8-25-2016

Toroidal Matrix Links: Local Matrix Homotopies and Soft Tori

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THESIS

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Mathematics

The University of New Mexico

Albuquerque, New Mexico

July, 2016

Dedication

To my mom Fidelia and my wife Mirna, for their love, support and their unbreakable faith in me.

“Mathematics is the alphabet with which God has written the universe.” – Galileo Galilei

Acknowledgments

I thank God, the divine mathematician, for the adventure of life.

I would like to thank my advisor, Professor Terry Loring, for many inspiring conversations, for many challenging and interesting questions, and for all his wise advice and outstanding support to my research work. I also thank Professor Loring for sharing Loring's connectivity technique, [25] and [24] with me, which together with [27] have been a powerful source of inspiration.

I would also like to thank Professors Alexandru Buium, Charles Boyer and Judith Packer, for kindly accepting to be part of my dissertation committee. I also thank Professor Buium, for very inspiring lectures on the algebraic structure of number systems and the analogies betweeen numbers and functions.

I am also grateful to Professors Stanly Steinberg and Concepción Ferrufino for so many stimulating conversations and for their great advice.

I am very grateful with the Erwin Schrödinger Institute for Mathematical Physics of the University of Vienna, for the outstanding hospitality during my visit in August of 2014, a good part of the research reported here was carried out while I was visiting the Institute.

I also want to thank Professor Moody Chu for his warm hospitality during my visit to the Department of Mathematics at NC State, for precise comments and challenging questions, and for sharing some interesting conjectures and problems with me.

I also thank Daniel Appelö for suggesting the study of the possibility of a connection between toroidal matrix links and fast Fourier transforms.

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Abstract

In this document we solve some local connectivity problems in matrix representations of the form $C(\mathbb{T}^N) \to M_n$ and $C(\mathbb{T}^N) \to M_n \leftarrow C([-1,1]^N)$ using the so called toroidal matrix links, which can be interpreted as normal contractive matrix analogies of free homotopies in algebraic topology.

In order to deal with the locality constraints, we have combined some techniques introduced in this document with several versions of the Basic Homotopy Lemma L.2.3.2, T.2.3.1 and C.2.3.1 obtained initially by Bratteli, Elliot, Evans and Kishimoto in [4] and generalized by Lin in [19] and [22].

We have also implemented some techniques from matrix geometry, combinatorial optimization and noncommutative topology developed by Loring [24, 27], Shulman [27], Bhatia [2], Chu [8], Brockett [5], Choi [7, 6], Effros [6], Exel [11], Eilers [11], Elsner [12], Pryde [31, 30], McIntosh [30] and Ricker [30].

Contents

Chapter 1

Introduction

In this document we study the solvability of some local connectivity problems via constrained normal matrix homotopies in C*-representations of the form

$$C(\mathbb{T}^N) \longrightarrow M_n, \tag{1.0.1}$$

for a fixed but arbitrary integer $N \geq 1$ and any integer $n \geq 1$. In particular we study local normal matrix homotopies which preserve commutativity and also satisfy some additional constraints, like being rectifiable or piecewise analytic.

We consider several versions of the Basic Homotopy Lemma L.2.3.2, T.2.3.1 and C.2.3.1 obtained initially by Bratteli, Elliot, Evans and Kishimoto in [4] and generalized by Lin in [19] and [22]. We combine the basic homotopy lemma with some techniques introduced here and some other techniques from matrix geometry and noncommutative topology developed by Loring [24, 27], Shulman [27], Bhatia [2], Chu [8], Brockett [5], Choi [7, 6], Effros [6], Exel [11], Eilers [11], Elsner [12], Pryde [31, 30], McIntosh [30] and Ricker [30], to construct the so called *toroidal matrix links*, which we use to obtain the main theorems presented in Chapter 4, and which consist on local connectivity results in matrix representations of the form 1.0.1 and also of the form

$$C(\mathbb{T}^N) \longrightarrow M_n \longleftarrow C([-1,1]^N). \tag{1.0.2}$$

Toroidal matrix links can be interpreted as noncommutative analogies of free homotopies in algebraic topology and topological deformation theory, they are introduced in Chapter 3 together with some other matrix geometrical objects.

In Chapter 5 we present a connectivity technique developed by T. Loring which provides us with very important imformation on the local uniform connectivity in matrix representations of the form $C(\mathbb{T}^2) \to M_n$.

Given $\delta > 0$, a function $\varepsilon : \mathbb{R} \to \mathbb{R}_0^+$ and two matrices x, y in a set $S \subseteq M_n$ such that $\|x - y\| \leq \delta$, by a $\varepsilon(\delta)$***-local matrix homotopy*** between x and y, we mean a matrix path $X \in C([0,1], M_n)$ such that $X_0 = x$, $X_1 = y$, $X_t \in S$ and $\|X_t - y\| \leq \varepsilon(\delta)$ for each $t \in [0,1]$. We write $x \rightsquigarrow_\varepsilon y$ to denote that there is a ε-local matrix homotopy betweeen x and y.

The motivation and inspiration to study local normal matrix homotopies which preserve commutativity in C*-representations of the form 1.0.1 and 1.0.2, came from mathematical physics [16, §3] and matrix approximation theory [9].

The problems from mathematical physics which motivated this study are inverse spectral problems, which consist on finding for a certain set of matrices $X_1, \ldots, X_N$ which *approximately satisfy* a set of polynomial constraints $\mathcal{R}(x_1, \ldots, x_N)$ on N NC-variables, a set of *nearby matrices* $\tilde{X}_1, \ldots, \tilde{X}_N$ which approximate $X_1, \ldots, X_N$ and exactly satisfy the constraints $\mathcal{R}(x_1, \ldots, x_N)$. The problems from matrix approximation theory that we considered for this study, are of the type that can be reduced to the study of the solvability conditions for approximate and exact joint diagonalization problems for N-tuples of normal matrix contractions.

Since the problems which motivated the research reported in this document can be restated in terms of the study local piecewise analytic connectivity in matrix representations of the form $C_\varepsilon(\mathbb{T}^2) \to M_n \leftarrow C(\mathbb{T}^N)$ and $C_\varepsilon(\mathbb{T}^2) \to M_n \leftarrow C([-1,1]^N)$, we studied several variations of problems of the form.

Problem 1.0.1 (Lifted connectivity problem) *Given $\varepsilon > 0$, is there $\delta > 0$ such that the following conditions hold? For any integer $n \geq 1$, some prescribed sequence of linear compressions $\kappa_n : M_{2n} \to M_n$, and any two families of N pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in M_n which satisfy the constraints $\|X_j - Y_j\| \leq \delta$, $1 \leq j \leq N$, there are two families of N pairwise commuting normal contractions $\tilde{X}_1, \ldots, \tilde{X}_N$ and $\tilde{Y}_1, \ldots, \tilde{Y}_N$ in M_{2n} which satisfy the relations: $\kappa(\tilde{X}_j) = X_j$, $\kappa(\tilde{Y}_j) = Y_j$ and $\|\tilde{X}_j - \tilde{Y}_j\| \leq \varepsilon$, $1 \leq j \leq N$. Moreover, there are N peicewise analytic ε-local homotopies of normal contractions $\mathbf{X}^1, \ldots, \mathbf{X}^N \in C([0,1], M_{2n})$ between the corresponding pairs $\tilde{X}_j$, $\tilde{Y}_j$ in M_{2n}, which satisfy the relations $\mathbf{X}^j_t \mathbf{X}^k_t = \mathbf{X}^k_t \mathbf{X}^j_t$, for each $1 \leq j, k \leq N$ and each $0 \leq t \leq 1$.*

By solving problem **P.**1.0.1 we learned about the local connectivity of arbitrary δ-close N-tuples of pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in M_n, which was the main motivation of the research reported here. We also obtained some results concerning to the geometric structure of the joint spectra (in the sense of [30]) of the N-tuples.

For a given $\delta > 0$, the study of the solvability conditions of problems of the form **P.**1.0.1 provided us with geometric information about local deformations of particular representations of the form $C(\mathbb{T}^N) \to A_0 := C^*(U_1, \ldots, U_N) \subseteq M_n$ and $C(\mathbb{T}^N) \to A_1 := C^*(V_1, \ldots, V_N) \subseteq M_n$, where $U_1, \ldots, U_N, V_1, \ldots, V_N \in \mathbb{U}(n)$ are pairwise commuting matrices such that $\|U_j - V_j\| \leq \delta$. By local deformations we mean a family $\{A_t\}_{t \in [0,1]} \subseteq M_n$ of abelian C*-algebras, with $A_t := C^*(\mathbf{X}^1_t, \ldots, \mathbf{X}^N_t)$ and where $\mathbf{X}^1_t, \ldots, \mathbf{X}^N_t \in C([0,1], U_n)$ are $\varepsilon(\delta)$-local matrix homotopies between $U_1, \ldots, U_N$ and $V_1, \ldots, V_N$ for some function $\varepsilon : \mathbb{R} \to \mathbb{R}^+_0$.

The main results are presented in Chapter 4, in section §4.2 we use toroidal matrix links to obtain some local piecewise analytic connectivity results which are non-uniform in dimension. In sections §4.3, §4.4 and §4.5 we derive some uniform local connectivity results.

Some applications to approximate solution of matrix equations on words are presented in Chapter 6.

Chapter 2

Preliminaries and Notation

2.1 Matrix Sets and Operations

Given two elements x, y in a C*-algebra A, we will write $[x, y]$ and $\mathrm{Ad}[x](y)$ to denote the operations $[x, y] := xy - yx$ and $\mathrm{Ad}[x](y) := xyx^*$.

Given any C*-algebra A and any element x in $M_n(A)$, we will denote by $\mathrm{diag}_n[x]$ the operation defined by the expression

$$\begin{array}{rcl} M_n(A) & \to & M_n(A) \\ x & \mapsto & \mathrm{diag}_n[x] \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} & \mapsto & \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{pmatrix}. \end{array}$$

Given a C*-algebra A, we write $\mathcal{N}(A)$, $\mathbb{H}(A)$ and $\mathbb{U}(A)$ to denote the sets of normal, hermitian and unitary elements in A respectively. We will write $\mathcal{N}(n)$, $\mathbb{H}(n)$ and $\mathbb{U}(n)$ instead of $\mathcal{N}(M_n)$, $\mathbb{H}(M_n)$ and $\mathbb{U}(M_n)$. A normal element u in a C*-algebra

A is called a partial unitary if the element $uu^* = p$ is an otrhogonal projection in A, i.e. p satisfies the relations $p = p^* = p^2$, we denote by $\mathbb{PU}(A)$ the set of partial unitaries in A and we write $\mathbb{PU}(n)$ instead of $\mathbb{PU}(M_n)$.

We write $\mathbb{I}$, $\mathbb{J}$, $\mathbb{T}^1$ and $\mathbb{D}^2$ to denote the sets $\mathbb{I} := [0,1]$, $\mathbb{J} = [-1,1]$, $\mathbb{T}^1 := \{z \in \mathbb{C} | \, |z| = 1\}$ and $\mathbb{D}^2 := \{z \in \mathbb{C} | \, |z| \leq 1\}$. For some arbitrary matrix set $S \subseteq M_n$ and some arbitrary compact set $\mathbb{X} \subset \mathbb{C}$, we will write $S(\mathbb{X})$ to denote the subset of elements in S described by the expression,

$$S(\mathbb{X}) := \{x \in S | \sigma(x) \in \mathbb{X}\}, \tag{2.1.1}$$

for instance we can write $\mathcal{N}(n)(\mathbb{D}^2)$ to denote the set of nomal contractions. We will denote by $\mathcal{M}_\infty$ the C*-algebra described by

$$\mathcal{M}_\infty := \overline{\bigcup_{n \in \mathbb{Z}^+} M_n}^{\|\cdot\|}. \tag{2.1.2}$$

In this document we write $\mathbb{1}_n$ to denote the identity matrix in M_n. The symbol $\mathbf{N}_n$ will be used to denote the diagonal matrices

$$\mathbf{N}_n := \mathrm{diag}\left[n, n-1, \ldots, 2, 1\right]. \tag{2.1.3}$$

We will write Ω_n and Σ_n to denote the unitary matrices defined by

$$\Omega_n := e^{\frac{2\pi i}{n}\mathbf{N}_n} \quad = \quad \mathrm{diag}\left[1, e^{\frac{2\pi i(n-1)}{n}}, \ldots, e^{\frac{4\pi i}{n}}, e^{\frac{2\pi i}{n}}\right] \tag{2.1.4}$$

and

$$\Sigma_n \quad := \quad \begin{pmatrix} 0 & \mathbb{1}_{n-1} \\ 1 & 0 \end{pmatrix}. \tag{2.1.5}$$

Remark 2.1.1 *The unitary matrices Ω_n and Σ_n are related, by the equation*

$$\Omega_n = \mathscr{F}_n^* \Sigma_n \mathscr{F}_n,$$

where $\mathscr{F}_N := \left(\frac{1}{\sqrt{N}} e^{\frac{2\pi i (j-1)(k-1)}{N}}\right)_{1 \leq j,k \leq N}$ is the discrete Fourier transform (DFT) unitary matrix.

Given an abstract object (group or C*-algebra) A we write A^{*N} to denote the operation consisting on taking the free product of N copies of A.

2.2 Joint Spectral Variation

2.2.1 Clifford Operators

Using the same notation as Pryde in [31], let $\mathbb{R}_{(N)}$ denote the Clifford algebra over $\mathbb{R}$ with generators $e_1, \ldots, e_N$ and relations $e_i e_j = -e_j e_i$ for $i \neq j$ and $e_i^2 = -1$. Then $\mathbb{R}_{(N)}$ is an associative algebra of dimension 2^N. Let $S(N)$ denote the set $\mathscr{P}(\{1, \ldots, N\})$. Then the elements $e_S = e_{s_1} \cdots e_{s_k}$ form a basis when $S = \{s_1, \ldots, s_k\}$ and $1 \leq s_1 < \cdots < s_k \leq N$. Elements of $\mathbb{R}_{(N)}$ are denoted by $\lambda = \sum_S \lambda_S e_S$ where $\lambda_S \in \mathbb{R}$. Under the inner product $\langle \mu, = \rangle_\lambda \sum_S \lambda_S \mu_S$, $\mathbb{R}_{(N)}$ becomes a Hilbert space with orthonormal basis $\{e_S\}$.

