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Spatial Decay of Rotating Waves and Restrictions on Finite Disks.

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DISSERTATION

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Dedication

To my Dad.

Acknowledgments

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Abstract

In this thesis, we consider the system of reaction-diffusion equations and the behavior of the solution of such a system. The focus is to concentrate on solutions which decay at ∞. Under suitable assumptions, we prove the solution and its derivatives decay exponentially in all space. We also attempt to show that the solution decays exponentially for the system of equations when posed on a finite disk. This result has been confirmed via numerical methods before, but has never been attempted through an analytic approach, like in this paper. We prove the exponential decay of the solution in a one dimensional case and also discuss the limitations we face when we extend the problem to a system of equations posed on a finite disk.

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Glossary

BVP	Boundary Value Problem
Df	Jacobian matrix of f
PDE	Partial Differential Equation
QCGL	Quintic-cubic Ginzburg-Landau equation

Chapter 1

Introduction

Wave motion is the key mechanism of interest to many fields of science, such as mechanics, acoustics, seismology, oceanography, etc. The study of nonlinear waves has quietly and steadily revolutionized the realm of science over recent years. Nonlinear waves are solutions of time dependent PDE's that are posed on an unbounded domain. There are different types of nonlinear waves, like travelling waves, rotating waves, solitons etc.

In this thesis, we want to study the behavior of the solutions of such PDE's when posed to a bounded domain. Mainly because problems posed on infinite domains are always idealizations, and formulations on finite regions may be more realistic and more accurate. Also, whenever we are doing numerical computations, we always reduce the PDE to a bounded domain, since they can only be performed on finite domains. For this reason, it is crucial to study the behavior of such problems when posed on a finite domain. Our main concern in this paper is to be able to understand the relation between the problems on unbounded domain to the problems on bounded domain. Two main questions arise while trying to understand that relation. Does the solution of the BVP on infinite domain decay exponentially? The main aim of this paper, is to prove that the problem in all space has a solution that decays exponentially. Does the BVP imposed on a bounded domain have a nearby solution? In doing so, we want to show that the error between the solution of the PDE on a bounded domain and the solution on an infinite domain is exponentially small as a function of domain size.

The motivation for this paper comes from the solution for the Quintic-Cubic Ginzburg-Landau equation (QCGL). The complex Ginzburg-Landau equation [6] was first derived in the studies of Poiseuille flow and reaction-diffusion systems and is one of the most studied equations in applied mathematics. Let us consider the QCGL equation

$$u_t = \alpha \Delta u + u(\mu + \beta|u|^2 + \gamma|u|^4) \tag{1.1}$$

where $x \in \mathbb{R}^2$, $u(x,t) \in \mathbb{C}$ and $\mu \in \mathbb{R}$, $\alpha, \beta, \gamma \in \mathbb{C}$. Since u is complex valued the equation corresponds to a real system with two variables. We will look at the numerical solution of this equation as discussed in the paper [1]. The following parameters were used to compute the solution.

$$\alpha = \frac{1}{2}(1+i),\ \beta = 2.5 + i,\ \gamma = -1 - 0.1i,\ \mu = \frac{-1}{2}.$$

For the numerical computations, Cosmol Multiphysics [3] has been used. The problem is discretized on balls $B_R(0) = \{x \in \mathbb{R}^2 : |x| \le R\}$ with Neumann boundary conditions. The real and imaginary parts of the solution are shown in Figure 1.1 and Figure 1.2.

Spinning solitons are solutions that rotate at constant speed and converge to 0 as $|x| \to \infty$. For more information on spinning solitons, see [4], [5].

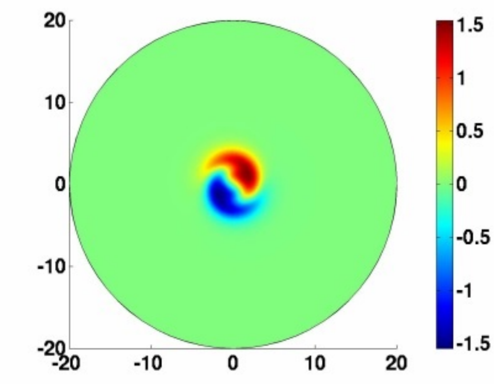

Figure 1.1: Real part of spinning solitons in the QCGL-system [1].

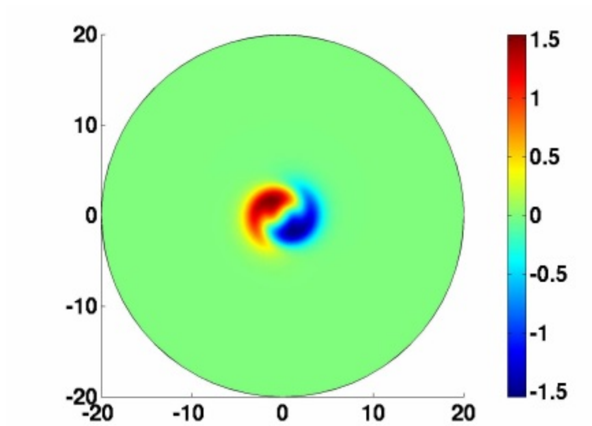

Figure 1.2: Imaginary part of spinning solitons in the QCGL-system [1].

1.1 Class of Problems

In recent years, systems of reaction-diffusion equations have received a great deal of attention, motivated by both their widespread occurrence in models of chemical and biological phenomena, and by the richness of the structure of their solution sets [8].

In the current thesis, we consider the following system of reaction-diffusion equations

$$u_t(x,t) = A\Delta u(x,t) + f(u(x,t)) \qquad ,t \geq 0, x \in \mathbb{R}^d, d \geq 2,$$
$$u(x,0) = u_0(x) \qquad ,t = 0, x \in \mathbb{R}^d, \tag{1.2}$$

where $A \in \mathbb{R}^{N\times N}$ is a diffusion matrix, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth nonlinearity, $u_0 : \mathbb{R}^d \to \mathbb{R}^N$ are the initial data and $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^N$ is the solution to be determined.

The rotating wave solutions of (1.2) are of the form

$$u_*(x,t) = v_*(e^{-tS}x), t \geq 0, x \in \mathbb{R}^d, d \geq 2, \tag{1.3}$$

with space dependent profile $v_* : \mathbb{R}^d \to \mathbb{R}^N$ and skew symmetric matrix $0 \neq S \in \mathbb{R}^{d\times d}$.

Transforming (1.2) via $u(x,t) = v(e^{-tS}x,t)$ we obtain the following evolution equation

$$\begin{cases} v_t(x,t) = A\Delta v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)), t \geq 0, x \in \mathbb{R}^d, d \geq 2, \\ v(x,0) = u_0(x) \qquad ,t = 0, x \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

with the term

$$\langle Sx, \nabla v(x)\rangle = \sum_{i=1}^{d} [(Sx)_i D_i v(x)] = \sum_{i=1}^{d}\sum_{j=1}^{d} [S_{ij} x_j D_i v(x)] \tag{1.5}$$

where $D_i = \dfrac{\partial}{\partial x_i}$.

