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ABC THEOREMS IN THE FUNCTIONAL CASE

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DEDICATION

I would like to dedicate this manuscript to my parents Ion, Ecaterina and to my lovely niece Maria Valentina.

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ABSTRACT

In this dissertation we will prove some ABC Theorems, namely for relatively prime by pairs p-adic entire functions in one variable, for p-adic meromorphic functions in several variables without common factors, under the hypothesis that no subsum vanishes, and also for pairwise relatively prime p-adic entire functions of several variables. In this thesis we will also prove a few generalizations of Buium's results that he used in order to prove his ABC Theorems for isotrivial abelian varieties, respectively with trace zero. We hope to be able to use these results in order to prove a version of an ABC Theorem for any abelian variety over a function field.

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ABC THEOREMS IN THE FUNCTIONAL CASE

1. **Introduction**

The abc-conjecture was first formulated by Masser and Oesterle in 1985 and it consists of the following:

Conjecture 1. *Given $\epsilon > 0$ there exists $k(\epsilon) > 0$ such that for any non-zero relatively prime integers a, b, c such that $a + b = c$ we have* $\max\{|a|, |b|, |c|\} \leq k(\epsilon)R^{1+\epsilon}$, *where R is the radical of the product abc and represents the product of distinct primes dividing abc.*

Masser was inspired by the Mason's theorem for three polynomials with coefficients in an algebraically closed field of characteristic 0.

Theorem 2 (Mason)**.** *Let $f(t)$, $g(t)$, $h(t)$ be three relatively prime polynomials with coefficients in an algebraically closed field of characteristic* 0, *not all constant such that $f + g = h$ then*

$$\max\{\deg f, \deg g, \deg h\} \leq \deg R(fgh) - 1,$$

where $\deg R(f)$ *is called the degree of the radical of f and represents the number of distinct zeros of f, or the degree of the square free part of f.*

Proof. We sketch the proof for monic polynomials. Let's assume $deg(f) \leq deg(g) \leq deg(h)$. Differentiating $f + g = h$ we get $f^{'} + g^{'} = h^{'}$, hence $fg' - f^{'}g = fh^{'} - f^{'}h$. By hypothesis $gcd(f, g, h) = 1$. But $R(f) = \frac{f}{gcd(f,f^{'})}$ and because $gcd(h, h^{'})$ divides $fh^{'} - f^{'}h$, we get that $gcd(h, h^{'})$ divides also $\frac{fg^{'}-f^{'}g}{gcd(f,f^{'})gcd(g,g^{'})}$, whose degree is less or equal to $\deg R(fg)$. Therefore $deg(h) < \deg R(fgh)$. $\square$

A consequence of Mason's theorem is the following theorem:

Theorem 3 (Fermat's Theorem for polynomials)**.** *Under the hypothesis of Mason's Theorem, the equation*

$$x^n(t) + y^n(t) = z^n(t)$$

has no solution for $n \geq 3$.

Proof. Assume by the way of contradiction that there exists $n \geq 3$ such that the given equation has a solution. Then, by Mason's theorem we get that

$$\deg(x(t)^n) = n\deg(x(t)) \leq \deg(x(t)) + \deg(y(t)) + \deg(z(t)) - 1.$$

Similarly relations for $y(t)$ and $z(t)$ hold.
Summing them up, we get that

$$n\deg(x(t) + \deg(y(t)) + \deg(z(t)) \leq 3[\deg(x(t)) + \deg(y(t)) + \deg(z(t))] - 3,$$

contradicting that $n \geq 3$. So our assumption is false, hence the given equation has no solution for $n \geq 3$.

$\square$

Proof. We also present a simpler proof using algebraic geometry ideas. After dividing by z^n, Fermat's Last Theorem says that there are no nontrivial rational points on the Fermat curve $x^n + y^n = 1$. But the Fermat equation has a polynomial solution which is a rational map from the projective line to the Fermat curve . They exist only if the curve has genus 0. It's known that the Fermat curve has genus $\frac{(n-1)(n-2)}{2}$, which is greater than 0 whenever $n > 2$.

$\square$

Remark 4. *Fermat's theorem for polynomials fails if char $p > 0$, by letting for example $x(t) = t, y(t) = 1$ and $z(t) = t + 1$.*

Mason's theorem for polynomials has been generalized in various directions like for example: to sums in one-dimensional function fields by Mason, by Voloch, by Brownawell and Masser [BM], to sums in higher dimensional function field by Hsia and Wang [HW], to sums of pairwise relatively prime polynomials of several variables by Shapiro and Sparer [SS]. The ABC theorem has also been proved for p-adic entire functions of one variable by Hu and Yang [HY 1].

We now recall some definitions related to p-adic valuations:

Let F denote an algebraically closed field of characteristic 0 complete with respect to a non-trivial non-Archimedean valuation v. Consider $|F^*| = \{|t|_v : t \in F^* = F \setminus \{0\}\}$. Let F^m denote the mth Cartesian product of F, which is the set of F-points of affine m-space $\mathbb{A}^m$. Fix $r > 0$. Then let $\mathbb{B}^m(r)$ be the "closed" ball of radius r in F^m, or in other words

$$\mathbb{B}^m(r) = \{(x_1, \dots x_m) \in F^m : |x_j|_v \leq r\}.$$

By an analytic function f on $\mathbb{B}^m(r)$ we mean a formal power series $\sum_\gamma a_\gamma x^\gamma$ in m variables $x_1, \dots, x_m$ with coefficients in F such that $\lim_{|\gamma|\to\infty} |a_\gamma|_v r^{|\gamma|} = 0$. By an entire p-dic function, we understand a formal power series with coefficients in F and with infinite radius of convergence. We will use $\mathcal{E}_m$ to denote the ring of entire functions on $\mathbb{A}^m$. If $f = \sum_\gamma a_\gamma x^\gamma$ is analytic then define $|f|_r = \sup_\gamma |a_\gamma|_v r^{|\gamma|}$.

The ring of analytic functions on $\mathbb{B}^m(r)$ when $r \in |F^*|$ is factorial (see Theorem 1, section 5.2.6 in [BGR]). Let f be an analytic function on the ball $\mathbb{B}^m(r)$, with a factorisation $f(x) = u \prod_{j=1}^{s} p_j(X)^{\alpha_j}$, where p_j are irreducible, distinct, the $\alpha_j > 0$ are integers and u is a unit in the ball. Then we define the radical of f to be $R(f) = v \prod_{j=1}^{s} p_j$ where v is another unit in $\mathbb{B}^m(r)$. Let us mention that although the ring of entire functions on F^m is not a factorial ring, the notion of "greatest common divisors" does make sens in such a ring , and it is only defined up to units, hence multiplicative constants (see [Ch]). We also say that two p-adic entire functions are relatively prime if their greatest common divisor is a unit (multiplicative constant). Then we can define a notion of the "radical" or the "square free part" of an entire p-adic function f in characteristic zero. The notion of the radical of an entire function in characteristic p was introduced in [ChTo], but we will not make use of it in this dissertation. Let f be an entire function in F^m. We define the radical of f, denoted by $R(f)$, to be the least common multiple of the functions $\frac{f}{gcd(f,\frac{\partial f}{\partial z_j})}$, where j is running from 1 to m. This radical is well-defined up to units.

In our second section we present the ABC Theorem for three p-adic entire functions, which is an analog of the Mason's Theorem:

Theorem 5 (ABC Theorem for three p-adic functions)**.** *Let $f_2 = f_0 + f_1$ be p-adic entire functions such that f_0 and f_1 are relatively prime p-adic entire functions in $\mathcal{E}_m$. If F has characteristic zero, assume that at least one of f_0 or f_1 is non-constant. Let $r_0 > 0$. Then, for $r \geq r_0$,*

$$\max_{0\leq i\leq 2} \log|f_i|_r \leq \log|R(f_0 f_1 f_2)|_r - \log r + O(1).$$

Actually, a part of this dissertation involves an analog of Mason's theorem for p-adic entire functions of one variable, respectively in several variables in an algebraically closed field of characteristic 0. An and Manh, ([AM 2]) gave an ABC type of theorem for p-adic entire functions of several variables $c = a + b$ under some rather restrictive hypotheses, including that a, b, c have no common zeros, which, in several variables, is a much stronger assumption than simply assuming that they are relatively prime in the ring of entire functions. In Section 5 a generalized ABC-Theorem for p-adic entire functions of several variables is proved by adapting the method of Shapiro and Sparer and thereby improving on the work of An and Manh:

Theorem 6. *Let $f_1, \ldots, f_s, s \geq 3$ be pairwise relatively prime p-adic entire functions of several variables on F^m such that at least one of the f_i is non-constant and such that $f_1 + \ldots f_s = 0$. Fix $r_0 \in |F^*|$. Then, for all $r \in |F^*|$ with $r \geq r_0$*

$$\max_{1\leq i\leq s}(\log|f_i|_r) \leq (s-2)[\log|Q|_r - \log r] + O(1),$$

where $Q = R(\prod_{i=1}^{s} f_i)$ *and the* $O(1)$ *is a constant independent of* r.

In Section 3, this theorem, corresponding to functions in one variable, will be improved by adapting an earlier work on polynomials in several variables and generalized Wronskians done by Bayat and Teimoori (see [BT]):

Theorem 7. *Let* $r > 1$ *and* $f_1 + \dots f_{n-1} = f_n$, *in which the* $f_i^{,}s$ *are relatively prime by pairs p-adic entire functions in one variable in* F, *and* k *out of the n-functions are constant (* $k \leq n-2$*). Then, for* $k = 0$, *we have*

$$(1) \qquad \max_{1\leq i\leq n} \log |f_i|_r \leq (n-2)\log |Q[f_1 \dots f_n]|_r - \frac{(n-1)(n-2)}{2}\log r + O(1),$$

and when $k \geq 1$, *we have*

$$(2) \quad \max_{1\leq i\leq n} \log |f_i|_r \leq (n-k-1)\log |Q[f_1 \dots f_n]|_r - \frac{(n-k)(n-k-1)}{2}\log r + O(1),$$

where $Q[f_1 \dots f_n] = R(\prod_{i=1}^{n} f_i)$ *and* $O(1)$ *is a constant independent of* r.

In Section 4 we prove a generalized ABC-Theorem for p-adic entire functions in several variables under the hypothesis that no subsum vanishes. Our method is essentially that of Hu and Yang ([HY 1]) for one variable, who adapted the method of Brownawell and Masser ([BM]).

Theorem 8. *Suppose that* $f_0, f_1, \dots, f_n$ *are non-constant, p-adic entire functions in* $\mathcal{E}_m$ *such that* $gcd(f_1, \dots f_n) = 1$ *and* $gcd(f_{s_{\alpha-1}}, \dots f_{s_\alpha - 1}) = 1$. *Let*

$$f_0 + f_1 \dots f_n = 0$$

such that no subsum vanishes. Then

$$\max_{j=0,\dots,n} \log |f_j|_r \leq l \log |R(\prod_{j=1}^{n_0} f_j) \prod_{\alpha=1}^{k} R(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j)|_r - l\log r + O(1)$$

for $r \geq r_0$ *for some fixed* $r_0 > 1$ *provided that* $n_\alpha \geq 2$ *for* $\alpha = 0, \dots k$.

In the case that all our p-adic entire functions are pairwise relative prime, then

$$\max_{j=0,\dots,n} \log |f_j|_r \leq l \log |R(\prod_{j=0}^{n} f_j)|_r - l \log r + O(1)$$

for $r \geq r_0$ for some fixed $r_0 > 1$ provided that $n_\alpha \geq 2$ for $\alpha = 0, \dots k$, where l, n_α are defined on page 30.

A related result, in the case of polynomials, could be found in [QT1]. But the authors don't consider the hypothesis that no subsum vanishes.