The *Clifford operator* of N elements $X_1, \ldots, X_N \in M_n$ is the operator defined in $M_n \otimes \mathbb{R}_{(N)}$ by

$$\mathrm{Cliff}(X_1, \ldots, X_N) := i \sum_{j=1}^{N} X_j \otimes e_j.$$

Each element $T = \sum_S T_S \otimes e_S \in M_n \otimes \mathbb{R}_{(N)}$ acts on elements $x = \sum_S x_S \otimes e_S \in \mathbb{C}^n \otimes \mathbb{R}_{(N)}$ by $T(x) := \sum_{S,S'} T_s(x_{S'}) \otimes e_S e_{S'}$. So $\mathrm{Cliff}(X_1, \ldots, X_N) \in M_n \otimes \mathbb{R}_{(N)} \subseteq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{R}_{(N)})$. By $\|\mathrm{Cliff}(X_1, \ldots, X_N)\|$ we will mean the operator norm of $\mathrm{Cliff}(X_1, \ldots, X_N)$ as an element of $\mathcal{L}(\mathbb{C}^n \otimes \mathbb{R}_{(N)})$. As observed by Elsner in [12, 5.2] we have that

$$\|\mathrm{Cliff}(X_1, \ldots, X_N)\| \leq \sum_{j=1}^{N} \|X_j\|. \tag{2.2.1}$$

2.2.2 Joint Spectral Matchings

It is often convenient to have N-tuples (or $2N$-tuples) of matrices with real spectra. For this purpose we use the following construction, initiated by McIntosh and Pryde. If $X = (X_1, \ldots, X_N)$ is a N-tuple of n by n matrices then we can always decompose X_j in the form $X_j = X_{1j} + iX_{2j}$ where the X_{kj} all have real spectra. We write $\pi(X) := (X_{11}, \ldots, X_{1N}, X_{21}, \ldots, X_{2N})$ and call $\pi(X)$ a partition of X. If the X_{kj} all commute we say that $\pi(X)$ is a commuting partition, and if the X_{kj} are simultaneously triangularizable $\pi(X)$ is a triangularizable partition. If the X_{kj} are all semisimple (diagonalizable) then $\pi(X)$ is called a semisimple partition.

We say that N normal matrices $X_1, \ldots, X_N \in M_n$ are *simultaneously diagonalizable* if there is a unitary matrix $Q \in M_n$ such that Q^*X_jQ is diagonal for each $j = 1, \ldots, N$. In this case, for $1 \le k \le n$, let $\Lambda^{(k)}(X_j) := (Q^*X_jQ)_{kk}$ the (k,k) element of Q^*X_jQ, and set $\Lambda^{(k)}(X_1, \ldots, X_N) := (\Lambda^{(k)}(X_1), \ldots, \Lambda^{(k)}(X_N)) \in \mathbb{C}^N$. The set

$$\Lambda(X_1, \ldots, X_N) := \{\Lambda^{(k)}(X_1, \ldots, X_N)\}_{1 \le k \le N}$$

is called the joint spectrum of $X_1, \ldots, X_N$. We will write $\Lambda(X_j)$ to denote the j-component of $\Lambda(X_1, \ldots, X_N)$, in other words we will have that

$$\Lambda(X_j) = \{\Lambda^{(1)}(X_j), \ldots, \Lambda^{(N)}(X_j)\}.$$

The following theorem was proved in McIntosh, Pryde and Ricker [30].

Theorem 2.2.1 (McIntosh, Pryde and Ricker) *Let $X = (X_1, \ldots, X_N)$ and $Y = (Y_1, \ldots, Y_N)$ be N-tuples of commuting n by n normal matrices. There exists a permutation τ of the index set $\{1, \ldots, n\}$ such that*

$$\|\Lambda^{(k)}(X_1, \ldots, X_N) - \Lambda^{(\tau(k))}(Y_1, \ldots, Y_N)\| \le e_{N,0}\|\mathrm{Cliff}(X_1 - Y_1, \ldots, X_N - Y_N)\| \quad (2.2.2)$$

for all $k \in \{1, \ldots, n\}$.

In this theorem, $e_{N,0}$ is an explicit constant depending only on N defined in [30, (2.4)].

2.3 Amenable C*-algebras and Basic Homotopy Lemmas

The following lemma is proved by H. Lin in [20].

Lemma 2.3.1 (H. Lin.) *For any $\varepsilon > 0$ and $d > 0$, there exists $\delta > 0$ satisfying the following: Suppose that A is a unital C^*-algebra and $u \in A$ is a unitary such that $\mathbb{T}^1 \backslash \sigma(u)$ contains an arc with length d. Suppose that $a \in A$ with $\|a\| \leq 1$ such that*

$$\|ua - au\| < \delta.$$

Then there is a self-adjoint element $h \in A$ such that $u = e^{ih}$,

$$\|ha - ah\| < \varepsilon \quad \text{and} \quad \|e^{ith}a - ae^{ith}\| < \varepsilon$$

for all $t \in \mathbb{I}$. If, furthermore, $a = p$ is a projection, we have

$$\left\| pup - p + \sum_{n=1}^{\infty} \frac{(iphp)^n}{n!} \right\| < \varepsilon.$$

The following lemma was proved by H. Lin in [22] using L.2.3.1, since for any integer $n \geq 1$ and any $u \in \mathbb{U}(n)$, we will have that $\mathbb{T}^1 \backslash \sigma(u)$ contains an arc with length at least $2\pi/n$.

Lemma 2.3.2 (H. Lin.) *Let $\varepsilon > 0$, $n \geq 1$ be an integer and $M > 0$. There exists $\delta > 0$ satisfying the following: For any finite set $\mathscr{F} \subset M_n$ with $\|a\| \leq M$ for all $a \in \mathscr{F}$, and a unitary $u \in M_n$ such that*

$$\|ua - au\| < \delta \quad \text{for all} \quad a \in \mathscr{F},$$

there exists a continuous path of unitaries $\{u(t)\}_{t\in\mathbb{I}} \subset M_n$ *with* $u(0) = u$ *and* $u(1) = \mathbb{1}_n$ *such that*

$$\|u(t)a - au(t)\| < \varepsilon \quad \text{for all} \quad a \in \mathscr{F}.$$

Morover,

$$Length(\{u(t)\}) \leq 2\pi.$$

2.3.1 Basic Homotopy Lemmas

Definition 2.3.1 *(***The obstruction** $Bott(u,v)$**.***) Given two unitaries in a* K_1*-simple real rank zero* C^**-algebra* A *that almost commute, the obstruction* $Bott(u,v)$ *is the Bott element associated to the two unitaries as defined by Loring in [24]. It is defined whenever* $\|uv - vu\| \leq \nu_0$*, where* ν_0 *is a universal constant. It is defined as the* K_0*-class*

$$Bott(u,v) = [\chi_{[1/2,\infty)}(e(u,v))] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right],$$

where $e(u,v)$ *is a self-adjoint element of* $M_2(A)$ *of the form*

$$e(u,v) = \begin{pmatrix} f(v) & h(v)u + g(v) \\ u^*h(v) + g(v) & 1 - f(v) \end{pmatrix},$$

where f*,*g*,*h *are certain universal real-valued continuous functions on* $\mathbb{T}^1$*.*

For details on the subject of K-theory for C^*-algebras the reader is referred to [32].

Let us use a similar convention to the one used by H. Lin in [19]. Let us write **B** to denote the class of unital C^*-algebras which are simple with real rank zero and stable rank one. The following results were proved by H. Lin in [19].

Theorem 2.3.1 (H. Lin [19], Generalized basic homotopy lemma) *Let X be a finite CW co-mplex of dimension 1. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there is $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: Let $A \in \mathbf{B}$ with $K_1(A) = \{0\}$, let $h : C(X) \to A$ be a unital homomorphism and let $u \in A$ be a unitary such that*

$$\|[h(g), u]\| < \delta \text{ for all } g \in \mathcal{G} \text{ and } Bott_1(h, u) = 0.$$

Then there exists a continuous rectifiable path of unitaries $\{u_t : t \in [0, 1]\}$ such that

$$u_0 = u,\ u_1 = 1_A \text{ and } \|[h(f), u_t]\| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

Moreover,

$$Length(\{u_t\}) \leq \pi + \varepsilon.$$

Corollary 2.3.1 (Bratteli, Elliot, Evans, Kishimoto and Lin. [4]) *Let $\varepsilon > 0$. We have that there is $\delta > 0$ satisfying the following: For any two unitaries u and v in a unital C^*-algebra $A \in \mathcal{B}$ with $K_1(A) = \{0\}$ and if*

$$\|[u, v]\| < \delta \quad \text{and} \quad Bott(u, v) = 0,$$

then there exists a continuous path of unitaries $\{u_t\}_{t \in \mathbb{I}}$ of A such that

$$u_0 = u, \ \ u_1 = 1 \text{ and } \ \|[u_t, v]\| < \varepsilon.$$

Morover,

$$\text{Length}(u_t) \leq \pi + \varepsilon.$$

Chapter 3

Matrix Varieties and Toroidal Matrix Links

Let us denote by $\mathcal{H}$ a universal separable Hilbert space, by $\mathbb{B}(\mathcal{H})$ the C*-algebra of bounded operators on $\mathcal{H}$, and for any given $S \subseteq \mathbb{B}(\mathcal{H})$ let us denote by $\mathbf{B}_r(S)$ the closed r-ball in S defined by $\mathbf{B}_r(S) := \{x \in S | \|x\| \leq r\}$.

Given some $N \in \mathbb{Z}^+$ and a set $\mathcal{R}(S) = \mathcal{R}(y_1, \ldots, y_N)$ of normed polynomial relations on the N-set $S := \{y_1, \ldots, y_N\}$ of NC-variables, we will call the set $\mathcal{Z}[\mathcal{R}]$ described by

$$\mathcal{Z}[\mathcal{R}] := \{x_1, \ldots, x_N | \mathcal{R}(x_1, \ldots, x_N)\} \tag{3.0.1}$$

with $x_1, \ldots, x_N \in \mathbf{B}_1(\mathbb{B}(\mathcal{H}))$, a noncommutative semialgebraic set.

Example 3.0.1 *As an example of normed NC-polynomial relations we can consider the set* $\mathcal{R}(x, y) := \{\|x^4 - 1\| \leq 10^{-10}, \|y^7 - 1\| \leq 10^{-10}, \|xy - yx\| \leq \frac{1}{8}, xx^* = x^*x = 1, yy^* = y^*y = 1\}$.

Given a NC-semialgebraic set $\mathcal{Z}[\mathcal{R}]$, we will use the symbol $\mathcal{E}\mathcal{Z}[\mathcal{R}]$ to denote the universal C*-algebra

$$\mathcal{E}\mathcal{Z}[\mathcal{R}] := C^* \langle x_1, \ldots, x_N \, | \mathcal{R}(x_1, \ldots, x_N) \rangle, \tag{3.0.2}$$

which we call the environment C*-algebra of $\mathcal{Z}[\mathcal{R}]$. For details on universal C*-algebras described in terms of generators and relations the reader is referred to [26].

Definition 3.0.2 (Semialgebraic Matrix Varieties) *Given $J \in \mathbb{Z}^+$, a system of J polynomials $p_1, \ldots, p_J \in \Pi_{\langle N \rangle} = \mathbb{C}\langle x_1, \ldots, x_N \rangle$ in N NC-variables $x_1, \ldots, x_N \in \Pi_{\langle N \rangle}$ and a real number $\varepsilon \geq 0$, a particular matrix representation of the noncommutative semialgebraic set $\mathcal{Z}_{\varepsilon,n}(p_1, \ldots, p_J)$ described by*

$$\mathcal{Z}_{\varepsilon,n}(p_1, \ldots, p_J) := \{ X_1, \ldots, X_N \in M_n \mid \|p_j(X_1, \ldots, X_N)\| \leq \varepsilon, 1 \leq j \leq J \}, \tag{3.0.3}$$

*will be called a ε, n-**semialgebraic matrix variety** (ε, n-SMV), if $\varepsilon = 0$ we can refer to the set as a **matrix variety**.*

Example 3.0.2 *As a first example, we will have that the set $\mathbf{Z_n} := \{X \in M_n | \mathbf{N}_n X - X\mathbf{N}_n = 0\}$ is a matrix variety defined by the set with one NC-polynomial relation $\{\mathbf{N}_n X - X\mathbf{N}_n = 0\}$. If for some $\delta > 0$, we set now $\mathbf{Z}_{n,\delta} := \{X \in M_n | \|[\mathbf{N}_n, X]\| \leq \delta\}$, the set $\mathbf{Z}_{n,\delta}$ is a matrix semialgebraic variety defined by the set with one normed NC-polynomial relation $\{\|\mathbf{N}_n X - X\mathbf{N}_n\| \leq \delta\}$.*

Example 3.0.3 *Other examples of matrix semialgebraic varieties, that have been useful to understand the geometric nature of the problems solved in this document, are described by the matrix sets $\mathbf{Iso}_\delta(x, y)$, $\mathbf{ST}_\delta(x, y, u)$ and $\mathbf{ST}_\delta^{(m)}(x, y, u)$, defined for some given $\delta \geq 0$, $m \geq 1$, any two normal contractions x and y and some fixed but arbitrary unitary u in M_n, by the expressions*

$$\mathbf{Iso}_\delta(x, y) := \left\{ (z, w) \in \mathcal{N}(n)(\mathbb{D}^2) \times \mathbb{U}(n) \,\middle|\, \begin{array}{c} \|xw - wz\| = 0 \\ \|[z, y]\| = 0 \\ \|z - y\| \leq \delta \end{array} \right\},$$

$$\mathbf{ST}_\delta(x,y,u) := \left\{ (z,v)\in\mathcal{N}(n)(\mathbb{D}^2)\times\mathbb{U}(n) \,\middle|\, \begin{array}{l} \|vx-zv\|=0, \\ \|[z,y]\|=0, \|z-y\|\leq\delta, \\ \|[u,x]\|=\|[u,y]\|=0. \end{array} \right\},$$

$$\mathbf{ST}^{(m)}_\delta(x,y,u) := \left\{ (z,v)\in\mathcal{N}(n)(\mathbb{D}^2)\times\mathbb{U}(n)^2 \,\middle|\, \begin{array}{l} \|vx-zv\|=0, \\ \|[z,y]\|=0, \|z-y\|\leq\delta, \\ \|[u,x]\|=\|[u,y]\|=0, \\ u^m=\mathbb{1}_n. \end{array} \right\}.$$

3.1 Toroidal Matrix Links

3.1.1 Finsler manifolds, matrix paths and toroidal matrix links

Definition 3.1.1 (Finsler manifold) *A Finsler manifold is a pair (M,F) where M is a manifold and $F:TM\to[0,\infty)$ is a function (called a Finsler norm) such that*

- *F is smooth on $TM\backslash\{0\}=\bigcup_{x\in M}\{T_xM\backslash\{0\}\}$,*
- *$F(v)\geq 0$ with equality if and only if $v=0$,*
- *$F(\lambda v)=\lambda F(v)$ for all $\lambda\geq 0$,*
- *$F(v+w)\leq F(v)+F(w)$ for all w at the same tangent space with v.*

Given a Finsler manifold (M,F), the length of any rectifiable curve $\gamma:[a,b]\to M$ is given by the length functional

$$L[\gamma]=\int_a^b F(\gamma(t),\partial_t\gamma(t))\,dt,$$

where $F(x, \cdot)$ is the Finsler norm on each tangent space $T_x M$.