Our interest is in skew-symmetric matrices $S = -S^T$, in which case (1.5) is a rotational term containing angular derivatives

$$\langle Sx, \nabla v(x)\rangle = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} [S_{ij}(x_j D_i - x_i D_j)v(x)] \tag{1.6}$$

Firstly, we would like to study the behavior of the steady state problem

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x)\rangle + f(v_*(x)) = 0, x \in \mathbb{R}^d. \tag{1.7}$$

In the infinite domain case, the problem is considered in [1, 7]. The main result of this paper has been stated as a conjecture in [1]. In paper [1], it was conjectured that one can show the solution of (1.7) decays exponentially as $|x| \to \infty$, but the details have never been carried out in [1]. However in [7], it has been shown that the solution v_* of (1.7) belongs to an exponentially weighted Sobolev space. This, for example is discussed in Section 1.2. We take this result and further show that the solution and its derivatives decay exponentially. However, comparing the solution of (1.7) posed on the infinite domain to the solution of (1.7) when posed on a finite disk with Dirichlet boundary conditions is still an open problem.

1.2 Previous Result

In [7], the behavior of the solution to (1.7) has been studied closely, under the following assumptions.

1. **Assumption 1** The matrix $A \in \mathbb{R}^{d\times d}$ is positive-definite, i.e. $A + A^T \geq 2C_A I$, for some positice constant C_A.

2. **Assumption 2** The matrix $S \in \mathbb{R}^{d\times d}$ is skew-symmetric, i.e. $S = -S^T$ (*rotational condition*).

3. **Assumption 3** The function $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ (*smoothness condition*).

4. **Assumption 4** $f(0) = 0$ (*constant asymptotic state*).

5. **Assumption 5** A and $Df(0) \in \mathbb{R}^{N\times N}$ are simultaneously diagonalizable.

6. **Assumption 6** $\sigma(Df(0)) \subset \{\lambda \in \mathbb{C} | Re\lambda < 0\}$.

Before stating the theorem; first let's introduce some necessary notations and definitions of [7]. In this paper, we let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 1. *For any given matrix $C \in \mathbb{K}^{N\times N}$, we define the following:*

1. *$\sigma(C)$ denotes the spectrum of C, i.e. the set of all eigenvalues of the matrix C.*
2. *$\rho(C) =: max_{\lambda\in\sigma(C)}|\lambda|$ denotes the spectral radius of C.*
3. *$s(C) =: max_{\lambda\in\sigma(C)} Re\lambda$ denotes the spectral abscissa of C.*

Using the above notation, we define the following constants for the matrix A in (1.7):

$$
\begin{aligned}
a_{min} &=: (\rho(A^{-1}))^{-1}, \\
a_{max} &=: \rho(A), \\
a_0 &=: -s(-A), \\
b_0 &=: -s(Df(0)).
\end{aligned}
$$

Definition 2. *[9] A function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a weight function of exponential growth rate $\eta \geq 0$ provided that*

$$\theta(x) > 0 \quad \forall x \in \mathbb{R}^d$$

$$\exists C_\theta > 0 : \theta(x+y) \leq C_\theta \theta(x) e^{\eta|y|} \quad \forall x, y \in \mathbb{R}^d.$$

Definition 3. *A weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geq 0$ is called radial provided that*

$$\exists \phi : [0, \infty) \to \mathbb{R} : \theta(x) = \phi(|x|) \quad \forall x \in \mathbb{R}^d.$$

Definition 4. *A radial weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geq 0$ is called non-decreasing (or monotonically increasing) provided that*

$$\theta(x) \leq \theta(y) \quad \forall x, y \in \mathbb{R}^d \quad \textit{with} \quad |x| \leq |y|.$$

Associated with the weight functions of exponential growth rate are exponentially weighted Lebesgue and Sobolev spaces

$$L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) =: \{u \in L^1_{loc}(\mathbb{R}^d, \mathbb{K}^N) \mid ||\theta u||_{L^p} < \infty\},$$
$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{K}^N) =: \{u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) \ \forall \ |\beta| \leq k\},$$

for every $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$. The main result of [7] is the following.

Theorem 1. *Under the above assumptions for some fixed p where $1 < p < \infty$, for every $0 < \vartheta < 1$ and for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geq 0$ with*

$$0 \leq \eta^2 \leq \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{max}^2 p^2},$$

there exists a constant $K_1 = K_1(A, f, d, p, \theta, \vartheta) > 0$ with the following property: Every classical solution v_* of

$$A\Delta v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0, x \in \mathbb{R}^d,$$

such that $v_* \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and

$$sup_{|x| \geq R_0} |v_*(x)| \leq K_1 \ for \ some \ R_0 > 0$$

satisfies

$$v_* \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$$

Theorem 1 states that, if one multiplies v_* by an exponentially growing function, then the product is still integrable. A similar result has been proven in the complex system case, when $f : \mathbb{C}^N \to \mathbb{C}^N$ in [7]. Using this result, we want to further prove that the solution v_* of (1.7) and its derivatives must decay exponentially. This is the main aim of this paper. And the motivation for this came from the following theorem in [7].

Theorem 2. *With all the assumptions as before, we consider some fixed* p*, where* $1 \leq p \leq \infty$*. Moreover, let* $0 < \vartheta < 1$ *and* $\lambda \in \mathbb{C}$ *with* $Re\lambda > 0$*. Then for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *of exponential growth rate* $\eta \geq 0$ *with* $0 \leq \eta^2 \leq \vartheta \frac{a_0 Re\lambda}{a_{max}^2 p^2}$ *and for every* $g \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ *we have* $v_* \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$ *with*

$$\begin{aligned} ||v_*||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} &\leq \frac{C_1}{Re\lambda}||g||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}, \\ ||D_i v_*||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} &\leq \frac{C_2}{(Re\lambda)^{1/2}}||g||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}, \quad i = 1, ..., d, \end{aligned}$$

where v_* *denotes the unique solution of* $(\lambda I - \mathcal{L}_0)v = g$ *in* $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ *with the* λ*-independent constants* C_1, C_2*. And* $[\mathcal{L}_0 v] =: A\Delta v(x) + \langle Sx, \nabla v(x)\rangle$*.*

Just as a note, we would like to mention here that all the constants that are introduced in the estimates in the following Sections, should be considered as some real constants which do not depend on other parameters; unless otherwise mentioned.

Chapter 2

Exponential Decay for the Infinite domain case

In order to investigate the exponential decay of (1.7) posed on the infinite domain first, we would like to break the problem down to a simpler case where we can handle the technicalities in detail. Once we understand these simple non-trivial cases; we would then like to understand the solution in a real, systems case. Now note that, in the one dimensional case we are already assuming that there is a solution to the problem which decays at ∞. But in the general scenario, we just assume that the solution is bounded. And the aim is to then show that the solution and its derivaties indeed decay exponentially.

So, the main focus of this chapter would be to discuss the exponential decay in the real case of the problem when posed on an infinite domain. In Section 2.1, we will discuss a half line 1-D problem. We show for the 1-D case, that if there is a solution that decays at ∞, then it has to decay exponentially. And to prove this, we use the result from Section 2.2. In Section 2.2, we discuss the exponential decay of (1.7) in the scalar case. Here, we use the Maximum principle approach to show that the solution decays exponentially. And then in Section 2.3 and 2.4, we discuss the all-space case. Here we first decompose the given system into an inhomogeneous system and apply a Theorem from [7] to prove that the solution and its derivatives decay exponentially using Sobolev inequalities.

2.1 A Simple 1-D Case

The problem (1.7) is quite complex. Hence, we will attempt to narrow down the problem and first study the behavior of the solution in a 1-D case.

Let's have a look at the BVP on unbounded domain.

$$\begin{cases} -\bar{u}'' + \bar{u} + \bar{u}^2 = 0 \\ \bar{u}(0) = 1 \\ \lim_{x\to\infty} \bar{u}(x) = 0 \end{cases} \tag{2.1}$$

where $\bar{u} : [0, \infty) \to \mathbb{R}$.