As is well known, the ABC Conjecture is a very strong conjecture that implies many theorems and conjectures in Diophantine equations (see e.g. [La] and [Go]) including the Fermat's Last Theorem for all sufficiently large exponents, proved for arbitrary exponent by Andrew Wiles in 1994. Let us mention some of the consequences of the ABC Conjecture:

Proposition 9. *The ABC Conjecture implies that there is only a finite number of positive integers n, x, y, z satisfying $n > 3, gcd(x, y, z) = 1$ and $x^n + y^n = z^n$.*

Proof. We prove that an explicit form of the weak ABC Conjecture: $a + b \leq R(ab(a + b))^2$ for an abc-triple (a, b, c) would imply a proof of the Asymptotic Fermat's Last Theorem. Assume x, y, z are positive integers that satisfy $x^n + y^n = z^n$ with $gcd(x, y, z) = 1$ and $x < y$. Then (x^n, y^n, z^n) is an abc-triple satisfying $R(x^n y^n z^n) \leq xyz < z^3$. By the explicit form of the weak ABC-Conjecture we get that $z^n < z^6$. But the Fermat equations has no non-trivial solutions for $n = 3$ (Euler), $n = 4$ (Fermat) and $n = 5$ (Dirichlet).

□

Another consequence of the ABC Conjecture is Catalan Conjecture, which was proved by P. Mihailescu in 2002 (see [Mi]) and states that the equation $x^p - y^q = 1$ where the unknowns x, y, p, q take integer values, all greater or equal to 2, has only one solution $(x, y, p, q) = (3, 2, 2, 3)$.

The ABC Conjecture implies the alternative of Pillai Conjecture as well, which states that the equation $x^p - y^q = k$ where the unknowns x, y, p, q take integer values, all ≥ 2, has only finitely many solutions (x, y, p, q), whenever k is a positive integer.

Remark 10. *By using the ABC Inequality for p-adic entire functions, presented in our second chapter, we conclude that Pillai's diophantine equation for p-adic entire functions*

$$f^n - g^m = c$$

where c is a constant function and $n, m \geq 2$ has no non-constant solutions.

Another application of the ABC-Conjecture is the Davenport's theorem proved in 1965 (see [Da]):

Theorem 11. *Let f, g be non-constant polynomials over $\mathbb{C}[x]$ such that $f^2 - g^3 \neq 0$, then we have*

$$\frac{1}{2} \deg g \leq \deg(f^2 - g^3) - 1.$$

Here is an extension of Davenport's theorem for p-adic entire functions, obtained by adapting the proof of the extension of Davenport's theorem for polynomials of Quang and Tuan (see [QT1]) and adding the hypothesis that the functions in any vanishing subsum be relatively prime. We will not provide the proof in our dissertation:

Theorem 12. *Let F be an algebraically closed field of characteristic 0. Given non-constant entire functions of several variables $f_1, \ldots, f_k$,$(k \geq 2)$ in $F[x_1, \ldots, x_l]$ be relatively prime in any vanishing subsum and positive integers $l_j (1 \leq j \leq k)$ such that $l_1 \leq l_2 \leq \ldots l_k$ and at least one of the following conditions is satisfied:*

(1) *The functions $f_1^{l_1}, \ldots, f_k^{l_k}$ have no common zeros.*

(2) $\sum_{j=1}^{k} l_j \leq k l_1 + k(k-1)$.

Suppose that $f_1^{l_1}, \ldots, f_k^{l_k}$ are linearly independent over F, and fix $r_0 \in |F^*|$ such *that* $r_0 > 1$. *Then for all* $r \in |F^*|$ *with* $r \geq r_0$ *we have*

$$(3) \qquad \{1 - \sum_{j=1}^{k} \frac{k-1}{l_j}\} \max_{1 \leq j \leq k} (\log |f_j^{l_j}|_r) \leq \log |\sum_{j=1}^{k} (f_j^{l_j})|_r - (k-1) \log r + O(1)$$

where $O(1)$ *is a constant independent of* r.

Let us recall a version of the ABC Conjecture for $n+1$ polynomials, called the Browkin-Brzeziński Conjecture (see [BB]):

Conjecture 13. *Let* F *be a fixed algebraically closed field of characteristic* 0 *and* $f_0, \ldots f_{n+1}$ *be* $n+2$ *polynomials not all constants in* $F[x]$ *that have no common zeros such that*

$$f_0 + \ldots + f_{n+1} = 0$$

Then

$$\max_{0 \leq j \leq n+1} deg(f_j) \leq (2n-1)(\deg R(f_0 \ldots . f_{n+1}) - 1)$$

Remark 14. *By Corollary* 6.5 *of the Generalized ABC Theorem for p-adic entire functions (see* [ChTo]*), if* F *has characteristic zero and in the case of polynomials when* $k = 3$*, and* $\bar{A} = 2n-3$*, we recover the Browkin-Brzeziński's Theorem of Quang and Tuan in* [QT2]*:*

$$\max_{0 \leq j \leq n} deg f_j \leq (2n-3)[\deg R(f_0 \ldots . f_{n+1}) - 1]$$

if $gcd(f_{i_1}, f_{i_2}, f_{i_3}) = 1$ *for all triples* $i_1 < i_2 < i_3$. *Note that in* [QT2] *and* [QT3] *(for the p-adic case) the authors neglected the necessary hypothesis that the functions in any vanishing subsum be relatively prime.*

The following theorem is another consequence of the ABC Conjecture (see [Ro]):

Theorem 15 (Roth's Theorem)**.** *Fix* $\epsilon > 0$. *For very algebraic number* α*, the diophantine inequality*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}}$$

has only finitely many solutions in coprime integers p, q.

The ABC Conjecture also implies the Mordell's Conjecture, which has been proved by G. Faltings in [Fa]:

Theorem 16 (Faltings' Theorem)**.** *Any curve of genus greater than* 1 *defined over a number field* K *has only finitely many rational points in* K.

The next theorem was proved by Elkies [El] in 1991:

Theorem 17. *The ABC Conjecture for number fields implies the Mordell Conjecture over an arbitrary number field.*

The following Mordel-Weil theorems corresponding to number fields and function fields are related to Falting's Theorem and the results of our last dissertation section:

Theorem 18 (Mordell-Weil)**.** *If* A *is an abelian variety over a number field* K*, then* $A(K)$ *is a finitely generated abelian group.*

Theorem 19 (Lang-Néron)**.** *If* A *is an abelian variety over a function field* K *and* A *has trace zero, then* $A(K)$ *is a finitely generated abelian group.*

In order to present the main result of our last dissertation section, we will make use of the following definitions:

Let k be a field and K be an extension field. Fix a non-zero k-derivation δ on K. By a D-scheme we understand a K-scheme V together with a lifting of δ to a derivation of $\mathscr{O}_V$ Then a D-group scheme is a group object in the category of D-schemes. Finally, an algebraic D-group is a D-group scheme which is of finite type over K. We also recall the notion of an isogeny. A homomorphism $\alpha : A \to B$ of abelian varieties is called isogeny if it is surjective and has finite kernel.

We now state the Bounded Multiplicity Theorems for isotrivial abelian varieties, respectively with trace zero over a function field proved by A. Buium in [Bu2] and [Bu3]:

Theorem 20 (Buium's Bounded Multiplicity Theorem-Isotrivial Case)**.** *Let* X *be a smooth projective complex curve ,* A *an abelian variety , and* Y *an effective divisor* *on A. Assume that Y contains no translate of a non zero abelian subvariety. Then there exists a real constant $C > 0$, depending only on X, A, Y with the property that for any morphism $f : X \to A$ with $f(X) \not\subset Y$, all points of the divisor f^*Y have multiplicity at most C.*

Before we state the next theorem, we need to explain what the bounded multiplicity property is in the non-isotrivial case. Let A be an abelian variety over a function field K over a field k of characteristic zero and $U \subset A$ an affine open subset. We say that a regular function $f \in \mathscr{O}(U)$ has the bounded multiplicity property if there exists a positive constant C depending on K, A, U, f such that for any $P \in U(K)$ and any $\nu \in M_K$ we have $\nu(f(P)) \geq -C$. Here we denote by M_K the set of places (by which we mean here discrete valuations) of K/k such that for all $\nu \in M_K, \nu(K^*) = \mathbb{Z}$.

Theorem 21 (Buium's Bounded Multiplicity Theorem-Trace Zero Case)**.** *The bounded multiplicity property holds for any regular function on any affine open subset of any abelian variety with trace zero.*

Before we make the next remark, let us recall the following definitions:

Let K be a field equipped with a family of absolute values $||_v : K \to [0, \infty), v \in M_K$, all of which, except finitely many, are non-archimedean. Set $v(x) = -\log |x|_v$ for $x \in K^\times$. Let (m_v) be a collection of positive integers such that the "product formula"

$$\sum_v m_v v(x) = 0, x \in K^\times$$

holds. Set

$$\gamma_v := \inf\{v(K^\times) \cap (0, \infty)\}$$

Also, for any $\mu = (\mu_1, \dots \mu_N) \in K^N$ and $v \in M_K$ set

$$v(\mu) = \min_j v(\mu_j)$$

Define the (affine, logarithmic) height

$$height_{A^N} : A^N(K) = K^N \to [0, \infty)$$

by the formula

$$height_{A^N}(\mu) = -\sum_{v(\mu)\leq 0} m_v v(\mu) = \sum_v m_v \max_j \log^+ |\mu_j|_v$$

where $\log^+ x := \max\{\log x, 0\}, x \in [0, \infty)$. Note that

$$height_{A^N}(\mu) = height_{P^N}(1 : \mu_1 \ldots : \mu_N)$$

where

$$height_{P^N}(x_0 : \ldots : x_N) = \sum_v m_v \max_j \log |x_j|_v$$

is the usual height in projective space. On the other hand define the (logarithmic) conductor

$$cond_{A^N} : A^N(K) \to [0, \infty)$$

by the formula

$$cond_{A^N}(\mu) = \sum_{v(\mu)\leq 0} m_v \gamma_v$$

Clearly, by the very definition of γ_v we have

$$cond_{A^N}(\mu) \leq height_{A^N}(\mu), \mu \in K^N.$$

It is easy to see that if $P : A^N(K) \to A^n(K)$ is a map given by an n-tuple of polynomials in N variables with K-coefficients then we have

$$height_{A^n} \circ P << height_{A^N} + O(1)$$

$$cond_{A^n} \circ P \leq cond_{A^N} + O(1)$$

This allows to define the height and conductor for any affine variety as follows. Let U be an affine variety over a function field K and N a positive integer. Let

$i : U \to \mathbb{A}^N$ be a closed immersion and define the height and conductor

$$height_U : U(K) \to [0, \infty)$$

$$cond_U : U(K) \to [0, \infty)$$

by the formulae

$$height_U(P) := height_{\mathbb{A}^N}(i(P)), P \in U(K)$$

$$cond_U(P) := cond_{\mathbb{A}^N}(i(P)), P \in U(K)$$

Definition 22. *We say that the "abc" estimate holds on an affine variety U if*

$$height_U << cond_U + O(1)$$

Remark 23. *The bounded multiplicity property implies an "abc" estimate. But the converse is not true, because the bounded multiplicity property fails in $U = \mathbb{P}^1 \backslash 3 points$, but Mason's theorem (i.e. an "abc" estimate in $U = \mathbb{P}^1 \backslash 3 points$) is true. To see that the bounded multiplicity property fails in $U = \mathbb{P}^1 \backslash 3 points$, consider $K = \mathbb{C}(t)$ and f to be the identity function. Then we could find a sequence of points $P_n \in \mathbb{A}^1(K)$, where $P_n = z_n = z_n(t) = t^n$ such that $f(P_n) = z_n = t^n$ has a zero at 0 of order $n \to \infty$.*

Let us now recall the definition of the Albanese variety $Alb(V)$, given by the following Universality Property: There is a morphism from the variety V to its Albanese variety $Alb(V)$, such that any morphism from V to an abelian variety (taking the given point to the identity) factors uniquely through $Alb(V)$. In order to prove the Bounded Multiplicity Theorem-Trace Zero Case, Buium made use of an equivalent form of the following result (see [Bu1]):