The pair $(\mathcal{N}, \|\cdot\|)$ is a Finsler manifold, where $\mathcal{N}$ denotes the set of normal matrices $\mathcal{N}$ (of any size) and $\|\cdot\|$ denotes the operator norm.

Definition 3.1.2 (Matrix path curvature) *Given a piecewise-C^2 matrix path $\gamma : [0,1] \to \mathcal{N}$, we define its curvature $\kappa[\gamma]$ to be*

$$\kappa[\gamma] := \frac{1}{\|\partial_t \gamma(t)\|} \left\| \partial_t \left(\frac{\partial_t \gamma(t)}{\|\partial_t \gamma(t)\|} \right) \right\|.$$

Definition 3.1.3 (Matrix flows) *Given $n \geq 1$, a mapping $\phi : \mathbb{R}_0^+ \times M_n \to M_n$, $(t, x) \mapsto x_t$ will be called a matrix flow in this document. If we have in addition that $\sigma(x_t) = \sigma(x_s)$ for every $t, s \geq 0$, we say that the matrix flow is isospectral.*

Definition 3.1.4 (interpolating path) *Given two matrices x and y in M_n and a matrix flow $\phi : \mathbb{I} \times M_n \to M_n$ such that $\phi_0(x) = x$ and $\phi_1(x) = y$, we say that the corresponding path $\{x_t\}_{t \in \mathbb{I}} := \{\phi_t(x)\}_{t \in \mathbb{I}} \subseteq M_n$ is a solvent path for the interpolation problem $x \rightsquigarrow y$.*

Definition 3.1.5 *($\ell_{\|\cdot\|}$) Given a matrix path $\{x_t\}_{t \in \mathbb{I}}$ in M_n we will write $\ell_{\|\cdot\|}(x_t)$ to denote the length of $\{x_t\}_{t \in \mathbb{I}}$ with respect to the operator norm which is defined by the expression*

$$\ell_{\|\cdot\|}(x_t) := \sup \sum_{k=0}^{m-1} \|x_{t_{k+1}} - x_{t_k}\|,$$

where the supremum is taken over all partitions of $\mathbb{I}$ as $0 = t_0 < \ldots < t_m = b$. If the function $x \in C(\mathbb{I}, M_n)$ is a piecewise C^1 function, then

$$\ell_{\|\cdot\|}(x_t) = \int_{\mathbb{I}} \|\partial_t x_t\| dt.$$

Definition 3.1.6 *(**$\|\cdot\|$-flatness**) A set $\mathcal{S}$ of M_n is said to be $\|\cdot\|$-flat if any two points $x, y \in \mathcal{S}$ can be connected by a path $\{x_t\}_{t \in \mathbb{I}} \subseteq \mathcal{S}$ such that $\ell_{\|\cdot\|}(x_t) = \|x - y\|$.*

Definition 3.1.7 (Toroidal matrix link) *Given any two normal contractions x, y in M_n, a toroidal matrix link is any normal path $x_t := \mathbb{K}[T_t(\mathbb{l}(x))]$ induced by a locally normal matrix flow $T : \mathbb{I} \times M_N \to M_N$ with $N \geq n$, together with a locally normal compression $\mathbb{K} : M_N \to M_n$ with relative lifting map $\mathbb{l} : M_n \to M_N$, which satisfy the interpolating conditions $\mathbb{K}[T_0(\mathbb{l}(x))] = x$ and $\mathbb{K}[T_1(\mathbb{l}(x))] = y$ together with the constraints $\|\mathbb{K}[T_t(\mathbb{l}(x))]\| \leq 1$ for each $t \in \mathbb{I}$.*

Remark 3.1.1 *In the particular case where $[\mathbb{K}(T_t(\mathbb{l}(x))), \mathbb{K}(T_t(\mathbb{l}(y)))] = 0$ for each $t \in \mathbb{I}$, whenever $[x, y] = 0$, we call T a toral matrix link.*

Remark 3.1.2 *The* **curved nature** *of the matrix varieties (***as Finsler sub-manifolds of $\mathcal{N}$***) whose local connectivity is studied in this document, induces an obstruction to local connectivity via entirely* **flat** *toroidal matrix links in general. The toroidal matrix links $\mathbf{T} \subset C([0,1], \mathcal{N})$ we have used to solve the connectivity problems which motivated this study satisfy the constraint*

$$0 \leq \kappa[T] \leq \frac{2}{\ell_{\|\cdot\|}(T)}, \ \ \forall T \in \mathbf{T}.$$

3.1.2 Embedded matrix flows in solid tori

Given some fixed but arbitrary $W \in \mathbb{U}(n)$, using the operation $\text{diag}_n : M_n \to M_n$ one can define the mapping $\mathscr{D} : \mathbb{U}(n) \times M_n \to \mathbb{D}^2$, determined by the expression.

$$\begin{aligned} \mathbb{U}(n) \times M_n &\to \mathbb{D}^2 && (3.1.1)\\ (W, x) &\mapsto \mathscr{D}_{\mathbb{T}}[W](x) && (3.1.2)\\ (W, x) &\mapsto \{(\text{diag}_n\,[WxW^*])_{k,k}\}_{1\leq k\leq n} && (3.1.3)\end{aligned}$$

It is clear that $\text{diag}\,[\mathscr{D}_{\mathbb{T}}[W](x)] = \text{diag}_n\,[WxW^*]$ and that $\text{diag}\,[\mathscr{D}_{\mathbb{T}}[\mathbb{1}_n](x)] = \text{diag}_n\,[x]$, because of this when $W = \mathbb{1}_n$ we will write $\mathscr{D}(x)$ instead of $\mathscr{D}_{\mathbb{T}}[\mathbb{1}_n](x)$.

Given a matrix flow $\mathbb{I} \times \mathcal{N}(n)(\mathbb{D}^2) \to \mathcal{N}(n)(\mathbb{D}^2), (t,x) \mapsto X_t(x)$, one can identify X with the set of flow lines in $\mathbb{D}^2 \times \mathbb{T}^1$ determined by $\{(\mathscr{D}(X_t(x)), e^{2\pi it})\}_{t\in\mathbb{I}}$. The geometric picture determined by the mapping cylinder $\mathcal{N}(n)(\mathbb{D}^2) \times \mathbb{I} \to \mathbb{D}^2 \times \mathbb{T}^1, (x,t) \mapsto (\mathscr{D}(X_t(x)), e^{2\pi it})$ will be called the embedded matrix mapping cylinder relative to the flow X. We can think of the embedded matrix mapping cylinder in topological terms as a deformation described by the expression $\mathcal{D}_{X,Z_2}$, which is defined as

$$\mathcal{D}_{X,Z_2}[Z_1 \times \mathbb{I}] := \frac{(Z_1 \times \mathbb{I}) \sqcup Z_2}{Z_1 \rightsquigarrow_{X_1} Z_2}, \tag{3.1.4}$$

where Z_1 and Z_2 are some prescribed matrix varieties such that $x \in Z_1$ and $y \in Z_2$.

Example 3.1.1 (Graphical example in M_3) *Let us set $\hat{u}_3 := e^{\frac{2\pi i}{3}\mathbf{N}_3}$. For some prescribed $W_3 \in \mathbb{U}(3)$, we can obtain a graphical example of a particular geometric picture of the computation of the embedded matrix mapping cylinder relative to the interpolating flow u which solves the problem $\hat{u}_3 \rightsquigarrow W_3\hat{u}_3W_3^*$.*

Let us set

$$\begin{aligned} Z_1 &:= \{z \in \mathbb{U}(3) | [\hat{u}_3, z] = 0\}, \\ Z_2 &:= \{z \in \mathbb{U}(3) | [W_3\hat{u}_3W_3^*, z] = 0\}. \end{aligned}$$

Using projective methods, we can trace specific flow lines along the matrix flows corresponding to the dynamical deformation $\mathcal{D}_{u,Z_2}[Z_1 \times \mathbb{I}]$, which solve the interpolation problem $\hat{u}_3 \rightsquigarrow W_3\hat{u}_3W_3^*$.

A particular (approximate) geometric picture of the matrix deformation induced by the toral matrix link $\{u_t\}_{t\in\mathbb{I}}$ in M_n, projected in $\mathbb{D}^2 \times \mathbb{T}^1$ via $\mathscr{D}_{\mathbb{T}}[\hat{u}_3](\cdot)$ is presented in figures F.3.1-F.3.3.

Alternative methods to trace particular flow lines on mapping cylinders can be otained using matrix homotopies, which can be done using similar methods to the ones implemented in [8].

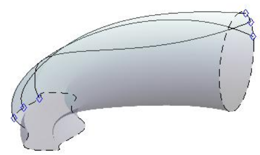


Figure 3.1: Projected matrix mapping cylinder corresponding to the orbit $u_{[0,\frac{1}{2}]}(\hat{u}_3)$ in M_3.

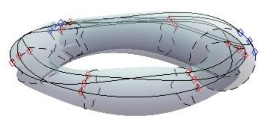


Figure 3.2: Projected matrix mapping cylinder corresponding to the orbit $u_{\mathbb{I}}(\hat{u}_3)$ in M_3.

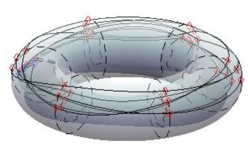


Figure 3.3: Embedded matrix mapping cylinder corresponding to the orbit $u_{\mathbb{I}}(\hat{u}_3)$ in M_3.

3.2 Environment algebras and localization

Definition 3.2.1 (Environment algebra (of a matrix algebra)) *Given a matrix algebra $A \subseteq M_n$, a universal C^*-algebra $\mathcal{E}_A := C_1^*\langle x_1, \ldots, x_m | \mathcal{R}(x_1, \ldots, x_m)\rangle$ for* *which there is a matrix representation* $\mathcal{E}_A \twoheadrightarrow \mathbf{E}_A \subseteq M_n$ *such that* $A \subseteq \mathbf{E}_A$*, will be called an* **environment algebra** *for* A*.*

Let us consider the universal C*-algebras $C(\mathbb{J})$, $C(\mathbb{T}^1)$, $C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1)$, $C_\delta(\mathbb{T}^2)$ and $C_\delta(\mathbb{J}\times\mathbb{T}^1)$, defined in terms of generators and relations by the expressions.

$$C(\mathbb{J}) := C_1^*\left\langle u \,\middle|\, h^* = h, \|h\| \leq 1 \right\rangle$$

$$C(\mathbb{T}^1) := C_1^*\left\langle u \,\middle|\, uu^* = u^*u = 1 \right\rangle$$

$$C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1) := C_1^*\left\langle u, v \,\middle|\, \begin{array}{l} uu^* = u^*u = 1, \\ vv^* = v^*v = 1 \end{array} \right\rangle$$

$$C_\delta(\mathbb{T}^2) := C_1^*\left\langle u, v \,\middle|\, \begin{array}{l} uu^* = u^*u = 1, \\ vv^* = v^*v = 1, \\ \|uv - vu\| \leq \delta \end{array} \right\rangle$$

$$C_\delta(\mathbb{J}\times\mathbb{T}^1) := C_1^*\left\langle h, u \,\middle|\, \begin{array}{l} h^* = h, \|h\| \leq 1 \\ uu^* = u^*u = 1, \\ \|hu - uh\| \leq \delta \end{array} \right\rangle$$

Let us consider now a local matrix representation result that we will use later in the construction of particular representation schemes.

Lemma 3.2.1 *For every integer $n \geq 1$, there are $s_2, u_n, v_n \in \mathbb{U}(\mathcal{M}_\infty)$ such that the diagram*

$$\begin{array}{ccccc} C(\mathbb{T}^1)^{*2} & \twoheadrightarrow & C^*\left\langle (\mathbb{Z}/n)^{*2} \right\rangle & \twoheadrightarrow & C^*_n(u_n, v_n) \\ \downarrow\!\!\!\downarrow & & & & \| \\ C^*\left\langle \mathbb{Z}/n * \mathbb{Z}/2 \right\rangle & \twoheadrightarrow & C^*_n(s_2, v_n) & = & M_n \end{array}$$

commutes, where $s_2 \in \mathbb{H}(n)$, u_n and v_n are unitary elements in M_n.

