2-14-2014

Using Bayesian Statistics to Model the Reliability of Complex Weapon Systems

Rebecca Lilley

Rebecca L. Lilley

Candidate

Statistics

Department

This thesis is approved, and it is acceptable in quality and form for publication:

Approved by the Thesis Committee:

Dr. Ronald Christensen, Chairperson

Dr. Shane Reese

Dr. Michael Sonksen

Using Bayesian Statistics to Model the Reliability of Complex Weapon Systems

by

Rebecca L. Lilley

B.S., Statistics, Utah State University, 2009

THESIS

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science
Statistics

The University of New Mexico

Albuquerque, New Mexico

December 2013

Dedication

This thesis work is first and foremost dedicated to my father, Dr. Ian Anderson. Without his encouragement from the beginning of my college career I may have never pursued an education in Statistics. I would also like to dedicate this work to my husband, Rob, who has supported and encouraged me throughout this process, especially towards the end when our time together had to be sacrificed to the completion of this manuscript.

Acknowledgments

I consider it an honor to have had the privilege of doing this research with Dr. Shane Reese, my advisor for this thesis. He has patiently guided me through this whole process and under his guidance I feel that I have developed into a statistician with a skill set that can contribute to future studies in my field.

I also wish to thank Dr. Michael Sonksen who has provided constant help and feedback through both the research and writing process of this thesis.

Finally, I owe an immense amount of gratitude to Mr. Richard Paulsen and Dr. Danny Huval, as well as others in the Air Force Nuclear Weapons Center. Without their belief in me as an analyst I would have never been given the opportunity to pursue a Master's degree; and therefore never been given the opportunity to complete this thesis.

Using Bayesian Statistics to Model the Reliability of Complex Weapon Systems

by

Rebecca L. Lilley

B.S., Statistics, Utah State University, 2009

M.S., Statistics, University of New Mexico, 2013

Abstract

This work describes a Bayesian model for assessing the reliability of complex systems using component tests, full system tests, and covariate information. Development of the model focused on understanding the relationship between component reliability and system reliability; and defining this relationship using mathematical expressions. The method in this thesis uses pass/fail data coupled with different levels of prior information about system reliability and covariate information to derive posterior distributions that model component and system reliability. This work provides insights on how the number of components, amount of prior information used, inclusion of covariates and spread of failures across components affects point estimates and densities for system reliability. The methodology in this paper is tested using simulated data weapons system test data.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The United States military has maintained a nuclear weapons program since the early 1940s. The U.S. built a large and diverse nuclear arsenal in an attempt to compete with the USSR during an arms race. In the 1950s, between the two countries, hundreds of nuclear tests were conducted. The pressure coming from protests surrounding those tests combined with the fall out from the Cuban Missile Crisis led to the signing of the Limited Test Ban Treaty in 1963, the first of three test ban treaties, which limited atmospheric nuclear tests. In 1974 the Threshold Test Ban Treaty, which limited the size of a nuclear explosion, was signed. Neither treaty was ratified in the hopes of negotiating a Comprehensive Test Ban Treaty (CTBT), a treaty banning all nuclear testing. The pursuit of a CTBT was blocked by the argument that testing was necessary to maintain the reliability of existing weapons. As the Cold War ended the pressure for a CTBT grew, and several nuclear states enacted testing moratoriums. Finally, on Sept 24 1996 the UN General Assembly opened the CTBT for signature.

The U.S. has signed, but not ratified, the CTBT; and continues to operate under a testing moratorium but maintains the right to exit the CTBT if high levels of confidence in the reliability of weapons requires testing. Under the CTBT computer models and non-nuclear experiments have taken the place of full nuclear testing. Maintaining the nuclear stockpile without testing is known as Stockpile Stewardship. Medalia (2010) has written a congressional report, Comprehensive Test Ban Treaty: Background and Current Developments, which details the Stockpile Stewardship Program. A science-based Stockpile Stewardship Program has been implemented in the U.S. to maintain a high level of confidence in the safety and reliability of the active stockpile. The ability of the Stockpile Stewardship Program to maintain the confidence needed was a topic of concern in the 1999 Senate debate. Those against the program argued stockpile stewardship offered no guarantee of maintaining weapon reliability and that experiments and models could offer no clues to the problems that may develop as the stockpile ages. Those in favor of the program argued that the program had already certified the stockpile 3 times. Since then, the Stockpile Stewardship Program has certified the stockpile 17 times using data obtained from non-nuclear testing and computer models.

The Stockpile Stewardship Program uses advanced physics-based and statistical models to use information gathered during non-nuclear tests to continually certify that the nuclear stockpile has maintained acceptable levels of safety and reliability. The physics-based models primarily use information from subcritical experiments to simulate how the components, aged materials, chemical explosives, properties of plutonium, radiation, and various environments affect the nuclear explosions of todays aged stockpile. Statistical models developed in support of Stockpile Health Assessment have had success in using limited field data to evaluate system reliability. The U.S. surveillance program has been implemented as part of the Stockpile Stewardship Program to collect data from each weapon system. Surveillance programs require a set number of weapons be pulled annually to have non-nuclear tests performed on individual components or destructive tests on the full system. Los Alamos National Laboratory (LANL) has been a leader in statistical modeling through the use of YADAS, a software program developed to address the statistical challenges presented by the limited data that can be collected by surveillance programs (Wilson et al., 2007).

The goal of this thesis is to expand the work done by LANL researchers. The model that will be developed in this thesis will be able to use the limited amounts of data collected through the surveillance program to obtain posterior distributions for system reliability. Comparisons of the system reliability densities will be made based upon variations in the input data: number of components, aging information, and inclusion of system test data. Bayesian statistics is a method that is growing in popularity within the nuclear community. An advantage of Bayesian statistics is the ability to incorporate prior information known about the system and evolve as additional information becomes available. This characteristic makes Bayesian statistics a natural fit for evaluating the reliability of systems where data may not be easily obtainable.

The model developed in this thesis will be a hierarchical Bayesian model with logistic regression. The hierarchical portion incorporates data from component and system tests. The logistic regression portion allows covariate information to enter the model. The model will be tested using simulated and real-world data. The simulated data will demonstrate how system reliability is related to the input data changes. The real-world data comes from an unknown weapons system and is presented as seven data sets. Each data set represents the same component tests but with varying definitions of what counted as a failure. The real data will demonstrate how system reliability can change based upon changes in the number/spread of failures.

The reliability model developed in this thesis was used to find posterior distributions for system reliability (and component reliability) under different scenarios. The simulated data highlighted the impact that increasing the complexity of a system can have on the point estimates for system reliability. Since system reliability is calculated as a product of the independent [1] component reliabilities the more components

[1]The choice to use independence in reliability theory needs to be considered carefully;

within a system the more important it is for each component to be highly reliable to keep the estimate for system reliability at an acceptable level. The first two data sets also made it apparent that the amount of data used in the simulated data analysis was enough to overwhelm the prior distributions; as the amount of data increases the results from different priors converge to the same conclusion.

The simulated data also allowed for system reliability to be calculated as a function of age; in both a one component and five component system. Incorporating this covariate in both of the systems provided insight into how reliability estimates degrade with an increase in uncertainty around that estimate as components/system ages. Combining the covariate information with the 5 component system and system tests into a full system increased the estimates for that component at every age from 96-90% up to 98-94%. System reliability increased with a decrease in uncertainty when the covariate information was included with the 5 components. The full system model proved that the more information, component tests, system tests, and covariate information, the more fidelity that will be present in the final estimates.

The reliability models were also used to determine system reliability from a real weapons system. Seven data sets were compiled from the weapons system tests and compared. This real data did not include any system tests making the difference between the system priors more pronounced. This highlights the impact that even a small number of system tests has on the likelihood. These data sets also highlighted how the uncertainty surrounding reliability estimates almost doubled as the priors changed from highly informative to non-informative and the number of failures moved towards zero. Besides introducing information into the model through the priors, uncertainty decreased for each component's reliability point estimate as the number of tests for that component increased. The impact of increasing the number of tests per component increases in importance if any failures have been detected in that component. The last important conclusion made from the real test data is the

---

future work in this field would explore the impact of dependent sub-systems.

spread of failures. Regardless of the number of failures, having the failures spread across the system provides a 57.8-72.2% chance of providing higher system reliability estimates over systems where all the failures occur on a single component.

The reliability models in this thesis have provided several different conclusions regarding the relationship between input data and system reliability. A broad understanding of how different types of data affect system reliability has been developed throughout this research. The conclusions made here may begin to allow decision makers to allocate surveillance resources to collecting data that results in high levels of fidelity surrounding estimates for system reliability. Expanding this model to include the cost associated with collecting each type of data will help inform decisions regarding the best way to allocate surveillance funding. Is it better to spend funding on collect a small number of system tests or use that funding to gather a larger number of component tests? Incorporating the cost of each type of test along with the knowledge already gained from this thesis can address questions such as this.

The reliability model in this thesis also uncovered how age affects reliability. Age is just one covariate that can impact the reliability of a system. The model has been written so that it can include an unlimited number of covariates. Knowing the impact of different covariates may help prioritize the components that need to have remedial measures performed to keep the reliability estimates from dropping. Future covariates of interest may include production facilities, storage facilities, and current refurbishment packages.

The focus of this thesis was to develop a reliability model that can model data collected through surveillance programs to provide distributions surround system reliability estimates. The analysis in this thesis has practical applications and can assist decision makers in determining how much and what type of data will provide the best insights into a stockpiles reliability. Knowing this information can help the U.S. government refine resource allocation to gather the most efficient types of data through the Stockpile Stewardship Programs surveillance programs.

Chapter 2

Literature Review

2.1 Historical Methods Evaluating Stockpile Reliability

Evaluating a stockpile using statistical methodology is not a new concept. Two important papers laid the foundation for stockpile evaluation. Derman and Solomon (1958) and Hillier (1962) both used the exponential distribution to model stockpile evaluation. The exponential model is used by both papers to determine the best surveillance plan out of a finite selection of plans. These papers determined the optimal time between inspections that resulted in the proportion of defectives in a stockpile being less than a predetermined acceptable level.

Derman and Solomon (1958) introduced three different methods for stockpile evaluation. These three methods where applied when (i) the stockpile was in a steady-state, (ii) the deterioration function, $p(t)$, was modeled as a step-function, and (iii) $p(t)$ was modeled with an exponential function. The method that models $p(t)$ as an exponential function spurred future research and will be briefly discussed.

The goal of Derman and Solomon (1958) was to develop a model that could be used to select the best surveillance plan out of a finite selection of plans. Only surveillance plans that maintained an acceptable proportion of defectives, π^*, were considered. Then among all possible surveillance plans, select those that maintained $\pi_\tau < \pi^*$ for all τ; where τ represents a specific plan. Derman and Solomon (1958) defined the decay function of the stockpile to be, $p(t) = 1 - \theta e^{-\alpha t}$. The time between inspections that minimized π_τ could be determined by the exponential decay function.

Similar to the previous paper, Hillier (1962) also modeled $p(t)$ as an exponential distribution. Optimizing the time between corrective actions would compensate for the deteriorating quality of a lot held in storage. Hillier (1962) expanded on Derman and Solomon (1958) by including the cost of each appropriate corrective action versus the cost of an imperfect lot when an emergency occurs. Hillier (1962) minimized the costs of a surveillance plan while minimizing the number of defective units. The objective of Hillier (1962) was to find the value of t, where t was the time between corrective actions that minimized total expected cost over the entire length of the program.

At the time of their publication, these two papers introduced new concepts about evaluating stockpile surveillance. Derman and Solomon (1958) and Hillier (1962) aimed to define the time between corrective actions that minimized the proportion of defectives in the stockpile. Hillier (1962) added the idea of minimizing the cost of a surveillance program. These are important papers since the idea of reviewing stockpile surveillance through analytical methods was introduced by them. These methods have recently been outgrown by the rapid advancements in Bayesian statistics and computer simulations.

2.2 Current Methods for Evaluating Stockpile Reliability

Previous literature reviews focused on early research attempts to evaluate stockpiles. Section 2.1 mentioned how these early research methods have been overcome by advances in Bayesian statistics and computer simulations. This section discusses Bayesian models and the Markov chain Monte Carlo (MCMC) simulations used to analyze them. Bayesian methods are able to formally incorporate supplementary information about the parameters of interest into an analysis (Martz, 1982). Because of this, Bayesian methods provide a natural approach to analyzing reliability data and will be the method used throughout this paper

2.2.1 Bayesian Model Building

Bayesian inference sets out to estimate unknown model parameters, Θ. To complete a Bayesian model two key distributions are needed. The first key distribution is the likelihood. The likelihood, $f(\mathbf{y}|\Theta)$, is the probability distribution that captures information provided by the observed data conditional on all parameters, where $\mathbf{y}$ denotes the collection of all observed data points, $y_1, y_2 \ldots, y_n$. The second key distribution is the prior distribution. The prior distribution, $\pi(\Theta)$, captures prior knowledge about Θ. Prior distributions are a joint distribution covering all unknown model parameters. [1] These distributions introduce various levels of prior information into the model and can be conjugate to the likelihood or not, informative, subjective or objective, or improper. Depending on the goal of the research and the resources available, different types of priors may be preferential. Yang and Berger (1996) provide a detailed list of various types of priors and the most common (useful) times to employ each type.

[1] Prior distributions can simply be the product of multiple independent distributions.

Bayes theorem is used to combine the likelihood and prior distributions; resulting in a third distribution referred to as the posterior distribution, $\pi(\Theta|\mathbf{y})$. The posterior distribution is used to make inferences on the unknown parameter given the observed data.

$$\pi(\Theta|\mathbf{y}) = \frac{f(\mathbf{y}|\Theta)\pi(\Theta)}{\int f(\mathbf{y}|\Theta)\pi(\Theta)d\Theta} \tag{2.1}$$
$$\propto f(\mathbf{y}|\Theta)\pi(\Theta). \tag{2.2}$$

Equation 2.1 can be difficult to solve analytically. The denominator can be challenging to solve and may not even be possible to solve as the model increases in complexity. The denominator of Equation 2.1 integrates out the parameters of interest so it becomes a constant in regard to Θ and can be dropped, resulting in Equation 2.2.

If the posterior distribution is in the same class of distributions as the prior it is a conjugate prior and can be written in the form of a known distribution (Gelman, 2006). A prior that is not conjugate may result in an unnormalized posterior probability density function. Unnormalized functions require alternative methods to analyze these functions before using them for inference about Θ. One approach is to employ Bayesian specific algorithms to solve. To use the algorithms, it's easiest to begin by deriving the complete conditionals of each unknown parameter.

Complete conditionals are the distributions of one parameter given all other parameters and data are in the model. Complete conditionals, $[\theta]$, are derived from the joint posterior distribution by keeping only terms involving θ, the single parameter of interest.

$$[\theta_i] = \frac{\pi(\theta_i, \Theta|\mathbf{y})}{\pi(\Theta|\mathbf{y})} \tag{2.3}$$
$$\propto \pi(\theta_i|\Theta, \mathbf{y}). \tag{2.4}$$

The complete conditionals of the unknown parameters, $[\theta_i]$, are distributions where only one parameter is considered random. The denominator of Equation 2.3 does not depend on θ_i; therefore proportionality can be used resulting in Equation 2.4. Complete conditionals can take the form of a standard probability density function or result in an unnormalized distribution. The class of MCMC simulations known as Gibbs sampling can be used to generate draws from each complete conditional. If the complete conditionals take the form of a known PDF, then closed-form Gibbs sampling can be used. If the complete conditional is an unnormalized density function, the class of MCMC simulations known as Metropolis-Hastings (M-H) can be used to simulate draws from each complete conditional (Gilks, Richardson, & Spiegelhalter, 1996).

2.2.2 Simulation Methods

Various user-friendly packages exist for implementing MCMC algorithms. One of the most widespread is WinBugs. Christensen, Johnson, Branscum, and Hanson (2011) describe WinBugs as a menu-driven program that draws samples from the posterior using Markov chain concepts. The user must only define the model structure. WinBugs eliminates the need to derive the complete conditionals and code the algorithm yourself. Another MCMC package comes from Los Alamos National Laboratory (LANL) researchers. This LANL developed package, referred to as YADAS, was built to address the statistical challenges specific to analyzing weapons data (Wilson et al., 2007). Neither of these packages will be used for the computations in this thesis; however, papers written about their development provide useful descriptions of the MCMC family of simulations. Graves (2007), one of the co-developers of YADAS, provides a useful overview to MCMC algorithms.

An MCMC algorithm's purpose is to draw samples from the joint posterior distribution of unknown parameters, Θ. Two MCMC based algorithms are common in Bayesian analysis. Gibbs sampling is used to sample from complete conditionals which take the form of known distributions, this method is not used in this thesis and will not be further explained. The Metropolis-Hastings (M-H) algorithm is used to generate a Markov chain that converges to a stationary distribution that is the joint posterior (Christensen et al., 2011).

Graves (2007) provides the following high-level, step-by-step description of the M-H algorithm.

1. The researcher must define three items. The first is a proposal distribution that will be used to select candidate values in Step 2. The second, is the number of iterations simulated, $t = 1...T$. The third are the initial values for Θ, denoted Θ^0.

2. Propose a new value for Θ' by sampling from the proposal density $T(.|\Theta^{t-1})$; where Θ^{t-1} is a vector of the previous iteration's values of Θ

3. Accept the new value with probability

$$min\left\{1, \frac{\pi(\Theta')}{\pi(\Theta^{t-1})}\frac{T(\Theta^{t-1}|\Theta)}{T(\Theta|\Theta^{t-1})}\right\}$$

If the new value is accepted, set $\Theta^t = \Theta'$, otherwise set $\Theta^t = \Theta^{t-1}$.

4. Return to step two and repeat T times.

Under achievable conditions, this algorithm will yield a Markov chain with a stationary distribution that converges to the desired posterior distribution, $\pi(\Theta|\mathbf{y})$. So the samples will eventually represent the joint posterior.

Together Hamada, Wilson, and Reese (2008) and Chib and Greenberg (1995) provide a more detailed description of the process defined by Graves (2007). Step 1 introduced the use of a proposal density for generating candidate values. These candidate values can be a vector of values representing Θ or one value representing the candidate value for θ_i. The M-H algorithms run in this thesis will update parameters one at a time and therefore the candidate values drawn will be univariate vectors represented by θ_i^t. A common proposal density is the Gaussian distribution with mean = θ^{t-1} and variance = σ_c^2. The Gaussian proposal densities are centered at the previous iteration's value. The variance of a Gaussian proposal density is a constant value that dictates the size of the sample space from which candidate values are drawn. The candidate value is a randomly selected value from the proposal density which is plugged into the unnormalized posterior distribution. The ratio of the complete conditional given the candidate value over the complete conditional given the previous iterations values is found. If this ratio is greater than 1, then the candidate value is rejected; otherwise the candidate value is accepted. The simulation repeats T times, selecting a new candidate value and storing each iterations value. The stored values represent draws taken from the marginal posterior density for each unknown parameter.

Step 1 also mentions selecting a starting value. This initial value, Θ^0, can be any value within the domain of Θ. Frequentist approaches, such as the maximum likelihood estimate, can be used to approximate the starting value for each parameter. The choice of Θ^0 may affect the number of iterations before the distribution converges. A poor choice for the starting point can increase the time it takes for the algorithm to converge. To accomodate this, Bayesians include an additional number of iterations called burn-in. This burn-in period is the time it takes for the algorithm to converge. Values obtained during burn-in iterations are discarded prior to making inference on the parameters.

The final note on MCMC simulations is the efficiency of the algorithm. This efficiency, determined by σ_c^2, is known as the acceptance rate. The acceptance rate is the percentage the algorithm accepts θ^t over θ^{t-1}, where $t-1$ is the accepted value for the previous iteration. If the spread is too large then the acceptance rate will be small. A large σ_c^2 means candidate values may be far from $\Theta^{[t-1]}$ and have a low probability of being accepted. If the spread is too small, then the acceptance rate will be large. A small σ_c^2 means the algorithm will take too long to move across the support of the density and lower probability regions will be undersampled. Both of these result in high autocorrelation among sampled values. Roberts, Gelman, and Gilks (1997) found the optimal rate of acceptance starts at 45% for one parameter models and moves closer to 23% as the number of parameters increases. In practice, acceptance rates ranging from 15-40% are common.

2.2.3 Bayesian Diagnostics

An important practice is to test how adequate the model works before beginning any analysis. Many diagnostic tests exist; but two will be used within this paper. The first diagnostics check to see that the MCMC chain has converged to a stationary distribution. The second set of diagnostics test model fit for the observed data and fitted model.

Cowles and Carlin (1996) compared numerous types of MCMC convergence diagnostics that have been developed. Convergence diagnostic tools are applied to the draws generated by the algorithm. Convergence diagnostics allow the researcher to see, either graphically or quantitatively, if the chain has converged. Cowles and Carlin (1996) discussed the bias of each diagnostic. These biases are beyond the scope of this thesis; but because of this possible bias no single diagnostic provides enough information to conclude convergene. To convince oneself that an MCMC chain has converged a combination of diagnostics should be used.

The most common convergence diagnostic is a traceplot. Traceplots are graphical representations of the marginal posterior distribution values produced by the algorithm over the iterations. These traceplots allow the researcher to detect unusual activity among the accepted candidate values. Evidence against convergence exists if strange shifts or long periods where no new values are accepted are detected. Remedial measures to fix this include increasing the burn-in period or transforming the parameters. A desirable traceplot closely resembles a scatterplot of whitenoise.

The next two convergence diagnostics are quantitative diagnostics that are computed and interpreted. Gelman and Rubin (1992) developed a convergence diagnostic that compares m different parallel chains; each with a different Θ^0. The weighted average of two statistics, W and B, are used to find the ratio of the current variance estimate to the within sequence variance.

$$\sqrt{\hat{R}} = \sqrt{\left(\frac{n-1}{n} + \frac{m+1}{mn}\frac{B}{W}\right)\frac{df}{df-2}}$$

$$W = \sum_{i=1}^{m} \frac{s_i^2}{m}$$

$$B = \frac{n}{m-1}\sum_{i=1}^{m}(\bar{x_{i.}} - \bar{x_{..}})$$

W is the average of the within-chain variances, s_i^2, each based on n values of x. B is a measure of the variance between m sequence means, $\bar{x_{i.}}$ based on n values of x. The result of this weighted average, $\sqrt{\hat{R}}$, is called the "shrink factor". The approaches 1 when the pooled within chain variance dominates the between chain variance. A shrink factor of 1 is interpreted to mean that the chains have escaped the influence of different starting points and traversed the entire target distribution (Cowles & Carlin, 1996).

The final convergence diagnostic in this paper is the Geweke distribution. Geweke et al. (1991) developed a convergence diagnostic that compares the mean of the beginning of a chain with the mean of the end of a chain. The Geweke diagnostic states that if the chain has converged than no significant differences should exist. Geweke uses a t-test to compare the mean of the first $p\%$ of the simulations to the mean of the last $p\%$. If the t-test produces a significant p-value than the null hypothesis is rejected and it can be concluded that a difference exists between the two means and a larger burn-in period should be explored.

Combining these three diagnostics: traceplots, Gelman and Rubin, and Geweke; can provide evidence that the MCMC has produced a stationary distribution. Once a stationary distribution has been obtained analysis can be done on the draws. Convergence diagnostics are only the first piece of the puzzle. Just because a distribution has become stationary does not mean that the model used to obtain that distribution fits the data. To evaluate the fit of the model, many Bayesian diagnostics have been developed. The Bayesian versions of the χ^2 goodness-of-fit test are popular.

Johnson (2004) discusses a Bayesian form of goodness-of-fit tests. The χ^2 goodness-of-fit diagnostic begins by randomly selecting a single value of the parameter vector from the posterior distribution, $\bar{\boldsymbol{\theta}}$. Let $0 < a_0 < \ldots < a_k = 1$ be quantiles from a Uniform(0, 1), and $p_j = a_j - a_{j-1}$. Also, define m_j as the number of observations from the data, y_i, for which $a_{j-1} < F(y_i|\bar{\boldsymbol{\theta}}) < a_j$. The χ^2 statistic is built using these values

$$R^B(\bar{\boldsymbol{\theta}}) = \sum_{k=1}^{K} \frac{m_k[\bar{\boldsymbol{\theta}} - np_k]^2}{np_k}. \tag{2.5}$$

Using $F(y_i|\bar{\boldsymbol{\theta}})$, the number of observations that fall into each bin are counted resulting in $\mathbf{m}$, a vector of bin counts. The value of the test statistic is computed using $\mathbf{m}$. If the test statistic is less than $\chi^2_{K-1,1-\alpha}$ then evidence exists to conclude the model fits the data.

2.3 Applied Reliability Research

This final section reviews articles that combine the framework of Bayesian analysis presented Section 2.2 with reliability research. Most of the papers follow a similar format: identify the problem; find the likelihood and priors; use algorithms to draw from the joint posterior, and conclude with inference on the marginal posterior distribution for each unknown parameter. Each paper adds useful information on how to apply Bayesian models to reliability research. The differences of each paper will be covered in-depth while the commonalities will be covered at a high level.

Hamada (2005) introduced a simple example of combining Bayesian methods with reliability analysis. Hamada (2005) used degradation data from a laser to assess the reliability function of a system. Degradation data is defined as the time it takes for a piece of equipment to degrade past an acceptable level.

Hamada (2005) used Bayesian methods to analyze the time for a laser to degrade past a certain level of performance. The chosen likelihood and priors were:

$$y_{ij}|\theta_i, \sigma^2 \sim \text{Normal}(\frac{1}{\theta_i}t_{ij}, \sigma^2);$$
$$\theta_i|\lambda, \beta \sim \text{Weibull}(\lambda, \beta)$$

λ, β, and σ^2 are also viewed as random parameters with non-informative priors. This type of model where the hyperparameters are random is known as a hierarchical model. Hamada (2005) used an MCMC with a burn-in period of 500 and post burn-in period of 10000 iterations. Draws were made for each of the four random parameters; θ_i, σ^2, λ, β. These draws make up marginal posterior distributions for each of the parameters. The draws for λ and β are used to find the posterior predictive distribution for reliability; where reliability has been defined:

$$R(t) = \exp\left[-\left(\frac{\lambda}{D^\beta}\right)t^\beta\right].$$

The next papers build on this model by increasing the complexity of the modeled system. A complex system is one that is made up of multiple components. Johnson, Graves, Hamada, and Reese (2003) and Anderson-Cook et al. (2007) both use hierarchical models to determine system reliability. The difference is the type of data used in the likelihood; Johnson et al. (2003) used failure time data and Anderson-Cook et al. (2007) used binary pass/fail data. Both papers collected data from component and system tests.

A difficult aspect of modeling system reliability, θ_s, is integrating multiple sources of information. Possible sources of information include component, subsystem, system data, and expert opinion. Anderson-Cook et al. (2007) discusses the importance of building a model that can incorporate these various data sources. The cost of system tests can be significantly higher than the cost of a component test. The cost differential of these two tests can mean a higher volume of data from component tests is available. Building a model that can include component tests may substantially enhance the final results.

To incorporate component and system test data the system structure must be understood. System structure defines the way components are interrelated, a detailed explanation is given in Section 3.1.1. Understanding the way a system operates as a function of its components allows the likelihood for system reliability, θ_s, to be written in terms of component reliability, θ_i. Including system data in this way minimizes the possible underestimation of θ_s that can occur as the number of components increases.

Johnson et al. (2003) also brings to light the challenge of incorporating expert opinion. Expert opinion on the reliability of each component is incorporated into their model as a pseudo-observation. These assessments are treated as observations from the likelihood. This approach is convenient from a practical standpoint. Experts may be more comfortable giving a value for a component's expected reliability as opposed to naming values for abstract parameters.

The models presented by Johnson et al. (2003) and Anderson-Cook et al. (2007) are analyzed using MCMC algorithms. Even though no draws were made for θ_s, the marginal posterior distribution for system reliability can still be computed. Since the system structure relates components to the system; the defined structure can use the marginal posterior densities for each θ_i and compute θ_s.

The last three papers discussed, Hamada et al. (2008), Johnson et al. (2003), and Anderson-Cook et al. (2007) gave demonstrations of how to apply the techniques from Section 2.2 to problems about system reliability. The final paper discussed in this section takes the leap between determining the marginal posterior distributions of parameters of interest to using that information to answer a more abstract problem.

Vanderwiel, Wilson, Graves, and Reese (2009) has developed a Bayesian model with the intent to answer broad policy questions about surveillance programs. Vanderwiel et al. (2009) is similar in purpose to the papers from Section 2.1.

Vanderwiel et al. (2009) looks at the relationship between stockpile reliability and surveillance programs. The model developed, RADAR, provides a probabilistic description of the uncertainties that accumulate if surveillance programs are stopped. RADAR is a Bayesian model that uses a Markov process to model the evolution of π_t, the reliability of a stockpile in year t. RADAR models the reliability to be 1 until a randomly onset time of degradation. This paper concludes that prior information, regardless of the hyperparameter values, is typically overwhelmed by successful system tests; as long as new test data is collected at least every 10 years. RADAR shows that the mean reliability estimate will remain relatively constant over time, but the uncertainty will immediately and progressively increase if surveillance is halted.

This section has covered a range of Bayesian models. Hamada et al. (2008) opened the section with a hierarchical model that used data collected from system tests to determine θ_s. Then Johnson et al. (2003) and Anderson-Cook et al. (2007) added complexity to the first model by using data collected from both component and system tests to determine θ_s. This section was concluded by the discussion from Vanderwiel et al. (2009) about applying a Bayesian model to an abstract problem.

Chapter 3

Reliability Models

A Bayesian model will be derived in this chapter that produces posterior densities for component and system reliability. A couple of simpler models will first be derived that when combined will produce a full-system reliability model that will be a flexible model that can take various types of input data. This chapter starts with a base model that uses component and system test data, similar to Johnson et al. (2003) and Anderson-Cook et al. (2007). A second model will be developed that uses component tests with covariate information. These two models will be combined to result in the full system reliability model.

3.1 Likelihood Distributions, $f(\mathbf{y}|\Theta)$

Hamada (2005) described two important pieces for building a Bayesian model. The first piece that needs to be identified is the likelihood. The likelihood is a function that describes the data given the parameters are known. The data of interest in this paper is the pass/fail results of individual component and system tests. Pass/fail data is best represented by the Binomial distribution, Appendix A.2 provides more information about the Binomial distribution.

The full expression of the likelihood, $f(y|\Theta)$, is the product of n_c Binomials.

$$f(\mathbf{y}|\Theta) \propto \prod_{i=1}^{n_c} \left(\theta_i^{\sum_j^{n_i} y_{ij}} (1-\theta_i)^{n_i - \sum_j^{n_i} y_{ij}} \right). \tag{3.1}$$

$$\text{where} = \begin{cases} \sum_{j=1}^{n_i} y_{ij} & = \text{Sum of successes, } (y_{ij}=1)\text{, for each component} \\ n_c & = \text{ \# of components in the system} \\ n_i & = \text{\# of tests for component i} \end{cases}$$

In Equation 3.1, $\mathbf{y}$ represents data that has been collected and Θ represents the parameters, θ_i; where θ_i represents the reliability for each component.

Anderson-Cook et al. (2007) discussed the importance of building a model capable of using all available data. The likelihood in Equation 3.1 models the data from component tests but not from system tests. The system test data of interest also comes in the form of the pass/fail results and can be modeled with another Binomial distribution.

As both Johnson et al. (2003) and Anderson-Cook et al. (2007) mention understanding the structure of the system is necessary to properly model the system data and write down the likelihood.

3.1.1 System Structure

To use system data with component data the system structure must be defined. Section 2.3 briefly introduced the importance of understanding system structure. This section gives a formal definition of system structure. System structure is a way to define a complex system by looking at how the subsystems (or components) are interrelated. Hamada et al. (2008) provides definitions of many different system structures. There is one system structure that is of interest for this paper. The system of interest in this paper follows a structure called a series system.

A system defined as a series system will function only if *all* components function. In other words the system reliability, $\theta_s = 1$ if $\theta_1 = \theta_2 = \ldots = \theta_{n_c} = 1$. If any θ_i has a known reliability less than 1, then

$$\theta_s = \prod_{i=1}^{n_c} \theta_i. \tag{3.2}$$

By Equation 3.2, the reliability of a series system is the product of the reliability of its components. This definition of system reliability is used to define the parameters of the system likelihood.

3.1.2 System Likelihood

A likelihood distribution that models system data as a function of component reliability can be written using Equation 3.2. From Equation 3.2 it is known that system reliability is the product of the component reliabilities. If the system test data, n_t, follows a Binomial distribution; then the likelihood can be written:

$$f(n_t|\theta_s) \propto \left(\prod_{i=1}^{n_c} \theta_i\right)^{n_s} \left(1 - \prod_{i=1}^{n_c} \theta_i\right)^{n_t - n_s}; \tag{3.3}$$

$$\text{where} = \begin{cases} n_t & = \text{\# of system tests} \\ n_s & = \text{\# of successful system tests} \end{cases}.$$

Combining Equations 3.1 and 3.3 results in the likelihood expression which captures all available pass/fail data.

$$f(\mathbf{y}, n_t|\Theta) \propto \prod_{i=1}^{n_c} \left(\theta_i^{\sum_j^{n_i} y_{ij}} (1-\theta_i)^{n_i - \sum_j^{n_i} y_{ij}}\right) \times \left(\prod_{i=1}^{n_c} \theta_i\right)^{n_s} \left(1 - \prod_{i=1}^{n_c} \theta_i\right)^{n_t - n_s}. \tag{3.4}$$

Defining Equation 3.4 as the likelihood for the data is the first step in building a Bayesian reliability model. The next piece of the base reliability model will be defining prior distributions for each unknown parameter.

3.2 Prior Distributions, $\pi(\boldsymbol{\Theta})$

The likelihood distribution for component and system data was written in the previous section, stated in Equation 3.4. The next step to building a Bayesian reliability model is defining prior distributions for all unknown parameters. Prior information for θ_i and θ_s can be incorporated into the model through the proper development of $\pi(\theta_i)$.

The relationship between θ_s and θ_i causes the component prior, $\pi(\theta_i)$, to be affected by prior information about the system. The simplest case involves using a Uniform distribution as the system prior; $\pi(\theta_s) \sim$ Uniform(0, 1). Now the relationship between a Uniform system prior and the component priors must be explored.

The first step is to select a prior distribution that reflects knowledge about the reliability of the components, θ_i. Using Anderson-Cook et al. (2007) as guidance, a Beta prior is the most common choice for modeling a parameter whose mass falls within 0 and 1; such as a reliability parameter. Appendix A.1 provides further detail on the Beta distribution. For this example $\pi(\theta_i) \sim$ Beta (a, b). For all the models developed in this paper the hyperparameters of both the component and system priors are known. The values, a and b, are found using the information from the system prior, in this case a Uniform(0, 1).

To obtain values for the component hyperparameters the expected value and variance of θ_s are used. Using the mean and variance equations for a Uniform(0, 1) the $\mathbb{E}[\theta_s] = .5$ and the $Var(\theta_s) = \frac{1}{12}$. Starting with $i = 1$; the $\mathbb{E}[\theta_1] = .5$ and the $\text{Var}(\theta_1) = \frac{1}{12}$. To find a and b values that result in those mean and variances, the mean and variance equations of a Beta distribution are set equal to .5 and $\frac{1}{12}$ respectively. Solving the system of equations results in $a = 1$ and $b = 1$. The Uniform(0, 1) is a special case of a Beta distribution with parameters, $a = 1, b = 1$, so these results are easily confirmed. In summary, for a one component system to have a Uniform system reliability the component prior will be a Beta(1, 1).