Theorem 24 (Buium's Theorem 2)**.** *Let W be any variety of general type over K. Let G be any algebraic D-group, $V \subset G$ a D-subvariety and $u : V \to W$ be a dominant morphism. Then the Albanese variety $Alb(W)$ descends to k.*

In the last chapter of this dissertation we will prove a generalization of this theorem. Let us denote by K the function field defined over an algebraically closed field k. Before we state such a generalization, let us make the following:

Remark 25. *Any abelian variety A is, up to isogeny and after replacing K by a finite extension of it, a product $B \times C$ with B isotrivial and C with trace zero. So, from now on, we can concentrate on the case when A is actually equal to $B \times C$.*

Theorem 26. *Let A be any abelian variety, $W \subset A$ subvariety of general type. Let G be any algebraic D-group, $V \subset G$ a D-subvariety and $u : V \to W$ be a dominant morphism. Then $W = (Z_K + Q) \times W'$ for some k-subvariety $Z_K = Z \otimes K$ of A, some $Q \in A(K)$ and $W' \subset A$ subvariety, such that $Alb(W')$ descends to k (where $Z_K := Z \otimes_k K$).*

In order to prove the ABC Theorem-Isotrivial Case, Buium made use of his lemma:

Lemma 27. *Let W be a projective variety of general type over K. Asume W is a closed subvariety of A_K, where A is an abelian k-variety (here k denotes the field of complex numbers). Let G be any algebraic D-group, $V \subset G$ an absolutely irreducible, reduced, D-subscheme and $u : V \to W$ be a dominant morphism of K-schemes. Then after replacing K by a finite extension of it, one may find a closed k-subvariety $Z \subset A$ and a point $Q \in A(K)$ such that $W = (Z_K + Q)$ in A_K.*

We will generalize the above Buium's Lemma (see [Bu2]) and get the following result for any abelian variety over a function field, not necessarily isotrivial or with trace zero:

Lemma 28. *Let W be a projective variety of general type over K. Asume W is a closed subvariety of A_K, where A is an abelian k-variety, $A = B \times C$ with B isotrivial and C with trace zero. Let G be any algebraic D-group, $V \subset G$ an absolutely irreducible, reduced D-schemes and $u : V \to W$ be a dominant morphism of K-schemes. Then after replacing K by a finite extension of it, one may find a closed* *k-subvariety Z of A and a point $Q \in A(K)$ such that $W = (Z_K + Q)$ (where $Z_K = Z \otimes SpecK$).*

Finally, we state our main result which is stronger than the previous lemma:

Proposition 29. *Let W be a projective variety of general type over K. Asume W is a closed subvariety of A_K, where A is an abelian k-variety, $A = B \times C$ with B isotrivial and C with trace zero. Let G be any algebraic D-group, $V \subset G$ an absolutely irreducible, reduced D-scheme and $u : V \to W$ be a dominant morphism of K-schemes. Then after replacing K by a finite extension of it, one may find an isotrivial k-subvariety $Z \subset B$ and an abelian k-subvariety $T \subset C$ with trace zero and points $Q \in B(K)$,$P \in C(K)$ such that $W = (Z_K + Q) \times (T + P)$(where $Z_K = Z \otimes SpecK$).*

Remark 30. *We hope to be able to use these above results in order to prove a version of an ABC Theorem for any abelian variety over a function field, not necessarily isotrivial or with trace zero.*

We conclude by noting:

Remark 31. *In 2012, Shinichi Mochizuki, announced the proof of the ABC Theorem for number fields.*

2. **Preliminaries**

In this section we give some definitions of p-adic absolute values and present some of their basic properties. We also present the ABC Theorem for three p-adic entire functions and conclude by using it in order to make a remark about the Pillai's diophantine equation involving such functions. A non-Archimedean absolute value $||$ over a commutative ring A is a function on A to the non-negative numbers satisfying the following three properties:

1) $|a| = 0$ if and only if $a = 0$;

2) $|ab| = |a||b|$ for all $a, b \in A$;

3) $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in A$.

Let F denote an algebraically closed field of characteristic 0 complete with respect to a non-trivial non-Archimedean absolute value $|.|$ Let

$$|F^*| = \{|t| : t \in F^* = F \setminus \{0\}\}.$$

Fix $r > 0$. Then let $\mathbb{B}^m(r)$ be the "closed" ball of radius r in F^m, or in other words

$$\mathbb{B}^m(r) = \{(x_1, \dots x_m) \in F^m : |x_j| \leq r\}.$$

If $x_1, \dots, x_m$ are F- valued variables, we use x to denote the m- tuple $(x_1, \dots, x_m)$.We use multi-index notation, so if $\gamma = (\gamma_1, \dots \gamma_m)$, where the γ_j are non-negative integers, then by definition

$$x^\gamma = x_1^{\gamma_1} \dots x_m^{\gamma_m}, |\gamma| = \gamma_1 + \dots + \gamma_m, \text{and} \gamma! = \gamma_1! \dots \gamma_m!.$$

By an analytic function f on $\mathbb{B}^m(r)$ we mean a formal power series

$$\sum_\gamma a_\gamma x^\gamma$$

in m variables $x_1, \dots, x_m$ with coefficients in F such that

$$\lim_{|\gamma| \to \infty} |a_\gamma| r^{|\gamma|} = 0.$$

If $f = \sum_{\gamma} a_\gamma x^\gamma$ is analytic on $\mathbb{B}^m(r)$, then define

$$|f|_r = \sup_{\gamma} |a_\gamma| r^{|\gamma|}.$$

Denote by $\mathcal{A}^m(r)$ the ring of analytic functions in $\mathbb{B}^m(r)$, which is factorial if $r \in |F^*|$(see [BGR]) Theorem 1 in section 5.2.6). By an entire p-dic function, we understand a formal power series with coefficients in F and with infinite radius of convergence. By a meromorphic function f on F^m (or on $\mathbb{B}^m(r)$) we will mean the quotient of two analytic functions $\frac{f}{g}$ such that f and g do not have common factors in the ring of analytic functions. Note that $| \, |_r$ is multiplicative, meaning that if f, g are two analytic functions then

$$|fg|_r = |f|_r |g|_r.$$

Let us now consider f to be an analytic function on the ball $\mathbb{B}^m(r)$, $r \in |F^*|$with the following factorisation $f(x) = u \prod_{j=1}^{s} p_j(X)^{\alpha_j}$, where p_j are irreducible, distinct, the $\alpha_j > 0$ are integers and u is a unit in the ball. Then we define the radical of f to be $R(f) = v \prod_{j=1}^{s} p_j$ where v is another unit in $\mathbb{B}^m(r)$. Let us mention that although the ring of entire functions on F^m is not a factorial ring, the notion of "greatest common divisors" does make sens in such a ring , and it is only defined up to units, hence multiplicative constants (see [Ch]). We also say that two p-adic entire functions are relatively prime if their greatest common divisor is a unit (multiplicative constant). Then we can define a notion of the "radical" or the "square free part" of an entire p-adic function f in characteristic zero. Let f be an entire function in F^m. For each j from 1 to m, define the radical of f, denoted by $R(f)$, to be the least common multiple of the functions $\frac{f}{gcd(f, \frac{\partial f}{\partial z_j})}$ as in [ChTo].

Proposition 32. *If f is an entire function then $|f|_r$ is a non-decreasing function of r.*

If f happens to be a polynomial of degree d, then we easily see that as $r \to \infty$,

$$\log |f|_r = d \log r + O(1).$$

Thus, in our ABC theorems for entire functions, $\log |f|_r$ will play the roll played by the degree in the case of polynomials in the left-hand side of the inequalities.

Corollary 33. *If f, g and h are p-adic entire functions such that $f = gh$ and if $r_0 > 0$, then for all $r \geq r_0$,*

$$\log |g|_r \leq \log |f|_r + O(1).$$

Proof. By the multiplicativity of $| \ |_r$ and Proposition 32,

$$\log |f|_r = \log |g|_r + \log |h|_r \geq \log |g|_r + \log |h|_{r_0},$$

which gives the first inequality. $\square$

Lemma 34 (p-adic Logarithmic Derivative Lemma)**.** *If f is a p-adic analytic function, then $|\frac{f'}{f}|_r \leq 1/r$.*

Proof. Let $f = \sum a_n z^n$. Since $|n|_r \leq 1$, we have

$$|f'|_r = \sup_{n \geq 1} |na_n|_r r^{n-1} = \frac{1}{r} \sup_{n \geq 1} |na_n|_r r^n \leq \frac{1}{r} \sup_{n \geq 0} |a_n|_r r^n = \frac{1}{r} |f|_r.$$

$\square$

If $f = \frac{g}{h}$ is a meromorphic function, then $|f|_r = \frac{|g|_r}{|h|_r}$.

We will make use many times of the following differential operator Δ of the form

$$\Delta = (\mu_1 \dots \mu_m)^{-1} \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}$$

where $\mu_i \geq 0$ are integers. Let's denote its rank by

$$\rho(\Delta) = \sum_{i=1}^{m} \mu_i = |\mu|.$$

Given $\Delta_0 \dots \Delta_n$ such that $\rho(\Delta_i) \leq i$, with $i = 0, 1 \dots n$ and $f_0, \dots, f_n$ are nonzero entire functions in F^l, then we define a generalized Wronskian as :

$$W[f_0, \dots, f_n] = \det |\Delta_i f_j|.$$

We will now prove the most basic version of an ABC theorem for non-Archimedean entire functions of several variables.

Theorem 35 (ABC Theorem for three p-adic entire functions)**.** *Let $f_2 = f_0 + f_1$ be entire functions such that f_0 and f_1 are relatively prime entire functions. If* **f** *has characteristic zero, assume that at least one of f_0 or f_1 is non-constant. Let $r_0 > 0$. Then, for $r \geq r_0$,*

$$\max_{0 \leq i \leq 2} \log |f_i|_r \leq \log |R(f_0 f_1 f_2)|_r - \log r + O(1).$$

Proof. We sketch the proof, by following the standard polynomial proof, as given for instance in [Va] and [ChTo], *mutatis mutandis.* Without loss of generality assume that f_0 is non-constant and that $\partial P_0/\partial z_1 \not\equiv 0$. Consider the Wronskian determinant,

$$W = \det \begin{pmatrix} f_0 & f_1 \\ \dfrac{\partial f_0}{\partial z_1} & \dfrac{\partial f_1}{\partial z_1} \end{pmatrix} = \det \begin{pmatrix} f_0 & f_2 \\ \dfrac{\partial f_0}{\partial z_1} & \dfrac{\partial f_2}{\partial z_1} \end{pmatrix} = \det \begin{pmatrix} f_2 & f_1 \\ \dfrac{\partial f_2}{\partial z_1} & \dfrac{\partial f_1}{\partial z_1} \end{pmatrix},$$

where the first equality defines W and the second two equalities follow from $f_2 = f_0 + f_1$.

We notice that $W \not\equiv 0$, because f_0 and f_1 were assumed relatively prime.

