

Example 3.1.5. (*The stabilizer of a subgroup*). Let G be a group that acts on a set S, and let $H < G$. We define the stabilizer of H at $s \in S$ to be $H_s = \{h \in H \mid hs = s\}$ (see Definition 1.5.5).

As per Proposition 1.5.6, $H_s < G$.

Proposition 3.1.3. *Fix $s \in S$. The map $H \mapsto H_s$ is an interior operation.*

Proof. *Show inclusion*: Clearly $H_s \subseteq H$.

Show idempotence: We want to see that $H_s \subseteq (H_s)_s$.

$(H_s)_s = \{h \in H_s \mid hs = s\} = H_s$ by definition.

So idempotence holds.

Show order-preserving: Let $K \subseteq H$.

Then $k \in K_s \implies ks = s$, and since $K \subseteq H$

it follows that $k \in H_s$.

So H_s is an interior operation. □

Example 3.1.6. (*Normal core*). Let G be a group, and take a subgroup H of G. Recall that the normal core of H in G is defined to be $\text{Core}_G(H) = \bigcap_{g \in G} g^{-1}Hg$ (see Definition 1.5.7).

Proposition 3.1.4. *The map $H \mapsto \text{Core}_G(H)$ is an interior operation.*

Proof. Show inclusion: Take $h \in \text{Core}_G(H)$.

Then $h \in g^{-1}Hg$ for all $g \in G$.

In particular, $h \in g^{-1}Hg$ for all $g \in H$.

$g \in H \implies g^{-1}Hg = H$, so $h \in H$.

So inclusion holds.

Show idempotence: This follows immediately by taking

Proposition 1.5.9(ii) and applying it to Proposition 1.5.9(i).

So idempotence holds.

Show extension: Take $J \subseteq H$.

Then $j \in \text{Core}_G(J) \implies j \in g^{-1}Jg$ for all $g \in G$.

$J \subseteq H \implies g^{-1}Jg \subseteq g^{-1}Hg$ for all $g \in G$.

So $j \in \text{Core}_G(H)$.

So the normal core is an interior operation. □

At first glance, it would seem that H^{int} is always normal in H, since nearly all these subgroups of H are normal (see Proposition A.3.1). However, the H^{int} is not always normal in H, and the counterexample is the following:

Example 3.1.7. Let $H = G$, and take a subgroup $K < G$. Then $G^{int} = G \cap K = K$ is normal in G iff $K \triangleleft G$.

Much like with closure operations, we also have the notion of a "pre-interior", defined as follows:

Definition 3.1.2. Let G be a group. A *pre-interior operation* $^{*}int$ on a set of subgroups S of G is a set map $^{*}int : S \to S$, $H \mapsto H^{*cl}$, satisfying the following two properties:

(i) (*Inclusion*). $H^{*int} \subseteq H$ for all $H \in S$;

(ii) (*Order-Preserving*). If $K \subseteq H$, then $K^{*int} \subseteq H^{*int}$.

If S is the set of all subgroups of G, we say $^{*}int$ is a *pre-interior operation on* G.

If a pre-interior operation $^{*}int$ also satisfies that $(H^{*int})^{*int} = H^{*int}$ (idempotence) for all $H < G$, then $^{*}int$ is an *interior operation* on G.

An example of a pre-interior is

Example 3.1.8. The map $H \mapsto [H, H]$, where $[H, H]$ is the commutator subgroup, is an pre-interior operation.

The proof of this is extremely straightforward to see, especially with Proposition 1.5.5 under our belts, and so it will be omitted.

3.2 A WAY IN WHICH INTERIOR OPERATIONS CAN DETERMINE CLOSURE OPERATIONS

The information in this section is inspired by [4], particularly the examples introducing section 2.1 (starting on pg. 2). We shall list the specific example that prompted this here:

Example 3.2.1. (*see [4], pg. 3*). Let G be a group, and $[G,G]$ its commutator subgroup. Then

$$c'_G(H) := [G,G] \cdot H$$

defines a closure operation for each subgroup H of G.

Although we have just seen that the commutator subgroup is not an interior operation, we can use this example to prove a stronger result. We will not, however, show that Example 3.2.1 is indeed a closure operation.

Construction 7. *Let G be a group, and int an interior operation on G such that $G^{int} \lhd G$ (see Proposition A.3.1 for some examples). Then the map*

$$H \mapsto H^{cl} := (G^{int}) \cdot H$$

defines a closure operation on G.

Proof. To see $H^{cl} < G$, consult Proposition A.3.2.

Show extension: This is clear, as Definition 1.1.8 points out that

$H \subseteq KH$.

So extension holds.

Show idempotence: This is also clear, as $(H^{cl})^{cl} = G^{int} \cdot H^{cl}$

$= G^{int} \cdot (G^{int} \cdot H) = (G^{int} \cdot G^{int}) \cdot H = G^{int} \cdot H = H^{cl}$.

So idempotence holds.

Show order-preserving: Suppose $J \subseteq H$.

Then $J^{cl} = G^{int} \cdot J \subseteq G^{int} \cdot H = H^{cl}$.

So this construction defines a closure operation. ☐

By choosing the right interior operation, we obtain constructions for Examples 2.4.1 and 2.4.2. In the case of Example 2.4.1, take *int* to be the trivial interior, so that $H^{cl} = G^{int} \cdot H = \{e\} \cdot H = H$. As for Example 2.4.2, by taking the identity interior $H \mapsto H$ we get that $H^{cl} = G^{int} \cdot H = G \cdot H = G$.

We close this section with an example of a closure operation induced by the Construction 7.

Example 3.2.2. (cf. [1], Section 1.6 Exercise 19). Let $G = \{z \in \mathbb{C} \mid z^n = 1$ for some $n \in \mathbb{Z}\}$. For any $k > 1$, define the map $\varphi : G \to G$ by $\varphi_k(z) = z^k$. It is clear that φ_k is a homomorphism for each k. So, we shall take our interior operation to be the kernel of φ_k.

Then $\ker \varphi_k = \{z \in G \mid z^k = 1\}$; i.e. $\ker \varphi_k$ is the set of kth roots of unity.

Let $k = 3$, and let $H = \{z \in \mathbb{C} \mid z^5 = 1\}$.

So, $H^{cl} = G^{int} \cdot H = \{gh \mid g \in \ker \varphi_k$ and $h \in H\}$.

Take $gh \in H^{cl}$ with $g = e^{2\pi i m/3}$ and $h = e^{2\pi i l/5}$, with $1 \leq m < 3$ and $1 \leq l < 5$.

