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The mathematics of the epicycloid : a historical journey with a modern perspective

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THE MATHEMATICS OF THE EPICYCLOID

A HISTORICAL JOURNEY WITH A MODERN PERSPECTIVE

BY

PRECIOUS ANDREW

B.S., MATHEMATICS, UNIVERSITY OF NEW MEXICO, 2007

THESIS

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Requirements for the Degree of

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ABSTRACT

The role of the epicycloid in both historical and mathematical contexts was studied through readings and research. Throughout history, circles have been considered perfect and thus very important shapes. Because epicycloids are constructed by combining circular motions, their historical value is evident. Epicycloids are traced through history, with special emphasis on their use in astronomy.

The epicycloid is also important from a purely mathematical perspective. The connection of the figure with Fourier series is analyzed and illustrated with various Matlab plots. Because of this connection, the power of the epicycloid as a modeling tool becomes clear.

The epicycloid has also made some more recent appearances and these are presented as well. Of note here is the Antikythera mechanism which is an ancient device incorporating epicycloids that was fairly recently discovered. Also of interest is a toy called the Spirograph, which uses epicycloids to create intricate patterns.

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1.1 Introduction to Epicycloids

In mathematics, the term epicycloid refers to the plane curve produced by letting a small circle roll along the circumference of a larger circle. Note that by increasing the size of the larger circle, we can produce the same curve by letting the small circle's center move around the circumference of the larger circle while the small circle itself rotates (see figure 1).

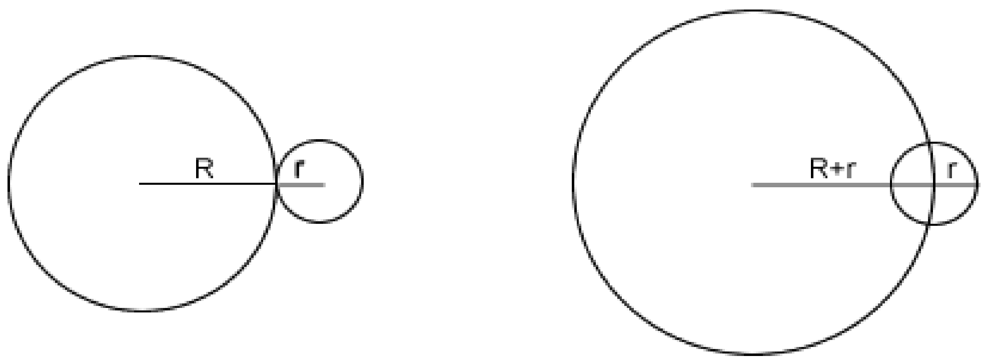


Figure 1: An epicycloid is the plane curve produced by letting a circle of radius r roll around a larger circle of radius R (illustrated above left). This process is equivalent to letting the center of the small circle move around a circle of radius R+r while the smaller circle itself rotates (illustrated above right).

In this construction, the smaller circle is called the epicycle and the larger circle is the deferent. There are basically only four parameters involved in producing an epicycloid: the radii of the respective circles and their rates of rotation (in the clockwise or counter-clockwise direction). With the simplicity of the geometry in mind, it is truly amazing what a wide array of figures we can produce. A few examples are shown in figure 2 below.

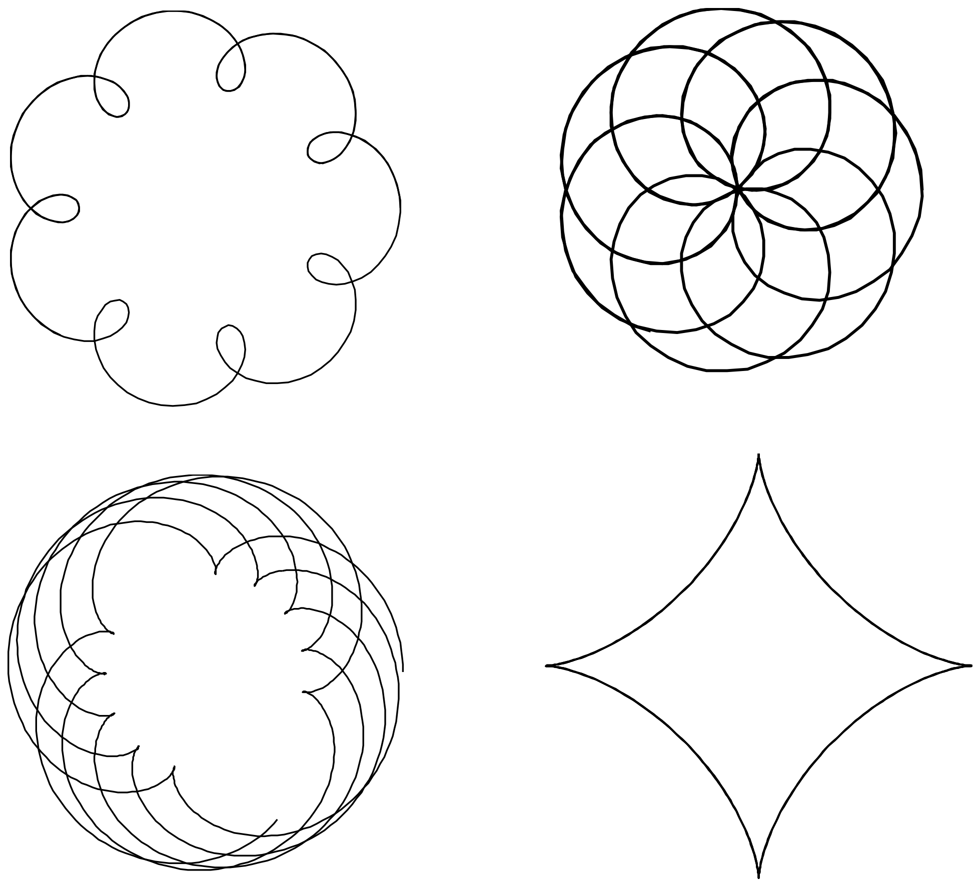


Figure 2: By varying the radii and angular velocities of the epicycle and the deferent, a variety of epicycloids can be produced. The four curves shown above are just a few interesting examples. Clockwise from top left, the angular velocities of the epicycles in the counter-clockwise direction are 8, 8, π, and -3. These are relative to a counter-clockwise angular velocity of 1 for the deferents. The radii of the deferents are 4, 1.01, 3, and 3, respectively. The radius of each epicycle is set at 1.

Because of the importance of the circle as a geometric figure, the ability to model different motions using circles alone was crucial in a historical context. The epicycloid was put to use by astronomers like Appolonius, Hipparchus. Ptolemy, and Copernicus, among others, to model the complex motion of the heavens. In addition to the historical value of the epicycloid, the curve has a surprising connection to approximation using Fourier series, one of the most commonly used approximation techniques today. While advanced mathematicians use epicycloids (disguised as Fourier series) more often than they may realize, the average person may also have encountered such curves when reading about recent research into the Antikythera Mechanism or even when playing with a child's toy called the Spirograph.

1.2 Why the Epicycloid?

What is perfection? Look around the room. There is a photo in a beautifully carved frame. The shape of the rectangle appeals to me. Looking from this chair the sides seem to align exactly, creating an illusion of perfect symmetry. Get up and move closer. Wait. From here I can see that the edge of the frame is not aligned with the shape of the white-stuccoed wall at its side. Adjust it slightly. Step back, look again. Hold on, now the bottom of the frame looks tilted compared with the horizontal horizon of the shiny red concrete floor. Adjust again. No good; this seems hopeless. The frame itself though, taken aside from its surroundings, appears a perfect shape. Step in to admire it. Exasperation! At this distance the sides of the frame itself do not seem perfectly in line with one another. There is a gap the size of a dime's thin side between the vertical and horizontal pieces of wood. This makes the right side tilt slightly towards the left. What is perfection? I am still admiring the beauty of the frame regardless. The ornate carvings in the ochre-stained oak are truly beautiful. Run my fingers over the wood, up and down, tracing the lines of the carving. At last, my fingers feel not smooth wood, but there are flaws. I see small chips in the stain, with dark brown peeking through. The wood feels rough to the touch, not smooth and slippery, as it appears.

What is perfection? Surely it is here, somewhere in this room. Aha, a shiny green bowl, its round top reflecting the light from the sun through the window. Look down at the bowl. The lip seems to create a perfect circle. The illusion is delightful. Run my fingers along the edge of the bowl. One complete rotation and I am back where I began. But alas, the potter's hands have flawed this piece. The grooves on the surface are not aligned exactly. Is this really a circle at all? Frustration. Lean in to admire the shiny depths of the sea green stain on the smooth ceramic. Looking into the bowl, concentric circles materialize, one inside the other, shrinking to a point at the depth of the bowl. Surely this is perfection. Eyes look deeper and deeper into the circles. But, none are quite perfect. Indeed, there are no circles at all, just human conceptions of circles. These are approximations of the circle made by the human eye.

What is perfection? Throughout the history of our world, humans have searched for an example of perfection on earth. The concept of perfection is an elusive one, however. It seems the paragon of perfection is an object that is completely even and completely balanced. This is an object that has been designed so precisely that when viewed from any angle and rotated in any direction it appears exactly as it did, unchanged based on perspective. The search for such an object on earth has proved fruitless. Nature tends from order to disorder and the human hand itself creates flaws, even when carefully trained and artistically skilled. While humanity's search for this object of complete symmetry here on earth may have come up empty-handed, that does not mean that such an object does not exist. The hunting grounds just need to be broadened.

Imagine a circle. Cut it down the center, completely across the diameter. The two sides are mirror images of one another. They are completely even and exactly balanced. Turn the circle a quarter turn and slice again. Alas, mirror images again! Now, turn an eighth of a rotation and slice. This is perfection. No matter how many slices are made along the circle's diameter, each piece is a perfect copy of every other. The circle has infinitely many lines of symmetry, unlike any other shape. The paragon of perfection, then, is not an object at all, but an abstraction. It is a shape created in one's mind. A perfect circle can never be duplicated in physical reality because all earthly creations are inherently flawed if one looks at them closely enough.

1.3 A Brief History of Modeling with the Circle

Being perceived as the epitome of perfection, the circle has drawn much interest throughout history. Greek philosopher and mathematician Plato, born in the 5th century BC, is usually given credit for giving the circle the important place in science which it would hold the next two millennia. Plato was an ideal thinker, believing that ideas were more important than reality. Thus, the circle was important to Plato because although it could not exist here on earth, it certainly did exist as an idea. More specifically, Plato believed that the circle "exists, but not in the physical world of space and time. It exists as a changeless object in the world of Form or Ideas."[1] Since the circle could not exist on earth, Plato put it to use in modeling the Heavens, where perfection was certainly achievable. Plato believed that the heavenly bodies, being examples of the divine, moved in circular orbits. These circular orbits were within crystalline spheres, since a sphere is the three dimensional version of a circle and is thus the perfect solid. In fact, the "stars, planets, sun, and moon moved around the earth attached to the surface of crystalline spheres which slid over one another," and as the spheres moved, "they created a sound in the cosmos called the music of the spheres."[2] With his model of the universe, Plato ingrained into science an idea that would dominate until the years of Kepler: the heavenly bodies move with uniform motion in circular orbits.

Aristotle, another Greek philosopher who was actually a student of Plato, was another major proponent of the circle's importance in modeling the heavens. Along similar lines as Plato, Aristotle believed that the heavenly bodies were perfect and thus must travel in circular orbits. Aristotle focused more on reality than on ideas, and hence his focus was on enhancing Plato's model so that it more accurately represented the observational data available at the time. Keeping with the principle of uniform circular motion, Aristotle added more spheres to Plato's model. The heavenly bodies orbited in circles inside of 55 concentric crystalline spheres, outside of which was the final sphere called the Prime Mover. It was this Prime Mover that "caused the outermost sphere to rotate at constant angular velocity, and this motion was imparted from sphere to sphere, thus causing the whole thing to rotate."[3] While slightly more advanced than Plato's model, neither construction could accurately account for the varying brightness of the planets as they make their way around their orbits. Another major flaw was that neither model could explain the retrograde motion of the planets, during which they appear to stop their motion across the sky, then travel backwards, stop, and then move forward again.

The question of how to accurately represent the observed motions of the heavenly bodies using only uniform circular motions was of much interest during this time. In fact, according to a later scientist and philosopher Simplicius, it was Plato himself who issued a challenge to mathematicians to find such a model of the heavens. He posed the problem to find "what circular motions uniform and regular, are to be admitted as hypotheses so that it might be possible to save the appearances presented by the planets."[4] Similar to the reaction to the Brachistochrone challenge much later, leading mathematicians began to make feverish attempts to answer Plato's challenge and thus prove their superiority among scientists.

It turns out that we have already encountered the shape that would provide the best answer to the challenge: it is the epicycloid. In the next chapter we present the parametric equations for the epicycloid and explain how the epicycloid can reproduce the retrograde motion observed in the planets.

Chapter 2 Mathematical Preliminaries

2.1 The Parametric Equations of the Epicycloid

Let us derive the parametric equations for the location of the planet p after time t, given the angular velocity of the deferent, ω_1, and that of the epicycle, ω_2, as well as the radius of the deferent, R, and of the epicycle, r. Consider a coordinate system in which the earth is placed at the origin. Let us start time at 0 when the epicycle is centered on the x-axis at (R,0), see figure 3 on the following page.

As the deferent rotates counter-clockwise by an angle Θ, the epicycle simultaneously rotates counter-clockwise over angle Φ.

The position of p relative to the center of the epicycle d is p_d: $(x,y)=(r\cos(\Phi), r\sin(\Phi))$ and the position of p relative to the earth e is

$$p_e: (x,y)=(r\cos(\Phi)+R\cos(\Theta), r\sin(\Phi)+R\sin(\Theta)) \qquad (2.1)$$

Now, let us relate the angles Θ and Φ. As the deferent rotates by Θ, a fixed point inside the deferent moves from A to B. Thus, the arc swept out is $(R-l)\Theta$, where l is the radius of the circle containing A and B, centered at d. In the same time, the point A can be considered as rotating with the epicycle from A to A', sweeping out angle Φ. Thus, the arc between A and A' is $l\Phi$. Since these two arcs come into contact during the same period of time, they must be of equal length.

So, we have $(R-l)\Theta = l\Phi$. Solving for Φ gives $\Phi = \left(\frac{R-l}{l}\right)\Theta$. By substitution into 2.1, the position of p relative to the earth is given by

$$p_e: (x,y)=\left(r\cos\left(\frac{R-l}{l}\right)\Theta + R\cos(\Theta), r\sin\left(\frac{R-l}{l}\right)\Theta + R\sin(\Theta)\right) \qquad (2.2)$$

Next, let us relate R, l, and Θ to the angular velocities ω_1 and ω_2. The deferent sweeps out the angle Θ in time Θ/ω_1. In the same time, the epicycle sweeps out angle Φ.

Thus, we have $\Theta/\omega_1=\Phi/\omega_2$. Solving for Φ, we get $\Phi= \omega_2\Theta/\omega_1$.

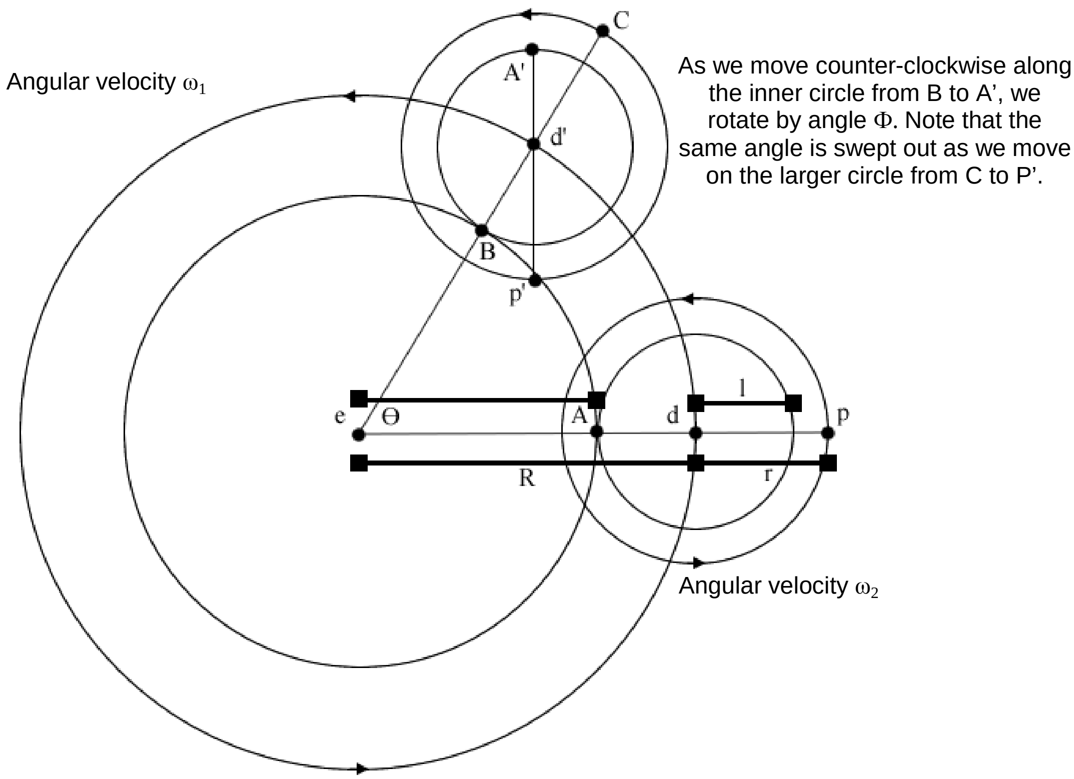


Figure 3: As the epicycle rotates, the arcs AB and AA' come into contact. The arc AB is $(R-l)\Theta$ and the arc AA' is $l\Phi$. The fact that these arcs come into contact over a given period of time allows us to relate the angles Θ and Φ.

Substituting this value into 2.1 gives the position of p as determined by the angular velocities as

$$(x,y)=(r\cos(\omega_2\Theta/\omega_1)+R\cos(\Theta),\ r\sin(\omega_2\Theta/\omega_1)+R\sin(\Theta)) \qquad (2.3)$$

As a final step, let us relate the angle Θ to the elapsed time t. Since the angular velocity of the deferent is ω_1, we have $\omega_1= \Theta/t$. Substituting this time into 2.3, we have the position of p with respect to the earth given parametrically as

$$(x,y)=(r\cos(\omega_2 t) + R\cos(\omega_1 t),\ r\sin(\omega_2 t) + R\sin(\omega_1 t)) \qquad (2.4)$$

2.2 Preliminary Lemma on Angles

The goal here is to show that given angle $\angle ADB = \gamma$, the angle between the center of the circle and A and B is $\angle ACB = 2\gamma$ (see figure 4).

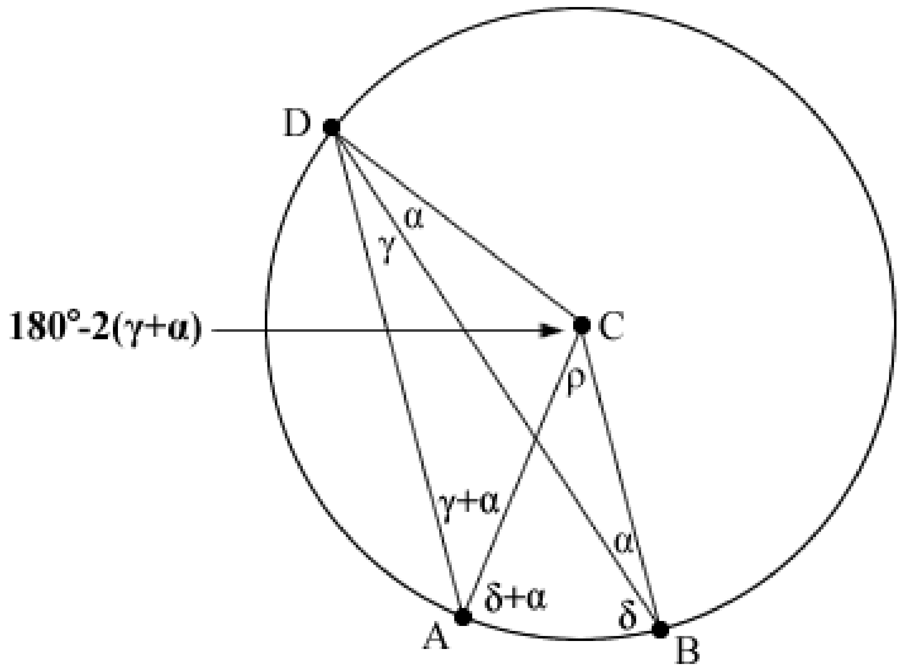


Figure 4: By properties of isosceles triangles presented below, it can be shown that the angle between A, C, and B is equal to twice the angle between A, D, and B.