This one component example provides the framework for developing the model of a more complex system. For this thesis, a complex system involves a system with $n_c > 1$ components. We begin with θ_s following a Uniform prior; $\pi(\theta_s) \sim$Uniform(0, 1). The prior for each θ_i will follow independent Beta priors; $\pi(\theta_i) \sim \text{Beta}(a, b)$, where $a_1 = a_2 = \ldots a_{n_c}$ and $b_1 = b_2 = \ldots = b_{n_c}$ for all n_c components [1].

Using equation 3.2, $\pi(\theta_s)$ will be the product of the n_c Beta distributions. Goodman (1962) derived an expression for relating variance to the product of n_c independent variables. Using Goodman (1962), the product of n_c Beta distributions will be set equal to the known variance, $\frac{1}{12}$ resulting in Equations 3.5 and 3.6 as expression for a and b. Knowing the value of n_c will allow specific values of a and b to be found for component priors based upon the number of components within the system.

$$a = \frac{\gamma}{(2+\gamma)\left[\left(\frac{1}{12\left(\frac{1}{(1+\gamma)^{2n_c}}\right)} + 1\right)^{1/n_c} - 1\right]}, \tag{3.5}$$

$$b = a\gamma, \tag{3.6}$$

$$\text{where} \quad \gamma = \frac{\left(1 - .5^{1/n_c}\right)}{.5^{1/n_c}}.$$

Simulations are used to verify that the product of n_c Beta(a, b) distributions approximate a Uniform(0, 1). 100,000 values were randomly selected from Beta(a, b) distributions and their product was plotted as histograms. The values for a and b are obtained using Equations 3.5 and 3.6 setting $n_c = 1, 7,$ and 42. The Beta distributions resulting from these three components are in Table 3.1.

Figure 3.1 shows the three resulting histograms with an overlay of the Uniform(0, 1). It can be concluded that the (a, b) values from Equations 3.5 and 3.6 result in a Uniform(0, 1) system reliability since the histograms closely mirror the Uniform

[1] This model places both the assumption of independence on $\pi(\theta_i)$ but also the assumption that each component follows the same prior distribution

Table 3.1: *Given that a system has a uniform system reliability, the hyperparameters for the component prior will vary based on n_c. This table shows different hyperparameter values based upon different values of n_c.*

n_c	$\pi(\theta_i) \sim Beta(a, b)$
1	Beta(1, 1)
7	Beta(1.18, .12)
42	Beta(1.20, .020)

density.

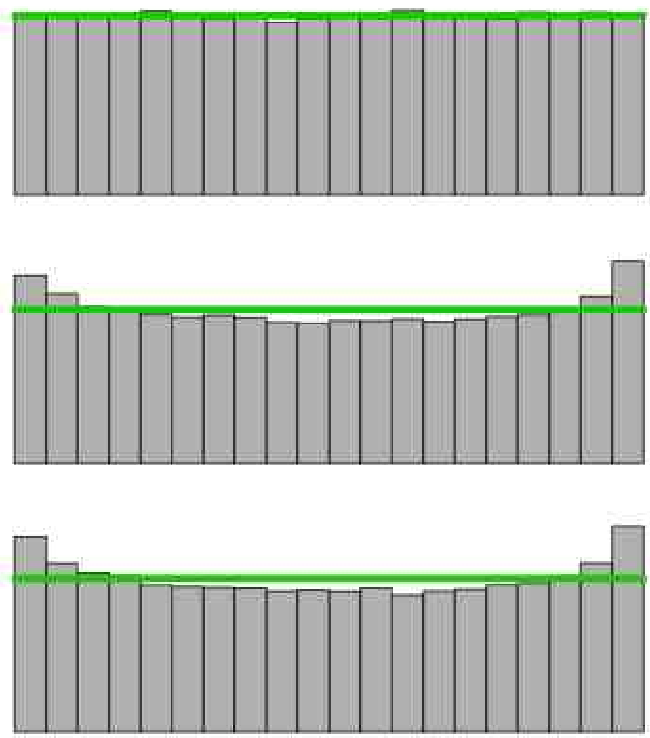


Figure 3.1: *A simulation was executed to verify that independent component reliabilities could have Beta priors and their product would approximate a Uniform system reliability. The three systems represented here are 1, 7, and 42 component systems. The green line represents the density of a Uniform(0, 1). The histograms represent draws taken from the product of n_c Beta(a, b) distributions using the values from Table 3.1. The histograms all closely approximate the Uniform density which confirms that the hyperparameters selected for each $\pi(\theta_i)$ approximate a Uniform system reliability using the series system definition.*

If additional information about θ_s is available, such as $\mathbb{E}$ and variance, it should be used in the model. Incorporating this prior knowledge about parameters into the model is one of the main advantages of Bayesian analysis. Changing $\pi(\theta_s) \sim Unif(0,1)$ to $\pi(\theta_s) \sim \text{Beta}(\alpha, \beta)$ allows prior information about the system to be incorporated into the model. The prior information available in this study is knowledge regarding the expected value of θ_s, R^*, with a known variance, VR^*. Knowing the mean and variance of θ_s allows α and β, to be solved for by setting the mean and variance equations of the Beta distribution equal to R^* and VR^*.

$$\begin{aligned}
R^* &= \frac{\alpha}{\alpha+\beta} \quad \text{therefore,} \\
\beta &= \frac{\alpha(1-R^*)}{R^*}. \qquad (3.7) \\
VR^* &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha\beta}{(\alpha^2+2\alpha\beta+\beta^2)(\alpha+\beta+1)} \qquad \text{Plugging in } \beta = \frac{\alpha(1-R^*)}{R^*} \\
&= \frac{\alpha^2\frac{1-R^*}{R^*}}{(\alpha^2+2\alpha^2\frac{1-R^*}{R^*}+\alpha^2(\frac{1-R^*}{R^*})^2)(\alpha+\alpha\frac{1-R^*}{R^*}+1)} \\
&= \frac{\frac{1-R^*}{R^*}}{(1+\frac{2(1-R^*)}{R^*}+\frac{1-R^*}{R^*}^2)(\alpha(1+\frac{1-R^*}{R^*})+1)} \\
&\vdots \\
\alpha &= \frac{\frac{1-R^*}{R^*}}{VR^*(1+\frac{2(1-R^*)}{R^*}+(\frac{1-R^*}{R^*})^2)(1+\frac{1-R^*}{R^*})} - \frac{1}{1+\frac{1-R^*}{R^*}}. \qquad (3.8)
\end{aligned}$$

The known values of R^* and VR^* can be plugged into Equations 3.7 and 3.8 to find the parameters of $\pi(\theta_s)$. Once the parameter values for $\pi(\theta_s)$ are known, component hyperparameters, a and b, can be solved for. Again, Goodman (1962) is used to understand the relationship between the variance of $\pi(\theta_s)$ and the product of n_c independent $\pi(\theta_i)$.

$$a = \frac{\gamma}{(2+\gamma)\left[\left(\frac{1}{\frac{1}{VR^*}\left(\frac{1}{(1+\gamma)^{2n_c}}\right)}+1\right)^{1/n_c}-1\right]}, \qquad (3.9)$$

$$\begin{aligned} b &= a\gamma, \\ \text{where} \quad \gamma &= \frac{\left(1 - {R^*}^{1/n_c}\right)}{{R^*}^{1/n_c}}. \end{aligned} \tag{3.10}$$

Equations 3.9 and 3.10 can be used to solve for the hyperparameters of the component priors. Simulations were executed to confirm that the product of the n_c Beta(a, b) distributions would result in a Beta(α, β). A total of 12 simulations were executed, three numbers of components, $n_c = 1$, 7, and 42, with four different values of R^* and VR^*. Data used for the simulations are shown in Table 3.2.

Table 3.2: *Four different sets of mean and variance values for system reliability are given in this table. Equations 3.7 and 3.8 were used to find the parameters of $\pi(\theta_s)$ given the mean and variance values. Equations 3.9 and 3.10 were used to solve for hyperparameter values of $\pi(\theta_i)$ given the value of the system variance.*

R*	VR*	$\pi(\theta_s) \sim Beta(\alpha, \beta)$	$\pi(\theta_i) \sim Beta(a, b)$		
			K = 1	K =7	K =42
.8	.01	(12, 3)	(7.11, 1.78)	(7.18, .23)	(7.19, .04)
.5	.05	(2, 2)	(1.66, 1.66)	(1.87, .20)	(1.90, .03)
.2	.10	(.12, 48)	(.27, 1.07)	(.58, .15)	(1.12, .09)
.6	.08	(1.12, .75)	(1.08, .72)	(.58, .15)	(1.21, .09)

Each histogram in Figure 3.2 represents 100,000 draws taken from the product of n_c Beta(a, b) distributions. The lines in Figure 3.2 represent the known densities of the different Beta(α, β) distributions. The histograms closely mirror the known distributions so it can be concluded that Equations 3.9 and 3.10 find parameter values for component prior distributions when $\pi(\theta_s) \sim$ Beta(α, β). Having defined various types of priors that could be useful in reliability research, the next step is to understand how both the likelihoods and priors will be used to determine posterior distributions.

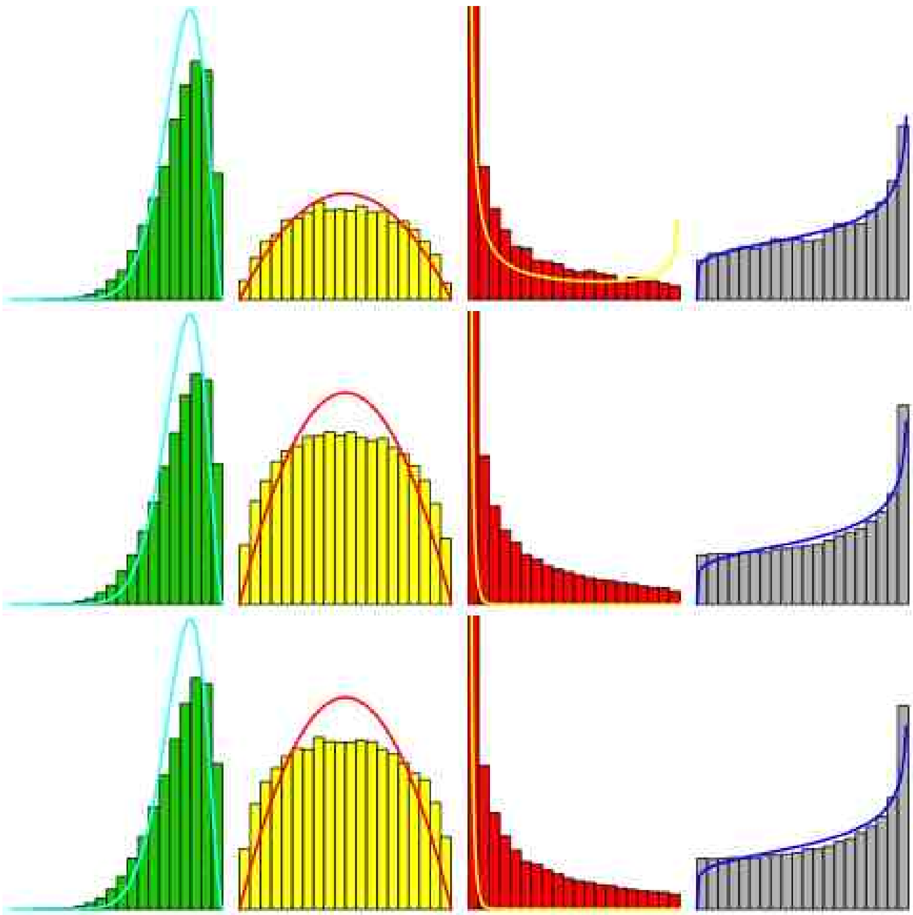


Figure 3.2: *Simulations were executed to verify that the product of independent Beta(a, b) distributions could represent draws from a different Beta(α, β) distribution. The values used for each simulation are shown in Table 3.2. The solid lines represent the known density of Beta(α, β). The histograms represent draws taken from the product of the n_c independent Beta(a, b) distributions.*

3.3 Posterior Distributions, $\pi(\Theta|\mathbf{y})$

From Section 2.2 the posterior distribution is the distribution from which inference can be made about all unknown parameters. The posterior distribution is proportional to the product of the likelihood, $f(\mathbf{y}|\Theta)$, and priors, $\pi(\boldsymbol{\theta})$. The likelihood describes data from both component and system tests resulting in Equation 3.4 from Section 3.1. Two different priors were described in Section 3.2. The first was a non-informative prior on θ_s and n_c independent Beta(a, b) priors for θ_i. The second was a Beta(α, β) prior on θ_s and n_c independent Beta(a, b) priors for θ_i. As Section 3.2 describes, the hyperparameter values for each of the Beta(a, b) priors is dependent on the mean and variance of θ_s and the number of components in the system.

3.3.1 Base Reliability Model

To find the joint posterior distributions for $\boldsymbol{\theta}$, the likelihoods and priors are plugged into Equation 2.2. Equation 3.11 is the posterior distribution for Θ. The parameters in Equation 3.11, a_p and b_p, are constants that are determined by the system prior chosen; either a Uniform(0, 1) or an arbitrary Beta (a, b).

$$\begin{aligned}
\pi(\Theta|\mathbf{y}) \propto & \prod_{i=1}^{n_c}\left(\theta_i^{\sum_j^{n_i} y_{ij}}(1-\theta_i)^{n_i-\sum_j^{n_i} y_{ij}}\right) \times \left(\prod_{i=1}^{n_c}\theta_i\right)^{n_s}\left(1-\prod_{i=1}^{n_c}\theta_i\right)^{n_t-n_s} \\
& \times \prod_{i=1}^{n_c}\frac{\Gamma(a_p+b_p)}{\Gamma(a_p)\Gamma(b_p)}\theta_i^{a_p-1}(1-\theta_i)^{b_p-1} \\
\propto & \prod_{i=1}^{n_c}\left(\theta_i^{\sum_j^{n_i} y_{ij}+a_p-1}(1-\theta_i)^{n_i-\sum_j^{n_i} y_{ij}+b_p-1}\right) \\
& \times \left(\prod_{i=1}^{n_c}\theta_i\right)^{n_s}\left(1-\prod_{i=1}^{n_c}\theta_i\right)^{n_t-n_s}. \qquad (3.11)
\end{aligned}$$

This posterior equation incorporates all of the information that is known or has been collected at both the component and system level. The likelihood incorporates data that have been collected from component and system tests. The priors bring in expert opinion on component/system reliability. The posterior distribution will change by different values of a_p and b_p depending on the system prior. These two terms can be interpreted to be additional successful, a_p, and additional failed, b_p, component tests. As the number of actual test data increases the information added by each prior will be overwhelmed by the data and the results from different priors will converge.

The posterior in this section has been written to accommodate component and system test data. Other types of data might be available and the more information capable of being used, the better. Expanding the base model to incorporate other sources of data is desired. But, before the base model is expanded a Bayesian model that only includes this additional information, known as covariates, will be built.

3.3.2 Covariate Reliability Model

The base model from Section 3.3.1 is useful for analyzing pass/fail data for component and system tests. Any information beyond the final test result does not get used in the base model. This additional information, known as covariates (or explanatory variables), can enhance the fidelity of the model. This section works on understanding how these covariates fit into a Bayesian model.

Each component i may have different types of covariate information available. Therefore each component will have have unique $m \times j$ matrix where m represents the number of covariates for that component and j represents the number of tests. This matrix of covariate information is denoted $X_{i\cdot}$, where X_{ij} is the j^{th} row of $X_{i\cdot}$.

Hamada et al. (2008) suggests using regression models to relate the covariates,

X_i, to θ_i. Regression models allow for the relationship between the likelihood and covariates to be expressed. The logistic regression model is most commonly used to model binary data. The response variable, y_{ij}, is the pass/fail result for each component i at the covariate levels of test j. The logistical regression model relates θ_{ij}, the reliability of component i at the covariate levels of test j, to the covariates through the logit link function

$$\text{logit}(\theta_{ij}) = \log\left[\frac{\theta_{ij}}{1-\theta_{ij}}\right] = {X_{ij}}^T \boldsymbol{\beta_i}. \tag{3.12}$$

$$\text{where} = \begin{cases} i & = 1, \cdot, n_c \\ j & = 1, \cdot, n_i \\ X_i & = n_i \times m_i \text{matrix} \end{cases}.$$

The ${X_{ij}}^T \boldsymbol{\beta_i}$ is the linear expression of relating the covariate matrix of X_i. to the regression parameters, $\boldsymbol{\beta_i}$. Inverting Equation 3.12 results in an expression for θ_{ij}

$$\theta_{ij} = \frac{1}{1+\exp[-{X_{ij}}^T \boldsymbol{\beta}]}. \tag{3.13}$$

Using the information presented in Section 2.2 a Bayesian model can be built to incorporate covariates. As before, the first step is identifying the likelihood. The data, y_{ij}, comes from a Bernoulli distribution. A Bernoulli describes the data since there is only one test for each combination of covariates. The expression for θ_{ij} from Equation 3.13 will be used as the parameter value of the Bernoulli distribution.

$$\begin{aligned} f(y_{ij}|\theta_{ij}, X_{ij}) &\sim \text{Bernoulli}(\theta_{ij}) \\ &= \prod_{i=1}^{n_c}\prod_{j=1}^{n_i}(\theta_{ij})^{y_{ij}}(1-\theta_{ij})^{1-y_{ij}} \\ &= \prod_{i=1}^{n_c}\prod_{j=1}^{n_i}\left(\frac{1}{1+\exp[-{X_{ij}}'\boldsymbol{\beta}_i]}\right)^{y_{ij}} \\ &\times \left(1-\frac{1}{1+\exp[-{X_{ij}}'\boldsymbol{\beta}_i]}\right)^{1-y_{ij}}. \end{aligned} \tag{3.14}$$

Equation 3.14 changes the parameter of interest from θ_{ij} to $\boldsymbol{\beta}_i$. Rather than defining the prior for θ_{ij} a prior for $\boldsymbol{\beta}_i$ must be defined. A common prior for $\boldsymbol{\beta}_i$, is the multivariate Normal distribution. The multivariate Normal distribution spans the real number line so the resulting values of $\boldsymbol{\beta}_i$ can be either positive or negative. This is a desired characteristic of a prior for $\boldsymbol{\beta}_i$ since these parameters represent slopes in a regression model. Zellner's prior is a common way to define the relationship between these parameters.

$$\begin{aligned}\pi(\boldsymbol{\beta}_i) &\sim MVN_p(\boldsymbol{\mu}, \sigma^2(X_i'X_i)^{-1}) \\ &\propto \exp[-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)'(\sigma^2(X_i'X_i)^{-1})(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)];\end{aligned}$$

where the hyperparameters, $\boldsymbol{\mu_i}$ and σ^2 are known and $(X_i'X_i)^{-1}$ represents the inverse of the design matrix for component i.

The full expression of the posterior distribution, given in Equation 3.15, of this covariate model is proportional to the product of the likelihood and prior.

$$\begin{aligned}\pi(\boldsymbol{\beta}_i|X_{ij}, y_{ij}, \sigma^2, \boldsymbol{\mu}_i) \propto \prod_i^{n_c}\prod_j^{n_i} &\left[\left(\frac{1}{1+\exp[-X_{ij}'\boldsymbol{\beta}_i]}\right)^{y_{ij}}\right. \\ &\times \left.\left(1 - \frac{1}{1+\exp[-X_{ij}'\boldsymbol{\beta}_i]}\right)^{1-y_{ij}}\right] \\ &\times \exp\left[-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)'(\sigma^2(X_i'X_i)^{-1})(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)\right]. \qquad (3.15)\end{aligned}$$

Using this posterior distribution inference can be done using different values for n_i and m_i across components. This model is flexible enough to accommodate different m and j values per component by allowing the vector $\boldsymbol{\beta}_i$ to differ in length among components. An advantage of this model is the ability to see how system reliability is affected as covariate information changes. Unfortunately this model does not include any information obtained from the system tests; and if component test data is available that does not have covariate information then it cannot be included. The prior information about θ_s derived in Section 3.2 is also not carried over to this model. The goal of this thesis is to build a Bayesian model that is flexible enough to accommodate multiple sources of data. Combining the base model with the covariate model provides that flexible model.

3.3.3 Full System Reliability Model

In this section the posterior distribution for the full system reliability model will be developed. So far, two models have been developed. The base model which takes data from component and system level tests and computes posterior densities for component and system reliabilities. The covariate reliability model uses covariate information available from component tests and computes component and system reliabilities as a function of the covariates. Each of these models has the ability to incorporate data from a specific source and find posterior distribution for both component and system reliabilities. What if the data available is a hybrid of these data sources? Some component tests may have covariate information while others may not; and we still want to be able to incorporate any information available about the system, using either prior knowledge of system tests. Rather than excluding information by limiting the analysis to using either the base or covariate model, a third model that combines elements of both the first two models will be derived. This third model will be flexible enough to accommodate data from the various covered sources. The posterior distribution of the full system reliability model will begin with the likelihoods and priors from the base and covariate models and slightly modify them.

The first source of data is component tests without covariate information. As in the base model; the data is modeled using a Binomial likelihood and the prior on θ_i is a Beta distribution with hyperparameters, a_p, b_p, determined by $\pi(\theta_s)$. The slight change in Equation 3.16 comes from the index for the number of components in the system. For the full system model, the number of components without covariate information will be denoted n_0.

$$\begin{aligned} f(y_{ij}|\theta_i) &\sim \text{Binomial}(n_i, \theta_i) \\ &= \prod_{i}^{n_0} (\theta_i)^{\sum_j y_{ij}} (1-\theta_i)^{n_i - \sum_j y_{ij}}. \end{aligned} \tag{3.16}$$

$$\begin{aligned} \pi(\theta_i) &\sim \text{Beta}(a_p, b_p) \\ &\propto \prod_{i=1}^{n_0} (\theta_i)^{a_p - 1} (1-\theta_i)^{b_p - 1}. \end{aligned} \tag{3.17}$$

The second source of data is component tests with covariate information. As in the covariate model; the data is modeled using a Bernoulli likelihood and the prior on $\boldsymbol{\beta}_i$ is a multivariate Normal distribution. The slight change to the likelihood in the full system reliability model is the index of i goes from 1 to n_p; where n_p is the number of components with covariate information. The total number of components in this final model is $n_0 + n_p = n_c$.

$$\begin{aligned} f(y_{ij}|\theta_{ij}, X_{ij}) &\sim \text{Bernoulli}(\theta_{ij}) \\ &= \prod_{i=1}^{n_p} \prod_{j=1}^{n_i} \left[\left(\frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]} \right)^{y_{ij}} \right. \\ &\quad \times \left. \left(1 - \frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]} \right)^{1-y_{ij}} \right]. \end{aligned} \tag{3.18}$$

$$\begin{aligned} \pi(\boldsymbol{\beta_i}) = MVN(\boldsymbol{\mu}, \sigma^2(X_i'X_i)^{-1}) \\ &\propto \exp[-\frac{1}{2}\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)'(\sigma^2(X_i'X_i)^{-1})(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)]. \end{aligned} \tag{3.19}$$

The final source of information in the final model is the expression representing the likelihood of system tests. This model uses a series system so the likelihood for system tests can be represented by a binomial with parameter $p = \theta_s = \prod_i^{n_c} \theta_i$; similar to the base model. The p statement gets complicated since information for each θ_i may not come from the same source. These sources need to be combined to properly represent p in the system likelihood. In Equation 3.20 the covariate matrix is denoted X_{il} rather than X_{ij}. The l represents an arbitrary number of tests for each component at equally spaced values across each covariate where l is the same values for all i. Algebraically, this slight modification is necessary when it comes to the computations for the system likelihood. As in the base model; there are two priors of interest for θ_s. Both the Uniform and Beta distributions will be considered to represent $\pi(\theta_s)$. The information from the different system priors is captured in Equation 3.16 as the hyperparameters a_p, and b_p. and therefore determine the hyperparameters of $\pi(\theta_i)$.

$$\begin{aligned}
f\left(\mathbf{y}|\prod_i \theta_i\right) &\sim \text{Binomial}\left(n_t, \prod_i^{n_c} \theta_i\right) \\
&= \text{Binomial}\left(n_t, \prod_i^{n_0} \theta_i \prod_i^{n_p} \prod_j^{n_i} \frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]}\right) \\
&= \left(\prod_i^{n_0} \theta_i \prod_i^{n_p} \prod_j^{n_i} \frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]}\right)^{n_s} \\
&\times \left(1 - \prod_i^{n_0} \theta_i \prod_i^{n_p} \prod_j^{n_i} \frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]}\right)^{n_t-n_s}. \qquad (3.20)
\end{aligned}$$

(3.21)

The joint posterior of the unknown parameters, θ_i and $\boldsymbol{\beta}_i$, for the full system reliability model is proportional to the product of Equations 3.16, 3.17, 3.18, 3.19, and 3.20. Equation 3.22 shows the full expression for the posterior of the full system model.

$$
\begin{aligned}
\pi(\boldsymbol{\theta}, \boldsymbol{\beta}|\mathbf{y}, X_i j) &\propto f(y_{ij}|\theta_{ij}) f(y_{ij}|X_{ij}, \boldsymbol{\beta}_i) f\left(\mathbf{y}|\prod_{i=1}^{n_c} \theta_i\right) \pi(\theta_i)\pi(\boldsymbol{\beta}_i) \\
&\propto \prod_{i}^{n_0} \left[(\theta_i)^{\sum_j y_{ij}+a_p-1}(1-\theta_i)^{n_j-\sum_j y_{ij}+b_p-1}\right] \\
&\times \prod_{i=1}^{n_p}\prod_{j=1}^{n_i}\left[\left(\frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]}\right)^{y_{ij}}\right. \\
&\times \left.\left(1-\frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]}\right)^{1-y_{ij}}\right] \\
&\times \left(\prod_{i}^{n_0}\theta_i \prod_{i}^{n_p}\prod_{j}^{n_i}\frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]}\right)^{n_s} \\
&\times \left(1-\prod_{i}^{n_0}\theta_i \prod_{i}^{n_p}\prod_{j}^{n_s}\frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]}\right)^{n_t-n_s} \\
&\times \exp[-\frac{1}{2}(\boldsymbol{\beta}_i-\boldsymbol{\mu}_i)'(\sigma^2(X_i'X_i)^{-1})(\boldsymbol{\beta}_i-\boldsymbol{\mu}_i)]. \qquad (3.22)
\end{aligned}
$$

Up to this point, several models have been developed to handle various types of data. But, no discussion has occurred on how to apply these models and compute values from the stated posterior distributions. The next section bridges the gap between the theoretical models and real-world application.

3.4 Computation

This section discusses how MCMC simulation techniques can be applied to the posterior models from Section 3.3. Section 2.2 introduced both Gibbs sampling and the Metropolis-Hastings algorithm as MCMC simulation techniques that generate draws from the marginal posterior distributions of each unknown parameter. To use Bayesian algorithms; the complete conditionals for all unknown parameters must be known. The complete conditional for each unknown parameter is proportional to the posterior distribution in terms of the unknown parameter. Any factors that do not involve the parameter of interest in the posterior may be dropped.

The base model has n_c unknown parameters; luckily, the complete conditionals for each θ_i are equal so only one statement needs to be derived. The base model posterior distribution is shown in Equations 3.11. Dropping any factors from this equation that do not involve θ_i result in $[\theta_i]$.

$$[\theta_i] \propto \left(\theta_i^{\sum_j^{n_i} y_{ij}+a_p-1}(1-\theta_i)^{n_i-\sum_j^{n_i} y_{ij}+b_p-1}\right) \times \left(\prod_{i=1}^{n_c}\theta_i\right)^{n_s}\left(1-\prod_{i=1}^{n_c}\theta_i\right)^{n_t-n_s}. \tag{3.23}$$

Equation 3.23 can not be expressed in closed form. Since $[\theta_i]$ does not resemble any known distributions the M-H algorithm will be used to generate draws from the marginal posterior distribution of each θ_i. The complete conditionals for the remaining two models are derived in the same way and result in unnormalized distributions that must be analyzed using M-H algorithms. Equation 3.24 shows the complete conditional for the covariate model in terms of $\boldsymbol{\beta}_i$.

$$[\boldsymbol{\beta}_i] \propto \frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]} \times \exp\left[-\frac{1}{2}(\boldsymbol{\beta}_i-\boldsymbol{\mu})'(\sigma^2(X_i'X_i)^{-1})(\boldsymbol{\beta}_i-\boldsymbol{\mu})\right]. \tag{3.24}$$

Equations 3.25 and 3.26 show the complete conditionals for the full system reliability model in terms of θ_i and $\boldsymbol{\beta_i}$.

$$[\theta_i] \propto \left[(\theta_i)^{\sum_j y_{ij}+a_p-1}(1-\theta_i)^{n_j-\sum_j y_{ij}+b_p-1}\right] \times \left(\prod_i^{n_c}\theta_i\right)^{n_s} \times \left(1-\prod_i^{n_0}\theta_i\prod_i^{n_p}\prod_j^{n_i}\frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]}\right)^{n_t-n_s} \tag{3.25}$$

$$
\begin{aligned}
[\boldsymbol{\beta}] \propto & \prod_{i=1}^{n_{cc}} \prod_{j=1}^{n_i} \left[\left(\frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]} \right)^{y_{ij}} \times \left(1 - \frac{1}{1+\exp[-X_{ij}{}'\boldsymbol{\beta}_i]} \right)^{1-y_{ij}} \right] \\
& \times \exp[-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)'(\sigma^2(X_i'X_i)^{-1})(\boldsymbol{\beta}_i - \boldsymbol{\mu}_i)] \\
& \times \left(\prod_{i}^{n_p} \prod_{j}^{n_i} \frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]} \right)^{n_s} \\
& \times \left(1 - \prod_{i}^{n_0} \theta_i \prod_{i}^{n_p} \prod_{j}^{n_i} \frac{1}{1+\exp[-X_{il}{}'\boldsymbol{\beta}_i]} \right)^{n_t - n_s} . \qquad (3.26)
\end{aligned}
$$

These complete conditionals will be used to analyze various data sets in Chapter 4. Regardless of the data being used, the settings for the M-H algorithms will be similar (if not identical) for each analysis. Section 2.2 gave a four step process of how M-H algorithms work. Step one of this four step process describes the information to be determined prior to running an MCMC algorithm.

The first decision is the number of iterations that each simulation will be run for. Using information gained from convergence diagnositcs, it was found that all algorithms converged after a burn-in of 10,000 iterations. Another 100,000 post burn-in iterations were run from which the marginal posterior densities will be derived. To get to this point where the algorithms all converged required the use of a transformation.

The data sets being analyzed in this paper all contain data for highly reliable components. There are several components within the data sets that contain zero failures. The M-H algorithm for these highly reliable components was showing a tendency to get"stuck" at values incredibly close to 1 for a large number of iterations. The acceptance rates were falling within the desired ranges; but inspection of the trace plots and Geweke diagnostics did not support convergence. A sample trace plot of what was being seen is in Figure 3.3. There were also instances when re-running the algorithm would result in a drastically different acceptance rate; sometimes going from 30% down to .05%. Circumstances such as these led to the conclusion that the algorithms were not converging to a stationary distribution.

To get the chains involving these zero failure data sets to converge a transformation on θ_i is suggested. The logit transformation in Equation 3.27 allows the algorithm to reach stationarity in a reasonable amount of time. Solving Equation 3.27 for θ_i allows θ_i to be expressed as a function of μ_i; as written in Equation 3.28. Writing the logit model in terms of μ_i allows for an easy substitution in the complete conditionals. The M-H algorithm will simulate values for μ_i; once all T iterations have been executed the μ_i's will be transformed back in θ_i's using Equation 3.28.[2]

$$\mu_i = \log\left(\frac{\theta_i}{1-\theta_i}\right). \tag{3.27}$$

$$\theta_i = \frac{\exp[\mu_i]}{1+\exp[\mu_i]}. \tag{3.28}$$

The diagnostics for models run using this transformation support concluding convergence and stationarity. Any data sets which contain zero failure components will be run using this transformation on θ_i.

The next decision to make involves the proposal distributions. Each unknown parameter will have its own proposal distribution. Candidate values for all θ_i's will be drawn from a Normal$(\theta_i^{[t-1]}, \sigma_c^2)\cdot I(0 \leq \theta_i \leq 1)$. Candidate values for all β_i's will be drawn from a Normal$(\beta_i^{[t-1]}, \sigma_c^2)$. Both of the proposal densities have Gaussian properties where the mean is the previous iterations value for the parameter of interest and variance determined by σ_c^2; where σ_c^2 is a user-defined value that determines the efficiency of the algorithm. The σ_c^2 values vary for each dataset and will be reported with each dataset in Appendix B Any other constants or hyperparameters will be determined by previously mentioned methods. The M-H settings used for the analysis in this paper are summarized in Table 3.3

[2]This transformation has the advantage of not allowing any simulated θ_i values being either 0 or 1.

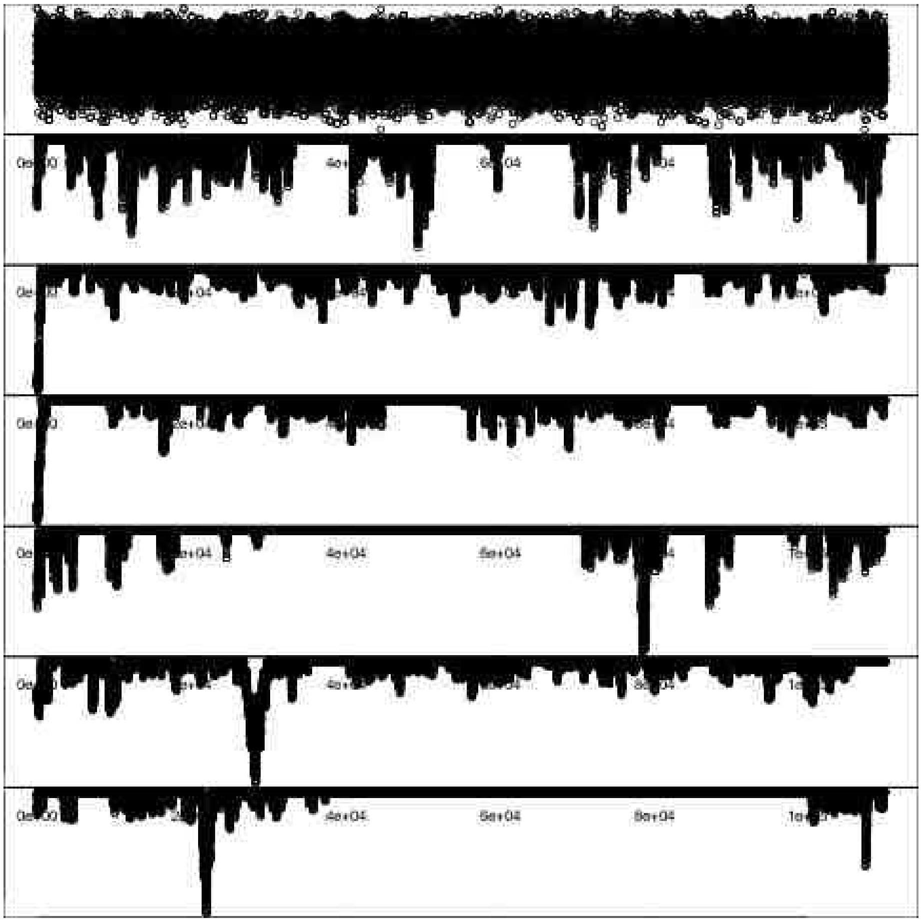


Figure 3.3: An example of trace plot for θ_i simulations that obtained a desired acceptance rate but has clearly not converged. This trace plot does have the desired characteristic of resembling white noise.

