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Optimal control problems, curves of pursuit

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Curves of Pursuit

by

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B.S. Mathematics,
Peoples' Friendship University of Russia, 2008

THESIS

Submitted in Partial Fulfillment of the
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Dedication

To my parents, Valentina and Nikolay Moiseev,

for their support and encouragement.

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ABSTRACT OF THESIS

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Abstract

We study a class of problems known as pursuit-evasion problems (PE). These problems can be understood as special cases of optimal control problems. After describing the two main principles to study optimal control problems, namely Pontryagin's maximum principle and Bellman's method of dynamic programming, this thesis focuses on specific examples of PE problems within the classes of pursuit problems, evasion problems, and pursuit-evasion problems.

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Chapter 1

Introduction

1.1 Overview

A pursuit-evasion (PE) problem refers to a family of mathematical problems in which one group (*Pursuers*) attempts to track down members of another group (*Evaders*) in an environment. There are different formulations of these problems and each formulation uses some specific features of the pursuit-evasion situation. Our objective in this thesis is to study a variety of problems that encompass the main pursuit-evasion problems from the point of view of the motion and strategy of the pursuer (Chapter 3), the evader (Chapter 4), and both (Chapter 5).

Pursuit-evasion (PE) problems can be approached stochastically or deterministically. With the stochastic approach (for a broader discussion see [22], [24], [25], [16] and [22]), it is more realistic to assume knowledge of the probability characteristics of target detection, whereas the deterministic approach has to contend with trajectories and control parameters that give the opportunity to find the minimum or maximum values. In this thesis we restrict the discussion to the deterministic approach, and state PE problems as optimal control problems, where we speak about optimality in the sense of rapidity of action, i.e., about achieving the target in the shortest time (Chapter 3), or avoiding the chaser as long as possible (Chapter 4), or both (Chapter 5).

Because the pursuit-evasion (PE) problem can be understood as a special case of the more general class of problems known as optimal control processes, we devote Chapter 2 to the formulation of the general optimal control problem and a discussion of the two main approaches to solve this problem, namely *Pontryagin's maximum principle* (Theorems 2.2.1, 2.2.2) and *Bellman's equation* (2.29). Pontryagin's maximum principle was discovered in the late 1950s ([24]) by the Russian mathematician Lev Semenovich Pontryagin.[1] The maximum principle is an effective tool in solving a broad range of control problems, we state it for the important time-optimal case (see sections 2.1, 2.2). Shortly before the appearance of Pontryagin's maximum principle in the late 1950s ([2], [3], [9], [16]), the American mathematician Richard Bellman published his *Dynamic Programming* [2], [3], [15], [16]. He constructed a partial differential equation for the functional that gives us the minimum time when we transfer the controlled object from the initial state to some other given point (see equation (2.6). This equation of Bellman's gives rise to another approach to the solution of optimal control problems (see section 2.3). It must be noted though that the assumption on the continuous differentiability of the functional (2.6) does not hold in the simplest cases. Thus, Bellman's consideration yields a good heuristic method, rather than a mathematical solution of the problem. The maximum principle, in

[1]Pontryagin's maximum principle gave birth to optimal control theory, which at present is a vital area in applied mathematics. Pontryagin was led to the formulation of the general time-optimal control problem by an attempt to solve a concrete fifth-order system of ordinary differential equations with three control parameters related to optimal maneuvers of an aircraft, which was proposed to him by the Russian Air Force in the early spring of 1955 [10]. Right after the formulation of the time-optimal control problem, during three days, or better to say, during three sleepless nights (Pontryagin suffered from severe insomnia and very often used to do math in bed all night long), the first and the most important step toward the final solution (the Pontryagin's maximum principle) was made by Pontryagin [10]. He derived the first version of the necessary conditions.

addition to its complete mathematical validity, also has the advantage that it results in a system of ordinary differential equations, whereas Bellman's approach requires the solution of a partial differential equation. Both approaches will be discussed and compared in Chapter 2.

Even though we are not going to literally apply Pontryagin's or Bellman's general approach to the specific examples discussed in this thesis, we introduce them because they provide the appropriate framework in the formulation of the pursuit-evasion problem.

We start Chapter 2 by discussing optimal control processes. A process is called controlled if it can be described by a vector differential equation with a control parameter and a phase point. The problem then is to choose such a control, as a function of time, so that the corresponding trajectory of the given differential equation is shifted from a given initial point to some other given point in minimum time. In this case the control and its corresponding trajectory are called *optimal*. The other important class of problems is a time-optimal control problem, which is defined in the same Chapter 2. In section 2.2 (Chapter 2) we introduce a hamiltonian function in order to state Pontryagin's maximum principle (Theorems 2.2.1, 2.2.2), which includes an important *Pontryagin's maximum condition*.

The rest of this thesis is organized as follows: in Chapter 3 we present a definition of the pursuit problem, and provide examples of the pursuit problem (Bouguer's problem (section 3.2), the plain and the wind problem (section 3.3), the tractrix (section 3.4), and Apollonius pursuit (section 3.5)). In Chapter 4 we state a definition of the evasion problem, and give examples of the evasion problem (Isaacs's problem (section 4.2), and the lady in the lake problem (section 4.3)). In Chapter 5 we present a definition of the pursuit-evasion problem, and examples of these problems (pursuit in the plane (section 5.2), one-dimensional rocket chase (section 5.2), and Kelly's game (section 5.4)). Moreover, we state the possible method of solving pursuit-evasion problems while using Pontryagin's maximum principle. Although this theorem gives the necessary conditions for optimality of pursuit-evasion problems (and it can be generalized to multiple pursuers and multiple evaders, as in [7]), the fact is that the PE problems studied in the current thesis can be analyzed directly by more elementary methods. Finally, in Chapter 6 we summarize the results of the thesis.

Chapter 2

Optimal Control Processes

Since the pursuit evasion (PE) problem can be understood as a special case of the more general class of problems known as optimal control processes, we are going to devote this chapter to the formulation of the general optimal control problem and a discussion of the two main approaches to solve this problem, namely *Pontryagin's maximum principle* (Theorems 2.2.1, 2.2.2) and *Bellman's equation* (2.29).

2.1 Formulation of the Optimal Control Problem

A desirable property of most technological processes is controllability, which roughly speaking means that a particular process can be realized by a proper adjustment of certain control parameters. Mostly important is the search, among all the controllable processes, of the control that optimizes a related function of this process. This problem is known as the optimal control problem. For example, one can speak about optimality in the way of spending the least possible time or using the minimum energy in order to reach the target. These problems can be formulated mathematically, and their solution is given by a general method known as *Pontryagin's maximum principle*

(Theorems 2.2.1, 2.2.2) ([10], [21], [22], [23]).

To start, we consider *control processes* which can be described by a system of ordinary differential equations

$$\frac{dx^i}{dt} = f^i(x^1, ..., x^n, u^1, ..., u^r) = f^i(x^k, u^j), \quad i, k = 1, ..., n, \quad j = 1, ..., r, \tag{2.1}$$

or in vector form,

$$\frac{dx}{dt} = f(x, u). \tag{2.2}$$

The variables $x^1, ..., x^n$ characterize the process, and they are known as the phase coordinates of the controlled object which define its state at each instant of time t. Giving a point $u = (u^1, ..., u^r) \in \mathcal{U} \subset \mathbb{R}^r$ is equivalent to giving a numerical system of parameters $u^1, ..., u^r$, and they are known as the control parameters which determine the course of the process. The functions f^i are defined for $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}^r$. They are assumed to be continuous in the variables $x^1, ..., x^n, u^1, ..., u^r$, and continuously differentiable with respect to $x^1, ..., x^n$. In other words, the functions

$$f^i(x^1, ..., x^n, u) \quad \text{and} \quad \frac{\partial f^i(x^1, ..., x^n, u)}{\partial x^j}, \quad i, j = 1, ..., n,$$

are defined and continuous everywhere on the direct product $\mathcal{X} \times \mathcal{U}$.

In order to find a solution of equation (2.1) and determine the course of the control process (2.1) in a certain time interval $t_0 \leq t \leq t_1$, it is sufficient for the control parameters $u^1, ..., u^r$ to be the functions of time on this time interval:

$$u^j = u^j(t), \quad j = 1, ..., r. \tag{2.3}$$

Then, for the given initial values

$$x^i(t_0) = x_0^i, \quad i = 1, ..., n, \tag{2.4}$$

the solution is uniquely determined, at least locally in time. Hence, we say that a *control*

$$U = (u^j(t), t_0, t_1, x_0^i), \quad j = 1, ..., r, \quad i = 1, ..., n \tag{2.5}$$

of equation (2.1) is given, if a function $u^j(t)$, its range of definition $t_0 \leq t \leq t_1$, and the initial value (2.4) of the solution $x^i(t)$ are given. Therefore, we only deal with piecewise continuous control functions $u^j(t)$ which admit discontinuities of the first kind, and *continuous* solutions of equation (2.2).

The *control problem* to be solved, which is related to the control process (2.1), *consists of the following.* We consider the integral function

$$L(U) = \int_{t_0}^{t_1} f^0\left(x^1, ..., x^n, u^1, ..., u^r\right) dt, \tag{2.6}$$

where $f^0(x^1, ..., x^n, u^1, ..., u^r)$ is a given function, continuous, together with its partial derivatives

$$\frac{\partial f^0}{\partial x^j}, \qquad j = 1, ..., n,$$

everywhere on the space $\mathcal{X} \times \mathcal{U}$. For each control (2.5), given on a certain interval $t_0 \leq t \leq t_1$, the course of the control processes is uniquely determined, at least locally in time, and the integral (2.6) takes on a definite value. Let us assume that there exists a control (2.5) which transfers the controlled object from a given initial phase state x_0^i (2.4) to a prescribed terminal phase state

$$x^i(t_1) = x_1^i, \qquad i = 1, ..., n. \tag{2.7}$$

It is required to find a control $u(t)$ which transfers the controlled object from state x_0^i to state x_1^i in such a way that the *functional* $L(U)$ *has a minimum value.* Thus, L is a function of the control U.

Let us summarize the above discussion and state the definition of the *optimal control problem* (equations (2.2), (2.4), (2.5), (2.6), (2.7)).

Definition (*Optimal Control Problem*) An ***optimal control problem*** is a problem given by the equations

$$\frac{dx}{dt} = f(x, u),$$

$$x(t_0) = x_0,$$

$$x(t_1) = x_1,$$

$$U = (u(t), t_0, t_1, x_0),$$

$$L(U) = \int_{t_0}^{t_1} f^0\,(x, u)\, dt,$$

where $x = (x^1, ..., x^n) \in \mathcal{X} \subset \mathbb{R}^n$, $u = (u^1, ..., u^r) \in \mathcal{U} \subset \mathbb{R}^r$ is some piecewise continuous function, $f = (f^1, ..., f^n)$ are continuous, together with its partial derivatives everywhere on the space $\mathcal{X} \times \mathcal{U}$, and $f^0(x, u)$ is a given function (also continuous, together with its partial derivatives, everywhere on the space $\mathcal{X} \times \mathcal{U}$).

Definition A control $U = (u(t), t_0, t_1, x_0^i)$ is called *optimal*, if, for any control

$$U^* = (u^*(t), t_0, t_1, x_0^i)$$

which transfers the point x_0^i to the point x_1^i, the inequality

$$L(U) \leq L(U^*)$$

is valid. The corresponding trajectory $x(t)$ is called an *optimal trajectory*.

Thus, an optimal control problem consists of finding the optimal controls and the corresponding optimal trajectories.

Remark 1. The times t_0 and t_1 are not fixed, we only require that the object should be in state (2.4) at the initial time, and in state (2.7) at the final time, and that the functional (2.6) should achieve a minimum. (The discussion of the case where the times t_0 and t_1 are fixed can be found in [24], §8 .)

Remark 2. If (2.5) is an optimal control of equation (2.2) corresponding to this control, and t_2, t_3 $(t_2 < t_3)$ are two points in the interval $t_0 \le t \le t_1$, then

$$U' = (u(t), t_2, t_3, x^i(t_2))$$

is also an optimal control.

Remark 3. If (2.5) is an optimal control of equation (2.2) that transfers the point x_0^i to the point x_1^i, and τ is an arbitrary number, then

$$U'' = (u(t-\tau), t_0 + \tau, t_1 + \tau, x_0^i)$$

is also an optimal control which transfers the point x_0^i to x_1^i.

Definition (*Time-Optimal Control Problem*) When the function $f^0(x^i, u^j)$ is defined by equation

$$f^0(x, u) \equiv 1, \tag{2.8}$$

the function of the control (2.5) in this case is

$$L(U) = t_1 - t_0,$$

and the optimality of the control $u(t)$ signifies *minimality of the transition time from* x_0 *to* x_1. The problem of finding optimal controls (and trajectories) in this case is called the *time optimal control problem.*

We should point out that up to know we have spoken about an optimal control which brought the object to a given point. However, the optimal control problem may consist of "optimality getting to" a *moving point* in phase space. Let us assume that there exists a moving point

$$x^i = \theta^i(t), \quad i = 1, ..., n, \tag{2.9}$$

in phase space. Then, there arises the problem of optimality bringing the object in coincidence with a moving point. This problem is easily reduced to the one considered above. It is sufficient to introduce new variables by setting

$$y^i = x^i - \theta^i(t), \quad i = 1, ..., n.$$

As a result of this transformation, the control system

$$\frac{dx^i}{dt} = f^i(x^i, u^j), \quad i = 1, ..., n, \quad j = 1, ..., r,$$

becomes a new system. The goal of the control process becomes that of bringing the new object $(y^1, ..., y^n)$ to the stationary point $(0, ..., 0)$ in phase space.

Of great importance is the case where $\mathcal{U} \subset \mathbb{R}^n$ is a compact domain. This is clearly the case in most practical applications, where the control parameters can only take values with predetermined upper and lower bounds. For example, $\mathcal{U}$ may be a *cube* defined by the inequalities

$$|u^j| \leq 1, \quad j = 1, ..., r.$$

In many instances it turns out that the optimal control (2.5) is realized by a piecewise constant control $(u^1(t), ..., u^r(t))$ with values switching between various vertices of $\mathcal{U}$.

It follows that the *class of admissible controls* (2.5) must include *piecewise continuous* functions. For the same reason, the phase coordinates $x^1, ..., x^n$ are assumed to be continuous and piecewise differentiable functions of time. Under these assumptions the necessary conditions for optimality are formulated in the form of *Pontryagin's maximum principle* (Theorems 2.2.1, 2.2.2) ([22], [23], [24]), which we will present in the next section.

2.2 Necessary Conditions for Optimality Pontryagin's Maximum Principle

In order to formulate the necessary optimality condition it will be convenient to reformulate our optimal control problem (for a broader discussion see [24]). Namely, let us adjoin a new coordinate x^0 to the phase coordinates $x^1, ..., x^n$, which vary according to (2.1). Let x^0 vary according to the law

$$\frac{dx^0}{dt} = f^0(x^1, ..., x^n, u^1, ..., u^r),$$

where f^0 is the function which appears in the definition of the functional $L(U)$ (see (2.6)). In other words, we shall consider the system of differential equations

$$\frac{dx^i}{dt} = f^i(x^1, ..., x^n, u^1, ..., u^r) = f^i(x, u), \quad i = 0, 1, ..., n, \tag{2.10}$$

whose right-hand sides do not depend on x^0. Introducing the vector

$$\mathbf{x} = (x^0, x^1, ..., x^n) = (x^0, x)$$

in the $(n + 1)$-dimensional vector space $\mathbf{X} = \mathbb{R} \times \mathcal{X} \subseteq \mathbb{R}^{n+1}$, we may rewrite system (2.10) in vector form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(x, u), \tag{2.11}$$

where $\mathbf{f}(x, u)$ is the vector in $\mathbf{X}$ with coordinates $f^0(x, u), ..., f^n(x, u)$. Note, that $\mathbf{f}(x, u)$ does not depend on the coordinate x^0 of the vector $\mathbf{x}$, that is $\vec{f}(x, u)$, not $\vec{f}(\vec{x}, u)$.

Now let $u(t)$ be an admissible control (2.5) (i.e., piecewise continuous) transferring x_0 to x_1, and let $x = x(t)$ be the corresponding solution of equation (2.2) with initial condition $x(t_0) = x_0$. Let us denote the point $(0, x_0)$ by $\mathbf{x}_0$, i.e., $\mathbf{x}_0$ is the point of $\mathbf{X}$ whose coordinates are $0, x_0^1, ..., x_0^n$, where $x_0^1, ..., x_0^n$ are the coordinates of x_0 in $\mathcal{X}$.

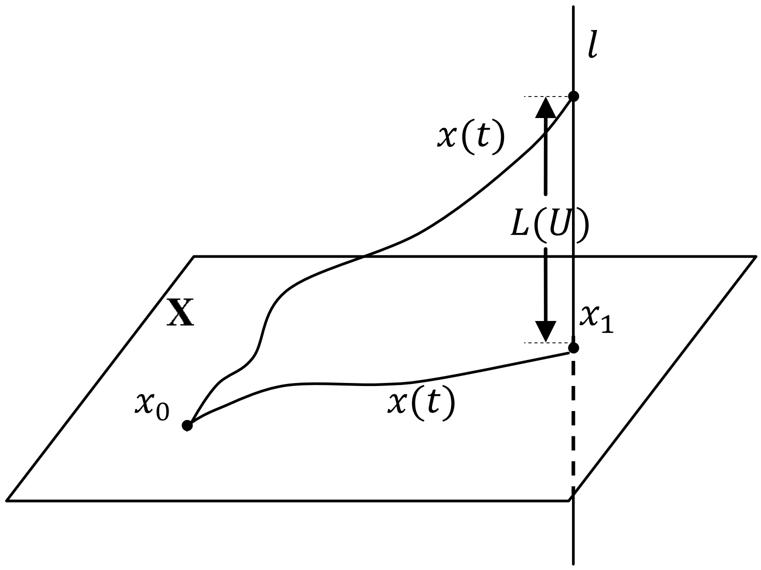

Figure 2.1: *Reformulation of the problem given by equation (2.1), where line l is passing through the point $(0, x_1)$ and is parallel to the x^0 axis, i.e., this line is made up of all the points (ξ, x_1) where the number ξ is arbitrary*

Then, it is clear that the solution of equation (2.11) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, corresponding to the control $u(t)$, is defined on the entire interval $t_0 \leq t \leq t_1$, and has the form

$$x^0 = \int_{t_0}^{t} f^0\left(x(t'), u(t')\right)dt', \qquad x = x(t).$$

In particular, when $t = t_1$

$$x^0 = \int_{t_0}^{t_1} f^0\left(x(t), u(t)\right)dt = L(U), \qquad x = x_1,$$

i.e., the solution $\mathbf{x}(t)$ of equation (2.11) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ passes through the point $\mathbf{x} = (L(U), x_1)$ at $t = t_1$. In other words, if we let l be the line in $\mathbf{X}$ passing through the point $\mathbf{x} = (0, x_1)$ and parallel to the x^0 axis (this line is made up of all the points (ξ, x_1) where the number ξ is arbitrary, see Figure 2.1), we can say that $\mathbf{x}(t)$ passes through a point on line l, with coordinate $x^0 = L(U)$, at the time $t = t_1$. Conversely, suppose that $u(t)$ is an admissible control (i.e., at least piecewise continuous) such that the corresponding solution $\mathbf{x}(t)$ of equation (2.11) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 = (0, x_0)$, at some time t_1 passes through a point $\mathbf{x}_1 \in l$, with coordinate $x^0 = L(U)$. Then, the control $u(t)$ transfers (in $\mathcal{X}$) the phase point from x_0 to x_1, and the functional (2.6) takes on the value $L(U)$.

Thus, we may formulate the above optimal problem (from 2.1) in the following equivalent form.

In the $(n+1)$-dimensional phase space $\boldsymbol{X}$ the point $\mathbf{x}_0 = (0, x_0)$ and the line l are given. The line l is assumed to be parallel to the x^0 axis, and to pass through the point $(0, x_1)$. Among all the admissible controls $u = u(t)$, having the property that the corresponding solution $\mathbf{x}(t)$ of (2.11) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ intersects l, find one whose point of intersection with l has the smallest coordinate x^0 (see [24]).

Let us now proceed to the formulation of the theorem which yields the necessary conditions of the problem. (The proof of this theorem can be found in [24], Chapter II.) To formulate the theorem, we shall consider, in addition to the fundamental system of equations (2.10) another system of equations in the auxiliary (supplementary) variables $\psi_0, \psi_1, ..., \psi_n$:

$$\frac{d\psi_i}{dt} = -\sum_{\alpha=0}^{n} \frac{\partial f^\alpha(x,u)}{\partial x^i}\psi_\alpha, \quad i = 0, 1, ..., n. \tag{2.12}$$

If we choose an admissible control $u(t)$, $t_0 \leq t \leq t_1$, and have the corresponding phase trajectory $\mathbf{x}(t)$ of system (2.10) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, system (2.12) takes the form

$$\frac{d\psi_i}{dt} = -\sum_{\alpha=0}^{n} \frac{\partial f^\alpha(x(t),u(t))}{\partial x^i}\psi_\alpha, \quad i = 0, 1, ..., n. \tag{2.13}$$

This system is linear and homogeneous. Therefore, for any initial condition, it admits the unique solution

$$\boldsymbol{\psi} = (\psi_0, \psi_1, ..., \psi_n)$$

for the ψ_i (which is defined on the entire interval $t_0 \leq t \leq t_1$ on which $u(t)$ and $\mathbf{x}(t)$ are defined). Similarly to the solution $\mathbf{x}(t)$ of system (2.11), the solution of system (2.13) consists of continuous functions $\psi_i(t)$ which have everywhere, except at a finite number of points (namely, at the points of discontinuity of $u(t)$), continuous derivatives with respect to t. Each solution of system (2.13) for any initial conditions will be called the solution of system (2.12) corresponding to the chosen control $u(t)$ and phase trajectory $\mathbf{x}(t)$.

Now we will combine systems (2.10) and (2.12) into one entry. We consider the following function $\mathcal{H}$ of the variables $x^0, x^1, ..., x^n, \psi_0, \psi_1, ..., \psi_n, u^1, ..., u^r$:

$$\mathcal{H}(\boldsymbol{\psi}, x, u) = (\boldsymbol{\psi}, \mathbf{f}(x, u)) = \sum_{\alpha=0}^{n} \psi_\alpha f^\alpha(x, u).$$

The above systems (2.10) and (2.12) can be rewritten with the aid of the function $\mathcal{H}$ in the form of the following *Hamiltonian system*:

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial \psi_i}, \qquad i = 0, ..., n, \tag{2.14}$$

$$\frac{d\psi_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x^i}, \qquad i = 0, ..., n. \tag{2.15}$$

For fixed (constant) values of $\boldsymbol{\psi}$ and x, the function $\mathcal{H}$ becomes a function of the parameter $u \in \mathcal{U}$. Let us now denote the least upper bound of the values of this function by $\mathcal{M}(\boldsymbol{\psi}, x)$:

$$\mathcal{M}(\boldsymbol{\psi}, x) = \sup_{u \in \mathcal{U}} \mathcal{H}(\boldsymbol{\psi}, x, u).$$

If the continuous function $\mathcal{H}$ achieves its upper bound on $\mathcal{U}$, then $\mathcal{M}(\boldsymbol{\psi}, x)$ is the maximum of the values of $\mathcal{H}$, for fixed $\boldsymbol{\psi}$ and x. Therefore, Theorem 2.2.1 below (a *necessary condition for optimality*) will be called the *maximum principle* (the principal content of the principle is in equation (2.16)) [10], [24].

Theorem 2.2.1 (Pontryagin's Maximum Principle) *Let $u(t)$, $t_0 \le t \le t_1$, be an admissible control such that the corresponding trajectory $\mathbf{x}(t)$ [see (2.14)] which begins at the point $\mathbf{x}_0$ at the time t_0 passes, at some time t_1, through a point on the line l. In order that $u(t)$ and $\mathbf{x}(t)$ be optimal it is necessary that there exists a nonzero continuous vector function $\boldsymbol{\psi}(\mathrm{t}) = (\psi_0(t), \psi_1(t), ..., \psi_n(t))$ corresponding to $u(t)$ and $\mathbf{x}(t)$ [see (2.15)], such that:*

(a) *for every t, $t_0 \le t \le t_1$, the function $\mathcal{H}(\boldsymbol{\psi}(t), x(t), u)$ of the variable $u \in \mathcal{U}$ attains its maximum at the point $u = u(t)$:*

$$\mathcal{H}(\boldsymbol{\psi}(t), x(t), u(t)) = \mathcal{M}(\boldsymbol{\psi}(t), x(t)), \tag{2.16}$$

(b) *at the terminal time t_1 the relations*

$$\psi_0(t_1) \le 0, \quad \mathcal{M}(\boldsymbol{\psi}(t_1), x(t_1)) = 0 \tag{2.17}$$

are satisfied. Furthermore, it turns out that if $\boldsymbol{\psi}(t)$, $\mathbf{x}(t)$, and $u(t)$ satisfy system (2.14), (2.15), and condition **(a)**, *the time functions $\psi_0(t)$ and $\mathcal{M}(\boldsymbol{\psi}(t), x(t))$ are constant. Thus, (2.17) may be verified at any time t, $t_0 \le t \le t_1$, and not just at t_1.*

The proof of the Theorem 2.2.1 can be found in [24], Chapter II.

To formulate the necessary condition for the time-optimal problem (equation (2.8)), where

$$f^0(x, u) \equiv 1,$$

let us form the Hamiltonian function

$$\mathcal{H} = \psi_0 + \sum_{\nu=1}^{n} \psi_\nu f^\nu(x, u).$$

Introducing the n-dimensional vector $\psi = (\psi_1, ..., \psi_n)$ and the function

$$H(\psi, x, u) = \sum_{\nu=1}^{n} \psi_\nu f^\nu(x, u),$$

we can rewrite equations (2.1) and (2.12) (with the exception for equation (2.12) for $i = 0$, which is now superfluous) in the form of the Hamiltonian system

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial \psi_i}, \quad i = 1, ..., n, \tag{2.18}$$

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x^i}, \quad i = 1, ..., n. \tag{2.19}$$

For fixed values of ψ and x, H is a function of u. We denote the upper bound of the values of this function by $M(\psi, x)$:

$$M(\psi, x) = \sup_{u \in \mathcal{U}} H(\psi, x, u).$$

Since

$$H(\psi, x, u) = \mathcal{H}(\boldsymbol{\psi}, x, u) - \psi_0,$$

we get

$$M(\psi, x) = \mathcal{M}(\boldsymbol{\psi}, x) - \psi_0,$$

and therefore (2.16) and (2.17) become

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) = -\psi_0 \geq 0.$$

Hence, we obtain the following theorem.