If $\angle ADB = \gamma$ and $\angle BDC = \alpha$, since the triangle ΔADC is isosceles, then $\angle DAC = \gamma + \alpha$. Also, $\angle DCA = 180^\circ - 2(\gamma + \alpha)$.

Then, ΔBDC is isosceles, so we have $\angle BDC = \angle DBC = \alpha$.

Now, ΔABC is also isosceles, so we know $\angle ACB = \delta + \alpha$.

Summing the angles in ΔABC gives $2(\delta + \alpha) + \rho = 180^\circ$. Similarly, for ΔABD we have $2(\delta + \alpha + \gamma) = 180^\circ$.

Combining these two equations, we have $2(\delta + \alpha) + \rho = 2(\delta + \alpha + \gamma)$. After simplification, we are left with $\rho = 2\gamma$.

Thus, we have shown that $\angle ACB$ is always twice $\angle ADB$.

2.3 The Epicycloid and Retrograde Motion

Next, let us present a proof that retrograde motion, as observed from earth, can indeed be produced by an epicycle moving upon a deferent. The proof incorporates a brief geometric result which we establish before presenting the details.

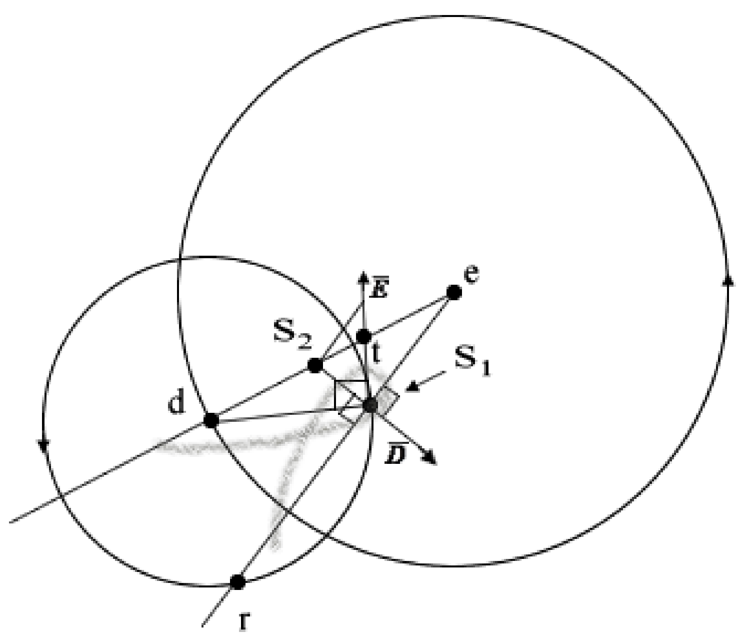
The goal here is to establish a condition relating the angular speeds of the deferent and epicycle with their respective radii such that if the condition is satisfied, retrograde motion is observed from earth. Then we will verify that the condition is sufficient.

The proof that we present follows the structure of Appolonius' Theorem on Stationary Points, but we use modern trigonometric methods which were not available to Appolonius to simplify the proof.

Here, ω_1 is again the angular velocity of the deferent and ω_2 is the angular velocity of the epicycle. In order for retrograde motion of the planet to be observed from the earth, there must be some point at which the planet appears stationary, call it S_1. At this point, the motion of the planet is given by the rotational motion of the deferent combined with the rotational motion of the epicycle.

Let $\overline{D}$ represent the motion of the planet at S_1 due to the rotation of the deferent and let $\overline{E}$ represent the motion of the planet at S_1 due to the rotation of the epicycle (see figure 5).

Figure 5: Due to the combined motions of the deferent and the epicycle, at the point S_1 the planet appears stationary from the earth. The planet then appears to travel backwards across the sky for a short period before resuming forward motion.



For the planet to appear still from the earth, the sum of the two velocity vectors $\overline{D}$ and $\overline{E}$ must lie along the line of sight from the earth to the planet. This line of sight is represented by the line segment eS_1.

Now, since $\overline{E}$ is tangent to the epicycle, it must be at a right angle to the radius. Thus, $\angle dS_1 t = 90^\circ$. Similarly, since $\overline{D}$ is tangent to the deferent at the moment when the planet is at S_1, the angle between $\overline{D}$ and $|eS_1|$ is also right. Hence, $\angle rS_1S_2 = 90^\circ$ since the two angles are opposite (see figure 6).

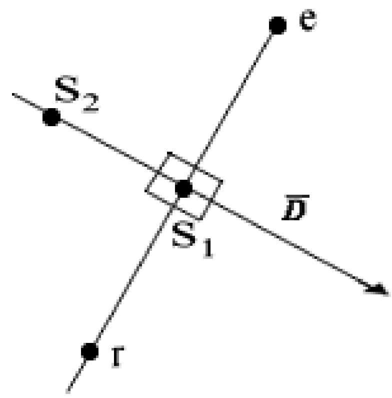


Figure 6: Since the angle between $\overline{D}$ and the earth is right, it is clear that the other three angles must also each be 90°.

Since we require $\overline{E} + \overline{D}$ to lie along the segment eS_1, we need to constrain $|\overline{D}|$ in relation to $|\overline{E}|$. We can do this by requiring that the component of $|\overline{D}|$ along the direction of $|\overline{E}|$ be equal in magnitude and opposite in direction to $|\overline{E}|$. Hence, we have $|\overline{D}| = |\overline{E}| \cos(\angle tS_1S_2)$.

Next, we need to determine the angular velocities of the deferent and the epicycle. Since the tangential velocity at a given point on a circle is directly proportional to the radius of the circle[5], we have $|\overline{D}| = \omega_1 |S_1 e|$ and $|\overline{E}| = \omega_2 |dS_1|$.

Since $|\overline{D}| = |\overline{E}| \cos(\angle tS_1S_2)$, we have $\omega_1 |S_1 e| = \omega_2 |dS_1| \cos(\angle tS_1S_2)$. Therefore,

$$\frac{\omega_1}{\omega_2} = \frac{|dS_1|}{|S_1 e|} \cos(\angle tS_1S_2).$$

Now, let us relate $\cos(\angle tS_1S_2)$ to the radii of the deferent and the epicycle.

Notice that $\angle tS_1S_2 + \angle dS_1S_2 = 90^\circ$ since $\angle dS_1t$ is right. Similarly, $\angle dS_1S_2 + \angle rS_1d = 90^\circ$ since $\angle rS_1S_2$ is right. Hence, we must have that $\angle tS_1S_2 + \angle dS_1S_2 = \angle dS_1S_2 + \angle rS_1d$. Subtracting $\angle dS_1S_2$ from both sides gives $\angle tS_1S_2 = \angle rS_1d$.

This further implies that $\cos(\angle tS_1S_2) = \cos(\angle rS_1d)$. But, we can use $\cos(\angle rS_1d) = \dfrac{|rS_1|}{2|dS_1|}$ as illustrated in figure 7.

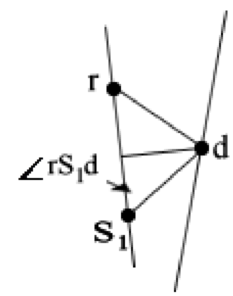


Figure 7: It is clear from the diagram that cos($\angle$ rS$_1$d) can be calculated as the above ratio.

From above then, we have $\dfrac{\omega_1}{\omega_2} = \left(\dfrac{|dS_1|}{|S_1e|}\right)\left(\dfrac{|rS_1|}{2|dS_1|}\right)$ which simplifies to $\dfrac{\omega_1}{\omega_2} = \dfrac{|rS_1|}{2|S_1e|}$.

Now, $|rS_1|$ and $|S_1e|$ are not extremely useful quantities for us. Instead, let us relate these to the radii $|de|$ and $|dS_1|$.

We have $\dfrac{|rS_1|}{2} < |dS_1|$ since r and S_1 lie on the epicycle. Also, $|dS_1| + |S_1e| > |de|$, assuming that the epicycle is indeed rotating (e.g. ω_2 is not zero). Hence, $|S_1e| > |de| - |dS_1|$. From above, we have that $\dfrac{\omega_1}{\omega_2} < \dfrac{|dS_1|}{|de| - |dS_1|}$. Equivalently,

$$\frac{\omega_1}{\omega_2} < \frac{Radius_{epicycle}}{Radius_{deferent} - Radius_{epicycle}} \qquad (2.5)$$

So, we have thus established a condition relating the radii of the deferent and the epicycle with their respective angular velocities necessary for a stationary point to occur.

Next, let us establish that retrograde motion is indeed observed from the earth given that 2.5 is satisfied. Referring again to figure 5, we need to show that the planet appears to be moving forward before reaching S_1, then backwards after passing S_1 on its orbit.

We will need a preliminary inequality before we begin, which will not be proven here. Refer to figure 8. Given $c \le d \prec a$, we have

$$\frac{d}{a-d} > \frac{\gamma}{\beta} \qquad (2.6)$$ [6]

Now, let us choose a point on the epicycle where forward motion should be observed from earth. Call this point f in figure 9 which is below. Notice that $|fr| < |rS_1| < |er|$, so we may apply 2.6, which gives

$$\frac{|rS_1|}{|eS_1|} > \frac{\gamma}{\beta} \quad (2.7)$$

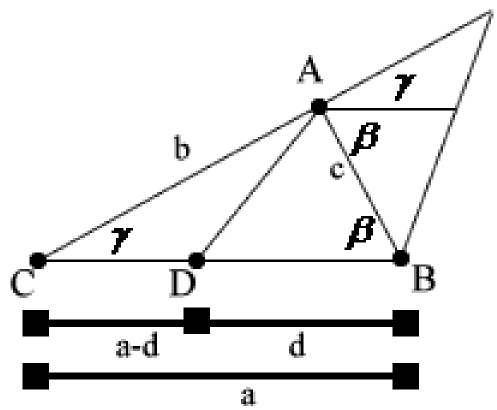


Figure 8: Given c $\le$ d < a, it can be shown that 2.6 above holds. See the above reference.

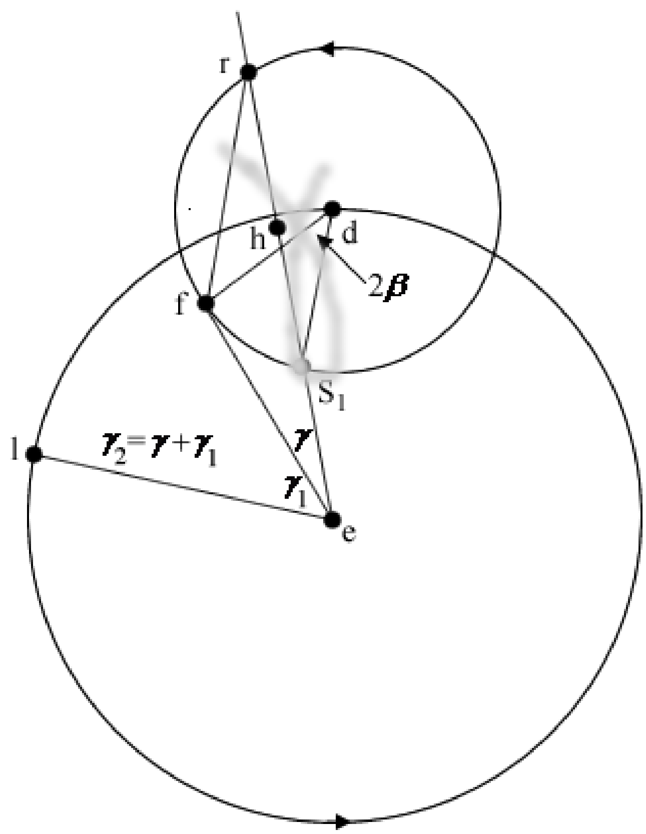


Figure 9: 2.6 can be applied here with a=|er|, d=|rS_1|, and a-d=|eS_1|.

Above we showed that $\frac{\omega_1}{\omega_2} = \frac{|rS_1|}{2|S_1e|}$. Combining this equality with 2.7, we have $\frac{2\omega_1}{\omega_2} > \frac{\gamma}{\beta}$ or equivalently $\frac{\omega_1}{\omega_2} > \frac{\gamma}{2\beta}$. Now, let us choose the point l to be the point on the deferent at which $\frac{\omega_1}{\omega_2} = \frac{\gamma_2}{2\beta}$ (see figure 9).

Then, based on the preceding lemma, the planet moves from f to S_1 as it rotates an angular distance of 2β. Since the angular velocity of the epicycle is ω_2, this rotation occurs in time $\frac{2\beta}{\omega_2}$. But, from above we have that $\frac{2\beta}{\omega_2} = \frac{\gamma_2}{\omega_1}$. This implies that as the planet progresses from f to S_1, the deferent simultaneously rotates from m to l (see figure 10).

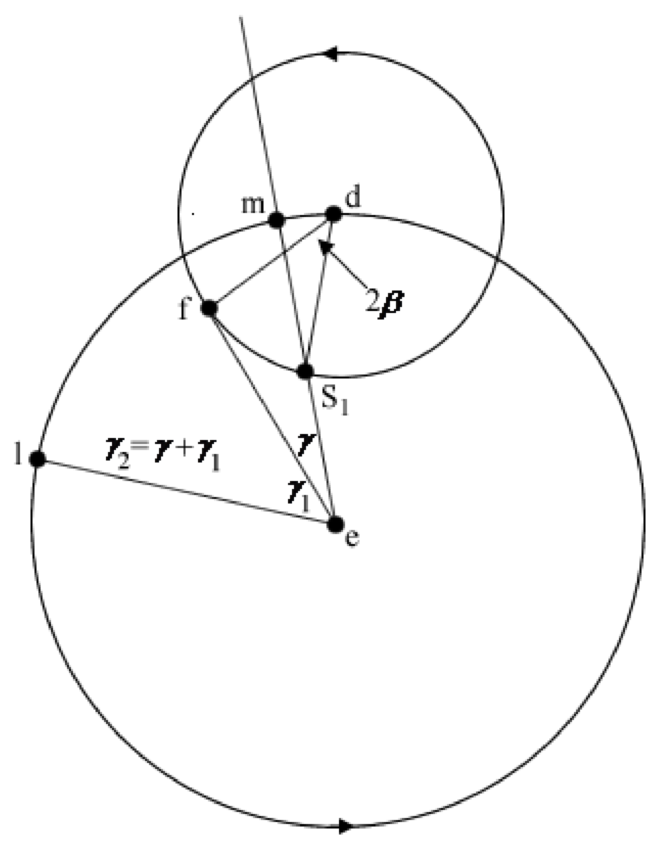


Figure 10: In the time it takes for the planet to appear to progress from f to S_1 on the epicycle, the deferent rotates so that the point m progresses to the point l.

So after time $\frac{2\beta}{\omega_2} = \frac{\gamma_2}{\omega_1}$ has elapsed, from the earth the epicycle appears to have rotated by angle γ and the deferent by γ_2. Thus, the planet appears to have moved forward by $\gamma_1 = \gamma_2 - \gamma$ since $\gamma_2 > \gamma$. Thus, we have established that before reaching the stationary point S_1, the motion of the planet appears forward from the earth.[7]

The next step is to establish that after passing S_1, there is some point b such that the planet appears to move backwards, from the perspective of earth, as the epicycle rotates from S_1 to b (see figure 11). If we let p be at the other stationary point, the point b is necessarily between S_1 and p.

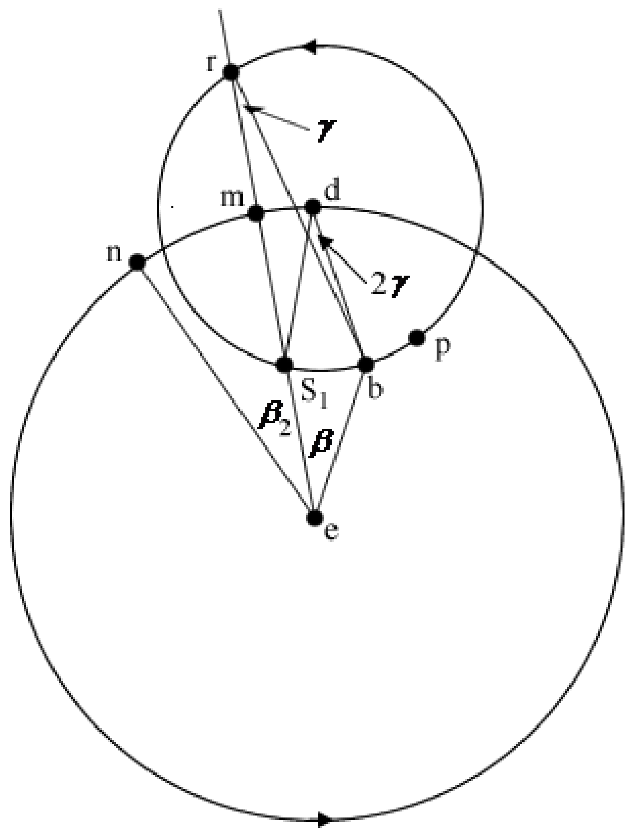


Figure 11: As the planet appears to progress from S_1 to b on the epicycle, the deferent rotates so that the point m progresses to the point n.

Here, let us apply 2.6 again. Since $|be| < |eS_1| < |er|$, we have

$$\frac{|eS_1|}{|rS_1|} > \frac{\gamma}{\beta} \quad (2.8)$$

Above we showed that $\frac{\omega_1}{\omega_2} = \frac{|rS_1|}{2|S_1 e|}$. Combining this with 2.8 gives $\frac{\omega_2}{2\omega_1} > \frac{\gamma}{\beta}$ or equivalently $\frac{\omega_2}{\omega_1} > \frac{2\gamma}{\beta}$. Now, let us choose the point n on the deferent at which $\frac{\omega_2}{\omega_1} = \frac{2\gamma}{\beta_2}$ (see figure 11 again here). Note that β_2 is necessarily greater than β for this equality to hold.

Now, as the planet rotates from S_1 to b on the epicycle, the angular rotation is 2γ (again by the preceding lemma on angles). Since the angular velocity of the epicycle is ω_2, this rotation takes time $\frac{2\gamma}{\omega_2}$. But, n has been chosen such that $\frac{2\gamma}{\omega_2} = \frac{\beta_2}{\omega_1}$. Therefore, as the planet rotates on the epicycle from S_1 to b, the deferent simultaneously rotates from m to n.

After time $\frac{2\gamma}{\omega_2} = \frac{\beta_2}{\omega_1}$ has elapsed, from the earth the epicycle appears to have rotated by β and the deferent by β_2. Thus, the position of the planet appears to have changed by $\beta - \beta_2 < 0$ since $\beta_2 > \beta$.

Indeed, we have established that between the two stationary points S_1 and p in figure 11, the motion of the planet appears backwards from the earth.

In this section we have shown that given Appolonius' condition on the angular velocities and radii of the epicycle and deferent is satisfied, the epicycloid model does produce retrograde motion from the perspective of the earth.

Chapter 3 The Early Work: Appolonius and Hipparchus

3.1 Appolonius' Alternative: The Movable Eccentric Circle

One important mathematician to respond to Plato's challenge was Appolonius of Perga, born in the third century BC. Appolonius did most of his work out of the intellectual center of Alexandria. He was the first to intensely study the use of epicycloids to model the heavens. In fact, it was Appolonius, in his Theorem on Stationary Points, who proved that the epicycloid model could reproduce the retrograde motions of the planets.[8] This was a major breakthrough since the inability to recreate this motion was a big problem with the previous models. Along with his studies on retrograde motion, Appolonius also looked into what kinds of figures could be produced by imposing certain conditions on the epicycle and the deferent. Appolonius was one of the first to consider the equivalence of using eccentric circles to model planetary motion and using the epicycloid. In an eccentric circle model, the earth is placed some distance, called the eccentricity, away from the center of the circle which carries the planet. Thus, the uniform circular motion occurs around the eccentric point instead of around the earth.

In Appolonius' model, the eccentric circle of radius $|AP|$ is rotating counterclockwise around the eccentric point A (see figure 12). Also, the eccentric point A remains at a fixed distance of $|AE|$ from the earth.