Then $gh = e^{2\pi i m/3} e^{2\pi i l/n} = e^{2\pi i(5m+3l)/15} = e^{2\pi i(8n/15)} = e^{16 n\pi i/15}$. An easy calculation shows us that $gh \in H^{cl}$ for all $n = 1, \ldots, 15$ (this is because $\gcd(3,5) = 1$). Thus $H^{cl} = \{z \in \mathbb{C} \mid z^{15} = 1\}$.

More generally: for any k, n we have that, for $H = \{z \in \mathbb{C} \mid z^n = 1\}$, $H^{cl} = \{z \in \mathbb{C} \mid z^r = 1\}$, where $r = \operatorname{lcm}(n, k)$.

3.3 CORRESPONDING OPERATIONS

Definition 3.3.1. Let *cl* be a closure operation on a group G, and *int* an interior operation. We say that *int* and *cl* are *corresponding operations* (*on a group* G) if, for all subgroups $H \in S$ (S some set of subgroups of G), we have that

(i) $(H^{cl})^{int} = H^{int}$, and

(ii) $(H^{int})^{cl} = H^{cl}$.

As always, let's look at a few examples of corresponding operations to get a feel for the idea.

Example 3.3.1. *A pretty trivial example of corresponding operations.* Let cl be as in Example 2.1.2 and int as in Example 3.1.2; i.e. $H^{cl} = G$ and $H^{int} = \{e\}$. Then cl and int are corresponding operations:

(i) $(H^{cl})^{int} = G^{int} = \{e\} = H^{int}$; and

(ii) $(H^{int})^{cl} = (\{e\})^{cl} = G = H^{cl}$.

So the trivial closure and trivial interior are corresponding operations.

Example 3.3.2. *The normal closure and normal core.* Let G be a group, and take $H < G$. Then the normal closure and normal core are corresponding operations $\iff H \triangleleft G$.

Proof. $(\Rightarrow)$: Suppose the normal closure and normal core are corresponding operations.

Then $\langle \text{Core}_G(H)^G \rangle = \langle H^G \rangle$ and $\text{Core}_G(\langle H^G \rangle) = \text{Core}_G(H)$.

By Propositions 1.5.9(i) and 1.5.7(ii), $\text{Core}_G(H) \triangleleft G \implies \text{Core}_G(H) = \langle \text{Core}_G(H)^G \rangle$.

Thus, by assumption, $\text{Core}_G(H) = \langle H^G \rangle$.

(Similarly, we could use the fact that $\langle H^G \rangle \triangleleft G \implies \langle H^G \rangle = \text{Core}_G(\langle H^G \rangle)$ to get that $\text{Core}_G(H) = \langle H^G \rangle$).

But $\text{Core}_G(H) \subseteq H \subseteq \langle H^G \rangle \implies H = \text{Core}_G(H)$ and $H = \langle H^G \rangle$.

So $H \triangleleft G$.

$(\Leftarrow)$: Now suppose $H \triangleleft G$.

Then $H = \langle H^G \rangle$ and $H = \text{Core}_G(H)$, hence $\langle H^G \rangle = \text{Core}_G(H)$.

By the same reasoning as above, $\langle H^G \rangle = \text{Core}_G(\langle H^G \rangle)$ and

$\text{Core}_G(H) = \langle \text{Core}_G(H)^G \rangle$.

$\implies \langle H^G \rangle = \langle \text{Core}_G(H)^G \rangle$ and $\text{Core}_G(H) = \text{Core}_G(\langle H^G \rangle)$.

Thus the normal closure and normal core are corresponding operations. □

Remark. Example 3.3.2 ultimately reduces down to the identity closure and interior, since $H \lhd G \implies H^{cl} = H^{int} = H$.

Example 3.3.3. *The obligatory non-example of corresponding operations.* Fix a subgroup K of G. Let cl be the join with respect to K, and int the intersection with respect to K. Then cl and int are not corresponding operations on G:

(i) $(H^{cl})^{int} = (H \vee K)^{int} = (H \vee K) \cap K$; and

(ii) $(H^{int})^{cl} = (H \cap K)^{cl} = (H \cap K) \vee K$.

Note that for (ii) we do not have that $(H \cap K) \vee K = H^{cl} = H \vee K$; indeed, if we take an element $h \in H \backslash K$, then $hk \in H \vee K$, but $hk \notin (H \cap K) \vee K$.

So this pairing fails as corresponding operations.

Example 3.3.2 (as well as the note following Example 3.1.1 in Section 3.1) give rise to this kind of notion of closure operations "fixing" subgroups. Here, we will give an official definition.

Definition 3.3.2. Let cl be a closure operation on G. If there is a set of subgroups S such that H is cl-closed $\forall\ H \in S$, we say that S is *cl-fixed* (or that S is *fixed by cl*).

Similarly, the set of all subgroups of G which are int-open is *int-fixed* (or *fixed by int*).

Note that every closure operation cl has at least one subgroup that is fixed with respect to cl; that subgroup is precisely H^{cl}. Similarly, H^{in} is always int-fixed.

3.4 A NOTION OF A "BOUNDARY"

You don't talk about open and closed sets without talking about the boundary. A standard definition of the boundary of a set S is

$$\partial S = \overline{S} \backslash S^{\circ},$$

where $\overline{S}$ denotes the closure of S, and S° the interior. Of course, the closure of ∂S is ∂S, and the interior of ∂S is the empty set.

In the context of subgroups, then, an analogous definition of the boundary of $H < G$ could be something along the lines of

$$\partial H = H^{cl} \backslash H^{int},$$

where *cl* and *int* are chosen appropriately.

Of course, by the way the "boundary" of H is defined it makes no sense to talk about its closedness or openness — since H^{cl}, H^{int} are subgroups of G, $\partial H = H^{cl} \backslash H^{int}$ does not contain the identity element, and hence cannot be a subgroup!

Instead, we will approach this from another angle: What elements should be appended to H^{int} in order to give us H^{cl}? It will be these elements that we will consider to be the "boundary" of H. We will formally define it below.

Definition 3.4.1. Let G be a group, and cl, int a pair of closure and interior operations on a set S of subgroups of G. For each $H \in S$, we define the *boundary of* H, ∂H, to be the set of all elements in $H^{cl} \backslash H^{int}$.

We will first focus on the examples given by the corresponding closure operations of Section 3.3.