Theorem 2.2.2 (Pontryagin's Maximum Principle for the time-optimal control problem (2.8)) *Let $u(t)$, $t_0 \leq t \leq t_1$ be an admissible control which transfers the phase point from x_0 to x_1, and let $x(t)$ be the corresponding trajectory (see (2.18)), so that $x(t_0) = x_0$, $x(t_1) = x_1$. In order that $u(t)$ and $x(t)$ be time-optimal it is necessary that there exist a nonzero, continuous vector function $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ corresponding to $u(t)$ and $x(t)$ (see (2.19)) such that:*

(a) *for all t, $t_0 \leq t \leq t_1$, the function $H(\psi(t), x(t), u)$ of the variable $u \in \mathcal{U}$ attains its maximum at the point $u = u(t)$:*

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)), \tag{2.20}$$

(b) *at the terminal time t_1 the relation*

$$M(\psi(t_1), x(t_1)) \geq 0 \tag{2.21}$$

is satisfied. Furthermore, it turns out that if $\psi(t)$, $x(t)$, and $u(t)$ satisfy system (2.18), (2.19), and condition **(a)**, *the time function $M(\psi(t), x(t))$ is constant. Thus, (2.21) may be verified at any time t, $t_0 \leq t \leq t_1$, and not just at t_1.*

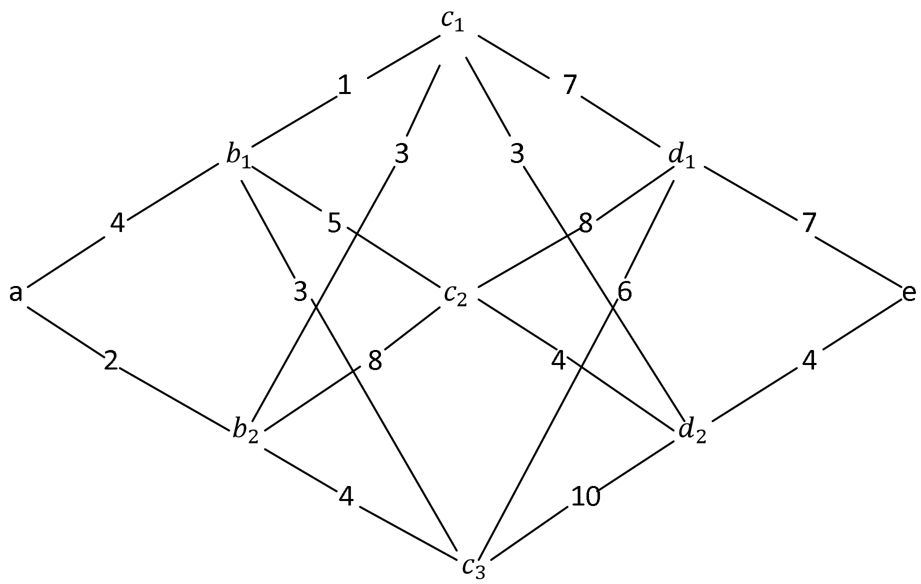

Figure 2.2: The route of the ship sailing from a to e

2.3 Bellman's Method of Dynamic Programming

Shortly before the appearance of Pontryagin's maximum principle in the late 1950s, R. Bellman published his *Dynamic Programming* [2], [3], [15], [16], which presents a related but different approach to the optimum design of control systems which is more efficient in some situations. The following simple example will illustrate some of the main ideas behind this dynamic programming approach.

Example Suppose a ship sailing from a and ending at e calls at three ports (at either of the two b's, at one of the three c's, and at one of the two d's) along the way (as shown in Figure 2.2) and picks up and delivers the amounts of cargo (in hundreds of tons). The objective is to deliver as much cargo as possible on the entire trip. Since there are only 12 different routes, it is a simple matter to list them all and choose the route that yields the maximum tonnage. However, we shall solve the problem differently and use the following reasoning. Suppose that, somehow we were to know the maximum tonnage values of the two shorter problems, one from b_1

to e and the other from b_2 to e, then it would be very easy to decide on the entire route. There are only two possible decisions left to be made at a: go to b_1 or go to b_2. To reach such a decision, simply add 4 to that maximum tonnage from b_1 to e that we somehow learned, add 2 to that maximum tonnage from b_2 to e, and choose the route that gives the larger value. In other words, we will have solved the original four-stage problem by first solving two three-stage problems. Similarly, each of these two three-stage problems (from b_1 to e or from b_2 to e) would be relatively easy to solve if we were to first solve three two-stage problems, namely, find the value given by the maximum tonnage path from each $c_i, i = 1, 2, 3$, to e. We continue this reasoning and reduce the process to two one-stage problems, from d_1 to e or from d_2 to e, at which stage the answer is obvious - go from d_1, because 7 is larger than 4.

Let us do the problem formally. We will break it into several n - stage problems, $n = 1, 2, 3, 4$. Notice that there are four stages: from a to b, b to c, c to d, and d to e. There are two possible terminal ports, or *states* as we will call them, namely b_1 and b_2 in stage one, three states c_1, c_2, and c_3 in stage two, two states d_1 and d_2 in stage three, and one state e in the last stage. Each of these states may also be thought of as the initial state for the following stages. For instance, b_1 may be considered the initial state of a three-stage problem, c_1 the initial state of a two-stage problem, *etc.* Let the variable x stand for the initial state for any n-stage problem, $n = 1, 2, 3, 4$. For instance, for a two-stage problem, x may be either c_1, or c_2, or c_3. Associated with each problem is also a *decision* or *control variable* u_n, n = *1, 2, 3, 4*, which chooses the immediate destination when there are n stages left to go. Thus, u_4 chooses b_1 or b_2, u_3 chooses c_1 or c_2 or c_3, u_2 chooses d_1 or d_2, and $u_1 = e$. Let $f_n(x, u_n)$ be the total number of tons delivered during the last n stages, given that the boat is in state x and the decision is u_n. If $\bar{u}_n$ is the decision which maximizes $f_n(x, u_n)$ for fixed n and x, let $\bar{f}_n(x)$ be that maximum value of f_n. Since $\bar{f}_n$ is the maximum value with respect to the decision variable u_n, it is now a function of the initial state variable x alone, hence, the notation $\bar{f}_n(x)$.

Problem	Initial state x	Max value $\overline{f_1}(x)$	Decision $u_1 = \overline{u_1}$
1	d_1	$7 = \overline{f_1}(d_1)$	e
2	d_2	$4 = \overline{f_1}(d_2)$	e

Figure 2.3: *Two one-stage problems in the first subproblem*

Problem	Initial state x	$f_2(x, u_2)$		Max value $\overline{f_2}(x)$	Optimal decision $\overline{u_2}$
		$u_2 = d_1$ $f_2(x, d_1) = s_{d_1} + \overline{f_1}(d_1)$	$u_2 = d_2$ $f_2(x, d_2) = s_{d_2} + \overline{f_1}(d_2)$		
1	c_1	7+7=14	3+4=7	$14 = \overline{f_2}(c_1)$	d_1
2	c_2	8+7=15	4+4=8	$15 = \overline{f_2}(c_2)$	d_1
3	c_3	6+7=13	10+4=14	$14 = \overline{f_2}(c_3)$	d_2

Figure 2.4: *Three two-stage problems*

In the first subproblem, there is only one stage left to go, and $\bar{u}_1 = u_1 = e$. The initial states are d_1 and d_2, as shown in Figure 2.3.

We move now to the three subproblems in each of which there are two stages to go, but we utilize the knowledge gained from the one-stage problem. If the boat is at c_1 ($x = c_1$), it can proceed to either d_1 ($u_2 = d_1$) or d_2 ($u_2 = d_2$). If $u_2 = d_1$, $f_2(c_1, d_1) = 7 + 7 = 14$. If $u_2 = d_2$, $f_2(c_1, d_2) = 3 + 4 = 7$. Since $14 > 7$, $\bar{u}_2$ should be $\bar{u}_2 = d_1$. Similarly, if the boat is at c_2 ($x = c_2$) and $u_2 = d_1$, $f_2(c_2, d_1) = 8 + 7 = 15$, while $f_2(c_2, d_2) = 4 + 4 = 8$ if $u_2 = d_2$. Let s_{u_n} be the number of tons of cargo delivered as a result of decision u_n. Then $f_2(x, u_2) = s_{u_2} + \bar{f}_1(u)$. Figure 2.4 shows the values for the different states and decisions.

Next, we move to the two subproblems in each of which there are three stages to go, and again we utilize the knowledge gained from the previous two-stage problems. If the boat is at b_1 ($x = b_1$) and it is decided to go to c_1 ($u_3 = c_1$), the total number of tons delivered would be 1, that between b_1 and c_1, plus 14, the maximum

Problem	Initial state x	$f_3(x,u_3)$			Max value $\overline{f_3}(x)$	Optimal decision $\overline{u_3}$
		$u_3 = c_1$ $f_3(x,c_1) = s_{c_1}+\overline{f_2}(c_1)$	$u_3 = c_2$ $f_3(x,c_2) = s_{c_2}+\overline{f_2}(c_2)$	$u_3 = c_3$ $f_3(x,c_3) = s_{c_3}+\overline{f_2}(c_3)$		
1	b_1	1+14=15	5+15=20	3+14=17	$20 = \overline{f_3}(b_1)$	c_2
2	b_2	3+14=17	8+15=23	4+14=18	$23 = \overline{f_3}(b_2)$	c_2

Figure 2.5: *Two three-stage problems*

Initial state x	$f_4(x,u_4)$		Max value $\overline{f_4}(x)$	Optimal decision $\overline{u_4}$
	$u_4 = b_1$ $f_4(a,b_1) = s_{b_1}+\overline{f_3}(b_1)$	$u_4 = b_2$ $f_4(a,b_2) = s_{b_2}+\overline{f_3}(b_2)$		
a	4+20=24	2+23=25	25	b_2

Figure 2.6: *Four-stage problem*

number of tons to be delivered between c_1 and e. That is, $f_3(b_1, c_1) = s_{c_1} + \bar{f}_2(b_1)$, or $f_3(x, u_3) = s_{u_3} + \bar{f}_2(x)$. See Figure 2.5.

The final, or four-stage problem should now be clear. So what is the optimal policy for the overall problem? Retrace the steps backwards starting with Figure 2.6. Starting at a, the optimal decision $\bar{u}_4$ is to go to b_2. At b_2, $\bar{u}_3$ tells us to go to c_2. At c_2, $\bar{u}_2$ tells us to go to d_1. At d_1, $\bar{u}_1$ says to go to e. Thus, the optimal route is $a \to b_2 \to c_2 \to d_1 \to e$, with a maximum tonnage of 25.

There are only four stages in this example, and each stage has very few states, so that the computational advantages of the dynamic programming approach over the direct, brute approach of listing all twelve possible routes may not be apparent. If a problem has many stages with many states, thus involving many decision processes, direct enumeration may require a phenomenal amount of work, and the computational savings of the dynamic programming approach are considerable. It has been shown that for a 20-stage problem with only 2 states in each stage, direct enumeration generates more than 1,000,000 additions, while dynamic programming requires only 220 additions.

The above example is a discrete multistage decision process problem, in which one chooses a decision from a finite set of decisions at each of a finite number of stages or times. Initially, the problem consisted of n stages, but we reduced it to a sequence of n single stage decision processes, for each of which there is an optimal policy. These problems are joined together by a functional equation. For this particular example, the functional is

$$f_n(x, u_n) = s_{u_n} + \bar{f}_{n-1}(u_n), \tag{2.22}$$

where

$$\bar{f}_n(x) = \max_{u_n} f_n(x, u_n), \qquad u = 1, 2, 3, 4.$$

Hence, we use two basic ideas, Bellman's principle of optimality and the principle of imbedding [16].

Summarizing the method discussed in this example yields ***Bellman's Principle of optimality***) ([16]) : *in control systems with a multistage decision process, given any current state, the remaining sequence of decisions forms an optimal policy with this given state regarded as the initial state. Thus, whatever the first state and decision that led to this current state, all future decisions are optimal.*

In our example, if we found ourselves at, say, state c_1 (regardless of what decision led us there), the policy $c_1 \to d_1 \to e$ is optimal with c_1 considered as the initial state. Similarly, if we found ourselves at, say, state b_1, the policy $b_1 \to c_2 \to d_1 \to e$ would be optimal with b_1 considered as the initial state. By applying this principle of optimality backwards step by step repeatedly, we obtain a policy which is optimal for the overall problem. In our example, in the one-stage problems, either decision

$d_1 \to e$ or $d_2 \to e$ is optimal (actually, the only possible decision), depending on whether d_1 or d_2 is the initial state. For the two-stage problems, if c_1 is the initial state, the decision $\bar{u}_2 : c_1 \to d_1$ is optimal, and the pair $\bar{u}_2, \bar{u}_1 : c_1 \to d_1 \to e$ constitutes an optimal policy with c_1 as the initial state. If c_2 is the initial state, the pair of decisions $\bar{u}_2, \bar{u}_1 : c_2 \to d_1 \to e$ is optimal. Similarly, for the three stage problems, if b_1 is the initial state, the optimal decision $\bar{u}_3 : b_1 \to c_2$, coupled with the optimal strategy from the two-stage problem $\bar{u}_2, \bar{u}_1 : c_2 \to d_1 \to e$, form the optimal strategy $\bar{u}_3, \bar{u}_2, \bar{u}_1 : b_1 \to c_2 \to d_1 \to e$, *etc.*

The other principle we used in the above example is *the principle of imbedding* ([16]) . The principle is working in the way that, instead of attempting to solve a difficult problem directly, one imbeds the problem in a family of simpler, easier to solve problems and obtains the solution to the original difficult problem as a result of the solutions to the problems in the family. By repeated use of the principle of optimality, each n-stage problem with $n > 1$ is converted into a one-stage problem with its own initial state and optimal policy. This is done through the use of some functional equation such as the relation given by (2.22), which, for each problem in the family with its initial state, assigns an optimum value to that problem and links that value with all immediately preceding states.

These two basic ideas - imbedding and principle of optimality - are also to be found in the dynamic programming approach to continuous cases.

Next, let us write *Bellman's equation* for a continuous time variable.

Proposition 2.3.1 *Suppose we have a time-optimal control problem (2.8). Let us fix some point x_1 of the space $\mathcal{X}$, and let $u(t)$, $t_0 \le t \le t_1$, be an optimal control which transfers (through the law of motion $x^i(t) = x^i$, $i = 1, ..., n$) the phase point from some position $x_0 \in \mathcal{X}$ to the position x_1, and let $x(t)$ be the corresponding optimal trajectory. The optimal transition time from the point x_0 to the point x_1, $t_1 - t_0$,* *will be denoted by $T(x_0)$. (The point x_1 does not enter into the notation for the transition time, since it does not vary). Thus, the function $T(x_0)$ is defined on the open set Ω of all points of $\mathcal{X}$ from which an optimal transition to x_1 is possible. We set $T(x) = -\omega(x)$, where $T(x)$ has continuous partial derivatives with respect to the coordinates of the point x, and derive that the function $\omega(x)$ satisfies the following nonclassical partial differential equation (which we shall call **Bellman's equation**) in the region Ω:*

$$\sup_{u \in \mathcal{U}} \sum_{\alpha=1}^{n} \frac{\partial \omega(x)}{\partial x^\alpha} f^\alpha(x, u) = 1. \tag{2.23}$$

Furthermore, the upper bound is attained at some point $u \in \mathcal{U}$ (namely, at the value of the optimal control at the time of departure from the point x), and the function $\omega(x)$ is nonpositive and vanishes only at the point x_1.

Proof It is given, that

$$\omega(x) = -T(x). \tag{2.24}$$

Since $x(t), t_0 \leq t \leq t_1$, is an optimal trajectory, and since each portion of an optimal trajectory is also an optimal trajectory,

$$\omega(x(t)) = -T(x_0) + t - t_0 \tag{2.25}$$

for every $t, t_0 \leq t \leq t_1$. Consequently,

$$\sum_{\alpha=1}^{n} \frac{\partial \omega(x(t))}{\partial x^\alpha} f^\alpha(x(t), u(t)) = \sum_{\alpha=1}^{n} \frac{\partial \omega(x(t))}{\partial x^\alpha} \frac{dx^\alpha}{dt} = \frac{d\omega(x(t))}{dt} = \frac{dt}{dt} = 1. \tag{2.26}$$

Now let v be an arbitrary point of the control region $\mathcal{U}$. We shall consider the motion of the phase point from the position $x(t)$ under the influence of a constant control which is equal to v. Here the problem can be imbedded into the family of problems, following the principle of imbedding discussed before. Mainly, we divide the whole process into two control processes. Thus, after an infinitesimal time interval $dt > 0$, the phase point will be in the position $x(t)+dx$, where the vector $dx = (dx^1, ..., dx^n)$ is defined by

$$dx^i = f^i(x(t), v)dt, \qquad i = 1, ..., n. \tag{2.27}$$

If we now move in an optimal manner from the point $x(t) + dx$ to the point x_1, the time spent in so doing will equal $T(x(t) + dx)$. Hence, the total time spent in a movement of this kind, while transferring from $x(t)$ to x_1, is equal to $T(x(t)+dx)+dt$. This time cannot be shorter than the optimal transition time $T(x(t))$, i.e.,

$$T(x(t) + dx) + dt \geq T(x(t)),$$

or equivalently,

$$\omega(x(t) + dx) - \omega(x(t)) \leq dt.$$

Multiply and divide the left side by dx^α, and since we know that

$$\sum_{\alpha=1}^{n} \frac{w\left(x(t) + dx\right) - w\left(x(t)\right)}{dx^\alpha} = \sum_{\alpha=1}^{n} \frac{\partial w\left(x(t)\right)}{\partial x^\alpha},$$

then because of (2.27), the last inequality may be rewritten in the form

$$\sum_{\alpha=1}^{n} \frac{\partial\omega(x(t))}{\partial x^\alpha} f^\alpha(x(t), v)dt \leq dt,$$

or

$$\sum_{\alpha=1}^{n} \frac{\partial\omega(x(t))}{\partial x^\alpha} f^\alpha(x(t), v) \leq 1, \qquad v \in \mathcal{U}. \tag{2.28}$$

Relations (2.26) and (2.27) show that

$$\sup_{v\in\mathcal{U}} \sum_{\alpha=1}^{n} \frac{\partial\omega(x(t))}{\partial x^\alpha} f^\alpha(x(t), v) = 1,$$

and the upper bound is achieved at $v = u(t)$.

Since an optimal trajectory leading to x_1 passes through each point x of Ω, we arrive at the conclusion that the function $\omega(x)$ satisfies the following nonclassical partial differential equation (**Bellman's equation**) in the region Ω:

$$\sup_{u \in \mathcal{U}} \sum_{\alpha=1}^{n} \frac{\partial \omega(x)}{\partial x^\alpha} f^\alpha(x, u) = 1. \tag{2.29}$$

Furthermore, the upper bound is attained at some point $u \in \mathcal{U}$ (namely, at the value of the optimal control at the time of departure from the point x), and the function $\omega(x)$ is nonpositive and vanishes only at the point x_1.

This is the principle of dynamic programming as applied to the optimal control problem (for simplicity we considered the time-optimal control problem (2.8)).

2.4 The relation between Pontryagin's Maximum Principle and Bellman's Method of Dynamic Programming

The main difference between the calculus of variations methods and dynamic programming lies in emphasis (see [2], [3], [9], [13], [16], [17], [24], [28]). The former considers variations of the candidate extremizing curve, whereas in dynamic programming the candidate curve varies over a small initial interval and the remainder of the curve is supposed to be optimal for the other part of the problem. In other words, the concept of variation is to be found in both approaches. Which of the two techniques is more desirable depends entirely on the needs and point of view of the user. The calculus of variations yields results whose analytical forms are useful to theorists, and its main appeal perhaps lies in solving deterministic control problems with time treated as continuous, although there are attempts to discretize time [30]. On the other hand, others claim that dynamic programming is the more promising and powerful tool with wider applications in a variety of subjects [2]. It is certainly much more efficient than the calculus of variations in dealing with stochastic control problems involving multistage decision processes [9], [17].

In what follows we shall consider the relation existing between the maximum principle and R.Bellman's method of dynamic programming (see in 2.3 the derivation of the *Bellman's equation* from Pontryagin's maximum principle for the time-optimal control problem (2.8)). For a fuller discussion, see Dreyfus [9].

The method of dynamic programming was developed for the needs of optimal control processes which are of a much more general character than those which are describable by systems of differential equations. Therefore, the method of dynamic programming carries a more universal character than the maximum principle ([3],

[24]). However, in contrast to the latter, this method does not have the rigorous logical basis in all those cases where it may be successfully made use of as a valuable heuristic tool.

The basis of the method of dynamic programming given by Bellman rests on the assumption that to the natural conditions of the problem (see our Theorems 2.2.1 and 2.2.2) another essential requirement has been added - the requirement that the function $w(x)$ defined in Proposition 2.3.1 be differentiable (for a broader discussion see [24]). This assumption does not follow from the statement of the problem, and is a restriction which, as we shall see below, is not satisfied even in the simplest examples.

However, after this assumption has been made, the method of dynamic programming leads to a certain partial differential equation, which we call *Bellman's equation.* This equation (under certain additional conditions) is equivalent to the Hamiltonian system (2.14), (2.15), and to the maximum condition (2.16), (2.17).

In section 2.3 we showed the relation of Bellman's method of dynamic programming to Pontryagin's maximum principle (for a broader discussion see [16], [24]). For the sake of simplicity we only considered the time-optimal problem (2.8).

Proposition 2.4.1 *Let us assume that the function $\omega(x)$ is twice continuously differentiable. Then Pontryagin's maximum principle can be derived from Bellman's principle of dynamic programming.*

Proof Since $\omega(x)$ is twice continuously differentiable, the function

$$g(x,u) = \sum_{\alpha=1}^{n} \frac{\partial \omega(x)}{\partial x^\alpha} f^\alpha(x,u), \tag{2.30}$$

which stands under the supremum in (2.29), has continuous first derivatives with respect to $x^1, ..., x^n$. It follows from Bellman's principle of dynamic programming (see (2.26) and (2.29)) that if $u(t)$ is an optimal control which transfers the phase point from the position x_0 to the position x_1, and $x(t)$ is the corresponding optimal trajectory, then for a fixed t, $t_0 \leq t \leq t_1$, the function $g(x, u(t))$ of the variable $x \in \mathcal{X}$ attains its maximum value (unity) at the point $x = x(t)$. From this it follows that

$$\frac{\partial g(x(t), u(t))}{\partial x^i} = 0, \qquad i = 1, ..., n, \qquad t_0 \leq t \leq t_1. \tag{2.31}$$

Taking the form of the function $g(x, u)$ (see (2.30)) into account, we obtain the relations

$$\begin{aligned} &\sum_{\alpha=1}^{n} \frac{\partial^2 \omega(x(t))}{\partial x^\alpha \partial x^i} f^\alpha(x(t), u(t)) \\ &\qquad + \sum_{\alpha=1}^{n} \frac{\partial \omega(x(t))}{\partial x^\alpha} \cdot \frac{\partial f^\alpha(x(t), u(t))}{\partial x^i} = 0, \qquad i = 1, ..., n, \end{aligned} \tag{2.32}$$

which are satisfied along the optimal trajectory. Furthermore, we have

$$\sum_{\alpha=1}^{n} \frac{\partial^2 \omega(x(t))}{\partial x^\alpha \partial x^i} f^\alpha(x(t), u(t)) = \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \omega(x(t))}{\partial x^i} \right) \frac{dx^\alpha(t)}{dt} = \frac{d}{dt} \left(\frac{\partial \omega(x(t))}{\partial x^i} \right),$$

so that relations (2.32) may be rewritten in the form

$$\frac{d}{dt} \left(\frac{\partial \omega(x(t))}{\partial x^i} \right) = - \sum_{\alpha=1}^{n} \frac{\partial f^\alpha(x(t), u(t))}{\partial x^i} \cdot \frac{\partial \omega(x(t))}{\partial x^\alpha}, \qquad i = 1, ..., n.$$

Thus, along each optimal trajectory, the variables

$$\psi_i(t) = \frac{\partial \omega(x(t))}{\partial x^i}, \qquad i = 1, ..., n, \tag{2.33}$$

satisfy the linear system of differential equations

$$\frac{d\psi_i(t)}{dt} = - \sum_{\alpha=1}^{n} \frac{\partial f^\alpha(x(t), u(t))}{\partial x^i} \psi_\alpha(t), \qquad i = 1, ..., n. \tag{2.34}$$

In addition, because of relation (2.26), Bellman's equation (2.29) can be written in the form

$$\sum_{\alpha=1}^{n} \psi_\alpha(t) f^\alpha(x(t), u(t)) = \sup_{u \in \mathcal{U}} \sum_{\alpha=1}^{n} \psi_\alpha(t) f^\alpha(x(t), u) = 1. \tag{2.35}$$

Relations (2.34) and (2.35) coincide with Pontryagin's maximum principle, and relation (2.33) points out the relation between $\psi_i(t)$ and the function $\omega(x)$ in an explicit form. We also note, as follows from (2.35), that the optimal motions can always be realized in such a way that

$$H(\psi(t), x(t), u(t)) \equiv 1 \tag{2.36}$$

along optimal trajectories. We remind that all of these results can be obtained provided that the function $\omega(x)$ is twice differentiable. Without this additional assumption the proof of relation (2.36) loses its validity.