To begin, we know that there is a solution to (2.1) by looking at the phase plane diagram as shown in Figure 2.1. We used MATLAB to get this result.

From the phase plane portrait, we see that the solution $\bar{u}$ which satisfies the above boundary conditions should be positive, meaning

$$\bar{u}(x) > 0, \quad x \in [0, \infty)$$

Let's define the linear operator $\tilde{\mathcal{M}}$ by

$$[\tilde{\mathcal{M}}w](x) = -w'' + \frac{1}{2}w + (\frac{1}{2} + \bar{u})w.$$

Now, consider the following function h defined on $[0, \infty)$, for some positive real

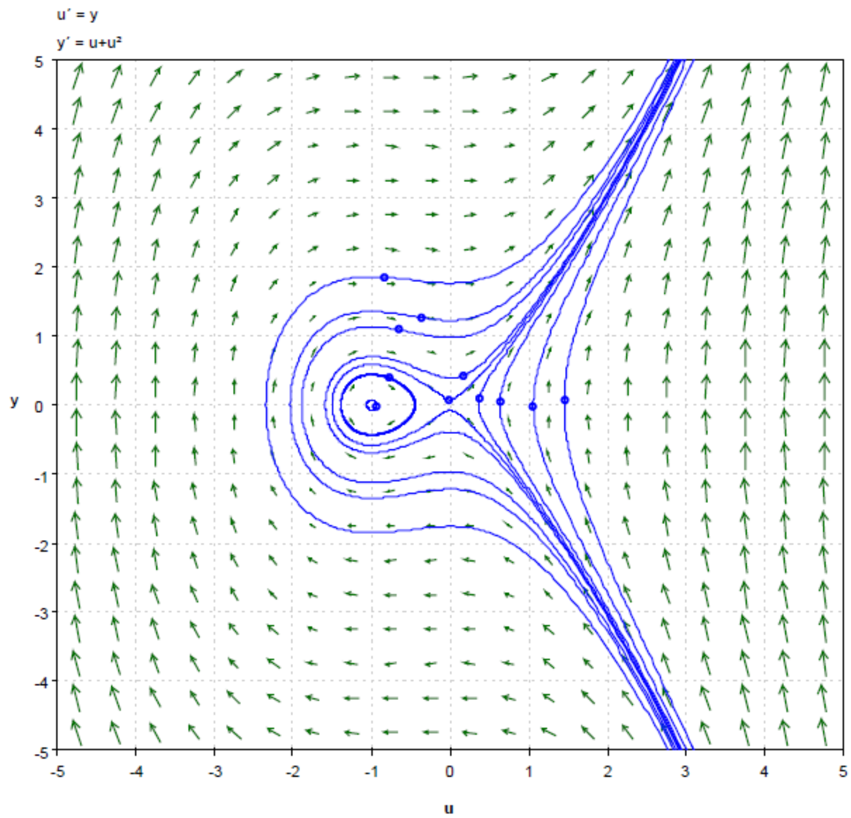

Figure 2.1: Phase plane portrait of the half-line BVP.

number C:

$$h(x) = C\, e^{\frac{-1}{2\sqrt{2}}x}$$

So, now to prove the exponential decay; we claim that $\delta = h - \bar{u} \geq 0$ for $x \geq R$, for some sufficiently large R.

Suppose not! Then $\delta(x) < 0$ for some $x > R$. This implies that the function δ has a negative minimum at some $x_0 > R$.

Now consider

$$\begin{aligned}\tilde{\mathcal{M}}\delta &= \tilde{\mathcal{M}}(h-\bar{u})\\ &= \tilde{\mathcal{M}}h - \tilde{\mathcal{M}}\bar{u}\\ &= -h'' + \frac{1}{2}h + (\frac{1}{2}+\bar{u})h, \quad \text{since} \quad \tilde{\mathcal{M}}\bar{u} = 0.\end{aligned}$$

Now, from the function h we have

$$\begin{aligned}h'' &= \frac{1}{8}C\, e^{-\frac{x}{2\sqrt{2}}}, \quad \text{which gives}\\ -h'' + \frac{1}{2}h &= \frac{3}{8}C\, e^{-\frac{x}{2\sqrt{2}}} > 0, \quad \text{for} \quad x \in [0,\infty).\end{aligned}$$

And since $\bar{u}(x) > 0$ for $x \in [0,\infty)$, we get

$$(\frac{1}{2}+\bar{u})h > 0$$

Therefore, we get

$$[\tilde{\mathcal{M}}\delta](x_0) > 0, \quad \text{for} \quad x_0 > R. \tag{2.2}$$

But,

$$[\tilde{\mathcal{M}}\delta](x_0) = -\delta''(x_0) + (1+\bar{u})\delta(x_0). \tag{2.3}$$

And, at x_0 where the function δ has a negative minimum we get

$$\begin{aligned}-\delta''(x_0) &\leq 0\\ (1+\bar{u})\delta(x_0) &< 0.\end{aligned}$$

Hence, from (2.3)

$$[\tilde{\mathcal{M}}\delta](x_0) < 0, \quad \text{for} \quad x_0 > R. \tag{2.4}$$

Therefore, from (2.2) and (2.4) we have a CONTRADICTION!

Hence, $\delta \geq 0$ which means $\bar{u} \leq h$.
Similarly, we can show that $-h \leq \bar{u}$ using a similar contradiction argument.

Hence, we get

$$-h \leq \bar{u} \leq h \implies |\bar{u}| \leq h$$

$$\text{And} \quad |\bar{u}| \leq C\, e^{\frac{-1}{2\sqrt{2}}x}.$$

Hence, this implies that $\bar{u}$ decays exponentially.

Therefore, from this we see that for (2.1) with solution $\bar{u}(x)$, if $\bar{u} \to 0$ as $x \to \infty$, then it goes to zero exponentially.

2.2 The Real, Scalar Case

Now that we understand the scenario in the case of a 1-D BVP, we want to proceed to the next case. We shall now look at the scalar case problem first, and try and understand the behavior of the solution just as in the 1-D example; instead of directly jumping to the more complicated systems case.

Consider the following steady state equation:

$$A\Delta v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0 \tag{2.5}$$

where $x \in \mathbb{R}^d$, $v : \mathbb{R}^d \to \mathbb{R}$, $A \in \mathbb{R}$ and positive.

Here, in this Section we aim to show that if (2.5) has a solution that decays at ∞ , then it decays exponentially under certain suitable assumptions which are listed below.

We begin, by assuming the following for (2.5):

1. $f(0) = 0$, $f'(0) = -k^2 < 0$; for some $k \in \mathbb{R}$.

   So with this assumption, we can write the function f as, $f(v) = -k^2 v + q(v)$, where $|q(v)| \leq C|v|^2$ for some positive constant C.

2. $q(v) = g(v)v$, with $g(0) = 0$ and $|g(v)| \leq C|v|$ for some positive constant C.

So, with the above two assumptions, we can transform (2.5) as follows:

$$A\Delta v(x) + \langle Sx, \nabla v(x)\rangle - \frac{k^2}{2}v(x) + (\frac{-k^2}{2} + g(v(x)))v(x) = 0. \tag{2.6}$$

Now, if there is a solution $v(x)$ to (2.5), which decays at ∞, then

$v(x) \to 0$ as $|x| \to \infty \implies g(v) \to 0$ as $|x| \to \infty$.

