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Closure operations on the submonoids of the natural numbers

Laurie Price

Candidate: Laurie Price

Department: Mathematics

This thesis is approved, and it is acceptable in quality and form for publication:

Approved by the Thesis Committee:

Dimiter Vassilev, Chairperson

Janet Vassilev

Alex Buium

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Laurie A Price

B.A., Art and Creative Writing, University of New Mexico, 1980
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Dedication

To my Dad, Tim and Tristan

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Abstract

We examine the algebraic structure of closure, semiprime and prime operations on submonoids of $\mathbb{N}_0$. We find that the closure operations under composition do not form a submonoid under composition. We also describe all the semiprime operations on $\mathbb{N}_0$ and show that they are a submonoid.

We investigate the relations among the semiprime operations on ideals of the sub-semi-group $\langle 2, 3\rangle$ and define which of these operations may form a monoid or a left act under composition.

We also consider the algebraic structure of monoids with multiple maximal ideals and generalize these results to higher dimensions.

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Chapter 1

Introduction

1.1 General Direction

In the following pages we will “translate” and explore some results from [13] on the structure of closure operations in a commutative ring in the new context of a monoid. We will investigate which theorems have analogs in this new realm and what structure gets preserved upon switching to this new sphere. In the next section we will review the pertinent mathematical definitions and concepts, but for now we note that the definition of a monoid involves fewer requirements than that of a ring, i.e., it has less structure, in a way. Somewhat counterintuitively, this actually expands our field of applicability, since having less rules gives us more “elbow room”. Just as studying, say, anyone who ever lived in New Mexico gives us a greater range of data to work with than setting our rules for subjects to those currently alive who live in the Heights, so here also constraining our field of study to monoids will expand the results to more mathematical objects.

1.2 Definitions and Concepts

Abstract or Modern Algebra often starts by taking the familiar properties of some set of numbers such as the integers and "abstracting" the rules that govern manipulating those numbers into a structure that can be applied to any sets sharing those numbers' properties. In the definition of a ring, some of the properties of integers become the axioms which define the new mathematical object. Thus a ring is based on an abstraction of the properties which apply to the integers. Formally, we recall from [2] that a ring is a nonempty set R along with 2 operations, + and · and which has the following properties for + :

closure: for $a, b \in R, a + b \in R$

commutative : $a + b = b + a$ for $a, b \in R$

associative: $(a + b) + c = a + (b + c)$ for $a, b, c \in R$

identity there exists an element 0 such that $0 + a = a$ for $a \in R$

inverse for every $a \in R$, there exists an element $b \in R$ such that $a + b = 0$.

and these properties for · :

closure: $a \cdot b = \in R$ for $a, b \in R$

associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for $a, b \in R$

along with a ***distributive*** property:

$a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for $a, b, c \in R$.

This base definition is then tweaked to generate different types of rings: the integers themselves are an example of a commutative ring with unit (the base definition does not require that the operation of multiplication be commutative, or specify that there is an element such that $1 \cdot a = a = a \cdot 1$, though it is common to assume that rings have a unit in general). Mathematical results are computed directly using the abstract structure; the integers may be brought out for concrete examples.

I is a ***subring*** of R if I is a subset of R in which the operations"$\cdot$" and "+" are just those of R applied to the elements of I.

Ideals may be thought of as infinite packets or classes of numbers. Following [7], [2], we define an an ***ideal*** (in a commutative ring) as a nonempty subset I of a ring R such that:

(a) I is a subring of R

(b) For any element "r" in R and element "a" in I, the products "ra" and "ar" are also in I.

Thus, an example of an ideal would be the set $\{\cdots - 4, -2, 0, 2, 4, \ldots\} = I$, in the integers where we see that any member of I times any integer gives another member of I.

A group may be thought of as a generalization of the rules for the set of integers, only allowing one operation, +. Here is the abstract definition [7], [2]:

A group is a non-empty set G together with an operation $*$ for which the following exist:

(a) ***Closure***: For a, b in $G, a * b$ is also in G.

(b) ***Associativity***: For a, b, c in $G, a * (b * c) = (a * b) * c$.

(c) ***Identity***: There is an element e in G such that for every a in G,
$a * e = e * a = a$.

(d) ***Inverse***: For every a in G, there is an element b in G such that
$a * b = b * a = e$.

In the integers, the additive identity is 0, and the inverse of any integer a

will be $-a$.

If we delete some of these properties, we obtain different sets of algebraic structures based on generalizations of different sets of numbers. Thus a set G that has only closure is called an algebraic structure with binary operation; if G has closure and associativity it is a *semi-group* (a generalization of the natural numbers, $\mathbb{N}$), and if G has closure, associativity and an identity element it is a *monoid* (a generalization of the natural numbers plus the number zero, denoted $\mathbb{N}_0$).

Our focus is on the monoid, whose stand-in will be $\mathbb{N}_0$ with the operation of addition. Since our operation is $+$, the ideals (defined later) will be sets such as $\{2, 3, 4, \ldots\}$ for the ideal $< 2 >$.

A *submonoid* will be a subset of a monoid in which the operation $+$ will be that of the monoid applied to the subset.

In this exploration we will also use the concept of an action. A group action of a group G on a set A is a map that takes a pair of elements (one each from the group G and the set A) and combines them using the rules for operations in G to obtain a new element in the set A. In symbols we write $G \times A \longrightarrow A$ and write $g \cdot a$ to mean that a specific element from G is acting on an element from A, where the new element will be some a in A. This map is defined to have the following properties:

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot (a)$, where on the right hand side of this equation $g_1 g_2$ combine to form a new element of G which in turn acts on an element of A to produce another element in A , and on the left hand side of the equation $g_2 \cdot a$ yields an element in A which is acted on by g_1 to produce another element in A.

(ii) $1 \cdot a = a$

The concept of an action is often useful for gaining additional information about a given group and set by observing the way the action of the group on the set gets carried out. In the above discussion we have actually described a left action, since the group elements appear on the left. A right group action may be defined similarly but with the group elements appearing on the right.

There exists a generalization of this notion of the action of a group on a set in the monoid setting, where instead of a group acting on a set we have an act of a monoid on the set. In our setting we will thus refer to left or right acts of monoids on sets.

1.3 Background/History

In the mid 1800's Kummer created a new kind of number, the ideal number, to allow extending the Fundamental Theorem of Arithmetic (that every integer greater than 1 may be uniquely factored as a product of primes) to certain types of rings. this was made necessary as part of his attempt to solve Fermat's "Last Theorem" ($x^n + y^n = z^n$ is impossible for rational integers x, y, z if $n > 2, n$ an integer). [1] [p. 473] Dedekind built on this concept to develop his ideals, which we have defined heuristically above. Krull [9] published the classic Idealtheorie in 1935, in which he developed the main abstract properties of ideals still used today. With the work of Wolfgang Krull and Emmy Noether (around 1930), the general ideal theory of commutative rings became an autonomous theory.[5] [p. iii]

The results we will be re-interpreting in the sphere of monoids involve the structure of closure operations (to be defined) on the set of ideals of a ring, which build on the properties first defined by Krull and later expanded on by Sakuma [11], Kirby [8], Heinzer, Ratliff, and Rush [6], and continued in papers such as those by Elliott [3].

1.4 Goals/Purpose

We will define and explore the structure on the set of closure operations on the ideals of the monoid defined by the natural numbers along with the identity zero, taking the results of [13] as a starting point. We will find out which theorems hold and examine whether the (potential) invariability of results can be predicted. We will also explore whether there are additional results that hold for monoids that weren't true in the arena of rings.

Chapter 2

Basics and closure operations on $\mathbb{N}_0$

A semigroup is a S set together with an associative binary operation. A monoid is a semigroup with an identity element. We will denote the monoid of the natural numbers with zero (and the binary operation addition), as $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An ideal in $\mathbb{N}_0$ will be defined as $< i > = \{i + a \mid i \in \mathbb{N}, a \in \mathbb{N}_0,\ i \text{ fixed}\}$.

Let $\mathscr{I} = \{< i > \subseteq \mathbb{N}_0 \mid < i > \text{ is an ideal of } \mathbb{N}_0\}$ and let $M_{\mathscr{I}} = \{f : \mathscr{I} \to \mathscr{I}\}$. The set $M_{\mathscr{I}}$ together with the binary operation of function composition and the identity map $e : \mathscr{I} \to \mathscr{I}$ is a monoid since function composition is associative.

$C_{\mathbb{N}_0} \subseteq M_{\mathscr{I}}$ will be the set of closure operations in $M_{\mathscr{I}}$, where a closure map $f_c \in C_{\mathbb{N}_0}$ such that $f_c(< i >) \mapsto < i >$ has the following properties:

(a) $< i > \subseteq f_c(< i >)$

(b) If $< i > \subseteq < j >$, then $f_c(< i >) \subseteq f_c(< j >)$

(c) $f_c \circ f_c = f_c$.

$S_{\mathbb{N}_0}$, the set of semiprime operations in $\mathbb{N}_0$, satisfies (a)-(c) and also:

(d) $f_c(< i >) + f_c(< j >) \subseteq f_c(< i + j >)$.

$P_{\mathbb{N}_0}$, the set of prime operations in $\mathbb{N}_0$, satisfies (a)-(d) and also:

(e) $f_c(< b > + < i >) =< b > + f_c(< i >), \forall\, b \in \mathbb{N}_0$.

Proposition 2.0.1 *$C_{\mathbb{N}_0}$, $S_{\mathbb{N}_0}$ and $P_{\mathbb{N}_0}$ are partially ordered sets where the partial ordering is defined as $f \leq g$ if $f(I) \subseteq g(I)$ for every I.*

Proof:

$f_c(I) \subseteq f_c(I)$, so that $f_c \leq f_c$ and the operations are reflexive.

The closure operations are antisymmetric since if $f_c \leq g_c$ and $g_c \leq f_c$ then $f_c = g_c$, and the operations are transitive since if $f_c \leq g_c$ and $g_c \leq h_c$, then $f_c \leq h_c$, therefore, $C_{\mathbb{N}_0}$, $S_{\mathbb{N}_0}$ and $P_{\mathbb{N}_0}$ are partially ordered sets. □

Definition 2.0.2 *A closure operation f is bounded on a numerical semigroup S if there is a proper ideal I such that for every ideal $J \subseteq I$, $f(J) = I$. If this does not hold, then f is unbounded.*

Proposition 2.0.3 *Under composition, $(C_{\mathbb{N}_0}, \circ)$ is not a submonoid.*

Proof:

Let $f_n : \mathscr{I} \to \mathscr{I}$ and $g_n : \mathscr{I} \to \mathscr{I}$ be defined as follows:

$$f_n(< i >) = \begin{cases} < i > \text{ if } i \leq n \\ < n > \text{ if } i > n \end{cases}$$

$$g_m(< i >) = \begin{cases} \mathbb{N}_0 \text{ if } i \leq m \\ < m > \text{ if } i > m \end{cases}$$

and $f_n(< 0 >) = < 0 > = g_m(< 0 >)$.

We want to show that property (c) does not hold, i.e.,

$f_c \circ f_c \neq f_c$, i.e., $(f_n \circ g_m) \circ (f_n \circ g_m)(< i >) \neq (f_n \circ g_m)(< i >)$

Letting $m > n$, we have that:

$$f_n \circ g_m(< i >) = \begin{cases} \mathbb{N}_0 \text{ if } i \leq m \\ < m > \text{ if } i > m \end{cases}$$

Thus,

$$(f_n \circ g_m) \circ (f_n \circ g_m)(< i >) = \begin{cases} (f_n \circ g_m)(\mathbb{N}_0) = f_n(\mathbb{N}_0) = \mathbb{N}_0 \text{ if } i \leq m \\ (f_n \circ g_m)(< n >) = f_n(\mathbb{N}_0) = \mathbb{N}_0 \text{ if } i > m \quad (5) \end{cases}$$

In equation (5) we did not get $< n >$, so that we have now shown that $f_c \circ f_c \neq f_c$. Thus $(C_{\mathbb{N}_0}, \circ)$ is not a submonoid, since property (c) does not hold. □

We next look at whether $S_{\mathbb{N}_0}$ and $P_{\mathbb{N}_0}$ (the sets of semiprime and prime operations of $M_{\mathscr{I}}$) are submonoids under composition. We will see that $S_{\mathbb{N}_0}$ is a submonoid and that $P_{\mathbb{N}_0}$ is {e} (the trivial submonoid of $M_{\mathscr{I}}$).