Let us give a simple example (see more examples in [16], [19], [24]) that shows that the function $\omega(x)$ does not have the first derivatives at the points which lie on the switching curves (this may be ascertained by direct calculations). Since every optimal trajectory passes along the switching curve during some time interval in this example, the assumption on the differentiability of $\omega(x)$ holds on none of the trajectories. Thus, even in the simplest examples, the assumptions which must be made in order to derive Bellman's equation do not hold.

Example *where Pontryagin's principle applies, but Bellman's fails because the control is discontinuous* (***Bang-Bang Problem***)

Consider the equation

$$\frac{d^2x}{dt^2} = u,$$

where u is a real control parameter constrained by the condition $|u| \leq 1$. The given equation can be rewritten using the phase coordinates $x^1 = x$ and $x^2 = dx/dt$. Hence, we get the following system:

$$\frac{dx^1}{dt} = x^2, \qquad \frac{dx^2}{dt} = u. \tag{2.37}$$

Let us consider (for a phase point moving in accordance with (2.37)) *the problem of getting to the origin* $(0, 0)$ *from a given initial state* x_0 *in the shortest time.* In other words, we shall consider the time-optimal problem for the case where the origin $(0, 0)$ is the terminal position x_1.

The Hamiltonian function $H(\boldsymbol{\psi}, x, u) = \sum_{\nu=1}^{n} \psi_\nu f^\nu(x, u)$ in this case has the form

$$H = \psi_1 x^2 + \psi_2 u. \tag{2.38}$$

Thus, since we know that

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x^i}, \qquad i = 1, ..., n,$$

(see equation (2.19)) we obtain the system of equations

$$\frac{d\psi_1}{dt} = -\frac{\partial H}{\partial x^1} = 0, \qquad \frac{d\psi_2}{dt} = -\frac{\partial H}{\partial x^2} = -\psi_1,$$

for the auxiliary variables ψ_1 and ψ_2. Hence, $\psi_1 = c_1$ and $\psi_2 = c_2 - c_1 t$ (c_1 and c_2 are arbitrary constants). Relation (2.20) yields (taking (2.38) and the condition $-1 \leq u \leq 1$ into account)

$$u(t) = sign\psi_2(t) = sign(c_2 - c_1 t). \tag{2.39}$$

It follows from (2.39) that *every optimal control* $u(t)$*,* $t_0 \leq t \leq t_1$*, is a piecewise constant function which takes on the values* ± 1*, and has at most two intervals on which it is constant* (since the linear function $c_2 - c_1 t$ changes sign at most once on the interval $t_0 \leq t \leq t_1$). Also, any such function $u(t)$ can be obtained from relation (2.39) for some values of c_1 and c_2.

From the system (2.37)

$$\frac{dx^1}{dt} = x^2, \qquad \frac{dx^2}{dt} = u$$

for the time interval on which $u \equiv 1$ we have

$$x^2 = t + s^2, \quad x^1 = \frac{t^2}{2} + s^2 t + s^1 = \frac{1}{2}\left(t + s^2\right)^2 + \left(s^1 - \frac{\left(s^2\right)^2}{2}\right)$$

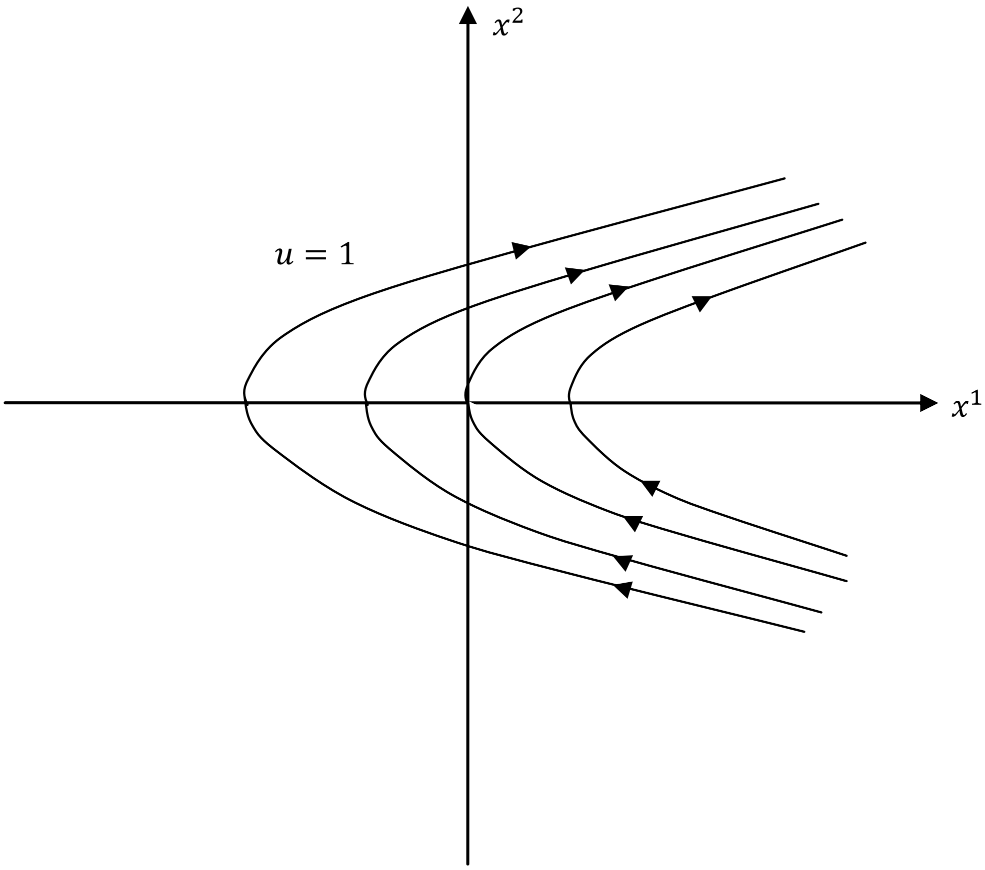

Figure 2.7: *Bang-bang time-optimal control: trajectories for $u = 1$ of parabolas given by equation (2.40)*

(s^1 and s^2 are constants of integration), from which we obtain

$$x^1 = \frac{1}{2}\left(x^2\right)^2 + s, \tag{2.40}$$

where $s = s^1 - \frac{1}{2}\left(s^2\right)^2$ is a constant. Thus, the portion of the phase trajectory for which $u \equiv 1$ is an arc of the parabola (2.40). The family of parabolas (2.40) is shown in Figure 2.7.

Analogously, for the time interval on which $u \equiv -1$, we have

$$x^2 = -t + s'^2,$$
$$x^1 = -\frac{t^2}{2} + s'^2 t + s'^1 = -\frac{1}{2}\left(-t + s'^2\right)^2 + \left(s'^2 + \frac{1}{2}\left(s'^2\right)^2\right),$$

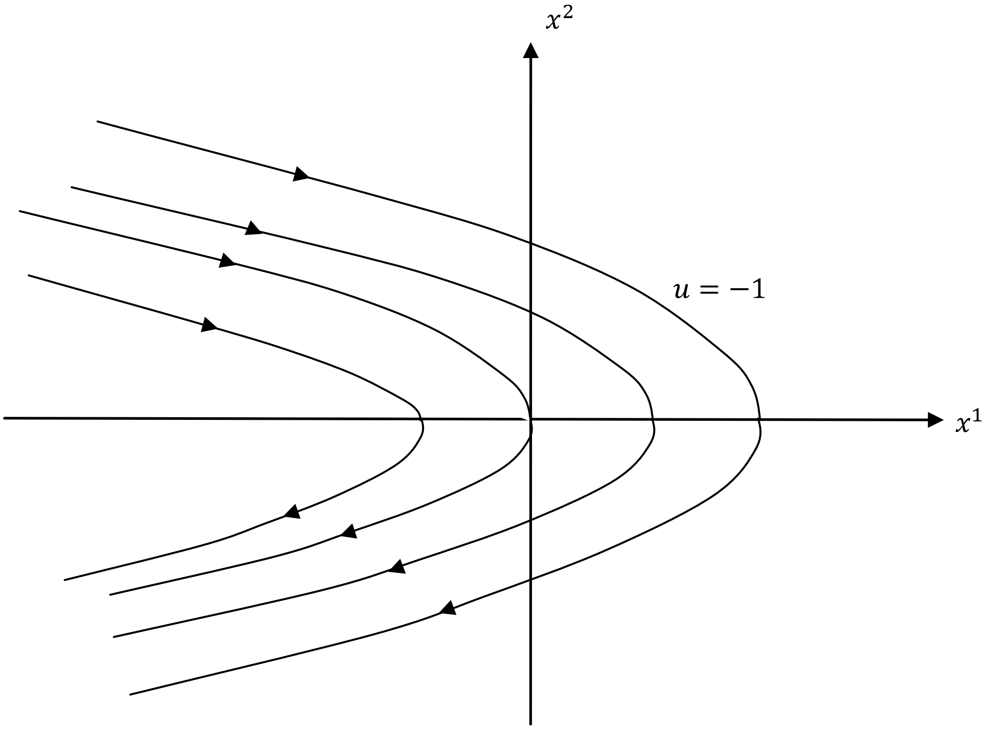

Figure 2.8: *Bang-bang time-optimal control: trajectories for $u = -1$ of parabolas given by equation (2.41)*

from which we obtain

$$x^1 = -\frac{1}{2}\left(x^2\right)^2 + s'. \tag{2.41}$$

The family of parabolas (2.41) is shown in Figure 2.8. The phase points move upwards along the parabolas (2.40) (since $dx^2/dt = u = +1$), and downwards along the parabolas (2.41) ($dx^2/dt = u = -1$).

As we said before, every optimal control $u(t)$ is a piecewise constant function, taking on the values ± 1, and having at most two intervals on which it is constant. If $u(t)$ is initially equal to $+1$, and then to -1, the phase trajectory consists of two adjoining parabolic segments (Figure 2.9). The second of these segments lies on that parabola defined by (2.41) which passes through the origin (since the desired trajectory must lead to the origin). On the other hand, if $u = -1$ first and $u = +1$

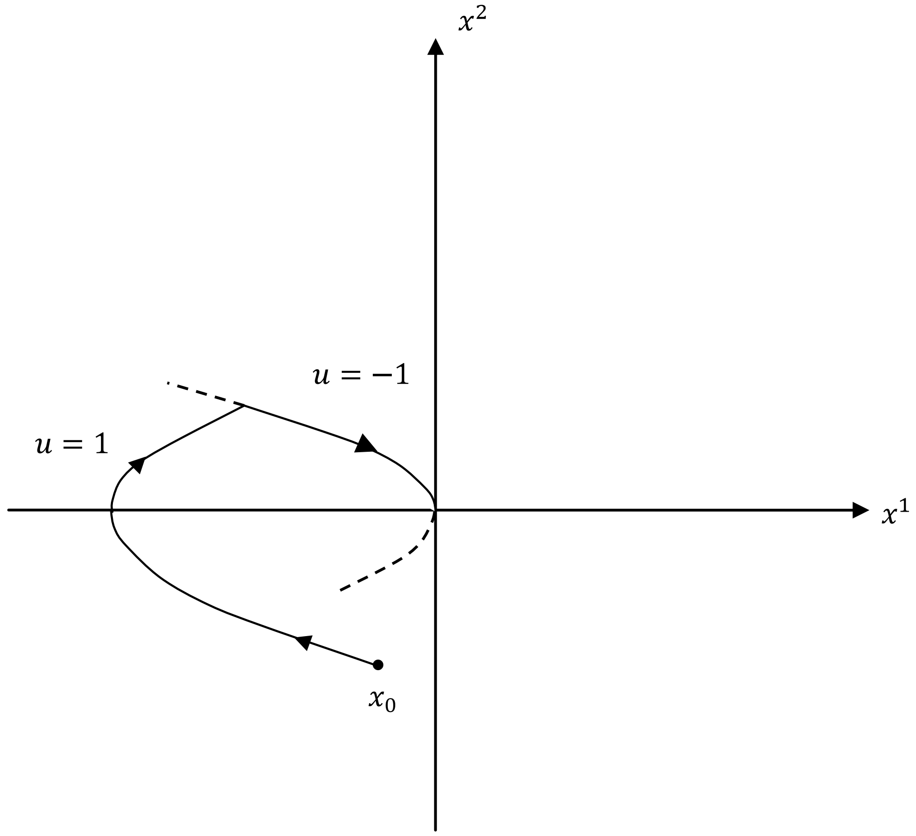

Figure 2.9: *Bang-bang time-optimal control: $u(t)$ is initially equal to $+1$, and then to -1, the phase trajectory consists of two adjoining parabolic segments given by equations (2.40) and (2.41), respectively*

afterwards, the phase curve is replaced by one which is symmetric with respect to the origin (Figure 2.10). In Figures 2.9, 2.10 the corresponding values of the control parameter u are written next to the parabolic arcs. Figure 2.11 shows the entire family of phase trajectories we obtained (AO is the arc of the parabola $x^1 = \frac{1}{2}\left(x^2\right)^2$ in the lower half-plane, BO is the arc of the parabola $x^1 = -\frac{1}{2}\left(x^2\right)^2$ in the upper half-plane). The phase point moves along an arc of the parabola (2.41) which passes through the initial points x_0, if x_0 is above the curve AOB; and along an arc of a parabola (2.40) if x_0 is below this curve. In other words, if the initial position x_0 is above the curve AOB, the phase point must move under the influence of the control

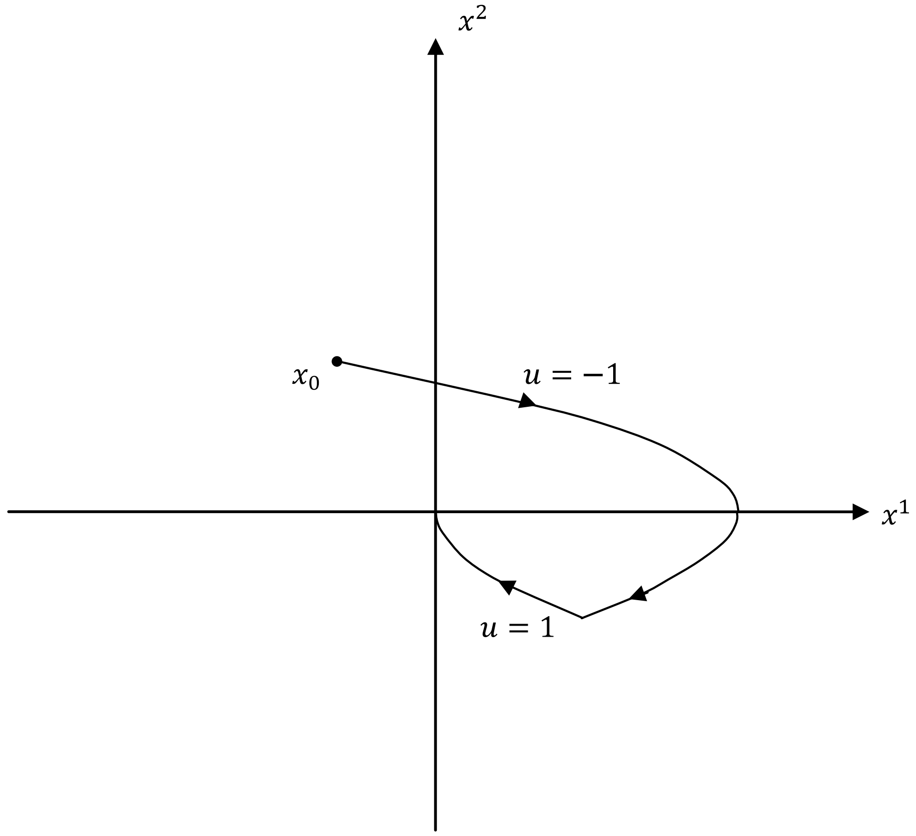

Figure 2.10: *Bang-bang time-optimal control: $u(t)$ is initially equal to -1, and then to $+1$, the phase trajectory consists of two adjoining parabolic segments given by equations (2.41) and (2.40), respectively*

$u = -1$ until it reaches the arc AO. At the instant it arrives, the value of u switches to $+1$ and remains at this value until the phase point reaches the origin. However, if the initial position x_0 is below AOB, u must equal $+1$ until the time it reaches the arc BO, and at that time the value of u changes to -1.

Definition A piecewise constant optimal control $u(t)$ that takes only two values on the boundary of the control space $\mathcal{U}$ is called a ***bang-bang control*** [19].

According to Theorem 2.2.2, *only the above described trajectories can be optimal.* Furthermore, it can be seen from the above investigation that from each point in

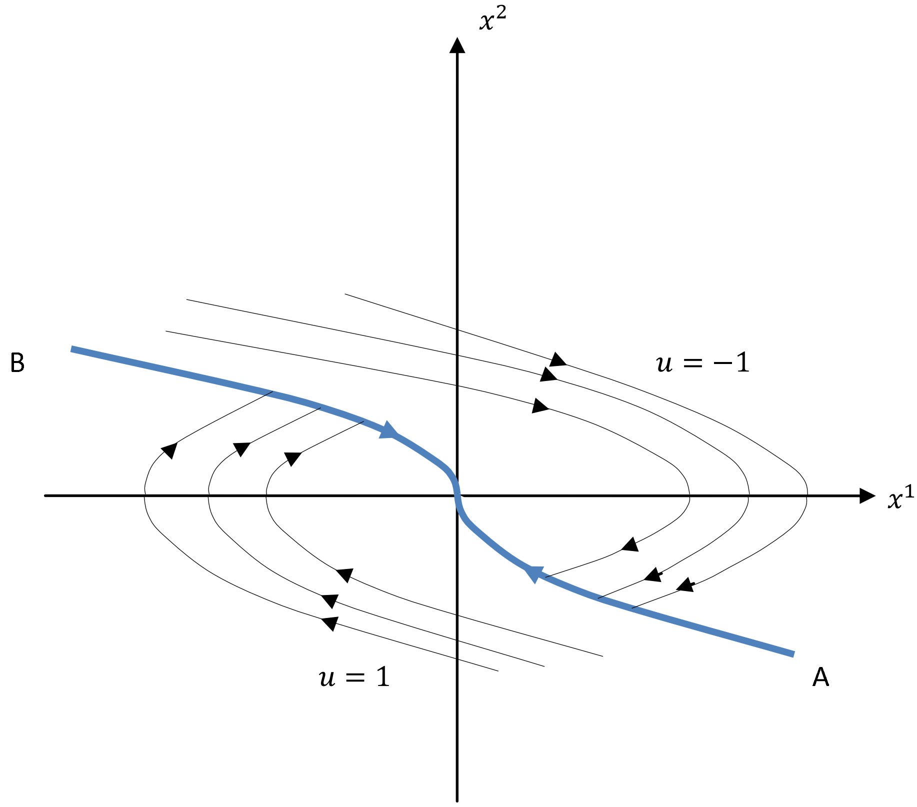

Figure 2.11: *Bang-bang time-optimal control: the switching curve and the family of phase trajectories we obtained (AO is the arc of the parabola* $x^1 = \frac{1}{2}(x^2)^2$ *in the lower half-plane, BO is the arc of the parabola* $x^1 = -\frac{1}{2}(x^2)^2$ *in the upper half-plane)*

the phase plane there is only one trajectory leading to the origin which can be optimal (i.e, once the initial point x_0 is given, the corresponding trajectory is uniquely determined). If we could be sure that the optimal trajectory always (i.e, for any initial point x_0) exists, we could confidently say that all the trajectories we have found are optimal (see [24], Chapter III for the formulation of the existence theorem for linear time-optimal systems). In particular, it follows from this theorem that in the present example there exists an optimal trajectory (see page 127 in [24]) for each initial point x_0. Thus, the trajectories we have found (Figure 2.11) are optimal, and there are no other optimal trajectories which lead to the origin.

Therefore, the solution of the optimal problem obtained in the above example can be interpreted as follows. Let $v(x^1, x^2) = v(x)$ be the function given in the x^1x^2 plane as follows:

$$v(x) = \begin{cases} +1 & \text{below the curve } AOB\text{, and on the arc } AO, \\ -1 & \text{above the curve } AOB\text{, and on the arc } BO. \end{cases}$$

Also, on each optimal trajectory the value $u(t)$ of the control parameter (at an arbitrary time t) is equal to $v(x(t))$, meaning that it equals the value of the function v at the point at which the phase point, moving along the optimal trajectory

$$u(t) = v(x(t)),$$

is located at the time t. This means that if we replace the variable u by the function $v(x)$ in the original system (2.37), we obtain the system

$$\begin{cases} dx^1/dt = x^2, \\ dx^2/dt = v\left(x^1, x^2\right). \end{cases} \tag{2.42}$$

We can find the optimal phase trajectory which leads to the origin from the solution of this system (2.42) (for an arbitrary initial state x_0). Therefore, we system (2.42) is the system of differential equations (with discontinuous right-hand side) for the determination of the optimal trajectories which lead to the origin.

Chapter 3

The Pursuit Problem

3.1 Statement of the Problem

Let us assume that two points, one of which we shall call "pursuing" (P) and the other "evading" (E), are moving in $\mathcal{X} \subset \mathbb{R}^n$:

$$x' = f(x, u, t), \quad y' = g(y, t), \tag{3.1}$$

where $u, \mathcal{U}$, and $x(t)$ are the control parameter, the control region, and the trajectory of the motion of the pursuing point P, respectively, and $y(t)$ is the trajectory of the motion of the evading point E.

Let $u(t)$ be a certain admissible control (i.e., piecewise continuous), and let $x(t)$ and $y(t)$ be the corresponding trajectories with initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{3.2}$$

If $x(t_1) = y(t_1)$ for some $t_1 > 0$, we shall call t_1 an *encounter time*, and the very occurrence that $x(t_1) = y(t_1)$ will be referred to as an *encounter*. If the control $u(t)$ is chosen arbitrarily, an encounter may not occur for any $t > 0$. If an encounter does occur, we shall call the control (which is an admissible control) $u(t)$ a *pursuing control*. Even then, for the given x_0, y_0, and the chosen control $u(t)$, more than one encounter may take place. We shall call the *smallest* positive number t_1, which is an encounter time, the *pursuit time* corresponding to the control $u(t)$. We shall denote the pursuit time by

$$T = \min_{u \in \mathcal{U}} T_u. \tag{3.3}$$

In what follows, the initial conditions (3.2) will be assumed to be fixed (in this connection, x_0 and y_0 do not enter into the notation for the pursuit time). Therefore, we get a statement of the ***pursuit problem***.

Definition The problem is called a ***pursuit problem*** if it is defined by equations (3.1) - (3.3)

$$x' = f(x, u, t), \quad y' = g(y, t),$$

$$x(0) = x_0, \quad y(0) = y_0,$$

$$T = \min_{u \in \mathcal{U}} T_u,$$

where x and y belong to $\mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^r$ and is admissible (piecewise continuous).

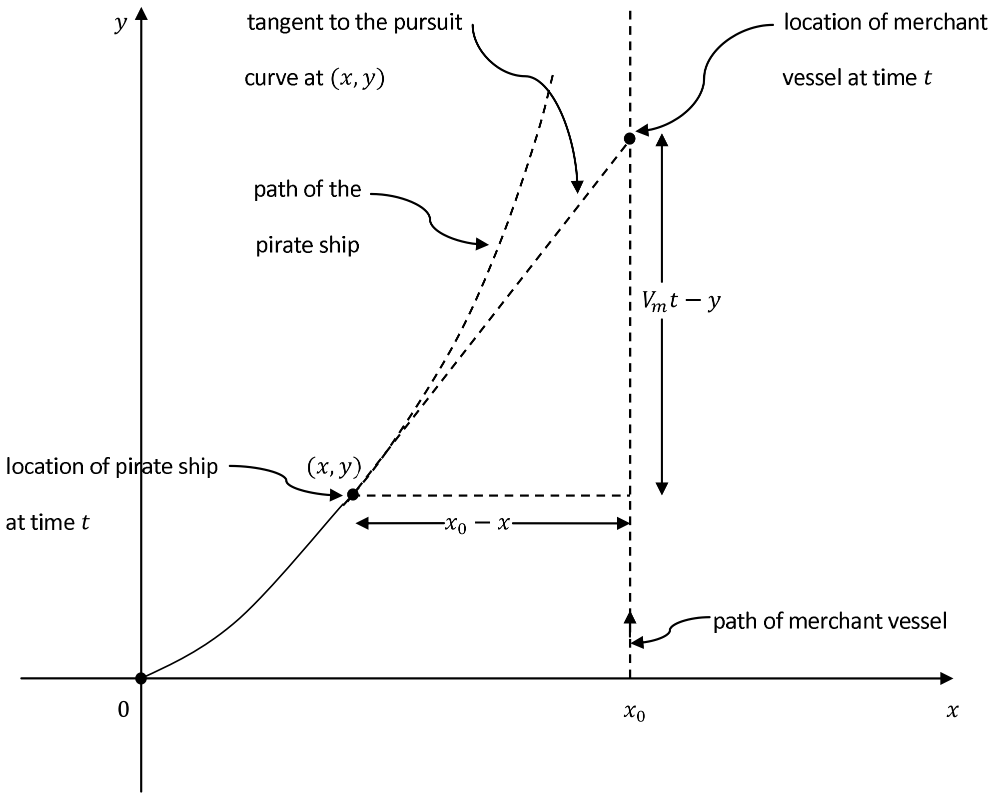

Figure 3.1: *The geometry of Bouguer's pursuit problem about a pirate ship moving directly toward the merchant vessel at constant speed* V_p *along a curved path and pursuing a merchant vessel travelling at constant speed* V_m *along the vertical line* $x = x_0$

3.2 Pierre Bouguer's Pursuit Problem

Modern mathematical pursuit analysis is generally assumed to begin with a problem posed and solved by the French mathematician and hydrographer Pierre Bouguer (1698-1758) in 1732 (see [5]). This general assumption is not quite correct, but Bouguer's problem is today nevertheless taken as the starting point of pursuit analysis in all modern textbooks. In his paper, Bouguer treated the case of pirate ship pursuing a fleeing merchant vessel, as illustrated in Figure 3.1. The pirate ship and the merchant vessel are taken to be at $(0, 0)$ and $(x_0, 0)$ at time $t = 0$, respectively, the instant the pursuit begins, with the merchant vessel travelling at constant speed V_m along the vertical line $x = x_0$. The pirate ship travels at constant speed V_p along a curved path such that *it is always moving directly toward the merchant*, that is, the velocity vector of the pirate ship points directly at the merchant vessel at every instant of time. Bouguer's problem was to determine the equation $y = y(x)$ of the curved path which he called the *line of pursuit*. The pursuit curve has its association with the path taken by a dog in following its master, and the falcon flying in its attack directly at the instantaneous location of its prey. This is the definition of what is now called *pure pursuit*.