Proof. Since we have that $C(\mathbb{T}^1)^{*2} \simeq C^*\left\langle \mathbb{F}_2 \right\rangle \simeq C^*(\mathbb{Z}^{*2})$, by universality of the C*-representa-tions

$$\begin{aligned} C^*(\mathbb{Z}^{*2}) &\simeq C^*\left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = \mathbb{1}, \\ vv^* = v^*v = \mathbb{1} \end{array} \right\rangle \\ C^*((\mathbb{Z}/n)^{*2}) &\simeq C^*\left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = \mathbb{1}, \\ vv^* = v^*v = \mathbb{1}, \\ u^n = v^n = \mathbb{1} \end{array} \right\rangle \\ C^*(\mathbb{Z}/n * \mathbb{Z}/2) &\simeq C^*\left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = \mathbb{1}, \\ vv^* = v^*v = \mathbb{1}, \\ u^n = v^2 = \mathbb{1} \end{array} \right\rangle, \end{aligned}$$

and by the structural properties of M_n, it is enough to find for any $n \in \mathbb{Z}^+$, up to unitary congruence in M_n, three unitaries $s_2, u_n, v_n \in \mathbb{U}(n)$ such that $C^*(s_2, v_n) = M_n = C^*(u_n, v_n)$ and $u_n^n = v_n^n = s_2^2 = \mathbb{1}_n$, this can be done by taking for any $n \in \mathbb{Z}^+$ the orthogonal projection $p := \mathrm{diag}\,[1, 0, \ldots, 0] \in \mathbb{H}(n)$ and the matrix $s_2 = \mathbb{1} - 2p \in \mathbb{H}(n)$, setting $u_n := \Omega_n$ and $v_n := \Sigma_n$ for $n \geq 2$ and $u_1 = v_1 = 1$ for $n = 1$, by functional calculus and direct computations it is easy to verify that $s_2, u_n, v_n \in \mathbb{U}(n)$ for every $n \in \mathbb{Z}^+$, and that $s_2 = s_2^*$, it is also easy to verify that the system of matrix units $\{ e_{i,j,n} \}_{1 \leq i,j \leq n}$ and u_n can be expressed as words in $C^*(s_2, v_n)$ for every $n \in \mathbb{Z}^+$, it is also clear that $p = e_{1,1,n}$ and hence, s_2 can be written as linear combinations of words in $C^*(u_n, v_n)$, we will then have that $C^*\left\langle \mathbb{Z}/n * \mathbb{Z}/2 \right\rangle \twoheadrightarrow$

$C^*(v_n, s_2)$ and $C^* \langle \mathbb{Z}/n^{*2} \rangle \twoheadrightarrow C^*(u_n, v_n)$ by the universal properties of $C^* \langle \mathbb{Z}/2 * \mathbb{Z}/n \rangle$ and $C^* \langle \mathbb{Z}/n^{*2} \rangle$ respectively, since it can be easily verified that

$$u_n^n = v_n^n = s_2^2 = \mathbb{1}_n,$$

from these facts and the universal property of $C(\mathbb{T}^1)^{*2} \simeq C^* \langle \mathbb{F}_2 \rangle \simeq C^* \langle \mathbb{Z}^{*2} \rangle$, the result follows. □

Remark 3.2.1 *It can be seen that for any matrix C^*-subalgebra $A \subseteq M_n$, there is $\delta > 0$ such that both $C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1)$ and $C_\delta(\mathbb{T}^2)$ are environment algebras of A. It can also be seen that for any abelian C^*-subalgebra $D \subseteq M_n$, $C(\mathbb{T}^1)$ is an environment algebra of D.*

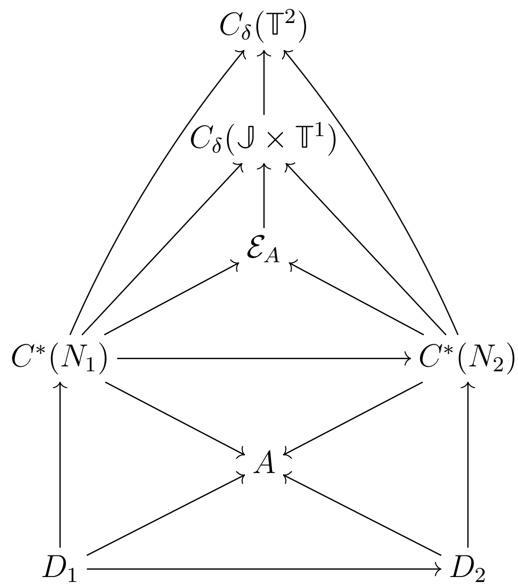
3.2.1 Localized matrix representations

Definition 3.2.2 (Localized N-tuples in $\mathcal{N}(n)(\mathbb{D}^2)^N$) *Given two integers $N \leq M$ and a universal C^*-algebra $\mathcal{L}(z_1, \ldots, z_M) := C^* \langle z_1, \ldots, z_M \,|\, \mathcal{R}_{\mathcal{L}}(z_1, \ldots, z_M) \rangle$, we say that two N-tuples $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ of normal contractions in M_n are δ-localized with respect to $\mathcal{L}(z_1, \ldots, z_M)$, if there are $Z_1, \ldots, Z_M \in \mathcal{N}(n)(\mathbb{D}^2)$ such that*

$$C^*(Z_1, \ldots, Z_N) \subseteq C^*(Z_1, \ldots, Z_M) \twoheadleftarrow \mathcal{L}(z_1, \ldots, z_M)$$

and $\max_{1 \leq j \leq N} \{\|X_j - Z_j\|, \|Y_j - Z_j\|\} \leq \delta$.

Given two abelian C*-subalgebras $D_1, D_2 \subseteq M_n$ and any two normal matrices $N_1, N_2 \in M_n$ with distinct eigenvalues such that $[N_1, X_1] = [N_2, X_2] = 0$ for every $X_1 \in D_1$ and every $X_2 \in D_2$, let us set $A := C^*(N_1, N_2) \subseteq M_n$, the situation in R.3.2.1 can be ilustrated via the following diagram of C*-representations.



(3.2.1)

3.3 Dimensionality Reduction of Matrix Semialgebraic Varieties

Definition 3.3.1 (Dimensionality reduction condition (DRC)) *Given any integer $n \geq 1$, any $\delta > 0$ and N pairwise commuting normal contractions $X_1, \ldots, X_N$ in M_n, we say that normal contractions have the dimensionality reduction condition (***DRC***), if there are a number $M \leq N$, M indices $j_1 < \cdots < j_M \in \{1, \ldots, N\}$ and a set of N functions $\mathbf{F}_N := \{f_1, \ldots, f_N\} \subset C(\mathbb{D}^2, \mathbb{D}^2)$, such that for each $1 \leq k \leq N$ we have $\|X_k - f_k(X_{j_1}, \ldots, X_{j_M})\| \leq \delta$. If the functions $f_1, \ldots, f_N$ and the number $\delta > 0$ do not depend on n, we will say $X_1, \ldots, X_N$ have the uniform dimensionality reduction condition (***UDRC***). We will write $\dim_\delta(\Lambda(X_1, \ldots, X_N))) = M$ or $\dim_\delta^{\mathbf{U}}(\Lambda(X_1, \ldots, X_N))) = M$ if the normal contraction $X_1, \ldots, X_N = N$ have that (***DRC***) or (***UDRC***) respectively, because of the corresponding geometric implications in the joint spectrum $\Lambda(X_1, \ldots, X_N)$ of $X_1, \ldots, X_N$.*

Example 3.3.1 (DRC in an approximate Matrix 2-Sphere) *Given $\delta > 0$, any three pairwise commuting matrices $X_1, X_2, X_3 \in \mathbb{H}(n)$ and two functions $f_2, f_3 \in C(\mathbb{J})$ such that*

$$\begin{cases} \|X_1^2 + X_2^2 + X_3^2 - \mathbb{1}_n\| \leq \delta, \\ \|f_2(X_1) - X_2\| \leq \delta, \\ \|f_3(X_1) - X_3\| \leq \delta. \end{cases}$$

We will have that $\dim_\delta(\Lambda(X_1, X_2, X_3)) = 1$.

Example 3.3.2 (DRC in an approximate Matrix 3-Sphere) *Given $\delta > 0$, any four pairwise commuting matrices $X_1, X_2, X_3, X_4 \in \mathbb{H}(n)$ and two functions $f_3, f_4 \in C(\mathbb{J}^2)$ such that*

$$\begin{cases} \|X_1^2 + X_2^2 + X_3^2 + X_4^2 - \mathbb{1}_n\| \leq \delta, \\ \|f_3(X_1, X_2) - X_3\| \leq \delta, \\ \|f_4(X_1, X_2) - X_4\| \leq \delta. \end{cases}$$

We will have that $\dim_\delta(\Lambda(X_1, X_2, X_3, X_4)) = 2$.

Example 3.3.3 (DRC in a Matrix 2-Torus) *Given $\delta > 0$, two pairwise commuting matrices $U_1, U_2 \in \mathbb{U}(n)$ and one function $f_2 \in C(\mathbb{T}^1, \mathbb{T}^1)$ such that $\|U_2 - f_2(U_1)\| \leq \delta$. We will have that $\dim_\delta(\Lambda(U_1, U_2)) = 1$.*

Lemma 3.3.1 (Non-derogatory normal approximants) *Given any $\delta > 0$ and any two normal matrices X, Y in M_n such that $\|X - Y\| \leq \delta$, there is a nonderogatory matrix $\tilde{X}$ such that,* $\max\{\|\tilde{X} - X\|, \|\tilde{X} - Y\|\} \leq \frac{3}{2}\delta$.

Proof. Let us set $\delta' := \frac{1}{2}\min\{\min_{x_j, x_k \in \sigma(X)}\{|x_j - x_k| \mid x_j \neq x_k\}, \delta\}$. By changing basis if necessary we can assume that X is diagonal, by applying permutation congruence transformations in necessary we can also assume that X has a diagonal block decomposition $X = X_1 \oplus X_2 \oplus \ldots \oplus X_r$. Given any diagonal block $X_{(j)} := \mathrm{diag}\,[x_{j_1}, \ldots, x_{j_m}] = \mathrm{diag}\,[x_{j_1}, \ldots, x_{j_1}]$ of m repeated eigenvalues of X, we can form the block $\tilde{X}_{(j)} := \mathrm{diag}\,[x_{j_1}, \ldots, x_{j_m}] + \mathrm{diag}\left[\left\{\frac{(k-1)\delta'}{m-1}\right\}_{1\leq k\leq m}\right]$, we will have that the matrix $\tilde{X} := \tilde{X}_{(1)} \oplus \cdots \oplus \tilde{X}_{(r)}$ satisfies the inequality $\|X - \tilde{X}\| \leq \frac{1}{2}\delta$, moreover, we will also have that $\|\tilde{X} - Y\| \leq \|\tilde{X} - X\| + \|X - Y\| \leq \frac{3}{2}\delta$ and we are done. $\square$

Definition 3.3.2 (Nearby generators) *Given $\delta > 0$ and N normal contractions $X_1, \ldots, X_N$ in M_n, we call a non-derogatory normal contraction $\tilde{X}$ like the one described in L.3.3.1 a δ-nearby generator of $X_1, \ldots, X_N$. The reason for this is that $C^*(X_1, \ldots, X_N) \subseteq C^*(\tilde{X})$.*

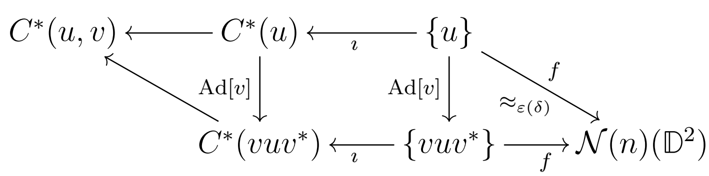
Remark 3.3.1 *It is important to notice that if we allow $\delta > 0$ to depend on n, any N normal contractions $X_1, \ldots, X_N \subseteq M_n$ will satisfy* $\dim_\delta(\Lambda(X_1, \ldots, X_N)) = 1$.

Chapter 4

Local Matrix Connectivity

4.1 Topologically controlled linear algebra and Soft Tori

Definition 4.1.1 (Controlled sets of matrix functions) *Given $\delta > 0$, a function $\varepsilon : \mathbb{R} \to \mathbb{R}_0^+$, a finite set of functions $F \subseteq C(\mathbb{T}^1, \mathbb{D}^2)$ and two unitary matrices $u, v \in M_n$ such that $\|uv - vu\| \leq \delta$, we say that the set F is δ-controlled by* $\mathrm{Ad}[v]$ *if the diagram,*



commutes up to an error $\varepsilon(\delta)$ for each $f \in F$.

Remark 4.1.1 *The C^*-homomorphism $C_\delta(\mathbb{T}^2) \to C^*(u,v)$ allows us to see that the Soft Torus $C_\delta(\mathbb{T}^2)$ provides an environment algebra for any δ-controlled set of matrix functions.*

Lemma 4.1.1 (Existence of δ-JMP) *Given any $\delta \geq 0$, for any 2 families of N pairwise commuting normal matrices $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ such that $\|x_j - y_j\| \leq \delta$ for each $1 \leq j \leq N$, there are a constant c_N (which depends only on the number of matrices in each family) and a permutation matrix $\mathcal{T} \in \mathbb{U}(n)$ such that, $\|\mathcal{T}\lambda(x_j)\mathcal{T} - \lambda(y_j)\| \leq c_N\delta$ for each $1 \leq j \leq N$.*

Proof. From T.2.2.1 we will have that there is a permutation τ of the index set $\{1, \ldots, n\}$ such that for each $1 \leq k \leq n$ we have that

$$\begin{aligned} |\Lambda^{(k)}(x_j) - \Lambda^{(\tau(k))}(y_j)| &\leq \|\Lambda^{(k)}(x_1, \ldots, x_N) - \Lambda^{(\tau(k))}(y_1, \ldots, y_N)\| \\ &\leq e_{N,0}\|\mathrm{Cliff}(x_1 - y_1, \ldots, x_N - y_N)\|. \qquad (4.1.1) \end{aligned}$$

Using 2.2.1 and as a consequence of 4.1.1 we can find a permutation matrix $\mathcal{T} \in \mathbb{U}(n)$ such that

$$\begin{aligned} \|\mathcal{T}^*\mathrm{diag}\,[\Lambda(x_j)]\,\mathcal{T} - \mathrm{diag}\,[\Lambda(y_j)]\,\| &\leq e_{N,0}\|\mathrm{Cliff}(x_1 - y_1, \ldots, x_N - y_N)\| \qquad (4.1.2) \\ &\leq e_{N,0}N\delta, \;\; 1 \leq j \leq N. \qquad (4.1.3) \end{aligned}$$

It can be seen that $c_N := e_{N,0}N$ and $\mathcal{T}$ satisfy the required conditions in the statement of the lemma and we are done. □

Remark 4.1.2 *Any permutation $\mathcal{T}$ satisfying the normed relation in the conclusion of L.4.1.1 will be called a joint matching permutation (**JMP**) for the matrices $x_1, \ldots, x_y$ and $y_1, \ldots, y_N$.*

Remark 4.1.3 *If for the matrices $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in L.4.1.1, we consider a basis in which $y_1, \ldots, y_N$ are diagonal, then there is a unitary $W \in M_n$ and a (**JMP**) $\mathcal{T} \in M_n$ such that*

$$\begin{aligned} \|W\mathrm{diag}\,[\Lambda(x_j)]\,W^* - \mathcal{T}\mathrm{diag}\,[\Lambda(x_j)]\,\mathcal{T}^*\| &\leq \|W\mathrm{diag}\,[\Lambda(x_j)]\,W^* - y_j\| \\ &\quad + \|y_j - \mathcal{T}\mathrm{diag}\,[\Lambda(x_j)]\,\mathcal{T}^*\| \\ &\leq (1 + c_N)\|x_j - y_j\| \leq (1 + c_N)\delta. \end{aligned}$$

It can be seen that the previous inequalities provide us with a particular representation of a δ-controlled set of matrix functions.