Proposition 2.0.4 *For the monoid* $\mathbb{N}_0$ *(whose maximal ideal is* $< 1 >$*), the set of semiprime operations* $S_{\mathbb{N}_0}$ *may be expressed as the submonoid*

$$M_0 = \{e\} \cup \{f_m \in M_{\mathscr{I}}\}$$

Where

$$f_m(< i >) = \begin{cases} < i > & \text{if } 0 \leq i < m \\ < m > & \text{if } i \geq m \end{cases}$$

We can get a similar proposition for $\mathbb{N}_0 \cup \{\infty\}$ if we use the following definition from [10].

Definition 2.0.5 *Let* S *be a monoid, and* A *be any set. Then* A *is a left (right)* S*-act if there is a map* $\delta\colon S \times A \longrightarrow A$ $(\delta\colon A \times S \longrightarrow A)$ *such that*
$\delta(st, a) = \delta(s, \delta(t, a))$ $(\delta(a, st) = \delta(\delta(a, s), t))$ *for every* $a \in A$
and for every $s, t \in S$. $\delta(e, a) = a$ $(\delta(a, e) = a)$.
In $\mathbb{N}_0$, e *is the identity map where* $e\colon \mathscr{I} \longrightarrow \mathscr{I}$ *such that* $e(< i >) = < i >$.

Proposition 2.0.6 *For the monoid* $\mathbb{N}_0 \cup \{\infty\}$*, 0 is the identity element and* ∞ *is an analog to the zero element since* $a + \infty = \infty$ *for every* $a \in \mathbb{N}_0$*.* $S_{\mathbb{N}_0 \cup \{\infty\}}$ *may be decomposed into the union of two submonoids, but* $S_{\mathbb{N}_0 \cup \{\infty\}}$ *is not a submonoid under composition.*

$$M_0 = \{e\} \cup \{f_m \in M_{\mathscr{I}}\}$$

Where

$$f_m(P_i) = \begin{cases} P_i & \text{if } 0 \leq i < m \\ P_m & \text{if } i \geq m \end{cases} \quad \text{and } f(<\infty>) = <\infty>$$

$$M_f = \{e\} \cup \{g_m \in M_{\mathscr{I}}\}$$

Where

$$g_m(<i>) = \begin{cases} P_i & \text{if } 0 \leq i < m \\ P_m & \text{if } i \geq m \end{cases} \quad \text{and } g(<\infty>) = <m>$$

Before proving these propositions, we will prove some lemmas which will be used in the proof.

Lemma 2.0.7 *Let* f_c *be a semiprime operation on* $\mathbb{N}_0$*. If* f_c *is constant for* $<i>$ *on a finite interval* $m \leq i \leq n$ *for* $m < n$*, then there exists a* $j \leq m$ *such that* $f_c(<i>) = <j>$ *for every* $i \geq j$*.*

Proof: Suppose that f_c is constant for $<i>$, where $m \leq i \leq n$ and $m < n$. Suppose also that $f_c(<i>) = <j>$. Then, using the closure property $f_c \circ f_c = f_c$, we have

$$<j> = f_c(<i>)$$

$$\Rightarrow f_c(<j>) = f_c(f_c(<i>)) = f_c(<i>)$$

$$\Rightarrow f_c(<j>) = <j>$$

and because $j \leq m$, then

$$< m > \subseteq \; f_c(< m >) = < j > .$$

Since the ideals of $\mathbb{N}_0$ are totally ordered and since f_c is a closure operation we have $f_c(< i >) \subseteq f_c(< j >)$ for $< i > \subseteq < j >$ by property (b), so that f_c is increasing on the ideals of $\mathbb{N}_0$.

We thus know $f_c(< n >) = < j >$. If we can also show that $f_c(< n+1 >) = < j >$ then it will follow by induction that $f_c(< n >) = < j >$ for every $i \geq j$.

Since f_c is increasing,

$$f_c(< n+1 >) = < k > \subseteq \; f_c(< j >) \; = < j >, \; j \leq k \leq n+1.$$

We may apply f_c to both sides of

$$f_c(< n+1 >) = < k >$$

and get

$$f_c(f_c(< n+1 >)) = f_c(< k >) = < k > \subseteq \; f_c(< j >) = < j > .$$

Since

$$f_c(f_c(< n+1 >)) = f_c(< n+1 >),$$

we then either have that

$$f_c(< n+1 >) = < j > \text{ or } f_c(< n+1 >) = < n+1 > .$$

Suppose that $f_c(< n+1 >) = < n+1 >$. Since f_c is a semiprime operation,

$$f_c(< i >) + f_c(< k >) \subset f_c(< i+k >) \; \forall \, i, k \in \mathbb{N}_0.$$

So,

$$f_c(< 1 >) + f_c(< n >) = f_c(< n+1 >)$$

and since we have $j < n$, then $j+1 < n+1$, so that

$$< n+1 > \nsubseteq < j+1 >$$

and thus

$$< j+1 > \subseteq \; f_c(< 1 >) + f_c(< n >) \subset f_c(< n+1 >) \; = < n+1 > \subset < j+1 > \Rightarrow$$

$$< n+1 > \subset < j+1 > \text{ and } < n+1 > \supseteq < j+1 >,$$

a contradiction. Thus, $f_c(< n+1 >) \neq < n+1 >$, so that $f_c(< n+1 >) = < j >$. $\square$

We will show that M_f is a left M_0-act but not a right M_0-act under composition in Lemma 2.0.8 and Lemma 2.0.9, respectively.

Lemma 2.0.8 *M_f is a left M_0-act under composition.*

Proof:

Applying Definition 2.0.4, we have $\delta\colon M_0 \times M_f \longrightarrow M_f$ such that $\delta(f_m \circ f_n, g_l) = \delta(f_m, \delta(f_n, g_l))$. Note that

$$f_m(f_n(g_l(\infty)) = f_m(f_n(< l >)) = g_{\min\{m,n,l\}}$$

under the definition of an act, so that

$$\delta(f_m \circ f_n, g_l) = \delta(f_{\min\{m,n\}}, g_l) = g_{\min\{m,n,l\}} \text{ and}$$

$$\delta(f_m, \delta(f_n, g_l)) = \delta(f_m, g_{\min\{n,l\}}) = g_{\min\{m,n,l\}}.$$

Thus, $\delta(f_m \circ f_n, g_l) = \delta(f_m, \delta(f_n, g_l))$ and hence M_f is a left M_0-act under composition. $\square$

Lemma 2.0.9 *M_f is not a right M_0-act under composition.*

Proof:

For M_f to be a right M_0-act, we would need to have:

$$\delta\colon M_f \times M_0 \longrightarrow M_f \text{ such that } \delta(g_l, f_m \circ f_n) = \delta(\delta(g_l, f_m), f_n).$$

But, since $f_n(<\infty>) = \infty \neq \min(n, \infty)$, we get:

$$\delta(g_l, f_m \circ f_n) = \delta(g_l, f_{\infty \,\text{or}\, \min\{m,n\}}) \neq g_{\min\{l,m,n\}}.$$

That is, since $f_n(<\infty>) = \infty \neq \min\{n, \infty\}$, we don't necessarily get the $\min\{l, m, n\}$ as the $\min\{m, n\}$ could be "buried" by ∞ as follows:

For $n < m < l$,

$$g_l(f_m(f_n(<\infty>))) = g_l(f_m(<\infty>)) = g_l(<\infty>) = <l> \neq \min\{l, m, n\}. \square$$

For case II, the above holds, and also

$$f_c(<\infty>) \subseteq \bigcap_{i \geq 0} f_c(<i>) = <m> .$$

Thus, $f_c(<\infty>) = <\infty>$ or $f_c(<\infty>) = <m>$ since $f_c(<n>) = <m>$ for $n \geq m$. So, $f_c = f_m$ or $f_c = g_m$ as stated in case II of Proposition 2.0.3. $f_c(<\infty>) \subseteq f_c(<i>) = <i>$ for $i \geq 0$.
Then $f_c(<\infty>) \subseteq \cap_{i \geq 0} f_c(<i>) = <\infty>$ so that f_c must be the identity map (case I), or the sets M_0, M_f in $S_{\mathbb{N}_0 \cup \{\infty\}}$ (case II) are submonoids of $M_{\mathscr{I}}$.

Proof of Proposition 2.0.4 and 2.0.6:

This proof will cover both $\mathbb{N}_0$ and $\mathbb{N}_0 \cup \{\infty\}$.

By Lemma 2.0.7, for any semiprime operation f_c on $\mathbb{N}_0$ or $\mathbb{N}_0 \cup \{\infty\}$ which is constant on some finite interval,

$$f_c(<i>) = <m> \text{ for every } i \geq m, \text{ for some m.}$$

Now, we will show that $f_c(< i >) = < i >$ for $i \leq m$.

Suppose, on the contrary, that $f_c(< i >) = < k >$ for some $k \leq i$ (we must have $k \leq i$ since f_c is increasing). Then for $k \leq j \leq i$,

$$f_c(< i >) = < k > \text{ (by hypothesis)}$$

$$f_c(f_c(< i >)) = f_c(< k >) \text{ (applying } f_c \text{ to both sides)}$$

$$f_c(< i >) = f_c(< k >) \text{ (by closure properties, } f_c \circ f_c = f_c)$$

$$< k > = f_c(< k >)$$

So, $< k > = f_c(< k >) \supseteq f_c(< j >) \supseteq f_c(< i >) = < k >$ If $k < i$, then $k < m$ also, and Lemma 2.0.7 implies that $f_c(< i >) = < k >$ on the interval $i \geq k$.

This contradicts $f_c(< i >) = < m >$ (for $i \geq m$, shown earlier). Thus, $f_c(< i >) = < i >$ for $i \leq m$, after all.

So, now we will have

$$f_m(< i >) = \begin{cases} < i > & \text{if } 0 \leq i < m \\ < m > & \text{if } i \geq m \end{cases}$$

Now, we take a semiprime operation f_c which is not constant for every interval $m \leq i \leq n$, where $m < n$. Suppose that $f_c(< i >) = k$ for $k < i$.

Then $< k > = f_c(< k >) \supseteq f_c(< j >) \supseteq f_c(< i >) = < k >$ for every $k \leq j \leq i$, which contradicts f_c not being constant on any interval.

Therefore, $f_c(< i >) = < i >$ so that f_c must be the identity map.

Since $f_m \circ f_n = f_{\min\{m,n\}}$ and $g_m \circ g_n = g_{\min\{m,n\}}$ we have closure on $S_{\mathbb{N}_0}$, so that the set M_0 of semiprime operations in $S_{\mathbb{N}_0}$

As shown earlier, in case II M_f is a left act but not a right act under composition, and $g_m \circ f_n$ is not a closure operation. So, $S_{\mathbb{N}_0 \cup \{\infty\}} = M_0 \cup M_f$ is not a submonoid under composition.

In case II, $n \geq 0, M_n = \{e\} \cup \{f_n\} \cup \{g_n\}$ are finite submonoids of $M_{\mathscr{I}}$ in $S_{\mathbb{N}_0 \cup \{\infty\}}$ $\forall n \geq 0$, so that M_0 and M_f are related. $\square$

Proposition 2.0.10 *The only element of* $P_{\mathbb{N}_0}$ *or of* $P_{\mathbb{N}_0 \cup \{\infty\}}$ *is the identity element* $\{e\}$*, where* $e : \mathscr{I} \Rightarrow \mathscr{I}$ *such that* $e(< i >) = < i >$.

Proof:

Let f_c be f_m where

$$f_m(< i >) = \begin{cases} < i > \text{ if } i \ < m \\ < m > \text{ if } i \geq m \end{cases}$$

$$< b > \ + f_m(< m >) =< b > \ + < m > \neq < m > = f_m(< b + m >)$$

which contradicts f_m being prime, since then we would have

$$< b > \ + f_m(< m >) \ \neq \ f_m(< b + m >)$$

(instead of $< b > \ + f_m(< m >) = f_m(< b + m >)$, as the definition requires). Thus, the only element of $P_{\mathbb{N}_0}$ or $P_{\mathbb{N}_0 \cup \{\infty\}}$ is $\{e\}$. $\square$

Chapter 3

Sub-semi-groups of $\mathbb{N}_0$

We now consider the sub-semi-group S, where $S = (2,3) = \{2\cdot\ i + 3\cdot\ j \mid\ i, j \in\ \mathbb{N}_0\}$. Here i and j indicate how many copies of 2 and 3 to take in forming each element of S. Multiplying the element 2 by a given $i \in\ \mathbb{N}_0$ is then analogous to raising the number 2 by a power, so that multiplication is a version of exponentiation in the monoids formed by $(\ \mathbb{N}_0, +\ )$ and its sub-semi-groups. S then includes the additive identity, zero, as the result of adding zero copies of the integers 2 and 3. Thus we get $S = \{0, 2, 3, 4, 5, \ldots\}$.