To find the curve of pursuit for Bouguer's problem, start by calling the location of the pirate ship, at arbitrary time $t \geq 0$, the point (x, y). At time t the merchant vessel has sailed to the point $(x_0, V_m t)$ and so, as shown in Figure 3.1, the slope of the tangent line to the pursuit curve (the value of dy/dx at (x, y)) is given by

$$\frac{dy}{dx} = \frac{V_m t - y}{x_0 - x} = \frac{y - V_m t}{x - x_0}. \tag{3.4}$$

We also know that, whatever the shape of the pursuit curve, the pirate ship has sailed along it at time t by a distance of $V_p t$. From calculus we know that this arc-length is also given by the expression on the right below, and so

$$V_p t = \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz, \tag{3.5}$$

where z is simply a dummy variable of integration. Solving (3.4) and (3.5) each for t, we can write

$$\frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz = \frac{y}{V_m} - \frac{x - x_0}{V_m} \cdot \frac{dy}{dx},$$

which, if we let $dy/dx = p(x)$, becomes

$$\frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z) dz} = \frac{y}{V_m} - \frac{x - x_0}{V_m} \cdot p(x). \tag{3.6}$$

Differentiating (3.6) with respect to x (using Leibniz's formula to differentiate an integral), we arrive at

$$\frac{1}{V_p}\sqrt{1+p^2(x)} = \frac{1}{V_m}\cdot\frac{dy}{dx} - \frac{x-x_0}{V_m}\cdot\frac{dp}{dx} - \frac{1}{V_m}p(x)$$

or, simplifying,

$$(x-x_0)\frac{dp}{dx} = -\frac{V_m}{V_p}\sqrt{1+p^2(x)} = -n\sqrt{1+p^2(x)}, \tag{3.7}$$

where $n = V_m/V_p$. (Ordinarily we'll have $n < 1$, the pirate ship sailing faster than the merchant. For $n > 1$ the problem is without interest as then the pirate ship is slower than the merchant and the concept of "pursuit" is meaningless. The $n = 1$ case, however, *does* offer us a curious mathematical problem with special interest that we'll go into later.) Separating variables,

$$\frac{dp}{\sqrt{1+p^2}} = -\frac{ndx}{x-x_0} = \frac{ndx}{x_0-x} \tag{3.8}$$

and, integrating (3.8) indefinitely, we have (with C as the constant of indefinite integration)

$$\ln\left(p+\sqrt{1+p^2}\right) + C = -n\ln\left(x_0-x\right). \tag{3.9}$$

From Figure 3.1 we see at $t = 0$ that $p = dy/dx = 0$ when $x = 0$, because at that instant both ships are on the x-axis (the fact that $dy/dx|_{t=0} = 0$ also follows *mathematically* from (3.4) since $y(t=0) = 0$). Inserting these initial conditions into equation (3.9), it follows that $C = -nln(x_0)$ and so (3.9) becomes

$$\ln\left(p+\sqrt{1+p^2}\right) - n\ln(x_0) = -n\ln\left(x_0-x\right),$$

which, after a few steps of algebra, reduces to

$$\ln\left[\left(p+\sqrt{1+p^2}\right)\left(1-\frac{x}{x_0}\right)^n\right] = 0,$$

which tells us that

$$\left(p+\sqrt{1+p^2}\right)\left(1-\frac{x}{x_0}\right)^n = 1. \tag{3.10}$$

Thus,

$$p + \sqrt{1+p^2} = \frac{1}{\left(1 - \frac{x}{x_0}\right)^n} = q, \tag{3.11}$$

where q has been introduced to keep the next few algebraic steps easy to follow. Solving (3.11) for p, we have

$$\sqrt{1+p^2} = q - p,$$

$$1 + p^2 = (q-p)^2 = q^2 - 2qp + p^2,$$

$$p = \frac{q^2-1}{2q} = \frac{1}{2}\left[q - \frac{1}{q}\right].$$

Thus, replacing q with its equivalent (from (3.11)) gives

$$p(x) = \frac{dy}{dx} = \frac{1}{2}\left[\left(1 - \frac{x}{x_0}\right)^{-n} - \left(1 - \frac{x}{x_0}\right)^{n}\right], \qquad n = \frac{V_m}{V_p}. \tag{3.12}$$

We can solve (3.12) for $y(x)$ by simple integration, writing C once more as the constant of integration,

$$y(x) + C = \frac{1}{2}\int \frac{dx}{\left(1 - \frac{x}{x_0}\right)^n} - \frac{1}{2}\int \left(1 - \frac{x}{x_0}\right)^n dx.$$

In both integrals change variable to $u = 1 - x/x_0$ (so $dx = -x_0 du$) to get

$$y(x) + C = \frac{1}{2}\int \frac{-x_0 du}{u^n} - \frac{1}{2}\int -x_0 u^n du, \tag{3.13}$$

which immediately integrates to

$$y(x) + C = -\frac{1}{2}x_0 \frac{u^{-n+1}}{-n+1} + \frac{1}{2}x_0 \frac{u^{n+1}}{n+1}$$

$$= \frac{1}{2}x_0\left[\frac{u \cdot u^n}{1+n} - \frac{u \cdot u^{-n}}{1-n}\right] = \frac{1}{2}x_0 u\left[\frac{u^n}{1+n} - \frac{u^{-n}}{1-n}\right].$$

That is,

$$y(x) + C = \frac{1}{2}x_0\left(1 - \frac{x}{x_0}\right)\left[\frac{\left(1 - \frac{x}{x_0}\right)^n}{1+n} - \frac{\left(1 - \frac{x}{x_0}\right)^{-n}}{1-n}\right],$$

or

$$y(x) + C = \frac{1}{2}(x_0 - x)\left[\frac{\left(1 - \frac{x}{x_0}\right)^n}{1+n} - \frac{\left(1 - \frac{x}{x_0}\right)^{-n}}{1-n}\right]. \tag{3.14}$$

Since $y(x = 0)$, then

$$C = \frac{1}{2}x_0\left[\frac{1}{1+n} - \frac{1}{1-n}\right] = -\frac{n}{1-n^2}x_0$$

and so inserting this result into (3.14) gives us our answer, the ***pursuit curve equation*** $y = y(x)$:

$$y(x) = \frac{n}{1-n^2}x_0 + \frac{1}{2}(x_0 - x) \times \left[\frac{\left(1 - \frac{x}{x_0}\right)^n}{1+n} - \frac{\left(1 - \frac{x}{x_0}\right)^{-n}}{1-n}\right], \qquad n = \frac{V_m}{V_p}. \tag{3.15}$$

"Capture" occurs when $x = x_0$ (the pirate ship pursuit curve intersects the merchant's course), which says capture occurs at the point $(x_0, n/(1-n^2)x_0)$. (This makes physical sense only if $n < 1$, of course, the case of the pirate ship being faster than the merchant.) For example, if the pirate ship sails twice as fast as the merchant, then $n = \frac{1}{2}$ and capture occurs at the point $(x_0, \frac{2}{3}x_0)$, while if the pirate ship sails only one-third faster than the merchant (i.e., $V_p = \frac{4}{3}V_m$), then $n = \frac{3}{4}$ and capture occurs at the point $(x_0, \frac{12}{7}x_0)$. As n approaches one, that is, as the sailing speeds of the pirate ship and the merchant vessel become equal, it is clear that the capture point moves ever father up the $x = x_0$ line and, in the limit $n = 1$, the capture point is at infinity (which is the physically obvious statement that capture does *not* occur). Figure 3.2 shows the pursuit curve up to the capture point for the case of $x_0 = 1$ and $n = \frac{3}{4}$. The analytical expression of (3.15) fails to make sense for the case of $n = 1$ ($V_m = V_p$), of course, because then we have a division by zero problem. To see what the correct analytical form of the pursuit curve is for $n = 1$,

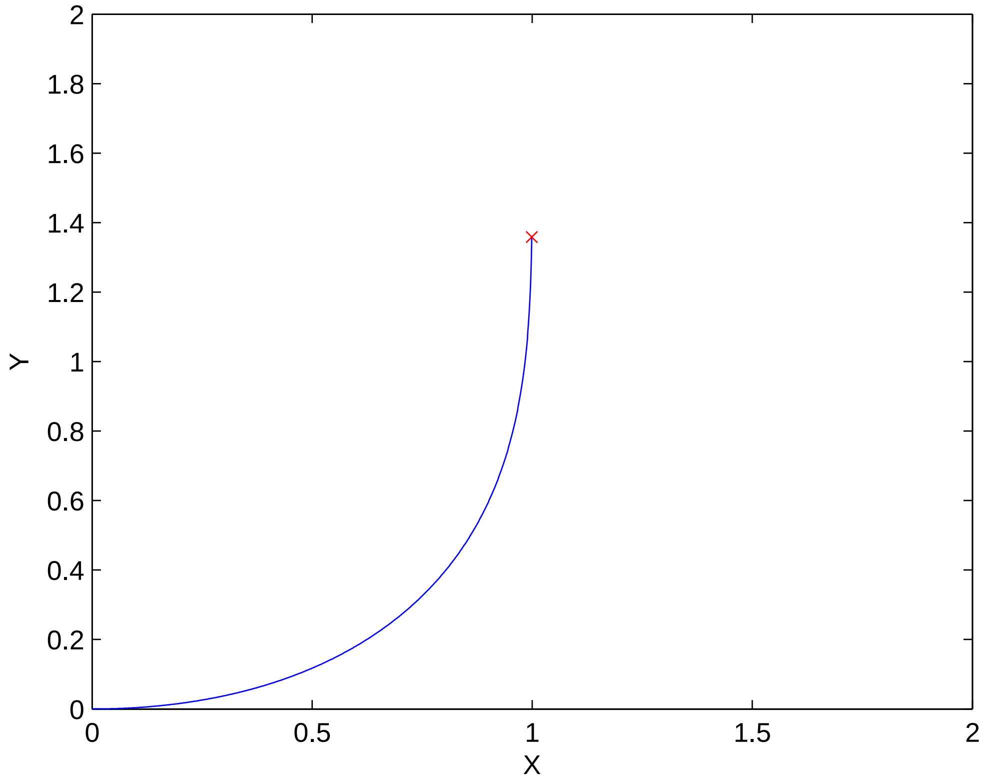

Figure 3.2: *The path of the pirate ship as given by equation (3.15) for* $n = 3/4$

return to (3.12), to just *before* we integrated dy/dx. Then

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(1 - \frac{x}{x_0}\right)^{-1} - \left(1 - \frac{x}{x_0}\right)\right] = \frac{1}{2}\left[\frac{1}{\left(1 - \frac{x}{x_0}\right)} - \left(1 - \frac{x}{x_0}\right)\right] \tag{3.16}$$

and so

$$y(x) + C = \frac{1}{2}\left[\int \frac{dx}{1 - \frac{x}{x_0}} - \int \left(1 - \frac{x}{x_0}\right) dx\right].$$

As before, change variables in both integrals to $u = 1 - x/x_0$ (and so $dx = -x_0 du$)

to get

$$
\begin{aligned}
y(x) + C &= \frac{1}{2}\int \frac{-x_0}{u} du - \frac{1}{2}\int u(-x_0) du \\
&= -\frac{1}{2} x_0 \ln u + \frac{1}{2} x_0 \cdot \frac{1}{2} u^2 \\
&= \frac{1}{2} x_0 \left[\frac{1}{2}\left(1 - \frac{x}{x_0}\right)^2 - \ln\left(1 - \frac{x}{x_0}\right)\right].
\end{aligned}
$$

Since $y(x = 0) = 0$, then $C = \frac{1}{4}x_0$, and so for $n = 1$ ($V_p = V_m$) the equation of the pursuit curve is

$$
y(x) = \frac{1}{2} x_0 \left[\frac{1}{2}\left(1 - \frac{x}{x_0}\right)^2 - \ln\left(1 - \frac{x}{x_0}\right)\right] - \frac{1}{4} x_0. \tag{3.17}
$$

When Bouguer's problem was included in the 1859 book *Treatise on Differential Equations* [4] by the famous British mathematician George Boole (1815 - 1864), the pursuit curve for $n = 1$ (pursuer and evader moving with equal speeds) case was declared to be a parabola, which is clearly wrong - as observed in Burton and Eliezer [1], whatever the pursuit curve is (for any value of n) it certainly must be asymptotic to the line $x = x_0$.

Now, for the $n < 1$ case let us calculate the total distance travelled by the pirate ship until its capture of the merchant vessel. As we discussed earlier, capture *does not* occur in the $n = 1$ case, and after a "long" time, the pirate ship will have sailed into a position directly behind the merchant and will simply chase, endlessly, after the merchant while remaining a *constant distance* behind it. It is an interesting mathematical problem to calculate the value of this so-called *tail chase lag distance.*

To calculate the distance sailed by the pirate ship until it captures the merchant vessel ($n < 1$), recall from (3.15) that capture occurs at $(x_0, n/(1 - n^2)x_0)$, i.e., the merchant vessel has travelled a distance of $n/(1-n^2)x_0$. Since the pirate ship travels

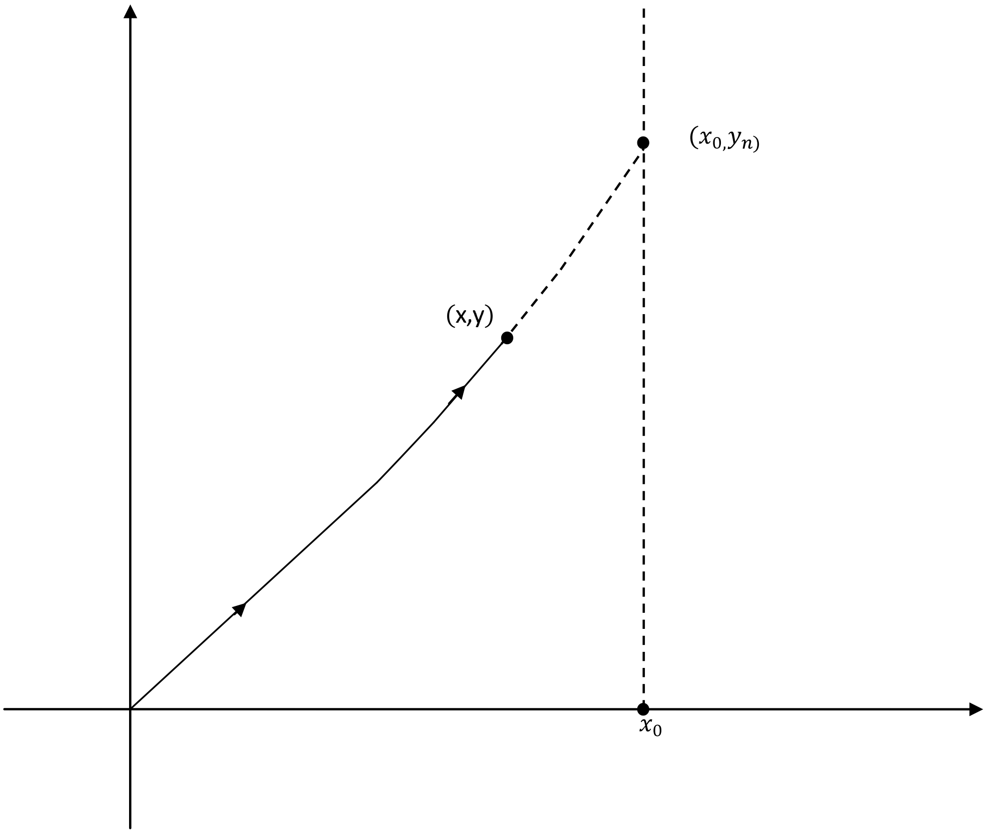

Figure 3.3: *The geometry of the tail chase as given by equation (3.17)*

$1/n$ times faster than does the merchant, the pirate travels $1/n$ times as far, that is, the pirate ship travels a total distance of $1/(1-n^2)x_0$.

To answer the second question, i.e., to determine the distance the pirate ship lags behind the merchant vessel after a long time has passed (for $n = 1$), refer to Figure 3.3. There we see the pirate ship at point (x, y), while the merchant vessel is at (x_0, y_n). Note, that this is for any arbitrary time t. The distance separating the pirate ship and the merchant vessel is D, where

$$D^2 = (y_m - y)^2 + (x_0 - x)^2 = (x_0 - x)^2 \left[1 + \left(\frac{y_m - y}{x_0 - x}\right)^2\right].$$

Now, here is an important fact: the line joining (x, y) to (x_0, y_m) is the tangent to the pirate's pursuit curve, because the chase is a *pure* pursuit, meaning that the pirate ship is moving directly at the instantaneous location of the merchant vessel (according to the statement of the problem), i.e., the velocity vector of the pirate ship points directly at the merchant vessel at every instant of time. Thus,

$$\frac{dy}{dx} = \frac{y_m - y}{x_0 - x}$$

and so

$$D^2 = (x_0 - x)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right].$$

Substituting (3.12) for dy/dx for the $n = 1$ case, that is, writing

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{\left(1 - \frac{x}{x_0}\right)} - \left(1 - \frac{x}{x_0}\right)\right],$$

we have

$$D^2 = (x_0 - x)^2 \left[1 + \frac{1}{4}\left\{\frac{1}{\left(1 - \frac{x}{x_0}\right)} - \left(1 - \frac{x}{x_0}\right)\right\}^2\right]$$

$$= x_0^2\left(1 - \frac{x}{x_0}\right)^2\left[1 + \frac{1}{4}\left\{\frac{1}{\left(1 - \frac{x}{x_0}\right)^2} - 2 + \left(1 - \frac{x}{x_0}\right)^2\right\}\right]$$

$$= x_0^2\left[\left(1 - \frac{x}{x_0}\right)^2 + \frac{1}{4} - \frac{1}{2}\left(1 - \frac{x}{x_0}\right)^2 + \frac{1}{4}\left(1 - \frac{x}{x_0}\right)^4\right].$$

As $t \to \infty$ we physically see the pirate ship pull into behind the merchant vessel and the pursuit becomes a vertically upward tail chase; thus, $x \to x_0$, and so

$$\lim_{t\to\infty} D^2 = \lim_{x\to x_0} D^2 = \frac{1}{4}x_0^2$$

or, at last,

$$\lim_{t\to\infty} D = \frac{1}{2}x_0.$$

Application of Bouguer's Pursuit Problem:

A merchant vessel, moving horizontal in a straight line, is b feet directly below one pirate ship "Black Pearl" and d feet directly above another pirate ship "Dead Men". Both pirate ships move directly toward the merchant vessel, reaching it simultaneously. We know that "Black Pearl" is slower than "Dead Men", and that "Dead Men" moves twice as fast as the merchant vessel. At what rate does the "Black Pearl" move?

We can see right away that the statements that "Black Pearl" is above the merchant vessel, and that "Dead Mean" is below, have nothing to do with the mathematics of the problem. Then, with no loss in the spirit of the problem, we can take the initial location of the "Black Pearl" as $(0, b)$ and of the "Dead Mean" as $(0, d)$. In our solution to Bouguer's problem, the initial separation between pursuer and pursued was x_0, and so b and d each play the role of x_0. We know from our earlier analysis that capture will occur after the evader has travelled distance of $n/(1-n^2)x_0$, where n equals the speed of the evader over the speed of the pursuer. For "Dead Men" we have $n = 1/2$, and for "Black Pearl" let's say it moves k times as fast as the merchant vessel (and so $n = 1/k$ for "Black Pearl"). Now, since both pursuers "capture" the vessel at the same instant (the same point) we have

$$\frac{1/2}{1-(1/2)^2}d = \frac{1/k}{1-(1/k)^2}b.$$

Hence,

$$\frac{1/2}{3/4}d = \frac{k}{k^2-1}b,$$

or

$$\frac{2}{3}d = \frac{k}{k^2-1}b,$$

where d, b, and k are some constants. Simplifying, we get

$$\frac{2}{3}dk^2 - bk - \frac{2}{3}d = 0,$$

$$k = \frac{b \pm \sqrt{b^2 + \frac{16}{9}d^2}}{\frac{4}{3}d}.$$

We realize that since "Black Pearl" starts closer to the vessel than does "Dead Men", k must be between one and two (the pirate ship "Black Pearl" must move faster than the merchant vessel to capture it, but slower than "Dead Men", according to the given condition). Hence, $k > 0$, and we use the plus sign. Therefore, "Black Pearl" moves $\frac{b+\sqrt{b^2+16/9d^2}}{4/3d}$ times as fast as the merchant vessel. Now, if we let, for example, $b = 50$ and $d = 100$, then we can find that "Black Pearl" moves 1.443 times as fast as the vessel.

Remark A different generalized form of Bouguer's problem was solved in Colman [8], in which the merchant vessel's straight sailing path is inclined from the vertical by angle α, i.e., the line $x = x_0$ is replaced by the straight line $y = (x - x_0) \cdot \cot \alpha$ for $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. Colman [8] does not give an explicit formula-equation for the flight path of the pursuer, but finds coordinates for the point of capture in the case when the ratio of pursuer's and evader's speeds is $n < 1$. The solution presented in this section is for $\alpha = 0$, while $\alpha = \pi/2$ radians would represent the merchant sailing directly *away* from the pirate ship (and $\alpha = -\pi/2$ radians would represent the merchant sailing directly *toward* the pirate ship). In both of these extreme cases the pursuit curve is, by inspection, simply $x = 0$ (the x-axis), but for $\alpha \neq \pm\pi/2$ or 0 the pursuit curve is pretty complicated, and its derivation is an exercise in nontrivial manipulation.

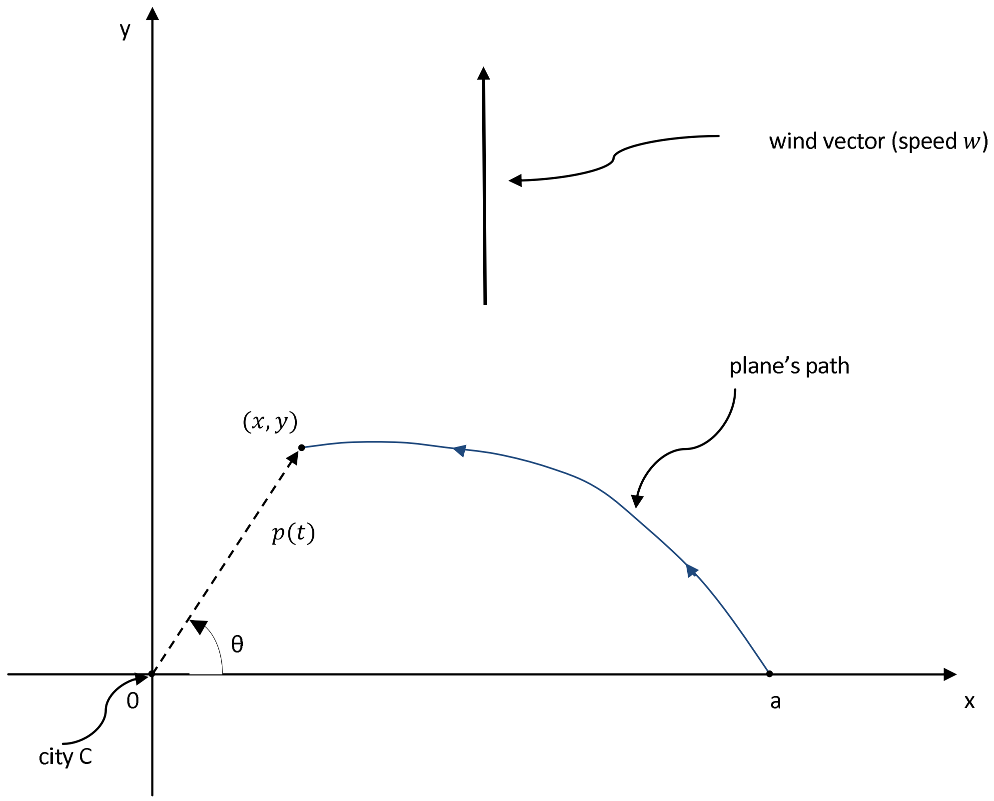

Figure 3.4: *The geometry of the wind-blown plane problem, where the plane's nose is always pointed toward a city C, the plane's speed is v mi/h, and a wind is blowing from the south at the rate of w mi/h*

3.3 Wind-Blown Plane Problem

Let us now present another important example (following [20]), where we use the analysis of the Bouguer's pursuit problem. It is similar to the problem solved in 1931 by E. Zermelo (see [31]).

A pilot always keeps the nose of his plane pointed toward a city C due west of his starting point at $(a, 0)$. Find equation of the plane's path if the plane's speed is v mi/h, and a wind is blowing from the south at the rate of w mi/h.

The pilot isn't really pursuing anything, of course, unless we consider this problem of "pursuit (with wind interference) of a stationary target", but the spirit of this problem is pure Bouguer.