Let $F = f_0 f_1 f_2$, $G = \gcd(F, \partial F/\partial z_1)$, and $H = F/G$. Then, by definition H divides $R(f_0 f_1 f_2)$, and so

$$\log |H|_r \leq \log |R(f_0 f_1 f_2)|_r + O(1)$$

for $r \geq r_0$ by Corollary 33. We notice that G divides W. Again applying Corollary 33, we see that for $r \geq r_0$,

$$\log |G|_r \leq \log |W|_r + O(1).$$

By the p-adic Logarithmic Derivative Lemma,

$$\left| f_i \frac{\partial f_j}{\partial z_1} \right|_r \leq \frac{|f_i f_j|_r}{r},$$

and hence using each of the three determinants defining W,

$$\log |W|_r \leq \log \min\{|f_0 f_1|_r, |f_0 f_2|_r, |f_1 f_2|_r\} - \log r.$$

Hence,

$$\begin{aligned}
\log \max |f_i|_r &= \log |f_0|_r + \log |f_1|_r + \log |f_2|_r - \log \min_{0 \leq i < j \leq 2} |f_i f_j|_r \\
&= \log |F|_r - \log \min_{0 \leq i < j \leq 2} |f_i f_j|_r \\
&= \log |H|_r + \log |G|_r - \log \min_{0 \leq i < j \leq 2} |f_i f_j|_r \\
&\leq \log |R(F)|_r + \log |W|_r - \log \min_{0 \leq i < j \leq 2} |f_i f_j|_r + O(1) \\
&\leq \log |R(F)|_r - \log r + O(1),
\end{aligned}$$

for $r \geq r_0$. □

Remark 36. *By the ABC Inequality for p-adic entire functions, because* $\frac{1}{n} + \frac{1}{m} \leq 1$, *we conclude that the Pillai's diophantine equation for p-adic entire functions* $f^n - g^m = c$ *where* c *is a constant function and* $n, m \geq 2$ *has no non-constant solutions.*

3. **A Generalized ABC-Theorem For P – Adic Entire Functions In One Variable**

In this section we will prove a generalized ABC-Theorem for relatively prime by pairs p-adic entire functions in one variable, by adapting the proof of the Theorem 5 (see [BT]). Note that the proof of their result is based on their Lemma 4 which is false for several variable polynomials, but true for univariate ones, as remarked by De Bondt in [DeBont]. However, their result is correct for several variable polynomials, but one needs another idea to prove it. In order to prove the theorem, we consider two cases as Bayat and Teimoori did. For both cases I apply Lemma 39 which is an adaption of Lemma 2.1 (see [SS]):

Lemma 37. *Let* $\Delta = (\mu_1 \ldots \mu_m)^{-1} \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \ldots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}$ *be an operator with* $\mu_i \geq 0$ *are integers and the rank of* Δ *is* $\rho(\Delta) = t = \sum_{i=1}^{m} \mu_i = |\mu|$. *Then for* $f \in \mathbb{C}[X]$ *not zero we have*

$$f^{-1}\Delta f = \frac{P(X)}{Q[f]^t}$$

where $P = P(X) \in \mathbb{C}[X_1, \ldots, X_n]$,

$$degP \leq t(degQ - 1)$$

and $Q(X) = Q[f] = R(f)$.

In order to prove Lemma 39, we use the following:

Lemma 38. *Let* g *be a meromorphic function on* F^m *and* $\Delta = \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}$ *be an operator with* $\mu_i \geq 0$ *are integers. Then* $|\frac{\Delta g}{g}|_r \leq \frac{1}{r^{|\rho(\Delta)|}}$, *where the rank of* Δ *is* $\rho(\Delta) = \sum_{i=1}^{m} \mu_i = |\mu|$.

Proof. See [CY, Lemma 4.1] □

Lemma 39. *Let* $r \in |F^*|$ *and let* $f \in \mathcal{A}^m(r)$. *Let* $Q = R(f)$ *and* $\Delta = (\mu_1 \ldots \mu_m)^{-1} \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}$ *be an operator with* $\mu_i \geq 0$ *are integers and the rank of* Δ *is* $\rho(\Delta) = \sum_{i=1}^{m} \mu_i = |\mu|$.

Then, there is an analytic function g on $\mathbb{B}^m(r)$ such that

$$f^{-1}\Delta f = \frac{g}{Q^{\rho(\Delta)}}$$

and such that

$$\log|g|_r \leq \rho(\Delta)[\log|Q|_r - \log r].$$

Proof. If $\rho(\Delta) = 0; \Delta f = f; g = 1$ so the lemma is true in this case.

As in [SS] if f has a factorisation

$$f(x) = u\prod_{j=1}^{s} p_j(X)^{\alpha_j}$$

where p_j are irreducible, distinct, the $\alpha_j > 0$ are integers and u is a unit in $\mathbb{B}^m(r)$. Then

$$Q = v\prod_{j=1}^{s} p_j$$

where v is another unit in $\mathbb{B}^r$. Then

$$f^{-1}\frac{\partial f}{\partial x_i} = \frac{\frac{\partial u}{\partial x_i}}{u} + \sum_{j=1}^{s}\alpha_j\frac{\frac{\partial p_j}{\partial x}}{p_j} = \frac{vg_1}{uQ}.$$

Set $g_2 = \frac{vg_1}{u}$. Then $\log\left|\frac{g_2}{Q}\right|_r = \log\left|\frac{\frac{\partial f}{\partial x_i}}{f}\right|_r \leq -\log r$ by Lemma 38, thus

$$\log|g_2|_r \leq \log|Q|_r - \log r.$$

Therefore

$$\begin{aligned}\log|gg_2|_r &= \log|g|_r + \log|g_2|_r \\ &\leq \rho(\Delta)(\log|Q|_r - \log r) + \log|Q|_r - \log r \\ &= [\rho(\Delta) + 1](\log|Q|_r - \log r)\end{aligned}$$

Let $t = \rho(\Delta)$ therefore

$$\log|gg_2|_r \leq (t+1)[\log|Q|_r - \log r]$$

Proceeding by induction on t so that assuming the lemma for Δ, it suffices to prove it for

$$\bar{\Delta} = \frac{\partial}{\partial x_i}\Delta,$$

as in [SS]. But as in [SS]

$$\begin{aligned} f^{-1}\bar{\Delta}f &= (f^{-1}\Delta f)\frac{f_{x_i}}{f} + \frac{g_{x_i}Q - tQ_{x_i}g}{Q^{t+1}} \\ &= \frac{g}{Q^t}\frac{g_2}{Q} + \frac{g_{x_i}Q - tQ_{x_i}g}{Q^{t+1}} \end{aligned}$$

where $g_{x_i} = \frac{\partial g}{\partial x_i}$ and $Q_{x_i} = \frac{\partial Q}{\partial x_i}$. If R is the numerator in this relation, then

$$|R|_r = |gg_2 + g_{x_i}Q - tQ_{x_i}g|_r \leq \max\{|gg_2|_r, |g_{x_i}Q - tQ_{x_i}g|_r\}$$

In order to show that

$$\log|R|_r \leq (t+1)[\log|Q|_r - \log r]$$

we use that

$$|R|_r \leq \max\{|gg_2|_r, |g_{x_i}Q|_r, |-tQ_{x_i}g|_r\}$$

where $|-tQ_{x_i}g|_r = |tQ_{x_i}g|_r$. By the p-adic Logarithmic Derivative Lemma we get

$$|g_{x_i}|_r \leq \frac{|g|_r}{r}$$

Similarly

$$|Q_{x_i}|_r \leq \frac{|Q|_r}{r}$$

Thus

$$\log|R|_r \leq \max\{(t+1)[\log|Q|_r - \log r], (t+1)[\log|Q|_r - \log r] + \log|t|_r\}$$

Because $|t|_r \leq 1$ then we get that

$$\log|R|_r \leq (t+1)[\log|Q|_r - \log r].$$

$\square$

We will now prove two lemmas, which will be used to prove our next theorem:

Lemma 40. *Let $\Delta_i (i = 1, \ldots, s)$ be a differential operator with $\rho(\Delta_i) \leq i$. Then for f nonzero entire function in several variables in F^l we have*

$$\log|f|_r - s\log|Q|_r \leq \log|(f, \Delta_1 f, \ldots, \Delta_s f)|_r + O(1) \tag{4}$$

for all $r \in |F^*|$, where $Q = R(f)$ and $(f, \Delta_1 f, \ldots, \Delta_s f)$ is the greatest common divisor of $f, \Delta_1 f, \ldots, \Delta_s f$ and $O(1)$ is a constant independent of r.

Proof. Suppose $p^e \mid f$ and $p^{e+1} \nmid f$. We consider the following two cases:

Case 1 : Suppose that $e \leq s$ Then $p^e \mid f$ and $p^e \mid Q^s$. Therefore $p^e \mid (f, Q^s)$, hence $p \nmid \frac{f}{(f,Q^s)}$.

Case 2 : Suppose $e > s$. Hence $p^s \mid (f, Q^s)$, also $p^{e-s} \mid (f, \Delta_1 f, \ldots, \Delta_s f)$, thus

$$p^{e-s+1} \nmid \frac{f}{(f, Q^s)}.$$

Because $p^e \mid f$ and $p^{e-i} \mid \Delta_i f$, also $\rho(\Delta_i) \leq i$ for all $i = 1, \ldots, s$ we get that

$$p^{e-s} \mid (f, \Delta_1 f, \ldots, \Delta_s f).$$

$$\begin{aligned} \log|f|_r - s\log|Q|_r &\leq \log\frac{|f|_r}{|(f, Q^s)|_r} \\ &\leq \log|(f, \ldots, \Delta_s f)|_r + O(1) \end{aligned}$$

□

Lemma 41. *Suppose $f_1, \ldots, f_n$ are nonzero p-adic entire functions in one variable in F. Then, for $W[f_1, \ldots, f_n] \neq 0$ we have*

$$\log|W[f_1, \ldots, f_n]|_r \leq \log|f_1 \ldots f_n|_r - \frac{n(n-1)}{2}\log r. \tag{5}$$

for any $r > 1$

Proof. The proof is done by induction on n. The initial step $n = 1$ is clear. Suppose it's true for $(n-1)$ functions. By expanding the Wronskian determinant$W[f_1, \ldots, f_n]$ with respect to the first row , we obtain as in [BT]:

(6)

$$\log |W[f_1, \ldots, f_n]|_r = \log |\sum_{i=1}^{n} (-1)^{i+1} f_i W[\Delta_1 f_1, \ldots, \Delta_1 f_{i-1}, \Delta_1 f_{i+1}, \ldots, \Delta_1 f_n]|_r$$

We have that:

$$\log |W[f_1, \ldots, f_n]|_r \leq \max_{1 \leq i \leq n} (\log |f_i|_r + \log |W[\Delta_1 f_1, \ldots, \Delta_1 f_{i-1}, \ldots, \Delta_1 f_n]|_r),$$

since $W[f_1, \ldots, f_n] \neq 0$, then there exists an i such that the right-hand side has the greatest absolute valuation in logarithm, namely

$$\log |W[f_1, \ldots, f_n]|_r \leq \log |f_i|_r + \log |W[\Delta_1 f_1, \ldots, \Delta_1 f_{i-1}, \ldots, \Delta_1 f_n]|_r.$$

We now get by the induction hypothesis that

$$\log |W[\Delta_1 f_1, \ldots, \Delta_1 f_{i-1}, \ldots, \Delta_1 f_n]|_r \leq \log |(\Delta_1 f_1 \ldots \Delta_1 f_{i-1} \Delta_1 f_{i+1} \ldots \Delta_1 f_n)|_r -$$

(7) $$\frac{(n-1)(n-2)}{2} \log r \leq \log |f_1 \ldots f_{i-1} f_{i+1} \ldots f_n|_r - \frac{n(n-1)}{2} \log r$$

since $\log |\Delta_1 f_i|_r = \log |f_i|_r + \log |g|_r - \rho(\Delta_1) \log |Q|_r$ for all i by Lemma 39. Hence $\rho(\Delta_1) \geq 1$, and $\log |g|_r \leq \rho(\Delta_1)[\log |Q|_r - \log r]$ so

$$\log |\Delta_1 f_i|_r \leq \log |f_i|_r - \log r.$$

Therefore we get by mixing the equations (6) and (7):

$$\log |W[f_1, \ldots, f_n]|_r \leq \log |f_1 \ldots f_n|_r - \frac{n(n-1)}{2} \log r.$$

for any $r > 1$. □

Theorem 42. *Let $r > 1$ and $f_1 + \dots f_{n-1} = f_n$, in which the $f_i^{,}s$ are relatively prime by pairs p-adic entire functions in one variable in F, and k out of the n-functions are constant ($k \leq n-2$). Then, for $k = 0$, we have*

$$\max_{1 \leq i \leq n} \log |f_i|_r \leq (n-2) \log |Q[f_1 \dots f_n]|_r - \frac{(n-1)(n-2)}{2} \log r + O(1), \tag{8}$$

and when $k \geq 1$, we have

$$\max_{1 \leq i \leq n} \log |f_i|_r \leq (n-k-1) \log |Q[f_1 \dots f_n]|_r - \frac{(n-k)(n-k-1)}{2} \log r + O(1), \tag{9}$$

where $Q[f_1 \dots f_n] = R(\prod_{i=0}^{n} f_i)$ and $O(1)$ is a constant independent of r.