4.2 Local piecewise analytic connectivity

In this section we will present some piecewise analytic local connectivity results in matrix representations of the form $C_\varepsilon(\mathbb{T}^2) \to M_n \leftarrow C(\mathbb{T}^N)$ and $C_\varepsilon(\mathbb{J} \times \mathbb{T}^1) \to M_n \leftarrow C(\mathbb{J}^N)$.

Theorem 4.2.1 (Local normal toral connectivity) *Given $\varepsilon > 0$ and any $n \in \mathbb{Z}^+$, there is $\delta > 0$ such that, for any $2N$ normal contractions $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, \quad 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, \quad 1 \leq j \leq N, \end{cases}$$

there exist N toroidal matrix links $X^1, \ldots, X^N$ in M_n, which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\begin{cases} [X_t^j(x_j), X_t^k(x_k)] = 0, \\ \|X_t^j(x_j) - y_j\| \leq \varepsilon, \end{cases}$$

for each $1 \leq j, k \leq N$ and each $t \in \mathbb{I}$. Moreover,

$$\ell_{\|\cdot\|}(X_t^j(x_j)) \leq \varepsilon, \quad 1 \leq j \leq N.$$

Proof. By changing basis if necessary, we can assume that $y_1, \ldots, y_N$ are diagonal. Let us set $M := \max_{1 \leq j \leq N} \|x_j - y_j\|$. By L.4.1.1 and R.4.1.3 we will have that there are diagonal matrices $\tilde{x}_1,\ldots,\tilde{x}_N$ in $\mathcal{N}(n)(\mathbb{D}^2)$, $W\in\mathbb{U}(n)$ and a constant c_N, such that $x_j:=W\tilde{x}_jW^*$ and

$$\|W\tilde{x}_j-\tilde{x}_jW\|\leq(1+c_N)M,\ \ 1\leq j\leq N. \tag{4.2.1}$$

By taking any diagonal hermitain matrix $\mathbf{H}$ with distinct eigenvalues and by setting $A:=C^*(W,\mathbf{H})$, we can now solve the interpolation problems using the generic matrix models $C^*(y_1,\ldots,y_N)\subseteq C^*(\mathbf{H})$ and $C^*(x_1,\ldots,x_N)\subseteq C^*(W\mathbf{H}W^*)$ related to the environment algebra $\mathcal{E}_A=C_\delta(\mathbb{J}\times\mathbb{T}^1)$ via a diagram of the form 3.2.1.

As a consequence of L.2.3.1, L.2.3.2 and the normed inequalities 4.2.1, we have that for any $\varepsilon_{c_N\delta}>0$, we can find $\delta>0$ and a unitary path $\mathcal{W}\in C(\mathbb{I},M_n)$ defined by the expression $\mathcal{W}_t:=e^{-itH_W}$ for each $t\in\mathbb{I}$, where $H_W\in M_n$ is a hermitian matrix such that $e^{iH_W}=W$, and is defined by $H_W:=h(W)$, for some function $h:\Omega^\alpha_{d,s}\to[-1,1]$, and where $\sigma(W)\subset\Omega^\alpha_{d,s}:=\{e^{i(\pi t+\alpha)}|-1+s<t<1-s\}\subset\mathbb{T}^1$, with $s,\alpha\in\mathbb{R}$ chosen in such a way that $\mathbb{T}^1\backslash\Omega^\alpha_{d,s}$ contains an arc of length d (with $d\geq 2\pi/n$). Using the path $\mathcal{W}$, we can construct N toroidal matrix links of the form

$$X^j_t:=\begin{cases}\mathrm{Ad}[\mathcal{W}_{2t}](x_j), & 0\leq t\leq\frac{1}{2},\\ (2-2t)\tilde{x}_j+(2t-1)y_j, & \frac{1}{2}\leq t\leq 1,\end{cases} \tag{4.2.2}$$

which solve the problems $x_j\rightsquigarrow y_j$, locally preserve normality and commutativity and for $\varepsilon:=(1+c_N)\max\{\varepsilon_{c_N\delta},\delta\}\geq 0$ satisfy the $\|\cdot\|$-distance constraints

$$\begin{aligned}\|X^j_t-y_j\| &\leq \|X^j_t-\tilde{x}_j\|+\|y_j-\tilde{x}_j\|\\ &\leq \varepsilon_{c_N\delta}+c_N\delta\\ &\leq \frac{\varepsilon}{(1+c_N)}+\frac{c_N\varepsilon}{(1+c_N)}=\varepsilon,\end{aligned}$$

together with the $\|\cdot\|$-length constraints

$$\begin{aligned}
\ell_{\|\cdot\|}(X_t^j) &\leq \ell_{\|\cdot\|}(\mathrm{Ad}[\mathcal{W}_t](x_j)) + \|\tilde{x}_j - y_j\| && (4.2.3)\\
&= \int_{\mathbb{I}} \|\partial_t \mathrm{Ad}[\mathcal{W}_t](x_j)\| dt + \|\tilde{x}_j - y_j\| && (4.2.4)\\
&= \|[H_W, \tilde{x}_j]\| + \|\tilde{x}_j - y_j\| && (4.2.5)\\
&\leq \varepsilon_{c_N\delta} + c_N\delta \leq \varepsilon, && (4.2.6)
\end{aligned}$$

which hold whenever $\|x_j - y_j\| \leq \delta$, $1 \leq j \leq N$, and we are done. □

Corollary 4.2.1 (Local hermitian toral connectivity) *Given $\varepsilon > 0$ and any integer $n \geq 1$, there is $\delta > 0$ such that, for any $2N$ hermitian contractions $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, \quad 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, \quad 1 \leq j \leq N, \end{cases}$$

there exist N toroidal matrix links $X^1, \ldots, X^N$ in M_n, which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\begin{cases} [X_t^j(x_j), X_t^k(x_k)] = 0, \\ X_t^j(x_j) = (X_t^j(x_j))^*, \\ \|X_t^j(x_j) - y_j\| \leq \varepsilon, \end{cases}$$

for each $1 \leq j, k \leq N$ and each $t \in \mathbb{I}$. Moreover,

$$\ell_{\|\cdot\|}(X_t^j(x_j)) \leq \varepsilon, \quad 1 \leq j \leq N.$$

Proof. Since for any $\alpha \in \mathbb{R}$, any pair of hermitian matrices $x, y \in \mathbb{H}(n)$ and any partial unitary $z \in \mathbb{PU}(n)$, we have that $x + \alpha(y - x)$ and zxz^* are also in $\mathbb{H}(n)$, the result follows as a consequence of L.4.1.1, R.4.1.3 and T.4.2.1. □

Theorem 4.2.2 (Local unitary toral connectivity) *Given any $\varepsilon \geq 0$ and any integer $n \geq 1$, there is $\delta \geq 0$ such that given any $2N$ unitary matrices $U_1, \ldots, U_N$,$V_1, \ldots, V_N$ in M_n which satisfy the relations*

$$\begin{cases} [U_j, U_k] = [V_j, V_k] = 0, \\ \|U_k - V_k\| \leq \delta, \end{cases}$$

for each $1 \leq j, k \leq N$, there are toroidal matrix links $u^1, \ldots, u^N$ in M_n which solve the interpolation problems

$$U_k \rightsquigarrow V_k, \quad 1 \leq k \leq N,$$

and also satisfy the relations

$$\begin{cases} [u_t^j, u_t^k] = 0, \\ (u_t^j)^* u_t^j = u_t^j (u_t^j)^* = \mathbb{1}_n, \\ \|u_t^j - V_j\| \leq \varepsilon, \end{cases}$$

for each $t \in \mathbb{I}$ and each $1 \leq j, k \leq N$. Moreover, $\ell_{\|\cdot\|}(u_t^j) \leq \varepsilon$, $1 \leq j \leq N$.

Proof. By changing basis if necessary we can assume that $V_1, \ldots, V_N$ are diagonal matrices. Let us set $M := \max_j\{\|u_j - v_j\|\}$. By L.4.1.1 and R.4.1.3 we will have that there are $W \in \mathbb{U}(n)$, diagonal unitaries $\tilde{V}_1, \ldots, \tilde{V}_N \in M_n$ and a constant c_N, such that $U_j = W\tilde{V}_j W^*$ and

$$\|W\tilde{V}_j - \tilde{V}_j W\| \leq (1 + c_N)M, \ 1 \leq j \leq N. \tag{4.2.7}$$

By taking any diagonal unitary $\mathbf{U}$ with distinct eigenvalues and setting $A := C^*(W, \mathbf{U})$, we can obtain generic matrix models $C^*(V_1, \ldots, V_N) \subseteq C^*(\mathbf{U})$ and $C^*(U_1, \ldots, U_N) \subseteq C^*(W\mathbf{U}W^*)$ to solve the interpolation problems, which are related to the environment algebra $\mathcal{E}_A = C_\delta(\mathbb{T}^2)$ via a diagram of the form 3.2.1.

As a consequence of the inequalities 4.2.7 and by applying L.2.3.1 and L.2.3.2, we have that for any $\varepsilon_{c_N\delta} > 0$, there are $\delta > 0$ and a unitary path $\mathcal{W} \in C(\mathbb{I}, M_n)$

defined by the expression $\mathcal{W}_t := e^{-itH_W}$ for each $t \in \mathbb{I}$, where $H_W \in M_n$ is a hermitian matrix such that $e^{iH_W} = W$, and is defined by $H_W := h(W)$, for some function $h : \Omega^{\alpha}_{d,s} \to [-1,1]$, and where $\sigma(W) \subset \Omega^{\alpha}_{d,s} := \{e^{i(\pi t+\alpha)} | -1+s < t < 1-s\} \subset \mathbb{T}^1$, with $s, \alpha \in \mathbb{R}$ chosen in such a way that $\mathbb{T}^1 \backslash \Omega^{\alpha}_{d,s}$ contains an arc of length d (with $d \geq 2\pi/n$).

Using the path $\mathcal{W}$, we can construct N local commutativity preserving piecewise smooth unitary paths, that will be defined for each $1 \leq j \leq N$ by

$$u^j_t := \begin{cases} \mathrm{Ad}[\mathcal{W}_{2t}](U_j), & 0 \leq t \leq \frac{1}{2}, \\ e^{(2t-1)\ln(\tilde{V}^*_j V_j)}\tilde{V}_j, & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{4.2.8}$$

It can be seen that the unitary toral matrix links 4.2.8 solve the interpolation problems $U_j \rightsquigarrow V_j$, $1 \leq j \leq N$, and locally preserve commutativity. As a consequence of the evident $\|\cdot\|$-flatness of $\mathbb{U}(C^*(\mathbf{U}))$, if we set $\varepsilon := (1 + c_N)\max\{\varepsilon_{c_N\delta}, \delta\}$ we can use L.2.3.1 and L.2.3.2 again to obtain for each $1 \leq j \leq N$ the $\|\cdot\|$-distance estimates

$$\begin{aligned} \|u^j_t - V_j\| &\leq \|u^j_t - \tilde{V}_j\| + \|\tilde{V}_j - V_j\| \\ &\leq \varepsilon_{c_N\delta} + c_N\delta \\ &\leq \frac{\varepsilon}{1+c_N} + \frac{c_N\varepsilon}{1+c_N} = \varepsilon. \end{aligned}$$

for each $t \in \mathbb{I}$, together with the $\|\cdot\|$-length estimates

$$\begin{aligned} \ell_{\|\cdot\|}(u^j_t) &\leq \ell_{\|\cdot\|}(\mathrm{Ad}[\mathcal{W}_t](U_j)) + \ell_{\|\cdot\|}(e^{t\ln(\tilde{V}^*_j V_j)}\tilde{V}_j) \\ &= \int_{\mathbb{I}} \|\partial_t \mathrm{Ad}[\mathcal{W}_t](U_j)\| dt + \|\tilde{V}_j - V_j\| \\ &= \|[H_W, \tilde{V}_j]\| + \|\tilde{x}_j - y_j\| \\ &\leq \varepsilon_{c_N\delta} + c_N\delta \leq \varepsilon, \end{aligned}$$

which hold whenever $\|U_j - V_j\| \leq \delta$ for each $1 \leq j \leq N$, and we are done. $\Box$

Theorem 4.2.3 (Locally symmetric toral linking) *Given $\varepsilon > 0$ and any $n \in \mathbb{Z}^+$, there is $\delta > 0$ such that, for any $2N$ normal contractions $x_1, \ldots, x_N$, $y_1, \ldots, y_N$*

and any unitary matrix u in M_n which satisfy the relations

$$\begin{cases} [u,x_j]=[u,y_j]=[x_j,x_k]=[y_j,y_k]=0, \quad 1\leq j,k\leq N, \\ \|x_j-y_j\|\leq \delta, \quad 1\leq j\leq N, \end{cases}$$

there exist N toroidal matrix links $X^1,\ldots,X^N$ in M_n, which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1\leq j\leq N,$$

and satisfy the constraints

$$\begin{cases} [u,X^j_t(x_j)]=[X^j_t(x_j),X^k_t(x_k)]=0, \\ \|X^j_t(x_j)-y_j\|\leq \varepsilon, \end{cases}$$

for each $1\leq j,k\leq N$ and each $t\in\mathbb{I}$. Moreover,

$$\ell_{\|\cdot\|}(X^j_t(x_j))\leq\varepsilon, \quad 1\leq j\leq N.$$

Proof. By changing basis if necessary, we can assume that $y_1,\ldots,y_N$ and u are diagonal. We will have that u commutes with both $x_1,\ldots,x_N$ and $y_1,\ldots,y_N$, this constraint will force the matrices $x_1,\ldots,x_N$ to have a block structure of the form,

$$x_j:=\begin{pmatrix} X^{(j)}_{1,1} & 0 & \cdots & 0 \\ 0 & X^{(j)}_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & X^{(j)}_{m,m} \end{pmatrix}, \quad 1\leq j\leq N,$$

for some $m\leq n$. Let us set $M:=\max_j\{\|x_j-y_j\|\}$, by L.4.1.1 and R.4.1.3 we will have that there are $W\in\mathbb{U}(n)$, diagonal normal contractions $\tilde{x}_1,\ldots,\tilde{x}_N\in M_n$ and a constant c_N, such that $x_j=W\tilde{x}_jW^*$ and

$$\|W\tilde{x}_j-\tilde{x}_jW\|\leq(1+c_N)M, \ 1\leq j\leq N. \tag{4.2.9}$$

For each $1\leq j\leq N$ and each $1\leq k\leq m$, let us consider the joint spectral decompositions $X^{(j)}_{k,k}=W_k\Lambda(X^{(j)}_{k,k})W_k^*$ and let us set $W:=W_1\oplus\cdots\oplus W_m$.