Let's define the linear operator $\tilde{\mathcal{L}}$ by

$$[\tilde{\mathcal{L}}w](x) = A\Delta w(x) + \langle Sx, \nabla w(x)\rangle - k^2 w(x) + g(v)w(x)$$

where we apply this operator to a particular function

$$h(x) = M\frac{\sqrt{R}}{\sqrt{|x|}}\ e^{-k(|x|-R)/\sqrt{8}}, \quad |x| \geq R$$

where M is the maximum norm of the function $h(x)$ and R is a sufficiently large number and A is a positive real number. For convenience, let's assume $|x|$ represents the maximum norm of x in $\mathbb{R}^d$.

So, now to prove the exponential decay; we claim that $\delta = h - v \geq 0$ for $|x| \geq R$, for some sufficiently large R.

Suppose not! Then $\delta(x) < 0$ for some x such that $|x| > R$. This implies that the function δ has a negative minimum at some x_0, where $|x_0| > R$.

Now consider

$$\begin{aligned}\tilde{\mathcal{L}}\delta &= \tilde{\mathcal{L}}(h-v)\\ &= \tilde{\mathcal{L}}h - \tilde{\mathcal{L}}v\\ &= A\Delta h + \langle Sx, \nabla h\rangle - \frac{k^2}{2}h + (\frac{-k^2}{2} + g(v))h, \quad \tilde{\mathcal{L}}v = 0 \quad \text{from} \quad (2.6)\end{aligned}$$

Now, when we evaluate $\tilde{L}\delta$ at x_0, we get

$$[\tilde{\mathcal{L}}\delta](x_0) < 0, \quad \text{for} \quad |x_0| > R. \tag{2.7}$$

We picked the function $h(x)$ such that the above inequality (2.7) holds true.

But,

$$[\tilde{\mathcal{L}}\delta](x_0) = A\Delta\delta(x_0) + \langle Sx, \nabla\delta(x_0)\rangle - (k^2 + g(v))\delta(x_0). \tag{2.8}$$

And, at x_0 where the function δ has a negative minimum we get

$$\begin{aligned}A\Delta\delta(x_0) &\geq 0\\ \langle Sx, \nabla\delta(x_0)\rangle &= 0\\ -(k^2 + g(v))\delta(x_0) &> 0.\end{aligned}$$

Hence, from (2.8)

$$[\tilde{\mathcal{L}}\delta](x_0) > 0, \quad \text{for} \quad |x_0| > R. \tag{2.9}$$

Therefore, from (2.7) and (2.9) we have a CONTRADICTION!

Hence, $\delta \geq 0$ which means $v \leq h$.

Similarly, we can show that $-h \leq v$ using a similar contradiction argument. Hence, we get $-h \leq v \leq h$ which implies $|v| \leq h$; which in turn implies v decays exponentially.

Therefore, from this we see that for (2.5) posed on infinite domain with solution $v(x)$, if $v \to 0$ as $|x| \to \infty$, then it goes to zero exponentially under suitable assumptions.

2.3 The Decomposed system

Let us revisit the original system of equations (1.7). We are just going to rewrite the PDE in a different fashion, for convenience.

$$-A\Delta v(x) - \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0, \qquad x \in \mathbb{R}^d. \tag{2.10}$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ denotes a C^∞ function with $f(0) = 0$ and set $B_0 = f'(0)$. Here we consider the following assumptions:

1. $A \in \mathbb{R}^{N\times N}$ is a positive definite matrix, i.e. there exists some $c_A > 0$ such that

   $$Re(w^* A w) \geq c_A |w|^2 \quad \text{for all} \quad w \in \mathbb{C}^N.$$

   In other notation, we can also say that

   $$A + A^T \geq 2C_A I \quad \text{where} \quad C_A > 0.$$

2. Assume that

   $$B_0 + B_0^T \geq 2\delta I \quad \text{where} \quad \delta > 0.$$

And all the other assumptions we had from Section 1.2 also hold.

Also, we assume that $v : \mathbb{R}^d \to \mathbb{R}^N$ solves (2.10).

For every, $v \in \mathbb{R}^N$, lets define a function $\varphi(s)$ as follows:

$$\varphi(s) = sv, \quad 0 \leq s \leq 1.$$

Now, for every $v \in \mathbb{R}^N$ we have

$$\begin{aligned} f(v) &= f(v) - f(0) \\ &= f(\varphi(1)) - f(\varphi(0)) \\ &= \int_0^1 \frac{d}{ds} f(\varphi(s))\, ds \\ &= \Big(\int_0^1 f'(sv)\, ds \Big) v \end{aligned}$$

We define the following function $B(x)$ using the solution $v(x)$ of the PDE (2.10) which satisfies

$$B(x) = \int_0^1 f'(sv(x))\, ds \quad \text{for} \quad x \in R^d$$

$$f(v(x)) = B(x)v(x) \quad \text{and} \quad B(x) \to B_0 = f'(0) \quad \text{as} \quad |x| \to \infty.$$

Now, set $Q(x) = B(x) - B_0$. So this gives,

$$B(x) = B_0 + Q(x) \quad \text{and} \quad Q(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

The PDE (2.10) becomes

$$-A\Delta v(x) - \langle Sx, \nabla v(x) \rangle + (B_0 + Q(x))v = 0, \quad x \in \mathbb{R}^d. \tag{2.11}$$

To decompose $Q(x)$, we use a cutoff function $\chi_1 \in C^\infty(\mathbb{R}^d, \mathbb{R})$ with

$$\chi_1(x) \begin{cases} = 1, & |x| \le 1, \\ \in [0,1], & 1 \le |x| \le 2, \\ 0, & |x| \ge 2, \end{cases}$$

and its scaled version

$$\chi_n(x) = \chi_1(\frac{x}{n}), \quad x \in \mathbb{R}^d.$$

Now, we rewrite $Q(x)$ using the cutoff function as follows:

$$Q(x) = Q_s(x) + Q_c(x)$$

where $Q_s(x) = (1 - \chi_n(x))Q(x)$ and $Q_c(x) = \chi_n(x)Q(x)$. So $Q_s(x)$ is small for large n since $Q(x) \to 0$ as $|x| \to \infty$ and $Q_c(x)$ has compact support. And we choose n in the definition of the function $\chi_n(x)$ so large that $Q_s(x)$ is very small.

Now, we rewrite the PDE (2.11) as

$$-A\Delta v(x) - \langle Sx, \nabla v(x)\rangle + (B_0 + Q_s(x))v = -Q_c(x)v(x), \quad x \in \mathbb{R}^d. \tag{2.12}$$

In Section 2.4, we use this equation to prove the main result of this thesis.

2.4 The Real, Systems Case

Consider the following system

$$-A\Delta v - \langle Sx, \nabla v\rangle + B(x)v = g, \quad x \in \mathbb{R}^d, \tag{2.13}$$

with the following assumptions:

(A1) $A \in \mathbb{R}^{N\times N}$ is positive, i.e. there exists some $c_A > 0$ such that

$$\mathrm{Re}(w^* A w) \geq c_A |w|^2 \quad \text{for all} \quad w \in \mathbb{C}^N.$$

(A2) $S \in \mathbb{R}^{N\times N}$ is skew symmetric, i.e. $S^T = -S$.