Proof. For different $k^{,}s$, we distinguish two cases as in [BT]:

Case1. Let $f_1, \dots, f_{n-1}$, be linearly independent over F Therefore $k = 0$ or $k = 1$ and $W[f_1, \dots f_{n-2}, f_{n-1}] \neq 0$. (see [Ro] and [Sc]) Without loss of generality, we assume that f_n is such that $\log |f_n|_r = \max_{i=1}^{n} (\log |f_i|_r)$, and therefore it is necessary to prove that

$$\log |f_n|_r \leq (n-2) \log |Q[f_1 \dots f_n]|_r - \frac{(n-1)(n-2)}{2} \log r.$$

We have $f_1 + \dots f_{n-1} = f_n$, so we get as in [BT] that:

$$W[f_1, \dots f_{n-2}, f_{n-1}] = W[f_1, \dots f_{n-2}, f_n].$$

But for any $i (i = 1, \dots, n)$, we have that

$$(f_i, \Delta_1 f_i, \dots, \Delta_{n-2} f_i) \mid W[f_1, \dots f_{n-2}, f_{n-1}].$$

Now, since the $f_i^{,}s$ are relatively prime by pairs, we obtain as in [BT]

$$\prod_{i=1}^{n} (f_i, \Delta_1 f_i, \dots, \Delta_{n-2} f_i) \mid W[f_1, \dots f_{n-2}, f_{n-1}],$$

and since $W[f_1, \dots f_{n-2}, f_{n-1}] \neq 0$, we conclude that

$$\sum_{i=1}^{n} \log|(f_i, \Delta_1 f_i, \dots, \Delta_{n-2} f_i)|_r \leq \log|W[f_1, \dots f_{n-2}, f_{n-1}]|_r + O(1).$$

Using equations (4) and (5) we obtain

$$\sum_{i=1}^{n}(\log|f_i|_r - (n-2)\log|Q[f_i]|_r) \leq \log|f_1 \dots f_{n-1}]|_r - \frac{(n-1)(n-2)}{2}\log r + O(1)$$

and since the $f_i^{,}s$ are relatively prime by pairs we get that

$$\log|f_n|_r \leq (n-2)\log|Q[f_1 \dots f_n]|_r - \frac{(n-1)(n-2)}{2}\log r + O(1),$$

Case2. Let $f_1, \dots, f_{n-1}$, be linearly dependent over F. The proof is done by induction on n. as in [BT]. For $n = 3$, by assumptions $k = 0, 1, 2, 3$, where $k = 3$ is trivial and $k = 2$ is impossible and cases $k = 0, 1$ are just Mason's theorem. Suppose that the theorem is true for all cases $m^{,}s, 3 \leq m \leq n$, and consider n entire functions. In equality $f_1 + \dots f_{n-1} = f_n$, as in [BT] assume the $f_i (i = 1, 2 \dots, n-1)$, are linearly dependent over F, and also k out of the n- functions ($k \geq n-2$) are constant, when $k = n$ inequality (9) is obvious and the case $k = n-1$ is clearly impossible, so $k \leq n-2$. Let $f_{i_1}, \dots, f_{i_q}, q \leq n-k$, be a maximal linearly independent subset of the $f_{i_1}, \dots, f_{i_{n-1}}$. Since $n-k \geq 2$, and the $f_j^{,}s$ are relatively prime by pairs, it follows that $q \geq 2$. So each $f_j, 1 \leq j \leq n-k; j$ not one of the $i_l^{,}s$, is a linear combination of the f_{i_l} of the form

$$f_j = \lambda_1 f_{i_1} + \dots + \lambda_q f_{i_q}, \tag{10}$$

where the $\lambda_l \in F$, and at least two of these $\lambda_l^{,}s$ are not zero. As in [BT] using our inductive hypothesis we apply case 1. This yield that $\lambda_l \neq 0$, then

$$\log|f_{i_l}|_r \leq (q-1)\log|Q[f_j(\prod_{l=1}^{q} f_{i_l})]|_r - \frac{q(q-1)}{2}\log r + O(1),$$

and so that

$$(11) \qquad \log|f_{i_l}|_r \leq (q-1)\log|Q[\prod_{l=1}^{n} f_i]|_r - \frac{q(q-1)}{2}\log r + O(1).$$

Now, since k out of the $f_i^{,}s$ are constant we obtain

$$(q-1)\log|Q[\prod_{l=1}^{n} f_i]|_r - \frac{q(q-1)}{2}\log r + O(1).$$

$$(12) \qquad \leq (n-k-1)\log|Q[\prod_{l=1}^{n} f_i]|_r - \frac{(n-k)(n-k-1)}{2}\log r + O(1).$$

Now, by using the equations (11) and (12) we get

$$(13) \qquad \log|f_{i_l}|_r \leq (n-k-1)\log|Q[\prod_{l=1}^{n} f_i]|_r - \frac{(n-k)(n-k-1)}{2}\log r + O(1).$$

Starting from (10) we get the same estimate (13) for $\log|f_j|_r$. Therefore the theorem is proved for such f_j and f_{i_l}. As in [BT] inserting all the relations of the form (10) into the right side of the equality $f_1 + \ldots f_{n-1} = f_n$ yields an equation of the form

$$(14) \qquad f_n = k_1 f_{i_1} + \ldots + k_q f_{i_q},$$

where the $k_j \in \mathbb{C}$. Moreover, if one of these $k_\nu = 0$, then the corresponding f_{i_ν} must appear in one of the equations (10) with a nonzero λ_ν. Hence (13) is proved for this f_{i_ν}. Finally for those $k_\nu \neq 0$, we treat (14) exactly as we did with (10), since $q + k \leq n$ and obtain the estimate (13) for $\log|f_{i_\nu}|_r$, and $\log|f_n|_r$. This completes the induction in this case. $\square$

4. **A First Generalized ABC- Theorem For P – Adic Entire Functions In Several Variables**

In this section we prove a second theorem for p-adic entire functions in several variables without common factors, under the hypothesis that no subsum vanishes.

Before proving the next theorem, we give the definitions of the characteristic and counting functions of a p-adic entire function in several variables f. The characteristic function of f, denoted by $T(f,r)$, measures the growth of f and is analogous to the degree of a polynomial. It is defined as:

$$T(f,r) = \log |f|_r,$$

where $r > 0$. Let

$$f = \sum_{\gamma} c_\gamma z^\gamma$$

be a p-adic entire function. Let $r > 0$ and let $\mathbf{r} = (r, \ldots, r)$ Then the unintegrated counting function of zeros of f, denoted by $n_f(0,r)$, is defined by Cherry and Ye in [CY] as

$$n_f(0,r) = \sup\{|\gamma| : |c_\gamma|\mathbf{r}^\gamma = |f|_r\}.$$

This is the number of zeros, counting multiplicity, that f has on a sufficiently generic line through the origin with $\max|z_j| \leq r$. They also define

$$n_f(0,0) = \lim_{r\to 0} n_f(0,r) = \min\{|\gamma| : a_\gamma \neq 0\}.$$

If $f = \frac{g}{h}$ is a meromorphic function and g and h have no common factors, then define the unintegrated counting function $n_f(a,r)$ by

$$n_f(\infty, r) = n_h(0,r).$$

For $a \in F$ and f a meromorphic function which is not identically equal to a, define

$$n_f(a,r) = n_{\frac{1}{(f-a)}}(\infty, r).$$

For $a \in \mathbb{P}^1(F)$ and a meromorphic function which is not identically equal to a, Cherry and Ye define the integrated counting function $N(f, a, r)$, which counts the number of times (as a logarithmic average) f takes the value a in $\mathbb{B}^m(r)$. They define it in [CY] as

$$N(f,a,r) = n_f(a,0)\log_\pi r + \frac{1}{\ln \pi}\int_0^r (n_f(a,t) - n_f(a,0))\,\frac{dt}{t}$$

Definition 43. *According to Brownawell, a finite function family $\mathbb{F} = \{f_i \ i \in J\}$ is called minimal if $\mathbb{F}$ is linearly dependent, and for any proper subset I of J the family $\{f_i \ i \in J\}$ is linearly independent. We also call the indices J minimal.*

Lemma 44. *Let $u_0, \ldots, u_n$ be $n+1$ vectors in a vector space over a given field. Assume $u_0 + \ldots + u_n = 0$ but no non-empty proper subsum vanishes. Then there exists a partition of indices*

$$\{0, 1, \ldots, n\} = I_0 \cup \ldots \cup I_k$$

satisfying the following properties: 1) I_α are non-empty disjoint sets;

2) There exist subsets $I_\alpha^{'}$ of $\{0, 1, \ldots, n\}$ with

$$I_0^{'} = \varnothing, \varnothing \neq I_\alpha^{'} \subseteq I_0 \cup \ldots \cup I_{\alpha-1} (\alpha = 1, \ldots, k)$$

such that the set $I_\alpha \cup I\alpha^{'}$ is minimal for each $\alpha = 0, \ldots, k$.

Proof. See [BM] $\square$

Next, let us consider the following equation:

$$f_0 + f_1 + \ldots + f_n = 0$$

. Assume that no proper subsum of the above equation is equal to 0. Then, by the above lemma, there exists a partition of indices

$$\{0, 1, \ldots, n\} = I_0 \cup \ldots \cup I_k$$

satisfying the two properties of the lemma. Set

$$n_0 + 1 = \sharp I_0; n_\alpha = \sharp I_\alpha (\alpha = 1, \ldots, k)$$

and write

$$s_\alpha = 1 + \sum_{\beta=0}^{\alpha} n_\beta, \alpha = 0, 1 \ldots, k.$$

Then

$$n_0 + n_1 + \ldots + n_k = n$$

Without loss of generality, we may assume that

$$I_0 = \{0, \ldots, n_0\}, I_\alpha = \{s_{\alpha-1}, \ldots, s_\alpha - 1\} (\alpha = 1, \ldots, k).$$

Since I_0 is minimal, then $f_1, \ldots f_{n_0}$ are linearly independent. Hence the Wronskian

$$\mathbb{W}_0 = \mathbb{W}(f_1, \ldots, f_{n_0}) \not\equiv 0$$

Similarly the functions $f_{s_{\alpha-1}}, \ldots, f_{s_\alpha - 1}$ are linearly independent, and so

$$\mathbb{W}_\alpha = \mathbb{W}(f_{s_{\alpha-1}}, \ldots, f_{s_\alpha - 1}) \not\equiv 0, \alpha = 1, \ldots, k$$

Define

$$l = \sum_{\alpha=0}^{k} \frac{n_\alpha(n_\alpha - 1)}{2}.$$

Theorem 45. *Using the above notations, suppose that $f_0, f_1, \ldots, f_n$ are non-constant, p-adic entire functions in $\mathcal{E}_m$ such that $gcd(f_1, \ldots f_n) = 1$ and $gcd(f_{s_{\alpha-1}}, \ldots f_{s_\alpha - 1}) = 1$. Let*

$$f_0 + f_1 \ldots f_n = 0$$

such that no subsum vanishes. Then

$$\max_{j=0,\ldots,n} \log |f_j|_r \leq l \log |R(\prod_{j=1}^{n_0} f_j) \prod_{\alpha=1}^{k} R(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j)|_r - l \log r + O(1)$$

for $r \geq r_0$ for some fixed $r_0 > 1$, provided that $n_\alpha \geq 2$ for $\alpha = 0, \ldots k$.