Since $\mathbb{T}^1\backslash\sigma(W)$ contains an arc of length at least $2\pi/n$, we can apply a similar functional calculus trick to the one implemented in the proof of L.2.3.1 to find a hermitian matrix $H_W := h(W)$, for some $h : \Omega^\alpha_{d,s} \to [-1,1]$ (with $\Omega^\alpha_{d,s}$ defined as in T.4.2.1), such that $W = e^{iH_W}$. By hypothesis and elementary functional calculus we will have that

$$uH_Wu^* = uh(W)u^* = h(uWu^*) = h(W) = H_W, \tag{4.2.10}$$

and this implies that if we set $X^j_t(x_j) := e^{itH_W}\tilde{x}_je^{itH_W}$, then $[u, X^j_t(x_j)]$ for each $t \in \mathbb{I}$. The result now follows by using a similar argument to the one used for the proof of T.4.2.1. $\Box$

4.3 Almost $\mathbb{Z}/m$-centralized normal matrices

In this section we will present an application of L.2.3.1 and L.2.3.2 which is less general than T.4.2.1 since it considers a much simpler geometric situation, we include this result because of its potential applications to the (uniform) approximate solution of matrix and operator equations and to the solution of constrained inverse eigenvalue problems.

Definition 4.3.1 *Given $\delta > 0$ any integer $n \geq 1$, some fixed but arbitrary integer $m \geq 1$, a matrix representation $\mathbb{Z}/m \to \mathbb{U}(n), \mathbb{Z}/m \ni \mathbf{1}_{\mathbb{Z}/m} \mapsto \mathbf{G} \in \mathbb{U}(n)$ (where $\mathbf{1}_{\mathbb{Z}/m}$ is the generator of $\mathbb{Z}/m$) and a subset $S \subseteq \mathcal{N}(n)$, we say that the set S is $(\mathbb{Z}/m, \delta)$-centralized with respect to $\mathbf{G}$, if $\|[\mathbf{G}, s]\| \leq \delta$ for each $s \in S$. In particular we say that a matrix $x \in \mathcal{N}(n)$ is $(\mathbb{Z}/m, \delta)$-centralized if $\|[\mathbf{G}, x]\| \leq \delta$. (The explicit reference to $\mathbf{G}$ will be omitted when it is clear from the context.)*

Lemma 4.3.1 (Approximate almost $\mathbb{Z}/m$-centralized normal matrices) *For any $\varepsilon > 0$ and a matrix representation $\mathbb{Z}/m \to \mathbb{U}(n), \mathbb{Z}/m \ni \mathbf{1}_{\mathbb{Z}/m} \mapsto \mathbf{G} \in \mathbb{U}(n)$,* *there is $\delta > 0$ such that, for any N pairwise commuting $(\mathbb{Z}/m, \delta)$-centralized matrices $X_1, \ldots, X_N \in \mathcal{N}(n)(\mathbb{D}^2)$ there are N pairwise commuting $(\mathbb{Z}/m, 0)$-centralized matrices $\tilde{X}_1, \ldots, \tilde{X}_N \in \mathcal{N}(n)(\mathbb{D}^2)$ and N toral matrix links $\mathcal{T}^1, \ldots, \mathcal{T}^N \in C(\mathbb{I}, M_n)$ which solve the problems $X_j \rightsquigarrow \tilde{X}_j$ and satisfy the constraints $\|\mathcal{T}^j_t - \tilde{X}_j\| \leq \varepsilon$ for each $t \in \mathbb{I}$ and $1 \leq j \leq N$. Moreover, $\ell_{\|\cdot\|}(\mathcal{T}^j_t) \leq \varepsilon$, $1 \leq j \leq N$.*

Proof. The correspondence $\mathbf{1}_{\mathbb{Z}/m} \mapsto \mathbf{G}$ implies that $\mathbf{G}^M = \mathbb{1}_n$. By elementary representation theory we have that for each $1 \leq j \leq N$, the matrices

$$\tilde{X}_j := \frac{1}{m}\sum_{j=0}^{m-1} \mathbf{G} X_j (\mathbf{G}^m)^*,$$

will satisfy the relations $[\mathbf{G}, \tilde{X}_j] = 0$. Using similar techniques to the ones implemented in the proof of [28, L.2.3] we can obtain the estimates

$$\|\tilde{X}_j\| = \frac{1}{m}\left\|\sum_{j=0}^{m-1} \mathbf{G} X_j (\mathbf{G}^m)^*\right\| \leq \|X_j\|$$

and

$$\begin{aligned}
\|\mathbf{G}X_j - X_j\mathbf{G}\| &\leq \|\mathbf{G}X_j\mathbf{G}^* - \tilde{X}_j\| + \|\tilde{X}_j - X_j\| \\
&= \|\mathbf{G}X_j\mathbf{G}^* - \mathbf{G}\tilde{X}_j\mathbf{G}^*\| + \|\tilde{X}_j - X_j\| \\
&\leq 2\|\tilde{X}_j - X_j\| \\
&\leq \frac{2m(m-1)}{2m}\|[\mathbf{G}, X_j]\| = (m-1)\|[\mathbf{G}, X_j]\|,
\end{aligned}$$

for each $1 \leq j \leq N$. It is clear now that we can find $\delta > 0$ small enough such that, by using L.2.3.1 and L.2.3.2 we can procced as in the proof of T.4.2.1 to find the solvent toral matrix links $\mathcal{T}^1, \ldots, \mathcal{T}^N \in C(\mathbb{I}, M_n)$, and we are done. □

4.4 Lifted local piecewise analytic connectivity

Definition 4.4.1 (Symmetry dilations) *Given any unitary $u \in M_n$, we will write $\hat{u}_s$ to denote its corresponding unitary dilation $\hat{u}_s := (\Sigma_2 \otimes \mathbb{1}_n)(u^* \oplus u)$ in M_{2n}.*

Definition 4.4.2 (Standard dilations) *We will denote by κ the matrix compression $M_{2n} \to M_n$ defined by the mapping*

$$\kappa : M_{2n} \to M_n, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x_{11}.$$

We say that $X \in M_{2n}$ is a **standard dilation** *of $x \in M_n$ if $x = \kappa(X)$.*

In this section we will derive some uniform connectivity results by using a dilation technique consisting on constructing for any normal contraction $x \in M_n$ and any unitary $w \in \mathbb{U}(n)$ which satisfy the relation $\|wx - xw\| \leq C\delta$ for some constants $C, \delta \geq 0$, the matrix dilations $\hat{w} := w \oplus w^*$ and $\hat{x} := \mathrm{Ad}[\hat{w}](x \oplus x)$ in M_{2n}. Using the symmetry dilation $\hat{w}_s$ of w we can obtain the normed relations $\|wx - xw\| = \|\hat{w}_s\hat{x} - \hat{x}\hat{w}_s\| = \|wxw^* \oplus w^*xw - x \oplus x\| \leq C\delta$.

Theorem 4.4.1 (Lifted local toral connectivity) *Given $\varepsilon > 0$, there is $\delta > 0$ such that, for any $2N$ normal contractions $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, & 1 \leq j \leq N, \end{cases}$$

there exist N pairwise commuting normal contractive standard dilations $x_j \mapsto \hat{X}_j \in M_{2n}$, $1 \leq j \leq N$ and N toroidal matrix links $X^1, \ldots, X^N$ in $C(\mathbb{I}, M_{2n})$, which solve the problems

$$\hat{X}_j \rightsquigarrow y_j \oplus y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\begin{cases} [X_t^j, X_t^k] = 0, \\ \|X_t^j - y_j \oplus y_j\| \leq \varepsilon, \end{cases}$$

for each $1 \leq j, k \leq N$ and each $t \in \mathbb{I}$. Moreover,

$$\ell_{\|\cdot\|}(X_t^j) \leq \varepsilon, \quad 1 \leq j \leq N.$$

Proof. By changing basis if necessary, we can assume that $y_1, \ldots, y_N$ are diagonal. Let us set $M := \max_{1\leq j\leq N} \|x_j - y_j\|$. By L.4.1.1 and R.4.1.3 we will have that there are diagonal matrices $\tilde{x}_1, \ldots, \tilde{x}_N$ in $\mathcal{N}(n)(\mathbb{D}^2)$, $W \in \mathbb{U}(n)$ and a constant c_N, such that $x_j := W\tilde{x}_jW^*$ and

$$\|W\tilde{x}_j - \tilde{x}_jW\| \leq (1 + c_N)M, \; 1 \leq j \leq N. \tag{4.4.1}$$

It can be seen that if for each $1 \leq j \leq N$, we set $\tilde{X}_j := \tilde{x}_j \oplus \tilde{x}_j$, $\hat{X}_j := \mathrm{Ad}[W \oplus W^*](\tilde{X}_j)$ and $\hat{y}_j := y_j \oplus y_j$, then we will have that $x_j \mapsto \mathrm{Ad}[\hat{W}_s](\tilde{X}_j) = \hat{X}_j$, $1 \leq j \leq N$ are N standard normal contractive dilations of $x_1, \ldots, x_N$, we will also have that

$$\|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \leq (1 + c_N)M, \; 1 \leq j \leq N. \tag{4.4.2}$$

Since $\hat{W}_s \in \mathbb{U}(2n) \cap \mathbb{H}(2n)$, we will have that $\hat{W}_s$ can be represented as $\hat{W}_s = e^{i\frac{\pi}{2}(\hat{W}_s - \mathbb{1}_{2n})}$ for any $n \geq 1$. We also have that there is a unitary path $\{\mathcal{W}_t\}_{t\in\mathbb{I}} \subset M_{2n}$ with $\mathcal{W}_t := e^{i\frac{\pi(1-t)}{2}(\hat{W}_s - \mathbb{1}_{2n})}$, which satisfies the conditions $\mathcal{W}_0 = \hat{W}_s$, $\mathcal{W}_1 = \mathbb{1}_{2n}$, together with the normed estimates,

$$\begin{aligned} \|\mathcal{W}_t\tilde{X}_j - \tilde{X}_j\mathcal{W}_t\| &= |\cos(\pi t/2)|\|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \\ &\leq \|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \leq (1 + c_N)M, \end{aligned}$$

for each $1 \leq j \leq N$ and each $0 \leq t \leq 1$. Moreover, for each $1 \leq j \leq N$ we have that

$$\begin{aligned} \ell_{\|\cdot\|}(\mathrm{Ad}[\mathcal{W}_t](\tilde{X}_j)) &= \int_{\mathbb{I}} \|\partial_t \mathrm{Ad}[\mathcal{W}_t](\tilde{X}_j))\| dt, \\ &= \frac{\pi}{2}\|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \leq \frac{\pi(1 + c_N)}{2}M. \end{aligned}$$

By using a similar argument to the one implemented in the proof of T.4.2.1, we can now construct the solvent toral matrix links $X^1, \ldots, X^N \in C(\mathbb{I}, M_{2n})$ in a uniform way, and we are done. □

Remark 4.4.1 *It can be seen that by using the technique implemented in the proof of T.4.4.1 one can obtain lifted versions of T.4.2.2, C.4.2.1 and T.4.2.3.*

Remark 4.4.2 *As a consequence of T.4.4.1 we can derive simple detection methods to identify families of pairwise commuting matrices in M_n that can be connected uniformly via piecewise analytic toral matrix links. The existence of these detection methods raises some interesting questions for further studies.*

Let us denote by $C^*_\varepsilon\langle\mathbb{Z}/2\times\mathbb{Z}\rangle$ the universal C^*-algebra defined by the expression

$$C^*_\varepsilon\langle\mathbb{Z}/2\times\mathbb{Z}\rangle := C^*_1\left\langle u,v \;\middle|\; \begin{array}{l} uu^* = u^*u = u^2 = 1, \\ vv^* = v^*v = 1, \\ \|uv - vu\| \leq \varepsilon \end{array} \right\rangle.$$

Remark 4.4.3 *We can interpret T.4.4.1 as an existence theorem of solutions to lifted connectivity problems defined on matrix representations of the form*

$$\begin{array}{ccccccc}
 & & C^*_\varepsilon\langle\mathbb{Z}/2\times\mathbb{Z}\rangle & \longrightarrow & C^*(\hat{U}_s,\hat{V}) & \longrightarrow & M_{2n}\;, \\
 & \nearrow & & & \uparrow & & \downarrow \\
C^*\langle\mathbb{F}_2\rangle & \longrightarrow & C_\delta(\mathbb{T}^2) & \longrightarrow & C^*(U,V) & \longrightarrow & M_n
\end{array}$$

with $\hat{U}_s = (\Sigma_2\otimes\mathbb{1}_n)(U^*\oplus U)$ and $\hat{V} = V\oplus V$.

Matrix Klein Bottles: Local matrix deformations and special symmetries

Using T.4.4.1 we can solve all connectivity problems (together with their softened versions) in M_n that can be reduced to connectiviy problems of the form $x \rightsquigarrow_\varepsilon x^*$ in $\mathcal{N}(n)(\mathbb{D}^2)$, with $xx^* = x^*x$, $\|x\|\leq 1$, $x^* = TxT$ and $T^2 = \mathbb{1}_n$.

Remark 4.4.4 *For each $\varepsilon\in[0,2]$, we can use the previously described symmetries and $\mathscr{D}_\mathbb{T}$ to interpret $\bigcup_{x\in M_n}\{x \rightsquigarrow_\varepsilon x^*\}$ as matrix analogies of the* Klein bottle.