A model that has the flexibility to handle data from multiple sources has been developed within this chapter. The full system reliability model in Section 3.3.3 can take pass/fail data from component and system tests, varying levels of covariate information from additional components, and expert opinion about system reliability. The final section of this chapter derived the complete conditionals for all unknown parameters. These complete conditionals bridge the gap between the theoretical models and applied analysis. Finally, this chapter concluds with a brief discussion of the MCMC settings which will be applied to all M-H algorithms in Chapter 4. The analysis will cover several different data sets from simulated data to actual test data

Table 3.3: *MCMC settings which are constant throughout the analysis section of this paper. The number of iterations and the proposal distributions for selecting candidate values will remain the same regardless of the data set being analyzed.*

Setting	Value
Burn-In Iteration	10000
Simulation Iterations	100000
β_i Proposal Distribution	$\text{Norm}(\beta_i^{[t-1]}, \sigma_c^2)$
μ_i Proposal Distribution	$\text{Norm}(\mu_i^{[t-1]}, \sigma_c^2)$
θ_i Proposal Distribution	$\text{Norm}({\theta_i}^{[t-1]}, \sigma_c^2) \cdot I(0 \leq \theta_i \leq 1)$

from an unknown weapon system.

Chapter 4

Results

This chapter presents analysis and results for simulated and weapons system data using the models defined in Chapter 3. Simulating data allows the model to be tested on a range of data that may not be available from real world tests. Four different sets of simulated data will be considered: an $n_c = 5$ system, an $n_c = 42$ system, an $n_c = 1$ component with covariates, and an $n_c = 5$ system where one component has covariate information and four do not. The weapon system test data contains seven different sets of data. Each data set uses the same test results but varies what is defined to be a "failure". The weapon system test data will allow the model to be applied to a real world problem.

The different data sets will be analyzed using three different system priors. A highly informative Beta prior for a highly reliable system, a less informative Beta prior for a highly reliable system, and a Uniform prior. The values of the component prior hyperparameters, a_p and b_p, will vary for each prior and data set. The $\pi(\theta_s)$ and $\pi(\theta_i)$ resulting from these three priors are in Table 4.1.

After running each model, diagnostics were checked to verify that the algorithm converged and that the model fits the data. Convergence diagnostics used in this paper include those listed in Section 3.4. Convergence was concluded for all the models in this thesis using the information obtained from these diagnostics. Acceptance rates fell between the desired 15-40%; the trace plots represented white noise; the majority of Geweke diagnostics showed no significant difference between the mean of the first 10% of the chain compared to the mean of the last 50% of the chain; and all Gelman and Rubin diagnostics over three chains resulted in a point estimate of 1 with an upper CI level also equal to 1. The specific diagnostics for each model are listed in Appendix B.

Model fit was checked through the Bayesian χ^2 lack-of-fit test. The largest χ^2 statistics obtained from all 11 data sets, came from the third weapons system data set, was 5.08. This value is less than the $\chi^2_{6,.95}$ of 12.59. Since 5.08 was the maximum χ^2 statistic from all models, there is little evidence to suggest that the models derived in Chapter 3 do not provide an adequate fit to any of the data sets. The final diagnostic used checked the $P(z^* < z^{obs})$; where z^* is the χ^2 statistic of simulated test data and z^{obs} is the χ^2 statistic of observed data. The χ^2 statistics were calculated using

$$\sum_{i=1}^{n_c} \frac{(y_i - \theta_i^{[t]})^2}{\theta_i};$$

where y_i is the total number of successful test for component i (either simulated or observed) and θ_i is the t^{th} reliability estimate for component i. This statistic is calculated over all t and the difference in the proportion of statistics for the simulated data minus the observed data is reported. If the model provides simulated values close to the observed values then histograms plotting the $z^* - z^{obs}$ should be centered around zero. The sixth weapons system data set resulted in the largest probability of seeing a simulated value greater than an observed value; where the probability was equal to 44.38%. The difference between the two χ^2 statistics for each $\theta_i^{[t]}$ are plotted as a histogram in Figure 4.1. This histogram depicts a fairly normal distribution centered around zero; so once again there is little evidence to support a lack-of-fit.

The diagnostics used in this thesis provide evidence in support of both algorithm convergence and model fit. No cause for concern was uncovered through the diag-

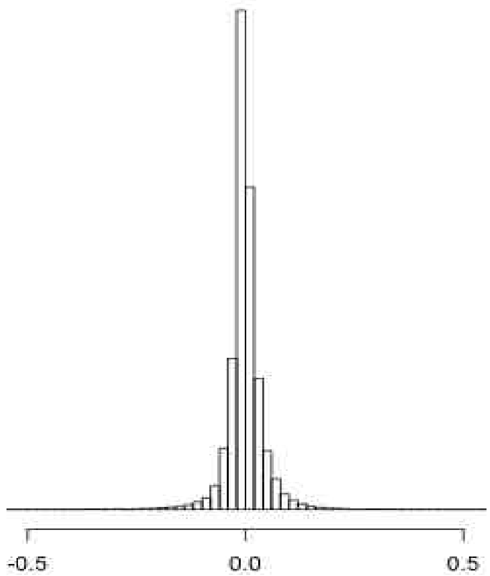


Figure 4.1: The difference between the χ^2 statistics that resulted from simulated data and the χ^2 statistics that resulted from observed data are plotted as a histogram. If the model fits the data appropriately, there should be minimal differences between the two statistics and the histogram should resemble a normal distribution centered around zero. This histogram shows that the χ^2 statistics for the sixth weapons system data set are not different enough to conclude a lack-of-fit.

nostic analysis. The positive results of the diagnostic analysis allow for analysis of each of the eleven data sets to commence.

Table 4.1: *The system priors, $\pi(\theta_s)$, used in this chapter are shown in the first two lines of this table. The resulting component priors, $\pi(\theta_i)$, make up the remainder of this table. The hyperparameter values for each $\pi(\theta_i)$ replace the a_p and b_p in the complete conditionals. $\pi(\theta_s)$ will take on three forms based upon the level of information known about the system. $\pi(\theta_i)$ will take on several different forms depending on $\pi(\theta_s)$ as well as the number of components in the system.*

	$R^* = .99, VR^* = .0025$	$R^* = .98, VR^* = .01$	Non-Informative
	$\pi(\theta_s) \sim$ Beta(2.9304, .0296)	$\pi(\theta_s) \sim$ Beta(.9408, .0192)	$\pi(\theta_s) \sim$ Uniform(0, 1)
Simulated Data			
$n_c = 5$	$\pi(\theta_i) \sim$ Beta(2.254, .0052)	$\pi(\theta_i) \sim$ Beta(1.108, .0051)	$\pi(\theta_i) \sim$ Beta(1.1685, 1.73)
$n_c = 42$	$\pi(\theta_i) \sim$ Beta(2.004, .0005)	$\pi(\theta_i) \sim$ Beta(.9902, .0005)	$\pi(\theta_i) \sim$ Beta(1.2005, .0199)
Weapon System Test Data			
Datasets 1-2, $n_c = 1$	$\pi(\theta_i) \sim$ Beta(3.807, .076)	$\pi(\theta_i) \sim$ Beta(1.813, .0725)	$\pi(\theta_i) \sim$ Beta(1, 1)
Datasets 3-7, $n_c = 7$	$\pi(\theta_i) \sim$ Beta(2.169, .0034)	$\pi(\theta_i) \sim$ Beta(1.068, .0034)	$\pi(\theta_i) \sim$ Beta(1.179, .1227)

4.1 Simulated Data

4.1.1 5 Component System – No Covariates

The data for this system was simulated using randomly generated values for n_i, and a random binomial with a high probability of success to determine the test results, y_{ij}, for each component. The test data is from component and system levels with no covariate information; complete data is shown in Table 4.2. The high reliability of the components and system caused convergence issues so the μ_i transformation described in Section 3.4 was used to ensure the algorithm converged to a stationary distribution. The algorithm was run using the MCMC settings from Table 3.3 and the σ_c values in Table B.2.

Table 4.2: *Simulated data for the 5 component system. This table shows the total number of tests for each component, the number of successes and failures, as well as the reliability of each component calculated using the expression* $\frac{Successes}{Total\ Tests}$.

	Total Tests	Successes	Failures	Reliability
1	11	11	0	1
2	47	46	1	0.98
3	34	34	0	1
4	17	16	1	0.94
5	45	40	5	0.89
System	10	9	1	0.90

The M-H algorithm generated draws from the marginal posterior densities for each θ_i. Using the draws for θ_i, the marginal posterior density of θ_s was calculated using Equation 3.2. Plots of the resulting marginal posterior densities are shown as boxplots in Figure 4.2. The first five boxplots represent the densities for each of the components and the sixth boxplot represents the posterior density for θ_s; this information is also provided as numerical summaries in Appendix C; these summaries are provided for all the following models in the appendix.

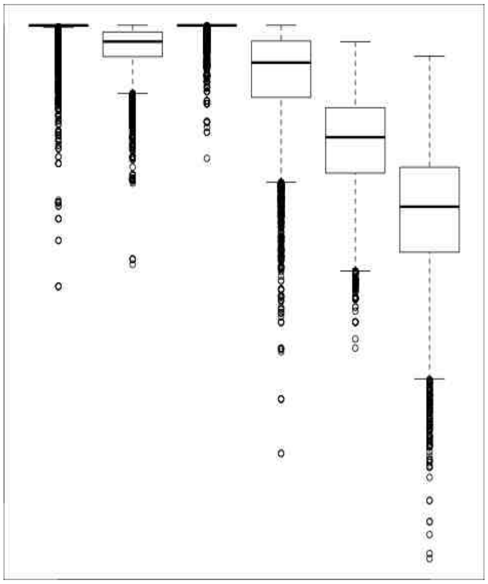
The posterior analysis focuses on the statistic of interest, system reliability, θ_s.

Table 4.3 gives the mean, standard deviation, and highest posterior density (HPD) intervals for each θ_s. The HPD is a Bayesian interval that consists of points such that $f(\theta_s|\mathbf{y}) \geq c$ where $f(.|\mathbf{y})$ is the posterior density of θ_s and c is chosen so that the region has the desired posterior probability, (Cox & Hinkley, 1979). For this thesis, the 95% HPDs will be reported for all θ_s estimates. The HPD is the shortest segment from $f(.|\mathbf{y})$ that contains 95% of the mass. This interval can be interpreted by saying that there is a 95% chance that the θ_s estimate will be within this interval given the observed data; the shorter the interval the less uncertainty that exists surrounding the point estimate.

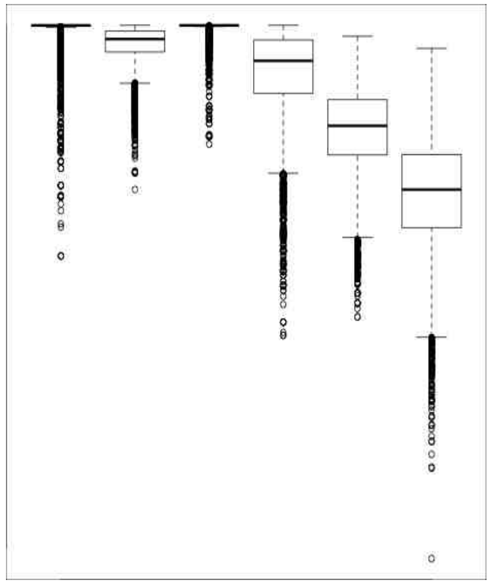
Table 4.3: *This table shows a comparison of the numerical summaries for θ_s given the different system priors used in this analysis.*

			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	Std Dev	LB	UB
Beta(2.931, .029)	0.8418	.0538	.7349	.9381
Beta(.941, .019)	0.8374	.0538	.7313	.9356
Uniform(0, 1)	0.8378	0.0543	.7335	.9347

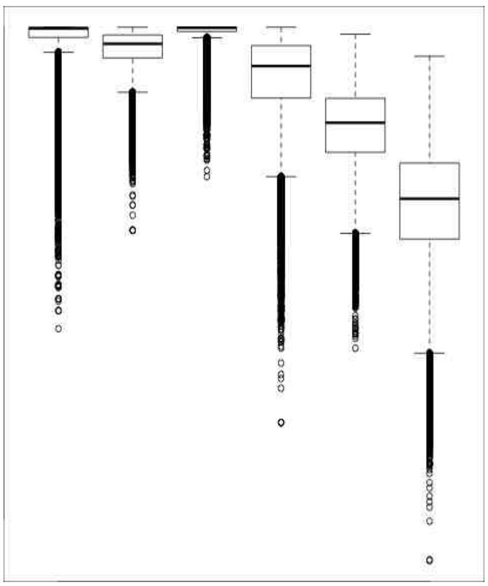
The differences in the posterior summaries for the three priors appears to be minimal. The HPD intervals have similar bounds with similar lengths for the three models. The fourth boxplot in Figure 4.2 shows the densities of the three θ_s plotted side-by-side. This figure highlights the similarities between the three different densities for each θ_s. Comparing the simulated values across the three marginal posterior distributions shows that there is a 50% chance that any of the three models will provide higher estimates for θ_s over any of the other models given the data. These results show that the amount of data introduced to the model through the component and system tests is enough to overwhelm any information brought into the model by the prior.



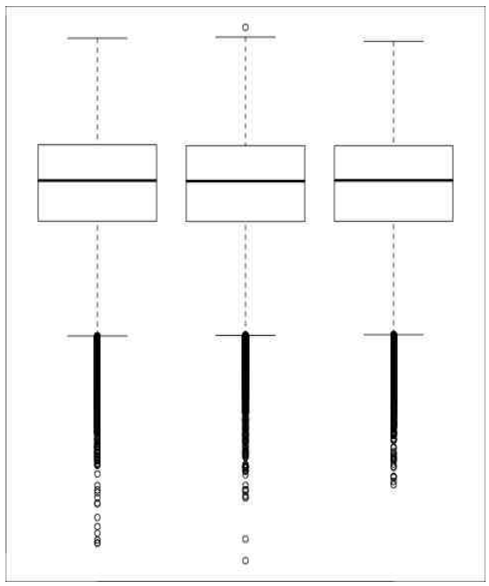
(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)



(c) $\pi(\theta_s) \sim$ Uniform(0, 1)



(d) System Reliabilities

Figure 4.2: The marginal posterior densities for θ_i and θ_s are plotted as boxplots. Each of the first three plots represent the densities obtained using one of the three system priors, as indicated in the subcaption. The fourth plot gives a side-by-side comparison of the θ_s marginal posterior densities.

4.1.2 42 Component System – No Covariates

The data for this $n_c = 42$ system comes from one of the weapon systems data sets that has been replicated six times. Since the data is replicated; only the first seven original data points are shown in Table 4.4 One of the components in the original data set has a relatively high level of failure; 3 of 12 tests. Since the 42 component system has replicated the original data set 6 times, this high-failure component shows up in the 42 component system 6 times. The analysis of this system will show how various priors affect the final system reliability given a relatively large number of low reliability components. Six of the components are considered "high-failure" but the remaining 36 have zero failures. Because of the high reliability of several components the μ_i transformation discussed in Section 3.4 will again be used to deal with the large number of highly reliable components. Posterior draws were simulated from the 42 components using an M-H algorithm with the MCMC settings from Table 3.3 and the σ_c values from Table B.3.

Table 4.4: *This data represents the test data for the first 7 components of the 42 component system. The data used in the analysis repeated this data set 6 times to result in a total of 42 components. The test results for component 1; are the same for component 8, 15, etc... This table shows the total number of tests for each component, the number of successes and failures, as well as the reliability of each component calculated using the expression* $\frac{Successes}{Total\ Tests}$

	Total Tests	Successes	Failures	Reliability
1	12	9	3	0.75
2	14	14	0	1
3	49	49	0	1
4	64	64	0	1
5	36	36	0	1
6	20	20	0	1
7	7	7	0	1
System	10	9	1	0.90

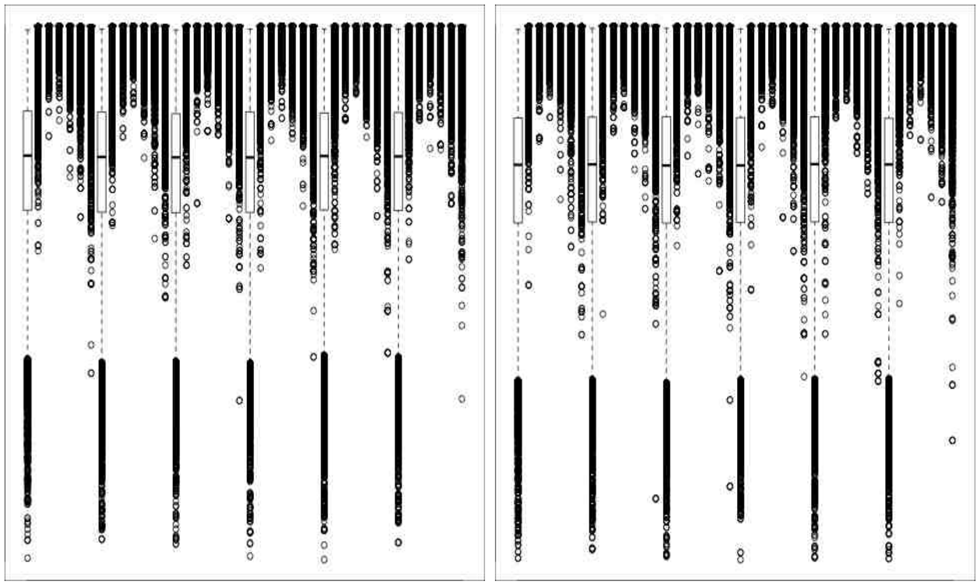
The draws from the M-H algorithm represent the marginal posterior densities of all θ_i. The marginal posterior density for θ_s was obtained using Equation 3.2. The marginal posterior densities for each of the 42 components are plotted as the first three boxplots in Figure 4.3. The fourth boxplot in Figure 4.3 shows the densities for θ_s plotted as side-by-side boxplots.

The main statistic of interest in this research is the reliability of the system, not necessarily the individual components, so side-by-side comparisons of θ_s based upon the different priors are given in Table 4.5 and plotted as the fourth boxplot in Figure 4.3. The estimates for θ_s compared to the 5 component system are much lower. The increase in the number of components appears to have had a significant impact on the posterior distribution for θ_s, regardless of any prior information. As the number of components in a system increases, i.e. the system gets more complex, the reliability of each component must approach 1 to keep the overall estimate for θ_s as a suitable level.

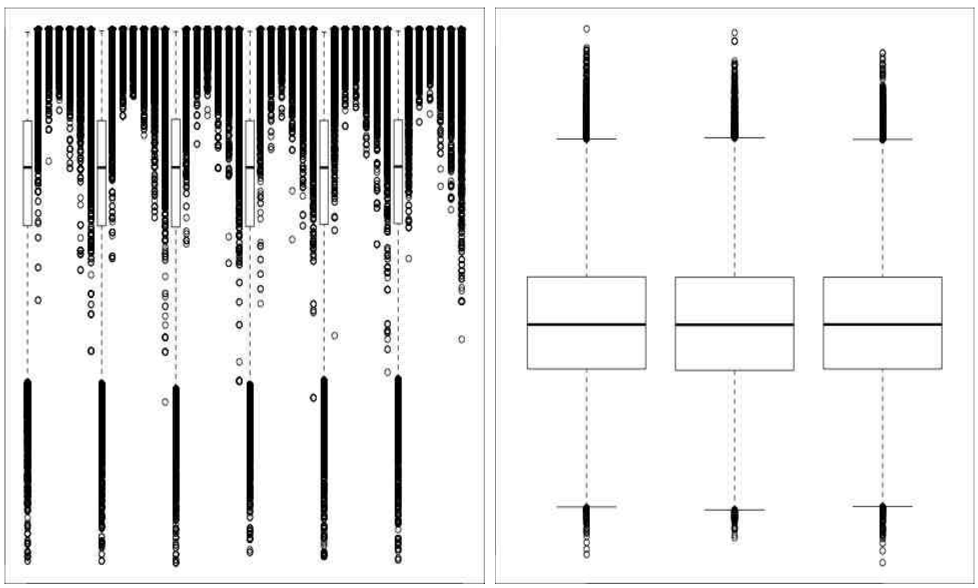
The highly informative Beta(2.9304, .0296) system prior gives the highest estimate for θ_s; however, there is no notable difference between the θ_s estimates or the range of the HPD intervals. Comparing the draws from each of the three marginal posterior distributions shows that the highly informative prior will result in better estimates for θ_s 52.3% of the time over both the low informative Beta prior and non-informative prior; which is not a notable difference between the two models. The conclusion that can be drawn from this data set is the impact that the number of components has on θ_s. As in the 5 component system, the amount of data overwhelms the prior information which is why the resulting summaries are so similar

Table 4.5: *This table shows a comparison of the numerical summaries for θ_s given the different priors used in this analysis.*

			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	σ^2	LB	.975%
Beta(2.931, .029)	0.3741	0.0791	0.2255	0.5299
Beta(.941, .019)	0.3738	0.0805	0.2208	0.5270
Uniform(0, 1)	0.3728	0.0785	0.2208	0.5293



(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296) (b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)



(c) $\pi(\theta_s) \sim$ Uniform(0, 1) (d) System Reliabilities

Figure 4.3: The first three plots show the marginal posterior densities of the 42 component reliabilities as boxplots. Each plot represents the results from using a different prior for system reliability. The fourth boxplot shows side-by-side comparisons of the three densities for each θ_s.

4.1.3 1 Component System – With Covariates

This section introduces determining system reliability, θ_s, as a function of covariates. The covariate of interest for this thesis is age. To use age as a covariate, a sequence of non-destructive tests are performed measuring the response variable of interest, y_{ij}, for each age, (Kelly & Vander Wiel, 2006). Fifty tests were simulated using a Binomial distribution with $p = \theta_{ij}$; where θ_{ij} was calculated using $\boldsymbol{\beta} = (3, -5)$ and Equation 3.13. The negative slope on β_1 was used to generate test data that had a higher propensity of failing at older ages..

The simulated test results are plotted in Figure 4.4. From this plot it is evident there is an increase in the number of failures as the component ages. Seeing this trend in real-life data would lead one to include the covariate, age, in the model.

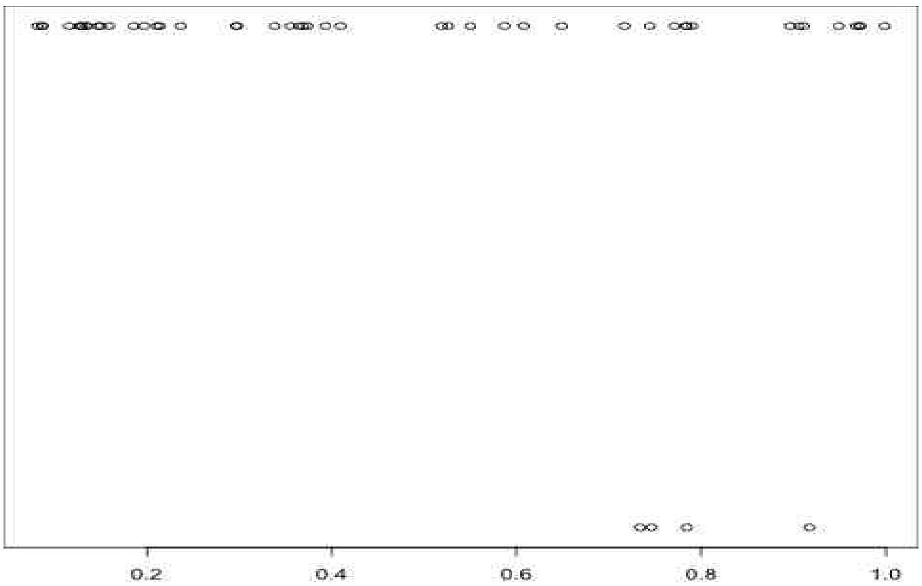


Figure 4.4: This plot shows the 50 test results of the component as a function of age, a success is plotted as a 1 while a failure is plotted as a 0. One data point was recorded for each age. The overall number of failures is small but all the failures occur at older ages. This plot provides evidence that the covariate of age might affect reliability and should be included in the model.

Not having any other component or system test results, the M-H algorithm was run using the complete conditional from Equation 3.24, the MCMC settings from

Table 3.3 and the σ_c in Table B.5. The prior for this model is on $\boldsymbol{\beta}$ and therefore the information provided from $\pi(\theta_s)$ is not included in this model and only one analysis will be run on this data set. The numerical marginal posterior summaries for $\boldsymbol{\beta}$ are given in Table 4.6 while the posterior densities are plotted as boxplots in Figure 4.5.

Table 4.6: *Marginal posterior distribution summaries of the $\boldsymbol{\beta}$ parameters for a 1 component covariate model.*

			Highest Posterior Density	
β_i	Expected Reliability	Std Dev	LB	UB
0	3.4223	0.5042	2.4990	4.4999
1	-1.0013	0.5216	-2.0484	0.0059

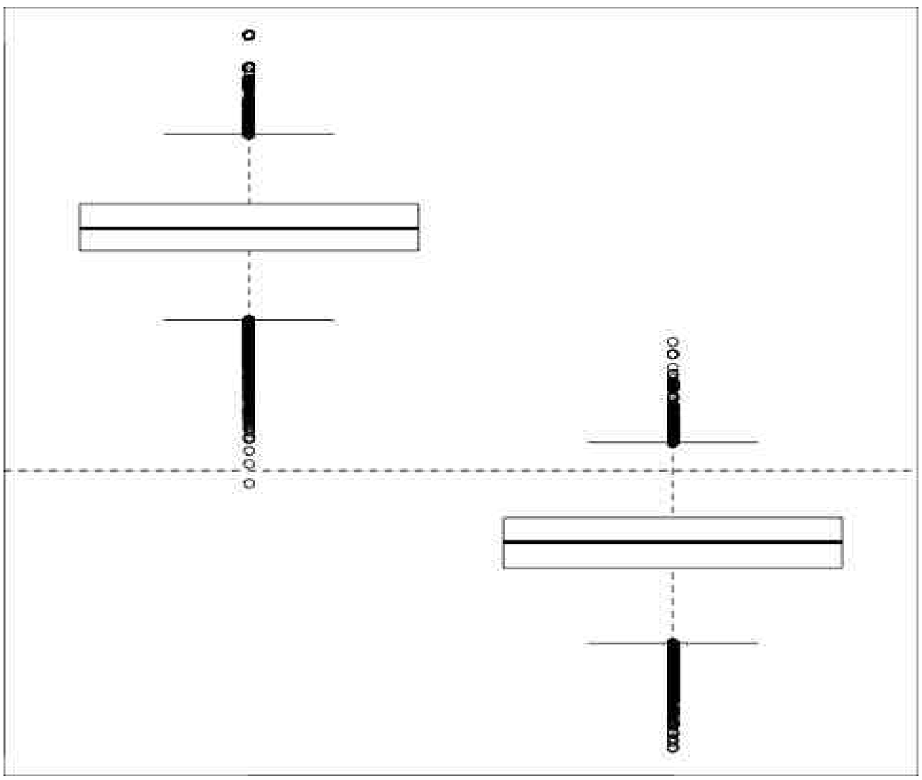


Figure 4.5: Posterior densities of the β_i parameters. The second boxplot represents the density of β_1; the parameter for age. The concentration of the second boxplot below zero, represented by the dotted line, confirms that age negatively affects θ_s.

The point estimate for the β_1 parameter is negative, -1.0013, with a 95% HPD interval that barely contains zero (upper limit of 0.0059). This provides evidence that there is a high probability confirming that age does have a negative effect on reliability. This is an important conclusion; however, the interest of this paper is using various models to determine θ_s. The $\boldsymbol{\beta}$ parameters need to be converted to reliabilities. Equation 3.13 can be used to convert the $\boldsymbol{\beta}$ parameters into the desired reliabilities. The estimate for θ_i will also represent θ_s because the system only has one component. The numerical posterior summaries of θ_s at different ages is shown in Table 4.7. In this analysis, age = 0 is considered the birth of the component; while age=1 is considered the age the component is scheduled to be retired.

Table 4.7: *This table shows how the reliability of a one component system degrades it ages. The ages shown here go from component conception (age=0) to component retirement (age=1).*

			Highest Posterior Density	
Age	Expected Reliability	Std Dev	LB	UB
0.00	0.9652	0.0195	0.9328	0.9925
0.25	0.9565	0.0224	0.9201	0.9886
0.50	0.9450	0.0275	0.9006	0.9835
0.75	0.9298	0.0355	0.8729	0.9811
1.00	0.9099	0.0476	0.8308	0.9791

Table 4.7 shows the reliability of the system degrading from 96.5% to 90.9% as it ages. This degradation is an expected result, what might not have been expected is the increase in the range of HPD intervals as the component ages. This increase of uncertainty becomes more apparent when plotted in Figure 4.6. Each consecutive age gives on average a better estimate 64.3% of the time when compared to the density for one age older. The density for age=0 will give a higher estimate for θ_s 95.0% compared to the density for age=1; this percent continues to decrease for each age interval compared to age=1 until it gets to age=.75 providing better estimate 63.3% of the time. From this model it can be concluded that as time passes and

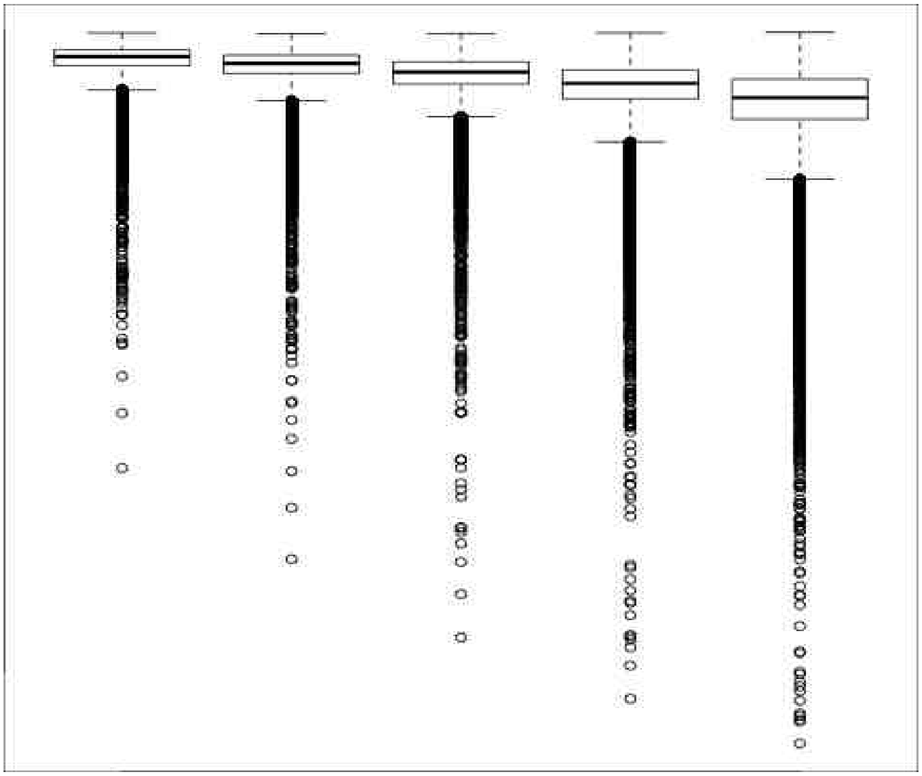


Figure 4.6: The five boxplots represent θ_s across various ages. The boxplots show a decrease in the expected reliability of the system at each age. The increasing spread across ages shows how uncertainty in θ_s estimates increases as the system ages and failures are introduced into the model.

failures are introduced to the model the estimates for θ_s decrease as the uncertainty in the estimate increases.

4.1.4 Full System

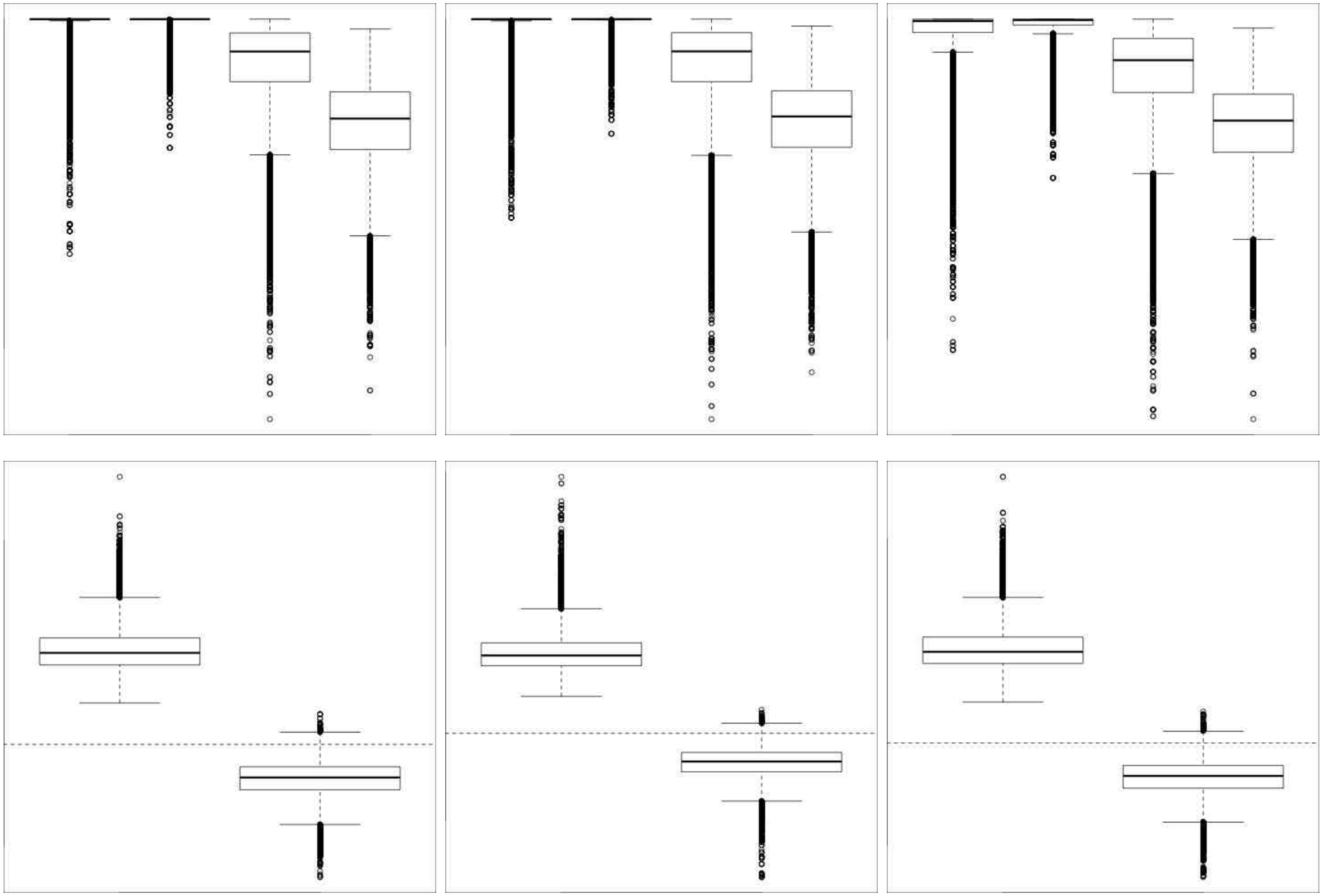
The fourth and last data set combines all three types of data discussed into the full system reliability model. The full system reliability model will be applied to a 5 component system, where one of the five components has covariate information, age, associated with each of its tests. The remaining four components have no covariate information associated with the test data. The data comes from Section 4.1.1 and replaces component 2 with the covariate data generated in Section 4.1.3. Using similar data sets will allow comparisons to be made across the three different data sets. This system also uses the same full system test data is in Section 4.1.1.