Following Gilmer [4], the sum $A + B$ of 2 nonempty subsets of a sub-semi-group will be defined as $\{a + b \mid a \in A, b \in B\}$

We will use the symbol $<>$ to indicate an ideal, so that $< 2, 3 >$ is the ideal generated by the elements 2 and 3, thus $< 2, 3 >$ is the set obtained by adding 2 to each element of S and then adding 3 to each element of S.

A principal ideal is one which is generated by a single element of S. In this case, a given principal ideal will consist of a single fixed element i of S together with the elements created by summing i with each element of S in turn, or:

$< i >= \{i + s \mid s \in S; i \in \mathbb{N};\ i \geq 2,\ i \text{ fixed }\}$

The non-principal or two-generated ideals are defined as:

$< i, j >= \{n \mid n = i + s \text{ or } n = j + s,\ s \in S; i, j \in \mathbb{N};\ i \geq 2,\ j > i,\ i \text{ fixed }\}$

Now, if $j \geq\ i+2$, the element j will already be included in the set generated by the ideal $< i >$, as we obtain it when i gets to the element $i + (j - i)$ as i is added to each element of S in turn, since S includes all the consecutive numbers after $i + 2$. Thus $< i, j > = < i >$ for $j \geq\ i + 2$.

For example, $< 6, 8 >= \{6, 8, 9, 10, \ldots\}$ and $< 6 >= \{6, 8, 9, 10, \ldots\}$,
so that $< 6, 8 >=< 6 >$.

Hence, the only two-generated non-principal ideals will be of the form

$< i, i + 1 >= \{i, i + 1, i + 2, \ldots\ \}$. For example, $< 6, 7 >= \{6, 7, 8, \ldots\}$

or of the form

$< i >= \{i, i + 2, i + 3, \ldots\}$. For example, $< 8 >= \{8, 10, 11, 12, \ldots\}$

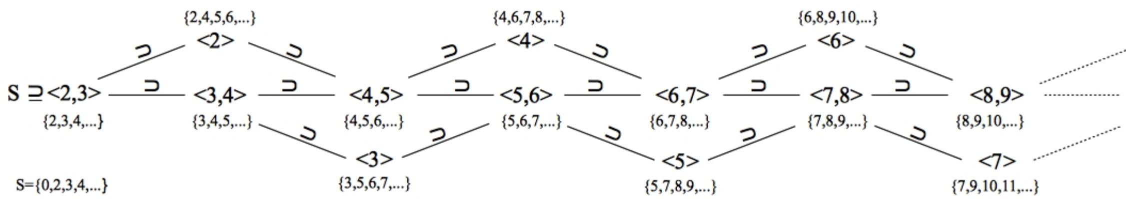
Note that $< i, i + 1 >$ gives the consecutive integers starting with i, whereas $< i >$ gives the consecutive integers with the element $i + 1$ deleted.

We have just shown the following proposition:

Proposition 3.0.11 *Every ideal of* $S = (2, 3)$ *can be expressed either as a principal ideal,* $< i >$*, or as a two-generated ideal of the form* $< i, i + 1 >$ *.*

We may form an infinite chain with these ideals, where each ideal is contained in the ideal to its left, as indicated in figure 3.1.

Figure 3.1:



In the following discussion, we will abbreviate the principal ideal $< i >$ as P_i, and the two-generated ideal $< i, i+1 >$ as M_i. An arbitrary, general ideal of either type will be denoted as I.

We will now see that a map that sends only one principal ideal to the main line will not be a semi-prime operation, and that mapping one ideal to the main line will end up forcing other ideals to be mapped to the main line in order to preserve semi-prime property (d).

For example, property (d) $f_i^u(I)+f_i^u(J) \subseteq\ f_i^u(I+J)$ is contradicted in the following example where f_3^u is a map which sends only the ideal $< 3 >$ to the main line, and maps all other ideals back to themselves:

$f_3^u(< 3 >) + f_3^u(< 4 >) =< 3, 4 > + < 4 >=< 7, 8 > \nsubseteq < 7 >= f_3^u(< 7 >) =$
$f_3^u(< 3 > + < 4 >)$

In general, we may state the following proposition:

Proposition 3.0.12 *For* $S = (2, 3) = \{0, 2, 3, 4, 5, \}$, *the following map defines a closure operation which is not semi-prime:*

$$f_i^u(I) = \begin{cases} M_i \text{ if } I = P_i \text{ or } M_i \\ I \text{ if } I \neq P_i \text{ or } M_i \end{cases}$$

i.e., we will show the following statements, which define a closure operation, are true:

(a) $I \subseteq\ f_i^u(I)$

(b) $I \subseteq J \Rightarrow f_i^u(I) \subseteq f_i^u(J)$

(c) $f_i^u(f_i^u(I)) = f_i^u(I)$

but that the property defining the semi-prime operation is not true:

(d) $f_i^u(I) + f_i^u(J) \subseteq f_i^u(I+J)$

Proof:

(a) $I \subseteq f_i^u(I)$ holds since $f_i^u(P_i) = M_i \supseteq P_i$ and

$f_i^u(M_i) = M_i \supseteq M_i$

(b) $I \subseteq J \Rightarrow f_i^u(I) \subseteq f_i^u(J)$ Take $P_i \subseteq M_i$ then $f_i^u(P_i) = M_i \subseteq M_i$ if $j = i$.

If $i > j + 2$ then $P_i \subseteq P_j$ so that

$f_i^u(I) = M_i \subseteq M_j$.

and if $I = M_i \subseteq P_j = J$, then $f_i^u(I) = M_i \subseteq M_j = f_i^u(J)$, since $P_j \subseteq M_j$.

(c) $f_i^u(f_i^u(I)) = f_i^u(I)$:

(d) $f_i^u(I) + f_i^u(J) \nsubseteq f_i^u(I+J)$:

$f_i(P_i) + f_i(P_j) = M_i + P_j = M_{i+j} \nsubseteq P_{i+j} = f_i(P_{i+j}) = f_i(P_i + P_j)$

This is true $\forall\, i, j \in \mathbb{N}$, thus semiprime property (d) $f_i^u(I) + f_i^u(J) \subseteq f_i^u(I+J)$ does not hold, and the closure operation is not semiprime. $\square$

If we changed this map to one in which all the principal ideals to the right of some particular principal ideal (on figure 3.1) were to map to the main line, we would then have a semiprime operation.

For instance,

$f_3^u(<3>) + f_3^u(<4>) =<3,4> + <4,5>=<7,8>\subseteq <7,8>= f_3^u(<7>) = f_3^u(<3> + <4>)$

In general, we may state:

Corollary 3.3 The following map defines a semiprime closure operation over $S = \{0, 2, 3, ...\}$ for $i \geq 2$:

$$f_n^u(I) = \begin{cases} I \text{ if } i < n \\ M_i \text{ if } I = P_i \text{ or } I = M_i \text{ and } i \geq n \end{cases}$$

Where $I =< i > or < i, i+1 >$, and $M_i =< i, i+1 >$.

This map causes an ideal on figure 3.1 to either stay where it is or bump to the nearest node on the main line. All the principal ideals P_i will bump to the main line whenever $i \geq n$.

Proof: $I \subseteq f_n^u(I)$ since $I \subseteq I$ and $< i > \subseteq < i, i+1 >$, so this map is extensive.

For $I \subseteq J$ we need $f_n^u(I) \subseteq f_n^u(J)$ for this map to be increasing. This clearly holds when the map keeps the ideal at same spot on the chain. Now if both I and J bump to the main line, we have $P_i \subseteq P_j \longmapsto M_i \subseteq M_j$

and if only one ideal bumps to the main line, we have

$P_i \subseteq P_j \longmapsto M_i \subseteq M_j$ or $P_i \subseteq M_j$

so this map is increasing.

If we take $f_n^u(f_n^u(I))$, it will equal $f_n^u(I)$, since once an ideal has bumped to the main line (which is the only case where it moves) it will stay there when we take f_n^u again (that is, $i \geq n$ will still hold once we've moved to M_i), so that the operation is idempotent.

To show that this map is semiprime, we also need $f_n^u(I) + f_n^u(J) \subseteq f_n^u(I + J)$ to be true. As we can see by generalizing the examples given earlier:

If I and J are M_i and M_j, then property (d) holds trivially.

If I and J are P_i and P_j, then if $i, j < n$ but $i + j > n$ we will have
$f_n^u(P_i) + f_n^u(P_j) = P_i + P_j = P_{i+j} \subseteq M_{i+j} = f_n^u(P_{i+j}) = f_n^u(P_i + P_j)$

If I and J are P_i and P_j, and if $i < n$ but $j \geq n$ we will have

$f_n^u(P_i) + f_n^u(P_j) = P_i + M_j = M_{i+j} \subseteq M_{i+j} = f_n^u(P_{i+j}) = f_n^u(P_i + P_j)$

The inclusion holds similarly if one ideal is principal and the other a main line ideal.

Thus the map $f_n^u(I)$ given above is indeed a semiprime operation. $\square$

In the next 4 lemmas, we will look at 4 mappings that will force our closure operation to be a bounded closure operation.

First we show that any semiprime map which moves a main line ideal 2 or more slots to the left on figure 3.1 must be a bounded semiprime map:

Lemma 3.0.13 *If* f^b *is a operation on* S *such that*

$f^b(M_{i+2}) = M_i$

then f^b *is a bounded semiprime operation.*

Proof: By the properties of closure operations, we have

$I \subseteq\ J \Rightarrow\ f^b(I) \subseteq\ f^b(J)$ and $f^b(f^b(I)) = f^b(I)$, thus since

$M_{i+2} \subseteq\ M_{i+1} \subseteq\ M_i$ we must also have

$f^b(M_{i+2}) \subseteq\ f^b(M_{i+1}) \subseteq\ f^b(M_i)$.

But $f^b(M_{i+2}) = M_i$, so that $f^b(M_i) = f^b(f^b(M_{i+2})) = M_i$, and thus

$M_i = f^b(M_{i+2}) \subseteq\ f^b(M_{i+1}) \subseteq\ f^b(M_i) = M_i$,

so that f^b of any of these ideals must be M_i.

In general, we would expect that $f^b(M_{i+m}) = M_i$, and we will show by induction that this is true: Suppose that $f^b(M_{i+k}) = M_i$. We want to show that $f^b(M_{i+k+1}) = M_i$.

Now $P_2 + M_{i+m-1} =< 2 > + < i + m - 1, i + m >$

$$= \{2, 4, 5, ...\} + \{i + m - 1, i + m, i + m + 1, ...\}$$

$$= \{i + m + 1, i + m + 2, i + m + 3, ...\}$$

$$= M_{i+m+1}.$$

So $M_{i+m+1} = P_2 + M_{i+m-1}$.

Thus $f^b(M_{i+m+1}) \supseteq f^b(P_2 + M_{i+m-1}) \supseteq f^b(P_2) + f^b(M_{i+m-1})$ (since we are dealing with a semiprime operation) and then $f^b(P_2) \supseteq P_2$ by definition of a closure operation.

We thus obtain the chain

$f^b(M_{i+m+1}) \supseteq P_2 + f^b(M_{i+m-1}) \supseteq P_2 + f^b(M_{i+k}) = M_{i+2} \supseteq M_{i+m+1}$.

If we apply f^b to the above chain, we get

$f^b(M_{i+m+1}) \supseteq f^b(M_{i+2}) = M_i \supseteq f^b(M_{i+m+1})$,

and since the left and right entries of the last chain are equal,

$f^b(M_{i+m+1}) = M_i$.

Therefore the semiprime operation f^b on S where $f^b(M_{i+2}) = M_i$ is a bounded semiprime operation, as M_i provides a bounds for f^b. □

We now show that if 2 main line ideals that are 2 or more slots apart map to the same ideal, then f^b is a bounded semiprime operation.