Example 3.4.1. (*Example 3.3.1 revisited*). Let *cl* and *int* be the trivial closure and interior, resp. Then for $H < G$, $H^{cl} = G$ and $H^{int} = \{e\}$.

Then the boundary of H, ∂H, is $H^{cl} \backslash H^{int} = G \backslash \{e\}$.

Example 3.4.2. (*Example 3.3.2 revisited*). Let S be the set of all normal subgroups of G, and take the normal closure and core of $H \in S$.

Then $\partial H = H^{cl} \backslash H^{int} = \langle H^G \rangle \backslash \operatorname{Core}_G(H) = H \backslash H = \emptyset$.

Example 3.4.3. (*Example 3.3.3 revisited*). Fix $K < G$, and take $H^{cl} = H \vee K$ and $H^{int} = H \cap K$. To really see what the boundary will look like, we focus our attention on a more concrete group G.

Let $G = \mathbb{Z}$, take $K = q\mathbb{Z}$ and let $H = p\mathbb{Z}$ with $p, q \in \mathbb{Z}$.

Then $H^{cl} = H \vee K = \langle p\mathbb{Z} \cup q\mathbb{Z} \rangle = \{np + mq \mid n, m \in \mathbb{Z}\}$.

$= \langle \gcd(p, q) \rangle$.

Also $H^{int} = H \cap K = p\mathbb{Z} \cap q\mathbb{Z} = \langle \operatorname{lcm}(p, q) \rangle$.

Then $\partial H = H^{cl} \backslash H^{int} = \langle \gcd(p, q) \rangle \backslash \langle \operatorname{lcm}(p, q) \rangle$

$= \{s \in \langle \gcd(p, q) \rangle \mid \operatorname{lcm}(p, q) \nmid s\}$.

As we can see by these examples, the idea of a boundary isn't intuitively comparable to that in analysis, topology, etc.

A

APPENDIX - ADDITIONAL PROOFS

It is here that we provide the proofs for some of the claims and propositions given in the main body of the text; these proofs are found here because they are too long or tedious to display at the time they are acknowledged.

A.1 PROOFS FROM CHAPTER 1

Proposition A.1.1. *(see Section 1.1, Proposition 1.1.1). Let $(G, \cdot)$ be a group. Then the inverse elements are unique.*

Proof. By contradiction.

Suppose $a \in G$ has two distinct inverses $b, c \in G$.

Then $ab = ba = e$ and $ac = ca = e$.

So, $ab = ac$.

Then, multiplying through by b from the left gives us

$b(ab) = b(ac)$.

Associativity $\implies$ $(ba)b = (ba)c$.

$ba = e \implies (ba)b = eb = b$ and $(ba)c = ec = c$.

Therefore $b = c$, which is a contradiction.

So a has precisely one inverse element in G. □

Proposition A.1.2. *(see Section 1.1, Proposition 1.1.3). Let H, K be subgroups of a group G. Then $H \cap K < G$.*

Proof. We use the Subgroup Criterion.

Take $a, b \in H \cap K$.

Then $a, b \in K$ and $a, b \in H$.

Then $b^{-1} \in K$ (and $b^{-1} \in H$) so ab^{-1} is in both H and K.

Therefore $ab^{-1} \in H \cap K$. □

Proposition A.1.3. *(see Section 1.1, Proposition 1.1.4). Let a_1, a_2 be elements in a group G. Then $\langle a_1 \rangle < \langle a_1, a_2 \rangle$ (likewise, $\langle a_2 \rangle < \langle a_1, a_2 \rangle$).*

Proof. We prove $\langle a_1 \rangle < \langle a_1, a_2 \rangle$.

Take $g \in \langle a_1 \rangle$. Then $g = a_1^m$, some $m \in \mathbb{Z}$.

Then $a_1^m = a_1^m \cdot e = a_1^m a_2^0 \in \langle a_1, a_2 \rangle$.

So $g \in \langle a_1, a_2 \rangle$ and hence $\langle a_1 \rangle < \langle a_1, a_2 \rangle$. □

Proposition A.1.4. *(see Section 1.1, Definition 1.1.8). Suppose $H, K < G$ and suppose further that $K < H$. Then $HK = H$.*

Proof. Know that $H \subseteq HK$. Need only see that $HK \subseteq H$.

Take $x \in HK$.

Then $x = h_1 k_1 h_2 k_2 \cdots h_n k_n$.

$h_i \in H$ and $k_i \in K \subseteq H$ for all i.

Thus $x \subseteq H$. □

Proposition A.1.5. *(see Section 1.2, Proposition 1.2.2). Let G be a finitely generated Abelian group. Then G is Noetherian.*

Proof. Let G be a finitely generated Abelian group with n generators,

and $\{e\} < H_1 < H_2 < \cdots$ a chain of subgroups of G.

By Proposition 1.2.1, each subgroup H_i is generated by no more than n elements.

So, each H_i is also finitely generated, and Abelian because G is.

Then if H_j is generated by $m \leq n$ elements ($j \in \mathbb{N}$), H_{j-1} is generated by no more than m elements.

Furthermore, if H_j and H_{j-1} both have m elements, then $H_j = H_{j-1}$ because $H_{j-1} < H_j$.

Thus any chain $\{e\} < H_1 < H_2 < \cdots$ has at most $n+1$ subgroups in it (counting the trivial subgroup $\{e\}$).

Therefore, G satisfies the ACC and hence is Noetherian. ☐

Proposition A.1.6. *(see Section 1.2, Example 1.2.1). Let $\mathbb{F}_2$ be the free group on two generators. Then $[\mathbb{F}_2, \mathbb{F}_2] = \langle [a^n, b^m] \mid m, n \in \mathbb{Z} \rangle$.*

Proof. The $\supseteq$ inclusion is clear.

Show $\subseteq$: Take $w \in [\mathbb{F}_2, \mathbb{F}_2]$.

Then w is of the form $xyx^{-1}y^{-1}$, where $x = a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k}$ and $y = a^{u_1}b^{v_1} \cdots a^{u_j}b^{v_j}$.