To determine θ_s as a function of both θ_i and age an M-H algorithm was run using the settings from Table 3.3 and the σ_c values shown in Table B.7. The μ_i transformation was used to compensate for the highly reliable components in this analysis.

The draws generated from the M-H algorithm allow posterior summaries to be derived for the four θ_i's, β_0 and β_1. The densities for each of the four θ_i parameters and two $\boldsymbol{\beta}$ are represented as boxplots in Figure 4.7. From this figure it an be noted that θ_4 has the lowest reliability estimates, which given that this component has 5 failures recorded out of 45 tests is expected. However, the posterior density for θ_3 appears to have the largest spread. Component 3 has only one failure, but the fewest tests, 17. This result leads to the conclusion that a smaller number of tests can increase the uncertainty surrounding the point estimates of θ_i; especially if a failure is present. There are no significant differences between these four densities than those from the Component 5 model; replacing component 2 with the covariate component did not appear to have any effect on the posterior distributions for the remaining 4 θ_i's.

The densities for the $\boldsymbol{\beta}$'s confirm the relationship between reliability and age; the expected value of β_1, the parameter for the covariate age averaged over priors is -1.7

with HPD interval of (-3.6, -0.23). There is a 95% chance that the point estimate for β_1 is negative regardless of the system prior chosen and negatively affects the reliability estimates. The β_1 estimate for the full system model is conclusively negative when the covariates are combined with additional test information as opposed to in the one component covariate model where a slight bit of ambiguity existed around the point estimate for β_1.

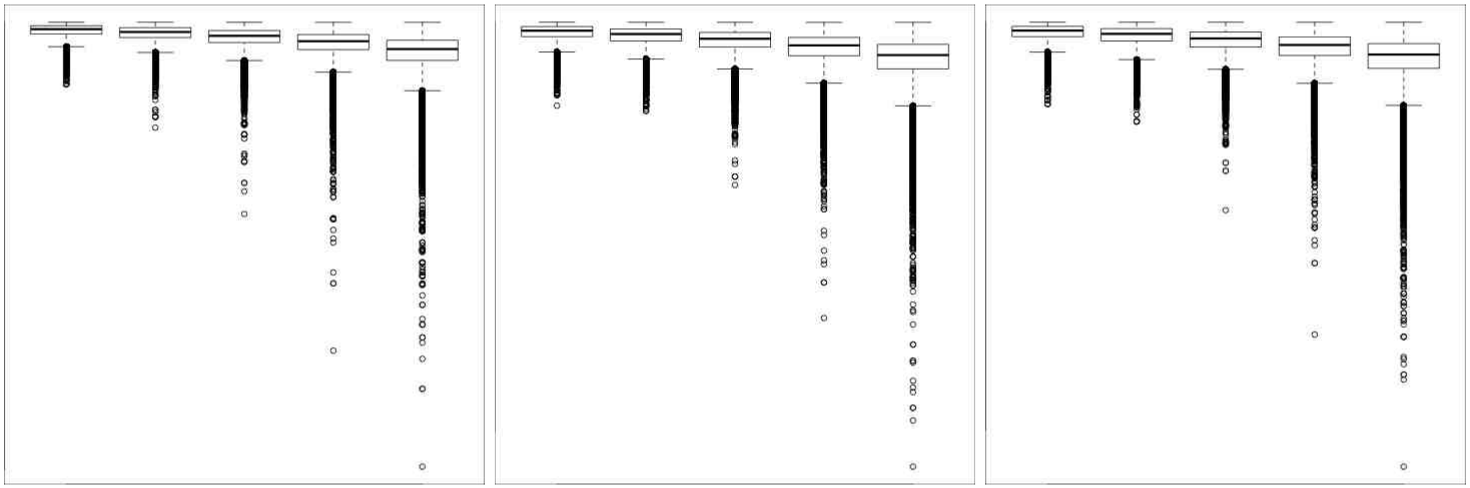


(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296) (b) $\pi(\theta_s) \sim$ Beta(.9408, .0192) (c) $\pi(\theta_s) \sim$ Uniform(0, 1)

Figure 4.7: The six plots show the posterior distributions as boxplots for θ_i and β_i by the different priors. The top plots represent the posterior densities for θ_i. The bottom plots represent the posterior densities for β_i where zero is represented by a dotted line.

Comparing the point estimates given in Table C.1 for θ_1-θ_4 to the point estimates for θ_1, $\theta_2 - \theta_5$ from the 5 component model show no difference. Comparing the posterior distributions for the two models there is only a 50.5-53.1% chance that the full system model will provide better estimates for any of the 4 components than the 5 component model. There is little evidence that incorporating the covariate information into one of the components has any effect on the estimates of the remaining components.

But does including the system information/tests in the model affect the estimates for the covariate component, θ_5? Using Equation 3.13 to relate $\boldsymbol{\beta}$ to θ_{ij} the marginal posterior distributions can be obtained for the fifth component as a function of age. The reliability of this component will be determined at it's youngest point, age = 0; it's oldest point, age = 1; and the quarter intervals in between, age = .25, .5, and .75. Using the estimates for θ_5 and the definition of a series system the reliability of the system can be obtained at the same 5 age points. The marginal posterior summaries for θ_5 based upon the three system priors are plotted as boxplots in Figure 4.8. As the component ages the expected reliability is shown to decrease from around 98.5% to 94.4% with an increase in the HPD interval length, regardless of the prior. The point estimates for θ_5 are higher than they were in the one component covariate model where they ranged from 96-90%. The higher estimates can be attributed to the including the system priors and the system tests in the model. The confidence from an obtained reliability point estimate will decrease the longer the component has been fielded.



(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296) (b) $\pi(\theta_s) \sim$ Beta(.9408, .0192) (c) $\pi(\theta_s) \sim$ Uniform(0, 1)

Figure 4.8: The densities shown in these three plots represent the reliability of θ_5 at 5 equally spaced ages. Each plot represents a different system prior.

It is evident that the reliability of θ_5 will degrade over time and the uncertainty in that reliability estimate increases as the component ages, but how does this affect the reliability estimates for θ_s? By combining the data from θ_1 through θ_4 with the data generated for the 5 data points of θ_5 a complete picture of θ_s can be drawn as a function of age.

The system posterior summaries are shown in Table 4.8. Across all three priors, Table 4.8 shows about a 4% decrease in reliability as the system ages and relatively stable HPD interval lengths; only increasing in about 2% over the age of the component. The θ_s summaries illustrate that the uncertainty gained about reliability degradation as a function of age from θ_5 is decreased when combined with the static reliability estimates from the remaining θ_i's.

Figure 4.9 presents this data in a different way. The three estimates for θ_s as a function of age are plotted as a line graph with 95% HPD intervals. This plot shows the decrease in reliability as the system ages and also shows a constant level of uncertainty across all ages; which confirms the findings from Table 4.8. The estimate for θ_s coming from the non-informative prior provides the overall lowest estimate with the smallest lower bound on the HPD interval. The two Beta priors provide near equal estimates for θ_s. Averaging over the priors, when the system is at age $= 0$ the model will give an estimate for system reliability higher than when age $=$ 1 74.1% of the time. As the age increases the probability of each consecutive age compared to age=1 decreases down to 58.2% when comparing the distribution for age=.75 to age=1. Finally, comparing this model to the model for the 5 component system - no covariates it can be seen which model will provide better estimates for θ_s. Comparing the 5 component system reliability distribution to the system reliability distribution for age=0 results in the first model only providing higher estimates for θ_s 44.4% of the time. As the system ages and the two models are compared when age=1; the first model will provide higher estimates for θ_s 64.6% of the time. As these comparisons show, the fidelity added to the model through the use of covariates

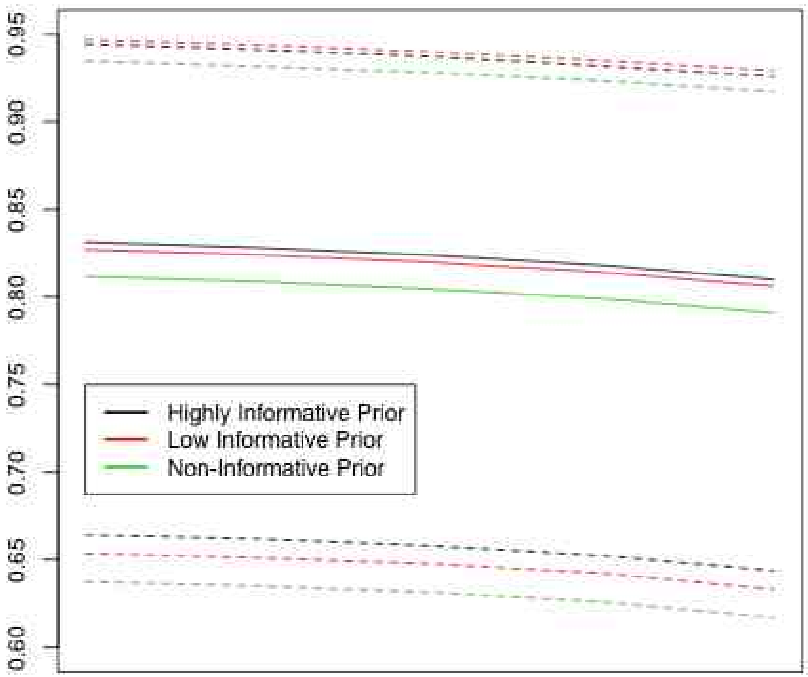


Figure 4.9: The three solid lines represent θ_s as a function of age depending on the prior reliability distribution chosen for θ_s. The dotted lines are the respective 95% credible intervals for each estimate of θ_s.

does provide better estimates; especially in the earlier years of a systems life. Even when the estimates for θ_s might be lower, the overall level of information obtained through using the full system model and including covariates is desired.

Table 4.8: *Numerical posterior distribution summary statistics are shown for θ_s using the posterior distributions for $\theta_1 - \theta_5$. Each table represents the results based upon different system priors. Each line represents the posterior system summary information for different ages.*

			Highest Posterior Density	
	Expected Reliability	Std Dev	LB	UB
0.00	0.8512	0.0518	0.7494	0.9449
0.25	0.8466	0.0518	0.7438	0.9401
0.50	0.8396	0.0522	0.7342	0.9322
0.75	0.8288	0.0534	0.7225	0.9251
1.00	0.8121	0.0573	0.6993	0.9147

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Highest Posterior Density	
	Expected Reliability	Std Dev	LB	UB
0.00	0.8520	0.0511	0.7510	0.9439
0.25	0.8473	0.0511	0.7479	0.9414
0.50	0.8404	0.0515	0.7391	0.9339
0.75	0.8297	0.0527	0.7238	0.9235
1.00	0.8131	0.0566	0.7017	0.9144

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Highest Posterior Density	
	Expected Reliability	Std Dev	LB	UB
0.00	0.8283	0.0569	0.7154	0.9318
0.25	0.8239	0.0569	0.7090	0.9256
0.50	0.8171	0.0571	0.6995	0.9173
0.75	0.8066	0.0580	0.6923	0.9143
1.00	0.7904	0.0613	0.6668	0.8992

(c) $\pi(\theta_s) \sim$ Uniform(0, 1)

4.1.5 Simulated Data Summary

Analysis of the four simulated data sets has proven that the models developed in this paper can address a variety of different data types. This section has shown how combinations of the three data types and different system priors discussed in this paper can be used by the reliability models derived in Chapter 3.

One of the biggest take-aways from this section is the importance of the volume of data. The four datasets used several data points collected from component and/or system tests. The volume of information available minimized the difference in estimates between the system priors. Having the data eventually overwhelm the priors is desirable because this ensures consistency of the posterior distributions. The influence of the prior distribution is overwhelmed as the volume of data increases. The results were not significantly different between the priors, but they did provide consistent results: highly informative giving the highest estimates for θ_s with the smallest HPDs and the non-informative giving the smallest estimates for θ_s with the largest HPDs; these differences would be even greater in the event of reduced volume of test data.

The level of uncertainty, or the spread of each posterior density, varied greatly under three different scenarios. The first scenario was if any failures were recorded. Comparing the distributions of θ_4 to the distributions of θ_1 and θ_2 in Figure 4.7 helps draw the conclusion that a component with failures will have a larger level of uncertainty associated with the estimate of its expected reliability; this uncertainty gets carried through when calculating θ_s estimates. From the same figure, looking at the distribution for θ_3 highlighted the increased spread when the number of tests were small. Even though θ_3 had less failures than θ_4; the amount of data available for θ_4 was able to keep the spread of the marginal posterior distribution for θ_4 smaller than that of θ_3. The third scenario involves the results from using the covariate age. The longer a component has been out in service; the wider the credible interval will be surrounding the point estimate for θ_s.

The final conclusion from the analysis of the simulated data sets comes from the full system reliability model. The analysis in this model showed an almost 6% decrease in expected reliability for θ_5 as it aged from 0 to 1 with increasing level of uncertainty. However; aggregating the information for θ_5 with the information from $\theta_1 - \theta_4$ into an estimate for θ_s causes a lot of that additional information to get lost. The amount of information from the static estimates for $\theta_1 - \theta_4$ minimize the impact of θ_5. The information for θ_5 is still present in the model and shows up in a decrease in θ_s across ages. Comparing this model to the first model run, 5 component system - no covariates, showed just how much more fidelity was gained by being able to view θ_s as a function of age. This example shows the importance of collecting and using as much information as possible.

The last four sections tested the models that were developed in Chapter 3. The results from the model have been consistent with what one would expect given the types of data and the types of system priors used in each analysis. Knowing that the model derived in this paper is providing useful and appropriate insights into the reliability of systems the model can be applied to a real-world problem set using test results from a weapons system.

4.2 Weapons System Test Data

The following section compares the results of seven data sets from component tests on an actual weapon system. The first five data sets represent test results from a 7 component weapon system, the final two data sets contain only one component. For the weapon test data there are no system tests or covariate information available to include in the model. The data sets were created from the same set of tests, but what constituted a failure was defined differently for each data set. The change in the number of components is due to the way researchers defined the system structure, from here referred to as the component block. The seven sets of data are shown in Table 4.9.[1] Besides calculating θ_s for the weapon system; the results will help find the best way to define a "failure", given that a government agencies intent is to report highly reliable systems.

Table 4.9: *The test results for the seven weapon data sets. Each data set represents the use of a different definition for what constitutes a failure. Each data set also represents a different definition of a component block; which is why five of the data sets have seven components and two have one component.*

		Data Set 1		Data Set 3		Data Set 3	
	Total Tests	Successes	Failures	Successes	Failures	Successes	Failures
1	12	9	3	6	6	12	0
2	14	14	0	14	0	13	1
3	49	49	0	49	0	49	0
4	64	64	0	64	0	64	0
5	36	36	0	36	0	35	1
6	20	20	0	20	0	20	0
7	7	7	0	7	0	6	1
		Data Set 4		Data Set 5			
		Successes	Failures	Successes	Failures		
		11	1	12	0		
		13	1	14	0		
		48	1	49	0		
		64	0	64	0		
		35	1	36	0		
		19	1	20	0		
		6	1	7	0		
		Data Set 6		Data Set 7			
	Total Tests	Successes	Failures	Successes	Failures		
	302	296	6	302	0		

As in the simulated section, posterior distributions for each of the parameters will be calculated using the same three system priors. The data sets for this section

[1] It is worth noting how difficult it would be to model this data with traditional logistic regression. The amount of zeros present will make MLE estimation quite difficult.

contain highly reliable components so the μ_i transformation described in Section 3.4 was used. The M-H algorithm obtained draws from the complete conditional for μ_i using the MCMC settings from Table 3.3 and the σ_c values from Table B.9.

4.2.1 Data Set 1

The first data set for the weapon test data is one of the seven component systems. Three failures were recorded on component 1 in this data set. At the end of this section this data set will be compared to data set 3 to see the differences between having all failures on one component versus having failures spread across multiple components.

The marginal posterior densities for each of the eight unknown parameters (7 θ_i's and θ_s) are plotted in the first three plots of Figure 4.10. The boxplots show the lower simulated values for θ_1 as well as the increased spread; things which are expected from Section 4.1 for a component with failures. The remaining six components all have high reliability estimates with smaller spreads. There are no surprising results from the analysis of the seven θ_i. The posterior distributions for θ_s were calculated using Equation 3.2 and the three θ_s given the system priors are plotted as side-by-side boxplots in the fourth plot of Figure 4.10.

The final comparison for data set 1 is to look at how the three priors affect system reliability with a side-by-side comparison. Both Table 4.10 and Figure 4.10 put the posterior summaries for θ_s side-by-side. The highly informative prior gave the largest point estimate for θ_s and also the smallest HPD interval. There is a 60% probability that the highly informative Beta prior will provide higher estimates for θ_s when compared to the model of the non-informative Uniform prior; this probability drops to 55% when compared to the low informative Beta prior. This is the first instance where the prior has had a noticeable affect when comparing the three resulting distributions. The amount of component test data is large; however, the impact of not including any system tests is allowing the information from the system priors to impact the final results.

Table 4.10: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.7756	0.0753	.5729	.9545
Beta(.9408, .0192)	0.7684	0.0777	.5500	.9472
Uniform(0, 1)	0.7341	0.0801	.5300	.9131

4.2.2 Data Set 2

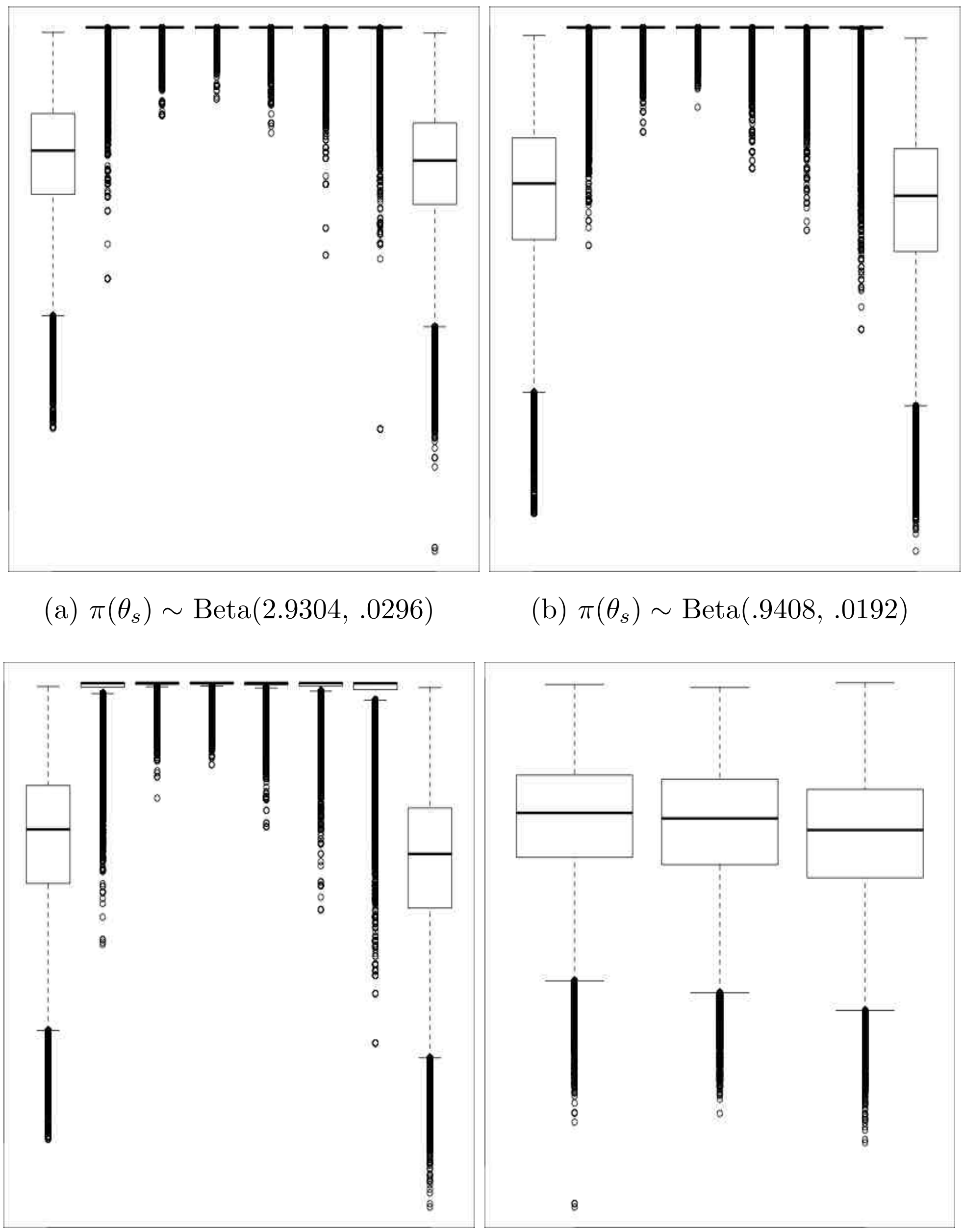
The second data set for the weapon test data contains seven different components. This data set has 6 failures occurring on component 1. The results from this data set can be compared to data set 4 to understand the impact of having all failures occur on one component versus having the failures spread out. The marginal posterior density summaries of the seven θ_i and θ_s for the three system priors are plotted as the first three plots in Figure 4.11. The boxplots presented show the lower simulated values for θ_1 as well as the increased spread; results that are expected from Section 4.1 conclusions. The remaining six components all have high reliability estimates with smaller spreads. The eighth boxplot in the first three plots shows how the density of θ_s is strongly influenced by the θ_1. Regardless of system prior, the maximum value for θ_1 does not reach 1. The addition of three extra failures on component 1 has impacted that components density by dragging the entire density down. The remaining six components have similar distributions with a majority of the density for each component close to 1.

Comparing the point estimates for θ_i from this data set to data set 1 using the information from Tables C.7 and C.8 show that $\theta_2 - \theta_7$ have similar estimates while the point estimate for θ_1 drops from 83.03% to 71.67% with the addition of the three additional failures; a drop of 11.36%.

The real interest of this thesis is in how the three system priors affect θ_s. Table 4.11 and the fourth plot in Figure 4.11 put the posterior distributions for θ_s side-by-side. The estimate for θ_s is highest from the highly informative Beta prior and lowest for the non-informative Uniform prior. The uncertainty surround the point estimates for each model are close to being equal with HPD intervals all covering a range of 31%. The highly informative Beta prior results in a model with a 58% probability of giving higher estimates for θ_s when compared to the non-informative Uniform prior; this probability drops slightly to 54% when compared to the less informative Beta prior. Including information about the system through the prior does have an impact on the final results for this data set; this impact can be attributed to the lack of any system information coming from actual tests.

Table 4.11: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

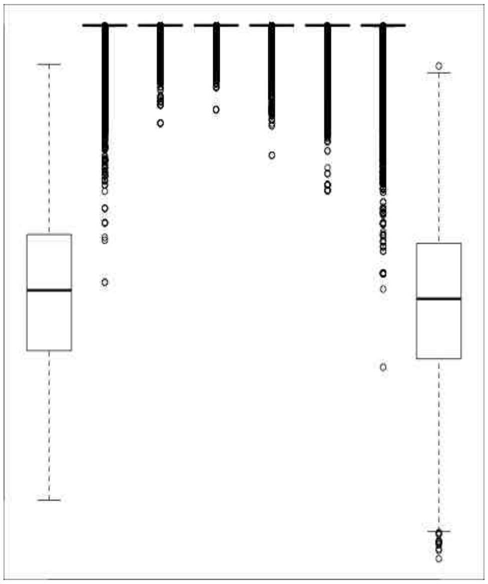
			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.7079	0.0859	.4855	.7988
Beta(.9408, .0192)	0.6945	0.0869	.4845	.7898
Uniform(0, 1)	0.6832	0.0872	.4629	.7780



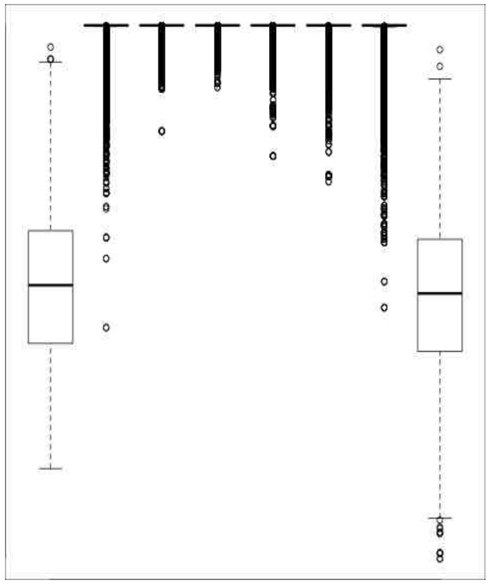
(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296) (b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

(c) $\pi(\theta_s) \sim$ Uniform(0, 1) (d) System Reliabilities

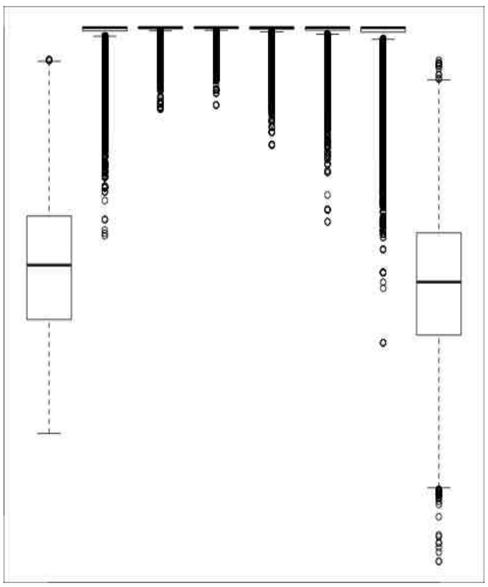
Figure 4.10: The densities shown in the first three plots represent the posterior reliabilities of the seven θ_i's and θ_s for weapon data set 1. Each of the first three plots represent the posterior densities obtained using one of the three different system priors. The fourth plot provides side-by-side comparisons of the three different θ_s densities.



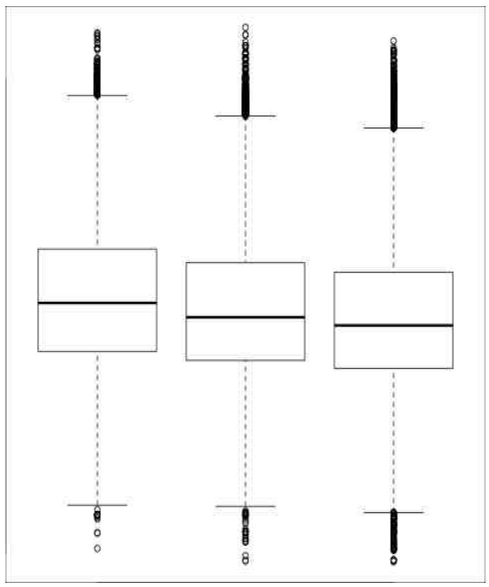
(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)



(c) $\pi(\theta_s) \sim$ Uniform(0, 1)



(d) System Reliabilities

Figure 4.11: The densities shown in the first three plots represent the posterior reliabilities of the seven θ_i's and θ_s for weapon data set 2. Each of the first three plots represent the posterior densities obtained using one of the three different system priors. The fourth plot provides side-by-side comparisons of the three different θ_s densities.

.

4.2.3 Data Set 3

The third data set for the weapon test data contains seven different components. This data set has 3 failures spread across the 7 components, where no component has more than one failure recorded. The marginal posterior densities for the unknown paramaters are plotted as boxplots in the first three plots of Figure 4.12. The boxplots show the lower estimates and wider densities for θ_2, θ_5, and θ_7; the components with one failure. These results are in line with what is known about the effect of a failure on posterior densities from Section 4.1.

The main interest of this thesis is the distribution of θ_s. The final analysis looks at how the three system priors affect θ_s with side-by-side comparisons given in Table 4.12 and the fourth plot of Figure 4.12. The highly informative Beta prior gives the highest point estimate for θ_s with the smallest HPD interval. Comparing the models, there is a 60.4% probability that the highly informative Beta prior will result in higher estimates for θ_s over the non-informative Uniform prior; this probability drops to 56.5% when comparing the low informative Beta prior to the non-informative Uniform prior. The two Beta priors provide similar densities with the highly informative Beta prior having a 53./6% chance of providing higher estimates. So far the results have been consistent in the last three data sets with the highly informative Beta prior providing higher estimates for θ_s with a 58-60% higher probability when compared to the non-informative prior.

4.2.4 Data Set 4

The fourth data set for the weapon test data contains seven components with 6 failures that are evenly spread across the components; with no component containing

Table 4.12: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.8039	0.1048	0.5905	0.9715
Beta(.9408, .0192)	0.7892	0.1124	0.5656	0.9742
Uniform(0, 1)	0.7649	0.1126	0.5395	0.9552

more than one failure. The marginal posterior densities of the eight unknown parameters are plotted as boxplots in the first three plots of Figure 4.13. The boxplots show the lower estimates and wider spreads for $\theta_1, \theta_2, \theta_3, \theta_5$, θ_6 and θ_7; all of the components with one failure recorded. The density for θ_s is not overly influence by any one component as it is in data sets 1 and 2. Again, these results are in line with what is known about the effect of a failure on posterior densities from Section 4.1.

The final analysis for data set 4 is to look at how the three priors affect θ_s with a side-by-side comparison. Table 4.13 and the fourth plot of Figure 4.13 provide side-by-side comparisons. The estimates for θ_s are again highest for the highly informative Beta prior, with a 2% decrease coming from each of the other two priors. The HPD interval is also the narrowest for the highly informative Beta prior. The highly informative Beta prior has a 60% chance of providing higher estimates for θ_s when compared to the non-informative Uniform prior, this value drops to 54.7% when compared to the low informative Beta prior; very similar values compare to the previous three data sets. Once again, even in spite of the large amount of information present, the choice of a prior is having a small impact on the posterior distributions of θ_s. If system tests are unavailable for inclusion in the model the more informative the system prior can be the better.

Table 4.13: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

			Highest Posterior Density	
	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.7046	0.1095	0.4841	0.9009
Beta(.9408, .0192)	0.6855	0.1152	0.4519	0.8883
Uniform(0, 1)	0.6611	0.1149	0.4361	0.8749

4.2.5 Data Set 5

The fifth data set for the weapon test data is the last data set that contains seven components. There are zero failures recorded in this data set. This data set can be compared to data set 7 to understand the impact of defining the block of components differently. The marginal posterior densities of the 8 unknown parameters are plotted as boxplots in the first three plots of Figure 4.14. The boxplots show wider densities for θ_1, θ_2, and θ_7. These three components have the lowest amount of data; 12, 14, and 7 tests respectively. The small number of tests are resulting in more uncertainty surrounding the point estimate for θ_i. The impact of a low number of component tests was commented on in the full system model. Having zero failures confirms that the conclusions made earlier about how a smaller number of component tests increases the uncertainty for that posterior density.

The final analysis for data set 5 is to compare how the three system priors affect θ_s with side-by-side comparisons in Table 4.14 and the fourth plot of Figure 4.14. The estimates for θ_s are fairly similar between the models using the two Beta priors, with a 3% decrease for the non-informative Uniform prior. The HPD interval for the non-informative prior is close to being double that of the HPD intervasl for either the Beta prior. The information provided about θ_s through the Beta system priors provides more certainty surrounding the point estimate for θ_s. Comparing the two

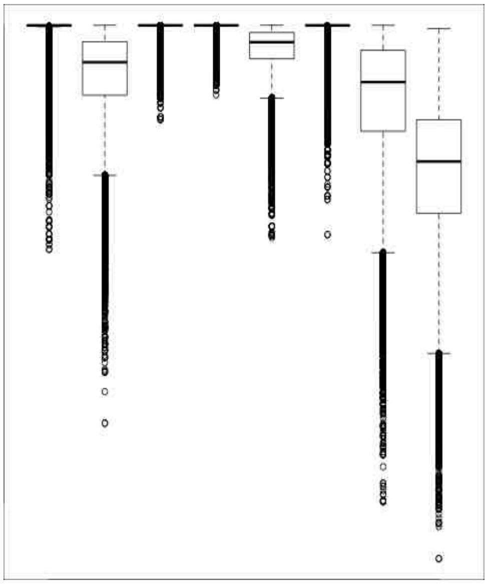
Beta priors shows that either of the models has a 50% chance of providing higher estimates for θ_s. There is a 73.5% probability that either of the Beta priors will provide higher estimates for θ_s when compared to the non-informative prior. The amount of data in this data set is the same as the four previous models; however, having zero failures for the entire system gives more weight to the priors.

with the highly informative providing the highest estimate, with a 3% decrease for the non-informative prior compared to either Beta prior. The HPD interval for the non-informative prior is close to being double that of the HPD intervals for either the Beta prior. Using a non-informative prior for this data set with zero failures resulted in quite a bit more uncertainty surrounding the θ_s estimates.

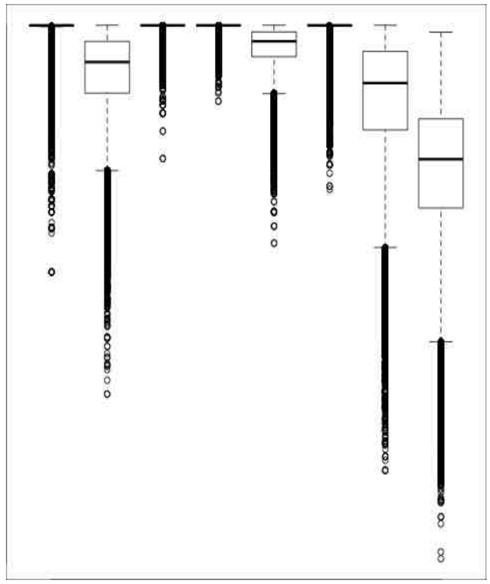
Both of the models that used a Beta prior provide similar output where each model is expected to provide higher estimates 50% of the time. There is a significant impact from the non-informative prior. Both of the Beta prior models will provide better estimates for θ_s about 73.5% of the time; with the highly informative prior giving slightly better estimates. The amount of data is still the same; however, having zero failures for the entire system appears to give more weight to the priors which bring in some level of information to the model.