Lemma 3.0.14 *If f^b is a semiprime operation on $S = \{0, 2, 3, ...\}$ and $f^b(M_i) = f^b(M_{i+2})$ for some i, then f^b is a bounded semiprime operation.*

Proof We will have 2 cases:

(1) $f^b(M_i) = M_k$ where $k \leq i$, or

(2) $f^b(M_i) = P_k$, where $k \leq i - 2$

Case (1): $M_k = f^b(M_k) \supseteq f^b(M_{k+1}) \supseteq f^b(M_{k+2}) \supseteq f^b(M_{i+2}) = M_k$, and then f^b is bounded by Lemma 3.0.4.

Case (2): $f^b(M_i) = P_k$, $k \leq i-2$. Here it suffices to show that for $I \subseteq P_k$, $f^b(I) = P_k$.

Now $P_k \supseteq I \supseteq M_{i+2}$, thus $f^b(P_k) \supseteq f^b(I) \supseteq f^b(M_{i+2})$ and since $f^b(P_k) = f^b(M_i)$, then $f^b(M_i) = f^b(M_{i+2})$, so $P_k = f^b(M_i) = f^b(M_{i+2}) = f^b(I)$.

We will show by induction that $f^b(M_{i+m}) = P_k$ for $m \geq 2$:

Suppose that $f^b(M_{i+j}) = P_k$ for $2 \leq j \leq m$.

Since $M_{i+m+1} = P_2 + M_{i+m-1}$, we have

$f^b(M_{i+m+1}) = f^b(P_2) + f^b(M_{i+m-1}) \supseteq P_{k+2} \supseteq M_{i+m+1}$ (1)

Now $M_{i+2} \subseteq P_{k+2} \subseteq P_k$, so $f^b(P_{k+2}) = P_k$, so when we take f^b of chain (1), we get that $f^b(M_{i+m+1}) = P_k$.

Thus $f^b(M_{i+n}) = P_k$ for $n \geq 0$.

Also, $M_{k+n} \supseteq P_{k+n} \supseteq M_{k+n+2}$. If we apply f^b to this chain, we get

$f^b(M_{k+n}) \supseteq f^b(P_{k+n}) \supseteq f^b(M_{k+n+2})$,

and since $f^b(M_{k+n}) = P_k$ for every $n \geq 2$, this chain becomes $P_k \supseteq f^b(P_{k+n}) \supseteq P_k$, so that $f^b(P_{k+n}) = P_k$.

Thus for $I \subseteq P_k$, $f^b(I) = P_k$ and f^b is bounded. So the map f^b where $f^b(M_i) = f^b(M_{i+2})$ provides a bounded semiprime operation. □

The following defines another bounded semiprime operation:

Lemma 3.0.15 *If f^b is a semiprime operation on $S = \{0, 2, 3, 4, ...\}$ and $f^b(M_j) = f^b(M_{j+1})$ for some j, then f^b is a bounded semiprime operation.*

Proof

We will have 2 cases:

(1) $f^b(M_i) = M_j$ where $j \leq i$, or

(2) $f^b(M_i) = P_j$, where $j \leq i - 2$

Case (1): If we take $j \geq 2$ and let $f^b(M_j) = S$, then $f^b(M_{2j}) = f^b(M_j + M_j)$,

since $M_j + M_j =< j, j+1 > + < j, j+1 >$

$$= \{j, j+1, j+2, ...\} + \{j, j+1, j+2, ...\}$$

$$= \{j+j, j+j+1, j+j+2, ...\}$$

$$=< 2j, 2j+1 >.$$

$$= 2 < j, j+1 >$$

$$= 2M_j = M_{2j}.$$

So $f^b(M_{2j}) = f^b(2M_j) \supseteq 2f^b(M_j) = 2S = S$.

Now $M_j \supseteq M_{j+2} \supseteq M_{2j}$, so $f^b(M_j) \supseteq f^b(M_{j+2}) \supseteq f(M_{2j})$, and since

$f^b(M_j) = S$ and $f^b(M_{2j}) = S$, this implies that $f^b(M_j) = f^b(M_{j+2})$, and thus f^b is bounded by lemma 3.0.5.

For case (2), $M_{j-1} \supseteq M_j$

so $f^b(M_{j-1}) \supseteq f^b(M_j) = P_{j-2}$

and $f^b(M_{j-1}) \supseteq M_{j-1}$.

Thus $f^b(M_{j-1}) \supseteq P_{j-1} \cup M_{j-1} = M_{j-2} \supseteq M_{j-1}$

so that $f^b(M_{j-1}) = f^b(M_{j-2})$.

Now If $f^b(M_{j-2}) = M_{j-2}$, f^b will be bounded from the result of case (1). If not, then we have

$f^b(M_{j-2}) = f^b(M_{j-1}) = P_{j-4}$.

Now $P_{2j-8} = f^b 2(M_{j-1}) \subseteq f^b(M_{2j-2}) \subseteq f^b(M_{2j-4}) \subseteq f^b(P_{2j-8})$. (5)

After applying f^b to chain (5), we get

$f^b(M_{2j-2}) = f^b(M_{2j-4})$

and f^b is then bounded by Lemma 3.0.14. Since f^b is bounded in each case, f^b is therefore a bounded semiprime operation. □

Lemma 3.0.16 *If f^b is a semiprime operation on $S = \{0, 2, 3, ...\}$ such that $f^b(M_j) = f^b(P_{j-2})$, for $j \geq 4$, then f^b is a bounded semiprime operation.*

Proof By the same reasoning used in Lemma 3.0.15,

$f^b(M_j) = f^b(P_{j-2})$ will imply that

$f^b(M_{j-1}) = f^b(M_{j-2})$, so that f^b is bounded by Lemma 3.0.15. □

We will now define the unbounded semiprime operations over $S = \{0, 2, 3, ...\}$.

Theorem 3.0.17 *If f is an unbounded semiprime operation over $S = \{0, 2, 3, 4, ...\}$ and $I =< i >$ or $< i, i+1 >$, then the function may be either the identity operation or we may have one of the following cases:*

$$f^u_n(I) = \begin{cases} I \text{ if } i < n \\ M_i \text{ if } I = P_i \text{ or } I = M_i \text{ and } i \geq n \end{cases}$$

$$g^u_n(I) = \begin{cases} I \text{ if } i \leq n+1 \text{ and } i \neq n \\ M_i \text{ if } I = P_i \text{ or } I = M_i \text{ and } i \geq n \text{ or } i \geq n+2 \end{cases}$$

That is, all the ideals below a given n map to themselves, and the principal ideals contained in P_n will map to the main line. The f^u_n function will map P_{n+1} to M_{n+1}, whereas the g^u_n function will map P_{n+1} to itself, so that the functions only differ in where they map P_{n+1}.

Proof: Suppose f^u_n is an unbounded semiprime operation over $S = \{0, 2, 3, 4, ...\}$ which is not the identity function. Then $f^u_n(I) \neq I$ for some I. If $I = M_j$ for some

$j \geq 2$, then by Lemma 3.0.14, f_n^u would be bounded, contradicting our assumption that f was unbounded, so I must be a principal ideal.

If $f_n^u(P_k) = f_n^u(P_{k+2})$ for some k, then $f_n^u(P_k) = f_n^u(M_{k+2}) = f_n^u(P_{k+2})$ and f_n^u is bounded by Lemma 3.0.16, and we have a contradiction again, so that this is not allowed, either.

If $f_n^u(P_k) = f_n^u(M_{k-1})$ for some k, then $f_n^u(P_k) = f_n^u(M_k) = f_n^u(M_{k-1})$, which is bounded by Lemma 3.0.15.

Thus $f_n^u(P_k) = M_k$.

Let $W = \{k \mid f_n^u(P_k) = M_k \text{ for some } k \geq 2\}$. Since W is a nonempty subset of $\mathbb{N}$, there exists a smallest $j \geq 2$ in W. Since $P_n = P_j + P_{n-j}$, for every $n \geq j+2$, then

$f_n^u(P_n) \supseteq f_n^u(P_j) + f_n^u(P_{n-j}) \supseteq M_j + P_{n-j} = M_n \supseteq P_n$. (1)

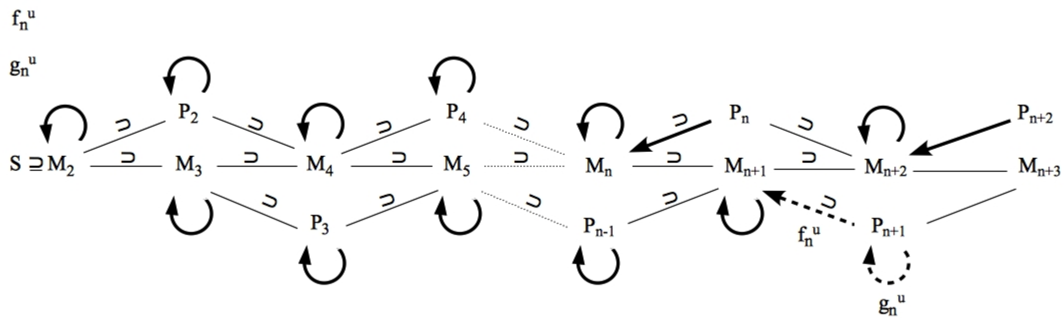
If we take f_n^u of chain (1), we get

$f_n^u(P_n) \supseteq f_n^u(P_j) + f_n^u(P_{n-j}) \supseteq f_n^u(M_j) + f_n^u(P_{n-j}) = f_n^u(M_n) \supseteq f_n^u(P_n)$.

And this implies that $f_n^u(P_n) = M_n$ for every $n = j$ or $n \geq j+2$.

For $n = k+1$ we have two possible mappings and thus obtain the two functions listed above, since P_{n+1} may map to itself (as it does with g_n^u) or P_{n+1} may map to M_{n+1} (as it does with f_n^u). The following diagram illustrates these possibilities:

Figure 3.2:



Thus all unbounded semiprime operations must take one of the forms in the diagram or be the identity function. □

Theorem 3.0.18 *All of the bounded semiprime operations on* $S = \{0, 2, 3, 4, ...\}$ *are of one of the following forms (where* $I =< i >$ *or* $< i, i+1 >$*):*

$$f_m^b(I) = \begin{cases} P_m \text{ if } I \subseteq P_m \\ M_m \text{ if } I = P_{m+1} \text{ or } I = M_{m+1} \\ M_{m-1} \text{ if } I = P_{m-1} \\ I \text{ if } I \supseteq P_m \end{cases}$$

$$f_{n,m}^b(I) = \begin{cases} M_m \text{ if } I \subseteq M_m \\ M_i \text{ if } I \subseteq M_n \\ I \;\text{ if } I \not\supseteq M_n \end{cases}$$

If $n < m - 2$ we will have:

$$g^b_{n,m}(I) = \begin{cases} M_m \text{ if } I \subseteq M_m \\ \\ M_{n+1} \text{ if } I = P_{n+1} \\ \\ M_i \text{ if } I \subseteq M_n \text{ and } I \neq P_{n+1} \\ \\ I \ \text{ if } I \nsubseteq M_n \end{cases}$$

$$\tilde{f}^b_{n,m}(I) = \begin{cases} M_m \text{ if } I \subseteq M_m \\ \\ M_{m-2} \text{ if } \ I = M_{m-1} \text{ or } I = P_{m-1} \text{ or } I = P_{m-2} \\ \\ M_i \text{ if } I \subseteq M_n \text{ and } I \nsubseteq M_{m-2} \\ \\ I \ \text{ if } I \nsubseteq M_n \end{cases}$$

For $n < m - 3$ we will have:

$$\tilde{g}^b_{n,m}(I) = \begin{cases} M_m \text{ if } I \subseteq M_m \\ \\ M_{m-2} \text{ if } \ I = M_{m-1} \text{ or } I = P_{m-1} \text{ or } I = P_{m-2} \\ \\ M_i \text{ if } I \subseteq M_n \text{ and } I \nsubseteq M_{m-2} \\ \\ M_{n+1} \text{ if } I = P_{n+1} \\ \\ M_i \text{ if } I \subseteq M_n \text{ and } I \neq P_{n+1} \\ \\ I \ \text{ if } I \nsubseteq M_n \end{cases}$$

Proof: If f^b is a bounded semiprime operation, we must have one of the following cases:

(1) $f^b_c(I) = P_m$ for $I \subseteq P_m$

(2) $f^b_c(I) = M_m$ for $I \subseteq M_m$

Case (1): In case 1, all ideals contained in P_m are mapped to P_m, and this mapping preserves the closure and semiprime properties for every $I \subseteq P_m$. We need to consider what can happen to the ideals P_{m-1}, M_{m+1}, and P_{m+1}, i.e., what sort of mappings will still preserve the semiprime properties, since they are not comparable to P_m.