In the notation of Figure 3.4, at an arbitrary time $t \geq 0$, the plane's location is the point $(x(t), y(t))$. Writing $\mathbf{u}_x$ and $\mathbf{u}_y$ as the unit vectors in the x and y directions (which are not functions of time), respectively, then we can write the *position vector* of the plane as

$$\mathbf{p}(t) = x(t)\mathbf{u}_x + y(t)\mathbf{u}_y,$$

and so the plane's *velocity vector* is

$$\frac{d}{dt}\mathbf{p}(t) = \frac{dx}{dt}\mathbf{u}_x + \frac{dy}{dt}\mathbf{u}_y.$$

Also, the plane's body axis (nose-to-tail) is always along the direction of $\mathbf{p}(t)$, at angle θ, toward C, where

$$\tan(\theta) = \frac{y}{x}.$$

The wind, blowing only along the y-axis, contributes nothing to the $\mathbf{u}_x$ component of the plane's velocity vector, that is, dx/dt is due only to the x-component of v.

$$\frac{dx}{dt} = -v\cos(\theta) = -\frac{vx}{\sqrt{x^2 + y^2}}, \tag{3.18}$$

where the minus sign is explicitly included, since as the plane flies toward C the value of x decreases with increasing t. The $\mathbf{u}_y$ component of the plane's velocity vector, on the other hand, is influenced by the wind, of course, as well as by the y-component of v,

$$\frac{dy}{dt} = w - v\sin(\theta) = w - \frac{vy}{\sqrt{x^2 + y^2}} = \frac{w\sqrt{x^2 + y^2} - vy}{\sqrt{x^2 + y^2}}. \tag{3.19}$$

Dividing (3.19) by (3.18), we eliminate explicit time and arrive at

$$\frac{dy}{dx} = \frac{vy - w\sqrt{x^2 + y^2}}{vx}. \tag{3.20}$$

Let us introduce a new variable z such that $y = zx$. Then (3.20) becomes

$$\frac{dy}{dx} = z + x\frac{dz}{dx} = \frac{vzx - w\sqrt{x^2 + z^2x^2}}{vx} = z - \frac{w}{v}\sqrt{1+z^2}$$

or, defining the *constant* $n = w/v$,

$$x\frac{dz}{dx} = -n\sqrt{1+z^2}, \tag{3.21}$$

from where we get

$$\frac{dz}{\sqrt{1+z^2}} = -n\frac{dx}{x}. \tag{3.22}$$

(Notice the similarity of (3.22) and (3.8).) Integrating indefinitely, with C as the constant of integration,

$$\ln[(z + \sqrt{1+z^2})] + C = -n\ln(x).$$

Since $y = 0$ when $x = a$, which means $z = y/x = 0$ when $x = a$, then we have $C = -n\ln(a)$, and so

$$\ln[(z + \sqrt{1+z^2})] = n\ln(a) - n\ln(x) = n\ln\left(\frac{a}{x}\right) = \ln\left(\frac{a}{x}\right)^n,$$

or,

$$z + \sqrt{1+z^2} = \left(\frac{a}{x}\right)^n. \tag{3.23}$$

Defining $q = (a/x)^n$, (3.23) becomes (similar to how we went from (3.11) to (3.12))

$$\sqrt{1+z^2} = q - z,$$

$$1 + z^2 = q^2 - 2qz + z^2,$$

$$z = \frac{q^2-1}{2q} = \frac{1}{2}\left[q - \frac{1}{q}\right].$$

Thus, replacing q with its definition,

$$z = \frac{1}{2}\left[\left(\frac{a}{x}\right)^n - \left(\frac{a}{x}\right)^{-n}\right] = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-n} - \left(\frac{x}{a}\right)^n\right]. \tag{3.24}$$

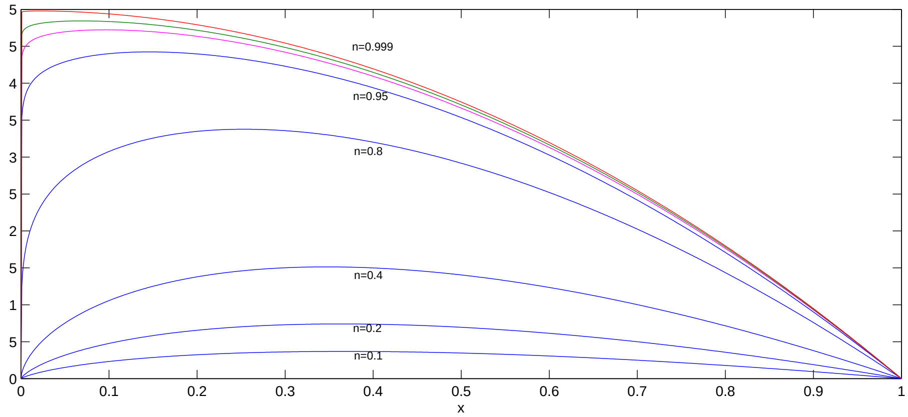

Figure 3.5: *Plots of the wind-blown plane's paths given by equations (3.25) for several values of* $n < 1$ *(n = 0.1, 0.2, 0.4, 0.8, 0.95, 0.99, 0.999)*

Since $y = zx$, then

$$y = \frac{1}{2}\left[\frac{x^{-n+1}}{a^{-n}} - \frac{x^{n+1}}{a^{n}}\right] = \frac{1}{2}\left[\frac{x^{-n+1}}{\frac{a^{-n+1}}{a}} - \frac{x^{n+1}}{\frac{a^{n+1}}{a}}\right]$$

or, at last, we have the ***equation of the wind-blown plane's path***:

$$y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{-n+1} - \left(\frac{x}{a}\right)^{n+1}\right], \qquad n = \frac{w}{v}. \tag{3.25}$$

When $n = 0$ - that is, when there is no wind - (3.25) collapses to the physically obvious $y(x) = 0$, which simply says that the plane moves directly to city C while always remaining on the x-axis. And when $n = 1$ (when the wind speed equals the plane's speed in still air), the plane's path is the *parabola*

$$y(x) = \frac{a}{2}\left[1 - \left(\frac{x}{a}\right)^{2}\right].$$

In this case when $x = 0$ we see that $y(0) = a/2$, that is, the plane does not reach city C. This probably makes intuitive sense, too, but it is interesting to see that the miss distance is so large. What happens, physically, in the $n = 1$ case, is that the plane arrives at the y-axis with a zero velocity component in the x-direction (notice that the plane's body axis has rotated through an angle of $\theta = 90°$, and then recall

(3.18)), and so there the plane remains, motionless at the point $(0, a/2)$, as it flies directly into the wind with the two equal magnitude but oppositely directed velocity vectors precisely cancelling each other. Figure 3.5 shows the plane's path for $a = 1$ for several different values of n, and it is clear that for $n < 1$ (≥ 1) the plane reaches (does not reach) city C.

Now, let us calculate the *total flight time* of the wind-blown plane for $n < 1$, and the *total distance* flown for the case of n "just less" than one.

For the total flight time T of the wind-blown plane, recall (3.18) and (3.25), where we showed that

$$\frac{dx}{dt} = -\frac{vx}{\sqrt{x^2+y^2}}$$

and

$$y = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{-n+1} - \left(\frac{x}{a}\right)^{n+1}\right], \qquad n = \frac{w}{v}.$$

So,

$$\int_0^T dt = -\int_a^0 \frac{\sqrt{x^2+y^2}}{vx}dx,$$

or

$$T = \frac{1}{v}\int_a^0 \sqrt{1+\frac{y^2}{x^2}}dx.$$

Also,

$$y^2 = \frac{a^2}{4}\left[\left(\frac{x}{a}\right)^{-2n+2} - 2\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^{2n+2}\right]$$

$$= \frac{a^2}{4}\left[\left(\frac{x}{a}\right)^{-2n}\frac{x^2}{a^2} - 2\frac{x}{a^2}^2 + \left(\frac{x}{a}\right)^{2n}\frac{x^2}{a^2}\right].$$

Thus,

$$\frac{y^2}{x^2} = \frac{1}{4}\left[\left(\frac{x}{a}\right)^{-2n} - 2 + \left(\frac{x}{a}\right)^{2n}\right],$$

and so

$$1 + \frac{y^2}{x^2} = 1 + \frac{1}{4}\left[\left(\frac{x}{a}\right)^{-2n} - 2 + \left(\frac{x}{a}\right)^{2n}\right]$$

$$= \frac{(x/a)^{-2n} - 2 + (x/a)^{2n} + 4}{4}$$

$$= \frac{(x/a)^{-2n} + 2 + (x/a)^{2n}}{4} = \left\{\frac{(x/a)^{n} + (x/a)^{-n}}{2}\right\}^2.$$

We can then write T as

$$T = \frac{1}{2v}\int_0^a \left[\left(\frac{x}{a}\right)^{n} + \left(\frac{x}{a}\right)^{-n}\right] dx$$

$$= \frac{1}{2v}\left[\int_0^a \left(\frac{x}{a}\right)^{n} dx + \int_0^a \left(\frac{x}{a}\right)^{-n} dx\right].$$

Letting $u = x/a$ ($dx = adu$), we then have

$$T = \frac{1}{2v}\left[\int_0^1 u^n adu + \int_0^1 u^{-n} adu\right] = \frac{a}{2v}\left[\frac{u^{n+1}}{n+1} + \frac{u^{-n+1}}{-n+1}\right]\Bigg|_0^1$$

$$= \frac{a}{2v}\left(\frac{1}{1+n} + \frac{1}{1-n}\right) = \frac{a/v}{1-n^2}, \qquad n = \frac{w}{v}.$$

This makes sense for $0 \le n < 1$. Notice that if $n = 0$ (no wind) then $T = a/v$, which is simply the time the plane requires to fly straight along the x-axis from $(a, 0)$ to $(0, 0)$ at a speed v. As n approaches one from below, of course, we see $T \to \infty$ as expected.

For the *total distance* flown by the plane when n is "just less" than one, that is, for the case where the plain "just managers" to reach city C, recall that at $n = 1$ the plane's path is the parabola

$$y = \frac{a}{2}\left[1 - \left(\frac{x}{a}\right)^2\right].$$

As n approaches one, then, the upward curved part of the flight path of the plane approaches this parabola, as illustrated in Figure 3.5. Let us look at the three plots which are for $n = 0.98$, $n = 0.99$, and $n = 0.999$, all for $a = 1$. From these curves it should be clear that the length of the longest flight path that just manages to reach city C is bounded from above by

$$\frac{a}{2} + \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

where the second term is the length of the parabolic arc. The first term, of course, is the length of the final leg of the journey back down along (almost along) the vertical axis to city C at the origin. On the parabolic arc we have

$$\frac{dy}{dx} = -\frac{a}{2} 2 \left(\frac{x}{a}\right) \frac{1}{a} = -\frac{x}{a},$$

and so our answer is

$$\frac{a}{2} + \int_0^a \sqrt{1 + \left(\frac{x}{a}\right)^2} dx.$$

If we change variables to $u = x/a$ ($dx = adu$), our answer becomes

$$\frac{a}{2} + \int_0^a \sqrt{1 + (u)^2} adu = a \left[\frac{1}{2} + \int_0^1 \sqrt{1 + u^2} du\right]$$

or,

$$a \left[\frac{1}{2} + \left\{\frac{u\sqrt{u^2+1}}{2} + \frac{1}{2} \ln\left(u + \sqrt{u^2+1}\right)\right\}\right]\Bigg|_0^1$$

$$= a \left[\frac{1 + \sqrt{2} + \ln(1 + \sqrt{2})}{2}\right] = 1.6478a.$$

This is the total distance flown by the plane when $n = 1 - \varepsilon$, where $\varepsilon > 0$, but is arbitrary small.

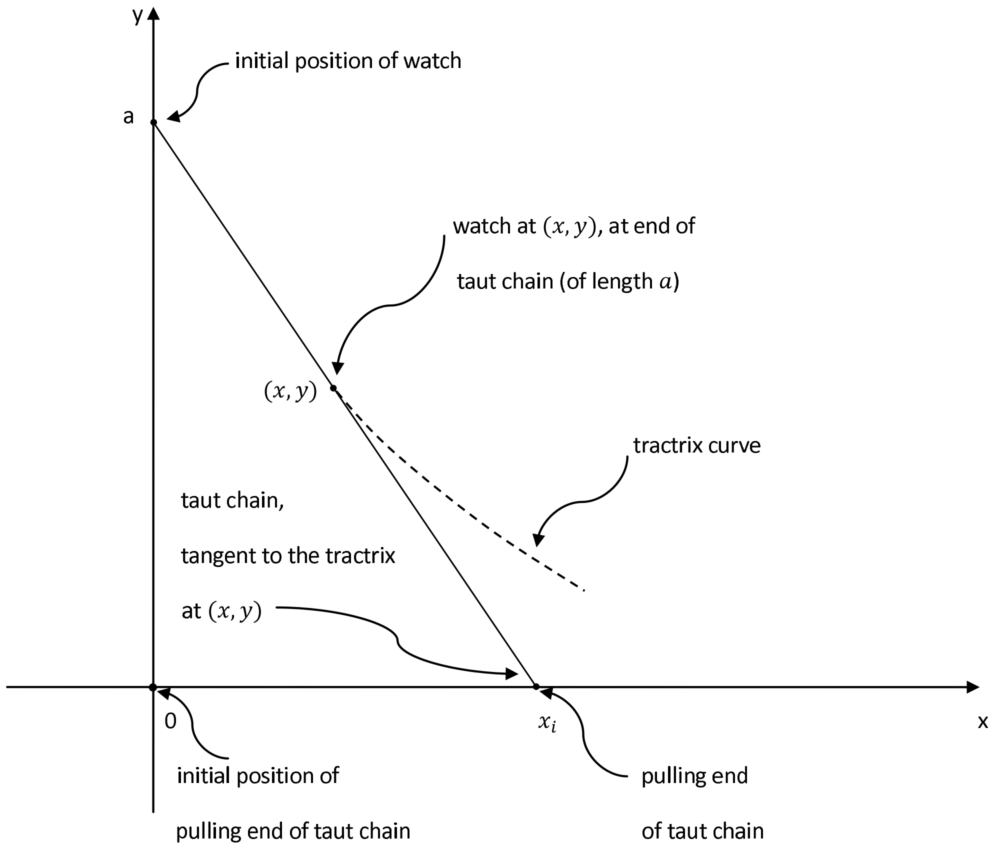

Figure 3.6: *The geometry of the tractrix problem, where a watch-on-a-chain with the chain of length a is initially on the y-axis, the end of the chain is pulled along the x-axis from the initial position on the origin*

3.4 The Tractrix

In the late seventeenth century there was also a different pursuit curve (as you will see, it is better to call this curve the *following* curve or the *tailing* curve) [20]. An example of such a problem (with the tailing curve) is illustrated in Figure 3.6, where a watch-on-a-chain has been laid out on a table-top with the chain (of length a) pulled out. In our thesis we follow the statement given in [20]. The watch is initially on the y-axis, and the other end of the chain is on the origin. If the end on the origin is then pulled along the x-axis, the watch will obviously be dragged along. We are interested in the *equation of the watch's path*, known as the *tractrix*. It was first introduced by Claude Perrault in 1670, and later studied by Sir Isaac Newton (1676) and Christian Huygens (1692) [18].

If (x, y) is the location of the watch at some arbitrary time $t \geq 0$, then it is clear that the taut chain is tangent to the tractrix at (x, y). This crucial observation allows us to calculate where the pulling end of the taut chain is (always on the x-axis), as follows. The slope of the tangent line is dy/dx and so, from analytic geometry, we have the equation of the tangent line as

$$y = x\frac{dy}{dx} + b, \tag{3.26}$$

where b is some constant. Let x_i be the value of x where the pulling end of the chain is located, by definition $y = 0$ there. So,

$$b = -x_i\frac{dy}{dx},$$

and therefore, the equation of the tangent line that intersects the x-axis at $x = x_i$ is

$$y = x\frac{dy}{dx} - x_i\frac{dy}{dx} = (x - x_i)\,\frac{dy}{dx}. \tag{3.27}$$

From the Pythagorean theorem we then have

$$(x - x_i)^2 + y^2 = a^2,$$

or, using (3.27) to solve for $(x - x_i)$, we have

$$\frac{y^2}{(dy/dx)^2} + y^2 = a^2,$$

or

$$\left[\frac{y}{dy/dx}\right]^2 = a^2 - y^2. \tag{3.28}$$

Taking the positive square root of both sides of (3.28), and noting that dy/dx is negative (look at Figure 3.6 again), we arrive at

$$-\frac{y}{dy/dx} = \sqrt{a^2 - y^2}, \tag{3.29}$$

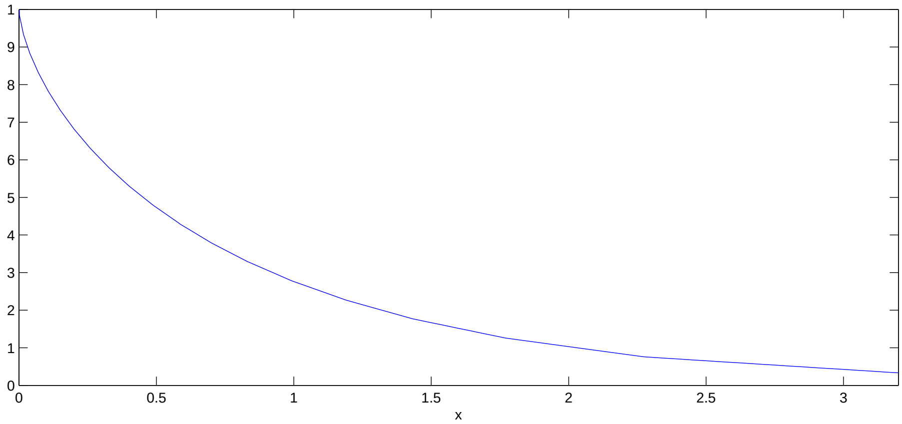

Figure 3.7: *A depiction of the tractrix given by equation (3.31) for* $a = 1$

a differential equation in which we can separate the variables. That is,

$$dx + \frac{\sqrt{a^2 - y^2}}{y} dy = 0. \tag{3.30}$$

Integrating indefinitely (with C as the arbitrary constant), we have

$$x + \sqrt{a^2 - y^2} - a \ln \left(\frac{a + \sqrt{a^2 - y^2}}{y} \right) = C.$$

Since $y(x = 0) = a$, we have $C = 0$ and so the equation of the watch's path as it is being dragged is

$$x = a \ln \left(\frac{a + \sqrt{a^2 - y^2}}{y} \right) - \sqrt{a^2 - y^2}. \tag{3.31}$$

Figure 3.7 shows the tractrix of (3.31) for the case of $a = 1$.

Finally, it is interesting to contrast the tractrix with Bouguer's pure pursuit curve for the special case of equal speeds for the pirate ship and the merchant vessel. The two curves seemingly have a common property, as the dragged watch is a constant distance from the pulled end of the chain, and the pirate ship ends up a constant distance behind the fleeing merchant vessel. The expression of (3.17) and (3.31) are quite different. The reason is that for the tractrix the constant lag of the watch is *always* the case, while the constant lag of the pirate ship is an *asymptotic* property that develops with the passage of time.

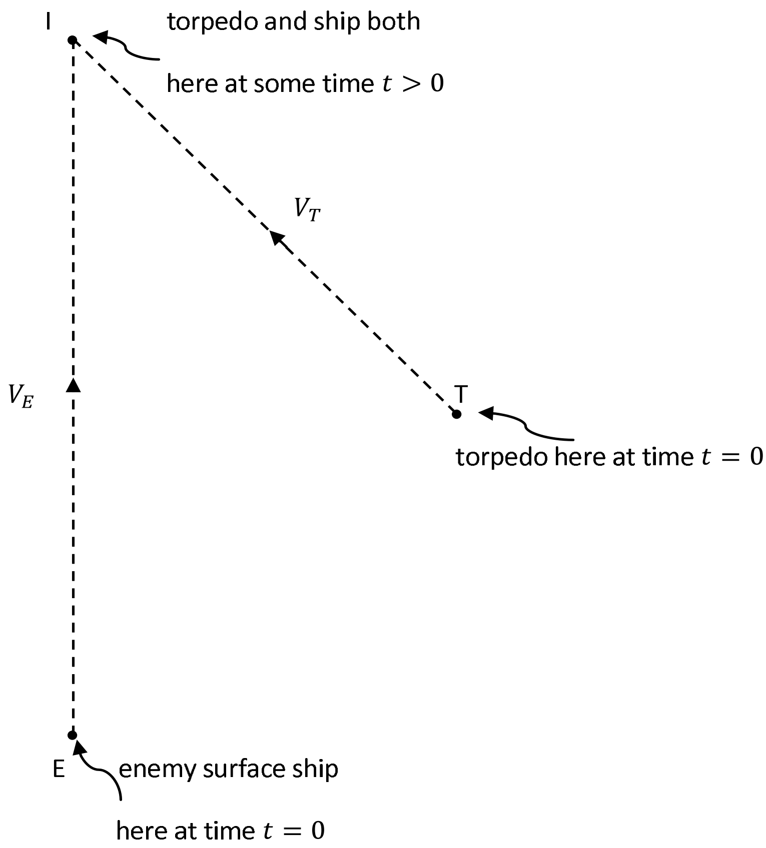

Figure 3.8: *Schematic of the pursuit by interception problem with pursuer* $\boldsymbol{T}$ *(Torpedo) and evader* $\boldsymbol{E}$ *(Enemy ship) moving with constant speeds* $V_{\boldsymbol{T}}$ *and* $V_{\boldsymbol{E}}$*, respectively*

3.5 Apollonius Pursuit Problem

In this section we talk about a question that you may have already thought about - since the merchant vessel being pursued by Bouguer's pirate ship always sails along a straight line, why does the pirate use *pure pursuit* (meaning that the pirate ship is moving directly at the instantaneous location of the merchant vessel) to run down his victim? Why doesn't the pirate ship simply sail along the straight line path that will *intercept* the merchant? Bouguer himself was not oblivious to that possibility. As Puckette [26] puts it, "[Bouguer] makes it quite clear that the pursuing ship could catch its quarry much more quickly by 'heading it off' than by merely following it (assuming the line of flight remains a straight line)".

There are at least two answers to that question (for a broader discussion see [20]). First, of course, the pure pursuit problem is simply interesting from a *mathematical* point of view. And second, if the merchant vessel deviates from its straight path and starts executing an active evasion plan, then the pirate ship is going to have to recalculate its intercept course continually anyway. A pure pursuit strategy is just one way to specify how to do repetitive new course calculations. And, in any case, even for the merchant vessel sticking to a straight line escape path, determining the intercept course for the pirate ship is a nontrivial calculation. In the days of submarine warfare in World War II [20], for example, this was a most practical problem - submarines fired their torpedoes on *intercept* courses at unsuspecting, that is, nonmaneuvering, enemy surface ships. Today, it isn't such an important problem because, unlike the torpedoes from yesteryear, modern torpedoes use what is called "active tracking", that is, they have onboard sensors and computers that continually locate the target no matter how that target moves. Still, the mathematics of interception remains elegant.

Let us suppose that the torpedo **T** is to intercept an enemy surface ship **E** (as shown in Figure 3.8), with **E** moving on a straight path and **T** moving on a straight path to intercept **E** at point **I**. If we assume that **E** and **T** move with constant speeds $V_{\mathbf{E}}$ and $V_{\mathbf{T}}$, respectively, then at the intercept point **I** the ratio of the two distances travelled from the instant of the torpedo firing must equal the ratio of the two speeds,

$$\frac{IT}{IE} = \frac{V_{\mathbf{T}}}{V_{\mathbf{E}}} = k, \tag{3.32}$$

where k is a constant ($k > 1$ is the usual case, but the $k < 1$ case will be of interest to us, too, before we are done).

Equation (3.32) is the mathematical statement of the physically obvious fact

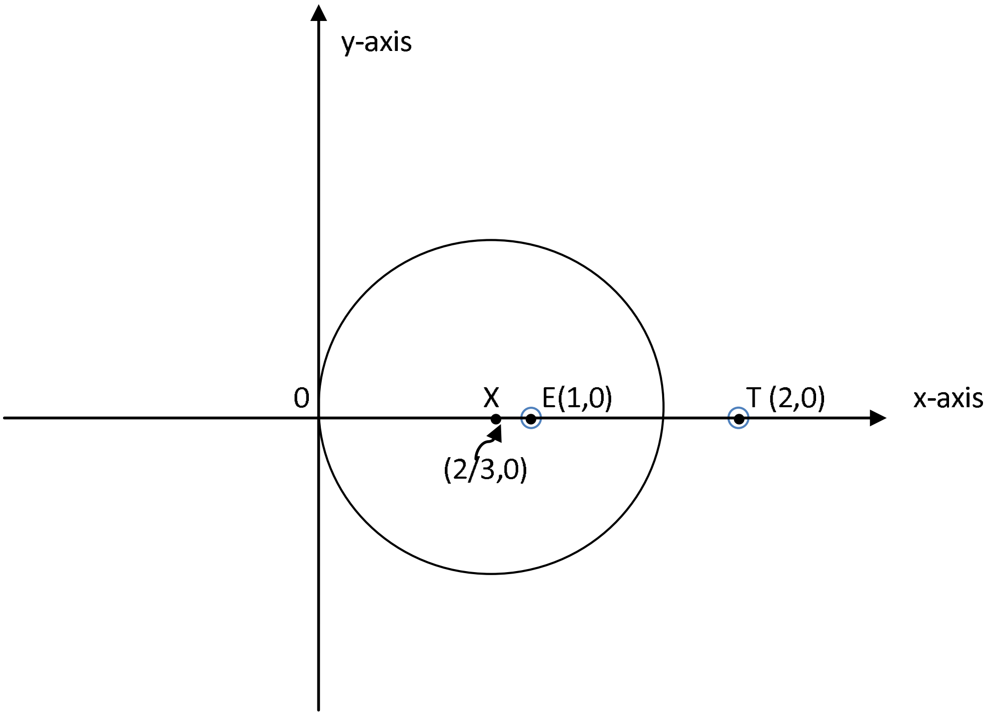

Figure 3.9: *The Apollonius circle centered on* $(2/3, 0)$ *with radius* $2/3$*, given by equation (3.34) for* $m = 1$*,* $p = 2$*, and* $k = 2$*, so that the torpedo is located at* $\boldsymbol{T}(2, 0)$ *and the enemy ship is at* $\boldsymbol{E}(1, 0)$

that, for an *interception* to occur, the torpedo and the ship must reach point **I** *simultaneously.* It is not enough for **E** and **T** to pass thorough **I** individually - *they must be at* ***I*** *at the same time.* To find where **I** is, given the locations of **E** and **T** at time $t = 0$, the two speeds $V_{\mathbf{E}}$ and $V_{\mathbf{T}}$, and the direction of **E**'s motion (the "heading" of **E**), what we must do first is find the set S of all the points in the plane such that (3.32) is satisfied. The point **I** can be any one of the points (there can be more than one) in S that also lie on the path of **E**.