(A3) $B \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N\times N})$ satisfies for some $c_B > 0$

$$\mathrm{Re}(w^* B(x) w) \geq c_B |w|^2 \quad \text{for all} \quad w \in \mathbb{C}^N.$$

For a given weight function $\theta \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$ satisfying $\theta(x) > 0$ for all $x \in \mathbb{R}^d$, we define the weighted spaces

$$
\begin{aligned}
L^2_\theta = L^2_\theta(\mathbb{R}^d, \mathbb{C}^N) &= \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) : \|u\|_\theta < \infty\}, \\
\|u\|^2_\theta &= \int_{\mathbb{R}^d} \theta(x)|u(x)|^2 dx, \\
H^k_\theta = H^k_\theta(\mathbb{R}^d, \mathbb{C}^N) &= \{u \in H^k_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) : \|u\|_{\theta,k} < \infty\}, \\
\|u\|^2_{\theta,k} &= \sum_{|\alpha| \le k} \|D^\alpha u\|^2_\theta.
\end{aligned}
$$

Further, we introduce an exponential weight function of rate ε,

$$\theta_\varepsilon(x) = \exp(-\varepsilon(1 + |x|^2)^{1/2}), \quad x \in \mathbb{R}^d.$$

Then, we have the following theorem from [2].

Theorem 3. *Let the assumptions (A1)-(A3) be satisfied. Then the following assertions hold for every*

$$|\varepsilon| \le \varepsilon_0 = \frac{1}{|A|}\left(\frac{c_A c_B}{d}\right)^{1/2}. \tag{2.14}$$

If $v \in H^2_{\mathrm{loc}} \cap L^2_{\theta_\varepsilon}$ *solves* (2.13) *for some* $g \in L^2_{\theta_\varepsilon}$, *then* $v \in H^1_{\theta_\varepsilon}$ *and the following estimate holds*

$$\|v\|_{\theta_\varepsilon,1} \le \frac{2}{c_B}(1 + \frac{c_B}{c_A})^{1/2}\|g\|_{\theta_\varepsilon}. \tag{2.15}$$

In Section 2.3, we have decomposed the system (1.7) into:

$$-A\Delta v(x) - \langle Sx, \nabla v(x)\rangle + (B_0 + Q_s(x))v = -Q_c(x)v(x), \quad x \in \mathbb{R}^d \tag{2.16}$$

where $v : \mathbb{R}^d \to \mathbb{R}^N$, $Q_s(x)$ is very small and $Q_c(x)$ has compact support.

The main goal in this Section, is to show that $v(x)$ and all order derivatives of $v(x)$ are in the following space $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$, for a fixed p. In other words we want to show that they are bounded by a constant in the $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ norm, i.e.

$$||D^\alpha v(x)||_{L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)} \le C_{\tilde{k}} \tag{2.17}$$

for some real constant $C_{\tilde{k}}$, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$ denotes the order of the derivative, with d being the dimensions of the vector x. And D^α is denoted by the multi-index notation:

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} ..... \partial {x_d}^{\alpha_d}}.$$

From here on, for the sake of convenience we write any D^α with $|\alpha| = j$, as D^j.

So, to prove (2.17) we use the method of mathematical induction.

First, we show that the result holds for $j = 0$.

Now, when we take a closer look at the equation (2.16), we see that the right hand side of the equation $-Q_c(x)v(x) \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. And since $Q_c(x)v(x)$ has compact support, let's assume $||Q_c(x)v(x)||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leq C_q$ for some real constant C_q. So, the equation (2.16) satisfies the requirements for Theorem 3. Hence, we get

$$\begin{aligned}
||v||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} &\leq \frac{2}{c_B}(1 + \frac{c_B}{c_A})^{1/2} ||Q_c(x)v(x)||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \\
&\leq \frac{2}{c_B}(1 + \frac{c_B}{c_A})^{1/2} C_q.
\end{aligned}$$

Assume the result is true for $j = m - 1$, where m is a natural number. That is $||D^{m-1}v||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leq C_{m-1}$ for some real constant C_{m-1}. Then, we want to prove it holds for $j = m$.

Let's take the m^{th} order derivative of the equation (2.16). This yields the following equation:

$$\begin{aligned}
&- A\Delta D^m v(x) - \langle Sx, \nabla D^m v(x)\rangle + (B_0 + Q_s(x) + Q_c(x))v = \\
&- D^m Q_c(x)v(x) + \langle D^m Sx, \nabla v(x)\rangle - D^m Q_s(x)v(x) =: \tilde{G}(x), \quad x \in \mathbb{R}^d.
\end{aligned}$$

Now, we see that the right hand side in the above equation $\tilde{G}(x)$ has compact support, so $\tilde{G}(x) \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. Hence, Theorem 3 holds. So for some constant C', we get

$$||D^m v||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leq C'.$$

Hence, we proved that $||D^j v||_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leq C_j$ by induction. In other words, we have shown that $D^\alpha v(x)\theta(x) \in L_p$, i.e. for some radially non-decreasing weight function θ we have:

$$||D^j v(x)\theta(x)||_{L^p} \leq C_{\tilde{k}} \tag{2.18}$$

Moving ahead, for convenience, we consider the following:

1. Let's take the exponentially weighted function to be $\theta(x) = e^{\tilde{C}(1+x^2)^{1/2}}$, for some positive constant $\tilde{C}$, for which (2.18) holds.
2. Let's consider $p = 2$, meaning let's just look at the L^2 space.

Now, let's define $v(x)e^{\epsilon(1+x^2)^{1/2}} =: h(x)$, for some real $\epsilon > 0$. So, from (2.18) for some positive constant C' we have:

$$\begin{aligned} ||h(x)||_{L^2} &= ||v(x)e^{\epsilon(1+x^2)^{1/2}}||_{L^2} \\ &\leq C'. \end{aligned}$$

And by taking the j-th order derivative of h, we get

$$D^j h(x) = \sum_{l=0}^{j} C_{lj}(D^l v)D^{j-l}e^{\epsilon(1+x^2)^{1/2}}$$

Now, by taking $\epsilon < \tilde{C}$, we can make sure the norm of any order derivative of $e^{\epsilon(1+x^2)^{1/2}}$ is very small. Because of this and from (2.18), we get

$$||D^j h(x)||_{L^2} \leq C''$$

for some constant C''.

Since, from above we see that $h(x)$ and all its derivatives are in L_2 space

$$\begin{aligned} ||h(x)||_{L^2} &\leq C', \\ ||D^j h(x)||_{L^2} &\leq C'' \end{aligned}$$

this implies that $h(x)$ belongs to the Sobolev space $W^{k,2}(\mathbb{R}^d, \mathbb{C}^N)$ by definition. Therefore, for some constant $\tilde{A}$, we get

$$||h||_{W^{k,2}} < \tilde{A}, \quad \text{for} \quad k > \frac{d}{2}. \tag{2.19}$$

So now from the Sobolev embedding theorem, we have

$$|h|_\infty \leq C||h||_{W^{k,2}}, \quad \text{for} \quad k > \frac{d}{2}$$

where C is a constant.

But, since $||h||_{W^{k,2}} < \tilde{A}$ from (2.19), we get the following from the Sobolev embedding theorem

$$||h||_\infty \leq C\tilde{A}, \quad \text{for} \quad k > \frac{d}{2}.$$

And this indeed implies that $v(x)$ decays exponentially pointwise. Its clear that the argument can be extended to all derivatives of v as well.

Hence, in conclusion we have shown that under certain assumptions the stationary solution $v(x)$ in all-space decays exponentially.

Chapter 3

Exponential Closeness to the solution of a Finite Domain problem

In this Chapter, we would like to discuss the closeness between the all-space solution and the solution of a finite domain problem. The first step in this process, is to show that there is a solution to the finite domain problem. And then, we show that the solution is exponentially close to the solution of the all-space problem.