By a *softened matrix Klein bottle* we mean that the symmetries are softened, in particular we can consider the connectiviy problems $x \rightsquigarrow_\varepsilon x^*$ and $y \rightsquigarrow_\varepsilon y^*$ in

$\mathcal{N}(n)(\mathbb{D}^2)$ subject to the normed constraints $\|xy - yx\| \leq \delta$, $yy^* = y^*y$, $\|x\|, \|y\| \leq 1$, $\|x^* - TxT\|, \|xT - Ty\| \leq \delta$ and $T^2 = \mathbb{1}_n$. The details regarding to the solvability of these local connectivity problems will be the subject of future work.

4.5 Dimensionality Reduction and Local $\mathbf{C}^0$ connectivity

Theorem 4.5.1 (UDRC constrianed unitary toral connectivity) *Given any $\varepsilon > 0$ and $d > 0$, there is $\delta \geq 0$, such that for any two families of N pairwise commuting unitaries $U_1, \ldots, U_N$ and $V_1, \ldots, V_N$ in M_n, such that $\dim_\delta^{\mathbf{U}}(\Lambda(U_1, \ldots, U_N)) = 1$ and $\mathbb{T}^1 \backslash \sigma(U_j)$ contains an arc of length d, $1 \leq j \leq N$, there are N toral unitary matrix links $Z^1, \ldots, Z^N \in C(\mathbb{I}, M_n)$ which solve the problems $U_j \rightsquigarrow V_j$ for each $1 \leq j \leq N$, and also satisfy the normed relations $\|Z_t^j - V_j\| \leq \varepsilon$, for each $t \in \mathbb{I}$ and $1 \leq j \leq N$. Moreover, $\ell_{\|\cdot\|}(Z_t^j) \leq 2\pi + \varepsilon$, for each $1 \leq j \leq N$.*

Proof. Given any $\varepsilon > 0$. Let us assume that $V_1, \ldots, V_N$ are diagonal. Since we have that $\dim_\delta^{\mathbf{U}}(\Lambda(U_1, \ldots, U_N)) = 1$, by using L.4.1.1 and R.4.1.3 we will have that there are $\delta \geq 0$, $k \in \{1, \ldots, N\}$, $W \in \mathbb{U}(n)$, $\tilde{V}_1, \ldots, \tilde{V}_N \in \mathbb{U}(n)$, $f_1, \ldots, f_N \in C(\mathbb{T}^1, \mathbb{T}^1)$ and a constant c_N such that, $\|f_j(U_k) - U_j\| \leq \delta$, $W\tilde{V}_j W^* = U_j$, $\|[\tilde{V}_j, W]\| \leq (1 + c_N)\delta$ and $[\tilde{V}_j, V_l] = 0$ for each $1 \leq j, l \leq N$.

Since $\mathbb{T}^1 \backslash \sigma(U_j)$ contains an arc of length $d > 0$, $1 \leq j \leq N$, by following a similar argument to the one used by Loring in [24, §6] we will have that $Bott(W, \tilde{V}_k) = 0$.

Since $\tilde{V}_k \in \mathbb{U}(n)$ and $f_1, \ldots, f_N \in C(\mathbb{T}^1, \mathbb{T}^1)$, $\delta > 0$ can be rescaled to $\delta \leq \frac{\varepsilon}{3+c_N}$ in such a way that by using the basic homotopy lemma (C.2.3.1), we can now find a unitary path $\mathcal{W} \in C(\mathbb{I}, M_n)$ such that for each $t \in \mathbb{I}$ and each $1 \leq j \leq N$

$$\|f_j(\mathcal{W}_t \tilde{V}_k \mathcal{W}_t^*) - f_j(\tilde{V}_k)\| \leq \frac{\varepsilon}{3 + c_N}, \tag{4.5.1}$$

and this implies that

$$\begin{aligned}\|\tilde{V}_j\mathcal{W}_t-\mathcal{W}_t\tilde{V}_j\| &\leq 2\|U_j-f_j(U_k)\|+\|\mathcal{W}_tf_j(\tilde{V}_k)-f_j(\tilde{V}_k)\mathcal{W}_t\|\\ &= 2\|U_j-f_j(U_k)\|+\|f_j(\mathcal{W}_t\tilde{V}_k\mathcal{W}_t^*)-f_j(\tilde{V}_k)\|\\ &\leq 2\delta+\frac{\varepsilon}{3+c_N},\end{aligned}$$

for each $t\in\mathbb{I}$ and each $1\leq j\leq N$. Moreover, $\ell_{\|\cdot\|}(\mathcal{W}_t)\leq 2\pi+\frac{\varepsilon}{3+c_N}$.

We can use the path $\mathcal{W}$ to construct N toral unitary matrix links $Z^1,\ldots,Z^N\in C(\mathbb{I},M_n)$ of the form

$$Z_t^j:=\begin{cases}\mathrm{Ad}[\mathcal{W}_{2t}](U_j), & 0\leq t\leq\frac{1}{2},\\ e^{(2t-1)\ln(\tilde{V}_j^*V_j)}\tilde{V}_j, & \frac{1}{2}\leq t\leq 1.\end{cases}\tag{4.5.2}$$

It can be seen that the unitary toral matrix links described by 4.5.2 solve the interpolation problems $U_j\rightsquigarrow V_j$, $1\leq j\leq N$, and locally preserve commutativity. For each $1\leq j\leq N$, we can obtain the $\|\cdot\|$-distance estimates

$$\begin{aligned}\|Z_t^j-V_j\| &\leq \|Z_t^j-\tilde{V}_j\|+\|\tilde{V}_j-V_j\|\\ &\leq 2\delta+\frac{\varepsilon}{3+c_N}+c_N\delta\leq\varepsilon.\end{aligned}$$

for each $t\in\mathbb{I}$, together with the $\|\cdot\|$-length estimates

$$\begin{aligned}\ell_{\|\cdot\|}(Z_t^j) &\leq \ell_{\|\cdot\|}(\mathrm{Ad}[\mathcal{W}_t](U_j))+\ell_{\|\cdot\|}(e^{t\ln(\tilde{V}_j^*V_j)}\tilde{V}_j)\\ &\leq 2\ell_{\|\cdot\|}(\mathcal{W}_t)+\|\tilde{V}_j-V_j\|\\ &\leq 2\pi+\frac{\varepsilon}{3+c_N}+c_N\delta\leq 2\pi+\varepsilon,\end{aligned}$$

and we are done. □

Chapter 5

Loring's unitary connectivity technique

Suppose U_t and V_t are unitary matrices in $\mathbf{M}_n(\mathbb{C})$ for $t = 0$ and $t = 1$ and we define

$$U_t = U_0 e^{t \ln\left(U_0^* U_1\right)} \tag{5.0.1}$$

and

$$V_t = V_0 e^{t \ln\left(V_0^* V_1\right)}. \tag{5.0.2}$$

For $t = 0$ or $t = 1$ the C^*-algebra generated by U_t and V_t is abelian, so select a MASA $C_t \cong \mathbb{C}^n$ in each case. Let

$$A(C_0, C_1) = \left\{ X \in C\left([0,1], \mathbf{M}_n(\mathbb{C})\right) \middle| \, X(0) \in C_0 \text{ and } X(1) \in C_1 \right\}.$$

Lemma 5.0.1 (Loring) *The C^*-algebra $A(C_0, C_1)$ has stable rank one.*

Proof. Starting with X continuous with $X(t)$ in C_t at the endpoints, we can adjust this by a small amount, leaving the endpoints in C_t, to get X piece-wise linear, with the endpoints of every linear segment having no spectral multiplicity and being invertible. Using Kato's theory of analytic paths, we can get a piece-wise continuous unitary U_t and piece-wise analytic scalar paths $\lambda_n(t)$ so that the new path $Y \approx X$ satisfies

$$Y(t) = U_t \begin{bmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_n(t) \end{bmatrix} U_t^*.$$

There may be finitely may places where $Y(t)$ is not invertible. These places will be in the interior of the segment so in an open interval where U_t is continuous. A small deformation of some of the λ_j will take the path through invertibles. We have not moved the endpoints in the second adjustment so the constructed element is in $A(C_0, C_1)$ and close to X.□

Lemma 5.0.2 (Loring) *The endpoint-restriction map $\rho : A(C_0, C_1) \to C_0 \oplus C_1$ induces an injection on K_0.*

Proof. The kernel of ρ is $C([0,1], \mathbf{M}_n(\mathbb{C}))$ which has trivial K_0-group. So this result follows from the exactness of the usual six-term sequence in K-theory.□

Lemma 5.0.3 (Loring) *Given unitaries U and V in $A(C_0, C_1)$, with $\|[U,V]\| \leq \nu_0$ as in D.2.3.1 (so the Bott index makes sense),* $\mathrm{Bott}(U,V)$ *is the trivial element of $K_0(A(C_0, C_1))$.*

Proof. By the previous lemma, we need only calculate $\mathrm{Bott}(\rho(U), \rho(V))$. These unitaries are in a commutative C^*-algebra so they have trivial Bott index.□

Theorem 5.0.2 (Loring) *Given $\epsilon > 0$, there exists $\delta > 0$ so that for all n, given unitary matrices U_0, U_1, V_0, V_1 in $\mathbf{M}_n(\mathbb{C})$ with $U_0V_0 = V_0U_0$, $U_1V_1 = V_1U_1$, $\|U_0 - U_1\| \leq \delta$ and $\|V_0 - V_1\| \leq \delta$, then there exists continuous paths U_t and V_1 between the given pairs of unitaries with each U_t and V_t unitary, and with $U_tV_t = V_tU_t$, $\|U_t - U_0\| \leq \epsilon$ and $\|V_t - V_0\| \leq \epsilon$ for all t.*

Proof. The paths U_t and V_t defined in equations 5.0.1 and 5.0.2 will be almost commuting unitary elements of $A(C_0, C_1)$. By Lemma 5.0.1 we may apply [10, Theorem 8.1.1] regarding approximating in $A(C_0, C_1)$ by commuting unitaries. Lemma 5.0.3 tells us there is no invariant to worry about, so we can find A_t and B_t close of U_t and V_t that are commuting continuous paths of unitaries with A_t and B_t in C_t for $t = 0, 1$. The unitary elements in the commutative C_t are locally connected, so we can find a short path from U_0 and V_0 to A_0 and B_0, and likewise at the other end. Concatenating, we get a paths of commuting unitary matrices from U_0 and V_0 to U_1 and V_1 so that at every point we are close to some pair (U_t, V_t). These then are all close to U_0 and V_0. □

Chapter 6

Local Deformation of Matrix Words

Given a finite set $\mathbf{C} := \{c_1, \ldots, c_M\} \subset M_n$ of normal contractions which contains the identity matrix $\mathbb{1}_n \in M_n$ and some fixed but arbitrary integer $L > 0$, by a mixed matrix word of length L we mean a function $W_L : M_n^M \times M_n^{2N} \to M_n, (c_1, \ldots, c_M, x_1, \ldots, x_{2N}) \mapsto c_{j_1} x_{j_1}^{k_1} \cdots c_{j_L} x_{j_L}^{k_L}$ on $2N$ matrix variables $\mathbf{X} := \{x_1, \ldots, x_{2N}\}$, where $c_{j_l} \in \mathbf{C}$ and $k_l \in \mathbb{Z}_0^+$, $1 \le l \le L$. The number $\deg(W_L) := \max_{1 \le l \le L}\{k_l\}$ will be called the degree of the word W_L. We call the sets $\mathbf{C}$ and $\mathbf{X}$ matrix coefficient and matrix variable sets respectively.

6.1 Matrix Equations on Words

By a system of matrix equations on words, we mean expressions of the form

$$\begin{cases} \mathbf{E_q}^{(1)}(x_1,\ldots,x_N) := \sum_{j=1}^{J_1} \alpha_{1,j} W_{1,L_j}(\mathbf{C}, x_1,\ldots,x_N, x_1^*,\ldots,x_N^*) = 0, \\ \qquad\qquad\vdots \\ \mathbf{E_q}^{(p)}(x_1,\ldots,x_N) := \sum_{j=1}^{J_p} \alpha_{p,j} W_{p,L_j}(\mathbf{C}, x_1,\ldots,x_N, x_1^*,\ldots,x_N^*) = 0. \end{cases} \tag{6.1.1}$$

Example 6.1.1 *For an example of a matrix equation on words we can consider an equation of the form*

$$UX - X^*U = R,$$

where X is a normal matrix variable in M_n and the matrice U and R are normal contractions in M_n. In this case the coefficient set is $\mathbf{C} := \{U, R, \mathbb{1}_n\}$ and the variable set is $\mathbf{X} := \{X\}$.

Example 6.1.2 *For an example of a system a matrix equations on words we can consider a system of matrix equations of the form*

$$\begin{cases} UX - X^*U = R, \\ VY + U^*XV = S, \end{cases}$$

where X and Y are normal matrix variables in M_n and the matrices U, V, R and S are normal contractions in M_n. In this case the coefficient set is $\mathbf{C} := \{U, V, R, S, \mathbb{1}_n\}$ and the variable set is $\mathbf{X} := \{X, Y\}$.