Table 4.14: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

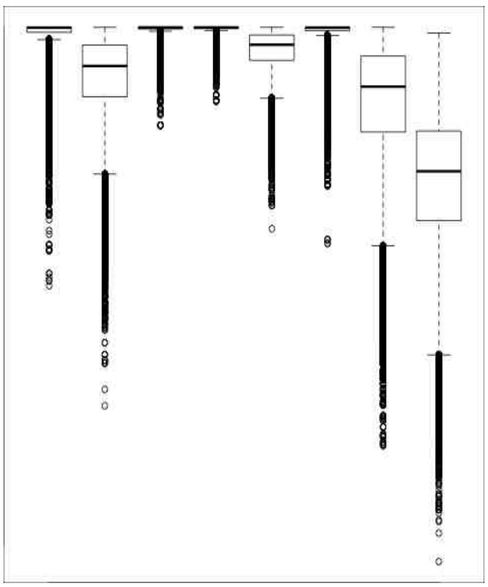
			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.9784	0.0354	0.9119	1.0000
Beta(.9408, .0192)	0.9768	0.0382	0.9044	1.0000
Uniform(0, 1)	0.9485	0.0567	0.8337	1.0000



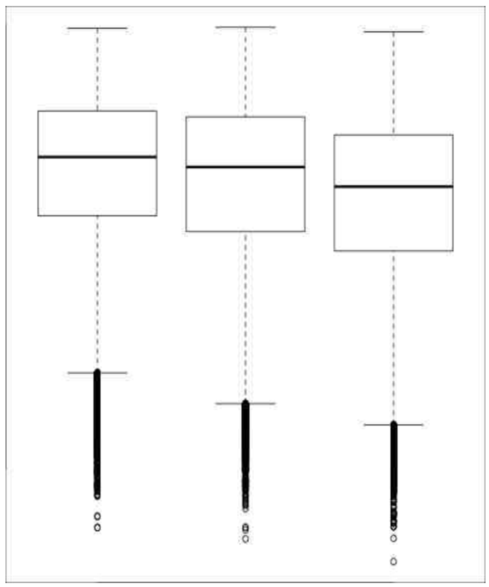
(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

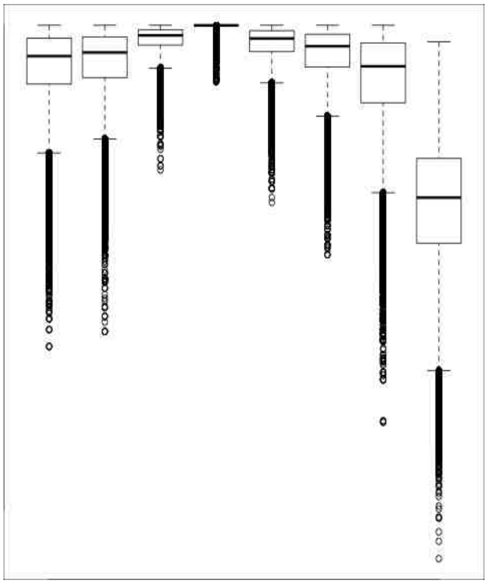


(c) $\pi(\theta_s) \sim$ Uniform(0, 1)

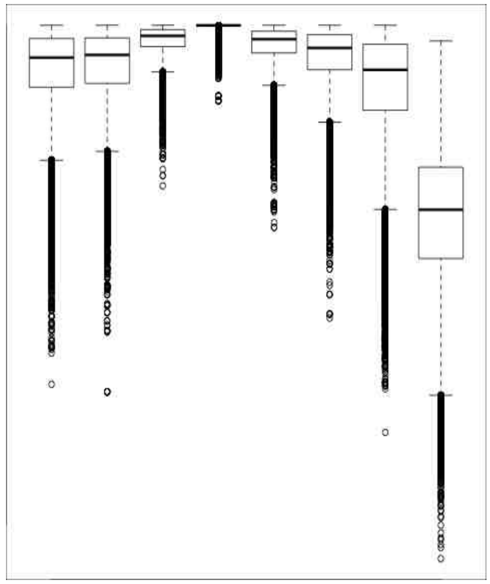


(d) System Reliabiliaties

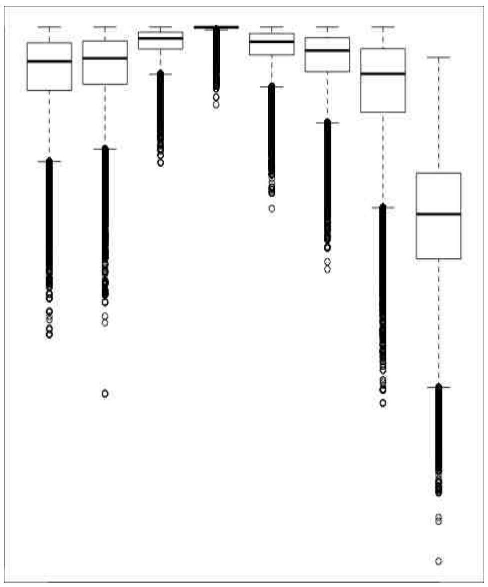
Figure 4.12: The densities shown in the first three plots represent the posterior reliabilities of the seven θ_i's and θ_s for weapon data set 3. Each of the first three plots represent the posterior densities obtained using one of the three different system priors. The fourth plot provides side-by-side comparisons of the three different θ_s densities.



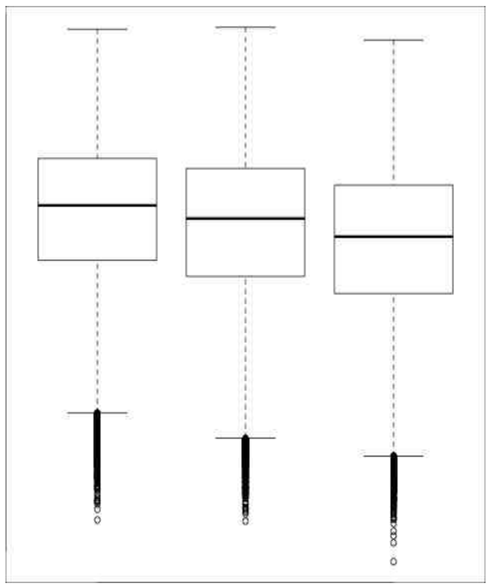
(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

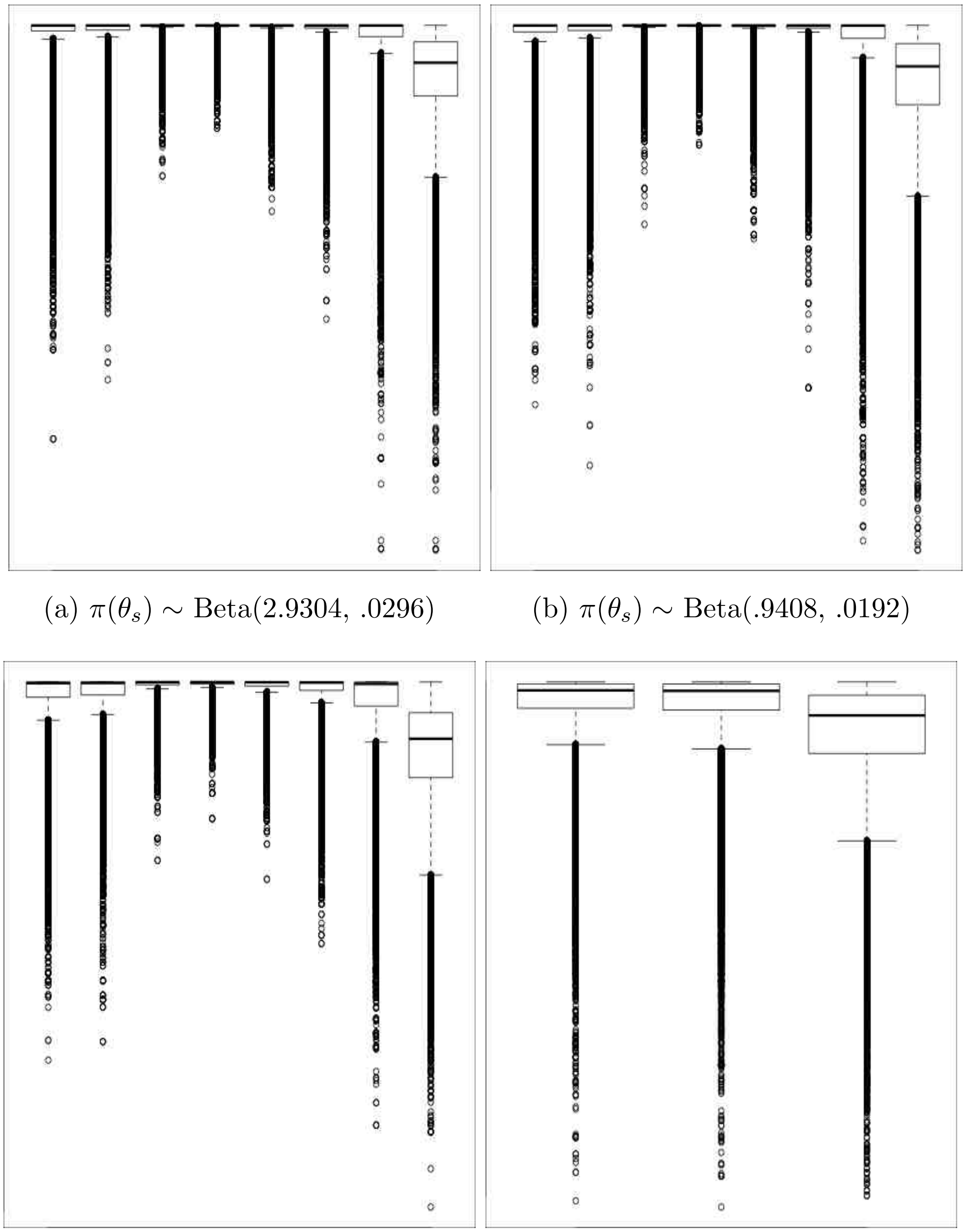


(c) $\pi(\theta_s) \sim$ Uniform(0, 1)



(d) System Reliabiliaties

Figure 4.13: The densities shown in the first three plots represent the posterior reliabilities of the seven θ_i's and θ_s for weapon data set 4. Each of the first three plots represent the posterior densities obtained using one of the three different system priors. The fourth plot provides side-by-side comparisons of the three different θ_s densities.



(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296) (b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

(c) $\pi(\theta_s) \sim$ Uniform(0, 1) (d) System Reliabiliaties

Figure 4.14: The densities shown in the first three plots represent the posterior reliabilities of the seven θ_i's and θ_s for weapon data set 4. Each of the first three plots represent the posterior densities obtained using one of the three different system priors. The fourth plot provides side-by-side comparisons of the three different θ_s densities.

4.2.6 Data Set 6

The sixth data set for the weapon test data is the first of two data sets that contain only one component. Six failures were recorded for this data set. Because there is only one component, the posterior distribution for $\theta_1 = \theta_s$. The posterior distributions for θ_s are plotted side-by-side in Figure 4.15 with the numerical summaries given in Table 4.15.

Table 4.15: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

			Highest Posterior Density	
$\pi(\theta_s)$	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.9803	0.0081	0.9640	0.9939
Beta(.9408, .0192)	0.9802	0.0079	0.9645	0.9938
Uniform(0, 1)	0.9770	0.0086	0.9597	0.9917

The point estimates for θ_s are similar among the three system priors with similar HPD intervals. Again, the highly informative Beta prior is giving the highest estimate for θ_s. In spite of the perceived similarities, there is a 60.9% probability of either Beta prior turning out a higher estimate for θ_s when compared to the non-informative Uniform prior. appear to be similar among the three priors. Again, the highly informative Beta prior does give the highest estimate for θ_s, which is expected. The HPD intervals are similar across the three different priors. But the two models which use a Beta prior do provide better estimates for θ_s 60.9% of the time when compared to the non-informative prior.

4.2.7 Data Set 7

The seventh and final data set for the weapon test data also contains only one component with a total of zero failures. The posterior distributions for θ_s are plotted

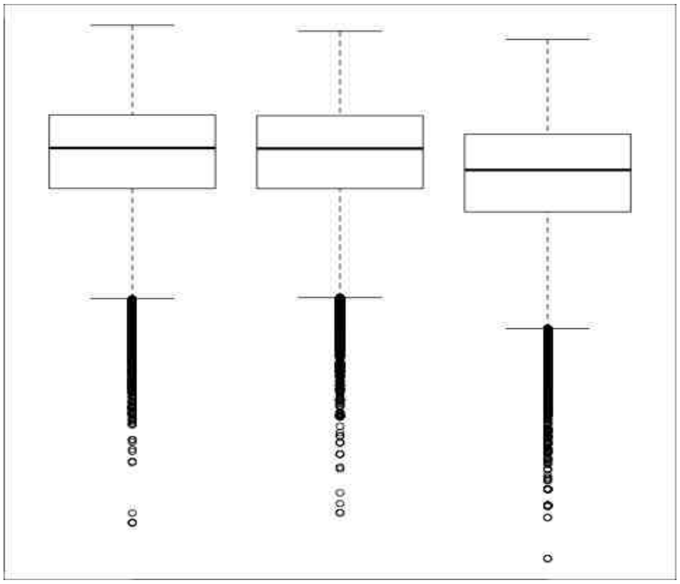


Figure 4.15: The densities shown in these three boxplots represent the reliability of θ_s for weapon data set 6 from each of the three different priors. The left hand boxplot shows the density of θ_s using a highly informative Beta prior; the middle boxplot shows the density of θ_s using a low informative Beta prior; the right hand boxplot shows the density of θ_s using a non-informative Uniform prior.

side-by-side in Figure 4.16 with the numerical summaries given in Table 4.16.

Table 4.16: *Comparison of the expected reliability and HPD intervals for θ_s given the different system priors.*

			Highest Posterior Density	
	Expected Reliability	Std Dev	.025%	.975%
Beta(2.9304, .0296)	0.9997	0.0009	0.9985	1.0000
Beta(.9408, .0192)	0.9997	0.0010	0.9971	1.0000
Uniform(0, 1)	0.9967	0.0032	0.9901	1.0000

The point estimates appear to be close to each other across all three priors, but the non-informative estimate of θ_s is 3 or more standard deviations away from the other two estimates. The HPD interval is also much wider for the non-informative prior when considering the size of the standard deviations. There is a 94.3% probability that either of the Beta priors will provide a higher estimate for θ_s. As with data set

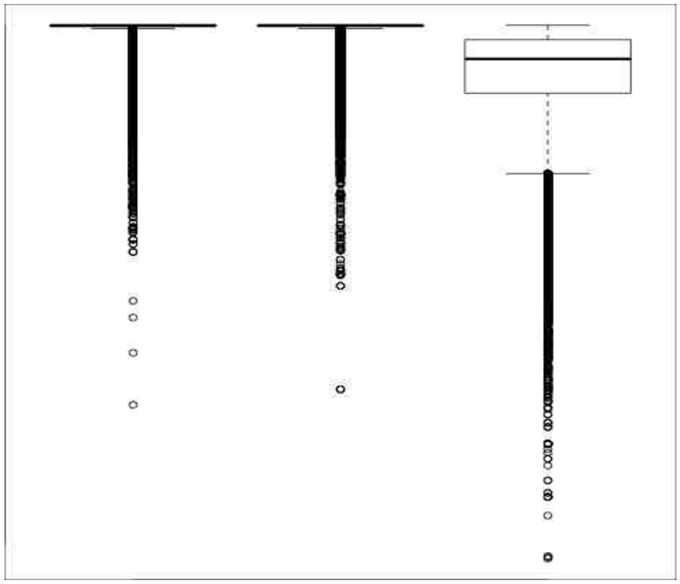


Figure 4.16: The densities shown in these three boxplots represent the reliability of θ_s for weapon data set 7 from each of the three different priors. The left hand boxplot shows the density of θ_s using a highly informative Beta prior; the middle boxplot shows the density of θ_s using a low informative Beta prior; the right hand boxplot shows the density of θ_s using a non-informative Uniform prior.

5, the absence of any failures gives more influence to the priors.

4.2.8 Weapons System Test Data Conclusion

The previous sections compared the effect of different priors to seven data sets of weapon test data. The highly informative Beta prior always provided the highest estimate for θ_s; which is expected given the information provided by that prior. However; there were no other notable conclusions made from comparing the posterior distributions of each data set under different priors. As mentioned earlier, the seven data sets contain information from the same tests. The differences in the seven data sets comes from the way researchers defined a failure and/or the configuration of components was defined. The differences between data sets is of more interest than the differences within each data set. This section will conclude the weapon test data section by making several of these comparisons.

Comparisons of θ_s between the seven different data sets will first be made. From the previous sections it is expected that the data sets with less components are expected to have the highest reliabilities; followed by the data sets with the fewest number of failures. Figure 4.17 shows the densities for the 21 different data sets grouped by system prior and ordered data set 1-7 in each group.

The densities from data sets 1-5 have similar spreads. This uncertainty can be attributed to deriving θ_s as the product of the θ_i's. Data set 5, with zero failures, gives a higher point estimate for θ_s then data set 6, which has 6 failures only for the two Beta priors. This comparison shows the importance of being able to include an informative system prior.

The overall comparison of the 21 densities does not shed light on which definition of a failure produces the highest estimate for θ_s; the focus will shift to comparing the densities between data sets that contain the same number of failures to try and answer this question. The densities for θ_s coming from the two data sets that contain 3 failures are plotted side-by-side in Figure 4.18; remember data set 1 has 3 failures from one component while data set 3 has 3 failures coming from 3 different

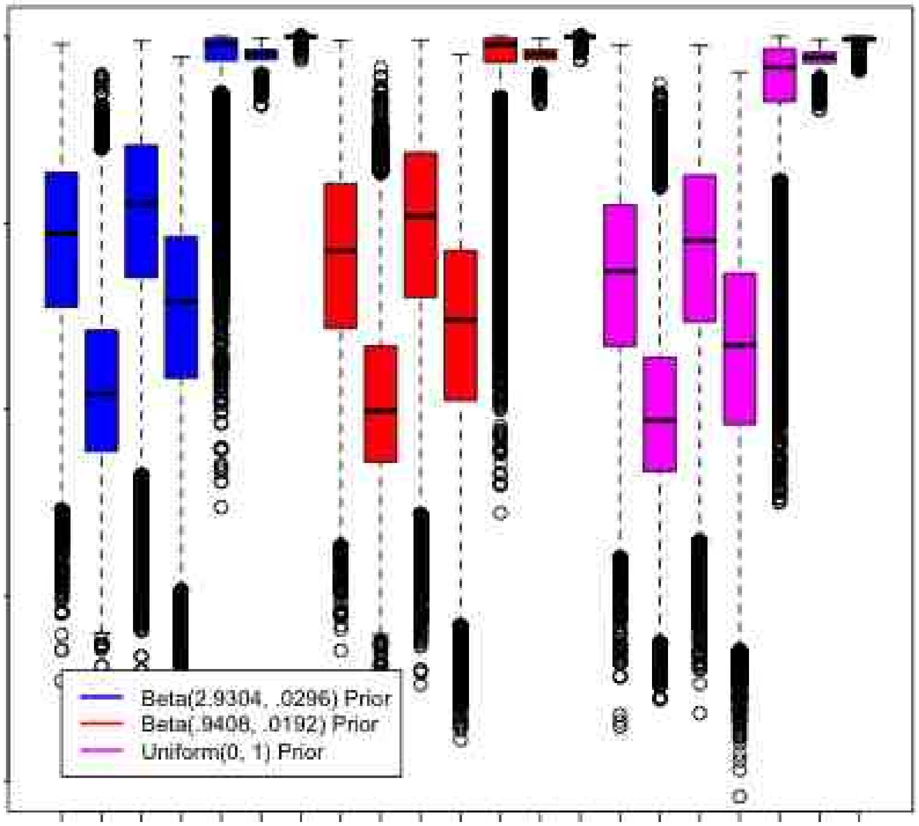


Figure 4.17: The densities of θ_s resulting from the 21 analyses of data set and prior combinations are plotted as boxplots. The first group of 7 boxplots come from the highly informative Beta prior, the second group of 7 boxplots comes from the low informative Beta prior, the third group of 7 boxplots comes from the non-informative Uniform prior.

components.

Figure 4.18 shows that the expected reliability for θ_s using data set 3 is consistently higher than the estimate from data set 1; regardless of the system prior. This comparison provides evidence for the conclusions that it should be preferred to have any failures spread across the system. Averaging over the priors, data set 3 has a 57.8% probability of resulting in a higher estimate for θ_s than data set 1.

The next set of comparisons comes from data sets with 6 failures recorded. There are three different datasets; each with six failures. Data set 2 has one of seven components with 6 failures; data set 4 has six of seven components with one failure

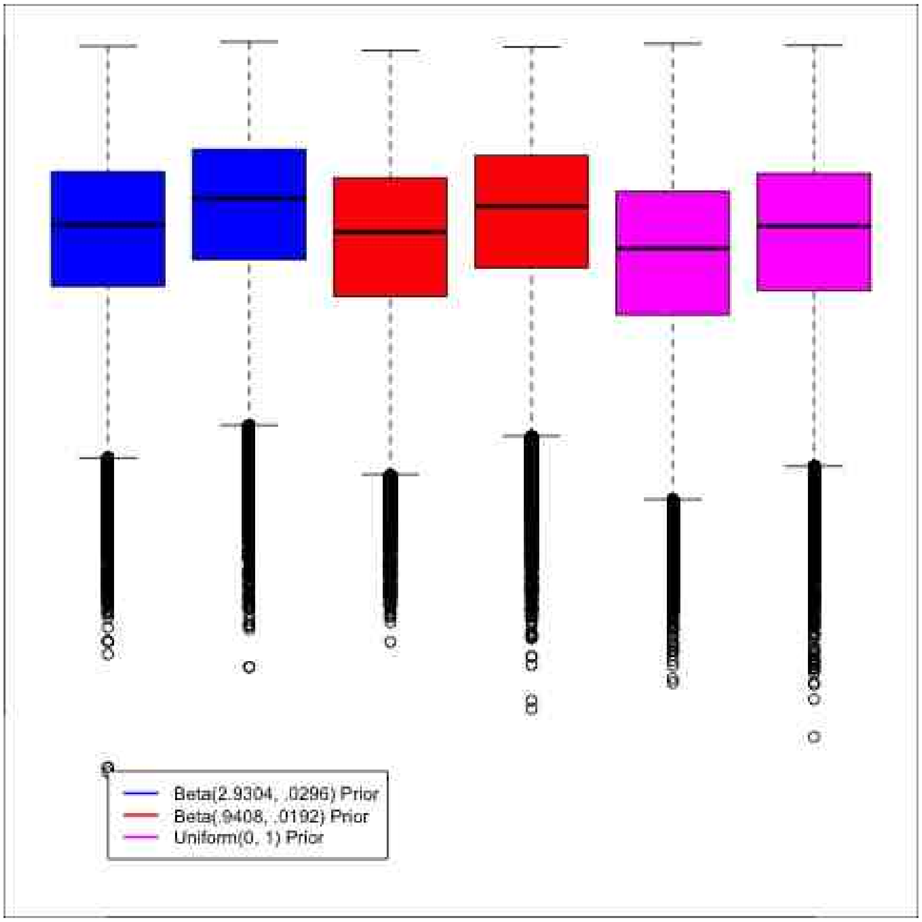


Figure 4.18: The densities of θ_s resulting from data set 1 and data set 3 are plotted as boxplots. Data set 1 has three failures all coming from component 1; data set 3 has three failures coming from three different components. Data set 1 resulted in the density plotted on the left of each prior comparison; data set 3 resulted in the density plotted on the right of each prior comparison.

each; data set 6 has one component with 6 failures. The densities for θ_s are plotted side-by-side by system prior in Figure 4.19. This comparison reiterates the same findings from the comparison of the three failure data sets. The difference in expected reliability between data set 2 and 4 is larger than it was with only three failures; with data set 4 resulting in a higher point estimate for θ_s. This comparison also highlights the advantage of a system with fewer components. Figure 4.19 shows that data set 6, where all the tests came from one component has a higher point estimate for θ_s as well as a smaller spread in the density. Data set 4 has a 72.2% probability of producing higher estimates for θ_s compared to data set 2. It can be concluded, as the overall number of failures increase it is beneficial to have the failures spread across the system. Data set 6 has a 99.99% probability of producing higher estimates compared to data set 2; and 99.4% of the time compared to data set 4. The fewer number of components that are included in the component block has a significant impact on the distributions of θ_s.

Data set 6 where all the tests came from one component has a higher point estimate of θ_s as well as a tighter posterior distribution. Data set 4 is expected to produce higher estimates for θ_s 72.2% of the time. Spreading the failures over the components does increase the estimate for θ_s. Data set 6 is expected to produce higher estimates for θ_s 100% of the time over data set 2, and 99.99% of the time over data set 4. Decreasing the number of components in a system has a huge impact on the estimates for θ_s.

The final comparison is between data sets with zero failures. The two data sets with zero failures recorded are data set 5 and 7. The densities for θ_s that resulted from these two data sets are plotted as side-by-side boxplots in Figure 4.20. This plot; like Figure 4.19, highlights the impact of having a system with a large amount of test data from a small number of components versus having the same amount of test data spread over a larger number of components. Fewer components result in higher estimates and smaller spreads for the densities of θ_s. Data set 7 has a 95.55% chance of producing higher estimates for θ_s when compared to data set 5 when the model uses the highly informative Beta prior, and that value drops down to 93.41% when the models use the non-informative Uniform priors. Regardless of the prior, condensing the system down to 1 component will result in better estimates for θ_s.

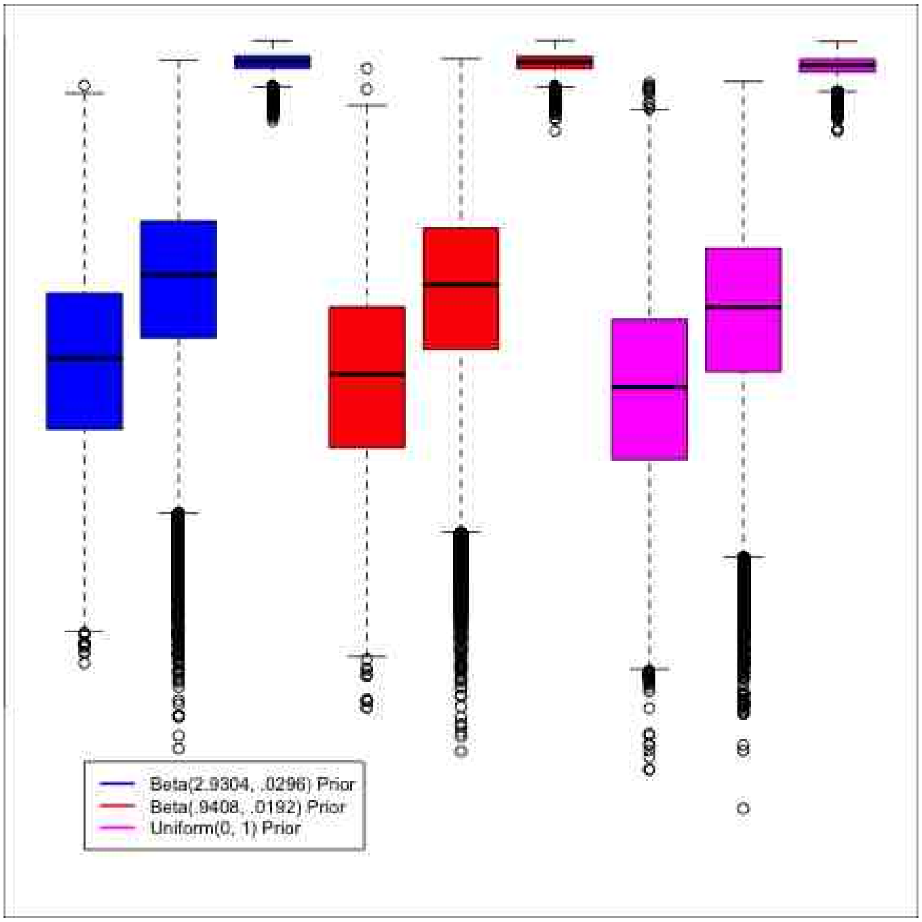


Figure 4.19: The densities of θ_s resulting from data set 2, data set 4, and data set 6 are plotted as boxplots. Data set 2 has six failures all coming from component 1; data set 4 has six total failures coming from six different components; and data set 6 has only one component with 6 failures. Data set 2 resulted in the density plotted on the left of each prior comparison; data set 4 resulted in the density plotted in the middle of each prior comparison; and data set 6 resulted in the density plotted to the right of each prior comparison.

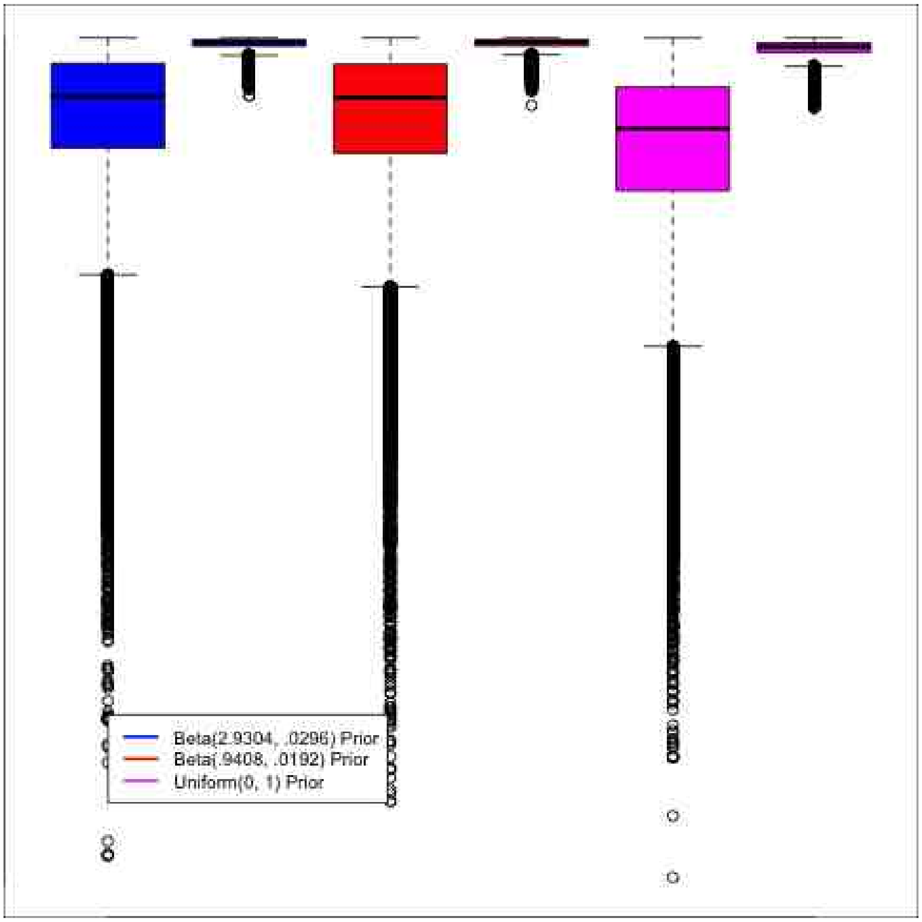


Figure 4.20: The densities of θ_s resulting from data set 5 and data set 7 are plotted as boxplots. Data set 5 has zero failures and seven components; data set 7 has zero failures and one component. Data set 5 resulted in the density plotted on the left of each prior comparison; data set 7 resulted in the density plotted to the right of each prior comparison.

Analyzing these different weapon data sets allowed for an understanding of how two component block schemes and spread of failures among components affect the posterior densities for θ_s. The conclusions made are given the assumption that the goal of an agency is to report a high level of system reliability with high confidence in those estimates. Figures 4.19 and 4.20 have shown that the best way to model a system is to define a block of components using the fewest components possible and increase the number of tests on those few components. The fewer number of components results in an increase in the estimates for θ_s. Allocating a large number of tests to few components increases the certainty one has in the estimates obtained. The positive impact of increasing test data has been seen throughout this analysis.

Once the block of components has been defined using the fewest components possible, the second way to increase estimates for θ_s is to minimize the number of overall failures. Figure 4.17 shows that regardless of how the failures are spread over the components; the less failures recorded the higher the estimate for θ_s; somewhat of an obvious conclusion. Figures 4.19 and 4.20 show that it is beneficial to have an even spread of failures across the components versus all the failures lumped onto any single component. If it only takes one component to fail for the whole system to fail a component with a lower point estimate for θ_i will have a bigger impact on θ_s.

The choice of priors has also given consistent estimates throughout. The amount of data available in each data set, 302 tests, was enough to negate some of the impact of each prior. The highly informative Beta prior consistently gave the highest estimates for θ_s while the non-inforamative Uniform prior consistently gave the lowest estimate for θ_s. The lack of any system tests in the weapons system data did make the prior information more important. As the number of failures went to zero, there was a noticeable difference in the results between system priors where the non-informative Uniform prior doubled the uncertainty. If the amount of test data was limited then the effect of these priors would be more pronounced and having the knowledge to use a highly informative prior would be desired.

Chapter 5

Conclusion

The United States Department of Defense is responsible for maintaining a large and diverse arsenal of weapons and delivery systems. Ensuring that all facets of this arsenal operate with high levels of reliability is a taxing job; especially for the nuclear arsenal which tight testing restrictions are enforced. This thesis developed a Bayesian model that can use a variety of available data sources to provide estimates of a systems reliability.

The Bayesian model in this paper has been built to be flexible enough to use different types of data so as to not restrict its use to one particular weapons system. The first type of data that this model incorporates is expert knowledge in the form of a prior distribution. Incorporating expert knowledge is one of the advantages of Bayesian statistics. The reliability model derived in this thesis has been written to incorporate different prior distributions. The first prior used in this thesis assumes that no reliable expert knowledge is available and therefore uses a non-informative Uniform distribution. The second prior assumes that information regarding the mean and variance of a systems reliability is known. This information is incorporated through a Beta system prior. Regardless of the knowledge available about system reliability it can easily be incorporated into this model.

Having found a way to incorporate expert opinion into the model the focus shifted to incorporating different sources of data. The reliability model is written using a Binomial likelihood to use pass/fail data from component and system level tests and also covariate information when available. The full expression of the model uses any combination of these data sources. The flexibility of this reliability model comes for it being written in a way to incorporate a range of prior information as well as its ability to use any combination of three data sources.

Once the reliability model was expressed in its complete form, the focus of this thesis shifted to running a range of data through the model. All of the data sets were run using three system priors: a highly informative Beta prior, a less informative Beta prior, and a non-informative Uniform prior. Simulated data was used to understand how all three types of data would affect reliability estimates. The four data sets looked at a system with a small number of components; a system with a large number of components; a system with only covariate information; and a full system with a small number of components, system tests, and covariate information.

The simulated data showed the impact of increasing system complexity. System reliability is found by taking the product of the component reliabilities. If a complex system has any low-reliable components the estimates for θ_s will reflect that with lower values. Including the covariate, age, brought a higher level of fidelity to the estimates for θ_s. Including this covariate allowed for system reliability to be reported as a function of age. In this full system model the estimates for θ_s ranged from 85-81% as the system aged. These estimates are about 2-3%higher than those obtained in the 5 component model without covariates. The nuclear arsenal has been aging over the past 80 years with minimal refurbishments so it is valuable to have a model that determines the rate of degradation as a function of age. Finally, the simulated data showed how large amounts of data overwhelm the prior information and estimates for θ_s will converge as the amount of data increases.

Moving from the simulated data to actual weapon test data allowed the model to analyze real-world information. The weapon test data can easily compare how the estimate for θ_s was affect by various changes within the same set of tests. Throughout the analysis of the seven data sets, the highly informative prior consistently gave higher estimates for θ_s, some models provided better estimates for θ_s 72.2% of the time when compared to the non-informative Uniform prior. The weapons test data showed a real difference between the three priors in spite of having 300+ tests. This difference between priors is a result of not having any system test data. Collecting data on the overall system can provide the model with more information than if the same amount of data came from component tests.