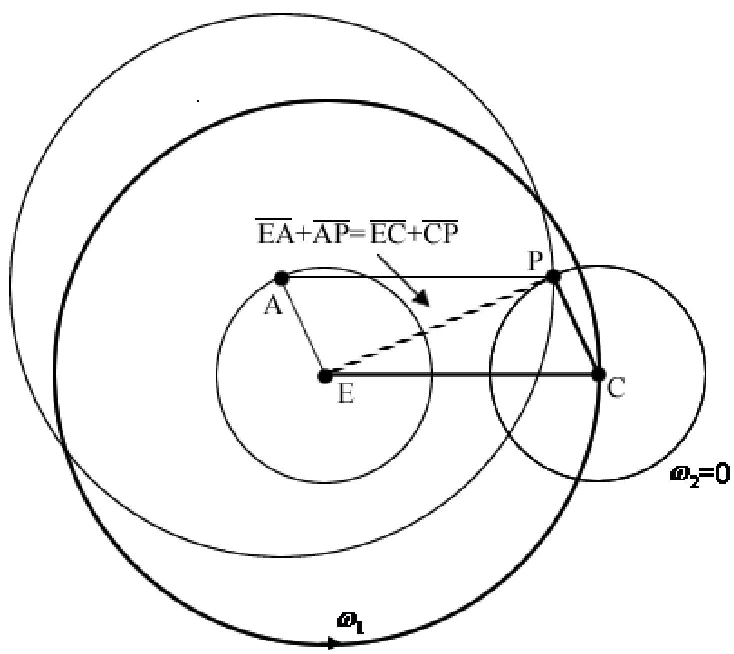


Figure 12: Since |AE|=|CP| and |AP|=|CE|, by requiring that |AE| remains parallel to |CP|, the planet P is observed from earth equivalently using either model.

Appolonius realized that this system is geometrically equivalent to the epicycloid model with the radius of the epicycle $|CP|$ equal to the eccentricity $|AE|$ and the radius of the deferent $|EC|$ equal to the radius of the eccentric circle $|AP|$. The only other condition that we must specify is that the segment $|AE|$ remains parallel to the segment $|CP|$, which is equivalent to requiring that the epicycle remains fixed $(\omega_2 = 0)$, while the deferent rotates with angular velocity ω_1.

Under these conditions, it is clear that the two systems are equivalent since the planet P can be reached from the earth by first moving from E to C and then to P or by moving first from E to A and then to P. Hence, the line of sight from earth to the planet is either along the vector $\overline{EC} + \overline{CP}$ or along $\overline{EA} + \overline{AP}$. But, since we have required that $\overline{EA}$ remains parallel to $\overline{CP}$ and we have $|\overline{EA}| = |\overline{CP}|$ and $|\overline{AP}| = |\overline{CE}|$, it is clear that the points A, C, E, and P are the vertices of a parallelogram. Thus, $\overline{EC} + \overline{CP} = \overline{AE} + \overline{AP}$ and the two systems are equivalent as observed from earth.[9]

Also, we can derive the parametric equations for the eccentric circle easily by setting $\omega_2 = 0$ in 2.4 (the parametric equations for the epicycloid). This gives

$$(x,y)=(r + R\cos(\omega_1 t), R\sin(\omega_1 t)) \qquad (3.1)$$

The eccentric circle model is of interest because later astronomers combined this model with epicycloids in order to represent complex planetary motions more accurately.

While Appolonius' contributions are immense, it seems that his focus was more on a qualitative, geometric study of the epicycloid itself, rather than on a quantitative application of the model to fit the observational data for the heavenly bodies. A more quantitative study would have to wait for Hipparchus, a mathematician and astronomer who lived in the 2nd century BC. Little is known of Hipparchus, and what we do know about his work is thanks to its description by later scientists. Hipparchus' goal was to determine the parameters needed for the epicycloid to accurately model the observational data and to make future predictions of the positions of the heavenly bodies.[10] In this spirit, he set about computing the constants needed to predict the future position of the Sun at a given time.

3.2 Hipparchus and the Sun

First, Hipparchus needed a value for the length of time it takes for the center of the epicycle of radius r to make a complete rotation around the deferent of radius R. For this, he used an approximation he had made of 365.2467 days.[11] Thus, each day the center of the epicycle sweeps out an angle of about $.9856^{\circ}$ or 59'8".

Hipparchus also had measured the lengths of the seasons by observation. He thus had the lengths of time necessary for the sun to rotate from Q_1, the vernal equinox, to Q_2, the summer solstice, and from Q_2 to Q_3, the autumnal equinox, then from Q_3 to Q_4, the winter solstice (see figure 13). With this information at hand, he was prepared to calculate numeric values for both the ratio of the radius of the epicycle to the radius of the deferent and the angular position of the sun's apogee F (the point at which the sun is farthest from the earth, a distance of R + r).

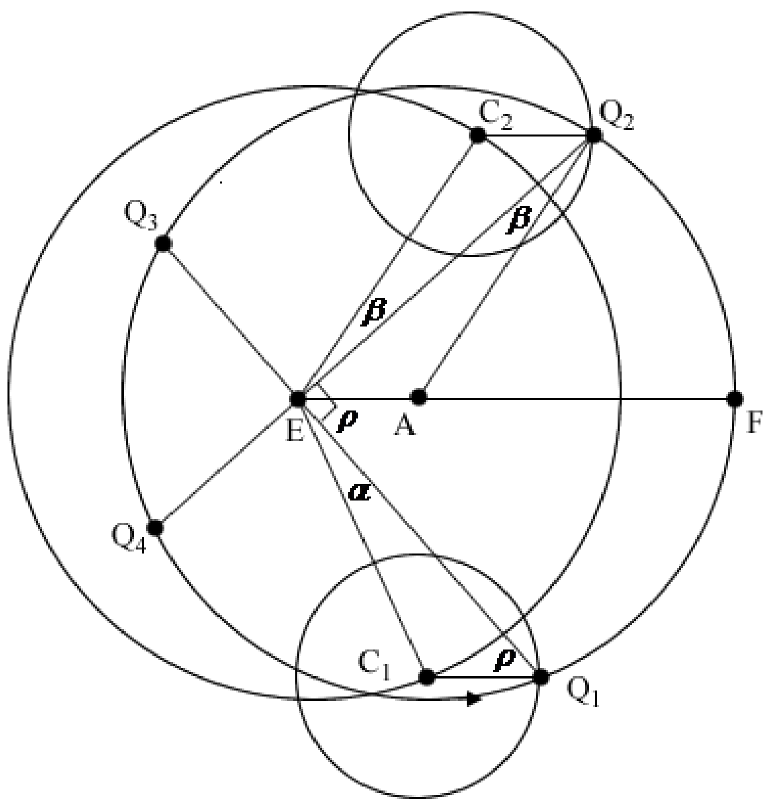


Figure 13: Hipparchus knew the lengths of time for the planet to progress between the equinoxes, Q_1 and Q_3, and the solstices, Q_2 and Q_4. He used this information to calculate the parameters for his solar model.

Hipparchus measured the position of the apogee F with respect to the vernal equinox, labeled Q_1 in figure 13. Thus, the angle he sought is $\angle Q_1EF = \rho$ in the figure.

Utilizing trigonometry, since $|EF|$ is parallel to $|C_1Q_1|$, $\angle C_1Q_1E = \rho$ also. Then, by the law of sines, $\frac{|C_1Q_1|}{\sin(\alpha)} = \frac{|EC_1|}{\sin(\rho)}$. Considering the sun at summer solstice Q_2, note that $|C_2Q_2|$ and the portion of the segment $|EF|$ of equal length always act as vertices in a parallelogram. Let us extend a line from the end of this segment (the point labeled A in figure 13) to Q_2. Then, if $\beta = \angle EC_2Q_2$, then we have that $\beta = \angle EQ_2A$ also. Since $|Q_2Q_4|$ and $|Q_1Q_3|$ meet at a right angle, we have $\angle FEQ_2 = 90^\circ - \rho$. Applying the law of sines again, we have $\frac{|EA|}{\sin(\beta)} = \frac{|AQ_2|}{\sin(90^\circ - \rho)}$. But, $|EA| = |C_2Q_2|$ and $|AQ_2| = |EC_2|$. Also, $\sin(90^\circ - \rho) = \cos(\rho)$ and hence $\frac{|C_2Q_2|}{\sin(\beta)} = \frac{|EC_2|}{\cos(\rho)}$.

Hipparchus then had the equation $\tan(\rho) = \left(\frac{|EC_1|}{|C_1Q_1|}\right)\left(\frac{\sin(\alpha)}{|EC_2|}\right)\left(\frac{|C_2Q_2|}{\sin(\beta)}\right)$, which can be simplified to $\tan(\rho) = \frac{\sin(\alpha)}{\sin(\beta)}$. Then we can compute $\frac{r}{R} = \frac{\sin(\alpha)}{\sin(\rho)}$.

To obtain numeric values for ρ and $\frac{r}{R}$, Hipparchus used the values he had computed for the lengths of the seasons. For the sun to travel from Q_1 to Q_2, Hipparchus measured the length of time as 94.5 days. From Q_2 to Q_3, the sun took 92.5 days. Hence, in 94.5 days, the center of the epicycle has rotated at constant angular velocity from from C1 to C2 (see figure 14 below). Thus, the angle swept out is $\angle C_1EC_2 = \alpha + \beta + 90^\circ$.

Then, in 92.5 days, C_2 has rotated to C_3 and hence the center of the epicycle has swept out angle $\angle C_2EC_3 = 90^\circ - \beta + \alpha$. So, we have $\alpha + \beta + 90^\circ = 94.5\left(\frac{360}{365.2467}\right)$, which gives $\alpha + \beta = 3.14253^\circ$. Also, $90^\circ - \beta + \alpha = 92.5\left(\frac{360}{365.2467}\right)$, which gives $\alpha - \beta = 1.17126^\circ$. Solving the above two equations for the two unknowns, we have $\alpha \approx 2.1569^\circ$ and $\beta \approx .9856^\circ$.

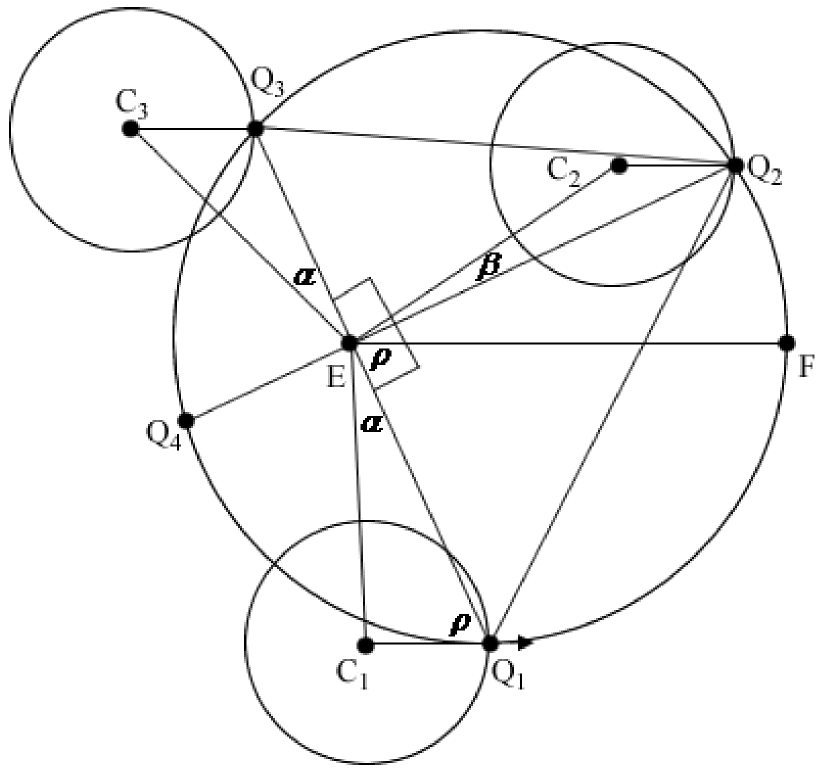


Figure 14: Using observed data for the lengths of the seasons, we obtain ρ≈65° and r/R=1/24.

Inserting the values for α and β into $\tan(\rho) = \frac{\sin(\alpha)}{\sin(\beta)}$, we obtain $\rho \approx 65.4378^\circ$ or equivalently $\rho \approx 65^\circ 26' 16''$. Since Hipparchus did not have modern trig methods, he obtained a slightly different value for ρ of $65^\circ 30'$.[12] Now, $\frac{r}{R} = \frac{\sin(\alpha)}{\sin(\rho)}$ so we have $\frac{r}{R} \approx .04138$ or $\frac{r}{R} \approx \frac{1}{24.165}$. Again, with Hipparchus' methods, he obtained $\frac{r}{R} \approx \frac{1}{24}$.

Thus, Hipparchus had found numeric values for the angle of the sun's apogee (measured from the vernal equinox) and the ratio of the radii of the epicycle to that of the deferent. Hipparchus' value for ρ was quite accurate as the actual value for the longitude of the sun's apogee was about 66°.[13] On the other hand, the ratio he calculated of $\frac{r}{R} \approx \frac{1}{24}$ is quite inaccurate due to observational errors. The actual value is approximately $\frac{r}{R} \approx \frac{1}{60}$. Hipparchus did not stop at modeling solar motion. He went on to study lunar motion as well.

3.3 Hipparchus and the Moon

Although the motion of the moon can also be modeled using an epicycloid, there is some variation from the solar model. Based on observations, Hipparchus knew that not only did the moon's speed vary as it rotated around the earth, but that the point at which the moon appeared to be moving the fastest was not at its perigee (the point at which it is closest to earth), as would be expected. In fact, the point of maximal speed varied over time.[14] Also based on the observational data, Hipparchus concluded that the moon's orbit lay on a plane tilted about 5° off the ecliptic. This tilt, however, does not affect the logistics of the epicycloid model because Hipparchus assumed that the 5° tilt did not affect the motion of the moon as observed from earth in order to simplify the calculations.[15]

In order for the location at which the moon achieves its maximal speed not to always be at perigee, Hipparchus used a rotating epicycle. He set about finding the parameters needed for the model to fit the observational data available to him. Choosing the angular velocity of the deferent was simple: after the deferent made one complete rotation, the moon appeared back at its starting point. As for the angular velocity of the epicycle, Hipparchus knew that each time the epicycle made a complete rotation, the moon appeared to move at the speed it began with. Thanks to the Babylonians and to his own observations, Hipparchus had data for the length of time required for the moon to make a complete revolution (called the sidereal month) and for the moon to return to its starting speed (called the anomalistic month). Using this data, Hipparchus concluded that the angular velocity of the deferent was 13.1764° per day and the angular velocity of the epicycle was 13.0650° per day.[16]

Since Hipparchus' work is now lost, it is not clear exactly what values he used for the radii of the deferent and the epicycle, although he did attempt to calculate these values. Because the observational data that Hipparchus used was obtained from observations of eclipses, his model worked well when the moon was near full but was not as accurate at other times.

Chapter 4 Ptolemy and the Golden Age of Epicycloids

4.1 From Hipparchus to Ptolemy

Hipparchus succeeded in putting together usable models for the sun and the moon, but not for the five known planets. He insisted that the motions of the planets should be modeled in a quantitative, mathematical manner, but apparently concluded that this was too complex a task to undertake.[17] Hipparchus is important for his models of the sun and the moon using epicycloids, and perhaps more for his insistence on a more quantitative approach to science which eventually led to a change in thinking overall. It became important to develop models that actually fit observational data and could predict future occurrences of phenomena, key features of the scientific method.

The modeling of planetary motions using epicycloids came with Greek astronomer Ptolemy, who lived approximately 85-165 AD in Alexandria.[18] Ptolemy is known as the most influential astronomer up to his time and his theories dominated science until the Renaissance 1400 years later. Ptolemy is best known for his 13 book treatise on mathematical astronomy titled the *Almagest*, which means the greatest when translated. The *Almagest* is a mathematically rigorous text like no other before it. In the first two of the 13 books, Ptolemy lays out proofs of the mathematical techniques he will use in his astronomical models, including some involving trigonometric theory and spherical geometry, quite groundbreaking studies for his time.

What is truly unique to the *Almagest* is that in it Ptolemy compiled the knowledge available up to his time and expanded upon it. His goal was to build upon previous work and he points this out in his description of the purpose of the *Almagest*, writing that "those topics which have not been dealt with by our predecessors at all, or not as usefully as they might have been, will be discussed at length to the best of our ability."[19] Here, Ptolemy was referring especially to the modeling of the planetary orbits, since no acceptable model had yet been developed.

While the models put forth in the *Almagest* were new in the respect of being presented in a systematic, quantified manner with accompanying rigorous proofs, many of the underlying ideas remained unchanged. Ptolemy believed strongly in the ideas of Aristotle, as passed down from Plato. Ptolemy built his entire system around Aristotle's model in which there is a "fixed earth around which the sphere of the fixed stars rotates every day, this carrying with it the spheres of the sun, moon, and planets."[20] All motions were to be circular and uniform, since Ptolemy again considered the circle to be the perfect shape and thus appropriate for modeling the heavens. Indeed, Ptolemy went so far as to separate the fields of physics and mathematics, with the former being applied to earthly, changing things, and the latter to the heavens, which are "eternal and impassible."[21]

Ptolemy's system uses epicycloids (in the form of deferents and epicycles or the equivalent eccentric circle) to model the motions of the heavenly bodies. In book 3 of the *Almagest*, Ptolemy set about studying solar motion.

4.2 Ptolemy and the Sun

Ptolemy expanded some on Hipparchus' solar theory, although he found it to be fairly accurate as presented by Hipparchus. Ptolemy studied the accuracy of the model by making his own observations to find the length of the seasons and the length of the year. Ptolemy found that the length of the seasons were unchanged from those values given by Hipparchus, hence he used 94.5 days for the length of summer and 92.5 days for the length of fall. When computing the length of the year, Ptolemy actually made a small error which led him to accept Hipparchus' value of about 365.2467 days as accurate.[22] Since he accepted this length, he concluded that Hipparchus' values for the parameters of the epicycloid were also accurate: $65^{\circ}30'$ for the longitude of the sun's apogee, and $\frac{1}{24}$ for the ratio of the radius of the epicycle to that of the deferent.

Using these values, Ptolemy went on to create tables that could be used to calculate the position of the sun at a given time. Consider figure 15.

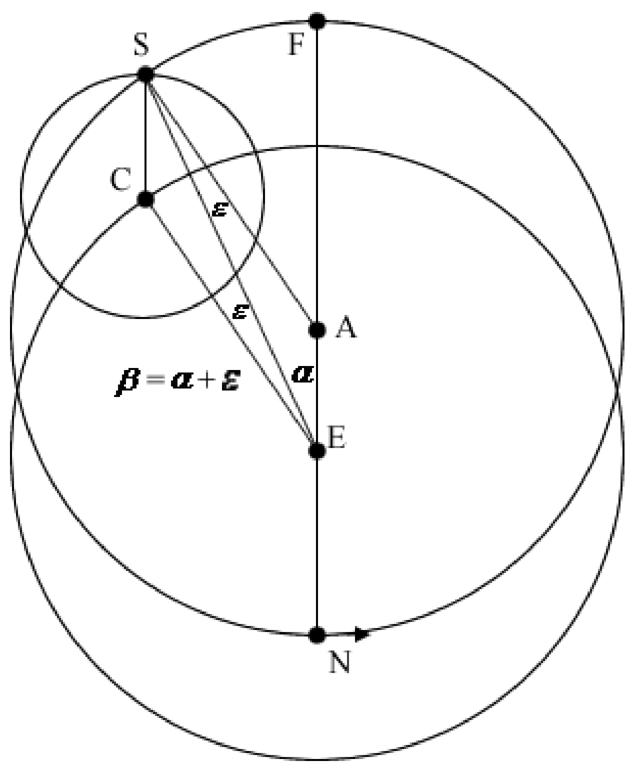


Figure 15: The sun's position as viewed from earth can be predicted by knowing only the angle β.

We wish to find the angle α, which gives the position of the sun as seen from earth. Since the center of the epicycle rotates uniformly around the deferent, the angle β is known. Call the angle β-α=ε. Since $|CS|$ and $|AE|$ are opposite sides of a parallelogram, $\angle ESA = \varepsilon$ also. Then, by the law of sines as applied to triangle ASE, we have $\frac{\sin(\varepsilon)}{|AE|} = \frac{\sin(\alpha)}{|AS|}$. Since $|AE| = |CS| = r$, $|AS| = |CE| = R$, and α= β-ε, this can be rewritten as $\frac{\sin(\varepsilon)}{r} = \frac{\sin(\beta - \varepsilon)}{R}$. Continuing to simplify, $\frac{\sin(\varepsilon)}{r} = \frac{\sin(\beta)\cos(\varepsilon) - \cos(\beta)\sin(\varepsilon)}{R}$ gives $\frac{R}{r} = \frac{\sin(\beta)}{\tan(\varepsilon)} - \cos(\beta)$. Thus,

$$\tan(\varepsilon) = \frac{\sin(\beta)}{\frac{R}{r} + \cos(\beta)} \qquad (4.1).$$

Then, $\varepsilon = \tan^{-1}\left(\frac{\frac{\sin(\beta)}{1}}{\frac{R}{r} + \cos(\beta)}\right)$. So, Ptolemy could now predict the position of the sun at a given time. Thanks to Hipparchus' calculation of the sun's apogee relative to the vernal equinox, Ptolemy knew it took about 66 days, 10 hours, and 55 minutes for the sun to travel from the vernal equinox to the apogee. With this value, he could calculate the angle β, and then using 4.1 he could find the angle ε.