Then
$$
\begin{aligned}
w &= xyx^{-1}y^{-1} \\
&= (a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k})(a^{u_1}b^{v_1} \cdots a^{u_j}b^{v_j}) \\
&\quad \cdot (a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k})^{-1}(a^{u_1}b^{v_1} \cdots a^{u_j}b^{v_j})^{-1} \\
&= (a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k})(a^{u_1}b^{v_1} \cdots a^{u_j}b^{v_j}) \\
&\quad \cdot (b^{-m_k}a^{-n_k} \cdots b^{-m_1}a^{-n_1})(b^{-v_j}a^{-u_j} \cdots b^{-v_1}a^{-u_1}) \\
&= a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k}a^{u_1}b^{v_1} \cdots a^{u_j}b^{v_j-m_k}a^{-n_k} \\
&\quad \cdots b^{-m_1}a^{-n_1}b^{-v_j}a^{-u_j} \cdots b^{-v_1}a^{-u_1}.
\end{aligned}
$$

Notice that the product $a^{n_1}b^{m_1}a^{n_2}b^{m_2}$ can be expressed
$$
\begin{aligned}
&a^{n_1}b^{m_1}(a^{-n_1}b^{-m_1})(b^{m_1}a^{n_1})a^{n_2}b^{m_2} \\
&= (a^{n_1}b^{m_1}a^{-n_1}b^{-m_1})b^{m_1}a^{n_1}a^{n_2}b^{m_2} \\
&= [a^{n_1}, b^{m_1}]b^{m_1}a^{n_1}a^{n_2}b^{m_2}
\end{aligned}
$$

$$= [a^{n_1}, b^{m_1}]b^{m_1}a^{n_1+n_2}b^{m_2}$$

$$= [a^{n_1}, b^{m_1}]b^{m_1}a^{n_1+n_2}(b^{-m_1}a^{-n_1-n_2})(a^{n_1+n_2}b^{m_1})b^{m_2}$$

$$= [a^{n_1}, b^{m_1}](b^{m_1}a^{n_1+n_2}b^{-m_1}a^{-n_1-n_2})a^{n_1+n_2}b^{m_1}b^{m_2}$$

$$= [a^{n_1}, b^{m_1}][a^{n_1+n_2}, b^{m_1}]^{-1}(a^{n_1+n_2}b^{m_1}b^{m_2})$$

$$= [a^{n_1}, b^{m_1}][a^{n_1+n_2}, b^{m_1}]^{-1}(a^{n_1+n_2}b^{m_1+m_2})$$

Likewise, for the product $b^{-v_2}a^{-u_2}b^{-v_1}a^{-u_1}$, we find that

$$b^{-v_2}a^{-u_2}b^{-v_1}a^{-u_1}$$

$$= b^{-v_2}a^{-u_2}a^{-u_1}b^{-v_1}[a^{-u_1}, b^{-v_1}]^{-1}$$

$$= b^{-v_2}a^{-u_2-u_1}b^{-v_1}[a^{-u_1}, b^{-v_1}]^{-1}$$

$$= b^{-v_2}b^{-v_1}a^{-u_2-u_1}[a^{-u_2-u_1}, b^{-v_1}][a^{-u_1}, b^{-v_1}]^{-1}$$

$$= (b^{-v_2-v_1}a^{-u_2-u_1})[a^{-u_2-u_1}, b^{-v_1}][a^{-u_1}, b^{-v_1}]^{-1}$$

It is clear to see that if we were to continue this, w is in fact the product of commutators $[a^n, b^m]$ and their inverses.

So $w \in \langle [a^n, b^m] \mid m, n \in \mathbb{Z} \rangle$.

So indeed $[\mathbb{F}_2, \mathbb{F}_2] \subseteq \langle [a^n, b^m] \mid m, n \in \mathbb{Z} \rangle$. □

Proposition A.1.7. *(see Section 1.2, Proposition 1.2.3). Let G be an infinitely generated group. Then under no circumstance is G Noetherian.*

Proof. By contradiction.

Suppose G is infinitely generated and Noetherian.

Take $g_1 \in S$, the generating set for G.

Then clearly $\{e\} < \langle g_1 \rangle$.

Take another element $g_2 \in S \backslash \{\langle g_1 \rangle\}$.

Then we have that $\{e\} < \langle g_1 \rangle < \langle g_1, g_2 \rangle$.

Continue taking elements $g_n \in S \backslash \{\langle g_1, g_2, \ldots, g_{n-1} \rangle\}$ to obtain the chain of subgroups

$$\{e\} < \langle g_1 \rangle < \langle g_1, g_2 \rangle < \cdots < \langle g_1, g_2, \ldots g_{n-1}, g_n \rangle$$

Since G is supposed to be Noetherian, this chain will terminate after, say, $m+1$ terms (accounting for the trivial subgroup).

But since S is infinite, there is always going to be an element $g_{m+1} \in S \backslash \{\langle g_1, g_2, \ldots, g_m \rangle\}$ such that

$$\{e\} < \langle g_1 \rangle < \cdots < \langle g_1, \ldots, g_m \rangle < \langle g_1, \ldots, g_m, g_{m+1} \rangle,$$

a contradiction since this chain was to end after $m+1$ terms (at the subgroup attained by appending g_m).

So this chain of subgroups does not satisfy the ACC, and hence G cannot be Noetherian. ☐

Proposition A.1.8. *(see Section 1.3, Proposition 1.3.2). For any group homomorphism $\varphi : G \mapsto G'$, ker $\varphi < G$.*

Proof. The first two group properties follow from the definition of ker φ.

All that needs to be shown is that ker φ contains all its inverse elements.

Let $a \in \ker \varphi$. We want to see that $a^{-1} \in \ker \varphi$.

$a \in \ker \varphi \implies \varphi(a) = e'$.

Know that $aa^{-1} = e$.

Then $\varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1}) = \varphi(a^{-1})$.

On the other hand, $\varphi(aa^{-1}) = \varphi(e) = e'$

(Note that a similar computation holds for $a^{-1}a = e$).

Thus $\varphi(a^{-1}) = e'$, and so $a^{-1} \in \ker \varphi$.

So ker φ is a subgroup of G. ☐

Proposition A.1.9. *(see Section 1.4.1, Proposition 1.4.2). For a subgroup $H < G$, then the number of left and right cosets of H in G are equal.*

Proof. Define the map $Ha \mapsto a^{-1}H$. This map is a bijection:

Show 1-1: Suppose $a^{-1}H = b^{-1}H$.

Then $H = ba^{-1}H \implies ba^{-1} \in H$.

H a subgroup $\implies ab^{-1} \in H$ (Proposition 1.1.2).

Therefore $H = Hab^{-1} \implies Hb = Ha$.

So the map is 1-1.

Show onto: Take $aH \subset G$.

G a group $\implies a^{-1} \in G$ and we then have that $Ha^{-1} \mapsto aH$.