In the case that all our p-adic entire functions are pairwise relative prime, then

$$\max_{j=0,\dots,n} \log |f_j|_r \leq l \log |R(\prod_{j=0}^{n} f_j)|_r - l \log r + O(1)$$

for $r \geq r_0$ for some fixed $r_0 > 1$, provided that $n_\alpha \geq 2$ for $\alpha = 0, \dots k$.

Proof. By using the previous notations of this section, let

$$\mathbb{W}_\alpha = \mathbb{W}(f_{s_{\alpha-1}}, \dots, f_{s_\alpha - 1}) = \begin{vmatrix} f_{s_{\alpha-1}} & \cdots & f_{s_\alpha-1} \\ \partial^{\gamma_1} f_{s_{\alpha-1}} & \cdots & \partial^{\gamma_1} f_{s_\alpha-1} \\ \cdots & \cdots & \cdots \\ \partial^{\gamma_{n_\alpha-1}} f_{s_{\alpha-1}} & \cdots & \partial^{\gamma_{n_\alpha-1}} f_{s_\alpha-1} \end{vmatrix}$$

with $|\gamma_i| = i$. If μ is a multi-index notation and F is a meromorphic function in m several variables then by $\partial^\mu F$ we mean $\frac{\partial^{|\mu|} F}{\partial x_1^{\mu_1} \dots \partial x_1^{\mu_m}}$ Then

$$\mathbb{W}_\alpha = \begin{vmatrix} P^{k_1} * P_{0,1} & \cdots & P^{k_t} * P_{0,t} & P_{0,t+1} & \cdots & P_{0,n_\alpha} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P^{k_1-[n_\alpha-(n_\alpha-i)]} * P_{n_\alpha-i,1} & \cdots & P^{k_t-[n_\alpha-(n_\alpha-i)]} * P_{n_\alpha-i,t} & P_{n_\alpha-i,t+1} & \cdots & P_{n_\alpha-i,n_\alpha} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P^{k_1-(n_\alpha-1)} * P_{n_\alpha-1,1} & \cdots & P^{k_t-(n_\alpha-1)} * P_{n_\alpha-1,t} & P_{n_\alpha-1,t+1} & \cdots & P_{n_\alpha-1,n_\alpha} \end{vmatrix}$$

where Q is the square-free part of $\prod f_{s_{\alpha-1}}, \dots, f_{s_\alpha-1}$ and P is a non-constant irreducible element in $\mathcal{E}_m$ such that $P \mid Q$. Let $P^{\hat{k}}$ to be the maximal power of P such that $P^{\hat{k}} \mid \prod_{i=s_{\alpha-1}}^{s_\alpha-1} f_i$, hence $k_i \geq 1$. We denote by $P_{i,j}$ where $0 \leq i \leq n_\alpha - 1$ and $1 \leq j \leq n_\alpha$ the expressions in $\mathcal{E}_m$ that are not divisible by P and $(k_1, \dots k_t, 0, \dots, 0)$ represents the partition of $\hat{k}$. The operation $*$ stands for the multiplication operation. Hence

$$\hat{k} = \sum_{i=1}^{t} k_i$$

where $t = n_\alpha - i; t < n_\alpha$, with $\alpha = 0, \ldots, \hat{k}$. Also i represents the number of columns with no element divisible by P. So $0 \leq i \leq n_\alpha - 1$. In order to avoid having any of these $k+1$ Wronskians to be 1×1, we need $n_\alpha \geq 2$ for each $\alpha = 0, \ldots k$.

The functions $f_{s_{\alpha-1}}, \ldots, f_{s_\alpha - 1}$ are linearly independent, and so

$$\mathbb{W}_\alpha = \mathbb{W}(f_{s_{\alpha-1}}, \ldots, f_{s_\alpha - 1}) \not\equiv 0, \alpha = 1, \ldots, \hat{k}.$$

The general term in $\mathbb{W}_\alpha$ with the smallest power of P is

$$P^{k_1 - (n_\alpha - 1)} P^{k_2 - (n_\alpha - 2)} \ldots P^{k_t - [n_\alpha - (n_\alpha - i)]} P_0 = P^{\hat{k} - \frac{n_\alpha(n_\alpha - 1) - i(i-1)}{2}} P_0.$$

where by P_0 we mean an expression that is not divisible by P. We have that

$$N(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, 0, r) - N(\mathbb{W}_\alpha, 0, r) \leq N\left(\frac{\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j}{gcd\left(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, \mathbb{W}_\alpha\right)}, 0, r\right) + O(1)$$

for any $r \geq r_0$ for some fixed positive real number r_0.

Hence $P^{\frac{n_\alpha(n_\alpha - 1) - i(i-1)}{2}}$ is the maximal power of P that divides $\frac{\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j}{gcd\left(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, \mathbb{W}_\alpha\right)}$

and

$$\frac{\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j}{gcd\left(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, \mathbb{W}_\alpha\right)} \mid Q^{\frac{n_\alpha(n_\alpha - 1) - i(i-1)}{2}}.$$

Therefore

$$N(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, 0, r) - N(\mathbb{W}_\alpha, 0, r) \leq N\left(Q_{\frac{n_\alpha(n_\alpha - 1)}{2}}, 0, r\right) + O(1)$$

for any $r \geq r_0$ for some fixed positive real number r_0,

since

$$\frac{n_\alpha(n_\alpha - 1) - i(i-1)}{2} = \frac{n_\alpha(n_\alpha - 1)}{2}$$

for $i = 1$ and

$$\frac{n_\alpha(n_\alpha - 1) - i(i-1)}{2} \le \frac{n_\alpha(n_\alpha - 1)}{2}$$

for every $i \ge 2$. Note that $i \ne 0$ since $gcd(f_{s_{\alpha-1}}, \dots, f_{s_\alpha - 1}) = 1$. In the above we used the following type of notation

$$f_{\hat{l}} = P_1^{\min(\hat{l}, e_1)} \dots P_h^{\min(\hat{l}, e_h)}$$

whenever $f = P_1^{e_1} \dots P_h^{e_h}$ with $P_1, \dots, P_h$ be irreducible elements in $\mathcal{E}_m$.

Note that in general, if $\Delta f \ne 0$ and $\rho(\Delta) = \hat{l}$ then

$$N^{(\hat{l})}(f, 0, r) = N\left(\frac{f}{gcd(f, \Delta f)}, 0, r\right)$$

and if $\Delta f = 0$ then $N^{(\hat{l})}(f, 0, r) = N(f, 0, r)$. So we have

$$N^{(\hat{l})}(f, 0, r) = N(\prod_{j=1}^{h} P_j^{\min\{e_j, \hat{l}\}}, 0, r).$$

In our case, in order to compute $\mathbb{W}_\alpha$ we used the tuple operator $\Delta = (\Delta_1, \dots, \Delta_{n_\alpha - 1})$ with $\rho(\Delta_i) = i$ for each $i = 1, \dots, n_\alpha - 1$. Then $\rho(\Delta) = 1 + 2 + \dots + n_\alpha - 1 = \frac{n_\alpha(n_\alpha - 1)}{2}$. So we just verified above that $N^{\left(\frac{n_\alpha(n_\alpha-1)}{2}\right)}(Q, 0, r) = N\left(Q_{\frac{n_\alpha(n_\alpha-1)}{2}}, 0, r\right)$

We apply the following version of the Second Main Theorem for p-adic entire functions (see Theorem 5.1. [CY]): Given a tuple consisting of $n + 1$ linearly independent p-adic entire functions, that are without common factors, $f = (f_0, \dots f_n)$, let $\gamma_1, \dots, \gamma_n$ be multi-indices with $|\gamma_i| \le i$ so that the generalized Wronskian $\mathbb{W} = W(f_0, \dots f_n) = \begin{vmatrix} f_0 & \cdots & f_n \\ \partial^{\gamma_1} f_0 & \cdots & \partial^{\gamma_1} f_n \\ \cdots & \cdots & \cdots \\ \partial^{\gamma_n} f_0 & \cdots & \partial^{\gamma_n} f_n \end{vmatrix} \ne 0$. Let $B = \sum_{i=1}^{n} |\gamma_i|$ and let r_0 be a positive real number then

$$T(f, r) \le N(\prod_{j=0}^{n} f_j, 0, r) - N(\mathbb{W}, 0, r) - B \log r + O(1)$$

for any $r \geq r_0$ and where $T(f,r) = \max_{j=0,\ldots,n} \log|f_j|_r + O(1)$ and $n < B < \frac{n(n+1)}{2}$.
For our purpose, we apply this theorem k+1 times, to each tuple $\hat{f} = (f_{s_{\alpha-1}}, \ldots, f_{s_\alpha - 1})$, respectively $\hat{f}_0 = (f_1, \ldots f_n)$ consisting of linearly independent functions such that $gcd(f_{s_{\alpha-1}}, \ldots, f_{s_\alpha-1}) = 1$ for each $\alpha = 1, \ldots, k$ and $gcd(f_0, \ldots, f_n) = 1$. Hence:

$$T(\hat{f_\alpha}, r) \leq N(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, 0, r) - N(\mathbb{W}_\alpha, 0, r) - B_\alpha \log r + O(1)$$

for each $\alpha = 1, \ldots, k$. and

$$T(\hat{f}_0, r) \leq N(\prod_{j=1}^{n_0} f_j, 0, r) - N(\mathbb{W}_0, 0, r) - B_0 \log r + O(1)$$

where $B_\alpha = 1 + 2 + \ldots n_\alpha - 1 = \frac{n_\alpha(n_\alpha - 1)}{2}$ by Cherry and Ye's definition. We sum up these $k+1$ inequalities, and we let $B = B_0 + \ldots B_k = \sum_{\alpha=0}^{k} \frac{n_\alpha(n_\alpha - 1)}{2} = l$ and get:

$$\begin{aligned}
\max_{j=0,\ldots,n} \log|f_j|_r \leq & N(\prod_{j=1}^{n_0} f_j, 0, r) + \sum_{\alpha=1}^{k} N(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j, 0, r) - \sum_{\alpha=0}^{k} N(\mathbb{W}_\alpha, 0, r) - B \log r + O(1) \\
\leq & \sum_{\alpha=0}^{k} N(Q_{\frac{n_\alpha(n_\alpha-1)}{2}}, 0, r) - l \log r + O(1) \\
\leq & \sum_{\alpha=1}^{k} \frac{n_\alpha(n_\alpha - 1)}{2} \log|R(\prod_{j=s_{\alpha-1}}^{s_\alpha-1}) f_j|_r + \frac{n_0(n_0-1)}{2} \log|R(\prod_{j=1}^{n_0} f_j)|_r - l \log r + O(1) \\
\leq & l \log|R(\prod_{j=1}^{n_0} f_j) \prod_{\alpha=1}^{k} R(\prod_{j=s_{\alpha-1}}^{s_\alpha-1} f_j)|_r - l \log r + O(1)
\end{aligned}$$

We used that Cherry-Ye's Poisson-Jensen-Green formula in [CY] says that there exists a constant C_f that depends on f but not on r, such that $N(f, 0, r) = log|f|_r + C_f$, for all r.

We made also use of the property of the counting function to be multiplicative and of the following facts:

$$N(\prod_{j=0}^{n} f_j, 0, r) = N(\prod_{j=1}^{n_0} f_j, 0, r) + \sum_{\alpha=1}^{k} N(\prod_{j=s_{\alpha-1}}^{s_\alpha-1} f_j, 0, r)$$

$N(\prod_{j=0}^{n} f_j, 0, r) = N(f_0, 0, r) + N(\prod_{j=1}^{n} f_j, 0, r)$ and $N(f_0, 0, r) \geq 0$ thus $N(\prod_{j=0}^{n} f_j, 0, r) \geq N(\prod_{j=1}^{n} f_j, 0, r)$. Therefore we conclude that

$$\max_{j=0,\ldots,n} \log |f_j|_r \leq l \log |R(\prod_{j=1}^{n_0} f_j) \prod_{\alpha=1}^{k} R(\prod_{j=s_{\alpha-1}}^{s_\alpha - 1} f_j)|_r - l \log r + O(1)$$

for any $r \geq r_0$ for some fixed real number $r_0 > 1$, provided that $n_\alpha \geq 2$ for $\alpha = 0, \ldots k$.