As a final step, Ptolemy need only subtract ε from β and he then had the angle α and hence the sun's position as observed from the earth.

Because Ptolemy did not have modern trigonometry at his disposal, calculating the amount to be subtracted (ε) was quite tedious. Because of this difficulty, Ptolemy went about creating a table to be used for quick calculations. Interestingly, Ptolemy chose not to use the time at which the sun passed apogee as a starting point for making these calculations. Rather, he chose to give the angle between the center of the epicycle and the sun's apogee measured at the time of the beginning of Babylonian King Nabonassar's reign corresponding to the year 747 BC.[23]

Ptolemy's ultimate concern was to accurately represent data and use this data to make future predictions. Since the data could not always be fit to sufficient accuracy using a simple epicycloid model, Ptolemy added more geometric devices to the model.

4.3 Ptolemy and the Equant

Before exploring Ptolemy's model further, it will be helpful to briefly consider a third geometric construction incorporated in addition to the simple epicycloid and the eccentric circle: the equant point. Ptolemy used the equant point to account for variations in the speed of the planets and the moon as they made their orbits. To explain the observations accurately, Ptolemy combined all three constructions at once: the epicycloid, the eccentric circle, and the equant point.

The equant point model is different from the others because here the center of motion is no longer the center of the circle around which the body rotates. Instead, the center of motion is about the equant point (labeled Q in figure 16), which is off-center. The body rotates around the equant point Q uniformly, so $\angle P_1P_2$ is swept out in the same amount of time as $\angle P_3P_4$ since both are right angles.

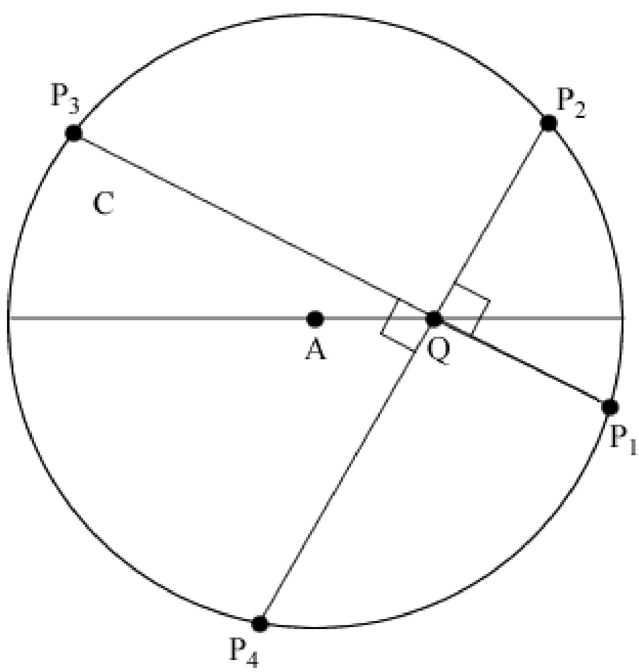


Figure 16: In the equant construction, the body rotates on a circle centered at a point which is not the center of motion. The motion is uniform around an off-center equant point.

From the center of the circle, the body appears to rotate faster when it is farther from the equant point and slower when it is near the equant point. This construction allowed Ptolemy to model some of the observed variation in lunar and planetary speeds.

The basic equant model outlined above was insufficient alone to model the observed data, so Ptolemy used it in combination with the eccentric circle and epicycloid.

A basic example of the models used in combination is shown in figure 17. Here the body B rotates around an epicycle which itself rotates around a deferent centered at A. The center of the deferent is neither the earth nor the center of rotation. The earth is off-center, as is the center of motion (the equant point), labeled Q in the figure.

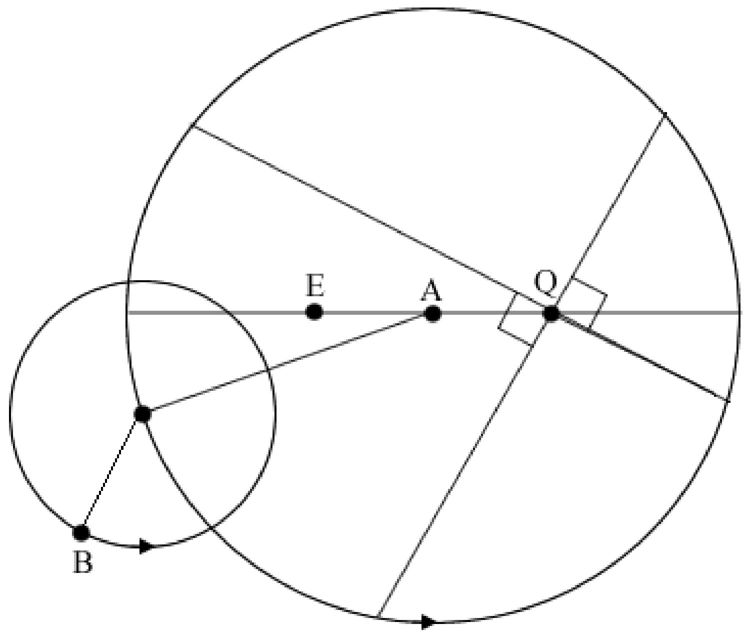


Figure 17: A basic combination of the epicycloid, eccentric circle, and equant point models.[24]

Even further modifications were necessary, depending on the particular celestial body being modeled.

The introduction of the equant construction represented a major shift in scientific study overall. By using the equant point, not only had Ptolemy violated the concept of strict circular motion about the earth by placing the earth off-center in the model, but he was no longer adhering to the idea that the motion of the heavenly bodies was uniform around the earth. Let us return briefly to the implications and criticisms of Ptolemy's model after taking a look at the rest of the system.

Books 4 and 5 of the *Almagest* are dedicated to the study of lunar motion, which is quite complex.

4.4 Ptolemy and the Moon

Ptolemy began with Hipparchus' lunar model and improved upon it. Since Hipparchus' lunar model was less accurate when the moon was not close to being full, Ptolemy altered the theory so that it better fit the data when this was the case. To account for the irregularities in lunar motion, Ptolemy found that the epicycloid, with either a fixed or rotating epicycle, was insufficient. Instead, he combined the epicycloid with an eccentric circle.

In Ptolemy's model, the moon travels on an epicycle rotating clockwise. But, the deferent is not centered on earth, but rather at an off-center point labeled A in figure 18. Also, the point A itself simultaneously rotates around the earth on a circle of radius $|AE|$ in a clockwise direction. The motion of the point A is not constant with respect to the earth, but rather as measured from the sun S as seen from the earth.

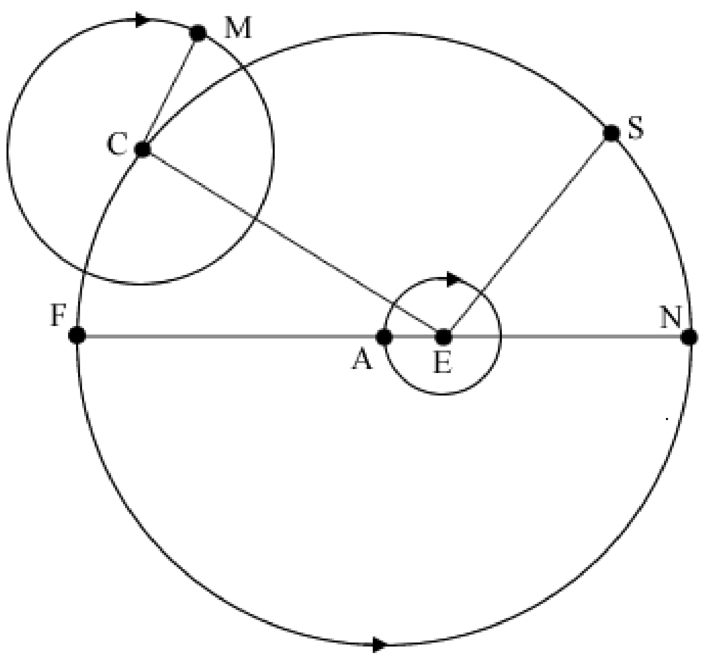


Figure 18: Ptolemy's model for lunar motion was quite complicated. Not only was the deferent not centered on the earth, but the center A itself rotated about the earth.

To fit the angular velocities and the radii for the deferent and the epicycle, Ptolemy again used the observational data he had for the length of the month. The details are not presented here, but some of the results are worth mentioning. Ptolemy set the length of $|FE|$ to 60 units, and then determined that the radius of the deferent should be

49.6833 units and the radius of the epicycle should be 5.25 units. So, the ratio of the radii of the epicycle to the deferent is approximately .1057. Ptolemy also calculated the eccentricity, $|AE|$ in figure 18, and the value he found here was 10.3167 units.[25] As for the angular velocities, Ptolemy found values that only differed slightly from those given by Hipparchus.

Although Ptolemy's lunar model is rather complex since it involves more than a basic epicycloid construction, it is interesting because it represents a shift from strict adherence to the ideas of Plato and Aristotle. The results predicted by Ptolemy's lunar model were better than those predicted by previous theories, but there were still flaws. One major difficulty was that the moon's "apparent diameter should vary by a factor of almost 2 during a single revolution."[26] This was obviously not realistic, even by naked eye observations. Ptolemy is said to have believed the model's purpose was to make predictions, which it did with sufficient accuracy, and thus this incorrect change in size was of secondary importance.

Ptolemy presents his model for the planets in the last 5 books of the Almagest. This model is perhaps his most important contribution of all since he was the first to create a comprehensive planetary model using epicycloids. Ptolemy's model is truly awe-inspiring. Not only is it mathematically rigorous, but it actually represents the observed data to an impressive degree of accuracy for its time. Since the motions of the planets are quite complicated, it follows that Ptolemy's model is in places quite complex, often incorporating combinations of the epicycloid, eccentric circle, and equant point constructions.

4.5 Ptolemy and the Planets, an Overview

Let us consider a simplified version of Ptolemy's planetary system, since our focus is on the incorporation of epicycloids rather than a detailed study of the physics behind the motions. Figure 19 below gives an overview of the model, neglecting the effects of placing the earth off-center and of making the center of rotation an equant point. Thus, we are studying only the epicycloid component of the model.

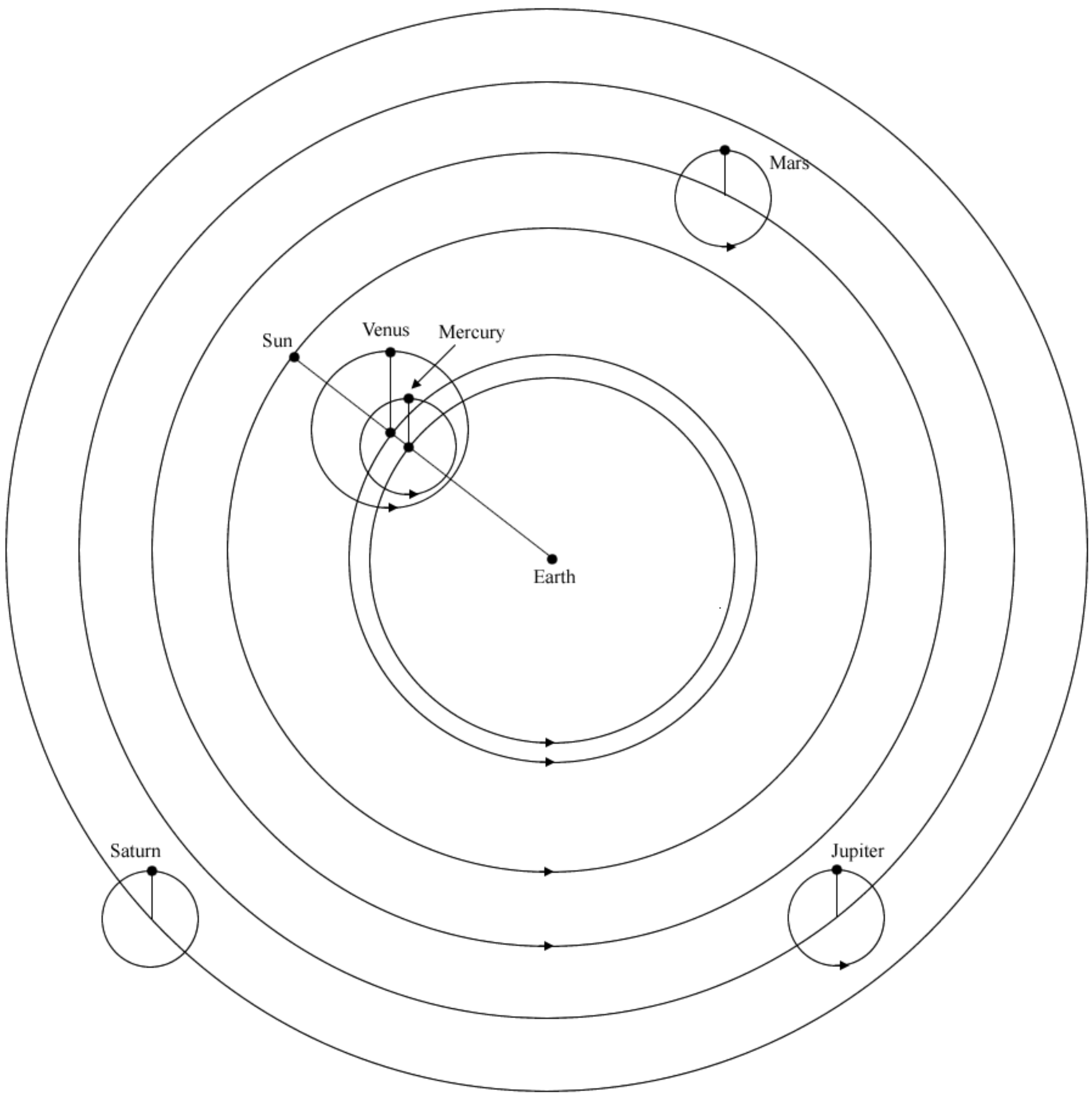


Figure 19: A simplified view of Ptolemy's planetary model. Each planet rotates on an epicycle which simultaneously moves around a deferent. The orbits of Mars, Jupiter, and Saturn are outside that of the Sun, while the orbits of Mercury and Venus are within the Sun's orbit.

In Ptolemy's time, only five planets were known: Saturn, Jupiter, Mars, Venus, and Mercury, respectively from outer to inner. Ptolemy arranged the five planets on deferents of increasing size, not centered on the earth but rather at eccentric points. Ptolemy placed the moon's sphere closest to earth and the sphere of the fixed background stars as the outermost. As for the five planets, Ptolemy did not doubt that Mars, Jupiter, and Saturn orbited beyond the sun, and hence these planets are often called the superior planets. Ptolemy was less sure about the correct placement of the spheres of Mercury and Venus, which are often called the inferior planets. His doubts arose because although Ptolemy knew these two planets always remained relatively close to the sun, he did not have sufficient data to conclude with certainty how the deferents of Mercury and Venus should be placed relative to the sun. As per Ptolemy, "the spheres of Venus and Mercury are placed by the earlier mathematicians below the sun's, but by some of the later ones above the sun's because of their never having seen the Sun eclipsed by them."[27]

Ptolemy decided to go with the earlier mathematicians and place the deferents of Mercury and Venus inside the Sun's orbit. In hindsight, had Ptolemy placed Mercury and Venus on a common deferent (the sun's sphere) and centered each of their epicycles on the sun, the entire system could easily be reckoned with a simplified version of Copernicus' heliocentric system. This equivalence will be discussed in a later section. Instead, Ptolemy centered the epicycles of the inferior planets on a line connecting the earth with the sun.[28] See figure 19 here.

Ptolemy's model is not without flaws. It is important to remember that Ptolemy intended his model to be used to predict the future locations of the heavenly bodies, which it did fairly well according to the standards of his time. With this ultimate goal in mind, Ptolemy was less concerned with whether the model was actually a physical reality or merely a mathematical construction. He was therefore willing to overlook some flaws that did not affect the model's capability to make predictions.

With this basic overview of Ptolemy's planetary system at hand, let us briefly consider his treatment of the inferior and superior planets and present some numeric values for the radii and velocities of the respective epicycles and deferents.

4.6 Ptolemy's Treatment of Venus and Mercury

Ptolemy had data indicating that Venus and Mercury remained close to the sun as they orbited. In fact, he knew Venus remained within 47° of the sun and Mercury within 29°, explaining why the planets appear to rise and set with the sun.[29] To account for this behavior, Ptolemy placed the centers of both epicycles on a fixed line joining the earth to the sun. As the sun traveled around the earth, Venus and Mercury followed along while also rotating on their epicycles. A simplified view is given in figure 20. This arrangement explained why the planets were obscured during most of the daylight hours, because their light was drowned out by the brightness of the sun. Also, close to sunrise and sunset, Venus and Mercury were visible from earth because the sun's light was blocked partially by the earth.

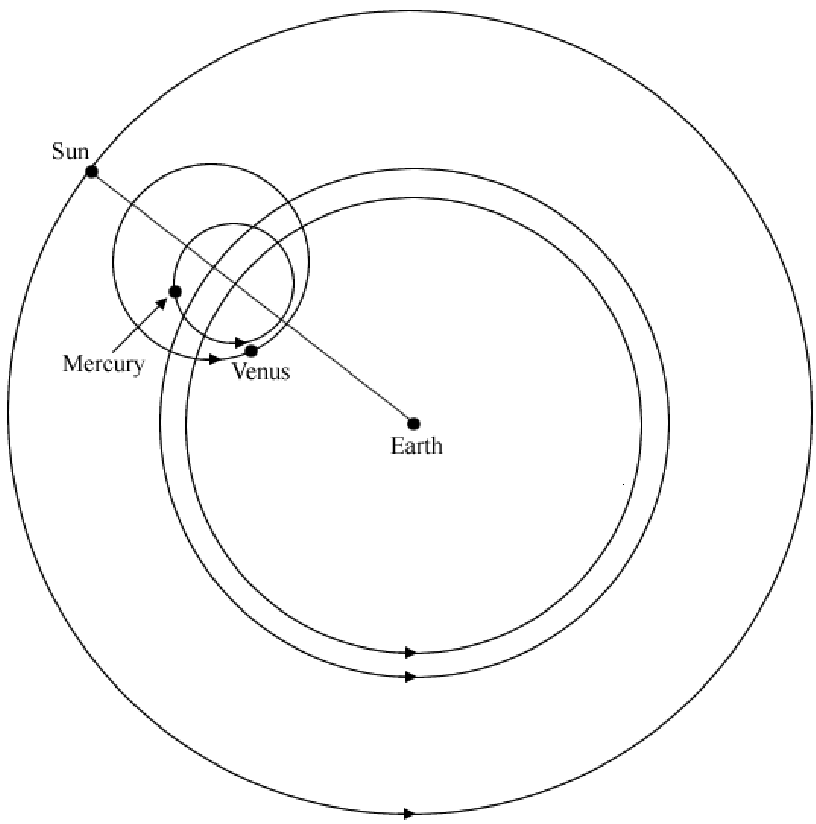


Figure 20: Because Venus and Mercury rise and set with the sun, Ptolemy centered their epicycles on a line joining the earth with the sun.

While we present an overview of Ptolemy's model, in reality he actually needed to make some modifications to accurately fit the data. For instance, to account for minor irregularities, the Sun, Earth, and center of the epicycle are not exactly aligned as they are in the simplified version above. The model for Venus is shown in more detail in figure 21.

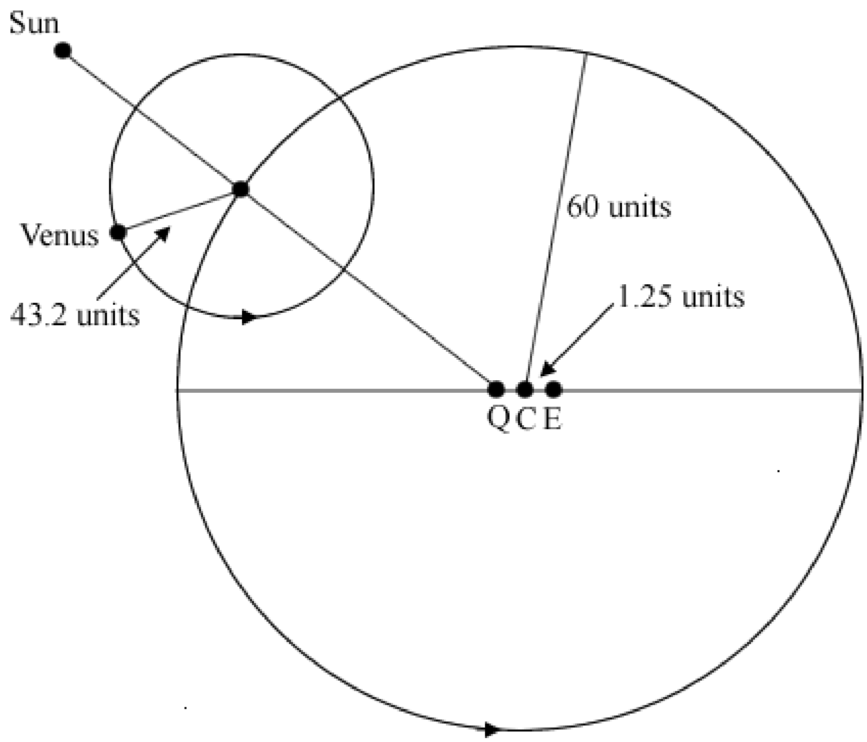


Figure 21: A more detailed look at Ptolemy's model for Venus. Both the earth E and the equant point Q are off-center.