In Section 3.1, we give the complete proof for a model problem in one-space dimension. In Section 3.2, we look at the finite dimensional problem in n-dimensions. And we also discuss the numerous difficulties that we face in generalizing the result.

3.1 The model problem

Now, let's take a close look at the model for the PDE on the bounded domain.

$$\begin{cases} -u'' + u + u^2 = 0 \\ u(0) = 1 \\ u(R) = 0 \end{cases} \tag{3.1}$$

where $x \in [0, R]$, for some large real number R.

Here, we aim to show that the solution u of the finite domain problem (3.1) is close to the solution $\bar{u}|_R$; where $\bar{u}$ denotes the exponentially decaying solution on half-line, as discussed in Section 2.1. And $\bar{u}|_R$ denotes the restriction of the decaying solution $\bar{u}$ to the bounded domain $[0, R]$.

Firstly, we want to prove that there is a locally unique solution to (3.1). Consider $u = \bar{u}|_R + h$, then the BVP is transformed to

$$-h'' + (1 + 2\bar{u}|_R)h + h^2 = 0, \quad h(0) = 0, \quad h(R) = -\bar{u}(R)$$

Using the simplified Newton's method, we get

$$-h''_{n+1} + (1 + 2\bar{u}|_R)h_{n+1} = -h_n^2$$

Rewriting the BVP from above, we arrive at the following linear inhomogeneous problem:

$$\begin{cases} -h''(x) + (1 + 2\bar{u}|_R)h(x) = F(x) \\ h(0) = 0 \\ h(R) = -\bar{u}(R) =: \beta \end{cases} \tag{3.2}$$

where $F(x) = -h_n^2$. We further define $1 + 2\bar{u}|_R(x) =: g(x) \geq 1$, since $\bar{u} > 0$ as seen in Section 2.1.

Now, we want to show that the inhomogeneous problem (3.2) has a unique solution $h(x)$ by first proving that the above problem with $F = 0$ and homogeneous boundary conditions has only a trivial solution. So we consider the following homogeneous problem corresponding to (3.2):

$$\begin{cases} -h''(x) + g(x)h(x) = 0 \\ h(0) = h(R) = 0 \end{cases} \tag{3.3}$$

We claim here that the above homogeneous problem (3.3) has only a trivial solution.

We prove this claim, by the method of contradiction. Suppose that there is a non-trivial solution, then we either have a positive maximum or a negative minimum. So, for some $x_0 \in (0, R)$ let $h(x_0) = |h|_\infty > 0$ be the positive maximum. Then

$$h''(x_0) \leq 0 \quad \text{and}$$
$$g(x_0)h(x_0) > 0.$$

Hence, this gives $-h''(x_0) + g(x_0)h(x_0) > 0$ which is a contradiction!

Therefore, the homogeneous problem (3.3) has only a trivial solution by the Maximum principle. Hence, the inhomogeneous problem (3.2) has a unique solution by Fredholm's alternative.

Now let $h(x) = l(x) + \eta(x)$, where $h(x)$ is the unique solution of the inhomogeneous problem (3.2) and $l(x) = -\bar{u}(R)\dfrac{x}{R}$. We have formulated the function $l(x)$, such that the problem has homogeneous boundary conditions. So, using $h(x) = l(x)+\eta(x)$ in (3.2) we get the following BVP in $\eta(x)$ is:

$$\begin{cases} [\tilde{\mathcal{L}}\eta] = -\eta''(x) + q(x)\eta(x) = -\eta^2(x) + \tilde{f}(x) \\ \eta(0) = \eta(R) = 0 \end{cases} \tag{3.4}$$

where $\quad q(x) = 1 + 2\bar{u}|_R(x) + 2l(x)$

$$\tilde{f}(x) = -l^2(x) - ((1 + 2\bar{u}|_R(x))l(x).$$

And define $\tilde{F}(x) =: -\eta^2(x) + \tilde{f}(x)$, for convenience.

Lemma 1. *Let q_0 be a real number, where $q(x) \geq q_0 > 0$. Let $-M \leq \tilde{F}(x) \leq M$, with M representing the maximum norm of $\tilde{F}(x)$. Then the solution η of (3.4) is bounded by $|\eta|_\infty \leq \dfrac{1}{q_0}|\tilde{F}|_\infty$.*

Proof. Let $\eta_0 = \dfrac{M}{q_0}$; then

$$[\tilde{\mathcal{L}}\eta_0](x) = q(x)\eta_0(x) = q(x)\frac{M}{q_0} \geq M$$

$$[\tilde{\mathcal{L}}(\eta_0 - \eta)](x) \geq M - \tilde{F}(x) \geq 0$$

Now let $\tilde{\eta} = \eta_0 - \eta$,then :

$$\begin{cases} [\tilde{\mathcal{L}}\tilde{\eta}](x) \geq M - \tilde{F}(x) \geq 0 \\ \tilde{\eta}(0) \geq 0 \\ \tilde{\eta}(R) \geq 0 \end{cases}$$

The claim here is that $\tilde{\eta} \geq 0$. Suppose not!

Then $\exists x_0 \in (0, R)$ such that $\tilde{\eta}(x_0) < 0$ (minimum). So, $\tilde{\eta}''(x_0) \geq 0$ which gives :

$$\begin{aligned}[\tilde{\mathcal{L}}\tilde{\eta}](x_0) &= -\tilde{\eta}''(x_0) + q(x_0)\tilde{\eta}(x_0) \\ &< 0\end{aligned}$$

CONTRADICTION!

Therefore $\tilde{\eta} \geq 0 \implies \eta \leq \dfrac{M}{q_0}$.

Similarly, if $\eta_1 = -\dfrac{M}{q_0}$; then

$$[\tilde{L}\eta_1](x) = q(x)\eta_1(x) = -q(x)\frac{M}{q_0} \leq -M$$

For $\bar{\eta} = \eta_1 - \eta$, we get:

$$\begin{cases} [\tilde{L}\bar{\eta}](x) \leq -M - \tilde{F}(x) \leq 0 \\ \bar{\eta}(0) \leq 0 \\ \bar{\eta}(R) \leq 0 \end{cases}$$

Again, the claim here is that $\bar{\eta} \leq 0$. Suppose not!

Then $\exists x_1 \in (0, R)$ such that $\bar{\eta}(x_1) > 0$ (maximum). So, $\bar{\eta}''(x_0) \leq 0$ which gives :

$$\begin{aligned} [\tilde{\mathcal{L}}\bar{\eta}](x_1) &= -\bar{\eta}''(x_1) + q(x_1)\bar{\eta}(x_1) \\ &> 0 \end{aligned}$$

CONTRADICTION!

Hence we get, $\bar{\eta} \leq 0 \implies \eta \geq \dfrac{-M}{q_0}$.

Therefore, if $-M \leq \tilde{F}(x) \leq M$ then $\dfrac{-M}{q_0} \leq \eta \leq \dfrac{M}{q_0}$. In other words :

$$|\eta|_\infty \leq \frac{1}{q_0}|\tilde{F}|_\infty \tag{3.5}$$

□

Now, we can write the BVP (3.4) as the fixed point equation

$$\eta = G[-\eta^2 + \tilde{f}]$$

where G is an integral operator corresponding to the Green's function of the linear BVP:

$$\begin{cases} -\eta''(x) + q(x)\eta(x) = F(x) \\ \eta(0) = \eta(R) = 0 \end{cases}$$

Consider the Banach space $(C[0,R],|.|_\infty)$ and a ball of radius r centered at '0' given by $B_r(0)=\{f(x)\in C[0,R]:|f|_\infty\le r\}$.