If we let the ideals P_{m-1}, M_{m+1}, and P_{m+1} just map back to themselves, then we run into a contradiction with the closure property $(b) I \subseteq J \Rightarrow I_c \subseteq J_c$.

For example, if we have $M_{m+2} \subseteq M_{m+1}$ then

$f^b_c(M_{m+2}) = P_m \nsubseteq M_{m+1} = f^b_c(M_{m+1})$, violating property (b).

The same problem occurs if we try to map P_{m+1} and P_{m-1} to themselves, so we need P_{m-1}, M_{m+1}, and P_{m+1} to map to some ideal I such that $P_m \subseteq I$.

If $f_c^b(M_{m+1}) = M_m$, then for $M_{m+2} \subseteq M_{m+1}$ we will get

$f_c^b(M_{m+2}) = P_m \subseteq M_m = f_c^b(M_{m+1})$,

which preserves property (b).

Then $P_{m+1} \subseteq M_{m+1}$ must imply $f_c^b(P_{m+1}) \subseteq f_c^b(M_{m+1})$, so that $f_c^b(P_{m+1}) = M_{m+1}$ would work, but then we would have

$f_c^b(f_c^b(P_{m+1}) = f_c^b(M_{m+1}) = M_m \neq M_{m+1} = f_c^b(P_{m+1})$,

contradicting closure property (c) $(I_c)_c = I_c$.

Thus $f_c^b(P_{m+1})$ must be M_m, i.e., mapping M_{m+1} to M_m also forces P_{m+1} to go to M_m.

P_{m-1} is similarly forced to map to M_{m-1}, and then for $M_{m+1} \subseteq P_{m-1}$,

$f_c^b(M_{m+1}) = M_m \subseteq M_{m-1} = f_c^b(P_{m-1})$, so property (b) still holds.

This we must have the ideals incomparable to P_m be mapped as follows:

$f_c^b(P_{m-1}) = M_{m-1}$, $f_c^b(M_{m+1}) = M_m$, $f_c^b(P_{m+1}) = M_m$.

We then need to see if this mapping also forces any of the ideals lower than P_{m-1}, M_{m+1}, and P_{m+1} to map to a new lower ideal, or if the lower ideals have to map back to themselves.

We already showed that P_{m-1} must map to M_{m-1}, so we need to see if any other P_{m-a} also can or need to map to the main line. Suppose $f_c^b(P_{m-a}) = M_{m-a}$, $a < 1$. Then we would have

$f_c^b(P_{m-5}) + f_c^b(P_{m-2}) = M_m \nsubseteq P_m = f_c^b(P_{m-5} + P_{m-2})$,

contradicting the semiprime property. Thus $f_c^b(P_{m-a}) = P_{m-a}$ after all.

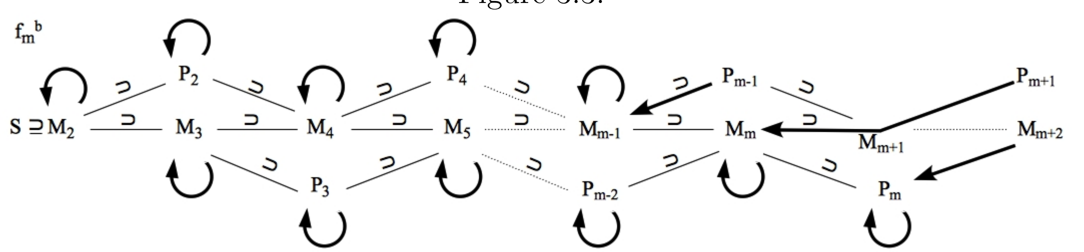
Suppose, for $a > 1$, we allow $f_c^b(M_{m-a}) = M_{m-a-1}$. Then we will run into the same contradiction with the semiprime property. For example,

$f_c^b(M_{m-4}) + f_c^b(M_{m-3}) = M_{m-5} + M_{m-4} = M_{m-2} \nsubseteq f_c^b(M_{m-4} + M_{m-3}) = f_c^b(M_m) = M_m$.

So M_{m-a} must map to itself in order to preserve the semiprime property.

The above results may be summarized in the following diagram:

Figure 3.3:



Case (2) We will now investigate the case where f_c^b is a bounded semiprime operation such that $f_c^b(I) = M_m$ for $I \subseteq M_m$. In general, we will want to look at any ideals incomparable to M_m and decide where they need to go to preserve the closure properties, and then determine the valid mappings for all ideals below those incomparable to M_m.

In this case all ideals contained in M_m are mapped to M_m, and this mapping preserves the closure and semiprime properties for every $I \subseteq M_m$. Only P_{m-1} is not comparable to M_m, so we need to examine what mapping of P_{m-1} will preserve the semiprime properties.

If $f_c^b(P_{m-1}) = P_{m-1}$, then we would contradict closure property (b) $I \subseteq J$ implies $I_c \subseteq J_c$, for we could then have $M_{m+1} \subseteq P_{m-1}$ but then the mapping would give that $f_c^b(M_{m+1}) = M_m \nsubseteq P_{m+1} = f_c^b(P_{m-1})$ since M_m and P_{m-1} are not comparable.

This may be solved by putting $f_c^b(P_{m-1}) = M_{m-1}$ or $M_{m-a}, a \geq 1$. We will see which cases are valid in the following discussion.

First we will examine where to map M_2, and show that $f_c^b(M_2) = M_2$. If $f_c^b(M_2) \neq M_2$, then it must equal S itself to preserve property (a) $I \subseteq I_c$.

Suppose $f_c^b(M_2) = S$. Then
$f_c^b(M_{m-2}) = f_c^b(M_2) + f_c^b(M_{m-2}) \subseteq f_c^b(M_2 + M_{m-2}) = f_c^b(M_m) = M_m$,
and if we apply f_c^b to this chain, we get that $f_c^b(M_{m-2}) \subseteq f_c^b(M_m)$. Combining this with the fact that $M_m \subseteq M_{m-2}$ implies that $f_c^b(M_m) \subseteq f_c^b(M_{m-2})$ we see that the following equality is true: $M_m = f_c^b(M_m) = f_c^b(M_{m-2})$. But since $M_{m-2} \nsubseteq M_m$, we have a contradiction. Thus, $f_c^b(M_2) = M_2$ after all.

Next, we look at the ideals on the main line between M_{m-2} and the lowest main line ideal, M_2. We will show that $f_c^b(M_n) = M_n$ for $2 < n \leq m-2$. Suppose M_n is mapped down by at least one slot, i.e. $M_n \subsetneq f_c^b(M_n) = I$.

Now, $M_m = M_n + M_{m-n}, P_{n-2} + M_{m-n} = M_{m-2}, M_{n-1} + M_{m-n} = M_{m-1}$
and $M_{m-1} \subseteq M_{m-2}$, so that
$M_{m-1} \subseteq f_c^b(M_n) + f_c^b(M_{m-n}) \subseteq f_c^b(M_n + M_{m-n}) = f_c^b(M_m) = M_m \subseteq M_{m-1}$. Thus we get once again that $M_m = f_c^b(M_m) = f_c^b(M_{m-1})$, a contradiction. Hence $f_c^b(M_n) = M_n$ for $2 \leq n \leq m-2$.

Now we examine where the $P_k, 2 \leq k \leq m-2$ may map. $P_k \subseteq M_k$ implies that $f_c^b(P_k) \subseteq f_c^b(M_k) = M_k$, and since $P_k \subseteq M_k$ and $M_k \subseteq M_k$, assigning $f_c^b(P_k)$ to either M_k or to P_k will satisfy the semiprime properties. Although both of the above assignments above are allowed for $P_k, 2 \leq k \leq m-2$, once a given P_k has been mapped to the main line this will force the P_j for $k < j \leq m-2$ to also map to the main line to avoid contradicting semiprime property (d). For instance, if $f_c^b(P_{m-4}) = M_{m-4}$ and $f_c^b(P_{m-2}) = P_{m-2}$, then we could get that
$f_c^b(P_{m-4}) + f_c^b(P_{m-6}) = M_{m-4} + M_{m-6} = M_{m-2} \nsubseteq P_{m-2} = f_c^b(P_{m-2})$, a contradiction.

We now need to examine the mapping possibilities for P_{m-1}, P_{m-2} and M_{m-2}. If $f_c^b(P_{m-1}) = M_{m-1}$, then the above reasoning implies that $f_c^b(P_{m-2}) = P_{m-2}$ or $f_c^b(P_{m-2}) = M_{m-2}$, while $f_c^b(M_{m-1}) = M_{m-1}$ to preserve property (c) $(I_c)_c = I_c$.

For $f_c^b(P_{m-1}) = M_{m-2}$, then if $f_c^b(P_{m-3}) = P_{m-3}$ we would have $P_{m-1} \subseteq P_{m-3}$ but that $f_c^b(P_{m-1}) = M_{m-2} \nsubseteq P_{m-3} = f_c^b(P_{m-3})$.

However, $f_c^b(P_{m-1}) = M_{m-2} \subseteq M_{m-3} = f_c^b(P_{m-3})$ is valid, so $f_c^b(P_{m-3}) = M_{m-3}$ in this case.

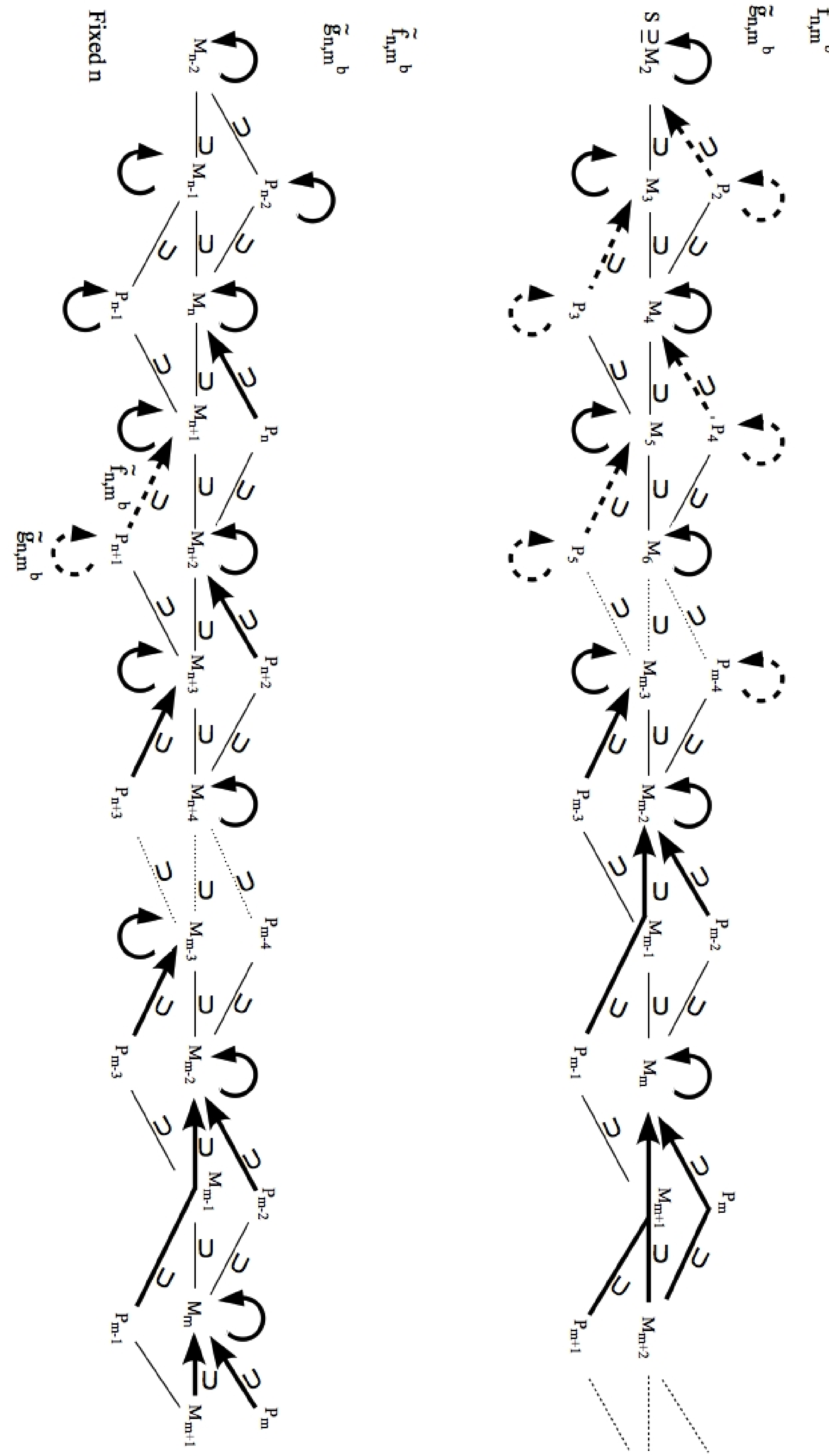
If $f_c^b(P_{m-1}) = M_{m-2}$ then the semiprime properties are satisfied when $f_c^b(P_{m-2})$ is equal to either M_{m-2} or P_{m-2}, although, as mentioned before, if some $P_k, 2 \leq k \leq m-2$ has mapped to the main line, then $f_c^b(P_{m-2})$ must also map to the main line.