So the number of left cosets equals the number of right cosets. ∎

Proposition A.1.10. *(see Section 1.4.1, Proposition 1.4.3). Let $H, K < G$. Then $HK < G \iff HK = KH$.*

Proof. $(\Leftarrow)$: Suppose $HK = KH$.

We invoke the Subgroup Criterion: take $a, b \in HK$.

Then $a = h_1k_1$, $b = h_2k_2$.

$ab^{-1} = (h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$

$k_1k_2^{-1} \in K$, so say $k_1k_2^{-1} = k_3$.

Then $ab^{-1} = h_1k_3h_2^{-1}$.

$KH = HK \implies k_3h_2^{-1} = h_3k_4$ for some $h_3 \in H$, $k_4 \in K$.

So $ab^{-1} = h_1h_3k_4 \in HK$.

Hence the Subgroup Criterion is satisfied and $HK < G$.

$(\Rightarrow)$: Now suppose $HK < G$. Want to see that $HK = KH$.

Show $HK \subseteq KH$: take $hk \in HK$.

Since $HK < G$, $hk = a^{-1}$ for some $a \in HK$.

Let $a = h_1k_1$.

Then $hk = a^{-1} = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$.

So $HK \subseteq KH$.

Show $KH \subseteq HK$:

Know $H < HK$ and $K < HK$.

Hence for all $h \in H$, $k \in K$ we have that $kh \in KH$.

So, for all $kh \in KH$ we have that $kh \in HK$.

So $KH \subseteq HK$.

Therefore $HK = KH$. □

Theorem A.1.11. *(see Section 1.4.3, Theorem 1.4.4). Let $N < G$. Then the following are equivalent:*

(i) $gN = Ng$ *for all* $g \in G$*;*

(ii) *For all* $g \in G$*,* $gNg^{-1} \subset N$*;*

(iii) $gNg^{-1} = N$*.*

Proof. *(i)* $\implies$ *(ii)*: $gN = Ng \implies \forall\, g \in G, \exists\, n, n' \in N$

such that $gn = n'g$.

Then $gng^{-1} = n' \in N$.

So $gNg^{-1} \subset N$.

(ii) $\implies$ *(iii)*: Suppose $gNg^{-1} \subset N$.

Show $N \subset gNg^{-1}$ for all g: Since $gNg^{-1} \subset N$,

then $g^{-1}Ng \subset N$ also.

Then $n = (gg^{-1})n(gg^{-1}) = g(g^{-1}ng)g^{-1} \subset gNg^{-1}$.

So $N = gNg^{-1}$ for all $g \in G$.

(iii) $\implies$ *(i)*: This is immediate. □

Theorem A.1.12. *(see Section 1.4.3, Theorem 1.4.5). Let $N \triangleleft G$ and let G/N denote the set of all (left) cosets of N in G. Then G/N is a group under the binary operation given by $(aN)(bN) = abN$ and $|G/N| = [G : N]$.*

Proof. *Show G/N is a group*: Associativity is clear.

N is our identity element; indeed, for all $g \in G$ we have

$(N)(gN) = (eN)(gN) = egN = gN$ and

$(gN)(N) = (gN)(eN) = geN = gN$.

The inverse of gN is $g^{-1}N$: $(gN)(g^{-1}N) = gg^{-1}N = N$

and $(g^{-1}N)(gN) = g^{-1}gN = N$.

Show $|G/N| = [G : N]$: This is clear by the definition of G/N.

□

Proposition A.1.13. *(see Section 1.4.3, Proposition 1.4.6). Let $\varphi : G \to G'$ be a group homomorphism. Then $\ker \varphi \triangleleft G$.*

Proof. Let $h \in \ker \varphi$, and take $g \in G$.

Then $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1})$

[since $h \in \ker \varphi$] $= \varphi(gg^{-1}) = e' \in G'$.

Thus $ghg^{-1} \in \ker \varphi$.

So $g(\ker \varphi)g^{-1} \subset \ker \varphi$.

Theorem 1.4.3 $\implies$ $\ker \varphi \triangleleft G$. □

Proposition A.1.14. *(see Section 1.4.3, Lemma 1.4.7). Let $K, N < G$ with $N \triangleleft G$. Then*

(i) $(N \cap K) \triangleleft K$;

(ii) $N \triangleleft (N \vee K)$;

(iii) $NK = N \vee K = KN$; *and*

(iv) If $K \triangleleft G$ and $K \cap N = \langle e \rangle$, then $nk = kn$ for all $k \in K$, $n \in N$.

Proof. *Show (i)*: Take $n \in N \cap K$ and $k \in K$.

$N \triangleleft G \implies knk^{-1} \in N$.

$n \in N \cap K \implies knk^{-1} \in K$.

Then $knk^{-1} \in N \cap K \; \forall \; n \in N \cap K$.

Thus $k(N \cap K)k^{-1} \subset N \cap K \; \forall \; k \in K$.

Hence $(N \cap K) \triangleleft K$.

Show (ii): Since $N < (N \vee K)$, we have that $N = N \cap (N \vee K)$.

Proposition 1.2.3 $\implies N \vee K < G$.

So by (i), $N = N \cap (N \vee K) \triangleleft (N \vee K)$.

Show (iii): $NK = \{nk \mid n \in N, k \in K\} \subset N \vee K$

(see Def. 1.1.8, 1.2.2).

Take $h = n_1 k_1 n_2 k_2 \cdots n_r k_r$, some $r \in \mathbb{N}$.

$N \triangleleft G \implies n_i k_i = k_j n_i'$, where $n_i' \in N$.

So $n_1 k_1 n_2 k_2 \cdots n_r k_r = n_1 (k_1 n_2)(k_2 n_3) \cdots (k_{r-1} n_r) k_r$

$= n_1 (n_2' k_{j_1})(n_3' k_{j_2}) \cdots (n_r' k_{j_{r-1}}) k_r$

$= n_1 n_2' (k_{j_1} n_3') \cdots k_{j_{r-1}} k_r$

$= \cdots$

$= nk$, where $n = n_1 n_2' \cdots$ and $k = \cdots k_{j_{r-1}} k_r$.

So $NK = N \vee K$.

Similarly, $KN = N \vee K$, so $NK = N \vee K = KN$.

Show (iv): If $K \triangleleft G$, then $\forall\, k \in K, n \in N$ we have that $nkn^{-1} \in K$.

$N \triangleleft G \implies knk^{-1} \in N$.

Then $nkn^{-1}k^{-1} = n(kn^{-1}k^{-1}) \in N$,

$nkn^{-1}k^{-1} = (nkn^{-1})k^{-1} \in K$.