We have the following fixed point equation:

$$\Phi(\eta)=\eta$$

where $\Phi(\eta)=G[-\eta^2+\tilde{f}]$.

To ensure that the Contraction Mapping Theorem applies, we will require

1. $\Phi:B_r(0)\to B_r(0)$

   Let $\eta\in B_r(0)$, and we have $|G|_\infty\le 2$ from Lemma 1 with $q_0=\frac{1}{2}$ since $|\bar{u}(R)|\le\frac{1}{4}$ holds for R sufficiently large.. Then

   $$\begin{aligned}|\Phi(\eta)|_\infty &\le 2|-\eta^2+\tilde{f}|_\infty\\ &\le 2r^2+2|\tilde{f}|_\infty, \quad |\tilde{f}|_\infty \quad \text{is small for R large}\\ &\le r.\end{aligned}$$

2. $\Phi:B_r(0)\to B_r(0)$ is a contraction mapping.

   Let $\eta_1,\eta_2\in B_r(0)$.Then

   $$\begin{aligned}\Phi(\eta_1)-\Phi(\eta_2)&=-G(\eta_1^2-\eta_2^2)\\ &=-G(\eta_1+\eta_2)(\eta_1-\eta_2)\end{aligned}$$

   which gives

   $$\begin{aligned}|\Phi(\eta_1)-\Phi(\eta_2)|_\infty&\le|G|_\infty 2r|\eta_1-\eta_2|_\infty\\ &\le k|\eta_1-\eta_2|_\infty.\end{aligned}$$

The above conditions are met subsequently by the choice of the radius 'r' of the ball $B_r(0)$. If we pick $r\le\frac{1}{4}$, we can ensure we have a contraction mapping. And for the sake of convenience, we pick $k = \frac{1}{2}$. That means, for any $\eta_1, \eta_2 \in B_r(0)$:

$$|\Phi(\eta_1) - \Phi(\eta_2)|_\infty \leq \frac{1}{2}|\eta_1 - \eta_2|_\infty$$

Therefore by the Contraction Mapping Theorem, η exists and is unique on $B_r(0)$.

Lastly, we want to show that the constructed solution u of the finite domain problem (3.1) is close to the solution $\bar{u}|_R$ where $\bar{u}$ is the solution of (2.1).

Let's take a look at the Fixed point equation : $\eta = \Phi(\eta) = G[-\eta^2 + \tilde{f}]$.

From here, we have $\Phi(0) = G\tilde{f}$. And since we assume $|G|_\infty \leq 2$, and also we can safely take $|\tilde{f}|_\infty \leq C_2 e^{-c_1 R}$, for some $c_1, C_2 \in \mathbb{R}$, because of the way we constructed $\tilde{f}$. Hence

$$|\Phi(0)|_\infty \leq 2C_2 e^{-c_1 R}$$

Since Φ is a contraction mapping with $k = \frac{1}{2}$, we have

$$\begin{aligned} |\eta - \Phi(0)|_\infty &= |\Phi(\eta) - \Phi(0)|_\infty \\ &\leq \frac{1}{2}|\eta - 0|_\infty \\ &= \frac{1}{2}|\eta|_\infty \end{aligned}$$

Now, we can write η as

$$\eta = \eta - \Phi(0) + \Phi(0)$$

By triangle inequality, we get:

$$\begin{aligned} |\eta|_\infty &\leq |\eta - \Phi(0)|_\infty + |\Phi(0)|_\infty \\ &\leq \frac{1}{2}|\eta|_\infty + |\Phi(0)|_\infty. \end{aligned}$$

Therefore $|\eta|_\infty \leq 2|\Phi(0)|_\infty$.

But since $u = \bar{u}|_R + \eta - \bar{u}(R)\dfrac{x}{R}$, we get:

$$|u - \bar{u}|_R|_\infty \leq \tilde{C}e^{-\tilde{c}R}.$$

for some positive real constants $\tilde{C}, \tilde{c}$.

Hence, this proves in the case of the 1-D BVP, that for large R the bounded interval problem (3.1) has a solution $u(x)$, which is locally unique and is exponentially close to the restriction $\bar{u}|_R$, where $\bar{u}(x)$ is the solution of the half line problem (2.1).

3.2 The Finite dimensional problem and its limitations

In this Section, we want to look at the solution of a finite dimensional problem. The first question that arises is, if there is a solution to the problem. The result is discussed in the following Proposition.

Proposition 1. *Consider a finite dimensional map $F : \mathbb{R}^N \to \mathbb{R}^N$ and consider the equation $F(u) = 0$. For this map, assume the following*

1. *There is an almost solution v_0 to the problem which satisfies $||F(v_0)||_\infty = \epsilon$, for a small ϵ.*
2. *The inverse $(F'(v_0))^{-1}$ exists, with F' representing the Jacobian of F. Also $||(F'(v_0))^{-1}|| \leq C$, where C is some real number.*
3. *F' is globally Lipschitz, with $L \geq 0$ as the Lipschitz constant.*

Then the finite dimensional problem $F(u) = 0$ has a solution if $\epsilon > 0$ is sufficiently small.

Proof. Consider $||.||$ to be the maximum norm.

Let v_0 be an almost solution. Then for some real $\epsilon > 0$

$$||F(v_0)|| = \epsilon$$

We want to solve $F(u) = 0$, where $u \in \mathbb{R}^N$. So, if u is the nearby solution and v_0 is an almost solution, then we would have $u = v_0 + \delta$, for some $\delta \in \mathbb{R}^N$. Hence, we get

$$0 = F(v_0 + \delta) = F(v_0) + F'(v_0)\delta + ...$$

Then by neglecting the higher order terms, we get $\delta = -(F'(v_0))^{-1}F(v_0)$, since $(F'(v_0))^{-1}$ exists.

Using simplified Newton's method, we get

$$v_1 = v_0 - (F'(v_0))^{-1}F(v_0),$$
$$v_2 = v_1 - (F'(v_0))^{-1}F(v_1), ...$$

This gives us the Fixed point iteration $v_{n+1} = \Phi(v_n)$ with

$$\Phi(v) = v - (F'(v_0))^{-1}F(v) \tag{3.6}$$

Let $w, v \in B_r(v_0)$ for some very small $r \in \mathbb{R}$ and $r > 0$. And let

$\theta(s) = v + s(w - v), \quad 0 \leq s \leq 1.$

Then by the Mean value theorem, for every $w, v \in B_r(v_0)$ we get

$$\begin{aligned} F(w) - F(v) &= F(\theta(1)) - F(\theta(0)) \\ &= \int_0^1 \frac{d}{ds} F(\theta(s))\, ds \\ &= \Big(\int_0^1 F'(\theta(s))\, ds\Big)(w - v) \end{aligned}$$

We define the following for convenience, $\Delta F =: \int_0^1 F'(\theta(s))\,ds$. So

$$F(w) - F(v) = \Delta F(w - v) \tag{3.7}$$

Since F' is globally Lipschitz, there exists a real constant $L \geq 0$ such that, for all $\tilde{w}, \tilde{v} \in \mathbb{R}^N$, we have

$$||F'(\tilde{w}) - F'(\tilde{v})|| \leq L||\tilde{w} - \tilde{v}||$$

Now

$$\begin{aligned}\Delta F &= \int_0^1 F'(v_0)\,ds + M \\ &= F'(v_0) + M\end{aligned}$$

where M is such that $||M|| \leq Lr$ since F' is globally Lipschitz and $v, w \in B_r(v_0)$.