Now $f_c^b(P_{m-1})$ may only be M_{m-1} orM_{m-2}, since if $f_c^b(P_{m-1}) = P_{m-3}$ we could have that $P_{m-1} \subseteq M_{m-2}$, whereas $f_c^b(P_{m-1}) = P_{m-3} \nsubseteq M_{m-2} = f_c^b(M_{m-2})$.

The above cases are summarized in the following diagrams, in which the thick dotted lines indicate the cases where the P_k may be assigned to themselves or to M_k (within the conditions mentioned above). The "fixed n" diagrams demonstrate how once a given n is chosen, then all the principal ideals contained in P_n and below the bounds must map to the main line, and that then all ideals to the left of P_n on the lattice will map to themselves.

Figure 3.4:

Figure 3.5:



We now let $M_o = \{e\} \cup \{f_n^u,\ g_n^u,\ f_m^b,\ g_m^b,\ f_{n,m}^b,\ g_{n,m}^b,\ \tilde{f}_{n,m}^b,\ \tilde{g}_{n,m}^b\}$, and consider the possible compositions that may be formed with functions from this set.

The following compositions show that the subset $M_1 = \{e\} \cup \{f_n^u,\ g_n^u,\ f_{n,m}^b,\ g_{n,m}^b\}$ will be a monoid under composition.

$$f_n^u \circ f_p^u = f_p^u \circ f_n^u = \begin{cases} f_n^u = f_p^u \text{ if } n = p \\ \\ f_n^u \text{ if } p > n \\ \\ f_p^u \ \text{ if } n > p \end{cases}$$

$$g_n^u \circ g_p^u = g_p^u \circ g_n^u = \begin{cases} f_p^u \text{ if } n = p + 1 \\ \\ g_p^u \text{ if } n \geq p + 2 \\ \\ f_n^u \ \text{ if } p = n + 1 \\ \\ g_n^u \text{ if } p \geq p + 2 \end{cases}$$

$$f_{n,m}^b \circ f_{p,q}^b = f_{p,q}^b \circ f_{n,m}^b = \begin{cases} f_{p,q}^b \ \ p \leq n \ \ , q \leq m \\ \\ f_{p,m}^b \ \ p \leq n \ \ , q \geq m \\ \\ f_{n,q}^b \ \ p \geq n \ \ , q \leq m \\ \\ f_{n,m}^b \ \ p \geq n \ \ , q \geq m \end{cases}$$

$$g^b_{n,m} \circ g^b_{p,q} = g^b_{p,q} \circ g^b_{n,m} = \begin{cases} f^b_{n,m} \;\; n = p-1 \;\; , m \leq q \\ \\ f^b_{p,m} \;\; n = p-1 \;\; , m \leq q \\ \\ f^b_{n,q} \;\; n = p-1 \;\; , m \geq q \\ \\ f^b_{p,q} \;\; n = p-1 \;\; , m \geq q \\ \\ g^b_{n,m} \;\; n \leq p-2 \;\; , m \leq q \\ \\ g^b_{p,m} \;\; n \leq p-2 \;\; , m \leq q \\ \\ g^b_{n,q} \;\; n \leq p-2 \;\; , m \geq q \\ \\ g^b_{p,q} \;\; n \leq p-2 \;\; , m \geq q \end{cases}$$

$$f^u_n \circ g^u_p = g^u_p \circ f^u_n = \begin{cases} f^u_n (= f^u_p) \text{ if } n = p \\ \\ f^u_n \;\; n \leq p+1 \\ \\ g^u_p \;\; n \geq p+2 \end{cases}$$

$$f^u_n \circ f^b_{p,m} = f^b_{p,m} \circ f^u_n = \begin{cases} f^b_{n,m} \;\; n \leq p \\ \\ f^b_{p,m} \;\; n \geq p \end{cases}$$

$$f_n^u \circ g_{p,m}^b = g_{p,m}^b \circ f_n^u = \begin{cases} f_{p.m}^b \; n = p \text{ or } n = p+1 \\ \\ f_{n,m}^b \;\; n \leq p-1 \\ \\ g_{p,m}^b \;\; n \geq p+2 \end{cases}$$

$$g_n^u \circ f_{p,m}^b = f_{p,m}^b \circ g_n^u = \begin{cases} f_{n,m}^b \;\; n \leq p \\ \\ f_{p,m}^b \;\; n \geq p \end{cases}$$

$$g_n^u \circ g_{p,m}^b = g_{p,m}^b \circ g_n^u = \begin{cases} f_{p,m}^b \text{ if } n = p+1 \\ \\ f_{n,m}^b \text{ if } n = p-1 \\ \\ g_{p,m}^b \text{ if } n = p \text{ or } n \geq p+2 \\ \\ g_{n,m}^b \text{ if } n = p \text{ or } n \leq p-2 \end{cases}$$

$$f^b_{n,m} \circ g^b_{p,q} = g^b_{p,q} \circ f^b_{n,m} = \begin{cases} f^b_{n,q} \;\; n \leq p \;\; , m \geq q \\ \\ f^b_{n,m} \;\; n \leq p \;\; , m \leq q \\ \\ f^b_{p,q} \;\; n = p+1 \;\; , m \geq q \\ \\ f^b_{p,m} \;\; n = p+1 \;\; , m \leq q \\ \\ g^b_{p,q} \;\; n \geq p+2 \;\; , m \geq q \\ \\ g^b_{p,m} \;\; n \geq p+2 \;\; , m \leq q \end{cases}$$

The following array shows that f^b_m can be composed with itself:

$$f^b_m \circ f^b_q = f^b_q \circ f^b_m = \begin{cases} f^b_m = f^b_q \text{ if } m = q \\ \\ f^b_{n,q} \text{ (where } n = q-1) \text{ if } m = q+1 \\ \\ f^b_q \;\; \text{if } m \geq q+2 \\ \\ f^b_{n,m} \text{ (where } n = m-1) \text{ if } q = m+1 \\ \\ f^b_m \;\; \text{if } q \geq m+2 \end{cases}$$

However, the function f^b_m could not be included in the monoid we defined, since it is possible to take $f^b_m \circ f^u_n$ of an ideal and get a function which is not defined as one our closure operations. The following case provides our counterexample, since the output function is not a valid closure operation:

Take $n = 7 = m$. Then

$$f_7^b \circ f_7^u(< i >) = \begin{cases} < i > & i \leq 5 \\ \\ < 6, 7 > & i = 6 \\ \\ < 7, 8 > & i = 7 \text{ or } 8 \\ \\ < 7 > & i \geq 9 \end{cases}$$

$$f_7^b \circ f_7^u(< i, i+1 >) = \begin{cases} < i > & i \leq 7 \\ \\ < 7, 8 > & i = 8 \\ \\ < 7 > & i \geq 9 \end{cases}$$

But if we form a set whose elements are

$\{f_m^b, f_{n,m}^b, g_{n,m}^b\} = S_0$

We may form a left act of $M_1 = \{e\} \cup \{f_n^u, g_n^u, f_{n,m}^b, g_{n,m}^b\}$ on S_0 if we can show that the composition of any function in M_1 with a function from S_0 yields another function in the set S_0, and the operation is then closed over the set S_0.

We have already shown in the work defining the monoid that the composition of any function from M_1 with $f_{n,m}^b$ or $g_{n,m}^b$ yields a function from S_0, and the following results show that any function from M_1 combined with f_m^b yields another function in S_0.

$$f_n^u \circ f_m^b = \begin{cases} f_{n,m}^b \text{ if } n \leq m \\ f_m^b \text{ if } n > m \end{cases}$$

$$g_n^u \circ f_m^b = \begin{cases} g_{n,m}^b \text{ if } n \leq m-3 \\ f_{n,m}^b \text{ if } n = m-2 \\ f_{n-1,m}^b \text{ if } n = m \\ f_m^b \text{ if } n = m-1 \text{ or } n > m \end{cases}$$

$$f_{p,q}^b \circ f_m^b = \begin{cases} f_{p,m}^b \text{ if } p \leq m-1 \\ f_{p-1,p}^b \text{ if } p = m \\ f_m^b \text{ if } p \geq m+1 \end{cases}$$

$$g_{p,q}^b \circ f_m^b = \begin{cases} g_{p,m}^b \ p \leq m-3 \\ f_{p,m}^b \ p = m-2 \\ f_{p-1,p}^b \ p = m \\ f_m^b \ p = m-1 \text{ or } p \geq m+1 \end{cases}$$

The function $\tilde{f}^b_{n,m}$ could also not been in the monoid, since we may take the composition of $\tilde{f}^b_{n,m}$ with itself and get a non-valid function, as shown below:

$$\tilde{f}^b_{5,10} \circ \tilde{f}^u_{5,9}(< i >) = \begin{cases} < i > & i \leq 4 \\ < 5,6 > & i = 5 \\ < 6,7 > & i = 6 \\ < 7,8 > & i = 7 \\ < 7,8 > & i = 8 \\ < 8,9 > & i \geq 9 \end{cases}$$

We also similarly get a non-valid function when composing $\tilde{g}^b_{n,m}$ with itself.

However, if we form a set of the functions $\{\tilde{f}^b_{n,m}, \tilde{g}^b_{n,m}\} = S_1$, and take the submonoid $\{e\} \cup \{f^u_n, g^u_n\} = M_2$, we will get a left act of M_2 on S_1, as the following compositions show:

$$f^u_n \circ \tilde{f}^b_{p,m} = \tilde{f}^b_{p,m} \circ f^u_n = \begin{cases} \tilde{f}^b_{n,m} & n \leq p \\ \tilde{f}^b_{p,m} & n \geq p \end{cases}$$

$$g^u_n \circ \tilde{f}^b_{p,m} = \tilde{f}^b_{p,m} \circ g^u_n = \begin{cases} \tilde{f}^b_{n,m} & n \leq p \\ \tilde{f}^b_{p,m} & n \geq p \end{cases}$$

$$f_n^u \circ \tilde{g}_{p,m}^b = \tilde{g}_{p,m}^b \circ f_n^u = \begin{cases} \tilde{f}_{p.m}^b \; n = p \; , n = p+1 \\ \\ \tilde{f}_{n,m}^b \;\; n \leq p-1 \\ \\ \tilde{g}_{p,m}^b \;\; n \geq p+2 \end{cases}$$

$$g_n^u \circ \tilde{g}_{p,m}^b = \tilde{g}_{p,m}^b \circ g_n^u = \begin{cases} \tilde{g}_{p.m}^b \;\; n = p \\ \\ \tilde{f}_{p,m}^b \;\; n = p+1 \\ \\ \tilde{f}_{n,m}^b \;\; n = p-1 \\ \\ \tilde{g}_{n,m}^b \;\; n \leq p-3 \\ \\ \tilde{g}_{p,m}^b \;\; n \geq p+2 \end{cases}$$