Now we need to identify, what is S. With no loss in generality we can draw a rectangular coordinate system such that **E** and **T** are both, at $t = 0$, on the positive horizontal axis with **T** to the right of **E** (see Figure 3.9). If we denote the coordinates of **E** and **T** by $(m, 0)$ and $(p, 0)$, respectively, with $p > m$ (we use m and p to retain a link with our original discussion of Bouguer's merchant vessel and pirate ship) and if (x, y) is any point in S, then (3.32) becomes

$$\frac{\sqrt{(x-p)^2+y^2}}{\sqrt{(x-m)^2+y^2}} = k. \tag{3.33}$$

If you now go through a few algebraic manipulations, then you should be able to confirm that (3.33) can be written as

$$\left[x - \frac{k^2m-p}{k^2-1}\right]^2 + y^2 = \left[\frac{k(p-m)}{1-k^2}\right]^2. \tag{3.34}$$

But this is the equation of a circle, with it center on the horizontal axis at $((k^2m - p)/(k^2-1), 0)$ and a radius of $k(p-m)/|1-k^2|$. The set S is a *circle*, called the *Apollonius circle* of the two points **E** and **T** (in their $t = 0$ locations on the horizontal axis), which is named after the third-century B.C. Greek mathematician Apollonius of Perga [20]. Apollonius realized (in his lost work Plane Loci) that (3.32) is a way to define a circle in a manner different from the usual Euclidean geometry definition (the path traced by a moving point that remains a fixed distance from a given point). The definition in (3.32) predates Apollonius, however, being known a century earlier to Aristotle. If $m = 1$, $p = 2$, and $k = 2$, for example, the Apollonius circle is centered on $(\frac{2}{3}, 0)$ with a radius of $\frac{2}{3}$; see Figure 3.9, where the center of the Apollonius circle is marked with an X and labeled small circles indicate the initial locations of the torpedo and the enemy ship. For the submarine to determine where to aim its torpedo (that is, to locate the point **I**), all that remains to do is to see where **E**'s path intersects the Apollonius circle. The intersection point is **I**. For example, you can see from Figure 3.9 that **I** is, approximately, at $(1, 0.58)$ if **E** has a heading angle of 90°.

Now, what if $k < 1$, meaning, what if the torpedo is *slower* than the surface ship? To be specific, let us now take $k = 1/2$, which reduces (3.34) (with $m = 1$ and $p = 2$)

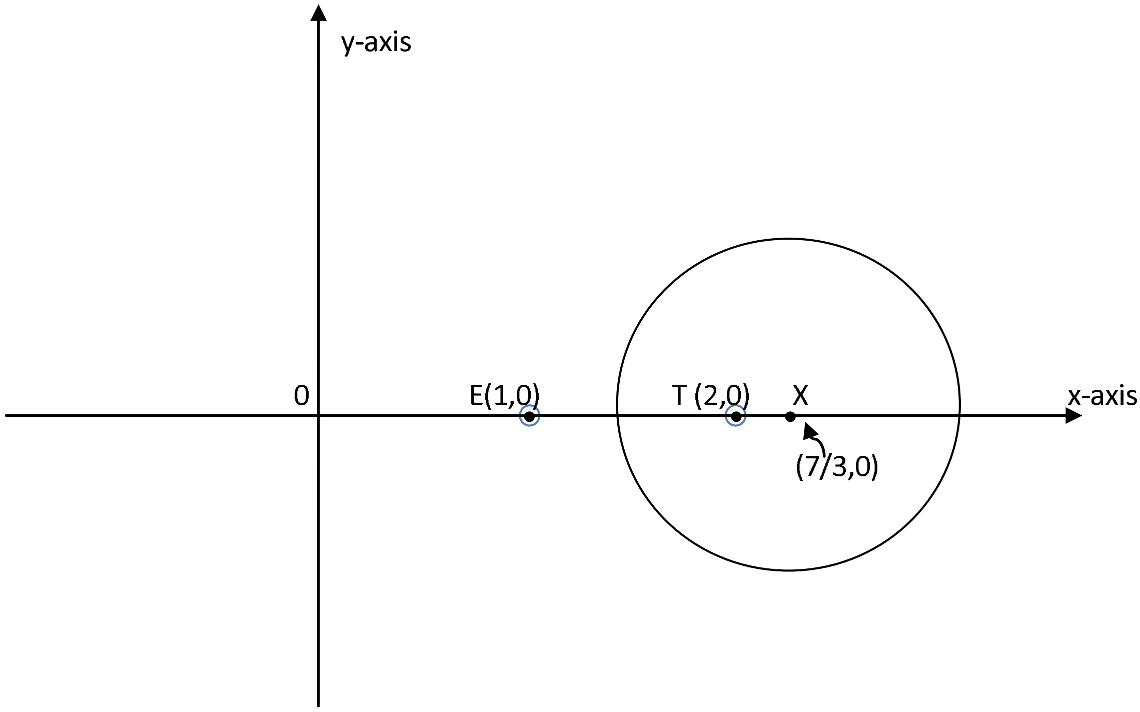

Figure 3.10: *The Apollonius circle centered on* $(7/3, 0)$ *with radius* $2/3$*, given by equation (3.34) for* $m = 1$*,* $p = 2$*, and* $k = 1/2$*, so that the torpedo is located at* $\boldsymbol{T}(2, 0)$ *and the enemy ship is at* $\boldsymbol{E}(1, 0)$

to

$$\left(x - \frac{7}{3}\right)^2 + y^2 = \left(\frac{2}{3}\right)^2.$$

That is, the Apollonius circle is still of radius $2/3$, but now is centered on $(7/3, 0)$, which means the center of the Apollonius circle is now to the right of the initial location of $\mathbf{T}$, as shown in Figure 3.10. You can see that now the torpedo may or may not be able to intercept the enemy ship - it is all a function of the heading angle of the ship. If the heading angle is sufficiently small that the ship's path crosses the Apollonius circle, then an interception by a slow torpedo of a fast enemy ship is possible (in fact, there will generally be two possible interception points), a result that often surprises.

Instead of considering specific values of k, m, and p, it is not at all difficult to

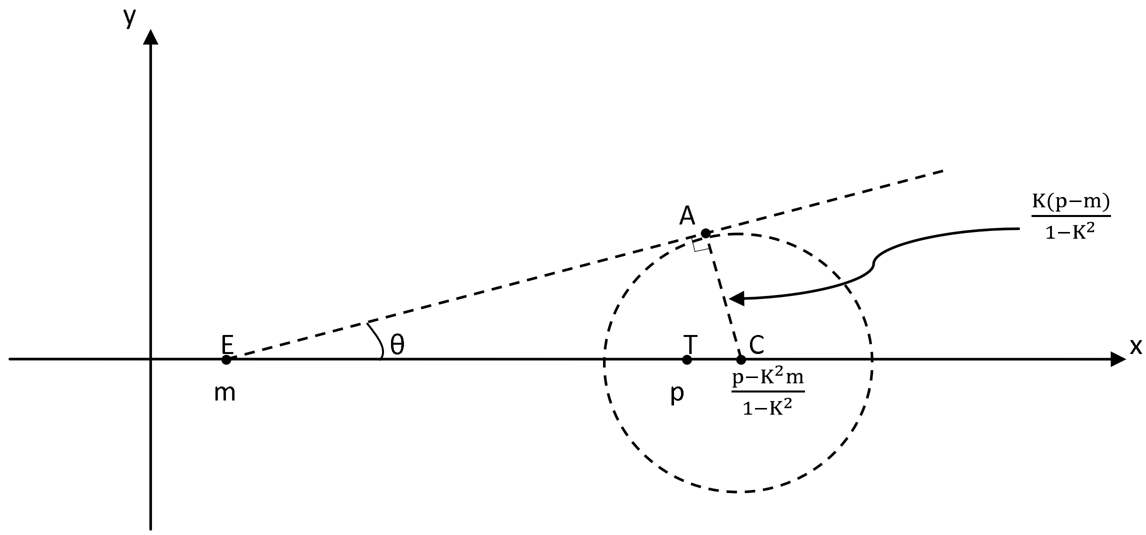

Figure 3.11: *The general geometry for a slow torpedo (**T**) interception of a fast enemy surface ship (**E**) (heading with an angle θ), where the Apollonius circle for the points $(m, 0)$ and $(p, 0)$, $p > m$, is given by equation (3.34) for $k < 1$*

be much more general and to derive an astonishingly simple condition that will tell us, for any $k < 1$, if a slow torpedo interception is, first, even possible and, if it is, where on the Apollonius circle the submarine should aim its slow torpedo. Equation (3.34) tells us that, for $k < 1$, the Apollonius circle for the points $(m, 0)$ and $(p, 0)$, $p > m$, is centered on the point C at $((p - k^2m)/(1 - k^2), 0)$ and has a radius of $k(p - m)/(1 - k^2)$, as illustrated in Figure 3.11. Now, imagine that the enemy ship's heading angle is θ, so that the ship just touches the Apollonius circle at A. If the absolute value of the heading angle is greater than θ then no interception is possible, and if the absolute value of the heading angle is less than θ then the enemy ship's path will cross the Apollonius circle *twice* and so there will be *two* possible interception points **I**. We can find a formula for θ, as follows.

The line AC, a radius of the Apollonius circle, is perpendicular to the tangent line EA, and so the triangle ECA is a right triangle. Thus,

$$\sin\theta = \frac{AC}{EC}. \tag{3.35}$$

The radius of the circle, as started before, is

$$AC = \frac{k(p-m)}{1-k^2},$$

while

$$EC = ET + TC = (p-m) + \left(\frac{p-k^2m}{1-k^2} - p\right) = \frac{(p-m)}{1-k^2}.$$

Inserting these expressions for AC and EC into (3.35) we arrive at

$$\sin\theta = k = \frac{V_{\mathbf{T}}}{V_{\mathbf{E}}},$$

that is,

$$\theta = \sin^{-1}\left(\frac{V_{\mathbf{T}}}{V_{\mathbf{E}}}\right). \tag{3.36}$$

If α is the heading angle of the enemy surface ship, then an interception using a slow torpedo is possible if $-\theta \leq \alpha \leq \theta$, and impossible otherwise.

Therefore, we can conclude that Apollonius circles can be used in the PE problems to analyze how to find a better strategy to escape or prolong the capture time whenever a successful escape is not possible.

Chapter 4

The Evasion Problem

4.1 Statement of the Problem

In this chapter we present PE problems with the emphasis on *evasion*. Let us assume that two points, one of which we shall call "pursuing" (P) and the other "evading" (E), are moving in $\mathcal{X} \subset \mathbb{R}^n$:

$$x' = f(x, t), \quad y' = g(y, v, t), \tag{4.1}$$

where v, $\mathcal{V}$, and $y(t)$ are the control parameter, the control region, and the trajectory of the motion of the evading point E, respectively, and $x(t)$ is the trajectory of the motion of the pursuing point P.

Let $v(t)$ be a certain admissible control (i.e., piecewise continuous), and let $x(t)$ and $y(t)$ be the corresponding trajectories with initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{4.2}$$

If $x(t_1) = y(t_1)$ for some $t_1 > 0$, we shall call t_1 an *encounter time*, and the very occurrence that $x(t_1) = y(t_1)$ will be referred to as an *encounter*. If the control $v(t)$

is chosen arbitrarily, an encounter may not occur for any $t > 0$. If an encounter does occur, we shall call the control (which is an admissible control) $v(t)$ an *evading control.* Even then, for the given x_0, y_0, and the chosen control $v(t)$, more than one encounter may take place. We shall call the *largest* positive number t_1, which is an encounter time, the *evading time* corresponding to the control $v(t)$. We shall denote the evading time by

$$T = \max_{v \in \mathcal{V}} T_v. \tag{4.3}$$

In what follows, the initial conditions (4.2) will be assumed to be fixed (in this connection, x_0 and y_0 do not enter into the notation for the evading time). Therefore, we get a statement of the ***evasion problem***.

Definition The problem is called an ***evasion problem*** if it is defined by equations (4.1) - (4.3)

$$x' = f(x,t), \quad y' = g(y,v,t),$$

$$x(0) = x_0, \quad y(0) = y_0,$$

$$T = \max_{v \in \mathcal{V}} T_v,$$

where x and y belong to $\mathcal{X} \subset \mathbb{R}^n$, $v \in \mathcal{V} \subset \mathbb{R}^r$ and is admissible (piecewise continuous).

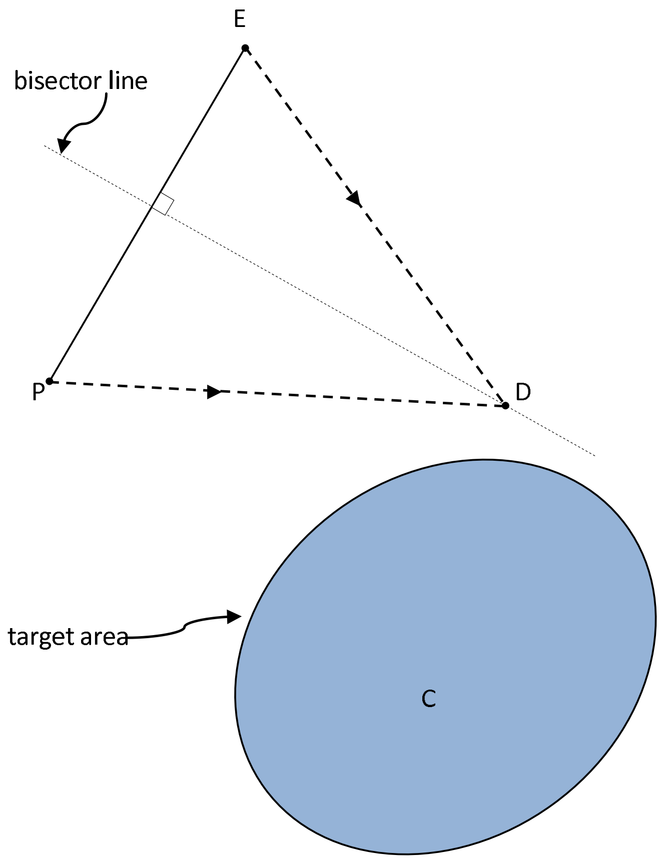

Figure 4.1: *P and E defending and attacking, respectively, the target area C*

4.2 Isaacs's Problem

One of the classic general evasion problems is Isaacs's guarding the target problem by the American mathematician Rufus Isaacs (1914 - 1981) [6], [14],[20], [27]. Isaacs states his problem, along with giving its general solution, as follows.

"*Both P and E (pursuer and evader) travel with the same speed. The motive of P is to guard a target C, which we take as an area in the plane, from attack by E. The optimal strategies for both P and E are: draw the perpendicular bisector of PE (where P and E denote starting positions). Any point in the half-plane above this line can be reached by E prior to P, and this property fails in the lower half-plane.*

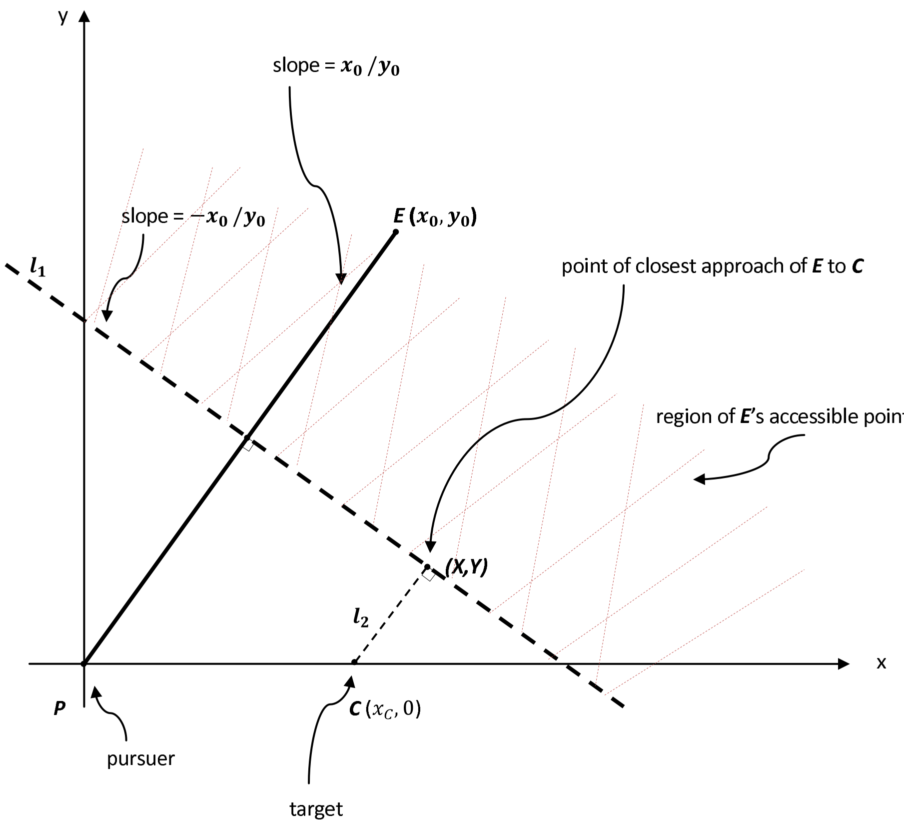

Figure 4.2: *The geometry of Isaacs's problem for P defending a point target C, and E attacking same target; l_1 is the perpendicular bisector of PE, l_2 is the perpendicular line segment from C to l_1*

Clearly, E should head for the best of his accessible points. Let D be the point of the bisector nearest C. The optimal strategies for both P and E decree that they travel toward D. When does the capture occur?"

The military conception of this problem by Isaacs is, E must reach at least the boundary of C to be successful in his attack. It may seem easy for E to reach an interior point of C (and thus, hit C), but it is not necessarily true. Here P is successful in defeating E if the capture point D (see Figure 4.1) is anywhere outside C. However, we need to know some additional information about the shape and dimensions of C to say more about the strategies of P and E, and the solution of the given problem.

Let us assume that C is, for example, the location of a specific enemy commander, or an enemy radio transmitter. Let us also consider that E is carrying an explosive device, which, when detonated, has a circular radius of destruction $R > 0$. For P let us say that it can stop E only by direct impact, i.e., P must intercept E. Therefore, for E to be successful, it must come within a distance less than or equal to R before P reaches E. Now the problem can be formulated in the following manner: "we have a crude model for an attacking missile versus a missile defense system that is supposed to protect an area, for example, a city, against a ballistic missile attack" [20].

Without any loss of generality we can place P at time $t = 0$ at the origin of an x-y coordinate system, and the point target C on the x-axis at $x = x_c > 0$. That is, the target is to the right of P. The case where the target is initially to the left of P is a mirror-image of our assumed case.

Let E be at (x_0, y_0) at time $t = 0$. We need to remark that we consider the case where C is not one of the E's accessible points, i.e., assume that y_0 is sufficiently large. If C is one of E's accessible points, E can destroy C by actually reaching C before P can reach E.

We can easily find the equation of the bisector line l_1

$$y = -\frac{x_0}{y_0}x + \frac{x_0^2 + y_0^2}{2y_0}, \tag{4.4}$$

which has the required slope and pass through the point midway between P and E at time $t = 0$, i.e., the point $(x_0/2, y_0/2)$. Then, l_2 is the perpendicular line segment from the point target C to l_1, and the length of l_2 is the closest approach distance of E to C. The slope of l_2 is y_0/x_0 and, since it passes through the point $(x_c, 0)$, we get the equation of l_2:

$$y = \frac{y_0}{x_0}x - \frac{y_0}{x_0}x_c. \tag{4.5}$$

Thus, the point of closest approach of E to C is the intersection of l_1 and l_2, which is the point (X, Y). Hence, we can find the values of X and Y from the equations (4.4) and (4.5):

$$\frac{y_0}{x_0}X - \frac{y_0}{x_0}x_c = -\frac{x_0}{y_0}X + \frac{x_0^2 + y_0^2}{2y_0},$$

which gives us

$$X = \frac{y_0^2}{x_0^2 + y_0^2}x_c + \frac{x_0}{2}, \tag{4.6}$$

and from any of the two equations we had above (4.4), (4.5) we get

$$Y = \frac{y_0}{x_0}\left(\frac{y_0^2}{x_0^2 + y_0^2}x_c + \frac{x_0}{2}\right) - \frac{y_0}{x_0}x_c. \tag{4.7}$$

Now we can find the length (squared) of l_2, and it is $(X - x_c)^2 + Y^2$, or

$$\left[\frac{y_0^2}{x_0^2 + y_0^2}x_c + \frac{x_0}{2} - x_c\right]^2 + \left[\frac{y_0}{x_0}\left(\frac{y_0^2}{x_0^2 + y_0^2}x_c + \frac{x_0}{2}\right) - \frac{y_0}{x_0}x_c\right]^2,$$

which after simplifying will be equal to

$$\frac{\left[x_0\left(x_0^2 + y_0^2\right) - 2x_c x_0^2\right]^2}{4x_0^2\left(x_0^2 + y_0^2\right)}.$$

Then, for E to achieve its mission goal of destroying C, the circular radius of destruction R (squared) of E's weapon must exceed the length (squared) of l_2, i.e.,

$$R^2 > \frac{\left[x_0\left(x_0^2 + y_0^2\right) - 2x_c x_0^2\right]^2}{4x_0^2\left(x_0^2 + y_0^2\right)},$$

or

$$R > \frac{x_0\left(x_0^2 + y_0^2\right) - 2x_c x_0^2}{2x_0\sqrt{x_0^2 + y_0^2}}.$$

Therefore, we get

$$\frac{R}{x_c} > \frac{\left(x_0/x_c\right)^2 + \left(y_0/x_c\right)^2 - 2\left(x_0/x_c\right)}{2\sqrt{\left(x_0/x_c\right)^2 + \left(y_0/x_c\right)^2}} = F\left(\frac{x_0}{x_c}, \frac{y_0}{x_c}\right). \tag{4.8}$$

Figure 4.3 shows several of the curves that represent the right-hand side of (4.8). Each curve gives the minimum value of R/x_c, as a function of x_0/x_c, for a given

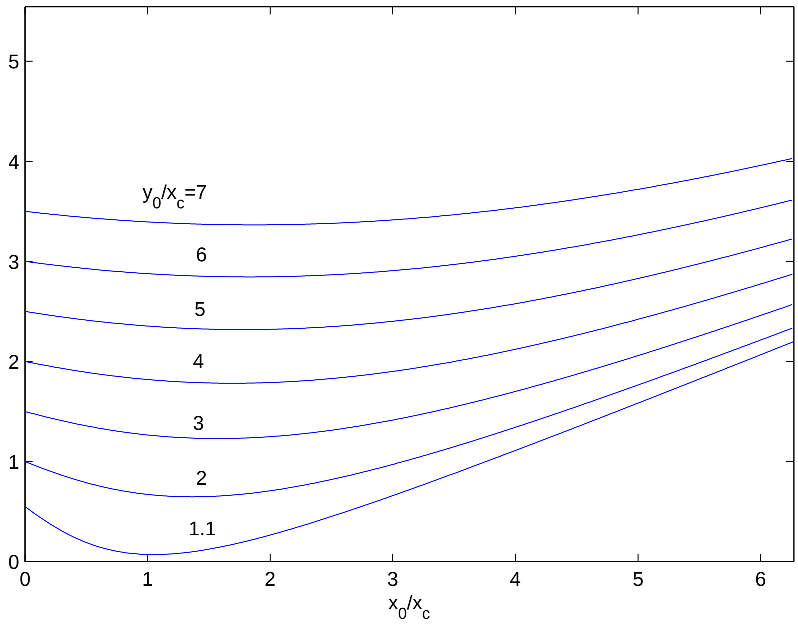

Figure 4.3: *Plot of $F(x_0/x_c, y_0/x_c)$ given in equation (4.8) as a function of x_0/x_c for a given fixed value of y_0/x_c ($y_0/x_c = 1.1, 2, 3, 4, 5, 6, 7$). Each curve gives the minimum value of R/x_c*

fixed value of y_0/x_c (the label-value next to each curve). From these curves E can determine the minimum value of R (the amount of explosive) required for success in destroying C as a function of both E's starting point and the location of the target.

4.3 Lady in the Lake Problem

The lady in the lake problem became famous decades ago, when it appeared in Martin Gardner's "Mathematical Games" column in Scientific American [11] in 1975. Gardner presented the problem as follows:

A young lady was vacationing on Circle Lake, a large artificial body of water named for its precisely circular shape. To escape from a man who was pursuing her, she got into a rowboat and rowed to the center of the lake, where a raft was anchored. The man decided to wait it out on shore. He knew she would have to come ashore eventually. Since he could run four times faster than she could row, he assumed that it would be a simple matter to catch her as soon as her boat touched the lake's edge. But the girl - a mathematics major at Radcliffe - gave some thought to her predicament. She knew that on foot she could outrun the man (which does raise the question of why such a smart lady got herself into this situation in the first place by rowing out into a lake!). It was only necessary to devise a rowing strategy that would get her to a point on shore before he could get there. She soon hit on a simple plan, and her applied mathematics applied successfully. What was the girl's strategy?