Comparing the 7 data sets showed up decreasing the size of a component block as well as spreading failures across the system resulted in higher point estimates for θ_s with higher levels of certainty in those estimates. The real world data led to the conclusion that the more tests performed on a component the more confident one can be in the results; regardless of the number of failures. There are increased levels of uncertainty for the point estimates if either a small number of tests are performed or failures are recorded. Out of these two, the one way to control the level of confidence in the point estimate for θ_s is to increase the number of tests performed.

This thesis has successfully developed a Bayesian reliability model that can be used by government agencies charged with the responsibility of determining system reliability. This model has concluded the impact that the number of tests, spread of failures, and age can have on estimates for system reliability. A wide range of topics has been covered by this thesis; but there are still topics that this research can be expanded to cover. This model has been written to include as many covariates as the researcher desires; but only one, age, was address in this paper. Other covariates could have an impact on the estimates of θ_i and θ_s. Future covariates of interest include: production facility, storage facility, and refurbishments. Knowing if these, or any other covariates, cause significant differences in θ_s can aide decision makers. If resources are limiting the number of component replacements that can be made;

knowing which covariates lower system reliability can ensure the components associated with those covariates are replaced first; therefore lowering the risk of a failure to the entire stockpile.

Resource allocation has been a constant theme throughout the introduction and conclusion. Using this model to address cost would be a worthwhile extension of the current efforts. Do the estimates for θ_s increase more from one system test or a handful of component tests? If a failure has been recorded, is it worth spending the money to obtain additional (hopefully failure free) tests? At what point does the fidelity of the results begin to diminish with added tests? These are questions that could be answered with minor modifications to the main framework of the model. These concepts are extensions to the current research that would further to enhance the U.S. weapons program.

This thesis has proven through the analysis that a reliability model can easily be applied to real-world problems. This paper set the stage for future work to be conducted in this field. Models such as the reliability model presented in this thesis are irreplaceable to the Stockpile Stewardship Program. The ability to answer questions regarding the future needs of the U.S. weapons program lie in the continuous use of reliability models such as the one developed in this thesis.

Appendix A

Distributions

A.1 Beta Distribution

$X|\ \alpha, \beta \sim Beta(\alpha, \beta)$

$$\begin{aligned} f(x|\alpha, \beta) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1, \quad \alpha > 0, \quad \beta > 0. \\ \mathbb{E}[X] &= \frac{\alpha}{\alpha+\beta}, \quad var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \end{aligned}$$

The parameters are α and β.

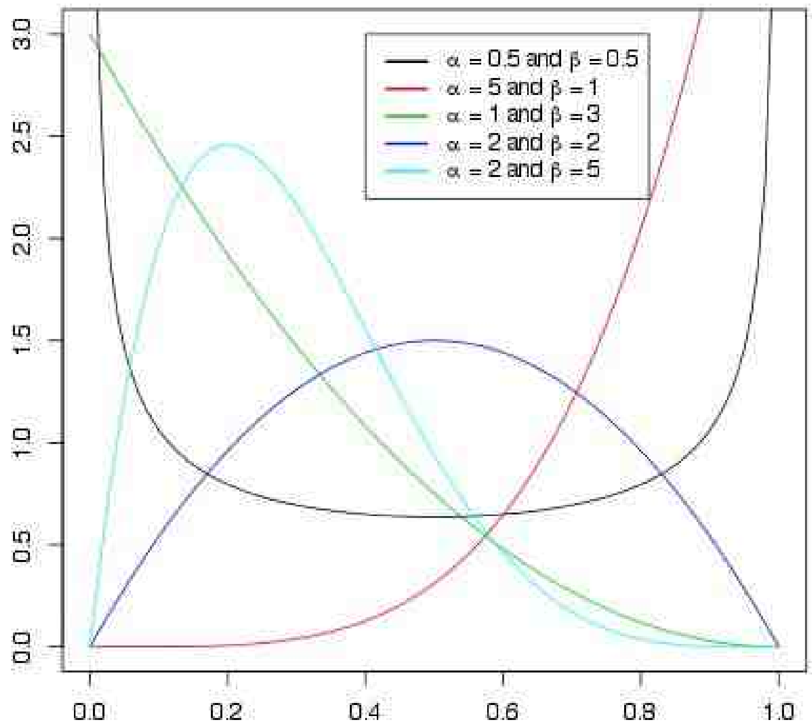


Figure A.1: *Probability density function of the Beta distribution using various parameter values.*

A.2 Binomial Distribution

$X|n,\ p \sim \textit{Binomial}(n, p)$

$$
\begin{aligned}
f(x|n,p) &= \binom{n}{x} p^x (1-p)^x, \quad x = 0, 1, 2, 3, \ldots, n, \quad 0 < p < 1. \\
\mathbb{E}[X] &= p, \quad var(X) = np(1-p).
\end{aligned}
$$

n is the number of tests and p is the probability of seeing a success.

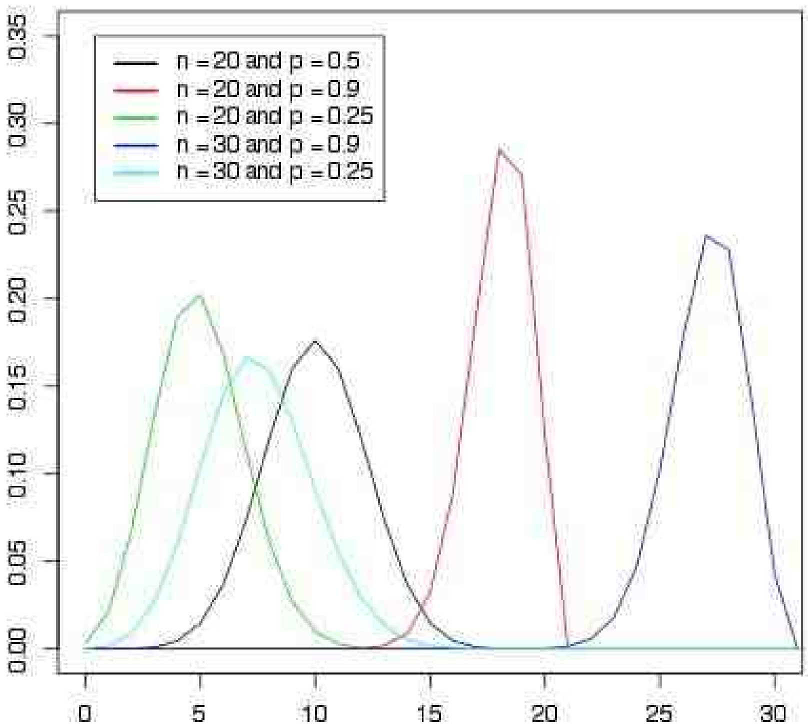


Figure A.2: *Probability density function of the Binomial distribution using various parameter values.*

A.3 Normal Distribution

$$X|\mu,\sigma^2 \sim N(\mu,\sigma^2)$$

$$\begin{aligned} f(x|\mu,\sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0 \\ \mathbb{E}[X] &= \mu, \quad var(X) = \sigma^2. \end{aligned}$$

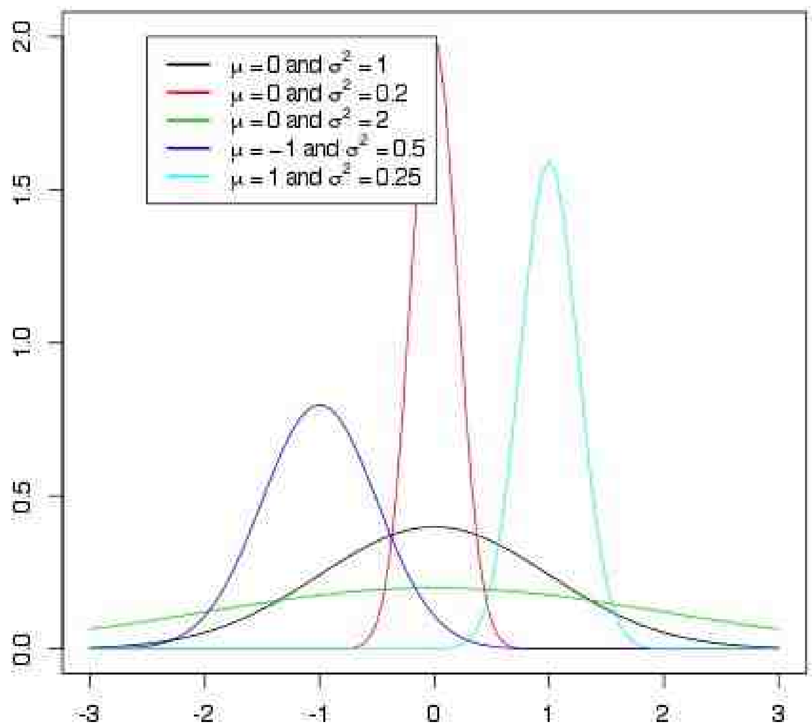


Figure A.3: *Probability density function of the Normal distribution using various parameter values.*

A.4 Uniform Distribution

$X \sim$ *Unif(a, b)*

$$
\begin{aligned}
f(x|a,b) &= \frac{1}{b-a}, \quad a < x < b \\
\mathbb{E}[X] &= \frac{a+b}{2} \quad var(X) = \frac{(b-a)^2}{12}
\end{aligned}
$$

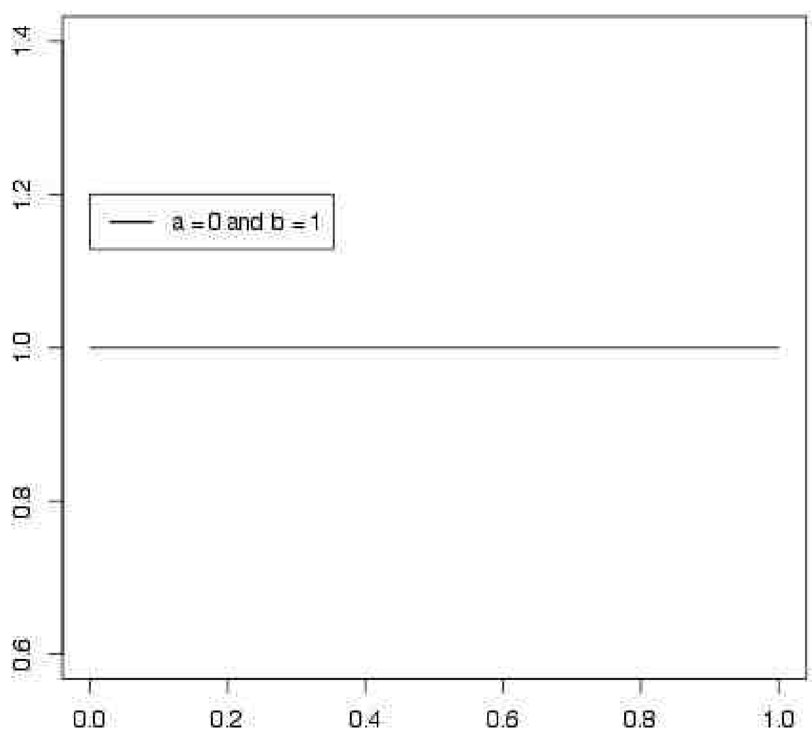


Figure A.4: *Probability density function of the Uniform distribution on the most common range (0, 1)*

Appendix B

Convergence Diagnostics

This section covers the diagnostics that were used to determine if the M-H algorithms were converging to stationary distributions. No one single diagnostic can confirm convergence; however, the combination of several diagnostics can lead one to conclude that the results appear stationary. All of the M-H algorithms run for the analysis in this paper were concluded to have converged by looking at a combination of acceptance rates, trace plots, Geweke diagnostics, and Gelman and Rubin diagnostics. The Gelman and Rubin diagnostic point estimates were 1 for the majority of the models; these diagnostics will not be individually displayed for the models in which this diagnostic equaled 1.

B.1 5 Component System

This section presents all of the convergence diagnostics for the 5 component system analyzed in Section 4.1. The σ_c and corresponding acceptance rates for the M-H algorithm run on the 5 component system data are shown in Table B.1.

The trace plots for the 5 unknown parameters are given in Figure B.1. The

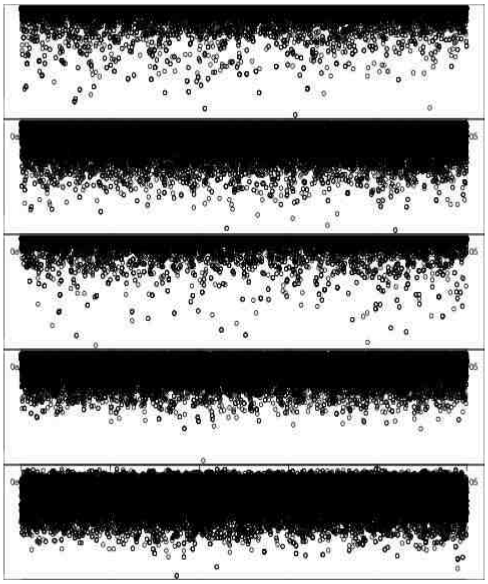
Table B.1: *This table gives the σ_c values used in the M-H algorithm for a 5 component system. Three different sets of σ_c and acceptance rates are given for each of the three system priors.*

$\pi(\theta_s) \sim$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform(0, 1)	
Acceptance Rates						
θ_i	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	14.00	0.4027	14.00	.4272	15.00	.4009
2	3.50	0.3634	3.50	.3658	3.50	.3641
3	14.00	0.3852	14.00	.4105	15.00	.3890
4	3.50	0.3641	3.50	.3614	3.50	.3593
5	2.50	0.2281	2.50	.2296	2.50	.2275

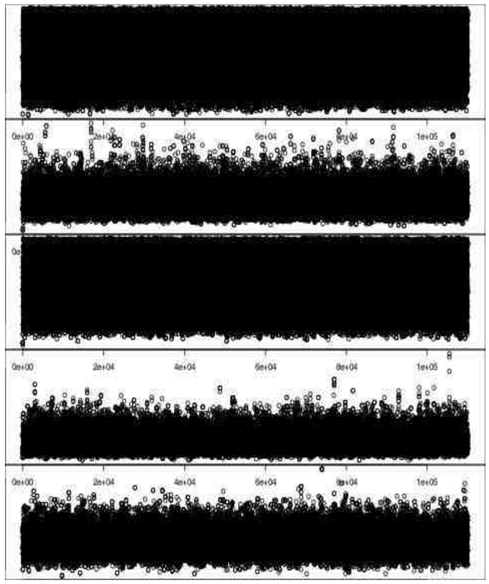
Geweke statistics are shown in Table B.2. Each column of the table represents the Geweke statistics for each component and each prior.

Table B.2: *The Geweke diagnostics are shown in this table. One Geweke diagnostics is given for each component for each system prior.*

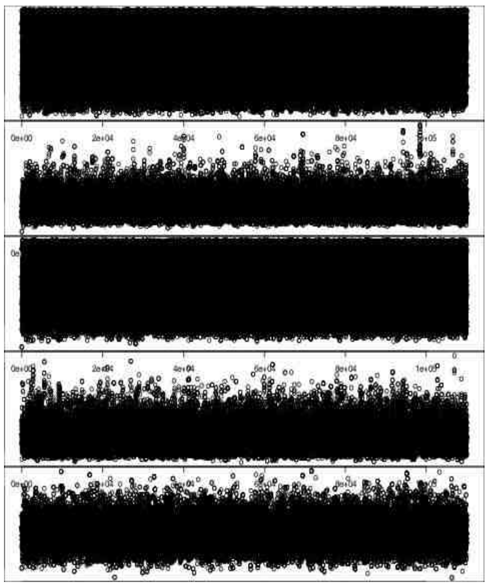
Component	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform(0, 1)
1	.6985	1.8088	-2.607
2	2.1138	-1.7903	0.4702
3	.4857	-0.4054	-0.8514
4	1.6258	-0.1214	-0.3453
5	1.0884	-0.8065	0.4993



(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)



(c) $\pi(\theta_s) \sim$ Unif(0, 1)

Figure B.1: Trace plots of the 5 unknown parameters in the 5 component system.

B.2 42 Component System

This section presents all of the convergence diagnostics for the 42 component system analyzed in Section 4.1. The σ_c and corresponding acceptance rates for the M-H algorithm run on the 42 component system data are shown in Table B.3.

Table B.3: *σ_c is the value that represents the spread of the proposal density for each unknown parameter. This spread affects the acceptance rate. Both the σ_c and the corresponding acceptance rates for the three different system priors are listed in this table. The middle portion of this table shows the statistics from the Geweke Diagnostic. The final portion of this table shows the statistics from the Gelman and Rubin diagnostic based off of three chains.*

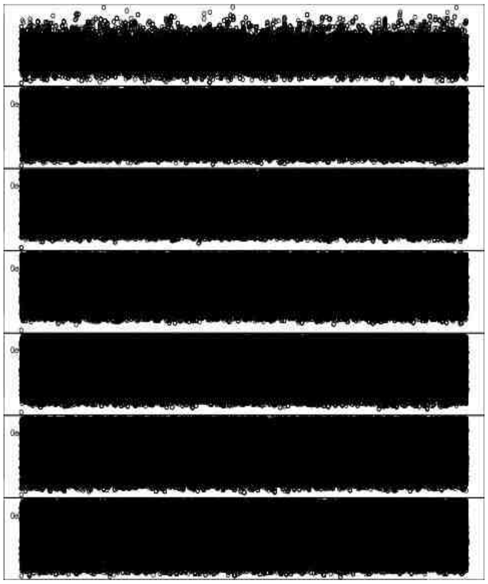
$\pi(\theta_s) \sim$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform(0, 1)	
θ_i	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	14	0.199	4	0.2018	4	0.2016
2	3.5	0.4189	14	0.4199	14	0.4129
3	14	0.3991	14	0.4	14	0.3975
4	3.5	0.3943	14	0.3941	14	0.3922
5	2.5	0.4058	14	0.4038	14	0.4002
6	14	0.4147	14	0.4145	14	0.4095
7	3.5	0.4257	14	0.4269	14	0.4204
8	14	0.1995	4	0.1997	4	0.1985
9	3.5	0.3722	16	0.3768	16	0.3715
10	2.5	0.3581	16	0.3578	16	0.3509
11	14	0.352	16	0.3521	16	0.3504
12	3.5	0.3621	16	0.3623	16	0.3558
13	14	0.3711	16	0.3732	16	0.3654
14	3.5	0.3811	16	0.3833	16	0.3811
15	2.5	0.1997	4	0.201	4	0.1977

Continued on Next Page . . .

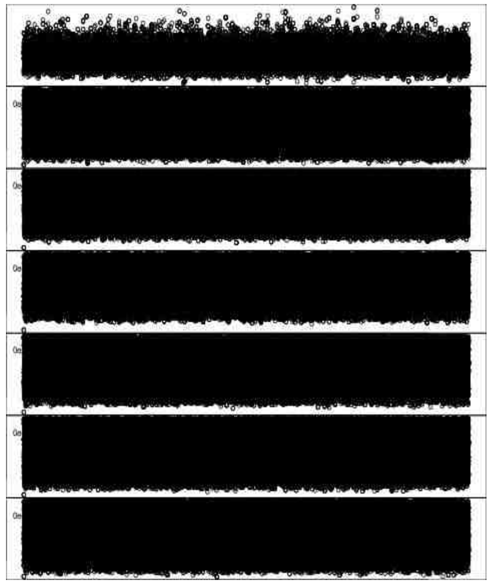
Table B.3 – Continued

θ_i	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
16	14	0.3769	16	0.3786	16	0.3712
17	3.5	0.3583	16	0.3565	16	0.3554
18	14	0.3544	16	0.3513	16	0.3494
19	3.5	0.362	16	0.3638	16	0.3582
20	2.5	0.3701	16	0.3725	16	0.3666
21	14	0.3819	16	0.3823	16	0.3773
22	3.5	0.1977	4	0.1989	4	0.1995
23	14	0.3757	16	0.3759	16	0.3701
24	3.5	0.3553	16	0.3582	16	0.3534
25	2.5	0.3533	16	0.3542	16	0.3516
26	14	0.3627	16	0.3614	16	0.3566
27	3.5	0.3696	16	0.3735	16	0.3669
28	14	0.3845	16	0.384	16	0.3726
29	3.5	0.1974	4	0.2008	4	0.1983
30	2.5	0.3729	16	0.3768	16	0.3722
31	14	0.3599	16	0.3577	16	0.3523
32	3.5	0.355	16	0.354	16	0.3494
33	14	0.3634	16	0.364	16	0.3606
34	3.5	0.3729	16	0.3716	16	0.3669
35	2.5	0.3822	16	0.381	16	0.3766
36	14	0.1979	4	0.1994	4	0.1987
37	3.5	0.3767	16	0.3767	16	0.3716
38	14	0.3556	16	0.3574	16	0.3552
39	3.5	0.3527	16	0.3522	16	0.3507
40	2.5	0.3617	16	0.3646	16	0.3582
41	14	0.3687	16	0.3692	16	0.3663
42	3.5	0.3805	16	0.3836	16	0.3779

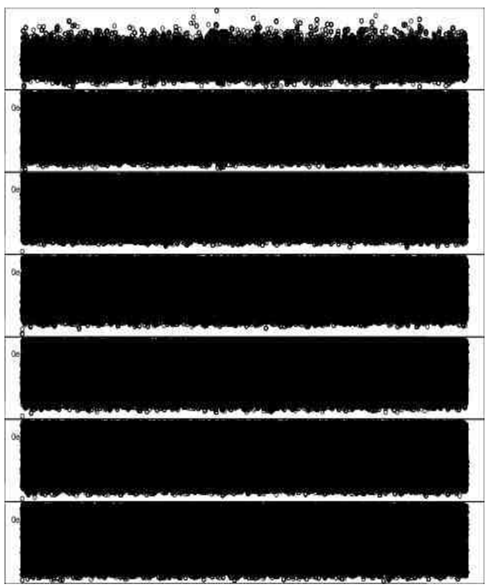
All 42 trace plots were inspected to ensure that the chains visually converged but because the data for this analysis was a seven component system replicated 6 times only the first 7 trace plots are shown in Figure B.2. The Geweke statistics for each of 42 component system are given in Table B.4.



(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)



(c) $\pi(\theta_s) \sim$ Unif(0, 1)

Figure B.2: Trace plots of the 5 unknown parameters in the 5 component system.

Table B.4: *The Geweke diagnostics are shown in this table. One Geweke diagnostics is given for each component for each system prior. The top line represents the system prior which resulted in each statistic; the component values are given in the first and fifth column.*

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)		Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	-0.13	0.00	0.32	22	-0.44	-0.17	-1.42
2	-0.18	-1.61	0.84	23	0.40	1.67	0.22
3	-0.85	-1.68	-0.07	24	1.27	0.40	0.05
4	-0.91	0.02	0.63	25	-0.16	1.63	1.90
5	0.38	-0.86	-0.74	26	-0.23	-0.90	-0.67
6	-0.33	0.91	1.27	27	0.44	-0.72	0.22
7	0.88	-0.76	-1.48	28	0.58	2.14	-1.40
8	0.88	-0.96	-0.19	29	2.13	0.53	0.16
9	1.29	-0.30	-1.10	30	-0.96	-0.36	-0.76
10	0.38	-0.11	-1.35	31	-0.09	-0.76	-1.00
11	0.11	0.95	2.14	32	0.10	-1.05	0.03
12	-1.08	-1.85	-1.61	33	0.37	0.71	0.49
13	-0.56	-0.24	-1.02	34	-0.45	0.14	-0.33
14	0.14	1.88	-1.12	35	1.32	0.62	-0.49
15	0.06	0.66	0.54	36	0.11	-0.48	0.92
16	-0.57	2.19	0.61	37	-0.17	0.67	1.19
17	-1.75	-0.51	0.44	38	1.36	1.66	-1.00
18	-0.34	0.55	0.14	39	-0.60	0.76	-0.05
19	-0.43	0.02	0.34	40	-1.08	0.20	1.12
20	1.53	0.60	-1.39	41	0.54	-1.47	0.08
21	-3.34	-0.48	-0.17	42	-1.01	0.01	1.02

B.3 1 Component System – With Covariates

This section presents all of the convergence diagnostics for the 1 component system analyzed in Section 4.1. The σ_c and corresponding acceptance rates for the M-H algorithm run on the covariate model are shown in Table B.5.

Table B.5: *The σ_c values and corresponding acceptance rate for each of the unknown parameters in the covariate model are given below.*

β_i	σ_c	Acc Rate
0	2	.188
1	1.25	.3749

The trace plots for the unknown parameters are given in Figure B.3; these trace plots represent the draws made on β_0 and β_1. The Geweke and Gelman and Rubin statistics for each of the unknown parameters are shown in the Table B.6.

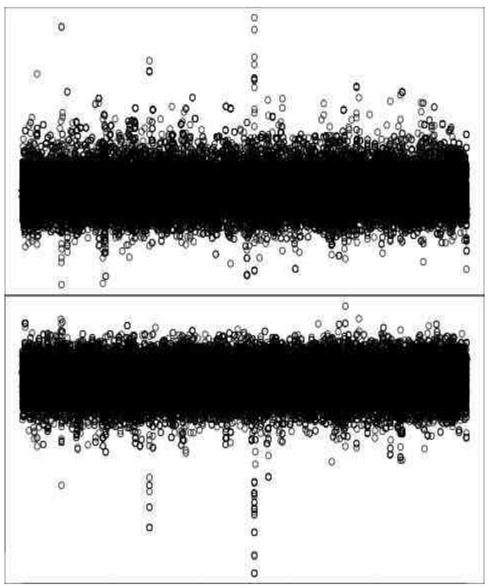


Figure B.3: Trace plots of the β_0 and β_1 parameters from the covariate model.

Table B.6: *The top half of this table gives the Geweke diagnostics for the $\boldsymbol{\beta}$ simulations. The second half of this table gives the Gelman and Rubin point estimate diagnostics.*

Geweke Diagnostics	
β_0	1.0275
β_1	.3556
Gelman and Rubin Diagnostics	
β_0	1.01
β_1	1.11

B.4 Full System Diagnostics

This section presents all of the convergence diagnostics for the full system model analyzed in Section 4.1. The σ_c and corresponding acceptance rates for the M-H algorithm run on the full system data are shown in Table B.7.

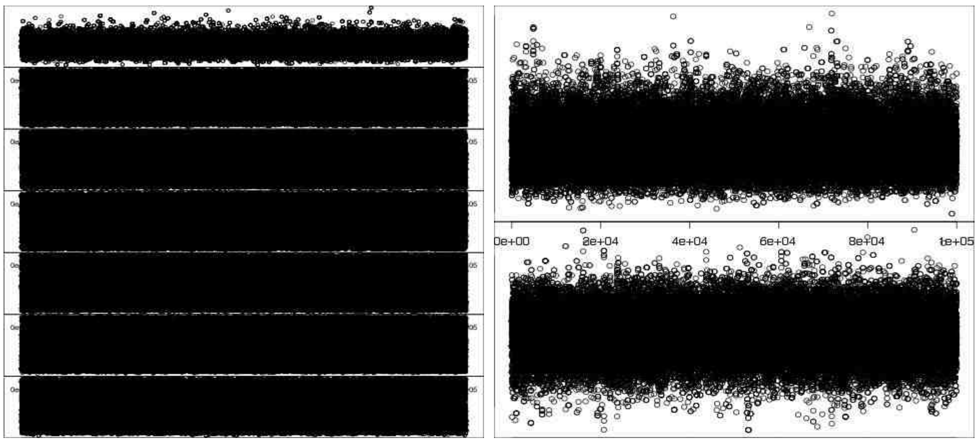
Table B.7: *The σ_c values determine the spread of the proposal density for each unknown parameter. This spread affects the acceptance rate of the algorithm. Both the σ_c and the acceptance rates for each of the three system priors are listed in this table.*

$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
μ_1	5.0	0.1795	5.0	0.1659	5.0	0.1642
μ_2	18.0	0.3856	20.0	0.3538	20.0	0.3554
μ_3	18.0	0.3853	20.0	0.3522	20.0	0.3543
μ_4	18.0	0.3859	20.0	0.3570	20.0	0.3560
μ_5	18.0	0.3861	20.0	0.3552	20.0	0.3560
μ_6	18.0	0.3863	20.0	0.3532	20.0	0.3539
μ_7	18.0	0.3885	20.0	0.3544	20.0	0.3549
β_0	1.5	0.3817	1.5	0.3801	1.5	0.3816
β_1	2.0	0.3254	2.0	0.3249	2	0.2859

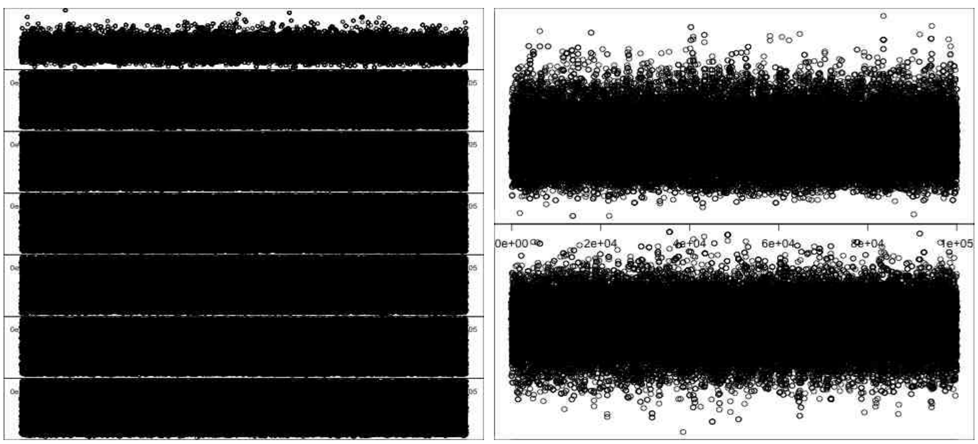
The trace plots for the unknown parameters are given in Figure B.3; trace plots for the simulated values of the μ_i's and the β_i's were inspected to ensure that the chains visually converged. The Geweke diagnostics are shown in Table B.8.

Table B.8: *The Geweke diagnostics are shown in this table. One Geweke diagnostics is given for each parameter and each system prior.*

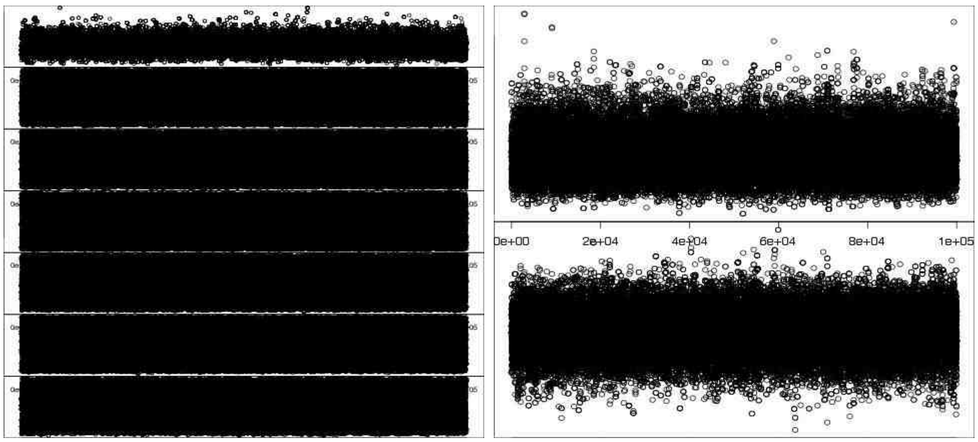
Parameter	Geweke Diagnostics		
μ_1	-0.1946	0.8863	0.3021
μ_2	0.1294	0.3613	-0.0685
μ_3	-2.3748	-0.8503	0.6715
μ_4	0.2044	0.6874	-0.5707
μ_5	0.7534	-0.3498	0.7299
μ_6	0.0509	-0.0671	-0.9328
μ_7	1.1839	-1.6387	-0.1720
β_0	-0.0491	-0.1248	0.9383
β_1	-1.1680	-0.3489	-0.6839



(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(b) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)



(c) $\pi(\theta_s) \sim \text{Beta}(2.9304, .0296)$

Figure B.3: Trace plots of the 9 unknown parameters in the full system model. The left hand side represents the trace plots for the 7 μ_i parameters; the right hand side represents the trace plots for the 2 β_i parameters.

B.5 Weapons System Test Data Diagnostics

This section presents all of the convergence diagnostics for the weapon test data analyzed in Section 4.2. The σ_c and corresponding acceptance rates for the M-H algorithm run on all seven data sets are shown in Table B.9.