We were not able to have a left act of the monoid $M_1 = \{e\} \cup \{f_n^u, g_n^u, f_{n,m}^b, g_{n,m}^b\}$ on $S_1 = \{\tilde{f}_{n,m}^b, \tilde{g}_{n,m}^b\}$ since when we compose $f_{n,m}^b$ with $\tilde{g}_{p,q}^b$ we get a function not in the set S_1:

$$
f^b_{n,m} \circ \tilde{g}^b_{p,q} = \tilde{g}^b_{p,q} \circ f^b_{n,m} = \begin{cases}
\tilde{g}^b_{p,m} & \begin{array}{l} m = q \\ m = q-1 \\ m \geq q+3 \end{array} \quad n \geq p+2 \\[2ex]
\tilde{f}^b_{p,m} & \begin{array}{l} m = q \\ m = q-1 \\ m \geq q+3 \end{array} \quad n = p+1 \\[2ex]
\tilde{f}^b_{n,m} & \begin{array}{l} m = q \\ m = q-1 \\ m \geq q+3 \end{array} \quad p \leq 1 \\[2ex]
\tilde{g}^b_{p,q} & \begin{array}{l} m = q+1 \\ or \\ m = q+2 \end{array} \quad n \geq p+2 \\[2ex]
\tilde{f}^b_{p,q} & \begin{array}{l} m = q+1 \\ or \\ m = q+2 \end{array} \quad n = p+1 \\[2ex]
\tilde{f}^b_{n,q} & \begin{array}{ll} m = q+1 & n = p \ \ or \ \ n = p-1 \\ m = p+2 & n = p \ \ or \ \ n = p-1 \end{array} \\[2ex]
\tilde{g}^b_{n,q} & \begin{array}{l} m = q+1 \\ m = q+2 \end{array} \quad n \leq p-2 \\[2ex]
g^b_{p,m} & \begin{array}{l} m = q-2 \\ m = q-3 \\ m = q-4 \end{array} \quad n \geq p+2 \\[2ex]
f^b_{p,m} & n \leq q-2 \quad n = p+1 \\[2ex]
f^b_{n,m} & \begin{array}{l} m = q-2 \\ m = q-3 \end{array} \quad n \leq p \\[2ex]
f^b_{n,m} & m \leq q-4 \quad \begin{array}{l} n = p \\ n = p-1 \end{array} \\[2ex]
g^b_{n,m} & n \leq q-4 \quad n \leq p-2
\end{cases}
$$

Since we may expect similar results upon taking $f^b_{n,m} \circ \tilde{f}^b_{p,q}$, $\tilde{g}^b_{n,m} \circ f^b_{p,q}$ and $g^b_{n,m} \circ \tilde{g}^b_{p,q}$, we would expect to also see a left act of $M_1 = \{e\} \cup \{f^u_n, g^u_n, f^b_{n,m}, g^b_{n,m}\}$ on $S_2 = \{f^b_{n,m}, g^b_{n,m}, \tilde{f}^b_{n,m}, \tilde{g}^b_{n,m}\}$, although these last 3 compositions have not yet been tested.

$f^b_m \circ \tilde{f}^b_{p,q}$ and $f^b_m \circ \tilde{g}^b_{p,q}$ both yield functions which are not valid. Although $\tilde{f}^b_{p,q} \circ f^b_m$ and $\tilde{g}^b_{p,q} \circ f^b_m$ do yield valid functions, we recall that $\tilde{f}^b_{p,q}$ and $\tilde{g}^b_{p,q}$ could not be composed with themselves and thus we cannot form a monoid with $\tilde{f}^b_{p,q}$ and $\tilde{g}^b_{p,q}$, so constructing a left act in this case is not possible.

Chapter 4

Monoids With Multiple Maximal Ideals

Consider the monoid $\mathbb{N}_0 \times \mathbb{N}_0 \cup \{\infty\}$ with maximal ideals given by $< (1,0) >$ and $< (0,1) >$. The principal ideals of $\mathbb{N}_0 \times \mathbb{N}_0$ correspond to a lattice point (i,j) with $i, j \in \mathbb{N}_0$, of $\mathbb{N}_0 \times \mathbb{N}_0$, and are of the form $< (i,j) >$ which we will denote more simply as $< (i,j) >$. Let the semiprime operation f_c on the principal ideals in $\mathbb{N}_0 \times \mathbb{N}_0 \cup \{\infty\}$ be:

$$f_c(< (i,0) >) = \begin{cases} < (i,0) > & \text{for } i < m \\ < (m,0) > & \text{for } i \geq m \end{cases}$$

$$f_c(< (0,j) >) = \begin{cases} < (0,j) > & \text{for } j < n \\ < (0,n) > & \text{for } j \geq n \end{cases},$$

$$\text{and } f_c(< \infty >) = < (m,n) >$$

In this section we will define the operation $\cap$ on the principal ideals to be such that

$$f_c(< (i,0) >) \cap f_c(< (0,j) >) \subseteq f_c(< (i,j) >).$$

So, for example,

$$f_c(<(4,0)>) \cap f_c(<(0,5)>) \subseteq f_c(<(4,5)>).$$

So, by a bumping up (see Remark 4.0.19 below) process we obtain:

$$f_c(<(i,j)>) = \begin{cases} <(i,n)> & \text{for } i < m, j \geq n \\ <(i,j)> & \text{for } i < m, j < n \\ <(m,j)> & \text{for } i \geq m, j < n \\ <(m,n)> & \text{for } i \geq m, j \geq n \end{cases}$$

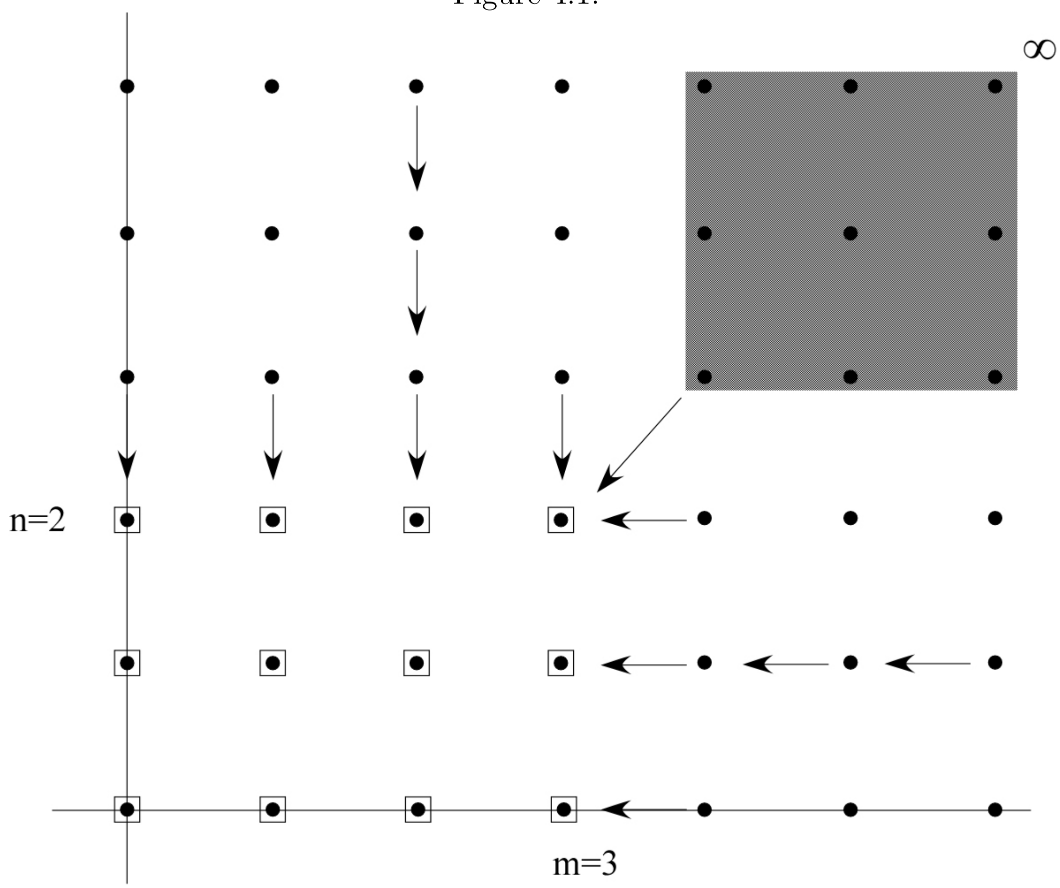
Remark 4.0.19 *Thus, we have bounds for our ideals. If an ideal is inside the bounds, it stays where it is. If it is outside of the bounds, it gets bumped up to the bound, and if the ideal is* $<\infty>$ *then it will be bumped up to the lattice point at the corner of the bounds to the ideal* $<(m,n)>$ *corresponding to the lattice point* (m,n) *in* $\mathbb{N}_0 \times \mathbb{N}_0$.

For example, if $m = 3$ and $n = 2$, in $\mathbb{N}_0 \times \mathbb{N}_0 \cup \{\infty\}$ we would have the situation shown in Figure 4.1.

Here the boxed lattice points represent the identity box, which is the set of all points (i,j) such that $f_c(<(i,j)>) = <(i,j)>$. Each column of lattice points above $n = 2$ bumps up (we say "up" since the resulting set is larger) to the first boxed point below it, as shown by the arrows. Similarly, the lattice points to the right of $m = 3$ bump up to the boxed lattice point directly to its left. Then all of the remaining points (those in the shaded area) bump to the corner lattice point $(3,2)$.

In the case of $\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \cup \{\infty\}$, with maximal ideals $<(1,0,0)>, <(0,1,0)>$,

Figure 4.1:

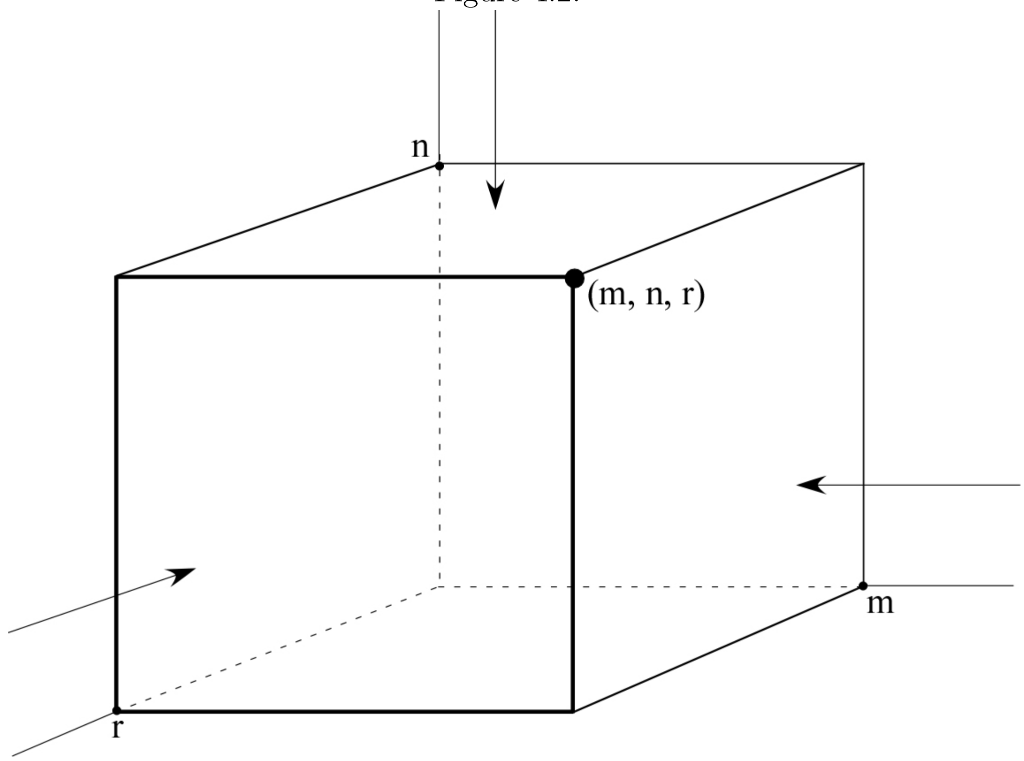


$<(0,0,1)>$ we obtain

$$f_c(<(i,j,k)>) = \begin{cases} <(m,n,r)> \\ <(i,n,r)> \\ <(m,j,r)> \\ <(m,n,k)> \\ <(i,n,k)> \\ <(m,j,k)> \\ <(i,j,r)> \\ <(i,j,k)> \end{cases}$$

by using similar bumping rules to those used in the $\mathbb{N}_0 \times \mathbb{N}_0 \cup \{\infty\}$ case. The corresponding diagram for this case is given in Figure 4.2.

Figure 4.2:



The lattice points outside the box are mapped to the nearest face of the box (as shown by the arrows) and the remaining lattice points (including $< \infty >$) are mapped to the corner of the box (the point (m, n, r) in Figure 4.2).