The lady's escape strategy consists of two stages. She first hops into her boat and rows away from the raft in such a way that she, the raft, and the man are always collinear. This first part of the lady's rowing path will clearly have to change direction constantly to continually maintain collinearity because the man will instantly begin running around the lake's edge in his attempt to intercept her at the shore. This is illustrated in Figure 4.4, where we assumed that the man runs counterclockwise around the lake. We will show later that the lady can maintain collinearity at least for a while. Let us assume that the man runs at speed v and that the lady rows with speed αv. Thus, in the original statement of the problem $\alpha = 0.25$. We see from Figure 4.4 that the man opens up the angle θ at the rate of $d\theta/dt = v/R$, where θ is

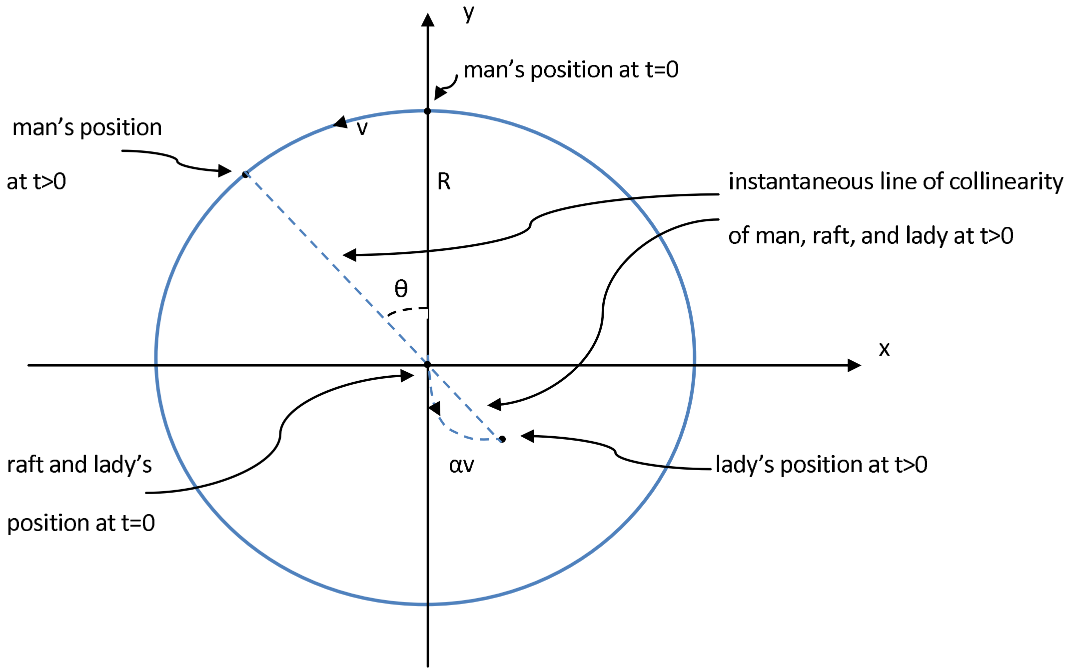

Figure 4.4: *The first stage of the lady's escape*

measured with respect to the line initially joining the man and the lady on the raft. Without any loss of generality we can assume that this initial line is the vertical axis of our coordinate system, as shown in Figure 4.4. Since the lady's angular speed component must be

$$v_\theta = r\frac{d\theta}{dt}$$

for her to maintain the raft between herself and the man, we can write her angular speed as

$$v_\theta = v\frac{r}{R}. \tag{4.9}$$

The farther she gets from the raft, then, (4.9) tell us, the greater must be her angular speed if she is to maintain collinearity.

Next, since the lady's total speed through the water is αv, her radial speed component (v_r) must be such that

$$v_r^2 + v_\theta^2 = (\alpha v)^2,$$

because her total speed is geometrically represented by the hypotenuse of a right triangle, with perpendicular sides v_r and v_θ. Thus,

$$v_r = \sqrt{\alpha^2 v^2 - v_\theta^2} = \sqrt{\alpha^2 v^2 - v^2 \frac{r^2}{R^2}},$$

or

$$v_r = \frac{dr}{dt} = v\sqrt{\alpha^2 - \frac{r^2}{R^2}}. \tag{4.10}$$

The lady has a positive v_r (that is, she moves ever closer to shore, all the while keeping half the lake between herself and the man) as long as $\alpha^2 - r^2/R^2 > 0$, that is, until $r = \alpha R$. At the instant her v_r drops to zero she switches to the second stage of her escape strategy, which we will describe below.

First, let us calculate, how long it takes her to arrive at the condition $v_r = 0$. Since $dt = dr/v_r$, then if we call $t = T$ the time at which $v_r = 0$, we have

$$\int_0^T dt = T = \int_0^{\alpha R} \frac{dr}{v_r} = \int_0^{\alpha R} \frac{dr}{v\sqrt{\alpha^2 - r^2/R^2}} = \frac{R}{v}\int_0^{\alpha R} \frac{dr}{\sqrt{(\alpha R)^2 - r^2}}$$
$$= \frac{R}{v}\left(\sin^{-1}\left(\frac{r}{\alpha R}\right)\right)\Big|_0^{\alpha R} = \frac{R}{v}\sin^{-1}(1),$$

or

$$T = \frac{\pi R}{2v}. \tag{4.11}$$

When the lady arrives at the circle with radius αR centered on the raft, at time $t = T$, she has arrived at what we call the “go-for-broke” circle, because now that she is no longer moving ever closer to shore with the first part of her escape strategy, she forgets about maintaining collinearity and simply rows straight for shore at her full water speed of αv. She has distance $R - \alpha R$ to row (at speed αv) and the man has distance πR (half the circumference of the lake) to run at speed v. She gets to shore before he gets to her if

$$\frac{R - \alpha R}{\alpha v} < \frac{\pi R}{v},$$

or

$$R(1 - \alpha) < \pi \alpha R,$$

or

$$1 - \alpha < \pi \alpha,$$

or

$$1 < \alpha(1 + \pi),$$

or, at last, if

$$\alpha > 1/(1 + \pi) = 0.241453. \tag{4.12}$$

Since $\alpha = 0.25$ in the Scientific American version of the problem, we see that this two-stage escape strategy works and that the lady's virtue is preserved.

Of course, if α is sufficiently large there is no need for a two-stage escape strategy. It is easy to see that if α is "big enough" then all the lady needs to do is immediately row directly to shore, to the point directly opposite the man's location. She gets to shore before he gets to her if

$$\frac{R}{\alpha v} < \frac{\pi R}{v},$$

that is, if $\alpha > 1/\pi = 0.3183099....$ Still, while not essential for her, the two-stage strategy will give the lady a little extra head start on the man, and it is interesting to calculate how much this head starts is for $\alpha = 1/\pi$. As before, in the two-stage strategy the man, the raft, and the lady remain collinear until the lady reaches the go-for-broke circle, with radius $\alpha R = R/\pi$. Then she rows straight for shore, now distance $R - R/\pi = R(1 - 1/\pi)$ away. Since her rowing speed is $\alpha v = v/\pi$, this requires a time (during her second stage) of

$$\frac{R(1-1/\pi)}{v/\pi} = \frac{R}{v}(\pi - 1).$$

The man reaches her landing point on the shore after running halfway around the lake, which requires a time (starting at the instant the lady "goes for broke") of

$$\frac{\pi R}{v} = \frac{R}{v}\pi.$$

So, she arrives at her landing point on the shore before he does by a time interval of

$$\frac{R}{v}\pi - \frac{R}{\pi}(\pi - 1) = \frac{R}{v}.$$

To put this head start (in time) in perspective, it is the time it takes the man to run distance R, the radius of the lake.

Let us suppose now that the lady does not have a big α. Suppose, in fact, that it is smaller than $(1+\pi)^{-1}$. Is it then impossible for her to escape from the man? Actually, if we make a plausible assumption about the man's reasoning (meaning, he is rational), then it is still possible for a slow-rowing lady to escape. Since the lady is a Radcliffe math major, and the man surely knows some math, too, therefore, let us assume that, as soon as the lady leaves the raft and begins to execute the first stage of her escape strategy, the man deduces what she is up to. That is, he observes that as he moves, she moves to keep the raft between him and her even as she moves ever closer to the shore. He then further deduces that as soon as she reaches her go-for-broke circle she will head straight for the shore. So, here is our assumption - as soon as he sees her go to the second stage of her escape strategy, that is, at the instant she makes straight for shore, he stops watching her carefully and simply runs around the lake to the point on the shore where he now knows she is heading. The only thing that will cause him to reevaluate matters is if the lady stops her go-for-broke rowing and, for whatever reason, begins to move back toward the raft.

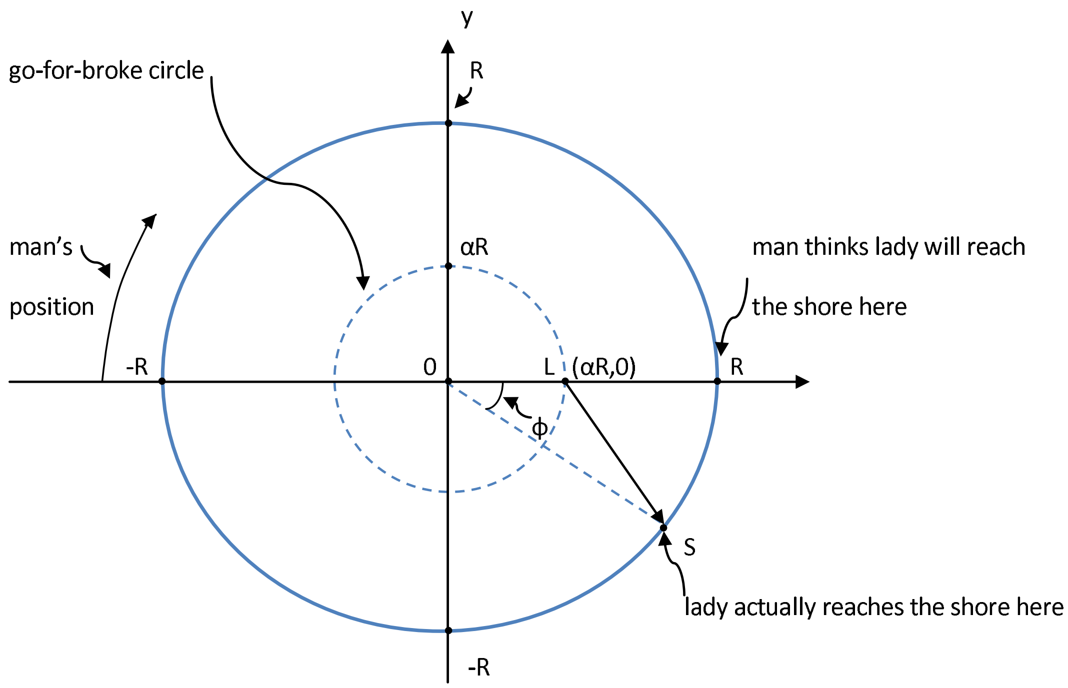

Figure 4.5: *The instant when the lady reaches her go-for-broke circle*

However, being a clever math major, and knowing her α is less than $(1 + \pi)^{-1}$, she has one last trick up her sleeve. She will, indeed, row a straight-line path to shore as soon as she reaches her go-for-broke circle, but it will not be the shortest distance straight-line path that the man thinks she will row. To see what she has in mind instead, look at Figure 4.5, where, with no loss in generality, we put the lady's position at the instant she reaches her go-for-broke circle at $(\alpha R, 0)$. The man's position at that instant is $(-R, 0)$. In the notation of the Figure 4.5, ϕ is the angle the straight line joining the raft to the lady's landing point on the shore (S) makes with the horizontal axis. The man is assuming that $\phi = 0$, but he is wrong, as you will see soon.

Let us talk about the lady's new escape strategy. First, we will simplify our calculations by noticing that the ratio of the radius of the go-for-broke circle to the radius of the lake is $\alpha R/R = \alpha$. If we next denote the radius of the go-for-broke circle as our unit distance, then $\alpha R = 1$, and so

$$\alpha = 1/R. \tag{4.13}$$

What this means is that if we wish to find the smallest value for α for which the lady can still escape, then an equivalent problem is that of finding the largest R for which the lady can still escape. And finally, since the lady rows at speed αv, we can write her rowing speed as $(1/R)v = v/R$. We can now set the problem up mathematically as follows. When the lady reaches her go-for-broke circle (point L in the figure), she is distance $\alpha R = 1$ from the raft, and the law of cosines tells us that the distance LS she has left to row to the shore to reach point S is

$$LS = \sqrt{1 + R^2 - 2R\cos\phi}.$$

This takes her a time interval of

$$\frac{LS}{v/R} = \frac{\sqrt{1 + R^2 - 2R\cos\phi}}{v/R} = \frac{R}{v}\sqrt{1 + R^2 - 2R\cos\phi} \tag{4.14}$$

to row.

The man is running clockwise around the lake to S (see Figure 4.5). We will quote Schuurman and Lodder [29] about what both the lady and the man conclude once she reaches her go-for-broke circle: "... she performs an infinitesimal radial feint (toward the shore that leads the man to start running clockwise). From the moment on, (the man's) best policy is to continue running clockwise if (the lady) goes to shore along a straight line not crossing the (go-for-broke) circle. If (the man) would return, a new diametrical mutual position, advantageous to (the lady) would be established." This last sentence is important to understand. It points out that the man should at any time reverse his running direction around the lake, then the lady could, at the least, start rowing directly away from him at the instant of his reversal and head straight for shore. That would have her starting the second stage of her original escape strategy from a point beyond the go-for-broke circle, and yet still leave the man with half the lake's circumference to travel. Even better (from the lady's point of view), would be for her to simply flip the sign of ϕ, and then the situation is just as it was before he switched. So, once the man has committed to a running direction, we see that he gains nothing by reversing his decision - he therefore will run through the angle $\pi + \phi$ to reach S. The time required for the man to run distance $(\pi + \phi)R$ around the lake to S is

$$\frac{R}{v}(\pi + \phi). \tag{4.15}$$

Thus, the lady will just escape the man if the two times given by (4.14) and (4.15) are equal, that is, if

$$\pi + \phi = \sqrt{1 + R^2 - 2R\cos\phi}.$$

Squaring both sides and solving for R gives

$$R = cos\phi \pm \sqrt{\cos^2\phi + (\pi + \phi)^2 - 1},$$

and since $R > 0$ we must use the plus sign,

$$R = cos\phi + \sqrt{\cos^2\phi + (\pi + \phi)^2 - 1}. \tag{4.16}$$

Figure 4.6 shows the behavior of $R(\phi)$, and it is obviously a nondecreasing function of ϕ. To find the smallest α for which the lady escapes we must use the largest possible value for R (see equation (4.13)). That is, we want to find the value of ϕ that maximizes $R(\phi)$. Now, even though R continually gets bigger with increasing ϕ, there is a limit on how big ϕ can be. If ϕ exceeds the value it has such that the line LS (in Figure 4.5) is tangent to the go-for-broke circle, then the lady's rowing path will take her back inside the go-for-broke circle, that is, she will have a radial speed component pointing back toward the raft (which is not a feature we expect in an escape strategy). That is, the lady should pick ϕ such that the line LS is perpendicular to the x - axis. From Figure 4.5 we see that this value of ϕ $(= \phi_t)$ satisfies the condition

$$\cos(\phi_t) = \alpha R/R = 1/R.$$

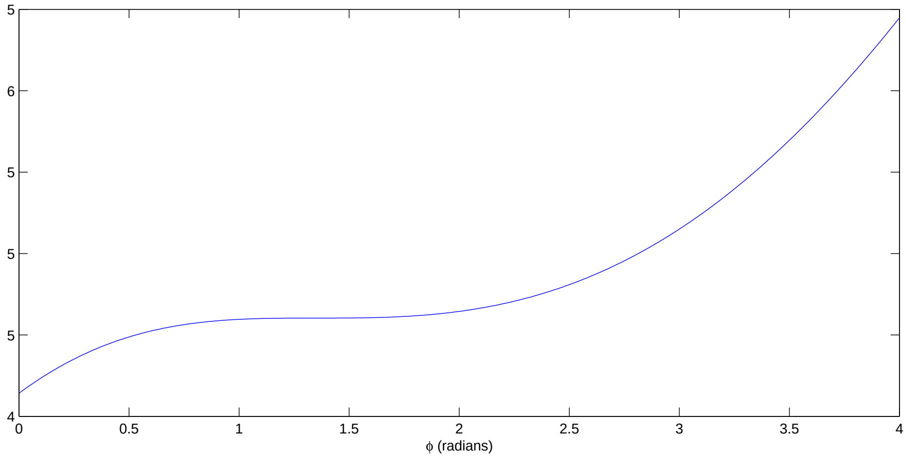

Figure 4.6: *The radius of the lake $R(\phi)$ (in radians) given by equation (4.12)*

If we substitute this condition in (4.16) we get

$$\frac{1}{\cos(\phi_t)} = \cos(\phi_t) + \sqrt{\cos^2\phi_t + (\pi + \phi_t)^2 - 1},$$

which reduces to the equation

$$\tan(\phi_t) = \pi + \phi_t. \tag{4.17}$$

It is clear, simply by sketching the curves for each side of (4.17), that there is a solution to (4.17) somewhere in the interval $(0, \pi/2)$. In [20] it was found (by numerical means) with the result $\phi_t = 1.3518168...$ radians, that

$$\cos(\phi_t) = 1/R_{max} = \alpha_{min} = 0.2172336....$$

Alternatively, the lady can escape even if the man runs $1/\alpha_{min} = 4.6033388...$ times as fast as she can row, which is significantly greater than the factor of four given in the Scientific American version of the problem.

Chapter 5

Pursuit-Evasion Problem as an Optimal Control Problem

5.1 Basic Concepts

Let us assume that two points, one of which we shall call "pursuing" and the other "pursued" or "evading", are moving in $\mathcal{X} \subset \mathbb{R}^n$. The motion of each of these points is subject to its own particular system of differential equations with its own particular control parameter. We shall denote the control parameter, the control region, and the trajectory of the motion of the pursuing point by $u, \mathcal{U} \subset \mathbb{R}^r$, and $x(t)$, respectively. We shall denote these quantities for the pursued point by the symbols v, $\mathcal{V} \subset \mathbb{R}^r$, and $y(t)$.

Let $u(t)$ and $v(t)$ be certain admissible controls (i.e., piecewise continuous), and let $x(t)$ and $y(t)$ be the corresponding trajectories with initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{5.1}$$

If $x(t_1) = y(t_1)$ for some $t_1 > 0$, we shall call t_1 an *encounter time*, and the very occurrence that $x(t_1) = y(t_1)$ will be referred to as an *encounter*. Generally speaking, if $u(t)$ and $v(t)$ are chosen arbitrarily, an encounter may not occur for any $t > 0$. If an encounter does occur, we shall say that $u(t)$ is a *pursuing control* (for a given control $v(t)$, and for given initial conditions x_0 and y_0). Even then, for the given x_0, y_0, $v(t)$, and the chosen control $u(t)$, more than one encounter may take place. We shall call the *smallest* positive number t_1, which is an encounter time, the *pursuit time* corresponding to the controls $u(t)$ and $v(t)$. We shall denote the pursuit time by $T_{u,v}$. In what follows, the initial conditions (5.1) will be assumed to be fixed (in this connection, x_0 and y_0 do not enter into the notation for the pursuit time).

Henceforth, we shall assume that the pursuing point has the following property: for every given control $v(t)$ there exists (for given initial conditions (5.1)) a pursuing control $u(t)$.

If the control $v(t)$ of the evading point has been chosen, we can pose the problem of finding a pursuing control $u(t)$ such that the corresponding pursuit time $T_{u,v}$ takes on a minimal value. We shall assume that there exists, for every admissible control $v(t)$, an admissible control $u(t)$ which brings about the minimum of the pursuit times. We shall denote the minimum by T_v:

$$T_v = \min_u T_{u,v}.$$

Furthermore, we shall assume that there exists an admissible control $v(t)$ which brings about the maximum of the values of T_v. We shall denote this maximum by T:

$$T = \max_v T_v = \max_v \, (\min_u T_{u,v}). \tag{5.2}$$

Similarly,

$$T = \min_u T_u = \min_u \, (\max_v T_{v,u}).$$

Moreover,

$$\min_u \, (\max_v T_{v,u}) = \max_v \, (\min_u T_{u,v}).$$

The problem consists of finding a pair of admissible controls $u(t)$ and $v(t)$ such that $T_{u,v} = T_{v,u} = T$. Such a pair $u(t)$ and $v(t)$ will be called an *optimal pair of controls*; the corresponding pair of trajectories $x(t)$ and $y(t)$ (with initial values (5.1)) will be called an *optimal pair of trajectories*. Thus, the control u (for a given control $v(t)$) is to be chosen in such a way that the encounter of the pursuing and pursued points will take place as soon as possible. The choice of the control v, on the other hand, is aimed at putting off the encounter as long as possible.

Remark Let us consider the case explained by equation (5.2). Note, that in choosing the control $u(t)$ (which defines the motion of the pursuing point), we shall always assume that the control $v(t)$ for the evading point is known beforehand.

In accordance with this fact, in order to determine T, first the minimum with respect to all possible controls $u(t)$ is taken for a certain fixed control $v(t)$, then the maximum with respect to all possible controls $v(t)$ is taken.

To solve the given problem, we shall assume that the motion of the pursuing point in X is described by the *linear* equation (in vector form)

$$\frac{dx}{dt} = f(x,u) \equiv Ax + Bu + c, \tag{5.3}$$

for which the corresponding control region $\mathcal{U}$ is a closed, convex, bounded polyhedron in $\mathbb{R}^r$, of the variable $u = (u^1, ..., u^r)$. Let the motion of the evading point be described by the equation (in vector form)

$$\frac{dy}{dt} = g(y,v,t) \tag{5.4}$$

and let the corresponding control region $\mathcal{V}$ be a set in the s-dimensional space $\mathbb{R}^s$ of the variable $v = (v^1, ..., v^s)$. We shall assume that the set of all piecewise continuous controls is the class of admissible controls (both for u and for v). We shall impose the usual conditions (continuity in y, v, and t, and continuous differentiability with respect to the coordinates $y^1, ..., y^n$ of y) on the coordinates of the vector function $g(y, v, t)$.

To solve the given problem we can use Pontryagin's maximum principle. We shall introduce two auxiliary vectors

$$\psi = (\psi_1, ..., \psi_n), \qquad \chi = (\chi_1, ..., \chi_n),$$

and two Hamiltonian functions

$$H_1(\psi, x, u) = \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x, u) = (\psi, f(x, u)),$$

$$H_2(\chi, y, v) = \sum_{\alpha=1}^{n} \chi_\alpha g^\alpha(y, v, t) = (\chi, g(y, v, t)),$$

corresponding to the pursuing and pursued objects. We can write the following two systems of equations for the auxiliary unknowns ψ_i and χ_i with the aid of H_1 and H_2:

$$\frac{d\psi_i}{dt} = -\frac{\partial H_1}{\partial x^i}, \ i = 1, 2, ..., n, \tag{5.5}$$

$$\frac{d\chi_i}{dt} = -\frac{\partial H_2}{\partial y^i}, \ i = 1, 2, ..., n, \tag{5.6}$$

Suppose that $u(t), x(t), v(t)$ and $y(t)$ are given. Then, if we substitute these functions in the right-hand sides of systems (5.5) and (5.6), we obtain linear systems in the unknowns ψ_i and χ_i. Every solution $\psi(t), \chi(t)$ of these systems will be said to *correspond* to the chosen functions $u(t), x(t), v(t)$, and $y(t)$. The following theorem gives a necessary condition for optimality in the problem under consideration.

Theorem 5.1.1 *Let u(t) and v(t) be an optimal pair of controls, let x(t) and y(t) be the corresponding optimal pair of trajectories, and let T be the pursuit time. Then,* *there exist nontrivial solutions $\psi(t)$ and $\chi(t)$ of systems (5.5) and (5.6) which correspond to u(t), x(t), v(t), and y(t) such that:*

1. ***the Maximum conditions***

$$\max_{u\in U} H_1(\psi(t), x(t), u) = H_1(\psi(t), x(t), u(t)), \tag{5.7}$$

$$\max_{v\in V} H_2(\chi(t), y(t), v) = H_2(\chi(t), y(t), v(t)) \tag{5.8}$$

hold for all t, 0 $\leq$ t $\leq$ T;

2. ***At the time t = T, the conditions***

$$H_1(\psi(T), x(T), u(T)) \geq H_2(\chi(T), y(T), v(T)), \tag{5.9}$$

$$\psi(T) = \chi(T) \tag{5.10}$$

hold.

The details of the proof of theorem 5.1.1 are long and involved, the reader can find them in [24]. Although this theorem gives the necessary conditions of optimality for PE problems (and it can be generalized to multiple pursuers and multiple evaders as in [7]), the fact is that the PE problems studied below can be analyzed directly by more elementary methods.

5.2 Simple Pursuit in the Plane

In the simple pursuit problem (we follow the representation given in [12]) two players move in the Euclidean plane R^2 with simple motion: each has a bound on his speed, but there are no further restrictions (e.g., abrupt directional changes are allowed). One player, the pursuer, wishes to capture the other, the evader, that is, attain perfect coincidence of their terminal positions. Here if $\alpha > \beta$ holds, for the pursuer's speed bound α and the evader's β, then termination is assured in finite time, whatever the initial positions and action of evader. On the other hand, in the case $\alpha \leq \beta$ the evader can avoid capture forever from any positions not in contact initially.

First, let us discuss briefly some aspects of simple motion for a single player. If the player's position at time $t \in R^1$ is denoted by $x(t) \in R^1$, then the velocity vector is $\dot{x}(t)$, and the speed $|\dot{x}(t)|$. Thus the dynamical constraint is $|\dot{x}| \leq \alpha$, and the following holds for the control u:

$$\dot{x} = u; \quad u : R^1 \to R^1, \quad |u(t)| \leq \alpha. \tag{5.11}$$

Hence,

$$x(t) = x(0) + \int_0^t u(s)ds$$

for some function $u(\cdot)$ as given above. For simplicity let us place the origin at $x(0)$, so that $x(0) = 0$.