The center of motion for Venus' orbit is the off-center equant point labeled Q in figure 21. The deferent is not centered on the earth or the equant point, but rather on the midpoint between these two, labeled C in the figure.

Ptolemy set the radius of Venus' deferent at 60 units and then went about determining what the radius of the epicycle should be in comparison. From observations, Ptolemy knew the maximum angle between the sun and Venus was about 47°, and he used this to conclude that the radius of Venus' epicycle should be about 43.2 units. He also determined that the distance between the earth and the center of the deferent should be about 1.25 units.[30]

Mercury's orbit was the hardest to model because it was not close to being circular and because the observational data available to Ptolemy was flawed. The data led Ptolemy to think there were two points on Mercury's orbit where the planet reached its minimal distance from earth. With this in mind, Ptolemy's model for Mercury was much more complex than the model for Venus. Figure 22 illustrates the model.

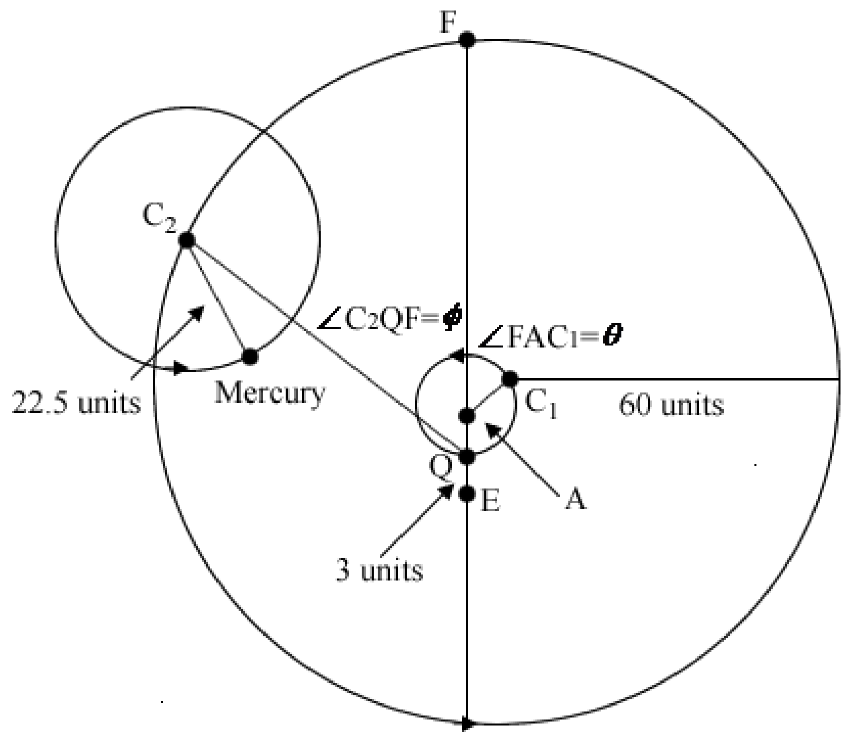


Figure 22: Ptolemy's model for Mercury is much more complicated than Venus', in part because his data was flawed. Not only do the epicycle and deferent rotate, but the center of the deferent itself rotates.

This time, the earth E and the equant point Q do not even lie on the diameter of the circle. Mercury orbits on an epicycle which rotates on the deferent centered on C_1. In addition, the center of the deferent C_1 rotates around the point A in the clockwise direction. For the required effect, Ptolemy also required the angles $\angle FQC_2 = \phi$ and $\angle FAC_1 = \theta$ to remain equal.

Again, Ptolemy set the deferent's radius at 60 units and then concluded that the epicycle's radius should be 22.5 units and that the distance between the earth and the equant point should be 3 units.[31]

Let us now look briefly at Ptolemy's model for the superior planets: Saturn, Jupiter, and Mars.

4.7 Ptolemy on Saturn, Jupiter, and Mars

Modeling the motion of Saturn, Jupiter, and Mars was simpler because these planets were not limited to within a certain angle of the sun. The basic scheme is the same for each of these three planets: the planet travels around an epicycle whose center rotates on a deferent which is not centered on the earth. The earth is placed eccentrically, with the center of the deferent halfway between the earth and the equant point (about which the motion is uniform). In the figure, the center of the epicycle C_2 rotates about the equant point Q uniformly. Based on the observational data, Ptolemy concluded that the lines connecting the earth with the sun, $|ES|$, and the planet with the center of the epicycle, $|PC_2|$, should always be parallel. Hence, $\angle NES = \angle AC_2P$ (both are labeled Θ in the figure). [32]

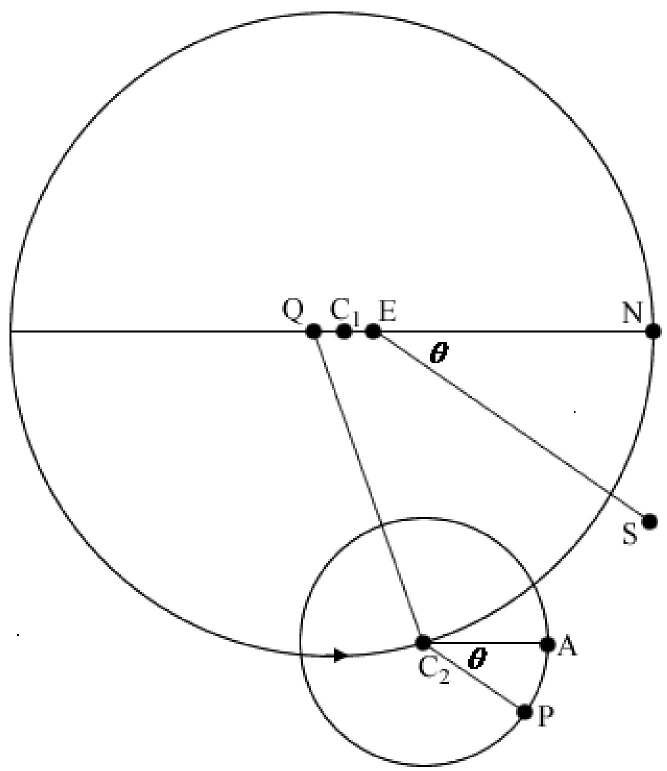


Figure 23: Ptolemy's model for each of the superior planets was based on the basic model to the left. The lines connecting the earth to the sun and the planet to the center of its epicycle always remain parallel.

Ptolemy used 60 units as the standard radius for the deferent of each of the three planets. He computed the radii of the respective epicycles to be 6.11, 11.5, and 39.5 units for Saturn, Jupiter, and Mars. The eccentricities (length of $|C_1E|$ in figure 23) he computed are 3.04, 2.75, and 6 units for Saturn, Jupiter, and Mars, respectively.[33]

Thus, Ptolemy had modeled the complete (as far as he was concerned) universe using epicycloids with slight modifications. Now that we have studied the model in some geometric detail, let us consider its implications in a historical context.

4.8 Reactions to Ptolemy's Model

Given the capabilities of the models put forth in the *Almagest*, it is not surprising that the work was controversial as well as awe-inspiring. Ptolemy's theories remained accepted as the most accurate representation of the known universe for the next 1400 years, a truly amazing feat. Indeed, "the *Almagest* was not only a work on astronomy, the subject was defined as what is described in the *Almagest*.[34]

The biggest stir created by the Almagest was due to Ptolemy's use of the equant point. Using the equant mechanism meant that the uniform motion of the heavenly bodies was not relative to the earth itself, but rather to the off-center equant point. In Ptolemy's opinion, this motion was still uniform and circular, and hence still met the standards of Plato and Aristotle. However, many astronomers believed that uniform motion about the equant point was not the strict uniform circular motion Plato had described.

Another major issue with the Ptolemaic model was its complexity. As we have discussed, the common belief at the time was that when studying the motions of the heavens, we are studying directly the work of God or the divine in its perfect, uncorrupted state. This being the case, many philosophers suggested that the universe should be simple and perfectly ordered. In fact, "God would create a harmonious and symmetrical universe, a simple universe absent of superfluous, ugly details."[35] Ptolemy's model, although quite accurate in making predictions, was not completely symmetrical due to the equant points, and certainly was not simple with its puzzling combinations of epicycloids, eccentrics, and equants.

An important consideration here is what the ultimate purpose of creating a model of the universe is. Ptolemy and his followers "were not concerned if his system did not describe the 'true' motions of the heavenly bodies; their concern was to 'save the phenomena,' that is, give a close approximation of where the heavenly bodies would be at a given point in time."[36] In most cases, the model did this very well and hence Ptolemy's followers were satisfied. Being the intellectual mastermind that he was, though, Ptolemy must have often wondered how the universe was physically constructed in reality, since there is no doubt that he was aware that his system could probably not operate in reality.

According to Plato and Aristotle, the heavenly bodies orbited the earth on spheres which were made of crystal. If the deferent for each planet was made of crystal, when the planet came around its epicycle, the crystal would shatter leading to certain catastrophe! Doubters of Ptolemy argued that even a predictive model should more accurately describe the physical realities of the universe. In the eyes of Islamic astronomer Ibn Rushd, Ptolemy's model was "contrary to nature," and "offers no truth, but only agrees with the calculations and not what exists."[37]

As the years passed, Ptolemy's model began to fall short even in making predictions. The inaccuracies increased as time went by since Ptolemy's system was based upon observations made and his time and prior to it. By the 16th century, there was serious interest in modifying Ptolemy's theories so as not only to simplify the details, but to more accurately fit with the astronomical phenomena of the time. This interest is justified considering that "in 1504 a Ptolemaic prediction for a conjunction of two planets was off by 10 days, and in 1563 another predicted conjunction was off by a month."[38] Even with so many doubts, modified versions of the model that would become widespread did not come until the 15th century.

Chapter 5 The Twilight of Epicycloids: Copernicus and Kepler

5.1 Why so Long?

Given that there were so many doubts and inaccuracies, it is valid to question why the Ptolemaic system remained the most used model of the universe for 1400 years. The answers to this question are rooted in the evolution of science and humanity over this period of time. The era between Ptolemy's lifetime and the Renaissance saw the fall of Roman civilization, the rise of the Arab Empire, the Dark Ages in the West, and finally the rebirth of Western intellectualism with the beginnings of the Renaissance. Thus, it is not in solitude that the story of the epicycloid unfolds, but rather it takes place within this historical framework by which it is shaped.

The symbolic end to the rule of the Roman Empire came with the sabotage of the library and museum at the intellectual capital of Alexandria. These iconic buildings were destroyed in the 4th century when Emperor Constantine took over the city and dedicated it to Christianity. The library and museum were considered pagan institutions and thus in 392, "the last fellow of the Museum was murdered by a mob and the Library was sabotaged."[39] The rise of Christianity in the West corresponded to the decline of scientific study. At first the Catholic Church felt it necessary to expound its position as ruler of humanity in all aspects of life and hence condemned much of the previous scientific knowledge since it was associated with paganism. In fact, during the early years of Catholic rule, the Church was "opposed to scientific endeavor, not unnaturally since the early Christians had had to fight for the survival of their religion by emphasizing the importance of its theology at the expense of pagan learning."[40]

During the period of Catholic rule, known as the Dark Ages, people in the West had limited access to most scholarly works such as those of Appolonius, Hipparchus, and Ptolemy. This is because the Greek language almost completely disappeared and was replaced by Latin. Most of the intellectual works had only been partially translated, if at all. In this sense it is almost as if they had never existed at all and would have to be redeveloped entirely from scratch. Of importance here, "the detailed Ptolemaic theory of the heavens appears to have been completely unknown," and its importance was "overshadowed largely by the conflict between Aristotle and Christianity."[41] In fact, the *Almagest* in its complete form was not available to the Latin-speaking world until the 15th century. While the study of epicycloids and astronomy slowed down in the West, the intellectual center shifted to the Islamic world.

The reign of Islam was strongest from the 7th to the 10th centuries. While the Muslims did attempt some astronomical study during this time, nothing notable in comparison to Ptolemaic theory was developed. This lack of consequential progress perhaps can be attributed to a lack of time and stability. The Roman Empire had flourished for centuries, providing philosophers and scientists with the opportunity to ponder in peace. The Islamic Empire was powerful for only a short period in comparison to the Roman Empire. By the 10th century, the Islamic Empire was broken up into "independent fragments, none of which provided the continuity over many generations that had made possible Alexandrian advances in geometrical astronomy."[42] This is not to say that the Arabs did not contribute to the story of astronomy and the epicycloid, however.

The Arabs are important for their preservation of ancient Greek works as well as for some original contributions. The Arabs had better access to astronomical texts since many had previously been translated into Arabic. While these translations were important because the works could be studied and built upon by the Muslims, they were perhaps more valuable in that they acted as preservations of the ancient texts. Since many of the Greek versions were destroyed during the reign of the Catholic Church, it was these Islamic translations that would be re-translated and would eventually re-introduce ancient knowledge to the Western world.

While the Arabs did study the Ptolemaic system and expand upon his model, their major contribution comes in the form of the advancement of mathematics as a whole. As far as direct relation to the *Almagest* and epicycloid, two Muslim astronomers are especially of note. In the 1200's, Nasir al-din al-Tusi wrote his *Memoir on Astronomy*, a "commentary on the *Almagest* which attempted to give Ptolemy's models a physical meaning."[43] This work is representative of a general shift in thinking; while it was important for a model to accurately describe data while staying true to Plato's principles, it was perhaps more so for the model to accurately represent physical reality. In this memoir, al-Tusi also presents the geometric discovery for which he is best known: the Tusi-Couple. Through this geometric construction, al-Tusi proves that straight lines (and hence rectilinear motion) can be represented by circular motion, basically in the form of an epicycloid.

The Tusi-Couple can be incorporated into the Ptolemaic model to make some simplifications and it also has philosophical implications. Islamic interest in the Tusi-Couple was primarily as a means of ridding the Ptolemaic model of the objectionable equant point. Removing the equant point was very important to the Muslims, and 14th century astronomer Ibn al-Shātir made a notable contribution to the story of the epicycloid when he created a model that attempted to remove the equant point from Ptolemy's model and also to model more data. While the Tusi-Couple is important as a geometric construction, it is perhaps more so as an example of the new way of thinking beginning to take hold. Was uniform circular motion of the heavens indeed a steadfast reality if there is really little distinction between the perfect figure of the circle, representing the heavenly, and the plain old straight line, representative of the earthly and imperfect? This kind of question opens the door to many more, one being if there is not that much distinction between the earthly and the heavenly, why must the universe revolve about the earth at all?

The Arabs also contributed by expanding and developing mathematical methods that would travel back to the West. Muslim achievements include the development of spherical geometry and trigonometric methods, the creation of trigonometry tables, and the contribution to observational data. Muslim works and the ancient texts began to be translated and make their way back to the West by the 10th century. By this time the Catholic Church felt secure in its position of power and began to allow the revival of intellectualism, giving rise to the first universities. The transition was slow, but "by the twelfth century, the study of cosmology and natural philosophy once again became acceptable."[44] This acceptance came with a new belief that understanding the world was an important component of understanding the divine. One influential supporter of this idea was Saint Thomas Aquinas, who stressed "that a complete understanding of the world could be obtained only through both revelation and reason."[45]

The doors were now open for the Renaissance to begin and a new way of thinking to flourish. The advent of the printing press allowed for the first widespread distribution of astronomy textbooks in the 1400's. By the 14^{th} century, the Renaissance had officially begun and over the 15^{th} and 16^{th} centuries it spread. Amidst new discussions of Aristotle, Ptolemy, epicycloids, and the possibility of a rotating earth, the astronomer whose model would alas replace Ptolemy's arose.

5.2 Copernicus' Significance in the Story of the Epicycloid

The mathematician and astronomer Copernicus lived from the late 15th to early 16th century and his model of the universe finally replaced Ptolemy's as the standard. Copernicus felt that a more extensive renovation of Ptolemy's model was in order, since the small changes made over the years had not been sufficient to correct the issue. Copernicus' objections to Ptolemy's model were much the same as those of astronomers before him, but his solution is definitely unique.

Like many others, Copernicus objected to Ptolemy's use of the equant point. He still believed in Plato's vision of uniform circular motion and so insisted that the true model of the universe would hold true to this notion. Copernicus indicated that one reason for his work was so that "there could perhaps be found a more reasonable arrangement of circles, from which every apparent irregularity would be derived while everything in itself would move uniformly, as is required by the rule of perfect motion."[46] Copernicus' model no longer requires use of the equant point, but it is still based on epicycloids. Another major objection Copernicus had was a seeming lack of flow or unification in Ptolemy's model. He felt that, the universe being divine as it was, should be tied together in a more elegant way than Ptolemy had presented.

Copernicus' solution to these problems was to place the sun at the center of the universe, rather than the earth. He was not the first to propose a heliocentric model, but his was the one that was finally accepted as the correct description of the universe. Copernicus first set forth his ideas in a paper titled *Commentariolus* sometime between 1510 and 1514. In *Commentariolus*, Copernicus laid out his assumptions about the universe, including the position of the earth and the fact that it rotates. Here, Copernicus correctly noted the truth about retrograde motion: this motion "was only apparent, but not real, and its appearance was due to the fact that the observers were not at rest in the center."[47] While he still used epicycloids to accurately fit the data, one major advantage of Copernicus' model was this explanation of the retrograde motion of the planets.

The *Commentariolus* was not widely distributed, but Copernicus' more extensive work, *On the Revolutions*, was. This 6 book work was published in 1543 and became the heliocentric counterpart to the *Almagest*. In the first book of the work, Copernicus presents an overview of his model. He stresses that this model has a major advantage over Ptolemy's version since in it all the pieces fit together more naturally. In Copernicus' system, the periods of rotation for the planets decrease from outer to inner. Saturn rotates around the sun in 30 years, Jupiter in 12 years, Mars in 2 years, then the earth in 1 year, carrying with it the moon upon an epicycloid. Inside the earth's orbit, Venus rotates about the sun in 9 months and Mercury in 80 days.[48] This explanation of the ordering was much more natural since the periods here decrease monotonically, while Ptolemy's model was odd in this respect since the sun, Mercury, and Venus all rotated in 1 year, a fact with no natural explanation.

5.3 Copernicus' Model

Copernicus was the last astronomer to create a comprehensive model of the universe based on epicycloids. While the advantages over the Ptolemaic system mentioned so far have some significance, Copernicus' major contribution was the shift from the geocentric universe to the heliocentric universe. While this is a major change in thinking, mathematically the change is not very significant. In fact, in general Copernicus' model does not significantly improve upon the abilities of Ptolemy's to make predictions. What is important to keep in mind, though, is that Ptolemy and Copernicus had different goals in creating models of the universe. Ptolemy's main focus was on creating a model with accurate, at least to the standards of his day, predictive capabilities. On the other hand, Copernicus' major ambition was to create a system which modeled the physical realities of the universe, rather than just a piece of mathematical machinery. In changing beliefs about the basic nature of our universe, Copernicus definitely succeeded.

Of course, Copernicus' model needed to also be able to make predictions and to incorporate sound mathematics, which it did. Indeed, while Copernicus' model, (a basic overview is given in figure 24), did not improve much upon Ptolemy's in ability to make predictions, it did explain some phenomena in a much simpler way.