So $nkn^{-1}k^{-1} \in N \cap K = \langle e \rangle$, hence $nk = kn$. □

Proposition A.1.15. *(see Section 1.4.3, Proposition 1.4.8). Let $\varphi : G \to G'$ be an isomorphism. If $N \triangleleft G$, then $\varphi(N) \triangleleft G'$.*

Proof. Let $N' = \varphi(N)$, where $N \triangleleft G$.

$N \triangleleft G \implies N = g^{-1} N g \;\forall\, g \in G$.

Then $N' = \varphi(N) = \varphi(g^{-1} N g) = (\varphi(g)^{-1}) N' (\varphi(g)) \;\forall\, g \in G$.

So $N' \triangleleft G'$. □

Proposition A.1.16. *(see Section 1.5, Proposition 1.5.1). $N_G(H) < G$, where $H < G$ (see Definition 1.5.1).*

Proof. We prove this using the Subgroup Criterion: Take $a, b \in N_G(H)$.

$b \in N_G(H) \implies b^{-1} \in N_G(H)$; indeed,

$b \in N_G(H) \implies bHb^{-1} = H$

$\implies H = b^{-1}H(b^{-1})^{-1} = b^{-1}Hb.$

Then $(ab^{-1})H(ab^{-1})^{-1} = (ab^{-1})H(ba^{-1}) = a(b^{-1}Hb)a^{-1}$

$= aHa^{-1} = H.$

So $N_G(H) < G$. □

Proposition A.1.17. *(see Section 1.5, Proposition 1.5.2). The following are all properties of the Normalizer:*

(i) $H < N_G(H)$*;*

(ii) $H \lhd N_G(H)$*; and*

(iii) If $H \subseteq K$ *and* $H \lhd K$*, then* $K \subseteq N_G(H)$*.*

Proof. *Show (i)*: This is clear, as for all $h, h_1 \in H$

we have that $h(h_1)h^{-1} \in H$.

Show (ii): This is also clear.

Take $g \in N_G(H)$. Then $gHg^{-1} = H \implies H \lhd N_G(H)$.

Show (iii): Suppose $H \lhd K$.

Then for all $k \in K$, $kHk^{-1} = H$.

$\implies k \in N_G(H)$, so $K \subseteq N_G(H)$. □

Proposition A.1.18. *(see Section 1.5, Proposition 1.5.3). $H \vee K < G$, where $H, K < G$ (see Definition 1.5.2).*

Proof. We use the Subgroup Criterion again.

Take $x, y \in H \vee K$.

Then $x = h_1^{r_1} k_1^{s_1} \cdots h_n^{r_n} k_n^{s_n}$, $y = \tilde{h}_1^{u_1} \tilde{k}_1^{v_1} \cdots \tilde{h}_m^{u_m} \tilde{k}_m^{v_m}$.

$xy^{-1} = (h_1^{r_1} k_1^{s_1} \cdots h_n^{r_n} k_n^{s_n})(\tilde{h}_1^{u_1} \tilde{k}_1^{v_1} \cdots \tilde{h}_m^{u_m} \tilde{k}_m^{v_m})^{-1}$

$= (h_1^{r_1} k_1^{s_1} \cdots h_n^{r_n} k_n^{s_n})(\tilde{k}_m^{-v_m} \tilde{h}_m^{-u_m} \cdots \tilde{k}_1^{-v_1} \tilde{h}_1^{-u_1})$

$= h_1^{r_1} k_1^{s_1} \cdots h_n^{r_n} k_n^{s_n} \tilde{k}_m^{-v_m} \tilde{h}_m^{-u_m} \cdots \tilde{k}_1^{-v_1} \tilde{h}_1^{-u_1} \in H \vee K$.

So $H \vee K < G$. □

Proposition A.1.19. *(see Section 1.5, Proposition 1.5.4). Take any $H, K < G$. Then $H \vee K = \bigcap \{ J < G \mid H, K < J\}$.*

Proof. Show $H \vee K \subseteq \bigcap \{ J \mid H, K < J\}$: Recall $H \vee K = \langle H \cup K \rangle$.

Take $J < G$ such that $H < J$ and $K < J$.

Hence $H \cup K \subseteq J \implies \langle H \cup K \rangle \subseteq \langle J \rangle = J$.

So $H \vee K \subseteq J$.

This holds for all such J, so $H \vee K \subseteq \bigcap \{ J \mid H, K < J\}$.

Show $H \vee K \supseteq \bigcap \{ J \mid H, K < J\}$: Take $g \in \{ J \mid H, K < J\}$.

Then $g \in J \ \forall\ J > H, K$.

In particular, $H > H$ and $K > K$.

$\implies g \in H \cap K \subseteq H \cup K$.

Thus $g \in \langle H \cup K \rangle = H \vee K$.

This holds $\forall\ g \in \bigcap \{ J \mid H, K < J\}$, so

$\bigcap \{ J \mid H, K < J\} \subseteq H \vee K$. □

Proposition A.1.20. *(see Section 1.5, Proposition 1.5.5). $[G, G] < G$ (see Definition 1.5.3).*

Proof. We always use the Subgroup Criterion.

Take $x, y \in [G, G]$.

Then $x = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $x_i = a_i b_i a_i^{-1} b_i^{-1}$, $i = 1, \ldots, n$.

Similarly, $y = y_1^{s_1} \cdots y_m^{s_m}$, with $y_j = c_j d_j c_j^{-1} d_j^{-1}$, $j = 1, \ldots, m$.

Then $xy^{-1} = (x_1^{r_1} \cdots x_n^{r_n})(y_1^{s_1} \cdots y_m^{s_m})^{-1}$

$= (x_1^{r_1} \cdots x_n^{r_n})(y_m^{-s_m} \cdots y_1^{-s_1})$

$\in [G, G]$ since $y_j^{-s_j} \in [G, G]$ for all $j = 1, \ldots, m$.

So $[G, G] < G$. □

Proposition A.1.21. *(see Section 1.5, Proposition 1.5.6 and Definition 1.5.5). The stabilizer G_s of G at $s \in S \subseteq G$ is a subgroup of G.*

Proof. We utilize the Subgroup Criterion.

Take $a, b \in G_s$.

Note that if $g \in G_s$, so is g^{-1}; indeed, $gs = s \implies s = g^{-1}s$,

so $g^{-1} \in G_s$.

Then $(ab^{-1})s = a(b^{-1}s) = a(s) = s$.