We want to know, where can the player get to at time t. The constraint on $u(\cdot)$ yields

$$|x(t)| \leq \int_0^t |u(s)|\, ds \leq \alpha t.$$

Any point y with $|y| \leq \alpha t$ can be "attained" by control u:

$$u(s) = \begin{cases} \alpha \frac{y}{|y|}, & \text{for } 0 \leq \text{s} \leq |y|; \\ 0, & \text{for } |y| < \text{s} \leq \text{t}. \end{cases}$$

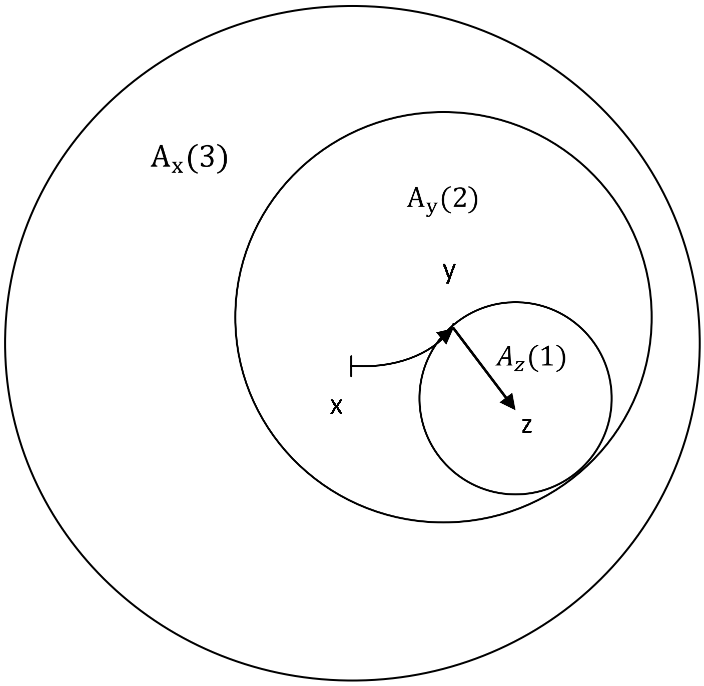

Figure 5.1: *Simple motion in the plane. Point x moves anywhere within $A_x(3)$ at time $t = 3$; if its position y at $t = 1$ or z at $t = 2$ is known, the possibilities are: reduce to $A_y(2)$ or $A_z(1)$.*

Thus, the attainability set ("reachable set") can be defined by $A_0 = \{y : |y| \leq \alpha t\}$. Figure 5.1 shows the simple motion in the plane, according to the defined attainability set rule.

Let us return to the game, in the case $\alpha > \beta$. If the pursuer's motion is x: $R^1 \to R^1$ and the evader's $y : R^1 \to R^1$, the equations of motion are

$$\dot{x} = u, \qquad \dot{y} = v \tag{5.12}$$

for suitable controls $u, v : R^1 \to R^2$ with all values $|u(t)| \leq \alpha$, $|v(t)| \leq \beta$. At any time denote the player's distance by $r = |x - y|$. Then

$$r\dot{r} = \frac{d}{dt} \cdot \frac{1}{2} |x - y|^2 = (x - y)' \cdot (\dot{x} - \dot{y}) = (x - y)'u - (x - y)'v. \tag{5.13}$$

The natural strategy for the pursuer is to take

$$u = \dot{x} = \alpha \frac{y - x}{|y - x|} = \frac{\alpha}{r}(y - x).$$

Then, in (5.13),

$$r\dot{r} = -\alpha \frac{r^2}{r} - (x - y)'v \leq -\alpha r + r\beta = -(\alpha - \beta)r$$

by Cauchy's inequality on the last term. Therefore, as long as $r > 0$, we have $\dot{r} \leq -(\alpha - \beta)$,

$$r(t) \leq r(0) - (\alpha - \beta)t = |x_0 - y_0| - (\alpha - \beta)t.$$

This shows that capture ($r = 0$) for the case $\alpha > \beta$ must occur at some time T with

$$T \leq \frac{|x_0 - y_0|}{\alpha - \beta}. \tag{5.14}$$

5.3 One-dimensional Rocket Chase

Next, we consider a general problem of the rocket chase and we base the discussion on [12].

Two players move on a straight line, the pursuer having a bound on his acceleration, the evader a bound on his speed. *The game ends when the pursuer attains a previously given distance from the evader.*

There is an obvious solution: the pursuer uses all his capabilities to move toward the evader, who is then captured within a bound time interval. (The precise time bound will depend on the parameters of the game, and on the initial positions.)

If $x : R^1 \to R^1$ describes the pursuer's motion, and $y : R^1 \to R^1$ describes the evader's motion, then the equations of motion are

$$\ddot{x} = u, \quad \dot{y} = v$$

for admissible $u, v : R^1 \to [-1, 1]$. Consider 1 as a bound for both controls, and ε as the evader's distance, $0 \leq \varepsilon < +\infty$. Thus, the evader moves on R^1 with simple motion, in the sense of what is given in the previous example. The pursuer's motion is described by

$$x(t) = x(0) + \dot{x}(0)t + \int_0^t \int_0^s u(r)drds = x(0) + \dot{x}(0)t + \int_0^t (t-s)u(s)ds \quad (5.15)$$

and suggested by the attainability sets in Figure 5.2, where

$$x_0 + \dot{x}_0 t - \frac{t^2}{2} \leq x(t) \leq x_0 + \dot{x}_0 t + \frac{t^2}{2},$$

$$y_0 - t \leq y(t) \leq y_0 + t,$$

since $u = v = 1$.

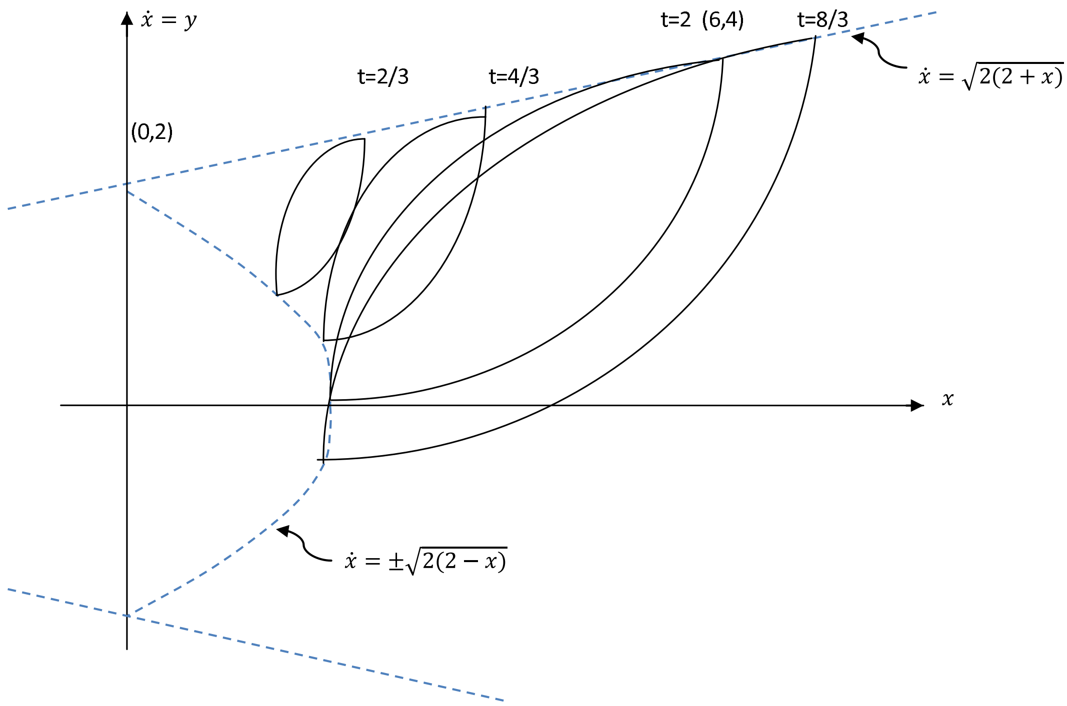

Figure 5.2: *Phase portrait of motion in* $\ddot{x} = u$ *in the* x*-*y *plane, where* $x(t)$ *is given by equation (5.15) for* $x(0) = 0$, $\dot{x}(0) = 2$*; attainability sets at* $t = 2/3,\ 4/3,\ 2,\ 8/3$ *for the same initial values* $x(0)$ *and* $\dot{x}(0)$*. The vertex loci are parabolas* $\dot{x} = y = \pm\sqrt{2(x+2)}$

The first-order version of the motion equation is the dynamical equation for the two-player system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad \dot{x}_3 = v.$$

Subsequently the matrix form of this will be treated,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

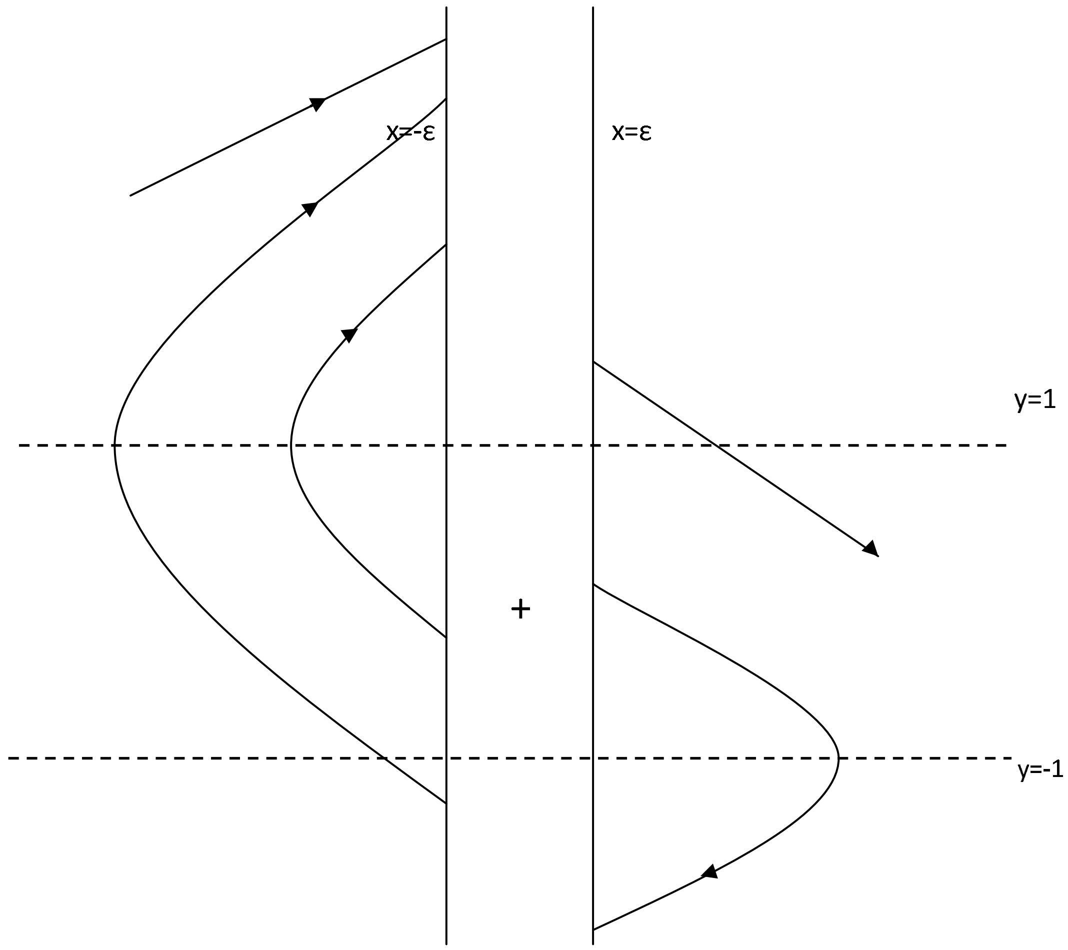

Figure 5.3: *Trajectories of* $\dot{x} = y - v$, $\dot{y} = u$ *in the x-y plane with* $u = v = \pm 1$ *outside target* $|x| \leq \varepsilon$

the termination condition $|x - y| \leq \varepsilon$ translating to

$$(1, 0, -1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in [-\varepsilon, \varepsilon].$$

Thus, the natural phase space is R^3. This can be reduced to R^2 by introducing new variables $x = x_1 - x_3, y = x_2$. The resulting equations and termination condition

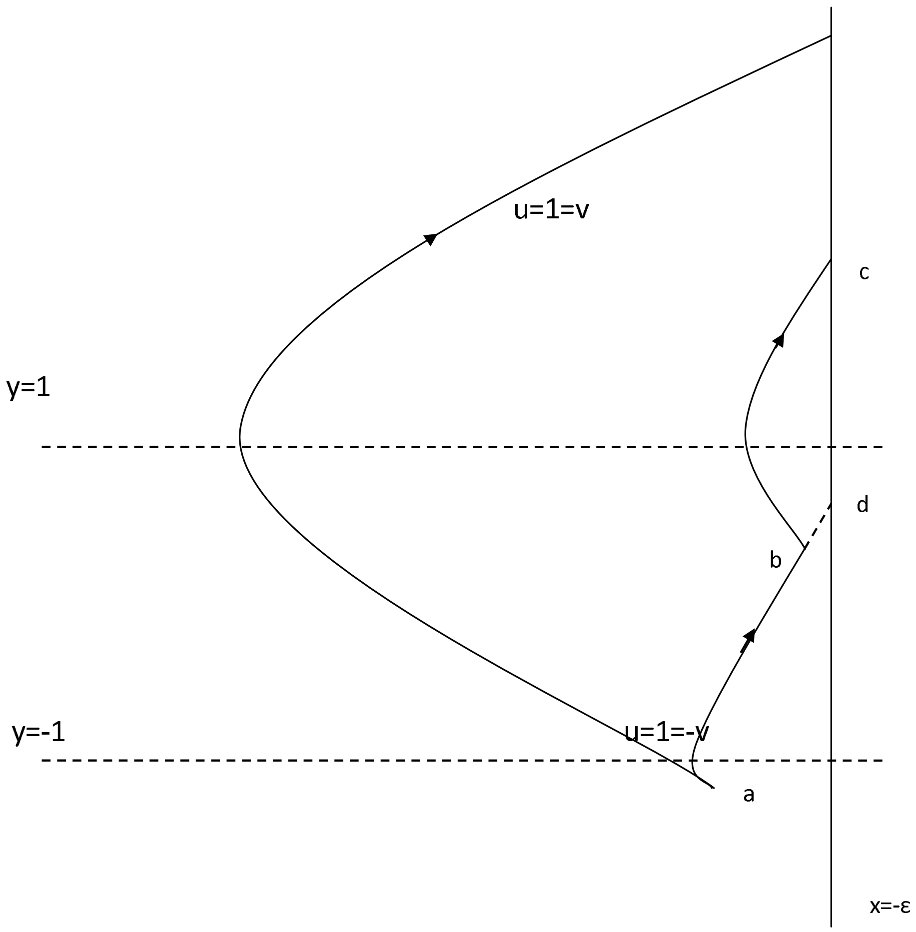

Figure 5.4: *Trajectories of $\dot{x} = y - v$, $\dot{y} = u$ in the x-y plane. From point a evader mistakenly chooses $v = -1$, but reverses his choice at b; capture occurs at c (later than it would have occurred at d)*

are

$$\dot{x} = y - v, \quad \dot{y} = u; \quad |x| \leq \varepsilon. \tag{5.16}$$

Let us assume (for a preliminary orientation), that both players' controls are constant on some time interval. The differential equation for the trajectories, with

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}},$$

is

$$(y - v)\frac{dy}{dx} = u, \qquad (y - v)^2 = 2ux + const. \tag{5.17}$$

The point then moves along the parabola, upward for $u > 0$ and downward for $u < 0$, (see Figure 5.3 with $u = v = 1$ in the left half plane, and $u = v = -1$ in the right).

Now let us suppose that, at some point to the left of the target, the evader chooses a control other than $v = 1$, and pursuer sticks with $u = 1$. The motion then proceeds along another parabola (with axis $y = v$, see Figure 5.4 for $v = -1$). Therefore, capture, even with $\varepsilon = 0$, can be ensured from all initial positions, for example, by taking $u = 1$ quite indifferent to evader's action.

5.4 Pursuit on a Sphere (Kelley's game)

Another example is given by pursuit on a sphere, originally formulated by Kelly. We follow the presentation of [12].

Two players move on the two-sphere S^2 in R^3, each with a fixed bound on his speed. *The game ends at the coincidence of positions.*

Here the idea is that "in a dogfight, the planes tend to move in a circular fashion". The simplification does away with one significant aspect of actual combat: that the roles of the pursuer and the evader are not fixed, but may well switch back and forth. The outcome is similar to the simple motion problem (section 5.2). Let us denote the pursuer's speed bound by α, the evader's by β. If $\alpha > \beta$, the pursuer can force termination from any initial position, within a bounded time interval. In the case $\alpha < \beta$ the evader can avoid capture at all times $t > 0$ (and the stand - off situation $\alpha = \beta$ is rather too sensitive to details in the specification of the players' strategies).

Let us talk about these different games. In the case $\alpha > \beta$, first assume that the players are not at diametrically opposite points initially. Then there is the unique shortest arc γ of a great circle joining their positions. By a parallel shift along γ, move a neighborhood of the evader's position to the pursuer's (this "action at a distance" serves to identify the control of the evader). The pursuer then uses the control $u = v + w$, where the first component neutralizes the evader's action, the second, w with magnitude $|w| = \alpha - \beta$ in the direction of γ, serves to decrease the player's distance (at a rate $\alpha - \beta$, see section 5.3) until capture occurs. If their initial positions are opposite, then any constant control u with $|u| = \alpha > \beta$, applied over a short interval, will achieve non-opposing positions. By a like reasoning, in the case $\alpha < \beta$ the evader can maintain forever an initial distance from the pursuer.

The idea is probably clear enough, and will apply equally well to simple pursuit on an n - dimensional Riemannian manifold (thus, the "diametrically opposite points" would be replaced by conjugate points). Consider the motion of a single player over the unit sphere S^{n-1} of $\mathbb{R}^n$. If its motion is described by $x : \mathbb{R}^1 \to \mathbb{R}^n$, $x(0) \in S^{n-1}$, then $x(t)$ will remain on S^{n-1} iff $|x(t)|^2$ is constant, i.e., x is perpendicular to $\dot{x} = \frac{dx}{dt}$. Further, the motion will be "simple" if the only further dynamical restriction is a magnitude bound on $\dot{x}$. We wish to express this as a relation between $\dot{x}$ and suitable arbitrary controls u.

Lemma 5.4.1 *(for a broader discussion see [12]) For any points $a \neq b$ on S^{n-1} there is a mapping $x, y \mapsto E(x, y)$, defined for x, y near a, b and with $(n, n-1)$ matrices as values, analytic in the coordinates of x, y and such that*

$$x'E(x,y) = 0, \quad E'(x,y)E(x,y) = I_{n-1}, \tag{5.18}$$

$$y'E(x,y) = |\sin\varphi|(1, 0, ..., 0), \tag{5.19}$$

where φ is the angle between x and y.

Proof By assumption, the vectors a, b are independent, so that there is a basis for R^n of the form $a, b, c_3, ..., c_n$. Then

$$x, y, c_3, ..., c_n \tag{5.20}$$

remain independent if x, y are close enough to a, b. Apply the Gram-Schmidt orthogonalization process to the sequence (5.20), obtaining orthonormal vectors $e_1, e_2, ..., e_n$. Since $|x| = 1$, we have $e_1 = x$, and, in the second step,

$$e_2 = \frac{y - (x'y)x}{|\sin\varphi|}, \tag{5.21}$$

since

$$|y - (x'y)x|^2 = 1 - (x'y)^2 = 1 - \cos^2\varphi.$$

Collect the column vectors $e_2, ..., e_n$ into the (n, n - 1) matrix $E(x, y)$. Equation (5.18) holds since $(x, E(x, y))$ is orthonormal. The first coordinate of $y'E(x, y)$ is $y'e_2 = |\sin\varphi|$ from (5.21). The remaining coordinates are 0, since e_k is perpendicular to both $e_1 = x$ and e_2, and hence to y also. This completes the proof.

Corollary 5.4.2 *On the neighborhood of any point on S^{n-1} there is an analytic mapping $x \mapsto E(x)$, whose values are $(n, n-1)$ matrices, and*

$$x'E(x) = 0, \quad E'(x)E(x) = I_{n-1}, \quad E(tx) = E(x) \;\; for \;\; t > 0. \tag{5.22}$$

Proof The corollary will follow on taking $y = b \neq \pm a$, and defining $E(x) = E(x, b)$. Positive homogeneity is ensured by extending $E(\cdot)$ in the obvious manner, namely $E(tx) = E(x)$ for $t > 0$.

Returning to Kelly's game (actually, for dimension's $n \geq 2$), we may choose the state space description

$$\dot{x} = E(x, y)u, \quad \dot{y} = E(y, x)v.$$

The control values in R^{n-1} are constrained by $|u(t)| \leq \alpha$, $|v(t)| \leq \beta$. The initial positions are on the unit sphere. If φ is the angle between x, y, and $r = |\sin\varphi|$, then

$$\begin{aligned} r\dot{r} &= \frac{1}{2}\frac{d}{dt}\sin^2\varphi = \frac{1}{2}\frac{d}{dt}(1 - (x'y)^2) \\ &= -(x'\dot{y} + y'\dot{x}) = -x'E(x, y)v - y'E(x, y)u. \end{aligned} \tag{5.23}$$

Write $u = -v + w$ with $w \in R^{n-1}$ to be chosen subsequently, subject to $|w| \leq \alpha - \beta$. Then, using (5.19),

$$\begin{aligned} r\dot{r} &= (-x'E(y, x) + y'E(x, y))v - y'E(x, y)w \\ &= 0 - |\sin\varphi|(1, 0, ..., 0)w \leq -r(\alpha - \beta) \end{aligned} \tag{5.24}$$

on taking $w' = (\alpha - \beta)(1, 0, ..., 0)$. Thus, $\dot{r} \leq -(\alpha - \beta) < 0$, and $\sin\varphi = 0$ can be attained in finite time, i.e., the capture is in finite time.

Chapter 6

Conclusions

In this thesis we studied a family of mathematical problems known as pursuit-evasion problems (PE). We presented PE problems within the classes of pursuit problems, evasion problems, and pursuit-evasion problems. We restricted the discussion to the deterministic approach of PE problems, and stated PE problems as optimal control problems. Because of that, we formulated the general optimal control problem, and discussed the two main approaches to solve this problem, namely, *Pontryagin's maximum principle* (Theorems 2.2.1, 2.2.2) (where we introduce a hamiltonian function) and *Bellman's method of dynamic programming* (equation (2.29)) (where we gave a simple example from dynamic programming to explain the main ideas of the method). To compare the two solutions we provided an example, where Pontryagin's principle applied, but Bellman's failed because the control was discontinuous (***Bang-Bang Problem***) . Thus, we proved that the assumption on the continuous differentiability of the functional (2.6), minimizing the transition time from the initial point to some other given terminal point, did not hold in the simplest cases. Therefore, we showed that in general Bellman's consideration yields a good heuristic method, rather than a mathematical solution of the problem.

Since there are different formulations of PE problems, we stated the definition of the PE problem from the point of view of the pursuer, the evader, and both. For the pursuit problems, we presented some main examples, namely, the Pierre Bouguer's pursuit problem, the wind-blown plane problem, the tractrix, and the Apollonius pursuit problem. In the Pierre Bouguer's pursuit problem, which we explained as a pure pursuit problem, we treated the case of a pirate ship pursuing a merchant vessel, and determine the equation of the the trajectory of the pirate ship (the pursuer), called the line of pursuit. We identified when did the capture occur, and talked about the cases where the pirate ship was slower than the merchant vessel (no capture), and where the pirate ship was faster than the vessel (there was a capture, and we calculated the total distance travelled by the pirate ship until its capture of the merchant vessel for that case). It was also interesting for us to find out that in the case where the pirate ship and the merchant vessel had equal speeds, the pursuit became a vertically upward tail chase, since the pirate ship pulled into behind of the merchant vessel.

In the other important example of the pursuit problem, we talked about the pursuit of a stationary target, namely, the problem where the plane went to the city C due west of his starting point, and the wind blew from the south. Here we identified that for the no wind case (when the ration of the wind's speed and the plane's speed is zero) the plane moved directly to the city C while always remaining on the x-axis. For the case where the wind and the plane had equal speeds we presented the plane's path and showed the different situation that happened according to the initial position of the plane. Also, when the ratio of the speeds was less that one, the plane did reach the city, and we calculate the time of the trip.

The other pursuit curve we talked about was the tractrix, which we showed got its name because of its path looking as the following curve, or the tailing curve. We compared the tractrix with Bouguer's pure pursuit curve for the special case of equal speeds for the pirate ship and the merchant vessel. Thus, we realized that the results were quite different, since for the tractrix the constant lag was always the case, while the constant lag of the pirate ship was an asymptotic property that developed with the passage of time.

Then, we presented Apollonius pursuit problem, where we discussed a method of interception, which is of great interest for the PE problems. We provided an example of a torpedo (T) trying to pursuer an enemy ship (E). Here we identified the three main cases of the location of T and E. We defined the set S of all the points in the plane so that the interception can be obtained. This set S is known as the Apollonius circle, and it is broadly used in the PE problems for analyzing how to find a better strategy to escape or prolong the capture time whenever a successful escape is not possible. Again, we talked about the cases of the speed differences, namely, for the fast torpedo we determined that the interception would occur, for the slow torpedo the interception might or might not occur.

Next, we defined the evasion problem and provided one of the general examples of those problems, called the Isaac's guarding the target problem, where we had P guarding the target area C from attack by E. We formulated the military conception of this problem, and identified the equation that gave us the evasion curves. Moreover, we showed how to determine the minimum value of the amount of explosive (that E carries) required for success in destroying the target area C, as a function of both E's starting point and the location of the target.

After that we provided a problem about the lady in the lake and the man who was trying to track her down. We talked about the strategies the lady had to have in order to escape the man, and explained each of the possible cases. Here we defined another interesting definition, known as a go-for-broke circle which we got by formulating the lady's strategies.

Finally, we formulated the PE problem, where there were both objects of the PE game, the pursuer P and the evader E. We stated the necessary conditions of optimality for those PE problems, which were similar to Pontryagin's maximum principle presented before. We provided examples of a simple pursuit in the plane, that we used as an example of the simple motion of P and E in the plane. We explained, what did we mean by the reachable set for this case, and compared the dynamical constraints of both P and E. In the one-dimensional rocket chase problem we presented the solution that could be used in other PE problems as an example of the problem, where the game ended when the pursuer attained a previously given distance from the evader. Here we discussed the different strategies the two objects had to pursuer there goal. Kelly's game is an example of a pursuit on a sphere, where the idea is that in a dogfight, the planes tend to move in a circular fashion. We expressed the relation between the objects' speeds and there controls. The outcome was important, and could be used as a simple motion of the PE problem on a sphere.

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