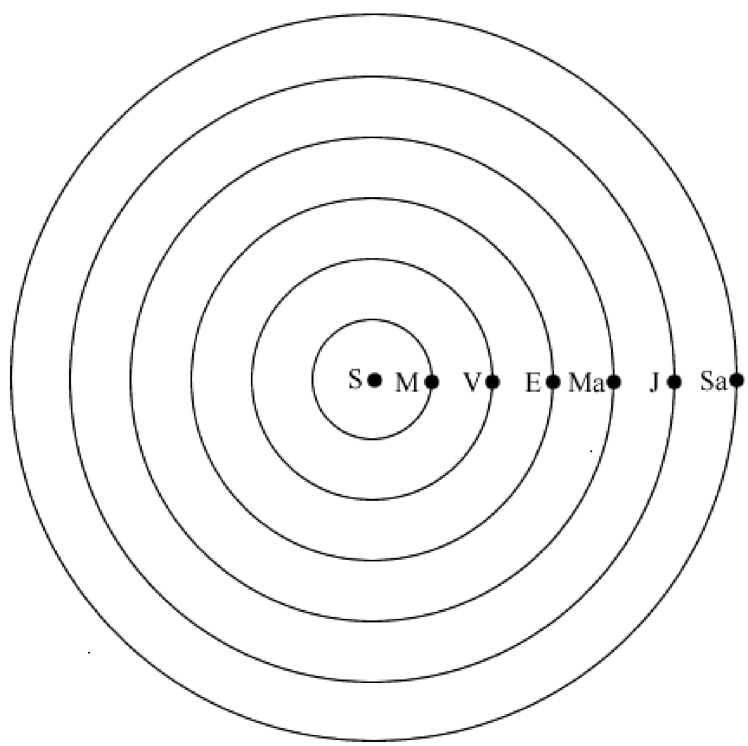


Figure 24: Simplified view of Copernicus' planetary model showing the ordering of the orbits.

First, Copernicus' system explained the fact that Venus and Mercury always appear close to the sun. By placing earth's orbit outside those of Venus and Mercury, it is clear that viewed from the earth, these planets never deviate far from the sun. Thus, Copernicus had the advantage over Ptolemy here since Ptolemy had to set the velocities and radii of these two planets at specific values in order to model this behavior. Second, Copernicus' model clearly can produce retrograde motion without the need for epicycloids.

While Copernicus' model explained these and some of the other phenomena in a simpler way than Ptolemy's did, it still required epicycloids to explain the variation in the speed of the planets as they orbit the sun. Indeed, the details of the Copernican model are comparable in complexity to that of the Ptolemaic system. Also, both models require a comparable number of circles. It is clear, then, that the major contribution Copernicus made was really to the progression of science as a whole rather than the specifics of his model.

Since we have studied the geometric details of Ptolemy's system in some detail, let us simply show here that with some simplification, the geocentric and heliocentric systems are mathematically equivalent. Consider the Ptolemaic system, but with planetary eccentricities all equal to zero. So, in this simplified version, all planetary orbits are centered on the earth, and the earth is also the center of the motion.

Let us take a look at the inner planets, Venus and Mercury, first. Here, one more slight simplification of the Ptolemaic model is necessary to establish equivalence. As we have seen, Ptolemy kept Venus and Mercury close to the sun by placing the centers of their epicycles on a line connecting the earth and the sun. Instead, consider a slight modification in which Venus and Mercury share a common deferent on which each epicycle rotates (see figure 25 below).

Figure 25: To the left, Ptolemy's actual handling of Mercury and Venus. Notice the planets are forced to remain close to the sun since their epicycles are centered on the line of sight from the earth to the sun. To the right, we have a slight modification in which this behavior is explained by placing Mercury and Venus on a common deferent.

Keeping this modification in mind, let us now compare this idealized Ptolemaic system with an idealized Copernican system in which the orbits are centered exactly on the sun. Consider figure 26 below.

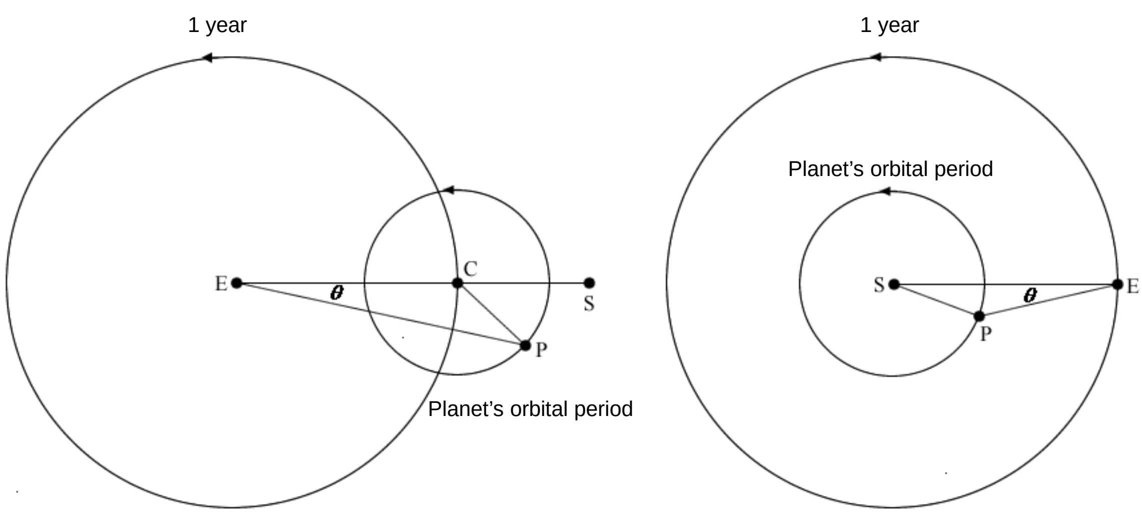


Figure 26: In the Ptolemaic system to the left, the radius of the deferent is equal to earth's distance from the sun in the Copernican system (right). Since the angular distance between the sun and the planet, as viewed from earth, must be the same in each system (angle ϴ), we conclude that the epicycle's radius is equal to the planet's distance from the sun in the Copernican system.

In the Ptolemaic system, we can reach the planet P from earth by first traveling to the center of the epicycle C, and then to the planet P. Note here the path of travel is along the vector $\overline{EC}$ and then along the vector $\overline{CP}$, so we can represent the path as the vector sum $\overline{EC}+\overline{CP}$. Now, consider the Copernican model. Here, to reach the planet from earth, we travel first from the earth to the sun, and then from the sun to the planet. So, here the path of travel is along $\overline{ES}+\overline{SP}$. Now, since the distance traveled from earth in the direction of the sun must be the same in both cases, vectors $\overline{EC}$ in the Ptolemaic model and $\overline{ES}$ in the Copernican model must have the same magnitude. Thus, in the Ptolemaic system, the radius of the deferent must be equal to the distance between the earth and the sun, which is 1 astronomical unit (AU), a distance of approximately 150 million kilometers.[49] Similarly, for the vector paths to be equivalent, we require $\angle CEP$ in the Ptolemaic system and $\angle SEP$ in the Copernican system (the angular position of the planet as viewed from earth relative to the sun) to be equal. This angle is labeled Θ in figure 26 above. Since the vector sums from earth to the planet are equivalent, we can conclude that $\overline{CP}$ in the Ptolemaic model and $\overline{SP}$ in the Copernican model are of equal magnitude. Thus, the radius of the epicycle is equivalent to the distance from the planet to the sun.

Now, since the earth orbits the sun once every year and the deferent's radius is 1 AU, it is clear that the center of the epicycle should rotate once around the deferent in 1 year. Also, since each planet orbits the sun in a fixed amount of time, call it T, we know that the epicycle makes one complete rotation in time T (the orbital period of the planet). Thus, for Venus the epicycle makes a rotation in about 225 days and for Mercury in about 27 days.[50] Since the radius of the deferent is equivalent to the distance from earth to the sun in the Copernican system, it is apparent that the assumption that Venus and Mercury share a common deferent is necessary.

For the outer planets, we can also show equivalence of the two systems in their simplified states. Refer to figure 27 here.

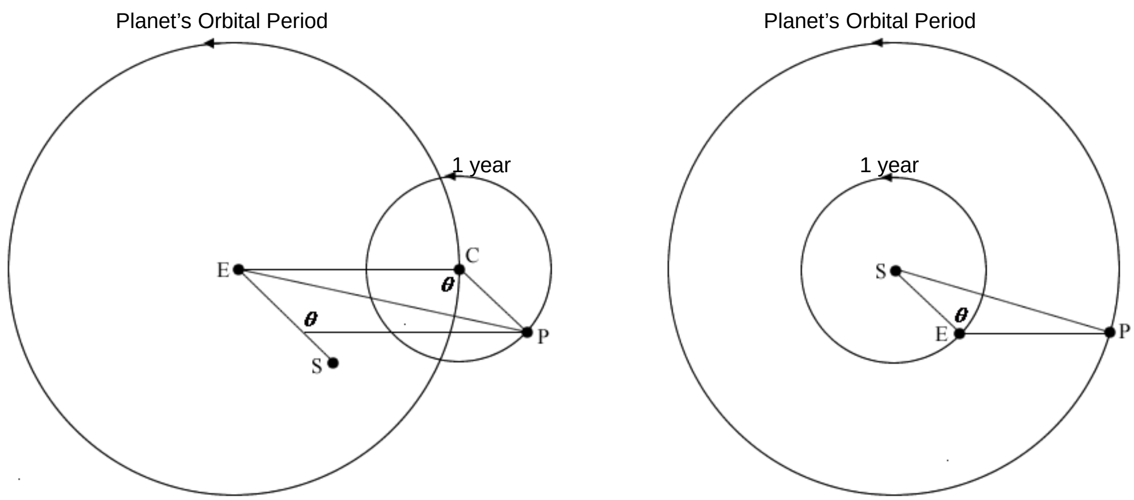


Figure 27: For the outer planets, the radius of the deferent is equal to the planet's distance from the sun in the Copernican model. From simple geometry, we conclude that the radius of the epicycle is equivalent to the distance from the earth to the sun (1 AU).

In the Ptolemaic system, to reach the planet P from the earth E, first we travel along $\overline{EC}$ and then along $\overline{CP}$, so along the vector sum $\overline{EC}+\overline{CP}$. In the Copernican model, first we travel from earth to the sun along $\overline{ES}$ and then from the sun to the planet along $\overline{SP}$. Hence, the path of travel is along $\overline{ES}+\overline{SP}$. For equivalence of the two models, the radius of the deferent is equal to the distance from the sun to the planet in the Copernican model. Then, note that $\angle ESP$ and $\angle ECP$ in the Ptolemaic model must be equal since they are opposite angles in a parallelogram (this angle is labeled θ in figure 27).

Now, θ is the angle between the sun and the planet, as viewed from the earth, and hence must be equivalent to $\angle SEP$ in the Copernican model. It is clear, then, that all the epicycle radius vectors for the superior planets ($\overline{CP}$ in figure 27) must point in the same direction. As for the length of the radii, since $\overline{EP}=\overline{EC}+\overline{CP}$ in the Ptolemaic system and $\overline{EP}=\overline{ES}+\overline{SP}$ in the Copernican system and since $\overline{EC}$ (Ptolemaic) = $\overline{SP}$ (Copernican), we have that $\overline{CP}$ (Ptolemaic) = $\overline{ES}$ (Copernican). So, it is clear that the radii of the epicycles are equal to the distance from the earth to the sun in the Copernican model. Thus, each outer planet has an epicycle of radius 1 AU.

Since earth orbits the sun once a year, each planet will make a complete rotation around its epicycle once a year. Similarly, the epicycle will make a complete rotation around the deferent in the amount of time it takes the planet to orbit the sun: about 29.5 years for Saturn, 12 years for Jupiter, and 2 years for Mars.[51]

We have shown that the Ptolemaic and Copernican models are geometrically equivalent with the modifications made above. Basically, for the inner planets the deferent represents the earth's orbit around the sun and the epicycle represents the planet's orbit around the sun. On the contrary, for the outer planets the deferent represents the planet's orbit around the sun and the epicycle the earth's orbit. So, while Copernicus' model revolutionized science, from a strictly mathematical standpoint, it does not differ much from Ptolemy's (aside from relatively minor changes).

5.4 Reactions to Copernicus

Copernicus' goal in placing the sun at the center of the universe does not seem to have been to revolutionize science. He simply felt that the heliocentric system was superior to the geocentric, and therefore he made the change. Initial reactions to On the Revolutions varied from heated disdain to lack of concern, but the new role of the earth as a body rotating about the sun eventually was accepted and Copernicus' work helped to set off the Scientific Revolution.

Copernicus objected that in Ptolemy's model the "sun, the moon, and the five planets seemed ironically to have different motions from the other heavenly bodies and it made more sense for the small earth to move than the immense heavens."[52] While this statement seems innocent enough, suggesting that the earth was not at the center of the universe and that it rotated had serious implications. The earth had long since held a special place at the center of the universe since many felt God gave humans the role of being superior beings. Suggesting the earth was not the center meant humans no longer held this superior role and were merely floating around on one of several planets. The idea that the earth was rotating seemed even more ridiculous when it seemed that if this was the case, we should feel the winds of rotation.

Perhaps surprisingly, it was not the Catholic Church that was the major objector to Copernicus' ideas, but rather the newly rising Protestant Church. Keeping in mind the ideas of the Reformation, this actually makes sense. Protestants were demanding a return to the word of the Bible taken literally, which they pointed out suggested an earth centered universe. Martin Luther was one of the biggest critics of Copernicus' work, stating that "the fool will turn the whole science of astronomy upside down. But, as the Holy Writ declares, it was the Sun and not the Earth which Joshua commanded to stand still."[53]

On the contrary, some religious believers felt that On the Revolutions fit well with faith. Some of the ideas could be interpreted to tie together religion and science, rather than to counter faith. Indeed, Copernicus' insistence that the universe was harmonious and symmetrical had appeal in the religious upheaval of the Reformation. One German mathematician Rheticus noticed how Copernicus' model related God, astronomy, and the ancient ideas of Plato. He remarked on Copernicus' six moving spheres that "the number six is honored beyond all others in the sacred prophecies of God…What is more agreeable to God's handwork than that this first and most perfect work should be summed up in this first and most perfect number?"[54] While strong opinions like these existed, a general feeling of indifference was more common.

While it is clear that Copernicus intended his system to be an actual physical representation of the universe, many accepted it merely as a predictive tool. This is partly due to a preface added to On the Revolutions without Copernicus' knowledge. While supervising the printing of the work, Lutheran clergyman Andrew Osiander added a preface suggesting that its purpose was to make predictions. The preface reads, in part, "it is not necessary that these hypotheses should be true, or even probable; but it is enough if they provide a calculus which fits the observations."[55] With this preface in mind, the most common initial reaction was acceptance of Copernicus' model as a mathematical tool, regardless of acceptance as a physical reality.

Copernicus was the last astronomer to create an all-encompassing model of the universe based on epicycloids. The next major innovation in modeling the universe came with Kepler's model which finally did away with the long-standing notion that the heavens were composed of objects moving in a uniform and circular manner. The eventual acceptance of the elliptical orbits of the planets meant that there was no longer a need for epicycloids to model the heavenly motions directly. However, the ellipses of Kepler, while new in idea, can be thought of as merely epicycloids with certain parameters.

5.5 The Ellipse as an Epicycloid

Let us establish that any ellipse can be created using an epicycloid. Consider an earth centered ellipse given by the parametric equations

$$\begin{cases} x = a\cos(\alpha) \\ y = b\sin(\alpha) \end{cases} \qquad (5.1)$$

We will consider the case where a > b and hence the x-axis is major. The goal is to compare 5.1 with the parametric equation derived earlier for the epicycloid:

$$\begin{cases} x = r\cos(\omega_2\Theta/\omega_1) + R\cos(\Theta) \\ y = r\sin(\omega_2\Theta/\omega_1) + R\sin(\Theta) \end{cases} \qquad (5.2)$$

Clearly we can rewrite 5.1 as

$$\begin{cases} x = (1/2)(a-b)\cos(\alpha) + (1/2)(a+b)\cos(\alpha) \\ y = (1/2)(b-a)\sin(\alpha) + (1/2)(a+b)\sin(\alpha) \end{cases}$$

Using the fact that cosine is an even function and sine is an odd function, the parametric equations can again be rewritten, this time as

$$\begin{cases} x = (1/2)(a-b)\cos(-\alpha) + (1/2)(a+b)\cos(\alpha) \\ y = (1/2)(a-b)\sin(-\alpha) + (1/2)(a+b)\sin(\alpha) \end{cases} \qquad (5.3)$$

We can now compare 5.2 with 5.3. If $R=(1/2)(a+b)$, $r=(1/2)(a-b)$, and $\omega_2=-\omega_1$, then the equations are identical.

Thus, given an ellipse with x the major axis, we can create it using an epicycloid with the above conditions on the radii and angular velocities of the epicycle and deferent. See figure 28 here. The same argument holds for an ellipse with y the major axis by simply reversing the roles of a and b.

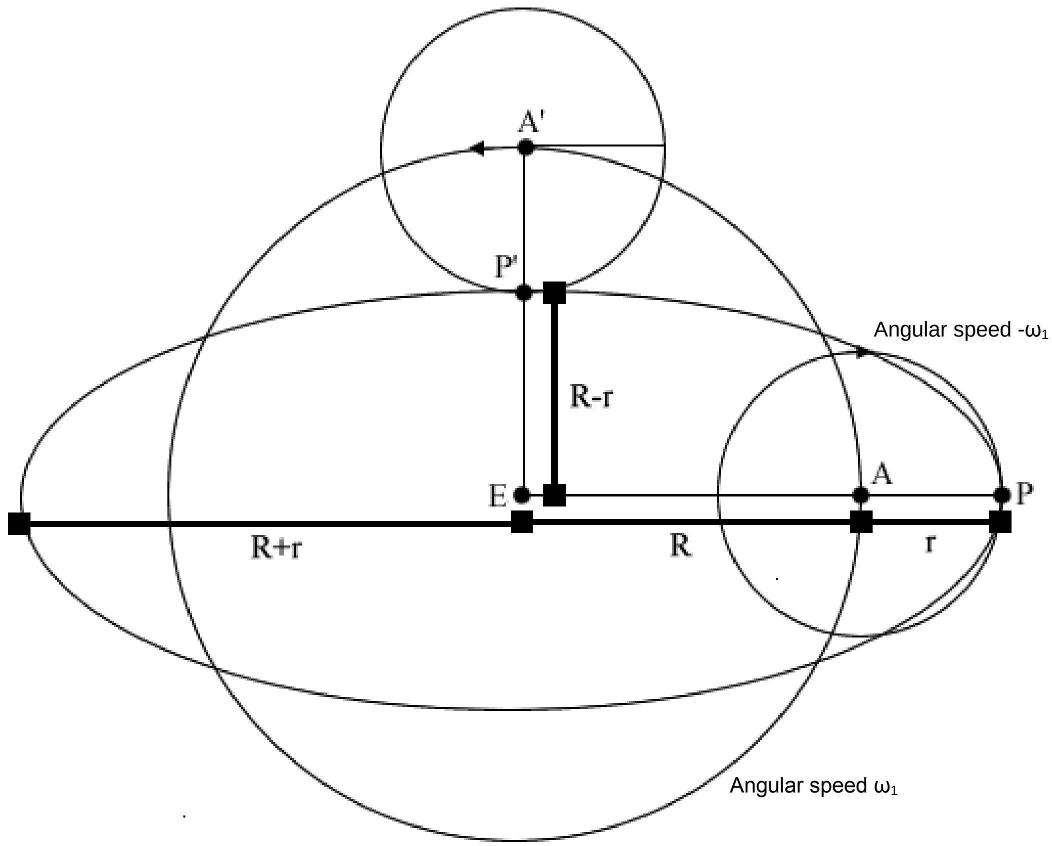


Figure 28: Assuming the angular velocity of the epicycle is the negative of the angular velocity of the deferent, the epicycloid traces out an ellipse with major axis of length 2(R+r) and minor axis of length 2(R-r).

We can rewrite our ellipse from 5.3 in terms of R and r as

$$\begin{cases} x = (R+r)\cos(\alpha) \\ y = (R-r)\sin(\alpha) \end{cases} \qquad (5.4)$$

Thus, the major axis of the ellipse is of length 2(R+r) and the minor axis is of length 2(R-r).

Most shapes traced out by epicycloids are not ellipses since in general ω_2 is not equal to $-\omega_1$, and this condition is necessary to create an ellipse. It is clear that since the circle is a special case of an ellipse, circles can also be created using the epicycloid. This is actually quite simple: simply set the length of the ellipse's major axis equal to the length of its minor axis, so we have R+r=R-r. Hence, this is equivalent to setting the radius of the epicycle to zero. The circle can be modeled by the deferent alone, which is an obvious fact.

We have shown here that Kepler's ellipses can be modeled using epicycloids. This was not really done in practice since the idea of elliptic motion of the planets was eventually accepted.

Chapter 6 Not the End for the Epicycloid: the Connection with Fourier Analysis

6.1 The Truth about Fourier Series

Thus far, we have focused mostly on epicycloids in a historical context. A common assumption is that the use of the epicycle on deferent to model phenomena is outdated and does not compare in accuracy to more modern methods. This is entirely false. In fact, as we will proceed to show, one of the most commonly used methods to approximate functions today, by expression as a Fourier series, is nothing more than approximating that function by epicycles upon epicycles. Hence, not only is this type of modeling not antiquated, but it is extremely powerful since the vast majority of functions can be modeled to a high level of accuracy by Fourier series and thus by stacking epicycle upon epicycle.