Given $\delta > 0$, an N-tuple $X_1, \ldots, X_N$ in M_n is called a δ-approximate solution of 6.1.1 if we have that

$$\begin{cases} \|\mathbf{E_q}^{(1)}(X_1,\ldots,X_N)\| \leq \delta, \\ \vdots \\ \|\mathbf{E_q}^{(p)}(X_1,\ldots,X_N)\| \leq \delta. \end{cases} \tag{6.1.2}$$

6.1.1 Perturbation and relative lifting of matrix words

Given any expression of the form

$$\mathbf{E_q}(x_1,\ldots,x_N) := \sum_{j=1}^{J} \alpha_j W_{L_j}(\mathbf{C}, x_1,\ldots,x_N, x_1^*,\ldots,x_N^*).$$

Let us restrict the variable subset $\mathbf{Y} := \{x_1,\ldots,x_N\}$ in such a way that its elements only take values on the unit ball of normal elements in M_n for some fixed but arbitrary integer $n \geq 1$, let us also impose the restriction $x_j x_k = x_k x_j$ $1 \leq j,k \leq N$, we will have that $\mathbf{E_q}(x_1,\ldots,x_N)$ can be represented by the expression

$$\mathbf{E_q}(x_1,\ldots,x_N) := \sum_{j=1}^{J} \alpha_j \tilde{W}_{L_j}(\mathbf{C}, x_1,\ldots,x_N),$$

where $\tilde{W}_{L_j}(\mathbf{C}, x_1,\ldots,x_N)$ is a mixed matrix word with respect to $\mathbf{C}$ and $\mathbf{Y}$. If we consider now two N-tuples of pairwise commuting normal matrices $X_1,\ldots,X_N$ and $\tilde{X}_1,\ldots,\tilde{X}_N$ in M_n, such that $\|X_j - \tilde{X}_j\| \leq \delta$ for some fixed but arbitrary number $\delta > 0$, it can be seen that if we define the numbers $d_j := \deg(\tilde{W}_{L_j})$, then there is a number $\tilde{C} := \tilde{C}(\alpha_1,\ldots,\alpha_J, d_1,\ldots,d_J, J) > 0$ such that

$$\|\mathbf{E_q}(X_1,\ldots,X_N) - \mathbf{E_q}(\tilde{X}_1,\ldots,\tilde{X}_N)\| \leq \tilde{C}\delta.$$

Let us now consider the mappings $\mathbb{L} : M_n \to M_{2n}$ and $\mathbb{K} : M_{2n} \to M_n$, defined by the expressions

$$\begin{aligned} \mathbb{L}(X) &:= \mathbb{1}_2 \otimes X, \\ \mathbb{K}\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) &:= x_{11}. \end{aligned}$$

It is clear that $\mathbb{L}$ is a relative lift of $\mathbb{K}$, if we write $\hat{\mathbf{E}}_{\mathbf{q}}(x_1, \ldots, x_N)$ to denote the function $M_{2n}^M \times M_{2n}^N \to M_{2n}$ described by the expression

$$\hat{\mathbf{E}}_{\mathbf{q}}(x_1, \ldots, x_N) := \sum_{j=1}^{J} \alpha_j \tilde{W}_{L_j}(\mathbb{L}(\mathbf{C}), x_1, \ldots, x_N),$$

we will have that

$$\begin{aligned} \|\hat{\mathbf{E}}_{\mathbf{q}}(\mathbb{L}(X_1), \ldots, \mathbb{L}(X_N)) - \hat{\mathbf{E}}_{\mathbf{q}}(\mathbb{L}(\tilde{X}_1), \ldots, \mathbb{L}(\tilde{X}_N))\| &= \|\mathbb{L}(\mathbf{E}_{\mathbf{q}}(X_1, \ldots, X_N) \\ &\quad -\mathbf{E}_{\mathbf{q}}(\tilde{X}_1, \ldots, \tilde{X}_N))\| \\ &\leq \tilde{C} \max_{1 \leq j \leq N} \|X_j - \tilde{X}_j\| \\ &\leq \tilde{C}\delta. \end{aligned} \tag{6.1.3}$$

A further reduction

Given any system of matrix equations on words like 6.1.1 with pairwise commuting normal contractive matrix variable set, using the unitaries $\Sigma_N \in M_N$ defined by 2.1.5 in §2.1 together with the mappings $\mathbf{\Sigma}_N := \mathrm{Ad}[\Sigma_N \otimes \mathbb{1}_n]$ we can lift any matrix equation on words in M_n to matrix equations in M_{Nn} of the form

$$\begin{cases} \mathbf{LE_q}^{(1)}(x_1, \ldots, x_N) := \sum_{j=1}^{J_1} \alpha_{1,j} \mathbf{W}_{1,L_j}(\mathbf{N}, \mathbf{X}_1, \ldots, \mathbf{X}_N) = 0, \\ \quad \vdots \\ \mathbf{LE_q}^{(p)}(x_1, \ldots, x_N) := \sum_{j=1}^{J_p} \alpha_{p,j} \mathbf{W}_{p,L_j}(\mathbf{N}, \mathbf{X}_1, \ldots, \mathbf{X}_N) = 0. \end{cases} \tag{6.1.4}$$

Where $\mathbf{N}$ is some set of pairwise commuting normal matrix contractions in M_{Nn} with contains $\mathbb{1}_{Nn}$ and $\mathbf{X}_1, \ldots, \mathbf{X}_N$ are pairwise commuting matrix variables in M_{Nn}, and where we have that any δ-approximate solution of 6.1.1 can be recovered from a corresponding lifted δ-approximation solution of 6.1.4.

6.2 Local Homotopies and Approximation Solution of Matrix Equations

Since $\mathbb{L}$ is a lift of $\mathbb{K}$, it can be seen that $X_1, \ldots, X_N$ are δ-approximate solutoins to 6.1.1, then $\mathbb{L}(X_1), \ldots, \mathbb{L}(X_N)$ will be δ-approximate solutions of the transformed system of matrix equations described by the expressions

$$\begin{cases} \hat{\mathbf{E}}_{\mathbf{q}}^{(1)}(x_1, \ldots, x_N) = 0, \\ \vdots \\ \hat{\mathbf{E}}_{\mathbf{q}}^{(p)}(x_1, \ldots, x_N) = 0. \end{cases} \tag{6.2.1}$$

Using the inequality 6.1.3 and T.4.4.1 we can derive the following result.

Theorem 6.2.1 *Given $\varepsilon > 0$, there is $\delta > 0$ such that the following holds: Given any integer $n \geq 1$ and any system of matrix equations on words in normal pairwise commuting matrix variables of the form:*

$$\begin{cases} \mathbf{E_q}^{(1)}(x_1, \ldots, x_N) := \sum_{j=1}^{J_1} \alpha_{1,j} W_{1,L_j}(\mathbf{C}, x_1, \ldots, x_N) = 0, \\ \vdots \\ \mathbf{E_q}^{(p)}(x_1, \ldots, x_N) := \sum_{j=1}^{J_p} \alpha_{p,j} W_{p,L_j}(\mathbf{C}, x_1, \ldots, x_N) = 0. \end{cases} \tag{6.2.2}$$

For any two N-tuples of pairwise commuting normal matrices $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in M_n such that, $\|\mathbf{E_q}^{(j)}(X_1, \ldots, X_N)\| \leq \delta$, $\|\mathbf{E_q}^{(j)}(Y_1, \ldots, Y_N)\| \leq \delta$ $1 \leq j \leq p$

and

$$\begin{cases} \|\mathbf{E_q}^{(1)}(X_1,\ldots,X_N)-\mathbf{E_q}^{(1)}(Y_1,\ldots,Y_N)\|\leq\delta, \\ \vdots \\ \|\mathbf{E_q}^{(p)}(X_1,\ldots,X_N)-\mathbf{E_q}^{(p)}(Y_1,\ldots,Y_N)\|\leq\delta, \end{cases} \tag{6.2.3}$$

there are, a mapping $\Phi : M_n \to M_{2n}$ and N picewise analytic pairwise commuting normal contractive matrix matrix paths $X^1,\ldots,X^N \in C([0,1],M_{2n})$, such that X^j is ε-local with respect to the pair $\Phi(X_j),\mathbb{L}(Y_j)$, $1\leq j\leq N$, and such that

$$\begin{cases} \|\hat{\mathbf{E}}_{\mathbf{q}}^{(1)}(X_t^1,\ldots,X_t^N)-\hat{\mathbf{E}}_{\mathbf{q}}^{(1)}(\mathbb{L}(Y_1),\ldots,\mathbb{L}(Y_N))\|\leq\varepsilon, \\ \vdots \\ \|\hat{\mathbf{E}}_{\mathbf{q}}^{(p)}(X_t^1,\ldots,X_t^N)-\hat{\mathbf{E}}_{\mathbf{q}}^{(p)}(\mathbb{L}(Y_1),\ldots,\mathbb{L}(Y_N))\|\leq\varepsilon, \end{cases} \tag{6.2.4}$$

for each $t\in[0,1]$.

Proof. Since by 6.1.3, given $\varepsilon>0$, there is $\delta>0$ such that

$$\begin{cases} \|\mathbf{E_q}^{(1)}(X_1,\ldots,X_N)-\mathbf{E_q}^{(1)}(Y_1,\ldots,Y_N)\|\leq\delta, \\ \vdots \\ \|\mathbf{E_q}^{(p)}(X_1,\ldots,X_N)-\mathbf{E_q}^{(p)}(Y_1,\ldots,Y_N)\|\leq\delta. \end{cases} \tag{6.2.5}$$

Moreover, by T.4.4.1 there are a unitary $W\in\mathbb{U}(n)$ and N picewise analytic pairwise commuting normal contractive matrix paths $X^1,\ldots,X^N\in C([0,1],M_{2n})$, such that and X^j is ε-local with respect to the pair $\Phi(X_j),\mathbb{L}(Y_j)$, $1\leq j\leq N$, with $\Phi := \mathrm{Ad}[\mathbb{1}_n\oplus W_n^{*2}]\circ\mathbb{L}$ and such that

$$\|\hat{\mathbf{E}}_{\mathbf{q}}^{(r)}(X_t^1,\ldots,X_t^N)-\hat{\mathbf{E}}_{\mathbf{q}}^{(r)}(\mathbb{L}(Y_1),\ldots,\mathbb{L}(Y_N))\| \;\leq\; \tilde{C}_r\max_{1\leq j\leq N}\|X_t^j-\mathbb{L}(Y_j)\|\leq\varepsilon, \tag{6.2.6}$$

where $\tilde{C}_r$ is some constant (that does not depend on the size of the matrices), for each $1\leq r\leq p$ and each $t\in[0,1]$, and we are done. ☐

Chapter 7

Hints and Future Directions

The detection (recognition) of almost localized matrix representations of universal C*-algebras that can be connected uniformly via piecewise analytic paths induces interesting problems which are topological/K-theoretical and computational in nature. Motivated by Loring's connectivity technique we consider that the study of uniform local matrix connectivity in C*-representations of the form $C(\mathbb{T}^N) \to M_n$ and $C_\varepsilon(\mathbb{T}^2) \to M_n \leftarrow C(\mathbb{T}^N)$ will present interesting challenges and questions that will be the subject of future study. In particular we are interested in the application of T.4.4.1 to the study of the question. Is $C^*\langle \mathbb{F}_2 \times \mathbb{F}_2 \rangle$ RFD? (This is equivalent to **Connes's embedding problem**.)

Let us consider now relative lifting problems of the form.

Problem 7.0.1 *Given any $\varepsilon > 0$, an integer $k \geq 1$ and a sequence of norm decreasing linear compressions $\kappa_n : M_{kn} \to M_n$. Is there $\delta > 0$ such that the following conditions hold? For any two families of N pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in M_{kn} and M polynomials $p_1, \ldots, p_M \in \mathbb{C}[x_1, \ldots, x_{2N}]$ such that $\|Y_j - X_j\| \leq \delta$,*

$$\|p_l(X_1, \ldots, X_N, Y_1, \ldots, Y_N)\| \leq \delta \tag{7.0.1}$$

and

$$\|p_l(\kappa_n(X_1),\ldots,\kappa_n(X_N),\kappa_n(Y_1),\ldots,\kappa_n(Y_N))\| \leq \delta \tag{7.0.2}$$

for each $1 \leq j \leq N$ *and* $1 \leq l \leq M$. *We have that there are* N *pairwise commuting normal contractions* $\tilde{X}_1,\ldots,\tilde{X}_N$ *in* M_{nk} *and* N *piecewise analytic normal contractive matrix paths* $\mathbf{X}^1,\ldots,\mathbf{X}^N \in C([0,1],M_{kn})$ *which satisfy the relations:* $\kappa_n(X_j) = \kappa_n(\tilde{X}_j)$, $\|\tilde{X}_j - Y_j\| \leq \varepsilon$, $\mathbf{X}^j_0 = \tilde{X}_j$, $\mathbf{X}^j_1 = Y_j$ *and* $\|\mathbf{X}^j_t - Y_j\| \leq \varepsilon$, *together with the normed constraints*

$$\|p_l(\kappa_n(\mathbf{X}^1_t),\ldots,\kappa_n(\mathbf{X}^N_t),\kappa_n(Y_1),\ldots,\kappa_n(Y_N))\| \leq \varepsilon \tag{7.0.3}$$

for each $1 \leq l \leq M$, $1 \leq j \leq N$ *and each* $t \in [0,1]$.

A better understanding of the geometric and combinatorial nature of problems of the form **P.**7.0.1, can lead to the solution of some conjectures related to matrix approximation problems, and which have been stated by Chu in [9], in the language of matrix homotopies. It would also allow one to find matrix based proofs of classical conjectures in matrix analysis and operator theory, restated in matrix terms by K. M. R. Audenaert and F. Kittaneh in [1].

Using a similar approach one can provide answers to some questions in topologically controlled linear algebra in the sense of [14], raised by M. H. Freedman.

Another area where a better understanding of the solvability conditions of problem **P.**7.0.1 could have important implications, is the study of solvability conditions of lifting problems related to the connectivity and asymptotic behavior of the matrix representations $C(\mathbb{T}^N) \to M_n \leftarrow C([-1,1]^N)$.

The construction and generalization of detection methods like the ones mentioned in the remark R.4.4.2 of theorem T.4.4.1 together with their implications on inverse spectral problems, will be the subject of future studies.

A better understanding of the geometric and approximate combinatorial nature of toroidal matrix links would provide a mutually benefitial interaction between matrix flows in the sense of Brockett [5] and Chu [9], topologically controlled linear algebra in the sense of Freedman and Press [14] and matrix geometric deformations in the sense of Loring [25], Hajac and Masuda [15], and Woronowicz [34]. This also may provide some novel generic numerical methods to study and compute normal matrix compressions, sparse representations and dimensionality reduction of large scale matrices.

Using the dynamical techniques presented in this document, it seems possible to find more mutually benefitial connections betweeen C^1-realizations of the toroidal matrix links studied here and the solution to some problems in the theory of matrix equations and matrix approximation. These connections will be the object of further studies.

Another interesting questions are motivated by the the possibility of using piecewise analytic toral matrix links to study the local deformation properties of matrix representations of the form $C_\varepsilon(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}/2 \to M_n$ (where α denotes the standard flip) via *softened matrix Klein bottles*. These problems are related to: *Galois correspondence* in the sense of Landstad, Olsen and Pedersen, spectral decomposition problems with *time reversal symmetry* in quantum theory and *deformation theory* for C^*-algebras in the sense of Loring, Dadarlat Hajac and Woronowicz.

The combination of toroidal matrix links with some matrix lifting techniques along the same lines of the proof of T.4.4.1, seem also promising on the solvability of some conjectures studied numerically on [28].

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