So $ab^{-1} \in G_s$ and hence $G_s < G$. □

Proposition A.1.22. *(see Section 1.5, Proposition 1.5.7 and Definition 1.5.6).*

(i) $\langle S^G \rangle$ is a normal subgroup of G; and

(ii) If $N \triangleleft G$, then $N = \langle N^G \rangle$.

Proof. *Show (i)*: To make things easier on the eyes, let $H = \langle S^G \rangle$.

We use Theorem 1.5.4 and show that $gH = Hg$.

Show $\subseteq$: $gH = \{gh \mid h \in H\}$, and take $gh \in gH$.

Then $gh = g(g_1^{-1}h_1g_1)(g_2^{-1}h_2g_2) \cdots (g_n^{-1}h_ng_n)$

$= g(g_1^{-1}h_1g_1(g^{-1}g))(g_2^{-1}h_2g_2) \cdots (g_n^{-1}h_ng_n)$

$= [(g_1g^{-1})^{-1}h_1(g_1g^{-1})]g(g_2^{-1}h_2g_2) \cdots (g_n^{-1}h_ng_n)$

$= [(g_1g^{-1})^{-1}h_1(g_1g^{-1})]g(g_2^{-1}h_2g_2(g^{-1}g) \cdots (g_n^{-1}h_ng_n)$

$= [(g_1g^{-1})^{-1}h_1(g_1g^{-1})][(g_2g^{-1})^{-1}h_2(g_2g^{-1})]g \cdots (g_n^{-1}h_ng_n)$

$= \cdots$

$= [(g_1g^{-1})^{-1}h_1(g_1g^{-1})] \cdots [(g_ng^{-1})^{-1}h_n(g_ng^{-1})]g$

$\in Hg$.

(Note: elements in H are really products of powers of

$g_i^{-1}h_ig_i$, but since, for example, $(g_i^{-1}h_ig_i)^{r_i} = g_i^{-1}h_i^{r_i}g_i$

and $h_i^{r_i} \in H$ we can represent this as just $g_i^{-1}h_ig_i$ for

some appropriately relabeled $h_i \in H$).

Show $\supseteq$: A similar computation shows that $Hg \subseteq gH$.

So $Hg = gH \implies H = \langle S^G \rangle \triangleleft G$.

Show (ii): That $N \subseteq \langle N^G \rangle$ is clear.

Show $\supseteq$: Take $n \in \langle N^G \rangle$.

Then $n = (g_1^{-1}n_1g_1)(g_2^{-1}n_2g_2)\cdots(g_m^{-1}n_mg_m)$.

But $g_i \in G$ and $N \triangleleft G \implies g_i^{-1}n_ig_i \in N$ for all i.

Thus $n \in N$, so $\langle N^G \rangle \subseteq N$.

So $N \triangleleft G \implies N = \langle N^G \rangle$. □

Proposition A.1.23. *(see Section 1.5, Proposition 1.5.8 and Definition 1.5.6). Let $S \subseteq G$. Then $\langle S^G \rangle = \bigcap H_i$, where $S \subseteq H_i$ and $H_i \triangleleft G\ \forall\ i$.*

Proof. Show $\langle S^G \rangle \subseteq \bigcap H_i$: Take $s \in \langle S^G \rangle$.

Then $s = g_1^{-1}s_1g_1 \cdots g_n^{-1}s_ng_n$, where $g_j \in G$ and $s_j \in S\ \forall\ j$.

$S \subseteq H_i\ \forall\ i$, so $s_j \in H_i\ \forall\ i, j$.

$\implies g_j^{-1}s_jg_j \in g_j^{-1}H_ig_j\ \forall\ i, j$.

But $H_i \triangleleft G \implies g_j^{-1}H_ig_j = H_i\ \forall\ i, j$.

Therefore $s \in \bigcap H_i$.

This holds $\forall\ s \in \langle S^G \rangle$, so $\langle S^G \rangle \subseteq \bigcap H_i$.

Show $\langle S^G \rangle \supseteq \bigcap H_i$: Take $s \in \bigcap H_i$.

Then $s \in H_i\ \forall\ i$.

In particular, $s \in \langle S^G \rangle$ by Proposition 1.5.7 (i).

This holds for all $s \in \bigcap H_i$, so $\langle S^G \rangle \supseteq \bigcap H_i$.

Therefore $\langle S^G \rangle$ is the intersection of all normal subgroups

containing S. □

Proposition A.1.24. *(see Section 1.5, Proposition 1.5.9 and Definition 1.5.7).*

(i) $\mathrm{Core}_G(H) \triangleleft G$;

(ii) If $N \triangleleft G$, then $N = \mathrm{Core}_G(N)$; and

(iii) $\mathrm{Core}_G(N)$ is the largest normal subgroup of G contained in N.

Proof. *Show (i)*: That $\mathrm{Core}_G(H) < G$ is clear, since $e \in g^{-1}Hg$

for all $H < G$ and $g \in G$. Also, if $g^{-1}hg \in \mathrm{Core}_G(H)$,

then $(g^{-1}hg)^{-1} = g^{-1}h^{-1}g \in \mathrm{Core}_G(H)$.

To see that $\mathrm{Core}_G(H) \triangleleft G$: We set out to prove

$g(\mathrm{Core}_G(H)) = (\mathrm{Core}_G(H))g$ (see Theorem 1.5.4).

Show $\subseteq$: Take $a \in g(\mathrm{Core}_G(H))$.

Then $a \in g(g'^{-1}Hg')$ for all $g' \in G$.

But $g(g'^{-1}Hg') = gg'^{-1}Hg'(g^{-1}g)$

$= [(g'g^{-1})^{-1}Hg'g^{-1}]g$ for all $g' \in G$.

So $a \in [(g'g^{-1})^{-1}Hg'g^{-1}]g \implies a \in (\mathrm{Core}_G(H))g$.

Show $\supseteq$: A similar approach gives us that $a \in (\mathrm{Core}_G(H))g$

$\implies a \in g(\mathrm{Core}_G(H))$.

So $\mathrm{Core}_G(H) \triangleleft G$.

Show (ii): Let N be a normal subgroup of G.

Then clearly $\mathrm{Core}_G(N) \subseteq N$, since $a \in \mathrm{Core}_G(N)$

$\implies a \in g^{-1}Ng = N$.

Show $\supseteq$: Take $n \in N$.

Then $n \in g^{-1}Ng$ for all $g \in G$ since $N = g^{-1}Ng$.