In 1807, work by Euler, D'Alembert, Bernoulli, and Joseph Fourier came to a culmination in Fourier's paper, "On the Propagation of Heat in Solid Bodies." In this work, Fourier introduced the representation of functions as sums of sines and cosines.[56] Today we know this as the Fourier series for a given function. While it would take even more work to determine exactly what conditions must be placed on a function for its Fourier representation to be valid and accurate, the basic ideas as we still invoke them today were introduced in the 1807 paper. It is interesting that the ideas put forth in 1807 were considered by many to be new, when in fact such approximations had been in use for centuries in the form of epicycle on deferent modeling. Let us now demonstrate the equivalence.

As is commonly known, a given function f(t) may be represented as a Fourier series in complex form using the relationship

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{\frac{in\pi}{L}t}, \text{ where } c_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-\frac{in\pi}{L}t}\,dt \qquad (6.1)$$

Here we implicitly assume that f(t) does have a Fourier series representation, and also that f(t) is periodic of period 2L. This is the case if f(t) is C^1 continuous and piecewise smooth. By choosing the period large enough, we can get an approximation to a given function on whatever interval we desire, assuming it is of finite length.

Using Euler's identity to expand the complex Fourier representation above, we have

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n (\cos(\frac{n\pi}{L}t) + i\sin(\frac{n\pi}{L}t)) \qquad (6.2)$$

Let us consider a partial sum of the terms, for n between -k and k. Then we have

$$f(t) \approx c_{-k}(\cos(\frac{k\pi}{L}t) - i\sin(\frac{k\pi}{L}t)) + ... + c_{-2}(\cos(\frac{2\pi}{L}t) - i\sin(\frac{2\pi}{L}t)) + c_{-1}(\cos(\frac{\pi}{L}t) - i\sin(\frac{\pi}{L}t)) + c_0$$

$$+ c_1(\cos(\frac{\pi}{L}t) + i\sin(\frac{\pi}{L}t)) + c_2(\cos(\frac{2\pi}{L}t) + i\sin(\frac{2\pi}{L}t)) + ... + c_k(\cos(\frac{k\pi}{L}t) + i\sin(\frac{k\pi}{L}t)) \qquad (6.3)$$

But, this is exactly equivalent to a stack of 2k-1 epicycles upon a deferent centered at c_0, with each circle having radius equal to c_n and appropriate orientation based on the sign of the index n. This is clear since we can represent a circle of period $2L/n_1$ and radius $\left|c_{n_1}\right|$, centered in the complex plane at the point c_0 as $C_1(t) = c_0 + c_{n_1}(\cos(\frac{n_1\pi}{L}t) \pm i\sin(\frac{n_1\pi}{L}t))$ where the sign depends on the direction of rotation.

Now consider the complex function $C_1(t)$ as itself the center for a new circle, this one of radius $\left|c_{n_2}\right|$ and period $2L/n_2$. This new construction, call it $E_1(t)$, represents one epicycle upon a deferent and is given by

$$E_1(t) = c_0 + c_{n_1}(\cos(\frac{n_1\pi}{L}t) \pm i\sin(\frac{n_1\pi}{L}t)) + c_{n_2}(\cos(\frac{n_2\pi}{L}t) \pm i\sin(\frac{n_2\pi}{L}t)) .$$

Continuing in this manner, it is clear that 6.2 is nothing more than a superposition of epicycles centered at the zero coefficient of the Fourier series of the function f(t). Hence, modeling f(t) with the partial sum of its Fourier series, where n runs from -k to k is nothing more than approximating that same function with 2k-1 epicycles and a deferent.

Thus, there is no doubt that the long established method of approximating orbits using epicycloids is still quite applicable today. Before considering a few examples, let us demonstrate the incredible capabilities of epicycle on deferent modeling by showing that any function f(t) (under the conditions mentioned above) can be approximated to any desired degree of accuracy using only a finite number of epicycles.

Since our function f(t) is well-behaved, it can be represented as a complex Fourier series using 6.1, where this series converges uniformly to the value of the function. Hence, given $\varepsilon > 0$ as small as we please, we have the inequality $\left| f(t) - f_N(t) \right| < \varepsilon$ for all $N > N_0$ (N_0 finite). Here $f_N(t)$ is the partial sum of 6.1 for n between -N and N. Invoking the equivalence established above, we have shown that the orbit traced out by 2N-1 epicycles on a deferent represented by the partial sum is epsilon-close to the graph of f(t) for all time t. Since $\varepsilon > 0$ can be chosen arbitrarily small, it is clear that f(t) can be represented to any desired degree of accuracy by the stacking of a finite number of epicycles.

The implications of this are truly astounding. Not only have we established the power of modeling with epicycloids, but we can now use the theory to model an infinitude of figures, many of which don't seem to lend themselves to circular modeling at all. Let us first consider the intriguing example of polygons and then take a look at tracing out a flower. For simplicity, we will consider the functions as periodic of period 2π, keeping in mind that other periods will produce similar results.

It has been established that for $n \geq 2$, the Fourier series for an n-gon in the complex plane is given by

$$f_n(t) = \sum_{k=1 \bmod (n)} \frac{e^{ikt}}{k^2} \qquad (6.4)$$ [57]

Using this representation, we can demonstrate a surprising fact: linear motion can be created using only circles.

A straight line can be thought of as a 2-gon in the complex plane and hence can be represented by 6.4 as

$$f_2(t) = \sum_{k=1\,\mathrm{mod}(2)} \frac{e^{ikt}}{k^2} \qquad (6.5)$$

where the length of the line will be half of the period (hence the length here will be π). Consider a partial sum of 6.5 for k between -N and N:

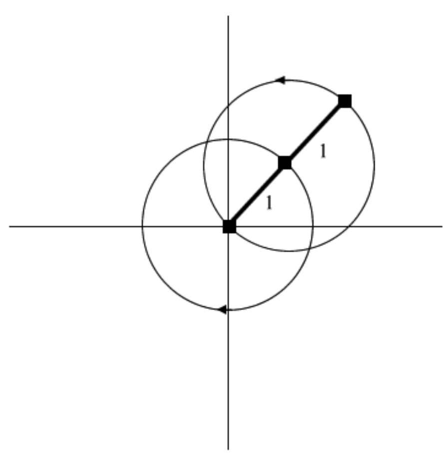
$$f_2(t) \approx (e^{-it} + e^{it}) + \frac{1}{9}(e^{-3it} + e^{3it}) + \frac{1}{25}(e^{-5it} + e^{5it}) + \ldots + \frac{1}{N^2}(e^{-Nit} + e^{Nit}) \qquad (6.6)$$

Since the numbers k=1mod(2) are all odd integers, it is clear that in 6.6, each value of k has a corresponding negative counterpart with the same coefficient. Hence, for the line, our Fourier series consists of a sum of paired terms all of the form

$$\frac{1}{k^2}(e^{-ikt} + e^{ikt}) \qquad (6.7)$$

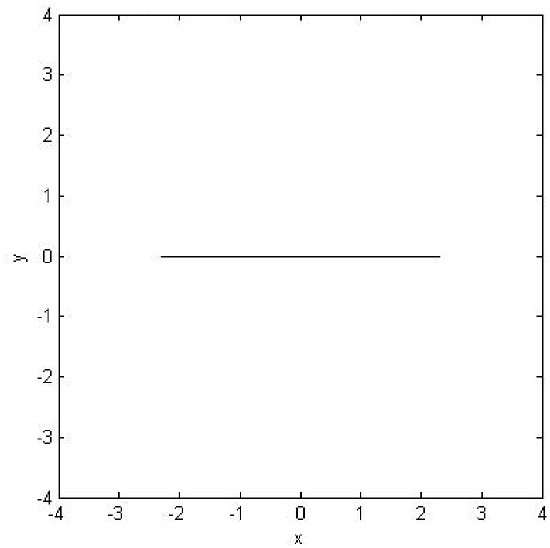
for a given value of k. Consider k=1. Expanding in terms of sines and cosines, 6.7 becomes just 2cos(t), which is completely real and of period 2π. Thus, the first complex pair produces a line segment. On the other hand, 6.7 is equivalent to $(\cos(t) - i\sin(t)) + (\cos(t) + i\sin(t))$ for k=1. This is the representation of an epicycle on a deferent centered at 0, both with radii equal to 1 and moving in opposite directions (see figure 29).

Figure 29: The figure is given by 2cos(t), or equivalently by (cos(t)-isin(t))+(cos(t)+isin(t)).



It is clear that such cancellations will occur for each respective pair of complex exponentials. This is equivalent to the cancellation of the vertical component of motion for the respective epicycles, since each pair represents a pair of epicycles of the same radius and period, moving in opposite directions. See the Matlab plot in figure 30.

Figure 30: A plot of 6.5 for $|k| \leq 6$ illustrating that in fact a straight line is drawn out by the epicycloids.



Let us consider the triangle next. The triangle's Fourier series is given by 6.4 as

$$f_3(t) = \sum_{k=1\,\mathrm{mod}(3)} \frac{e^{ikt}}{k^2} \qquad (6.8)$$

Expanding as sines and cosines for values of k between -4 and 4, we have

$$f_3(t) \approx \frac{1}{4}(\cos(2t) - i\sin(2t)) + (\cos(t) + i\sin(t)) + \frac{1}{16}(\cos(4t) + i\sin(4t)) \qquad (6.9)$$

Thus, we have a triangle in the complex plane represented as two epicycles on a deferent of radius 1. See figure 31.

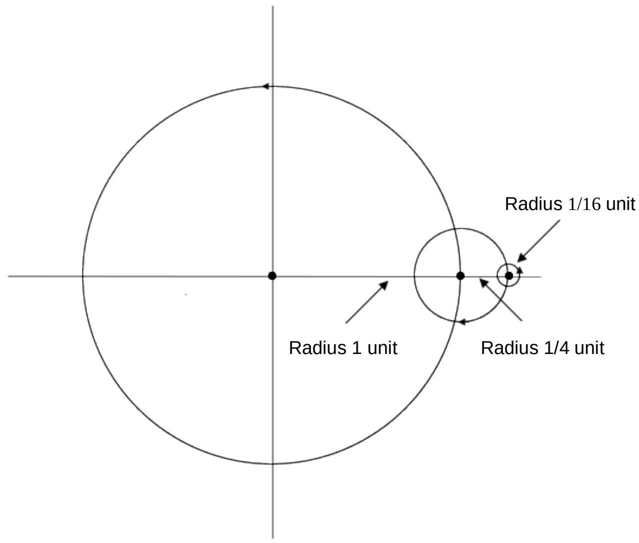


Figure 31: A triangle is traced out in the complex plane by two epicycles with radii 1/4 and 1/16, rotating about a deferent of radius 1. The periods, from largest circle to smallest, are 2π, π, and π/2.

Here the vertical component of motion is preserved since the indices, k=1mod(3), do not come in precise positive and negative pairs that cancel. Plots of 6.8 are given below for $|k| \le 4$ (the three circles shown above), and $|k| \le 100$.

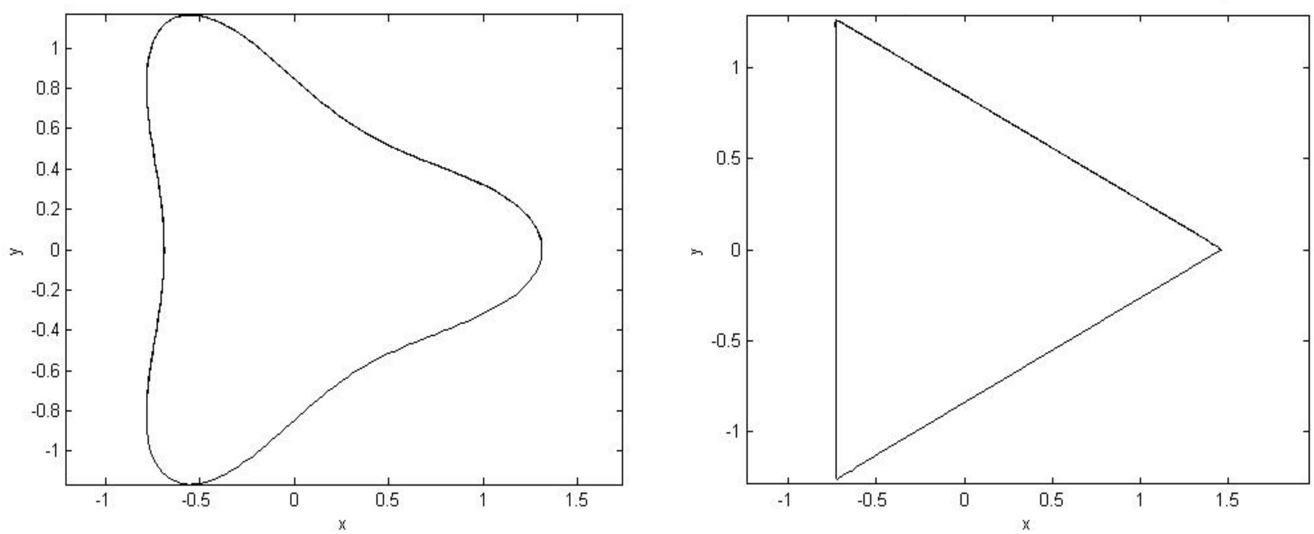


Figure 32: A triangle is traced out in the complex plane to any desired degree of accuracy by stacking epicycloids. The results are shown here for 3 circles and 67 circles, respectively.

As a final example for polygons, let us look briefly at the square. Here, 6.4 gives the Fourier representation as

$$f_4(t) = \sum_{k=1\,\mathrm{mod}(4)} \frac{e^{ikt}}{k^2} \qquad (6.10)$$

Here, let us expand in sines and cosines for values of k between -5 and 5:

$$f_4(t) \approx \frac{1}{9}(\cos(3t) - i\sin(3t)) + (\cos(t) + i\sin(t)) + \frac{1}{25}(\cos(5t) + i\sin(5t)) \qquad (6.11)$$

Here we have the square approximated by 3 epicycles on a deferent of radius 1. Again, the vertical component of motion is preserved due to the nature of the indices. Plots are shown for $|k| \leq 5$ and $|k| \leq 100$.

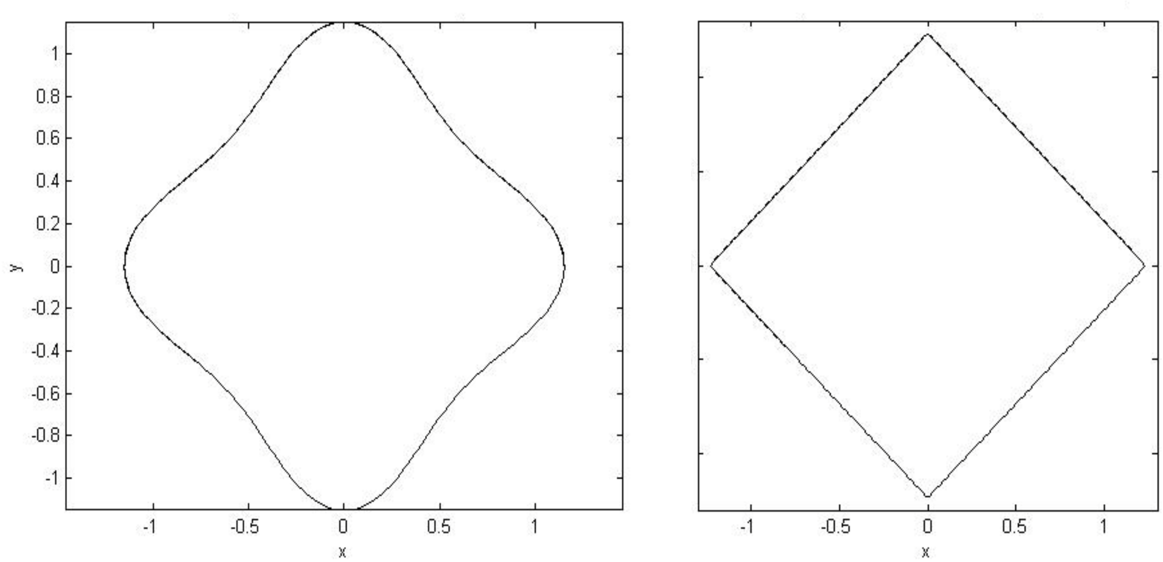


Figure 33: A square may also be created to any desired degree of accuracy by stacking epicycloids. The results are shown here for 3 circles and 47 circles, respectively.

From our examination of polygons, some similarities are evident. First, since 1=1mod(n) for any number of vertices n, any polygon will have a standard unit circle (radius 1, period 2π, positive orientation) in its Fourier representation. In addition, this circle will always be the deferent (the largest circle) since the coefficients are $\frac{2\pi}{k}$ and these represent the respective radii. Thus, as |k| increases, the radii decrease. Also, the polygons will always have periods of 2π since the successive periods $\frac{2\pi}{k}$ decrease as |k| increases. The polygons are especially interesting because they are made of *lines* and so it is surprising that they can be represented by the superposition of *circles*.

Keeping in mind that almost any figure can be represented with epicycloids, let us consider just one further example as an illustration of the possibilities. We can represent a 20-petaled flower in the complex plane by the parametric equations

$$\begin{cases} x = \sin(10t)*\cos(t) & (6.12) \\ y = \sin(10t)*\sin(t) & \end{cases}$$

for a given t-interval, or equivalently as $z = \sin(10\mathrm{t})*\cos(\mathrm{t}) + i\sin(10\mathrm{t})*\sin(\mathrm{t})$ for the same interval. Now, by expanding 6.12 using Euler's identity, we have the equivalent parametric equations

$$\begin{cases} x = 1/2*(\sin(11t)+\sin(9t)) & (6.13) \\ y = -1/2*(\cos(11t)-\cos(9t)) & \end{cases}$$

This is the representation of the flower as a Fourier series. We have 6.1, but with only two terms:

$$\sin(10\mathrm{t})*\cos(\mathrm{t}) + i\sin(10\mathrm{t})*\sin(\mathrm{t}) = \sum_{n=-\infty}^{n=\infty} c_n e^{\mathrm{int}} \text{ where } c_{-11} = \frac{-1}{2},\ c_9 = \frac{1}{2}, \text{ and } c_n = 0\ \ \forall\ n \neq -11, 9\,.$$

Hence, the flower is traced out by one epicycle upon a deferent where both circles have radius $\frac{1}{2}$. One circle has period $\frac{2\pi}{11}$ and the other has period $\frac{2\pi}{9}$. Below is a plot of 6.13.

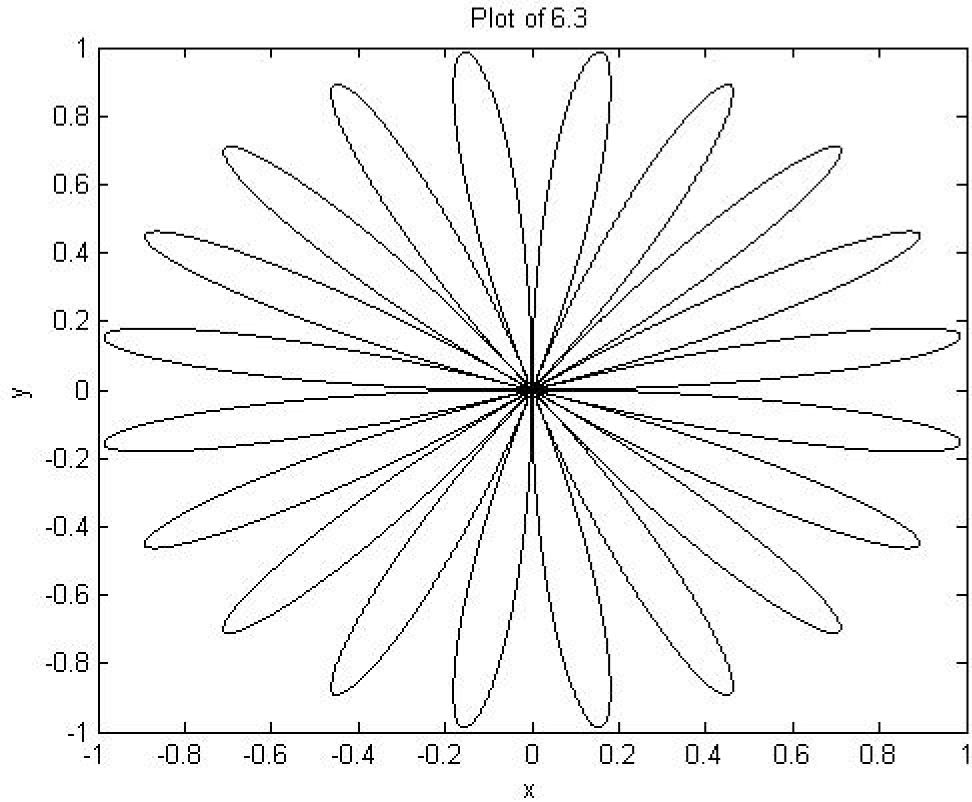


Figure 34: A plot of the Fourier series of the flower. The flower is traced out by one epicycle rotating upon a deferent.