In the case that all our p-adic entire functions are pairwise relative prime, then

$$R(\prod_{i=0}^{n} f_i) = \sum_{i=0}^{n} R(f_i)$$

and we get that

$$\max_{j=0,\ldots,n} \log |f_j|_r \leq l \log |R(\prod_{j=0}^{n} f_j)|_r - l \log r + O(1)$$

for any $r \geq r_0$ for some fixed real number $r_0 > 1$,provided that $n_\alpha \geq 2$ for $\alpha = 0, \ldots k$.

$\square$

5. **A Second Generalized ABC- Theorem For P – Adic Entire Functions In Several Variables**

Theorem 46. *Let $f_1, \ldots, f_s$, with $s \geq 3$, be pairwise relatively prime p-adic entire functions of severable on F^m such that at least one of the f_i is non-constant and such that $f_1 + \ldots f_s = 0$. Fix $r_0 \in |F^*|$. Then, for all $r \in |F^*|$ with $r \geq r_0$*

$$\max_{1 \leq i \leq s} (\log |f_i|_r) \leq (s-2)[\log |Q|_r - \log r] + O(1)$$

where $Q = R(\prod_{i=1}^{s} f_i)$ and the $O(1)$ is a constant independent of r.

Proof. We consider two cases as in [SS]). The first case is based on the second case. If more than one f_i is constant, we may combine all the constant f_i and assume without loss of generality that at most one f_i is constant.

Case 1. Let $-f_s = f_1 + \ldots + f_{s-1}$. Assume that the $f_i, i = 1, \ldots, s-1$ are linearly dependently over F. By the above considerations, at most one of the $f_i, i = 1, \ldots, s-1$ is constant. Let $f_{i_1}, \ldots, f_{i_q}; q < s-1$ be a maximal linearly independent subset of the $f_j, j = 1, \ldots, s-1$. Since $s-1 \geq 2$ and the f_j are relatively prime by pairs, we get $q \geq 2$. Each $f_j, 1 \leq j \leq s-1, j$ that differs from one of the i_k, is a linear combination of the f_{i_k} of the form

$$f_j = \lambda_1 f_{i_1} + \ldots + \lambda_q f_{i_q} \tag{15}$$

where $\lambda_k \in F$ and at least two of these λ_k are not zero.

As in [SS] by the second case we get:

If $\lambda_k \neq 0$ then by the second case we get:

$$\log |\lambda_k f_{i_k}|_r \leq (q-1)(\log |Q_1|_r - \log r) + O(1)$$

where Q_1 is the square-free part of $f_j \prod_{k=1}^{q} \lambda_k f_{i_k}$

But $q < s-1$ thus $q-1 < s-2$.

Therefore

$$|Q_1|_r < |Q|_r + O(1).$$

So

$$\log|f_{i_k}|_r \le \log|\lambda_k f_{i_k}|_r$$
$$\le (s-2)(\log|Q|_r - \log r) + O(1)$$

because $|Q_1|_r \mid |Q|_r$ thus $\log|Q_1|_r \le \log|Q|_r + O(1)$ We get the same bound for $\log|f_j|_r$, because, as in [SS]:

$$\log|f_j|_r = \log|\lambda_1 f_{i_1} + \ldots + \lambda_q f_{i_q}|_r + O(1)$$
$$\le \log \max_{1\le k\le q} |\lambda_i f_{i_k}|_r + O(1)$$
$$\le (s-2)(\log|Q|_r - \log r) + O(1)$$

So the theorem is proved for such f_j and f_{i_k}.

As in [SS] inserting all equations (15) into the right side of the equation

$$-f_s = f_1 + \ldots + f_{s-1}$$

yields the following equation

$$f_s = k_1 f_{i_1} + \ldots + k_q f_{i_q}$$

where $k_j \in F$.

If one of these $k_t = 0$ then f_{i_t} appears in one equation of type (15) with a non-zero λ_t. Hence

$$\log|f_{i_t}|_r \le (s-2)[\log|Q|_r - \log r] + O(1).$$

Finally for those $k_t \ne 0$ applying the same reasoning as we did for f_j and f_{i_k} (we have $q+1 < s$) and we get the desired estimates for $\log|f_{i_t}|_r$ and for $\log|f_s|_r$.

Therefore the theorem is proved in this case.

Case 2. Let us denote $R(f_i)$ by $Q[f_i]$. The $f_1, \ldots, f_{s-1}$ are linearly independent over F.

Then there exists a generalized Wronskian of the following form which doesn't vanish.

$$W[f_1, \dots f_{s-1}] = \det |\Delta_i f_j|$$

where operators Δ_i are such that $\mu_i \geq 0$ are integers, and $\rho(\Delta_i) \leq i$ and $1 \leq \rho(\Delta_i)$ for $i \geq 2$.

As in [SS] applying the operators Δ_i, $i = 1, \dots, s-1$ to

$$-f_s = f_1 + \dots + f_{s-1}$$

yields

$$-f_s^{-1} \Delta_i f_s = \sum_{j=1}^{s-1} (f_j^{-1} \Delta_i f_j)(\frac{f_j}{f_s})$$

for $i = 1, \dots, s-1$. Solving $s-1$ linear equations in the $\frac{f_j}{f_s}$ we get:

(16) $$\frac{f_j}{f_s} = \frac{\det |\frac{(\Delta_i f_k)}{f_k} \quad \text{for} k \neq j; \frac{(-\Delta_i f_k)}{f_s} \quad \text{in the } j-\text{th col.}|}{\det |\frac{(\Delta_i f_k)}{f_k}|}$$

We have $\det |\frac{(\Delta_i f_k)}{f_k}|$ equals $W[f_1, \dots, f_{s-1}]$ divided by the product of the f_k and hence is not zero.

As in [SS] let N and D be the numerator and denominator respectively of the right hand side of (16).

Fix j then we have:

$$\frac{f_j}{f_s} = \frac{N}{D}.$$

D is a determinant, so D equals a sum of the form

(17) $$\pm \frac{(\Delta_{i_1} f_1) \dots (\Delta_{i_{s-1}} f_{s-1})}{f_1 \cdots f_{s-1}}$$

where $i_1, \dots, i_{s-1}$ is a permutation of $1, \dots, s-1$.

Now fix $r \in |F^*.|$ There exist $P_{i,k}$ such that

$$\frac{\Delta_i f_k}{f_k} = \frac{P_{i,k}}{Q[f_k]^{\rho(\Delta_i)}}$$

and by Lemma 39, (17) is of the form

$$\pm \frac{P_{i_1,1}}{Q[f_1]^{\rho(\Delta_{i_1})}} \cdots \frac{P_{i_{s-1},s-1}}{Q[f_{s-1}]^{\rho(\Delta_{i_{s-1}})}} \tag{18}$$

where $k = 1, \dots, s-1; P_{i_k,k} \in \mathbb{B}^m(r)$ and

$$\log |P_{i_k,k}|_r \leq \rho(\Delta_{i_k})(\log |Q[f_k]|_r - \log r).$$

So there exists a function P such that

$$D = \frac{P}{Q[f_1, \dots, f_{s-1}]^{s-2}}$$

because

$$Q[f_1]^{\rho(\Delta_{i_1})} \dots Q[f_{s-1}]^{\rho(\Delta_{i_{s-1}})} \mid Q[f_1, \dots f_{s-1}]^{s-2},$$

as the f_k are relatively prime by pairs.

Thus (18) has the form $\pm \frac{P}{Q[f_1 \cdots f_{s-1}]^{s-2}}$

So

$$\log |P|_r = \log(|P_{i_1,1}|_r \dots |P_{i_{s-1},s-1}|_r) + \log \left| \frac{Q[f_1 \dots f_{s-1}]^{s-2}}{Q[f_1]^{\rho(\Delta_{i_1})} \dots Q[f_{s-1}]^{\rho(\Delta_{i_{s-1}})}} \right|_r.$$

But

$$Q[f_1] = u_1 \prod_{i=1}^{k} p_{1,i}.$$

And so on,

$$Q[f_{s-1}] = u_{s-1} \prod_{i=1}^{m} p_{s-1,i}.$$

where u and u_i for all $i = 1, \ldots s-1$ are units in $\mathbb{B}^m(r)$ and

$$Q[f_1 \ldots f_{s-1}] = u \prod p_{1,i} \ldots \prod p_{s-1,i}.$$

Because $f_1, \ldots f_{s-1}$ are relatively prime we get that

$$u = \prod_{i=1}^{s-1} u_i$$

So

$$\begin{aligned}
\log |P|_r &= \sum_{k=1}^{s-1} \log |P_{i_k,k}||_r + \log \left| \frac{u^{s-2}}{\prod_{k=1}^{s-1} u_i^{\rho(\Delta_{i_k})}} \frac{Q[f_1]^{s-2-\rho(\Delta_{i_1})} \ldots Q[f_{s-1}]^{s-2-\rho(\Delta_{i_{s-1}})}}{\prod_{i=1}^{s-1} u_i^{s-2-\rho(\Delta_i)}} \right|_r \\
&= \sum_{k=1}^{s-1} \log |P_{i_k,k}|_r + \sum_{i=1}^{s-1} \log |Q[f_i]^{s-2-\rho(\Delta_{i_i})}|_r \\
&\leq \sum_{k=1}^{s-1} \rho(\Delta_{i_k})(\log |Q[f_k]|_r - \log r) + \sum_{i=1}^{s-1} (s-2-\rho(\Delta_{i_i})) \log |Q[f_i]|_r \\
&= -\sum_{k=1}^{s-1} \rho(\Delta_k) \log r + (s-2) \sum_{i=1}^{s-1} \log |Q[f_i]|_r \\
&\leq (s-2)[\log \prod_{i=1}^{s-1} |Q[f_i]|_r - \log r]
\end{aligned}$$

Therefore

$$\log |P|_r \leq (s-2)[\log \prod_{i=1}^{s-1} |Q[f_i]|_r - \log r] \tag{19}$$

We used that $\rho(\Delta_{i_k}) \geq 1$ for all Δ_{i_k} except one. But

$$\prod_{i=1}^{s-1} |Q[f_i]|_r = \left| (\prod_{i=1}^{s-1} u_i Q[f]) \frac{1}{u} \right|_r = |Q[f]|_r$$

where

$$f = \prod_{i=1}^{s-1} f_i.$$

So (19) becomes

$$\log |P|_r \leq (s-2)[\log |Q[f]|_r - \log r].$$

Thus D has the form

$$D = \frac{d}{Q[f_1 \dots f_s]^{s-2}}$$

where $d \in \mathcal{A}^m(r)$ and

$$\log |d|_r \le (s-2)(\log |Q[f_1 \dots f_s]|_r - \log r).$$

Analogously

$$N = \frac{n}{Q[f_1 \dots f_{s-1}]^{s-2}}$$

where $n \in \mathcal{A}^m(r)$ and

$$\log |n|_r \le (s-2)(\log |Q[f_1 \dots f_s]|_r - \log r)$$

So we get $\frac{f_j}{f_s} = \frac{n}{d}$ hence $f_j d = f_s n$. Thus $f_j \mid f_s n$. Because f_j and f_s are relatively prime then $f_j \mid n$. So there exists g_j such that $n = f_j g_j$. Therefore

$$\begin{aligned}
\log |f_j|_r &= \log |n|_r - \log |g_j|_r \\
&\le \log |n|_r - \log |g_j|_{r_0} \\
&= \log |n|_r - O(1) \\
&\le (s-2)(\log |Q[f_1 \dots f_s]|_r - \log r) + O(1)
\end{aligned}$$

Similarly we get:

$$\begin{aligned}
\log |f_s|_r &\le \log |d|_r - O(1) \\
&\le (s-2)[\log |Q[f_1 \dots f_s]|_r - \log r] + O(1)
\end{aligned}$$

So we get that

$$\begin{aligned}
\max_{i=1,\dots,s} \log |f_i|_r &\le (s-2)(\log |Q[f_1 \dots f_s]|_r - \log r) + O(1) \\
&= (s-2)[\log |Q|_r - \log r] + O(1)
\end{aligned}$$

since $Q = R(\prod_{i=1}^{s} f_i)$.