Table B.9: *The σ_c values determine the spread of the proposal density for each unknown parameter. This spread affects the acceptance rate of the algorithm. Both the σ_c and the acceptance rates for each of the three system priors are listed in the columns of each table. Each table shows the σ_c and acceptance rates for one of the seven different data sets*

$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	3	0.2328	3	0.2351	3	0.235
2	15	0.4005	15	0.4004	15	0.3177
3	15	0.3803	15	0.3793	15	0.3108
4	15	0.3725	15	0.3732	15	0.3085
5	15	0.3865	15	0.3886	15	0.3128
6	15	0.2944	15	0.3947	15	0.3156
7	15	0.4085	15	0.4134	15	0.32207

(a) Weapon Data Set 1

$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	3	0.1813	2.5	0.2161	3	0.1793
2	18	0.3397	18	0.3385	18	0.2717
3	18	0.3235	18	0.3216	18	0.2625
4	18	0.3216	18	0.3203	18	0.2627
5	18	0.3291	18	0.3286	18	0.2653
6	18	0.3364	18	0.3367	18	0.2706
7	18	0.3487	18	0.3486	18	0.2749

(b) Weapon Data Set 2

$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	5	0.3874	5	0.3544	5	0.2778
2	20	0.2335	20	0.2297	20	0.3467
3	20	0.3398	20	0.3248	20	0.2676
4	20	0.3287	20	0.3287	20	0.2653
5	20	0.2315	20	0.2358	20	0.3589
6	20	0.3458	20	0.3487	20	0.2783
7	20	0.1764	20	0.2288	20	0.3386

(c) Weapon Data Set 3

The trace plots for the unknown parameters are given in Figures B.4, B.5, and

$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	6	0.2306	6	0.2338	6	0.2173
2	6	0.2329	6	0.2317	6	0.2197
3	6	0.2307	6	0.2324	6	0.2176
4	18	0.3188	18	0.3192	18	0.2617
5	6	0.23386	6	0.2331	6	0.2188
6	6	0.2328	6	0.2322	6	0.2177
7	6	0.1762	6	0.2305	6	0.2184

(d) Weapon Data Set 4

$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	15	0.3801	15	0.3797	15	0.2977
2	15	0.38	15	0.3804	15	0.2975
3	15	0.3798	15	0.3786	15	0.3066
4	18	0.3176	18	0.3241	18	0.2594
5	15	0.383	15	0.3828	15	0.3053
6	15	0.3821	15	0.3852	15	0.3019
7	15	0.3748	15	0.3713	15	0.2889

(e) Weapon Data Set 5

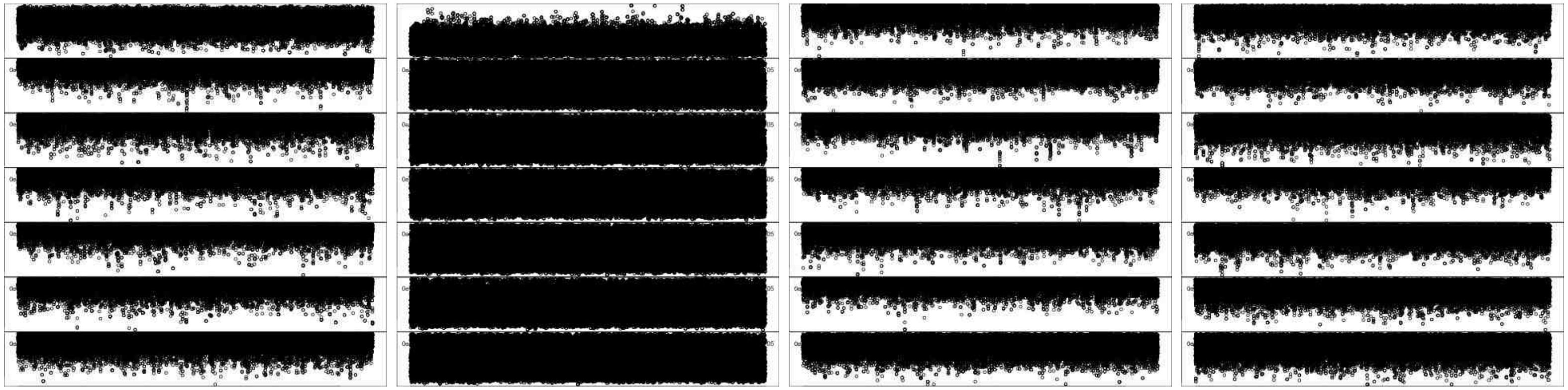
$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	1.5	0.3038	2.5	0.1923	1.5	0.2878

(f) Weapon Data Set 6

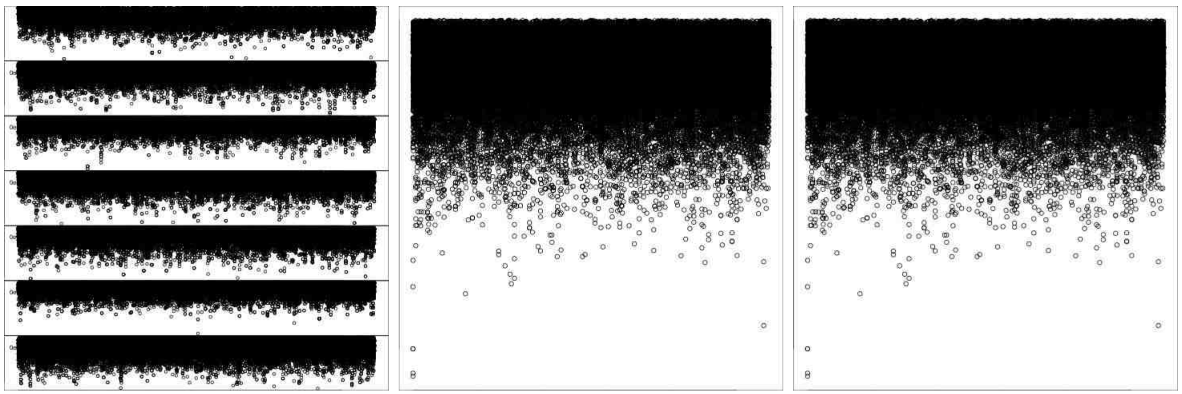
$\pi(\theta_s)$	Beta(2.9304, .0296)		Beta(.9408, .0192)		Uniform (0, 1)	
	σ_c	Acc Rate	σ_c	Acc Rate	σ_c	Acc Rate
1	4.5	0.28031	4.5	0.2857	4.5	0.2083

(g) Weapon Data Set 7

B.6. Trace plots are for the μ_i values that were simulated during the M-H algorithm. All trace plots were inspected to ensure that the chains visually converged.

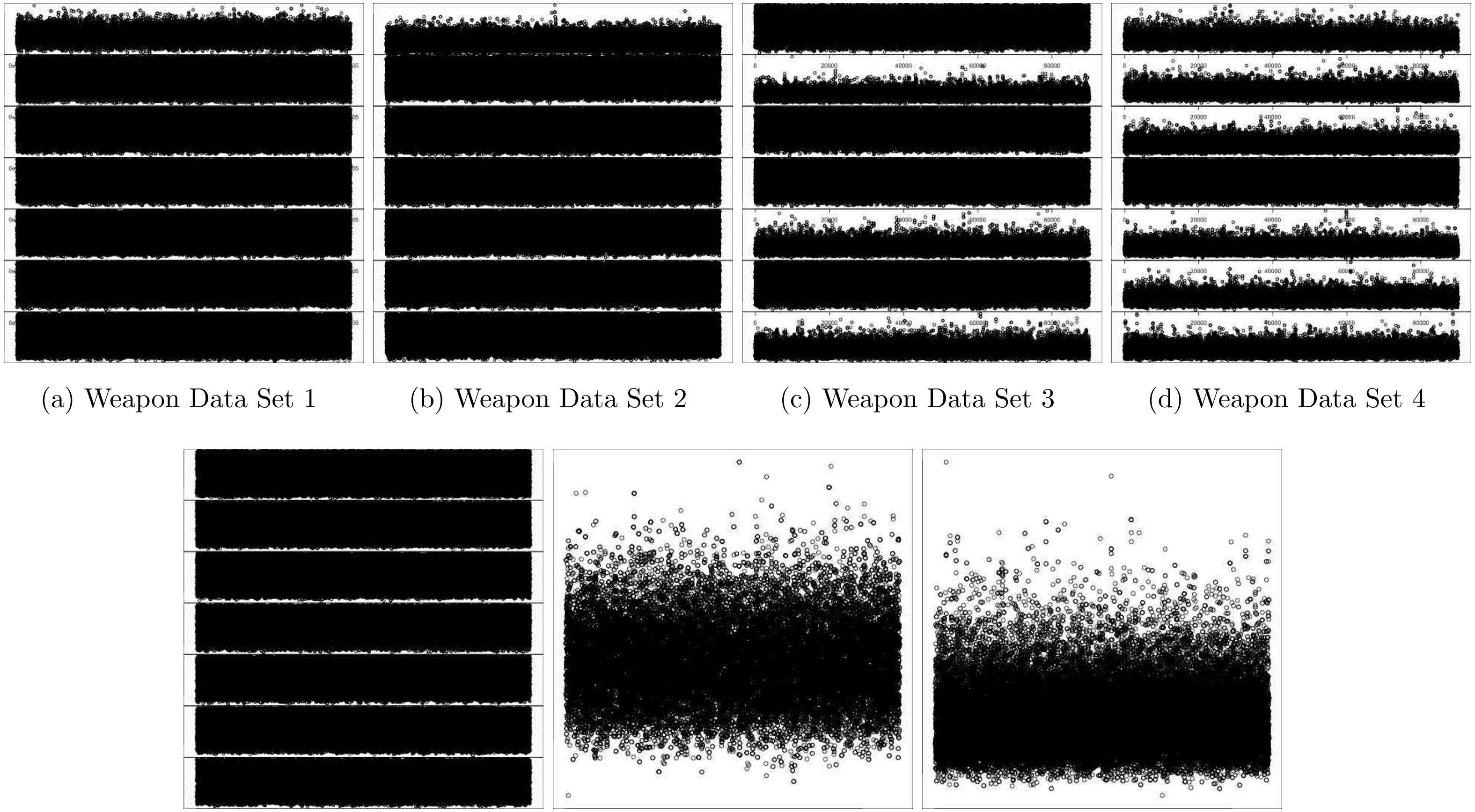


(a) Weapon Data Set 1 (b) Weapon Data Set 2 (c) Weapon Data Set 3 (d) Weapon Data Set 4



(e) Weapon Data Set 5 (f) Weapon Data Set 6 (g) Weapon Data Set 7

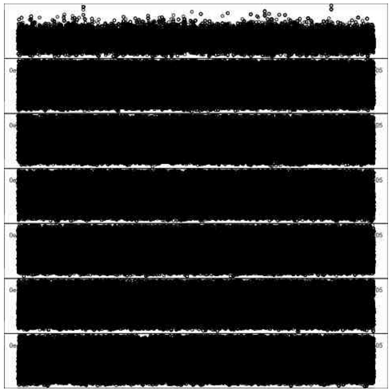
Figure B.4: The seven trace plots represent the trace plots for the μ_i draws from the M-H algorithm where the system prior was a Beta(2.9304, .0296). Each trace plot represents a distribution that has converged. Under the system prior Beta(2.9304, .0296) the M-H algorithm converged to a stationary distribution.



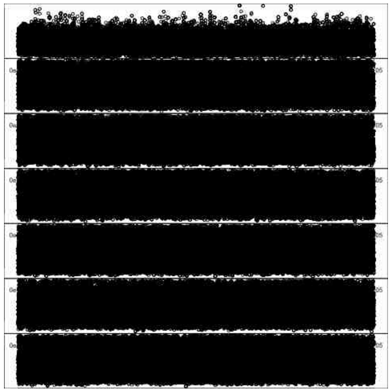
(a) Weapon Data Set 1 (b) Weapon Data Set 2 (c) Weapon Data Set 3 (d) Weapon Data Set 4

(e) Weapon Data Set 5 (f) Weapon Data Set 6 (g) Weapon Data Set 6

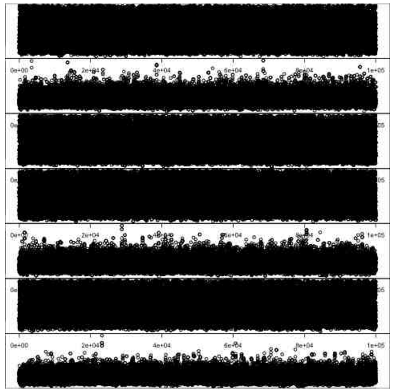
Figure B.5: The seven trace plots represent the trace plots for the μ_i draws from the M-H algorithm where the system prior was a Beta(.9408, .0192). Each trace plot represents a distribution that has converged. Under the system prior Beta(.9408, .0192) the M-H algorithm converged to a stationary distribution.



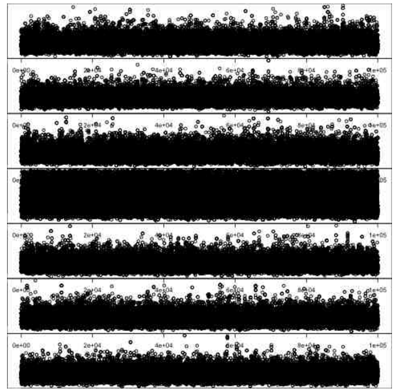
(a) Weapon Data Set 1



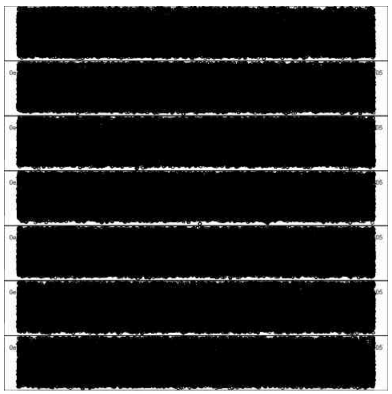
(b) Weapon Data Set 2



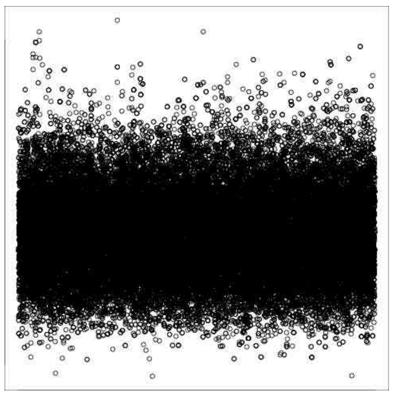
(c) Weapon Data Set 3



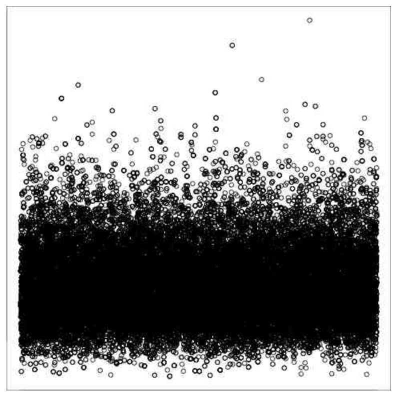
(d) Weapon Data Set 4



(e) Weapon Data Set 5



(f) Weapon Data Set 6



(g) Weapon Data Set 7

Figure B.6: The seven trace plots represent the trace plots for the μ_i draws from the M-H algorithm where the system prior was a Uniform(0, 1). Each trace plot represents a distribution that has converged. Under the system prior Uniform(0, 1) the M-H algorithm converged to a stationary distribution.

The Geweke diagnostic statistics for the weapons data sets are shown in the Table B.9.

Table B.9: *The Geweke diagnostics for the weapons system are shown in this table. One Geweke diagnostics is given for each component for each system prior.*

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	0.9925	1.0326	-1.7745
2	-1.1829	-1.9139	0.6532
3	-0.8969	-0.4063	-0.7893
4	0.9765	-0.6696	0.0660
5	-0.3033	0.8482	-1.7007
6	-0.0444	-1.0775	-0.2595
7	0.5360	0.4635	0.2956

(a) Weapon Data Set 1

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	-0.9037	2.6354	-1.1485
2	0.6242	-1.0851	1.1389
3	0.6806	-0.5422	0.2155
4	-0.4196	-0.8299	-0.4443
5	0.5489	-0.4140	-0.7143
6	-0.3613	-0.8700	0.2883
7	0.2424	1.1015	-0.5357

(b) Weapon Data Set 2

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	0.2970	-0.1687	1.1093
2	-0.6309	0.7616	-1.6109
3	0.2228	1.1606	-0.7199
4	-0.3025	0.7400	-1.3869
5	-0.9670	-0.6739	-0.1472
6	-1.7636	-0.9048	1.2862
7	-0.1823	-1.9673	-1.7418

(c) Weapon Data Set 3

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	0.2827	0.9340	1.6689
2	0.2069	1.0569	-1.5809
3	-0.0520	0.3087	-1.2898
4	2.1167	1.0243	-0.4535
5	1.1342	-0.2573	1.1180
6	1.0400	0.1515	-1.5721
7	0.9605	0.6248	-1.3462

(d) Weapon Data Set 4

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	0.1148	-0.2330	0.8897
2	-1.0618	-0.7931	-3.2348
3	-1.3793	-0.6897	1.2391
4	-2.4769	-0.1768	0.3602
5	1.5716	2.1468	0.7764
6	0.6595	1.4855	-0.5765
7	0.3934	-1.9377	0.3117

(e) Weapon Data Set 5

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	-0.9829	0.9846	1.2330

(f) Weapon Data Set 6

$\pi(\theta_s)$	Beta(2.9304, .0296)	Beta(.9408, .0192)	Uniform (0, 1)
1	0.2535	0.9458	0.3928

(g) Weapon Data Set 7

Appendix C

Quantile Summaries

This appendix provides summaries of the mean, standard deviation, and 5 number quantile summary of each data set. Numerical summaries are for the component marginal posterior densities. Each table displays the results obtained from all three system priors; starting with the highly informative Beta prior on top and the non-informative Uniform prior at the bottom.

C.1 5 Component System

Table C.1 gives the numerical summaries for marginal posterior distributions of the 5 θ_i's for this model.

C.2 42 Component System

Table C.2 gives the numerical summaries for marginal posterior distributions of the 42 θ_i's for this model.

Table C.1: *The numerical summaries are given for each of the θ_i as well as for θ_s. Each table represents the posterior summaries obtained using one of the three system priors.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9964	0.0131	0.7613	0.9993	1.0000	1.0000	1.0000
2	0.9809	0.0183	0.8233	0.9732	0.9865	0.9943	1.0000
3	0.9983	0.0061	0.8709	0.9997	1.0000	1.0000	1.0000
4	0.9574	0.0393	0.5557	0.9395	0.9691	0.9868	1.0000
5	0.9013	0.0402	0.6419	0.8771	0.9059	0.9308	0.9920

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Quantile Summary				
	Expexted Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9963	0.0135	0.7320	0.9994	1.0000	1.0000	1.0000
2	0.9805	0.0188	0.8196	0.9726	0.9861	0.9943	1.0000
3	0.9982	0.0063	0.8379	0.9996	1.0000	1.0000	1.0000
4	0.9549	0.0413	0.6511	0.9361	0.9665	0.9857	1.0000
5	0.8994	0.0408	0.6559	0.8748	0.9043	0.9291	0.9897

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9961	0.0142	0.7350	0.9993	1.0000	1.0000	1.0000
2	0.9805	0.0189	0.8200	0.9728	0.9863	0.9942	1.0000
3	0.9983	0.0064	0.8554	0.9997	1.0000	1.0000	1.0000
4	0.9554	0.0410	0.6331	0.9369	0.9670	0.9859	1.0000
5	0.8995	0.0404	0.6940	0.8750	0.9041	0.9291	0.9955

(c) $\pi(\theta_s) \sim$ Unif(0, 1)

Table C.2: *Numerical summaries for 42 θ_i are given in the following three tables. Each table represents the summaries obtained using the different system priors.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.8682	0.0691	0.5020	0.8275	0.8786	0.9203	0.9963

Continued on Next Page . . .

Table C.2 – Continued

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
2	0.9973	0.0101	0.7896	0.9995	1.0000	1.0000	1.0000
3	0.9989	0.0039	0.8968	0.9998	1.0000	1.0000	1.0000
4	0.9991	0.0033	0.9117	0.9998	1.0000	1.0000	1.0000
5	0.9986	0.0054	0.8587	0.9997	1.0000	1.0000	1.0000
6	0.9979	0.0077	0.8211	0.9996	1.0000	1.0000	1.0000
7	0.9962	0.0140	0.6751	0.9994	1.0000	1.0000	1.0000
8	0.8668	0.0701	0.5197	0.8256	0.8773	0.9194	0.9971
9	0.9974	0.0097	0.8133	0.9995	1.0000	1.0000	1.0000
10	0.9989	0.0041	0.8962	0.9998	1.0000	1.0000	1.0000
11	0.9991	0.0033	0.9267	0.9998	1.0000	1.0000	1.0000
12	0.9986	0.0052	0.8766	0.9997	1.0000	1.0000	1.0000
13	0.9979	0.0077	0.8010	0.9996	1.0000	1.0000	1.0000
14	0.9964	0.0132	0.7459	0.9994	1.0000	1.0000	1.0000
15	0.8663	0.0691	0.5154	0.8252	0.8770	0.9178	0.9963
16	0.9976	0.0093	0.7762	0.9996	1.0000	1.0000	1.0000
17	0.9988	0.0043	0.8340	0.9997	1.0000	1.0000	1.0000
18	0.9990	0.0036	0.9017	0.9998	1.0000	1.0000	1.0000
19	0.9986	0.0052	0.8982	0.9997	1.0000	1.0000	1.0000
20	0.9979	0.0080	0.8199	0.9996	1.0000	1.0000	1.0000
21	0.9965	0.0129	0.6494	0.9993	1.0000	1.0000	1.0000
22	0.8669	0.0685	0.5039	0.8254	0.8769	0.9193	0.9970
23	0.9975	0.0096	0.7735	0.9995	1.0000	1.0000	1.0000
24	0.9988	0.0043	0.8928	0.9998	1.0000	1.0000	1.0000
25	0.9991	0.0034	0.9136	0.9998	1.0000	1.0000	1.0000
26	0.9986	0.0052	0.8948	0.9997	1.0000	1.0000	1.0000
27	0.9979	0.0078	0.8314	0.9996	1.0000	1.0000	1.0000
28	0.9964	0.0136	0.6903	0.9994	1.0000	1.0000	1.0000
29	0.8675	0.0689	0.5006	0.8277	0.8782	0.9185	0.9972
30	0.9973	0.0104	0.7909	0.9995	1.0000	1.0000	1.0000
31	0.9988	0.0042	0.9122	0.9997	1.0000	1.0000	1.0000
32	0.9991	0.0033	0.9366	0.9998	1.0000	1.0000	1.0000

Continued on Next Page . . .

Table C.2 – Continued

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
33	0.9986	0.0051	0.8959	0.9997	1.0000	1.0000	1.0000
34	0.9979	0.0079	0.8221	0.9996	1.0000	1.0000	1.0000
35	0.9964	0.0136	0.6941	0.9994	1.0000	1.0000	1.0000
36	0.8675	0.0687	0.5166	0.8273	0.8779	0.9189	0.9971
37	0.9973	0.0099	0.7815	0.9995	1.0000	1.0000	1.0000
38	0.9989	0.0041	0.9116	0.9998	1.0000	1.0000	1.0000
39	0.9991	0.0035	0.8853	0.9998	1.0000	1.0000	1.0000
40	0.9986	0.0051	0.8843	0.9997	1.0000	1.0000	1.0000
41	0.9978	0.0081	0.8338	0.9996	1.0000	1.0000	1.0000
42	0.9963	0.0135	0.6510	0.9993	1.0000	1.0000	1.0000

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.8593	0.0731	0.5020	0.8156	0.8700	0.9138	0.9966
2	0.9974	0.0098	0.7574	0.9996	1.0000	1.0000	1.0000
3	0.9989	0.0042	0.8919	0.9998	1.0000	1.0000	1.0000
4	0.9991	0.0035	0.9145	0.9998	1.0000	1.0000	1.0000
5	0.9986	0.0051	0.8376	0.9997	1.0000	1.0000	1.0000
6	0.9978	0.0084	0.7934	0.9997	1.0000	1.0000	1.0000
7	0.9963	0.0139	0.7110	0.9994	1.0000	1.0000	1.0000
8	0.8601	0.0722	0.5105	0.8165	0.8704	0.9147	0.9976
9	0.9973	0.0102	0.7303	0.9995	1.0000	1.0000	1.0000
10	0.9989	0.0043	0.9009	0.9998	1.0000	1.0000	1.0000
11	0.9991	0.0034	0.9237	0.9998	1.0000	1.0000	1.0000
12	0.9986	0.0053	0.8655	0.9998	1.0000	1.0000	1.0000
13	0.9979	0.0078	0.8288	0.9996	1.0000	1.0000	1.0000
14	0.9963	0.0143	0.5576	0.9994	1.0000	1.0000	1.0000
15	0.8590	0.0742	0.5038	0.8154	0.8693	0.9150	0.9977
16	0.9974	0.0101	0.7944	0.9996	1.0000	1.0000	1.0000

Continued on Next Page . . .

Table C.3 – Continued

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
17	0.9989	0.0042	0.8784	0.9998	1.0000	1.0000	1.0000
18	0.9991	0.0034	0.8616	0.9998	1.0000	1.0000	1.0000
19	0.9986	0.0053	0.8549	0.9997	1.0000	1.0000	1.0000
20	0.9980	0.0079	0.7707	0.9997	1.0000	1.0000	1.0000
21	0.9963	0.0142	0.5691	0.9995	1.0000	1.0000	1.0000
22	0.8592	0.0730	0.5005	0.8159	0.8694	0.9141	0.9973
23	0.9974	0.0100	0.7532	0.9996	1.0000	1.0000	1.0000
24	0.9989	0.0042	0.8866	0.9998	1.0000	1.0000	1.0000
25	0.9991	0.0034	0.9262	0.9998	1.0000	1.0000	1.0000
26	0.9986	0.0053	0.8208	0.9997	1.0000	1.0000	1.0000
27	0.9979	0.0082	0.7892	0.9996	1.0000	1.0000	1.0000
28	0.9962	0.0145	0.6718	0.9994	1.0000	1.0000	1.0000
29	0.8597	0.0728	0.5027	0.8166	0.8706	0.9148	0.9971
30	0.9974	0.0103	0.7117	0.9996	1.0000	1.0000	1.0000
31	0.9988	0.0046	0.9139	0.9998	1.0000	1.0000	1.0000
32	0.9991	0.0034	0.9306	0.9998	1.0000	1.0000	1.0000
33	0.9986	0.0053	0.8811	0.9998	1.0000	1.0000	1.0000
34	0.9978	0.0081	0.8432	0.9996	1.0000	1.0000	1.0000
35	0.9961	0.0148	0.6677	0.9994	1.0000	1.0000	1.0000
36	0.8590	0.0734	0.5016	0.8159	0.8703	0.9135	0.9968
37	0.9974	0.0098	0.7402	0.9996	1.0000	1.0000	1.0000
38	0.9989	0.0043	0.8976	0.9998	1.0000	1.0000	1.0000
39	0.9991	0.0034	0.8934	0.9998	1.0000	1.0000	1.0000
40	0.9986	0.0052	0.8396	0.9998	1.0000	1.0000	1.0000
41	0.9978	0.0081	0.8352	0.9996	1.0000	1.0000	1.0000
42	0.9964	0.0138	0.6122	0.9995	1.0000	1.0000	1.0000

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.8609	0.0724	0.5020	0.8182	0.8720	0.9149	0.9974
2	0.9969	0.0108	0.7945	0.9993	1.0000	1.0000	1.0000
3	0.9987	0.0045	0.8751	0.9997	1.0000	1.0000	1.0000
4	0.9990	0.0036	0.8985	0.9997	1.0000	1.0000	1.0000
5	0.9984	0.0057	0.8792	0.9996	1.0000	1.0000	1.0000
6	0.9975	0.0086	0.8390	0.9995	1.0000	1.0000	1.0000
7	0.9957	0.0152	0.6771	0.9991	1.0000	1.0000	1.0000
8	0.8606	0.0721	0.5058	0.8183	0.8714	0.9141	0.9983
9	0.9969	0.0109	0.7711	0.9993	1.0000	1.0000	1.0000
10	0.9987	0.0047	0.8800	0.9997	1.0000	1.0000	1.0000
11	0.9990	0.0035	0.9195	0.9997	1.0000	1.0000	1.0000
12	0.9983	0.0061	0.8592	0.9996	1.0000	1.0000	1.0000
13	0.9976	0.0086	0.8293	0.9995	1.0000	1.0000	1.0000
14	0.9958	0.0145	0.7144	0.9990	1.0000	1.0000	1.0000
15	0.8609	0.0720	0.5055	0.8188	0.8714	0.9145	0.9966
16	0.9969	0.0109	0.7498	0.9994	1.0000	1.0000	1.0000
17	0.9987	0.0045	0.8862	0.9997	1.0000	1.0000	1.0000
18	0.9989	0.0040	0.8839	0.9997	1.0000	1.0000	1.0000
19	0.9983	0.0058	0.8826	0.9996	1.0000	1.0000	1.0000
20	0.9975	0.0089	0.8146	0.9994	1.0000	1.0000	1.0000
21	0.9958	0.0147	0.5744	0.9991	1.0000	1.0000	1.0000
22	0.8602	0.0720	0.5019	0.8171	0.8709	0.9144	0.9982
23	0.9969	0.0107	0.7391	0.9993	1.0000	1.0000	1.0000
24	0.9987	0.0045	0.9056	0.9997	1.0000	1.0000	1.0000
25	0.9989	0.0038	0.9198	0.9997	1.0000	1.0000	1.0000
26	0.9984	0.0058	0.8461	0.9996	1.0000	1.0000	1.0000
27	0.9975	0.0090	0.8027	0.9995	1.0000	1.0000	1.0000
28	0.9957	0.0146	0.7604	0.9990	1.0000	1.0000	1.0000
29	0.8608	0.0722	0.5001	0.8194	0.8714	0.9138	0.9974
30	0.9970	0.0106	0.7869	0.9994	1.0000	1.0000	1.0000
31	0.9987	0.0044	0.8799	0.9997	1.0000	1.0000	1.0000

Continued on Next Page . . .

Table C.4 – Continued

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
32	0.9990	0.0036	0.9386	0.9997	1.0000	1.0000	1.0000
33	0.9984	0.0055	0.8698	0.9996	1.0000	1.0000	1.0000
34	0.9976	0.0086	0.7808	0.9994	1.0000	1.0000	1.0000
35	0.9957	0.0150	0.6308	0.9990	1.0000	1.0000	1.0000
36	0.8603	0.0722	0.5047	0.8177	0.8707	0.9138	0.9989
37	0.9969	0.0112	0.7079	0.9993	1.0000	1.0000	1.0000
38	0.9987	0.0047	0.8464	0.9997	1.0000	1.0000	1.0000
39	0.9989	0.0037	0.9154	0.9997	1.0000	1.0000	1.0000
40	0.9983	0.0059	0.8675	0.9996	1.0000	1.0000	1.0000
41	0.9975	0.0088	0.7629	0.9994	1.0000	1.0000	1.0000
42	0.9958	0.0147	0.7041	0.9991	1.0000	1.0000	1.0000

C.3 Full System Model

Table C.5 gives the numerical summaries for marginal posterior distributions of the 4 θ_i's for this model.

Table C.5: *Numerical posterior summaries for the four θ_i's without covariate information of the full system model. Each table gives the simulation results using a different prior for θ_s.*

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
θ_1	0.9966	0.0122	0.7859	0.9994	1.0000	1.0000	1.0000
θ_2	0.9985	0.0057	0.8826	0.9997	1.0000	1.0000	1.0000
θ_3	0.9593	0.0382	0.6353	0.9429	0.9705	0.9874	1.0000
θ_4	0.9047	0.0394	0.6615	0.8811	0.9093	0.9337	0.9910

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
θ_1	0.9966	0.0120	0.8173	0.9993	1.0000	1.0000	1.0000
θ_2	0.9985	0.0055	0.8946	0.9997	1.0000	1.0000	1.0000
θ_3	0.9593	0.0383	0.6320	0.9424	0.9704	0.9876	1.0000
θ_4	0.9056	0.0390	0.6751	0.8821	0.9105	0.9341	0.9937

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
θ_1	0.9880	0.0244	0.6891	0.9875	0.9982	0.9999	1.0000
θ_2	0.9946	0.0111	0.8508	0.9945	0.9992	1.0000	1.0000
θ_3	0.9504	0.0424	0.6267	0.9311	0.9614	0.9818	1.0000
θ_4	0.8998	0.0405	0.6240	0.8749	0.9046	0.9295	0.9916

(c) $\pi(\theta_s) \sim$ Uniform (0, 1)

Table C.6 shows the numerical summaries for the fifth component as a function of age.

Table C.6: *Numerical posterior distribution summary statistics are shown for θ_5. Each table represents the results based upon different system priors. Each line represents the posterior summary information for various ages.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
0.00	0.9857	0.0120	0.8844	0.9802	0.9889	0.9947	1.0000
0.25	0.9804	0.0144	0.8599	0.9731	0.9837	0.9911	1.0000
0.50	0.9723	0.0182	0.7017	0.9632	0.9760	0.9854	1.0000
0.75	0.9599	0.0249	0.4556	0.9482	0.9646	0.9768	0.9999
1.00	0.9405	0.0366	0.2294	0.9244	0.9470	0.9647	0.9998

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
0.00	0.9856	0.0120	0.8929	0.9799	0.9888	0.9945	1.0000
0.25	0.9803	0.0143	0.8361	0.9731	0.9834	0.9909	1.0000
0.50	0.9722	0.0181	0.6988	0.9632	0.9757	0.9853	1.0000
0.75	0.9599	0.0247	0.5135	0.9483	0.9643	0.9768	0.9998
1.00	0.9407	0.0364	0.3244	0.9244	0.9471	0.9647	0.9994

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
0.00	0.9859	0.0117	0.8820	0.9803	0.9888	0.9946	1.0000
0.25	0.9805	0.0141	0.8322	0.9734	0.9836	0.9910	1.0000
0.50	0.9725	0.0180	0.7073	0.9634	0.9759	0.9854	1.0000
0.75	0.9601	0.0248	0.4979	0.9484	0.9645	0.9771	0.9998
1.00	0.9407	0.0369	0.2893	0.9244	0.9474	0.9652	0.9996

(c) $\pi(\theta_s) \sim$ Uniform (0, 1)

C.4 Weapons System Data Set 1

Table C.7 gives the numerical summaries for marginal posterior distributions of the 7 θ_i's for this model.

Table C.7: *Numerical posterior distribution summary statistics are shown for the first weapon data set with three failures occurring in the first component.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.8303	0.0769	0.5033	0.7829	0.8403	0.8870	0.9916
2	0.9968	0.0116	0.7772	0.9995	1.0000	1.0000	1.0000
3	0.9987	0.0046	0.8932	0.9998	1.0000	1.0000	1.0000
4	0.9991	0.0034	0.9184	0.9998	1.0000	1.0000	1.0000
5	0.9984	0.0062	0.8560	0.9997	1.0000	1.0000	1.0000
6	0.9975	0.0091	0.7926	0.9996	1.0000	1.0000	1.0000
7	0.9952	0.0171	0.6913	0.9992	1.0000	1.0000	1.0000

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.8300	0.0784	0.5021	0.7813	0.8403	0.8882	0.9960
2	0.9928	0.0179	0.7149	0.9953	0.9997	1.0000	1.0000
3	0.9974	0.0066	0.8743	0.9983	0.9999	1.0000	1.0000
4	0.9979	0.0054	0.9108	0.9986	0.9999	1.0000	1.0000
5	0.9965	0.0090	0.8432	0.9977	0.9998	1.0000	1.0000
6	0.9946	0.0137	0.7528	0.9963	0.9998	1.0000	1.0000
7	0.9894	0.0256	0.6074	0.9925	0.9995	1.0000	1.0000

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.8300	0.0784	0.5021	0.7813	0.8403	0.8882	0.9960
2	0.9928	0.0179	0.7149	0.9953	0.9997	1.0000	1.0000
3	0.9974	0.0066	0.8743	0.9983	0.9999	1.0000	1.0000
4	0.9979	0.0054	0.9108	0.9986	0.9999	1.0000	1.0000
5	0.9965	0.0090	0.8432	0.9977	0.9998	1.0000	1.0000
6	0.9946	0.0137	0.7528	0.9963	0.9998	1.0000	1.0000
7	0.9894	0.0256	0.6074	0.9925	0.9995	1.0000	1.0000

(c) $\pi(\theta_s) \sim$ Unif(0, 1)

C.5 Weapons System Data Set 2

Table C.8 gives the numerical summaries for marginal posterior distributions of the 7 θ_i's for this model.

Table C.8: *Numerical posterior distribution summary statistics are shown for the second weapon data set with six failures occurring in the first component.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.7167	0.0858	0.5001	0.6575	0.7210	0.7798	0.9590
2	0.9972	0.0104	0.7295	0.9995	1.0000	1.0000	1.0000
3	0.9989	0.0042	0.8970	0.9998	1.0000	1.0000	1.0000
4	0.9991	0.0035	0.9112	0.9998	1.0000	1.0000	1.0000
5	0.9985	0.0055	0.8631	0.9997	1.0000	1.0000	1.0000
6	0.9977	0.0083	0.8261	0.9996	1.0000	1.0000	1.0000
7	0.9962	0.0142	0.6400	0.9994	1.0000	1.0000	1.0000

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.7035	0.0869	0.5000	0.6413	0.7069	0.7683	0.9756
2	0.9970	0.0114	0.6591	0.9995	1.0000	1.0000	1.0000
3	0.9989	0.0043	0.8807	0.9998	1.0000	1.0000	1.0000
4	0.9991	0.0034	0.9301	0.9998	1.0000	1.0000	1.0000
5	0.9985	0.0055	0.8525	0.9997	1.0000	1.0000	1.0000
6	0.9978	0.0079	0.8237	0.9996	1.0000	1.0000	1.0000
7	0.9959	0.0153	0.6816	0.9993	1.0000	1.0000	1.0000

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.7032	0.0872	0.5001	0.6404	0.7074	0.7677	0.9599
2	0.9934	0.0168	0.7435	0.9957	0.9997	1.0000	1.0000
3	0.9975	0.0066	0.8993	0.9983	0.9999	1.0000	1.0000
4	0.9980	0.0052	0.9045	0.9986	0.9999	1.0000	1.0000
5	0.9967	0.0086	0.8548	0.9978	0.9999	1.0000	1.0000
6	0.9951	0.0124	0.7607	0.9966	0.9998	1.0000	1.0000
7	0.9908	0.0233	0.6118	0.9941	0.9996	1.0000	1.0000

(c) $\pi(\theta_s) \sim$ Uniform (0, 1)

C.6 Weapons System Data Set 3

Table C.9 gives the numerical summaries for marginal posterior distributions of the 7 θ_i's for this model.