Of course, there is nothing special about the flower shown above aside from it having a relatively simple Fourier series representation that makes a nice example. We could have considered a wide variety of functions as examples using the same procedure.

Just with this short demonstration, it is clear that approximating data or figures using epicycloids is not at all an outdated method. Indeed, we use the method quite often, only disguised as what is today known as modeling with Fourier series. Since both methods are equivalent, the accuracy and power of modeling with epicycloids is now clear. With this conclusion, the reasoning of the astronomers of antiquity becomes quite impressive and the astronomers themselves now seem much wiser than their time would lead us to think they were.

6.2 Epicycloids and non-Periodic Motion

So far we have considered using epicycloids to model periodic motion. In a historical context, periodic motions are important because they often occur in natural settings such as the planets orbiting through the heavens. However, the capabilities of epicycloids in modeling are not limited to periodic motions alone; in fact, we can use a similar process to model non-periodic motions as well.

Since the Fourier series representation of an orbit always involves only integer indices, each sine and cosine pair has period $\frac{2\pi}{k}$ for the appropriate value of k. Since the index here has initial value 1 (index 0 can be thought of as corresponding to the center of the deferent), the largest period will always be 2π. Since each of the other periods T_k can be multiplied by k to get 2π, all the periods are commensurable and hence the entire orbit has period 2π. Since a periodic motion in the complex plane is always a closed curve, the Fourier series interpretation of epicycloids always produces a periodic, closed orbit.

On the other hand, we can produce non-periodic motions using epicycloids as well. Consider one epicycle rolling on a deferent, represented by the parametric equations

$$\begin{cases} x = r*\cos(\omega_2 t)+R*\cos(\omega_1 t) \\ y = r*\sin(\omega_2 t)+R*\sin(\omega_1 t) \end{cases} \qquad (6.14)$$

where r is the epicycle's radius, ω_2 is its angular velocity, R is the deferent's radius, and ω_1 is the angular velocity of the deferent. Now, let us choose one of the velocities, say ω_1, equal to 2π, for example. Then set ω_2 equal to $2\pi\sqrt{2}$. Thus, the parametric equations for this particular case are

$$\begin{cases} x = r*\cos(2\pi\sqrt{2}\,t)+R*\cos(2\pi t) \\ y = r*\sin(2\pi\sqrt{2}\,t)+R*\sin(2\pi t) \end{cases} \qquad (6.15)$$

The terms r*cos($2\pi\sqrt{2}$ t) and r*sin($2\pi\sqrt{2}$ t) are periodic with period $\frac{1}{\sqrt{2}}$, obviously an irrational number. On the other hand, the terms R*cos(2πt) and R*sin(2πt) are periodic of period 1, which is clearly rational. Now, since $\sqrt{2}$ is is not a multiple of any two rational numbers, the two periods will never match up and hence the sum of the two terms (6.15) will not be periodic. Below are plots of 6.15 for R and r equal to 1, with 11 values of t and 101 values of t, respectively. Notice that the curves do not close and become more dense as we increase the number of t values.

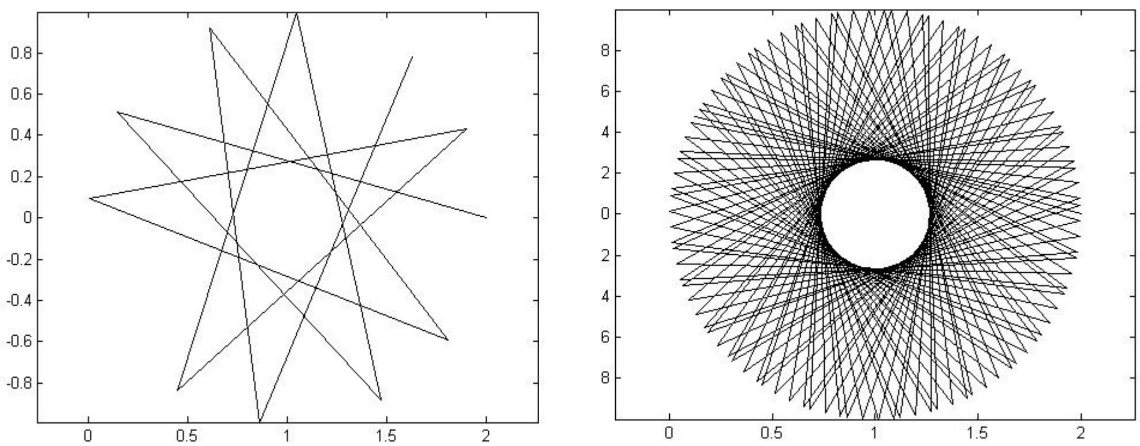


Figure 35: Plots of 6.15 are shown. On the left, we have the equations plotted for integer values of t between 0 and ten, and on the right for integer values of t between 0 and 100. Due to the non-commensurable periods, the curves never close, but rather become denser as we increase the number of values of t.

Here, specific numbers were chosen for illustrative purposes, but the results hold in general. If some of the terms have irrational periods and others do not, then the periods will never match up and the overall function will not be periodic. We considered specifically one epicycle upon a deferent, but the result can be extended to stacked epicycles since each epicycle that we add just corresponds to adding another sine and cosine term in the parametric equations.

Thus, using irrational periods we can model motion that is not even periodic using epicycloids. In the complex plane, the orbit we produce is not closed since it is not periodic. The ability to model non-periodic motion is just another illustration of the power of modeling using epicycloids.

Chapter 7 Recent Sightings of Epicycloids

7.1 A Modern Discovery: Epicycloids and the Antikythera

In 1901, divers off the coast of a Mediterranean Island called Antikythera made an unexpected discovery when they encountered the remains of an ancient shipwreck. At first unnoticed among the treasures recovered, including vases, bronze statues, and glassware, was perhaps the most valuable treasure of all. What looked to be just lumps of bronze corroded over the years turned out to be a complex device for modeling the position of the sun, moon, and perhaps the planets using the astronomical theory of ancient times. The device is the earliest example of the use of gears discovered as of yet and has given modern scientists a new perspective on the sophistication of ancient thinkers and designers alike. Although some aspects of the device, which has come to be called the Antikythera Mechanism, remain mysterious, we do have many answers thanks to extensive research that has been done over the past century. We now have reputable theories on the origins of the device and on its workings and structure, including the incorporation of the theory of epicycloids in line with Hipparchus' models.

There has been much speculation on where the mysterious, remarkably complex device came from. The existence of a similar contraption had been suggested by Marcus Cicero a Roman philosopher, in the first century BC. According to his writings, a bronze planetarium had been recovered after the defeat of Syracuse by Roman general Marcellus in 212 BC.[58] This planetarium was among the creations thought to be made by Archimedes, who is known for both his mathematical abilities and his skill in incorporating these ideas into mechanical models. Prior to the discovery of the Antikythera mechanism, Cicero's claim that "the invention of Archimedes deserves special attention because he had thought out a way to represent accurately by a single device for turning the globe those various and divergent movements with their different rates of speed,"[59] was thought to be an overstatement. Although no one doubted the ingenuity of Archimedes' creations, there was simply no evidence that something of this degree of complexity had ever existed. The discovery of the Antikythera permanently altered this assumption. To be clear, the device and the one that inspired Cicero's awe are not one and the same.

The ship carrying the Antikythera is estimated to have sunk in approximately 85 BC, and the device itself is thought to have been built about 20 years prior to the shipwreck.[60] Hence, this device could not have been built by Archimedes, but could perhaps be modeled after one that was. Among the inscriptions on the Antikythera, there is no signature, nor is there a clear indication of where the device was constructed. The other items recovered from the shipwreck, including Greek luxury items, suggest that the ship may have been heading for Rome with looted Greek treasures. One intriguing clue is the discovery of vases in the rubble designed in the style being used on the Greek island of Rhodes.[61] Perhaps the ship had made a stop at Rhodes and acquired the mechanism while there.

While the origins of the device are still somewhat unclear, research has provided much insight into the structure and workings of the Antikythera mechanism. The device was hand-cranked and is thought to have been inside a wooden box with a handle on the side that was used for operation. When a person turned the handle, time passed before his or her eyes. The positions of the sun and moon were accurately displayed and while it is still questionable, there is evidence that the device may also have modeled the orbits of the five planets known at the time. In addition, the device provided the date, predicted the positions of certain stars, tracked cycles important in predicting eclipses and creating calendars, and may have even been used to track the dates of Olympic Games.[62] All of this was achieved by at least 30 gear wheels, ranging in size, each of which had gear teeth shaped like equilateral triangles, with the number of teeth ranging from 15 to 223. See figure 36 below for an overview of the gearing.

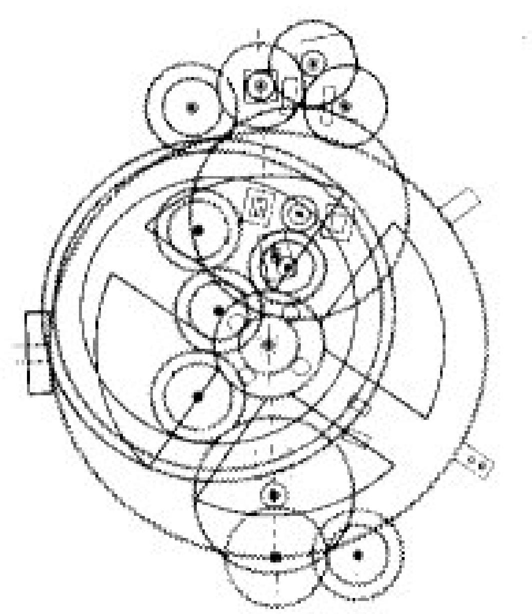


Figure 36: A detailed look at the gearing inside the Antikythera mechanism. As can be seen, the device is quite complex, with circles rolling upon and within other circles.[63]

It was these gears that allowed the device to make accurate predictions, for the wheels "multiplied the speed of rotation by precise mathematical ratios depending on the number of teeth on each wheel."[64] It is in the details of the gearing that the connection with Hipparchus becomes clear. While the part of the mechanism thought to model the motion of the five planets has never been found, there is a fair amount of certainty in how the portion controlling the lunar motion operated. Hipparchus accounted for the variation in the moon's speed by having it orbit the earth on a circle which was slightly off-center from the earth. As he showed, this eccentric circle model is equivalent to an epicycle on deferent model, with the respective radii and velocities chosen appropriately. Researchers have found that the "gears inside the Antikythera mechanism precisely model this theory. One gearwheel sits on top of another, but on a slightly different axis."[65] Here, the design is truly ingenious, with the top and bottom wheels connected with a pin-in-slot device designed so as the wheels turn, the "pin slides back and forth in the slot. This causes the speed of the top wheel to vary, even though the speed of the bottom wheel is constant."[66] It is amazing to find complex examples of epicycloids in action in such an ancient device. While there is still room for debate, experts think the Antikythera mechanism also modeled the orbits of the planets, accounting even for their varying speeds and brightness, by using the theory of epicycloids. In fact, there is evidence that the mechanism modeled the motion of the planets "using what is still known today as epicyclic gearing - small wheels riding around on bigger wheels."[67]

It is now clear that the ancient Greeks were capable of producing an extremely complex mechanical model which brought to life the mathematical theories of the time. The Antikythera mechanism is evidence that the Greeks were more technologically advanced than was ever thought. Questions remain, however. It is not clear exactly what the device was used for. Suggestions include predicting the positions of the heavenly bodies (although it is not necessary to have a physical model to do this), astrology, adjusting or creating calendars, setting the dates of events like festivals or Olympic Games, predicting eclipses, or perhaps as a model for display.[68] Perhaps an even more perplexing question to the modern thinker is why the Greeks never used gearing devices like those in the mechanism to create clocks or steam engines, as would be done much later. Since we now know the Greeks had the capabilities, why was there not an industrial revolution in the first century BC? This question is more easily answered than might be thought. The ancient thinkers did not think in terms of creating devices to do work here on earth, but rather created models to recreate the otherworldly, the most divine dimension of human life. There is a major difference in goals: "where we see the potential of that technology to measure time accurately and make machines do work, the Greeks saw a way to demonstrate the beauty of the heavens and get closer to the gods."[69] While the Greeks may not have used gears to advance industry, the Antikythera mechanism remains an astounding example of complex epicycle-on-deferent theory put to use in a mechanical device.

7.2 Epicycloids for Fun: the Spirograph

While the average person may not have come in contact with epicycloids in astronomy, put them to use while computing a Fourier series, or even read about the discovery of the Antikythera mechanism, most of us encountered the figure as children playing with a classic toy called the Spirograph (see figure 37 below). The toy is made up of plastic gears of different sizes, each with holes strategically placed at various locations. By assembling the gears appropriately using the included colored pins, one can produce a vast array of complex figures. What is amazing is the beauty and complexity of these figures which are produced using only simple circular gears. But this fact is not surprising when we consider that many of the shapes produced by the Spirograph came simply from varying the parameters of an epicycle rolling on a deferent.

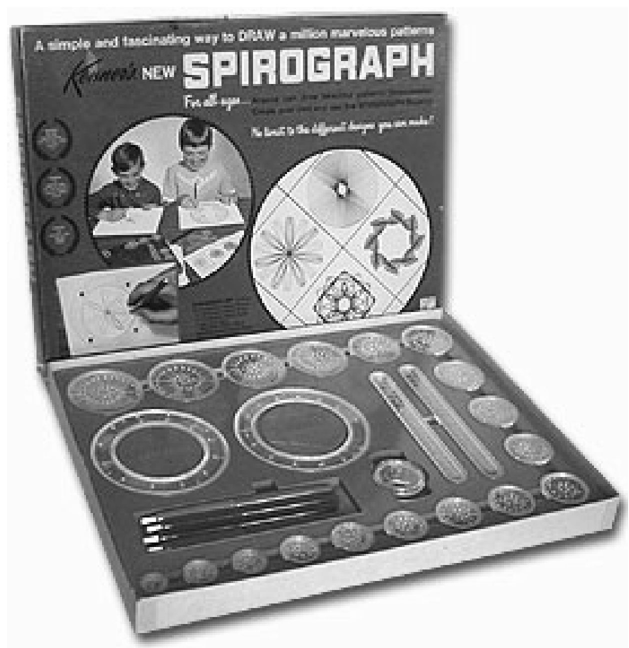


Figure 37: The contents of the box containing the classic Spirograph toy. Notice that a variety of plastic gears (epicycles and deferents) are included.[70]

By placing a small circular gear on the outside of a larger ring-like gear and rotating them, we produce an epicyclic orbit (see figure 38 below).

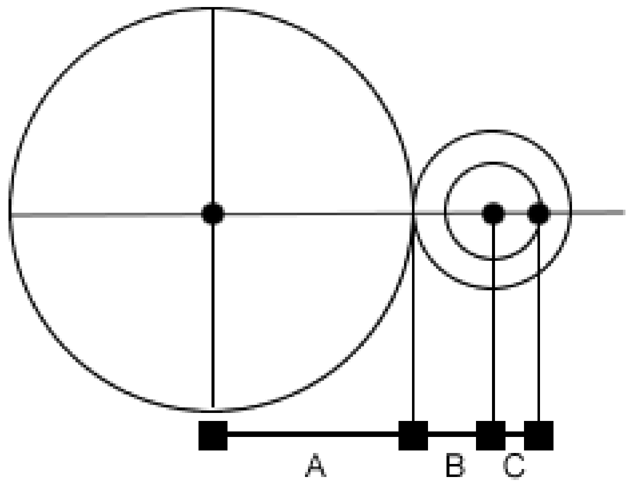


Figure 38: The basics of the Spirograph. Here, the radius of the epicycle (smaller gear) is B and the radius of the deferent is A+B, where the larger ring gear has radius A. The pencil is inserted into a hole a distance of C from the center of the smaller gear.

Here, if the larger gear has A gear teeth and the smaller gear has B teeth, by an appropriate scaling of the axis, the situation can be represented by letting the radius of the epicycle be B and the radius of the deferent be A+B. Now, the pencil is positioned in a hole in the body of the epicycle gear a distance C from its center. The parametric equations for the position of the pencil tip after time t are

$$\begin{cases} x = (A+B)*\cos(t)-C*\cos(\frac{A+B}{B}t) & (7.1)^{1} \\ y = (A+B)*\sin(t)-C*\sin(\frac{A+B}{B}t) \end{cases}$$

Below is typical pattern that might be produced using the Spirograph:

[1] See Ippolito for a derivation.

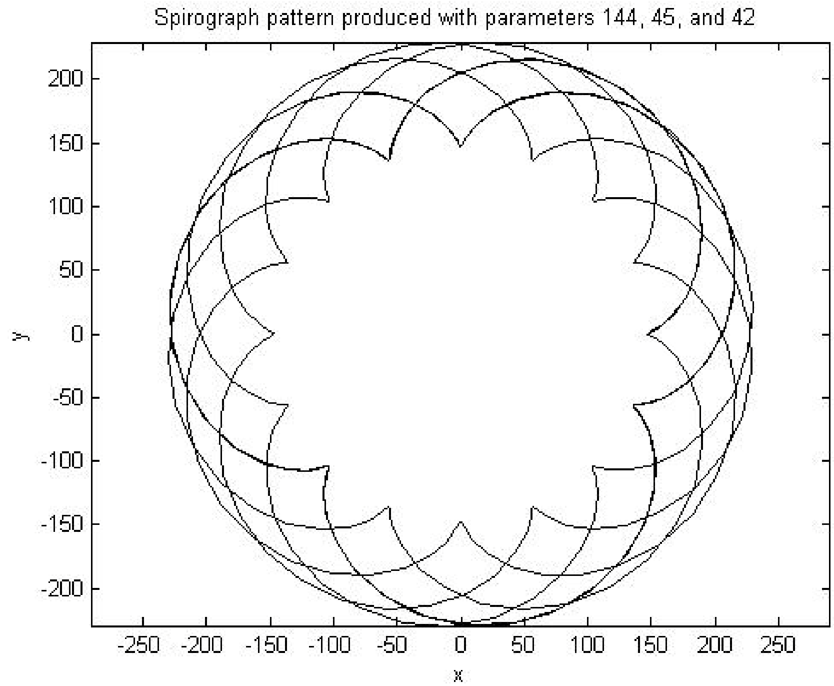


Figure 39: Pattern produced using a ring gear with 144 gear teeth and a circle gear with 45 teeth and hole positioned 42 units (with appropriate scaling) from its center. Hence, in 7.1, B=45, A=99, and C=42.

The capabilities of the Spirograph are limited in comparison to epicycle on deferent modeling in general since the toy only comes with a few wheels. Of course, since the gears all have a rational number of teeth and rational radii, after some number of rotations, the orbits produced by the Spirograph will all be closed. Thus the Spirograph is capable of producing only periodic orbits. Even with these limitations, the toy can create an awe-inspiring array of figures incorporating epicycles. The childhood memories of the simple toy that many of us have make it clear that we all have more experience with epicycloids than we may think.

Chapter 8 Conclusion

The journey through the history of the epicycloid is intriguing. Astronomers such as Appolonius, Hipparchus, Ptolemy, and Copernicus put the epicycloid to use in modeling the universe in truly impressive ways. Each of these scientists, among others, contributed to the story of the figure as an astronomical model by adding new details to previous versions or in some cases completely renovating them. The use of the epicycloid allowed the complex motions of the heavenly bodies to be modeled and predicted to an incredible degree of accuracy while still conforming to Plato's dictum of uniform circular motion.

The story of the epicycloid in the context of the history of science is fairly well known. Much more obscure is the mathematical potential of epicycloids in fitting data and curves. When we consider that the procedure of stacking epicycles upon one another is mathematically identical to adding terms in a Fourier series representation, the true power of the epicyloid becomes clear. This connection with Fourier series makes the innovation of the astronomers in using the epicycloid even more incredible. The fact is that the use of the epicycloid in astronomy was a genius idea for the time.

Not only does the Fourier connection shed light on this fact, but it also clears up the common misconception that the epicycloid is really only important in a historical context. Since the importance of the Fourier series in mathematics can hardly be questioned, neither then can the importance of the epicycloid since the two are one and the same. Not only is the epicycloid impressive as a geometric figure alone, but it is so for its role in the history of science and its mathematical power as well.

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