Therefore $n \in \bigcap_{g \in G} g^{-1}Ng = \mathrm{Core}_G(N)$.

So $N = \mathrm{Core}_G(N)$ for all $N \triangleleft G$.

Show (iii): Suppose $\exists\ M \triangleleft G$ and $\mathrm{Core}_G(N) \subseteq M \subsetneq N$.

$M \subsetneq N \implies g^{-1}Mg \subsetneq g^{-1}Ng\ \forall\ g \in G$.

But $M \triangleleft G$, so $g^{-1}Mg = M \; \forall \; g \in G$.

$\implies M \subseteq \bigcap_{g \in G} g^{-1}Ng = \text{Core}_G(N)$.

Therefore $\text{Core}_G(N) = M$. ∎

A.2 PROOFS FROM CHAPTER 2

Proposition A.2.1. *(see Section 2.1, Example 2.1.5). Let $H < G$, where G is a Noetherian Abelian group. Then $rad(H) < G$.*

Proof. Recall $\text{rad}(H) = \{g \in G \mid g^n \in H \text{ some } n > 0\}$.

We will use the Subgroup Criterion.

Take $a, b \in \text{rad}(H)$.

Then $\exists \; n, m \in \mathbb{N}$ such that $a^n, b^m \in H$.

$\implies a^{mn}, b^{mn} \in H$.

Subgroup Criterion $\implies a^{mn}b^{-mn} = (ab^{-1})^{mn} \in H$

So $ab^{-1} \in \text{rad}(H)$ since $mn > 0$. ∎

Proposition A.2.2. *(see Section 2.2, Proposition 2.2.1). Let $f : G \to H$ be a nonzero homomorphism mapping a group G to a simple group H. Then f is an epimorphism $\iff \langle K^{f(G)} \rangle = f(G)$ for all nontrivial subgroups K of $f(G)$.*

Proof. $(\Rightarrow)$: Suppose f is an epimorphism.

Then $f(G) = H$.

Clearly $\langle K^H \rangle \subseteq H$.

H simple $\implies$ H contains no nontrivial normal subgroups.

By Proposition 1.5.7(i), $\langle K^H \rangle \triangleleft H$ so $\langle K^H \rangle = H$.

$(\Leftarrow)$: Now suppose $\langle K^{f(G)} \rangle = f(G)$ for all $K < G$.

$f(G) < H$ and H simple $\implies$ either $\langle K^{f(G)} \rangle = \{e\}$ or

$\langle K^{f(G)} \rangle = H$ by Proposition 1.5.7(ii).

Then $\langle K^{f(G)} \rangle = f(G) = H$ since this holds for all $K < G$.

Therefore f is an epimorphism. □

Proposition A.2.3. *(see Section 2.2, Proposition 2.2.4). Let $\varphi : G \to G'$ be a group homomorphism. Then for $H, K < G$, $\varphi(H \vee K) = \varphi(H) \vee \varphi(K)$.*

Proof. Show $\subseteq$: Take $x \in H \vee K$.

Then $x = h_1k_1h_2k_2 \cdots h_nk_n$, where $h_i \in H$ and $k_i \in K$.

So $\varphi(x) = \varphi(h_1k_1h_2k_2 \cdots h_nk_n) = \varphi(h_1)\varphi(k_1) \cdots \varphi(h_n)\varphi(k_n)$

since φ homomorphism.

Hence $\varphi(x) \in \varphi(H) \vee \varphi(K)$.

Show $\supseteq$: This is clear by reversing the steps for "$\subseteq$". □

Proposition A.2.4. *(see Section 2.5, Construction 5). Let H be a finitely generated subgroup of a group G, with generating set $\{h_1, h_2, \ldots, h_n\}$. Then $H = \bigvee_i \langle h_i \rangle$.*

Proof. *Show* $\subseteq$: This is clear.

Show $\supseteq$: Take $g \in \bigvee_i \langle h_i \rangle$.

By definition, $\bigvee_i \langle h_i \rangle = \langle \bigcup_i \langle h_i \rangle \rangle = \langle \langle h_1 \rangle \cup \langle h_2 \rangle \cup \ldots \cup \langle h_n \rangle \rangle$.

Since $g \in \bigvee_i \langle h_i \rangle$, g is of the form $g = h_1^{r_1} h_2^{r_2} \cdots h_n^{r_n}$, $r_i \in \mathbb{Z}$.

But then $g \in H$, and hence $\bigvee_i \langle h_i \rangle \subseteq H$.

So $H = \bigvee_i \langle h_i \rangle$. □

A.3 PROOFS FROM CHAPTER 3

Proposition A.3.1. *(see Section 3.1, Examples 3.1.1 - 3.1.6). The following are all normal subgroups of $H < G$: H, $\{e\}$, $ker_H(\varphi)$, H_s, and $Core_G(H)$.*

Proof. The normality of all but H_s are either clear or proved

elsewhere.

Show $H_s \triangleleft H$: We show $gH_sg^{-1} \subseteq H_s \ \forall\, g \in H$.

Take $ghg^{-1} \in gH_sg^{-1}$.

Then $(ghg^{-1})(s) = g(h(g^{-1}(s))) = g(g^{-1}(s))$ since $h \in H_s$.

$\implies (ghg^{-1})(s) = g(g^{-1}(s)) = s$, so $ghg^{-1} \in H_s$.

Thus $H_s \triangleleft H$. □

Proposition A.3.2. *(see Section 3.3, Construction 7).* $H^{cl} < G$*, where* $H^{cl} := G^{int} \cdot H$ *for any interior operation on* G *such that* $G^{int} \triangleleft G$*.*

Proof. $G^{int} \triangleleft G \implies G^{int} \cdot H = G^{int} \vee H$ by Lemma 1.4.7 (iii).

$\implies G^{int} \cdot H < G$ by Proposition 1.5.3. □

LIST OF REFERENCES

[1] Dummit, David S. and Foote, Richard M. *Abstract Algebra*. 3rd Ed. John Wiley & Sons, 2004.

[2] Epstein, Neil. ”A guide to closure operations in commutative algebra”. *Progress in Commutative Algebra* 2 (2012): 1-37.

[3] Epstein, Neil. ”Semistar operations and standard closure operations”. *Comm. Algebra* 43 (2015): 325-336.

[4] Gautam, Vishvajit V. S. ”A Note on Closure Operators in Category of Groups”. arXiv:math/0212152v1 [math.GR] 11 Dec 2002.

[5] Hungerford, Thomas W. *Algebra*. New York: Springer-Verlag, 1974.