4.1 Generalizing to Higher Dimensions

The coproduct of $\mathbb{N}_0$ will be the monoid defined by

$$\coprod_{\lambda \in \Lambda} \mathbb{N}_0 = \{\phi \colon \Lambda \longrightarrow \mathbb{N}_0 \mid \phi(\lambda) = 0 \text{ for all but finitely many } \lambda \in \Lambda\},$$

i.e. the coproduct is the set of all functions mapping all but finitely many elements to zero.

Suppose $\phi(\lambda_j) = m_j$ for $\lambda_1, \lambda_2, \ldots, \lambda_s$ and $\phi(\lambda_j) = 0$ for all other λ_j. This would correspond to the ideal $< \phi^{m_1\, m_2\, \ldots\, m_r}_{\lambda_1\, \lambda_2\, \ldots\, \lambda_r} >$. The function $\phi \equiv 0$ in $\coprod_{\lambda\in\Lambda} \mathbb{N}_0$ corresponds to the ideal $< \phi^0 > = < 0 > = \mathbb{N}_0$.

We will use certain subsets of the monoid $\coprod_{\lambda\in\Lambda} \mathbb{N}_0$ to determine the semiprime operations of a monoid with maximal ideals $< \phi^{m_1\, m_2\, \ldots\, m_r}_{\lambda_1\, \lambda_2\, \ldots\, \lambda_r} >$, $\lambda \in \Lambda$ in a manner similar to that used to identify semiprime operations in $\mathbb{N}_0$. All of the non-zero ideals in a monoid are of the form $< \phi^{m_1\, m_2\, \ldots\, m_r}_{\lambda_1\, \lambda_2\, \ldots\, \lambda_r} >$. Note: we are still only doing operations on the principal ideals in the coproduct.

Definition 4.1.1 *A generalization of the identity box B into higher dimensions is the identity λ-box B_λ of a semiprime operation f_c on some monoid.*

For ease of notation, we are denoting an element $\phi^j_\lambda \in \coprod_{\lambda\in\Lambda} \mathbb{N}_0$ where $\phi(\lambda) = j$ unless $\lambda \notin \Lambda$, then $\phi(\lambda) = 0$. The elements of $\coprod_{\lambda\in\Lambda} \mathbb{N}_0$ are of the form $\phi^{j_1\, j_2\, \ldots\, j_r}_{\lambda_1 \lambda_2\, \ldots\, \lambda_r} =: \phi^{j_1 +\, j_2\, +\cdots+\, j_r}_{\lambda_1 +\, \lambda_2\, +\cdots+\, \lambda_r}$ if the λ's are distinct. This corresponds to the set of all $\phi^{m_1\, m_2\, \ldots\, m_r}_{\lambda_1\, \lambda_2\, \ldots\, \lambda_r}$ in $\coprod_{\lambda\in\Lambda} \mathbb{N}_0$, where $f_c(< \phi^{m_1\, m_2\, \ldots\, m_r}_{\lambda_1\, \lambda_2\, \ldots\, \lambda_r} >) =< \phi^{m_1\, m_2\, \ldots\, m_r}_{\lambda_1\, \lambda_2\, \ldots\, \lambda_r} >$.

Example 4.1.2 *Consider the following example where Λ is $\mathbb{N}_0$, so that our map will assign values to finitely many of the slots in the element $(0, 0, 0, ...)$. If the identity box is*

$$\phi^{7\ 11\ 3}_{2\ 5\ 10}(\lambda) = \begin{cases} 7 & \textit{for } \lambda = 2 \\ 11 & \textit{for } \lambda = 5 \\ 3 & \textit{for } \lambda = 10 \\ 0 & \textit{for } \lambda \neq 2, 5, 10 \end{cases}$$

Then we get (0,7,0,0,11,0,0,0,0,3...) as the lattice point element. To get 7 for $\lambda = 2$ means 7 goes in the 2nd slot, then 11 goes in the 5th slot, etc.

Thus $f_{\phi^{7\ 11\ 3}_{2\ 5\ 10}}(<\phi^{8\ 6\ 1\ 8}_{2\ 5\ 10\ 21}>) = <\phi^{7\ 6\ 1}_{2\ 5\ 10}> = (0,7,0,0,6,0,0,0,0,1,0,0,\dots)$ *after applying the closure operation.*

That is, the closure operation takes $<\phi^{8\ 6\ 1\ 8}_{2\ 5\ 10\ 21}>$ *to* $<\phi^{7\ 6\ 1}_{2\ 5\ 10}>$*. The 21st slot is assigned to 0, since the identity box assigns no value to the 21st slot. Note that the 5th slot stays at 6 since 6 < 11, and the 10th slot stays at 1 since 1 < 3.*

The λ-box B_λ is bounded if for every $\lambda \in \Lambda$ there exists a finite m such that

$$f_c(<\phi^{j_1\ j_2\ \dots\ j_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}>) = <\phi^{m_1\ m_2\ \dots\ m_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}> \text{ for } j \geq m.$$

For every $\lambda_h \in \Lambda$ define

$$m_h = \begin{cases} m \text{ if } f_c(<\phi^{j}_{\lambda_h}>) = <\phi^{m}_{\lambda_h}> \\ \infty \text{ otherwise} \end{cases}$$

In fact, all semiprime operations on the ideals of $\coprod_{\lambda\in\Lambda}\mathbb{N}_0$ satisfy

$$f_{B_\Lambda}(<\phi^{j_1\ j_2\ \dots\ j_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}>) = \begin{cases} <\phi^{j_1\ j_2\ \dots\ j_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}> \quad \text{if } <\phi^{j_1\ j_2\ \dots\ j_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}> \in B_\Lambda \\ <\phi^{k_1\ k_2\ \dots\ k_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}> \quad \text{if } <\phi^{j_1\ j_2\ \dots\ j_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}> \notin B_\Lambda \text{ and} \\ \qquad k_l = m_l \neq \infty \text{ for some } l \text{ and } k_h = j_h \\ \qquad \text{for all } h \text{ with } k_h \leq m_h. \end{cases}$$

If we have two identity Λ-boxes C_Λ and D_Λ, then $C_\Lambda \cap D_\Lambda$ will also be an identity Λ-box. The effect of applying the composition $f_{C_\Lambda} \circ f_{D_\Lambda}$ to any given ideal (except $<\infty>$) is the same as that of the mapping $f_{C_\Lambda} \cap f_{D_\Lambda}$ since in each case the resulting ideal will be the result of bumping up from the previous value.

The semiprime elements of $\mathbb{N}_0^\Lambda \cup \{\infty\}$ ($\mathbb{N}_0^\Lambda$ is $\mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \times \mathbb{N}_0, \Lambda$ times) correspond to elements of $\coprod_{\lambda\in\Lambda}\mathbb{N}_0$ under partial ordering. Thus, when B_Λ is bounded by a finite $\lambda \in \Lambda, m_\lambda \neq 0$, there exist two types of semiprime operations, f_{B_Λ} and g_{B_Λ} where $f_{B_\Lambda}(<\infty>) = <\infty>$ and $g_{B_\Lambda}(<\infty>) = <\phi^{m_1\ m_2\ \dots\ m_r}_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}>$ and $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$ is the set of all $\lambda_j \in \Lambda$ with $m_j < \infty$.

We may define two subsets of $M_{\mathscr{I}}$ as the following:

$$M_f = \{e\} \cup \{g_{B_\Lambda} \in M_{\mathscr{I}} \mid g_{B_\Lambda}(<\infty>) = <\phi_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}^{m_1\ m_2\ \dots\ m_r}>\},$$

the set of closure operations whose ideal $<\infty>$ is closed, and

$$M_0 = \{e\} \cup \{f_{B_\Lambda} \in M_{\mathscr{I}} \mid f_{B_\Lambda}(<\infty>) = <\infty>\},$$

the set of closure operations whose ideal $<\infty>$ is not closed.

Suppose that C_Λ and D_Λ are two identity Λ-boxes such that both C_Λ and D_Λ are bounded. Then, since $C_\Lambda \cap D_\Lambda$ is also a bounded identity Λ-box and $f_{C_\Lambda} \circ f_{D_\Lambda} = f_{C_\Lambda \cap D_\Lambda}, g_{C_\Lambda} \circ g_{D_\Lambda} = g_{C_\Lambda \cap D_\Lambda}$, we have closure on $M_{\mathscr{I}}$ of M_0 and M_f. Thus, M_0 and M_f are submonoids of $M_{\mathscr{I}}$.

Now suppose that C_Λ and D_Λ are two distinct identity Λ-boxes and that D_Λ is bounded. Then, $C_\Lambda \cap D_\Lambda \subseteq D_\Lambda$ is also a bounded identity Λ-box.

We also have for $f_{C_\Lambda} \in M_0$ and $g_{D_\Lambda} \in M_f$ that $f_{C_\Lambda} \circ g_{D_\Lambda} = g_{C_\Lambda \cap D_\Lambda}$ so that M_f is a left M_0-act. But, $g_{D_\Lambda} \circ f_{C_\Lambda} \neq g_{C_\Lambda \cap D_\Lambda}$ (i.e. does not necessarily equal), because

$$g_{D_\Lambda} \circ f_{C_\Lambda}(<\infty>) = g_{D_\Lambda}(<\infty>) = <\phi_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}^{m_1\ m_2\ \dots\ m_r}> \in D_\Lambda \neq C_\Lambda \cap D_\Lambda$$

so that $g_{D_\Lambda} \circ f_{C_\Lambda}$ is not a closure operation. Thus, M_f is not a right M_0-act.

Proposition 4.1.3 *The set of semiprime operations on $\coprod_{\lambda \in \Lambda} \mathbb{N}_0$ can be decomposed into two submonoids, M_f and M_0, where M_f is a left but not a right M_0-act under composition.*

Proposition 4.1.4 *The only element of the set of prime operations on $\coprod_{\lambda \in \Lambda} \mathbb{N}_0$ is the identity map $\{e\}$ where $e(<\phi_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}^{m_1\ m_2\ \dots\ m_r}>) = <\phi_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}^{m_1\ m_2\ \dots\ m_r}>$.*

Proof:

Let $<b> = <\phi_{\lambda_0'}^{m_0'}>, f_c = f_{\phi_{\lambda_1\ \lambda_2\ \dots\ \lambda_r}^{m_1\ m_2\ \dots\ m_r}}$

If f_c is prime, then

$$< b > +f_c(< \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >)$$

$$= f_c(< b > \ + < \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >)$$

$$= f_c(< \phi^{m_0'}_{\lambda_0'} > \ + < \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >)$$

$$= f_c(< \phi^{m_0'+m_1}_{\lambda_0'+\lambda_1} \ + \ \phi^{m_2\ ...\ m_r}_{\lambda_2\ ...\ \lambda_r} >)$$

since f_c could be $f_{\phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r}}$, we would have

$$< b > +f_c(< \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >)$$

$=< b > \ + < \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} > \subsetneq < \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} > f_c(< \phi^{m_0' + m_0\ m_1\ m_2\ ...\ m_r}_{\lambda_0' + \lambda_0\ \ \lambda_1\ \lambda_2\ ...\ \ \lambda_r} >;$
$= f_c(< b > \ + < \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >),$

which implies

$$< b > +f_c(< \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >) \ \subsetneq \ f_c(< b > \ + < \phi^{m_1\ m_2\ ...\ m_r}_{\lambda_1\ \lambda_2\ ...\ \lambda_r} >),$$

contradicting property (e). Thus, $\{e\}$ is the only member of the set of prime operations on $\coprod_{\lambda\in\Lambda} \mathbb{N}_0$. $\square$

Chapter 5

Future Research Directions

We may investigate the effect on the results of chapter 3 of adding the ∞ element to the subsemigroup S.

This work could also by continued by analyzing the algebraic structure of closure operations on $\mathbb{N}_0$ act $\mathbb{Z}$ and on $\mathbb{N}_0 \cup \{\infty\}$ act $\mathbb{Z} \cup \{\infty\}$. In this case, the recent paper "A Look at the Prime and Semiprime Operations of One-dimensional Domains" [12] should have some results that would have analogs that could be applied to $\mathbb{N}_0$ act $\mathbb{Z}$. In these new cases, we would introduce ideals generated by negative integers, see if the ideals may be totally or partially ordered, find what closure operations are valid, and investigate the structure of the relations among the new closure operations.

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