Now from the fixed-point iteration (3.6) we have

$$\begin{aligned}\Phi(w) - \Phi(v) &= (w - v) - (F'(v_0))^{-1}(F(w) - F(v) \\ &= (w - v) - (F'(v_0))^{-1}(\Delta F)(w - v) \\ &= [I - (F'(v_0))^{-1}(\Delta F)](w - v).\end{aligned}$$

And since $\Delta F = F'(v_0) + M$, we get

$$(F'(v_0))^{-1}\Delta F = I + (F'(v_0))^{-1}M$$

This gives

$$\begin{aligned}\Phi(w) - \Phi(v) &= -(F'(v_0))^{-1}M(w - v) \quad \text{and} \\ ||\Phi(w) - \Phi(v)|| &\leq Lr||(F'(v_0))^{-1}||\ \ ||w - v|| \\ &\leq CLr||w - v||.\end{aligned}$$

Now, Φ is a contraction mapping if

$$CLr < 1 \tag{3.8}$$

Also, we need to show that $\Phi : B_r(v_0) \to B_r(v_0)$. Consider $v \in B_r(v_0)$. Then $||v - v_0|| \leq r$.

And from (3.6), we get $\Phi(v_0) = v_0 - (F'(v_0))^{-1}F(v_0)$. Now since v_0 is an almost solution, we have that $||(F'(v_0))^{-1}F(v_0)|| \leq C\epsilon$ from the assumptions.

Now to show that Φ maps $B_r(v_0)$ into itself, consider

$$\begin{aligned}
||\Phi(v) - v_0|| &= ||\Phi(v) - \Phi(v_0) + (F'(v_0))^{-1}F(v_0)|| \\
&\leq Lr||v - v_0|| + C\epsilon \\
&\leq Lr^2 + C\epsilon
\end{aligned}$$

Note that Φ maps $B_r(v_0)$ into itself if and only if for every $v \in B_r(v_0)$, $||\Phi(v) - v_0|| \leq r$, meaning we need to have

$$Lr^2 + C\epsilon \leq r \tag{3.9}$$

So, Φ is a contraction mapping on $B_r(v_0)$, when both the inequalities (3.8) and (3.9) are satisfied. And this can be achieved if $\epsilon > 0$ and $r > 0$ are sufficiently small. This shows that Φ is a contraction mapping on $B_r(v_0)$ and hence the contraction mapping theorem applies.

Therefore, by the contraction mapping theorem, we know that the fixed point equation $\Phi(v) = v$ has a unique solution in $B_r(v_0)$. And this implies that the finite dimensional problem $F(u) = 0$ has a locally unique solution u which is close to the almost solution v_0.

Hence, this proves that under certain assumptions; if a Finite dimensional problem has an almost solution, then it has a nearby solution.

□

Now, we have to show that this solution is close to the all-space solution. This task turned out to be a lot more complicated than we anticipated. There are many issues we encountered in the process, and we would like to discuss a few of these limitations that we faced.

Let us explain some of the issues involved:

1. Recall the stationary equation

$$A\Delta v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0, x \in \mathbb{R}^d$$

   with solution $v_*(x)$. We have shown in Section 2.4 that the solution $v_*(x)$ and the corresponding derivatives of $v_*(x)$ all decay exponentially. Now, since the above stationary equation has a solution in all-space, then there are numerous more solutions to the equation because of rotations.

   Now consider the non-linear stationary equation on a finite disk

$$\begin{aligned} A\Delta v + \langle Sx, \nabla v\rangle + f(v(x)) = 0, &\quad \text{in} \quad B_R \\ v(x) = 0, &\quad \text{for} \quad |x| = R \end{aligned}$$

   for large R. We expect that the function $v_*(x)$ satisfies the Dirichlet boundary conditions upto an exponentially small term and that there is a solution near $v_*(x)$ which is locally unique except for rotations. And because of these rotations, when we linearize the equation, we get zero eigenvalue.

   Consider the linear operator

$$\begin{aligned} \mathcal{L}w = A\Delta w + \langle Sx, \nabla w\rangle + Df(v(x))w, &\quad \text{in} \quad B_R \\ w(x) = 0, \quad \text{for} \quad |x| = R & \end{aligned}$$

The operator $\mathcal{L}$ has the eigenvalue zero. Not just that, the operator also has exponentially small eigenvalues (close to 0) . This rises the question, if the inverse of the operator exists. And this could make the solution of the stationary equation on a finite disk unstable.

2. The next question that we need to ask is, if there really is a solution for the stationary equation on a finite disk that is exponentially close to the stationary solution in all space. And, if there is a solution, it is not unique because of the rotations.

3. And to deal with the issue of the zero eigenvalue, we contemplate that we may need to introduce one more variable of freedom into the existing problem. And that calls for a whole different formulation of the problem.

Chapter 4

Conclusions and Future Work

The main motivation for this thesis comes from the conjecture stated in [1]. The conjecture states that the solution to the stationary problem of the reaction-diffusion system decays exponentially as $|x| \to \infty$. In this thesis, we have mainly looked at the stationary solution of the reaction-diffusion system with two goals. The main one being to prove that the solution of the all space problem decays exponentially. And we have proved this result, starting from a model problem on the half-line and we were able to then generalize it to all-space. And for this, we have used the result from [7] and Sobolev theory extensively.

Having achieved the first goal, we wanted to extend this result to the problem posed on a finite disk. And the motivation for this came from the numerical computations and the fact that formulations on finite region are more realistic. And also since, whenever we do numerical computations we reduce the given differential equation to a bounded domain. For the problem posed on a finite disk, we have looked at the model problem in one-space dimension and carried out all the details. The solution in this case is close to the solution of the all space problem. The exponential decay of the all space solution suggests that one can generalize the result to the finite domain problem. However, we have not carried out the detailed arguments. Our part of future work is to show the closeness between the all-space solution and the solution of a finite domain problem. Also, to do that we need to address the issue of the zero eigenvalue. And hence, prove that the solution of the finite domain problem decays exponentially.

Bibliography

[1] W.-J. Beyn and J. Lorenz, *Nonlinear stability of rotating patterns*, Dynamics of Partial Differential Equations, **5**, No. 4, (2008), pp. 349–400.

[2] W.-J. Beyn and J. Lorenz, *Rotating Patterns on Finite Disks*, Unpublished.

[3] Comsol, *Comsol Multiphysics 3.3*, Comsol Inc., www.comsol.com, 2007.

[4] L.-C. Crasovan, B. A. Malomed, and D. Mihalache, *Stable vortex solitons in the two dimensional Ginzburg–Landau equation*, Phys. Rev. E, 63 (2001), 016605.

[5] L.-C. Crasovan, B. A. Malomed, and D. Mihalache, *Spinning solitons in cubic-quintic nonlinear media*, Pramana J. Phys., 57 (2001), pp. 1041–1059.

[6] L. D. Landau and V. L. Ginzburg, *On the theory of superconductivity*, Journal of Experimental and Theoretical Physics (USSR), 20:1064, 1950.

[7] Denny Otten, *Spatial decay and spectral properties of rotating waves in parabolic systems*, Dissertation,Universität Bielefeld, 2013.

[8] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer, 1994.

[9] S. Zelik and A. Mielke, *Multi-pulse evolution and space-time chaos in dissipative systems*, Mem. Amer. Math. Soc., 198(925):vi+97, 2009.