Table C.9: *Numerical posterior distribution summary statistics are shown for the third weapon data set where three failures occurred; one failure in three different components.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9966	0.0126	0.7659	0.9994	1.0000	1.0000	1.0000
2	0.9487	0.0460	0.5838	0.9269	0.9614	0.9829	1.0000
3	0.9988	0.0046	0.9016	0.9997	1.0000	1.0000	1.0000
4	0.9990	0.0034	0.9273	0.9998	1.0000	1.0000	1.0000
5	0.9753	0.0237	0.7781	0.9650	0.9824	0.9925	1.0000
6	0.9975	0.0094	0.7811	0.9996	1.0000	1.0000	1.0000
7	0.9231	0.0657	0.5019	0.8892	0.9407	0.9739	1.0000

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9966	0.0126	0.7248	0.9994	1.0000	1.0000	1.0000
2	0.9467	0.0469	0.5888	0.9244	0.9590	0.9821	1.0000
3	0.9987	0.0049	0.8514	0.9997	1.0000	1.0000	1.0000
4	0.9990	0.0037	0.9155	0.9998	1.0000	1.0000	1.0000
5	0.9750	0.0241	0.7571	0.9650	0.9822	0.9926	1.0000
6	0.9974	0.0096	0.8171	0.9995	1.0000	1.0000	1.0000
7	0.9176	0.0698	0.5035	0.8833	0.9356	0.9710	1.0000

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9919	0.0198	0.7003	0.9943	0.9997	1.0000	1.0000
2	0.9425	0.0491	0.5612	0.9196	0.9552	0.9795	1.0000
3	0.9973	0.0069	0.8865	0.9982	0.9999	1.0000	1.0000
4	0.9979	0.0053	0.9142	0.9986	0.9999	1.0000	1.0000
5	0.9729	0.0245	0.7665	0.9617	0.9799	0.9911	1.0000
6	0.9944	0.0144	0.7495	0.9962	0.9997	1.0000	1.0000
7	0.9142	0.0691	0.5153	0.8787	0.9313	0.9667	1.0000

(c) $\pi(\theta_s) \sim$ Uniform (0, 1)

C.7 Weapons System Data Set 4

Table C.10 gives the numerical summaries for marginal posterior distributions of the 7 θ_i's for this model.

Table C.10: *Numerical posterior distribution summary statistics are shown for the fourth weapon data set where six failures occurred; one failure in six different components.*

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9281	0.0670	0.5022	0.8980	0.9483	0.9783	1.0000
2	0.9370	0.0590	0.5120	0.9108	0.9545	0.9809	1.0000
3	0.9805	0.0193	0.7632	0.9725	0.9865	0.9943	1.0000
4	0.9990	0.0038	0.9203	0.9998	1.0000	1.0000	1.0000
5	0.9734	0.0259	0.7366	0.9628	0.9811	0.9922	1.0000
6	0.9542	0.0435	0.5596	0.9358	0.9672	0.9862	1.0000
7	0.8906	0.0952	0.5004	0.8420	0.9174	0.9645	1.0000

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9230	0.0704	0.5070	0.8911	0.9431	0.9758	1.0000
2	0.9328	0.0624	0.5032	0.9054	0.9507	0.9794	1.0000
3	0.9800	0.0194	0.7986	0.9720	0.9858	0.9941	1.0000
4	0.9990	0.0038	0.9292	0.9998	1.0000	1.0000	1.0000
5	0.9728	0.0264	0.7230	0.9619	0.9809	0.9920	1.0000
6	0.9521	0.0454	0.5880	0.9334	0.9658	0.9854	1.0000
7	0.8779	0.1036	0.5000	0.8225	0.9066	0.9596	1.0000

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

			Quantile Summary				
	Expected Reliability	Std Dev	0%	25%	50%	75%	100%
1	0.9158	0.0722	0.5027	0.8806	0.9359	0.9699	1.0000
2	0.9269	0.0639	0.5015	0.8973	0.9445	0.9742	1.0000
3	0.9778	0.0203	0.7902	0.9690	0.9834	0.9926	1.0000
4	0.9978	0.0058	0.8915	0.9985	0.9999	1.0000	1.0000
5	0.9698	0.0276	0.6975	0.9579	0.9775	0.9898	1.0000
6	0.9471	0.0475	0.5530	0.9250	0.9599	0.9828	1.0000
7	0.8690	0.1053	0.5006	0.8098	0.8960	0.9526	1.0000

(c) $\pi(\theta_s) \sim$ Uniform (0, 1)

C.8 Weapons System Data Set 5

Table C.11 gives the numerical summaries for marginal posterior distributions of the 7 θ_i's for this model.

Table C.11: *Numerical posterior distribution summary statistics are shown for the fifth weapon data set where zero failures occurred.*

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
1	0.9900	0.0256	0.6412	0.9953	1.0000	1.0000	1.0000
2	0.9913	0.0229	0.6560	0.9963	1.0000	1.0000	1.0000
3	0.9975	0.0079	0.8628	0.9994	1.0000	1.0000	1.0000
4	0.9982	0.0058	0.9064	0.9995	1.0000	1.0000	1.0000
5	0.9965	0.0107	0.7892	0.9990	1.0000	1.0000	1.0000
6	0.9935	0.0180	0.7595	0.9978	1.0000	1.0000	1.0000
7	0.9849	0.0350	0.5833	0.9897	0.9999	1.0000	1.0000

(a) $\pi(\theta_s) \sim$ Beta(2.9304, .0296)

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
1	0.9893	0.0275	0.6009	0.9947	0.9999	1.0000	1.0000
2	0.9904	0.0247	0.6424	0.9953	0.9999	1.0000	1.0000
3	0.9975	0.0078	0.8670	0.9993	1.0000	1.0000	1.0000
4	0.9982	0.0060	0.8899	0.9996	1.0000	1.0000	1.0000
5	0.9968	0.0097	0.8165	0.9990	1.0000	1.0000	1.0000
6	0.9936	0.0179	0.7376	0.9978	1.0000	1.0000	1.0000
7	0.9841	0.0368	0.5527	0.9884	0.9999	1.0000	1.0000

(b) $\pi(\theta_s) \sim$ Beta(.9408, .0192)

	Expected Reliability	Std Dev	Quantile Summary 0%	25%	50%	75%	100%
1	0.9860	0.0287	0.5495	0.9862	0.9990	1.0000	1.0000
2	0.9873	0.0264	0.6632	0.9876	0.9991	1.0000	1.0000
3	0.9964	0.0088	0.8516	0.9974	0.9998	1.0000	1.0000
4	0.9973	0.0068	0.8875	0.9980	0.9999	1.0000	1.0000
5	0.9950	0.0119	0.7922	0.9962	0.9997	1.0000	1.0000
6	0.9906	0.0205	0.6565	0.9921	0.9995	1.0000	1.0000
7	0.9784	0.0411	0.5351	0.9757	0.9979	1.0000	1.0000

(c) $\pi(\theta_s) \sim$ Uniform (0, 1)

Appendix D

Glossary

Table D.1: *This is the notation used in this document with definitions of what each notation represents.*

Notation	Definition
Basic Notation	
$\boldsymbol{\Theta}$	Denotes all unknown parameters
θ_i	Reliability of component i
i	Index for components; $i = 1 \ldots n_c$
n_c	Total number of components in the system
n_0	Number of components without covariates (full system model only)
n_p	Number of components with covariates (full system model only)
θ_s	Reliability of the system
$\mathbf{y}$	Vector of 0's and 1's representing the test data
y_{ij}	Represents the j^{th} test of component i
y_{sj}	Represents the j^{th} test of the system
j	Index for component tests; $j = 1 \ldots n_i$
n_i	Number of tests for component i
n_s	Number of successful system tests
n_t	Number of overall system tests
Covariate Notation	

Continued on Next Page…

Table D.1 – Continued

Notation	Definition
X_{ij}	The j^{th} set of covariate values for component i
m_i	Number of covariates for component i
X_i	An $m \times j$ matrix of all covariate information for component i
$\boldsymbol{\beta_i}$	Vector of all parameter estimates for component i
β_{ik}	The k^{th} parameter estimate for component i
θ_{ij}	The reliability of the i^{th} component at the j^{th} covariate values
$\boldsymbol{\mu}$	Vector of known mean parameter values of the $\boldsymbol{\beta_i}$ prior MVN distribution
Σ	The known variance parameter value of the $\boldsymbol{\beta_i}$ prior MVN distribution; can also be written $\sigma^2(X_i'X_i)^{-1}$
$(X_i'X_i)$	Design matrix of the covariates for component i
Simulation Notation	
σ_c	Candidate sigma value for proposal distribution of the MCMC simulations
$\theta_i^{[t]}$	t^{th} simulated value for the i^{th} component
T	Number of MCMC iterations

Appendix E

R Code

E.1 R Code for Uniform and Beta Simulations

This section gives the code that was used for simulating the product of n_c Beta distributions that resulted in either a Uniform or Beta system reliability that was used in Section 3.2.

```
> ###################
> ###
> ###  Solving Parameters for a System with a Uniform Prior Distribution
> ###        (Beta Distribution on Components)
> ###        Rebecca L. Lilley
> ###        April 24, 2013
> ###################
>
> ###Beta Parameters (a, b) for K=1
> m <- .5
> k <- 1
```

```
> con <-(1- m^(1/k))/m^(1/k)
> a.fun <- con/( (2+con) * ((1/( 12 *(1/(1+con))^(2*k)) +1)^(1/k) -1 ) )
> b.fun <-a.fun*con

> nint <- 100000
> theta.1 <- numeric(nint)
> for(i in 1:nint){
+         theta.1[i] <- rbeta(1, a.fun, b.fun)
+ }
> ###################
> ###
> ###        Solving Parameters for a System with a Beta Prior Distribution
> ###        (Beta Distribution on Components)
> ###
> ###################
>
> ###K = 1
> ###Select the mean and variance of the system...
>###solve for system parameters (a, b)
> r <- .8
> vr <- 1/100
> a <- ((1-r)/r)/(vr*(1+2*(1-r)/r+((1-r)/r)^2)*(1+(1-r)/r))-1/(1+(1-r)/r)
> b <- a*(1-r)/r

> ###Solve for the component prior hyperparameters (c, d)
> m <- r
> k <- 1
> con <-(1- m^(1/k))/m^(1/k)
> c.fun <- con/( (2+con) * ((1/( (1/vr) *(1/(1+con))^(2*k)) +1)^(1/k)-1))
```

```
> d.fun <-c.fun*con


> ####Simulation for R* = .8, VR*=.01
> nint <- 10000
> theta.1 <- numeric(nint)
> for(i in 1:nint){
+                 theta.1[i] <- rbeta(1, c.fun, d.fun)
+ }
>
> ##Repeat above process for various mean/variance
>##combinations at different K's
```

E.2 R Code of Functions For All Models

This section gives the functions that were used to run the 5 component, 42 component, and 7 weapons system data sets. The code includes functions for generating the a_p, b_p parameters, performing the μ_i transformation and running the Metropolis-Hastings algorithm. The second half of the code gives the functions that are used for generating the summary statistics. The final part of the code gives the functions that were used to generate the χ^2 lack-of-fit diagnostics.

```
> ###Functions to estimate priors based upon different system priors
> estBetaParams <- function(mu, var) {
+   a <- ((1 - mu) / var - 1 / mu) * mu ^ 2
+   b <- a * (1 / mu - 1)
+   return(params = list(a=a, b=b))
+ }
> estCompPriors <- function(m, ncomp, vr){
```

```
+   con <- (1- m^(1/nc))/(m^(1/nc))
+         a <- con/( (2+con)*((1/( (1/vr) *(1/(1+con))^(2*nc)) +1)^(1/nc)-1))
+     b <-a*con
+         return(priors = list(a=a, b=b))
+ }
> #############
> #          Mu_i Transformation for MCMC (ALL MODELS)
> #############
> log.mu.post <- function(j, mu, sumy.ij, n.t, c.fun, d.fun){
+         dbeta((exp(mu[j])/(1+exp(mu[j]))), c.fun+sumy.ij[j],
        d.fun+n.t[j]-sumy.ij[j], log=T) + xs[2]*log(exp(mu[j])
        /(1+exp(mu[j]))) + xs[3]*log(1-prod(exp(mu)/(1+exp(mu))))
        + (mu[j]) -2*log(1+exp(mu[j]))
+ }
> #########
> #                Acceptance Function for Transformation
> #########
> r <- function(j, p.cand, mu, sumy.ij, n.t, a, b){
+         log.mu.post(j, p.cand, sumy.ij, n.t, a, b)
        - log.mu.post(j, mu, sumy.ij, n.t, a, b)
+ }
> ###########
> #                Mu Metropolis Function
> ###########
> mu.metrop.fun <- function(TT, ncomp, mu.init, csig, sumy.ij, n.t, a, b){
+         mu <- matrix(1, nrow=TT, ncol=ncomp)
+         mu[1,] <- mu.init
+         acc <- rep(0, ncomp)
+ for(i in 2:TT){
```

```
+         for(j in 1:ncomp){
+         mu[i, j] <- mu[i-1, j]
+         cand <- rnorm(1, mu[i-1, j], csig[j])
+         if(cand>0){
+                 if(cand<20){
+                 ttt<-mu[i-1,]
+                 ttt[j]<-cand
+                 accept <- r(j, ttt, mu[i-1,], sumy.ij, n.t, a, b)
+                 u <- runif(1, 0, 1)
+                 if(log(u) < accept){mu[i, j] <- cand;
                        acc[j] <- acc[j]+1}
+                 }
+                 }
+         }
+ }
+ list(mu=mu, acc=acc)
+ }
> #####
> #                Theta Complete Conditional
> #####
> log.post <- function (ind, theta, sumy.ij, n.t, xs, a, b){
+         (sumy.ij[ind] + a-1)*log(theta[ind]) +
        (n.t[ind] - sumy.ij[ind] + b-1)*log(1-theta[ind])+
        (xs[2])*sum(log(theta))+(xs[3])*log(1-prod(theta))
+ }
> #####
> #                Theta Acceptance Function
> #####
> r.theta <- function(ind, theta, ttt, sumy.ij, n.t, xs, a, b){
```

```
+         log.post(ind, ttt, sumy.ij, n.t, xs,a, b)
        - log.post(ind, theta[i-1,], sumy.ij, n.t, xs, a, b)
+ }
> ####
> #                Theta Metropolis-Hastings Function
> ####
> theta.metrop.fun <- function(TT, ncomp, theta.init, csig,
sumy.ij, n.t, a, b){
+         theta <- matrix(1, nrow=TT, ncol=ncomp)
+         theta[1,] <- theta.init
+         acc <- rep(0, ncomp)
+ for(i in 2:TT){
+ for(j in 1:ncomp){
+         theta[i, j] <- theta[i-1, j]
+         cand <- rnorm(1, theta[i-1, j], csig[j])
+         if(cand>0){
+                 if(cand<1){
+                 ttt<-theta[i-1,]
+                 ttt[j]<-cand
+                 accept <- r(j, ttt, theta[i-1,], sumy.ij, n.t, a, b)
+                 u <- runif(1, 0, 1)
+                 if(log(u) < accept){theta[i, j] <- cand;
                acc[j] <- acc[j]+1}
+                 }
+                 }
+         }
+ }
+ list(theta=theta, acc=acc)
+ }
```

```
##FINAL MODEL FUNCTIONS

##########
#Beta Complete Conditional for MCMC (Covariates Only)
##########
beta.cc <- function(beta.sims, sigma2, X, beta.mu, X.sim, mu){
        convert2theta <- 1/(1+exp(-X.sim \%*\% beta.sims))
        beta.post <- sum(-y.j*log(1+exp(-X\%*\% beta.sims))
        + (1-y.j)*(-X\%*\%beta.sims)-log(exp(-X\%*\%beta.sims)+1))
        - 1/2*t(beta.sims-beta.mu)\%*\% (sigma2*solve(t(X)\%*\%X))
        \%*\% (beta.sims-beta.mu) + (xs[2])*sum(-log(1+exp(-X.sim
        \%*\% beta.sims))) + (xs[3])*log(1-prod(exp(mu)/(1+exp(mu)))
        *prod(convert2theta))
        return(beta.post)
}

#############
#  Mu_i Transformation for Final Model
#############
log.mu.post <- function(j, mu, sumy.ij, n.t, c.fun, d.fun, X.sim, beta.sims){
        convert2theta <- 1/(1+exp(-X.sim \%*\% beta.sims))
        mu.post <-dbeta((exp(mu[j])/(1+exp(mu[j]))), c.fun+sumy.ij[j],
        d.fun+n.t[j]-sumy.ij[j], log=T) + xs[2]*log(exp(mu[j])/(1+exp(mu[j])))
        + xs[3]*log(1-prod(convert2theta)*prod(exp(mu)/(1+exp(mu)))) + (mu[j])
        return(mu.post)
}

### ACCEPTANCE FUNCTIONS FOR FINAL MODEL
```

```
r.mu <- function(j, ttt, mu, sumy.ij, n.t, a, b, X.sim, beta.sims){
        log.mu.post(j, ttt, sumy.ij, n.t, a, b, X.sim, beta.sims)
        - log.mu.post(j, mu, sumy.ij, n.t, a, b, X.sim, beta.sims)
}

r.beta <- function(ttt, beta.sims, sigma2, X, beta.mu, X.sim, mu){
        beta.cc(ttt, sigma2, X, beta.mu, X.sim, mu) - beta.cc(beta.sims, sigma2,
}

####
#        Metropolis Hastings Algorithm for Final Model
####

full.model.metrop <- function(niter, ncomp, mu.init, csig, r.mu,
nparam, beta.init, csigb, r.beta, sumy.ij, n.t, a, b, X.sim, beta.sims, sigma2,
mu <- matrix(0, nrow=(niter), ncol=ncomp)
mu[1,] <- mu.init
acc <- rep(0, ncomp)
beta.sims <- matrix(0, nrow=(niter), ncol=nparam)
beta.sims[1,] <- beta.init
accb <- rep(0, nparam)
for(i in 2:niter){
####Update Mus
        for(j in 1:ncomp){
                mu[i,j] <- mu[i-1, j]
                cand <- rnorm(1, mu[i-1, j], csig[j])
                if(cand>0){
                        if(cand<20){
                        ttt<-mu[i-1,]
```

```
                        ttt[j]<-cand
                        accept <- r.mu(j, ttt, mu[i-1,], sumy.ij,
                        n.t, a, b, X.sim, beta.sims[i-1,])
                        u <- runif(1, 0, 1)
                        if(log(u) < accept){mu[i, j] <- cand;
                        acc[j] <- acc[j]+1}
                        }
                }
        }
        ####Update Betas
                for(k in 1:nparam){
                beta.sims[i,k] <- beta.sims[i-1, k]
                cand <- rnorm(1, beta.sims[i-1, k], csigb[k])
                bbb <- matrix( beta.sims[i-1,], ncol=nparam)
                bbb[k] <- cand
                        acceptb <- r.beta(t(bbb), beta.sims[i-1,],
                        sigma2, X, beta.mu, X.sim, mu[i,])
                        u <- runif(1, 0, 1)
                        if(log(u) < acceptb){beta.sims[i, k] <- cand;
                        accb[k] <- accb[k]+1}
        }
}
return(list(mu=mu, acc=acc, beta.sims=beta.sims, accb=accb))
}

> ###Functions for posterior summary information
> param.summary <- function(sims, burn, niter){
+         param.mean <- apply(sims[(burn+1):niter,], 2, mean)
+         param.sd <- apply(sims[(burn+1):niter,], 2, sd)
```

```
+         param.q <- apply(sims[(burn+1):niter,], 2, quantile)
+         param.95 <- apply(sims[(burn+1):niter,],2, quantile,
        probs=c(.025, .975))
+         return(p.summary = list(param.mean=param.mean,
        param.sd=param.sd, param.q=param.q, param.95=param.95))
+ }
> sys.summary <- function(sys, burn, niter){
+         sys.mean <- mean(sys[(burn+1):niter])
+         sys.sd <- sd(sys[(burn+1):niter])
+         sys.q <- quantile(sys[(burn+1):niter])
+         sys.95 <- quantile(sys[(burn+1):niter], probs=c(.025, .975))
+         return(sys.summary = list(sys.mean=sys.mean,
        sys.sd=sys.sd, sys.q=sys.q, sys.95=sys.95))
+ }
> ###HPD SUMMARY
> hpd.int <- function(p, burn, sim, system){
+         sort.sys <- sort(system[(burn+1):(burn+sim)])
+         HPD <- p*sim
+         bound <- matrix(0, ncol=3, nrow=HPD)
+         for(i in 1:HPD){
+                 bound[i, 1] <- sort.sys[i]
+                 bound[i, 2] <- sort.sys[i+(1-p)*sim]
+                 bound[i, 3] <- bound[i,2]-bound[i,1]
+         }
+         ind2 <- which(bound[,3] == min(bound[,3]), arr.ind=T)
+         interval <- bound[ind2,]
+         return(interval)
+ }
> tables <- function(comp.r, csig, acc, niter, psummary, ssummary){
```

```
+         f.table1 <- cbind(comp.r[,1], comp.r[,2], compr[,3], comp.r[,4])
+         f.table2 <- cbind(csig, acc/niter)
+         f.table3 <- cbind(psummary$param.mean, psummary$param.sd,
        t(psummary$param.q))
+         f.table4 <- cbind(psummary$param.mean, psummary$param.sd,
        t(psummary$param.95))
+         f.table5 <- cbind(ssummary$sys.mean, ssummary$sys.sd,
        t(ssummary$sys.q))
+         f.table6 <- cbind(ssummary$sys.mean, ssummary$sys.sd,
        t(ssummary$sys.95))
+         data.table <- xtable(f.table1, digits=4)
+         acc.table <- xtable(f.table2, digits=4)
+         param.table1 <- xtable(f.table3, digits=4)
+         param.table2 <- xtable(f.table4, digits=4)
+         sys.table1 <- xtable(f.table5, digits=4)
+         sys.table2 <- xtable(f.table6, digits=4)
+         return(list(data.table=data.table, acc.table=acc.table,
         param.table1=param.table1, param.table2=
         param.table2, sys.table1=sys.table1, sys.table2=sys.table2))
+ }
>

###MODEL FIT DIAGNOSTICS
> zobs <- rep(0, sim)
> zsim <- rep(0, sim)

> simdraws <- function(diag.yij, n.t, theta, burn, sim, ncomp){
 for(t in 1:sim){
for(i in 1:ncomp){
```

```
        diag.yij[t, i] <- rbinom(1, n.t[i], theta[burn+t, i])
        }
        }
        return(diag.yij)
}

> z.values <- function(diag.yij, n.t, theta, burn, sim, ncomp, sumy.ij){
for(t in 1:sim){
for(i in 1:ncomp){
        zobs[t] <- sum(((sumy.ij[i]-n.t[i]*theta[burn+t, i])^2)
        /(n.t[i]*theta[burn+t, i]))
        zsim[t] <- sum(((diag.yij[t, i]-n.t[i]*theta[burn+t, i])^2)
        /(n.t[i]*theta[burn+t, i]))
}
}
        return(list(zobs=zobs, zsim=zsim))
}

>compare <- function(zsim, zobs){
        mean(zsim>zobs)
}
```

E.3 R Code for Executing Functions for Component and/or System Test Data

This section gives the code that was used for executing the 5 component system for one of the priors. It shows how the test data needs to be formatted to run through the functions given in the previous section.

```
##################
#
#           LOAD MODEL_FUNCTIONS.R
#
#################
####################
#         Simulated Data Analysis
#          5 Component System
#          Three Different Priors
#           Rebecca L. Lilley
#           May 26, 2013
####################

#Simulated Data
#Number of tests per component
> n.t <- c(11, 47, 34, 17, 45)
#Number of successes were generated from an random binomial
> sumy.ij <- c(11, 46, 34, 16, 40)
> failures <- n.t-sumy.ij
> prior.rel <- sumy.ij/n.t
> comp.r <- cbind(n.t, sumy.ij, failures, prior.rel)

> ncomp <- dim(comp.r)[[1]]

#System tests
> xs <- c(10, 9, 1, .9)
```

```
####################
> r.star1 <- .99
> vr.star1 <- .0025
####################

> model5.1 <- estCompPriors(r.star, ncomp, vr.star)
> burn <- 10000
> sim <- 100000
> niter <- burn + sim
> csig1 <- c(14, 3.5, 14, 3.5, 2.5)
> muinit5.1 <- c(1, 1, 1, 1, 1)
> muinit5.2 <- c(5, 5, 5, 5, 5)
> muinit5.3 <- c(3, 3, 3, 3, 3)

> chainmodel5.1 <- mu.metrop.fun(niter, ncomp, muinit5.1,
csig1, sumy.ij, n.t, model5.1$a, model5.1$b, xs)
> chainmodel5.2 <- mu.metrop.fun(niter, ncomp, muinit5.2,
csig1, sumy.ij, n.t, model5.1$a, model5.1$b, xs)
> chainmodel5.3 <- mu.metrop.fun(niter, ncomp, muinit5.3,
csig1, sumy.ij, n.t, model5.1$a, model5.1$b, xs)

> write.table(chainmodel5.1$mu, "/Users/Utah/Desktop/
MH_Results/chain1_model51.txt", sep=',')

#####
#                DIAGNOSTICS
#####
```

```
> jpeg("betahh_trace5.jpg")
> par(mfrow=c(5, 1), mar=c(0,0,0,0))
> for(i in 1:ncomp){
        plot(chainmodel5.1$mu[,1], ann=F)
> }
> dev.off()

> geweke.diag(chainmodel5.1$mu[(burn+1):niter])

> all.chains.model5 <- mcmc.list(as.mcmc(chainmodel5.1[[1]]),
as.mcmc(chainmodel5.2[[1]]),  as.mcmc(chainmodel5.3[[1]]))
> gelman.diag(all.chains.model5)


####ANALYSIS
> theta5.1 <- apply(chainmodel5.1$mu, 2, function(x) exp(x)/(1+exp(x)))
> sysrel5.1 <- apply(theta5.1, 1, prod)

> thetasum5.1 <- param.summary(theta5.1, burn, niter)
> syssum5.1 <- sys.summary(sysrel5.1, burn, niter)
> hpdtheta5.1 <- hpd.int(.05, burn, sim, sysrel5.1)
> theta5.1tables <- tables(comp.r, csig1, chainmodel5.1$acc,
niter, thetasum5.1, syssum5.1)

####After running through this code for the different system priors:
####SYSTEM COMPARISON
> mean(sysrel5.1[(burn+1):niter]<sysrel5.2[(burn+1):niter])
> mean(sysrel5.1[(burn+1):niter]<sysrel5.3[(burn+1):niter])
> mean(sysrel5.2[(burn+1):niter]<sysrel5.3[(burn+1):niter])
```

```
###MODEL FIT DIAGNOSTICS
mdraws.model5 <- simdraws(diag.yij, n.t, theta5.1, burn, sim, ncomp)
mzvalues.model5 <- z.values(mdraws.model5, n.t, theta5.1,
 burn, sim, ncomp, sumy.ij)
compare(mzvalues.model5$zsim, mzvalues.model5$zobs)
```

E.4 R Code for Full Model

This section gives the code that was used for executing the full model. This code differs from the previous sections because it has an extra section for converting the β_i to θ_i and calculating system reliability as a function of the covariates.

```
#####################
#####################
r.star <- .99
vr.star <- .0025
#####################

fullmodel.param1 <- estCompPriors(r.star, ncomp, vr.star)
burn <- 10000
sim <- 100000
niter <- burn+sim
```

```
####For Mu Updates
csig <- c(30, 30, 4.5, 2.5)
mu.init <- c(5, 5, 5, 5)

####For Beta Updates
beta.init <- c(3, -.5)
csigb <- c(2, 2)
sigma2 <- .5

chain1 <- full.model.metrop(niter, ncomp, mu.init, csig, r.mu,
nparam, beta.init, csigb, r.beta, sumy.ij, n.t, fullmodel.param1\$a,
fullmodel.param1\$b, X.sim, beta.sims, sigma2, beta.mu)
chain1\$acc/niter
chain1\$accb/niter

###
#                  DIAGNOSTICS
###

jpeg("fullmubetaHH_trace.jpg")
par(mfrow=c(4, 1), mar=c(0, 0, 0, 0))
for(i in 1:ncomp){
plot(chain1$mu[(burn+1):niter, i])
}
dev.off()

jpeg("fullmubetaHH_trace_beta.jpg")
par(mfrow=c(2, 1), mar=c(0, 0, 0, 0))
plot(beta.sims[(burn+1):niter, 1])
```

```
plot(beta.sims[(burn+1):niter, 2])
dev.off()

write.table(mu, "/Users/Utah/Desktop/full1_mu_0520.txt", sep=",")
write.table(beta.sims, "/Users/Utah/Desktop/full1_beta_0520.txt", sep=",")

###Converting Mu's to Theta
thetafull.sims1 <- apply(chain1$mu, 2, function(x) exp(x)/(1+exp(x)))

###Functions for posterior summary information
fulltheta1.1 <- param.summary(thetafull.sims1, burn, niter)
fulltheta1.1

fullbeta1.1 <- param.summary(chain1$beta.sims, burn, niter)
png("fullbetaHH_thetamu_boxplot.#png")
boxplot(thetafull.sims1[(burn+1):niter,1:5])
dev.off()

png("fullbetaHH_betamu_boxplot.#png")
boxplot(chain1$beta.sims[(burn+1):niter,1],
chain1$beta.sims[(burn+1):niter,2])
dev.off()

####Posterior Summaries for Covariate Component and
Overall System Reliability Using 5 Covariate Points
post.age <- seq(0, 1, length=5)
intercept <- c(rep(1, 5))
post.X <-cbind(intercept, post.age)
post.theta1 <- t(convert2theta(post.X, t(chain1$beta.sims)))
```

```
postthetasum1 <- param.summary(post.theta1, burn, niter)
postthetasum1

png("theta6mubetaHH_boxplot.#png")
boxplot(post.theta1[1,], post.theta1[2,], post.theta1[3,],
post.theta1[4,], post.theta1[5,])
dev.off()

sysrel.sub <- apply(thetafull.sims1, 1, prod)
sysrel.full1 <- sysrel.sub*(post.theta1)

syssum.full1 <- param.summary(sysrel.full1, burn, niter)
syssum.full1

tables.full1 <- tables(comp.r, csig, acc, niter, thetasum.full1, syssum.full1)
tables.full1

 hpd.int(.05, burn, sim, post.theta1[,1])
 hpd.int(.05, burn, sim, post.theta1[,2])
 hpd.int(.05, burn, sim, post.theta1[,3])
 hpd.int(.05, burn, sim, post.theta1[,4])
 hpd.int(.05, burn, sim, post.theta1[,5])

png("finalbetaHH_systemmu_boxplot.#png")
boxplot(sysrel.full1[,1], sysrel.full1[,2], sysrel.full1[,3],
sysrel.full1[,4], sysrel.full1[,5])
dev.off()
```

References

Anderson-Cook, C., Graves, T., Hamada, M., Hengartner, N., Johnson, V., Reese, C., et al. (2007). Bayesian stockpile reliability methodology for complex systems. *Military Operations Research*, *12*(2), 25–37.

Chib, S., & Greenberg, E. (1995). Understanding the metropolis-hastings algorithm. *The American Statistician*, *49*(4), 327–335.

Christensen, R., Johnson, W. O., Branscum, A. J., & Hanson, T. E. (2011). *Bayesian ideas and data analysis: An introduction for scientists and statisticians.* London: CRC Press.

Cowles, M. K., & Carlin, B. P. (1996). Markov chain monte carlo convergence diagnostics: a comparative review. *Journal of the American Statistical Association*, *91*(434), 883–904.

Cox, D. R., & Hinkley, D. V. (1979). *Theoretical statistics.* London: Chapman & Hall/CRC.

Derman, C., & Solomon, H. (1958). Development and evaluation of surveillance sampling plans. *Management Science*, *5*(1), 72–88.

Gelman, A. (2006). Prior distributions for variance parameters in hierarchical models (comment on article by browne and draper). *Bayesian analysis*, *1*(3), 515–534.

Gelman, A., & Rubin, D. B. (1992). Inference from iterative simulation using multiple sequences. *Statistical science*, 457–472.

Geweke, J., et al. (1991). *Evaluating the accuracy of sampling-based approaches to the calculation of posterior moments.* Minneapolis: Federal Reserve Bank of Minneapolis, Research Department.

Gilks, W. R., Richardson, S., & Spiegelhalter, D. J. (1996). *Markov chain monte carlo in practice* (Vol. 2). London: Chapman & Hall/CRC.

Goodman, L. A. (1962). The variance of the product of k random variables. *Journal of the American Statistical Association*, *57*(297), 54–60.

Graves, T. (2007). Design ideas for markov chain monte carlo software. *Journal of

Computational and Graphical Statistics, *16*(1), 24–43.

Hamada, M. (2005). Using degradation data to assess reliability. *Quality Engineering*, *17*(4), 615–620.

Hamada, M., Wilson, A. G., & Reese, C. S. (2008). *Bayesian reliability*. Manhatten: Springer.

Hillier, F. (1962). Surveillance programs for lots in storage. *Technometrics*, *4*(4), 515–530.

Johnson, V. (2004). A bayesian $\chi 2$ test for goodness-of-fit. *The Annals of Statistics*, *32*(6), 2361–2384.

Johnson, V., Graves, T., Hamada, M., & Reese, C. (2003). A hierarchical model for estimating the reliability of complex systems. *Bayesian statistics*, *7*, 199–213.

Kelly, E., & Vander Wiel, S. (2006). Planning and evaluating sampling designs to determine aging effects. *Los Alamos National Laboratory, Los Alamos, NM, USA, Tech. Rep. LA-UR-06-7244*.

Martz, H. F. (1982). *Bayesian reliability analysis*. Hoboken: Wiley Online Library.

Medalia, J. (2010). *Comprehensive nuclear-test-ban treaty: Background and current developments*. Washington D.C.: DIANE Publishing.

Roberts, G. O., Gelman, A., & Gilks, W. R. (1997). Weak convergence and optimal scaling of random walk metropolis algorithms. *The Annals of Applied Probability*, *7*(1), 110–120.

Vanderwiel, S. A., Wilson, A. G., Graves, T. L., & Reese, C. S. (2009). A random onset model for degradation of high-reliability systems. *Technometrics*, *53*(2), 163-172.

Wilson, A., Hamada, M., Anderson-Cook, C., Graves, T., Vander Wiel, S., Sims, B., et al. (2007). Statistical models for stockpile health assessment. *ADTSC nuclear weapons highlights; Los Alamos National Laboratory, Los Alamos, NM, USA, Tech. Rep. LALP-07-041*.

Yang, R., & Berger, J. O. (1996). *A catalog of noninformative priors*. Duke University: Institute of Statistics and Decision Sciences.