$\square$

6. **A Result On Abelian Varieties**

In our last section we will prove a few generalizations of Buium's results that he used in order to prove his ABC Theorems for isotrivial abelian varieties, respectively with trace zero. Before we state and prove our results, we first recall the following definitions. Let C be a smooth projective curve defined over a closed field k with function field $K = k(C)$.

Definition 47. *An abelian variety is a projective algebraic variety that is also an algebraic group, i.e. has a group law that can be defined by regular functions.*

Definition 48. *A homomorphism $\alpha : A \to B$ of abelian varieties is called isogeny if it is surjective and has finite kernel.*

Remark 49. *We write $A \sim B$ if there is an isogeny $A \to B$.*

Definition 50. *Let F be a field of characteristic zero with a derivation on it and $\mathscr{C}$ be the field of constants. An F-variety is said to descends to constants if it comes, via base change, from a variety over $\mathscr{C}$.*

Definition 51. *Let A be an abelian variety over the function field K. Then A is isotrivial if and only if there exists A_0 abelian variety $/k$ such that $A \otimes_K \bar{K} = A_0 \otimes_k \bar{K}$, i.e. A is descending to k (by $\bar{K}$ we understand the algebraic closure of K).*

Definition 52. *An abelian variety A over the function field K is said to have trace zero if $A \otimes_K \bar{K}$ has no nonzero abelian subvarieties descending to k (by $\bar{K}$ we understand the algebraic closure of K).*

We also need to recall the following definitions (see [Bu2]):

Let k be a field and K be its extension field. Fix a non-zero k-derivation δ on K. By a D-scheme we understand a K-scheme V together with a lifting of δ to a derivation of $\mathscr{O}_V$. Then a D-group scheme is a group object in the category of D-schemes. Finally, an algebraic D-group is a D-group scheme which is of finite type over K.

Now we recall the construction of Buium's jet spaces $jet_\infty(X)$, by following his notations and terminology (see [Bu4]):

Let X and Y be two K-schemes. By a kernel (over K)

$$(f,\delta) : X \to Y$$

we will understand a pair (f,δ) consisting of a morphism of K-schemes $f : X \to Y$ and a derivation $\delta \in Der(\mathscr{O}_Y, f_*\mathscr{O}_X)$whose restriction to K coincides with the derivation δ on K.

By a prolongation sequence one understands two kernels of the form

$$X_1 \xrightarrow{(f_1,\delta_1)} X_0 \xrightarrow{(f_0,\delta_0)} X_{-1}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{O}_{X_{-1}} \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & f_{0*}\mathscr{O}_{X_0} \arrow[d, "f_1"] \\
f_{0*}\mathscr{O}_{X_0} \arrow[r, "\delta_1"] & f_{0*}f_{1*}\mathscr{O}_{X_1}
\end{tikzcd}$$

One says that (f_1,δ_1) is a prolongation of (f_0,δ_0). Let

$$P = P(f,\delta) := SpecS(\Omega_X)/J$$

where Ω_X is the $\mathscr{O}_X$-module of Kähler differentials (i.e. Kähler differentials over $\mathbb{Z}$), $S()$ is the symmetric algebra of an $\mathscr{O}_X$-module and J is the ideal generated by local sections of the form $d(f(a)) - \delta a$, where a are local sections of $\mathscr{O}_Y$ and

$$d = d_X : \mathscr{O}_X \to \Omega_X$$

is the universal Kähler derivation. Let $(f, \delta) : X \to Y$ be a kernel, then

$$(\tilde{f}, \tilde{\delta}) : P = P(f, \delta) \to X$$

is called its canonical prolongation. Also the composition $f : X \xrightarrow{\delta} Spec\mathrm{K} \to \mathscr{O}_X$ is a derivation of K into $\mathscr{O}_X$, still denoted by δ. Then we get a kernel

$$(f, \delta) : X \to SpecK$$

called the tautological kernel of X. By an infinite prolongation sequence we will understand a projective system

$$\ldots \to X^r \xrightarrow{(f_r, \delta_r)} X^{r-1} \to \ldots \xrightarrow{(f_0, \delta_0)} X^{-1}$$

of kernels such that each sequence of theree terms contained in it is a prolongation sequence. Now we are ready to construct Buium's Jet Spaces: If X is any K-scheme define inductively a sequence of kernels

$$(f_r, \delta_r) : X^r \to X^{r-1}, r \geq 0$$

where

$$(f_0, \delta_0) : X^0 \to X^{-1}$$

is the tautological kernel

$$(f, \delta) : X \to SpecK$$

and

$$X^{r+1} \to X^r$$

is the canonical prolongation of

$$X^r \to X^{r-1}$$

i.e

$$X^{r+1} \to X^r$$

identifies with $P(f_r, \delta_r) \to X^r$.Therefore we produce an infinite prolongation sequence, called the canonical infinite prolongation sequence of X. Finally, we define $jet_\infty(X)$(Buium's Infinite Jet Space of X) to be the D-scheme obtained by taking the projective limit of this infinite prolongation sequence.

Before we recall Buium 's Theorem 1 (see [Bu1]), let us fix an integral scheme S over an algebraically closed field k of characteristic zero on which is given a derivation $\delta \in Der\mathcal{O}_S$ such that $\{x \in Q(S); \delta x = 0\} = k$, where $Q(S)$ is the quotient field of S. Let G/S be a group scheme of finite type, $X \subset G$ a closed subscheme, W/S a projective scheme and $X \to W$ an S-morphism. Let H/S be a horizontal, closed subgroup scheme of finite type of $jet_\infty(G)/S$. Let us also recall the definition of the Albanese variety $Alb(V)$, where V is any variety, given by the following Universality Property: There is a morphism from the variety V to its Albanese variety $Alb(V)$, such that any morphism from V to an abelian variety (taking the given point to the identity) factors uniquely through $Alb(V)$. Then the following is equivalent to Buium's Theorem 2 in the Introduction:

Theorem 53 (Buium's Theorem 1)**.** *Assume that the generic fiber of W/S is an irreducible variety of general type, whose Albanese variety does not descend to k. Then the image of the morphism $H \bigcap jet_\infty(X) \to W$ is not Zarinski dense in W.*

We will make use of the following results:

Theorem 54 (Kobayashi-Ochiai's Theorem)**.** *(see* [KO]*) Let $f : V \to W$ be a holomorphic map between $\mathbb{C}$-varieties, where W is projective (possibly singular) of general type, and assume that $f(V)$ contains some open subset of W in the complex topology; then f is rational.*

Theorem 55 (Ueno's Theorem)**.** *(see* [Ueno]*) Let A be an abelian variety and $W \subset A$ subvariety. Then there exists $B \subset A$ abelian subvariety such that $B + W = W$(so B acts on W) such that W/B is of general type.*

Lemma 56. *Let P/S be an abelian scheme, where S is a smooth $\mathbb{C}$-variety, and let δ be a vector field on S without zeros such that $\{f \in Q(S); \delta f = 0\} = \mathbb{C}$. Assume* *that, for any analytic disk Δ embedded into S and tangent to δ, the fibers of P/S above points in Δ fall into, at most, countably many isomorphism classes. Then the geometric generic fiber of P/S descends to $\mathbb{C}$.*

Result 57 (Hamm's Result)**.** *(see* [Ha]*) Let $H_\Delta \to \Delta$ be a local submersion of analytic manifolds, where Δ is a disk in $\mathbb{C}$, and let $\delta, \tilde{\delta}$ be nowhere vanishing vector fields on Δ and H_Δ, respectively, with $\tilde{\delta}$ lifting δ. Given holomorphic maps $\mu : H_\Delta \times_\Delta H_\Delta \to H_\Delta, H_\Delta \to H_\Delta$ and $\Delta \to H_\Delta$ satisfying the "usual" axioms of multiplication, inverse and unit. Assume that these maps are horizontal (with respect to the vector fields $\delta, \tilde{\delta}$ and $\tilde{\delta} \otimes 1 + 1 \otimes \tilde{\delta}$ on Δ, H_Δ and $H_\Delta \times_\Delta H_\Delta$,respectively). Assume also that each fiber of $H_\Delta \to \Delta$ has finitely many components. Then there exists a holomorphic Δ-isomorphism $\sigma : H_\Delta \to H_0 \times \Delta$ where H_0 is some Lie group such that σ transports the map μ into $\mu_0 \times 1_\Delta$ (where $\mu_0 : H_0 \times H_0 \to H_0$ is the multiplication on H_0) and σ transports $\tilde{\delta}$ into $1 \otimes \delta$.*

Before we state and prove a generalization of Buium's Theorem 2 for any abelian varieties, let us make the following:

Remark 58. *Any abelian variety A is, up to isogeny and after replacing K by a finite extension of it, a product $B \times C$ with B isotrivial and C with trace zero. So, from now on, we can concentrate on the case when A is actually equal to $B \times C$.*

Theorem 59. *Let A be any abelian variety, and $W \subset A$ subvariety of general type. Let G be any algebraic D-group, $V \subset G$ a D-subvariety and $u : V \to W$ be a dominant morphism. Then $W = (Z_K + Q) \times W'$ for some k-subvariety Z_K of A, some $Q \in A(K)$ and $W' \subset A$ subvariety, such that $Alb(W')$ descends to k (where $Z_K := Z \otimes_k K$).*

Proof. By Ueno's Theorem, there exists an abelian subvariety T of A such that W is, up to isogeny, $T \times (W/T)$ and W/T is a subvariety of general type. Because $T = T + 0$ there exists some k-subvariety Z_K of A and some $Q \in A(K)$ such that $T = (Z_K + Q)$. It is left to show that $Alb((W/T))$ descends to k. We give a sketch proof of this fact by following some notations and arguments of Buium's proof of his Theorem 1 in the case where $k = \mathbb{C}$ and S is a smooth, affine, irreducible $\mathbb{C}$-variety. We don't mention anything about the general case, because it's similar to his general case proof (see [Bu1]). Exactly as in [Bu1], it will suffice to prove the theorem after S is replaced by $S^{'}$, an *étale* covering of a Zariski-opet set of S. Then, we may assume that H/S is smooth,W/S is flat with geometrically integral fibers and δ doesn't cancel anywhere on S. Suppose that $H \bigcap jet_{\infty}(X)$ dominates W and choose a component V of $H \bigcap jet_{\infty}(X)$ that dominates W. Because $H \bigcap jet_{\infty}(X)$ is a horizontal subscheme of $jet_{\infty}(G)$, so V will be, too and therefore we get the diagram

$$W \overset{\alpha}{\leftarrow} V \rightarrow H,$$

where $V \rightarrow H$ is a closed immersion with V horizontal in H. Let Δ be an analytic disk embedded in S, which is tangent to δ and via the change base $\Delta \rightarrow S$ we get the diagram

$$W_{\Delta} \xleftarrow{\alpha_{\Delta}} V_{\Delta} \rightarrow H_{\Delta}$$

Hence V_{Δ} is horizontal in H_{Δ} for $\tilde{\delta}$. By Hamm's Result , there exists an analytic isomorphism $\sigma : H_{\Delta} \rightarrow H_0 \times \Delta$. Thus $\sigma(V_{\Delta}) = H_1 \times \Delta$ since any closed analytic subset of $H_0 \times \Delta$, which is horizontal for $1 \otimes \delta$, is of the form $H_1 \times \Delta$, for H_1 closed analytic subset of H_0.

By composing the morphisms $\alpha : V \rightarrow W$ and $\beta : W \rightarrow W/T$ we get the morphism $\gamma : V \rightarrow W/T$. This is a rational dominant morphism, because α is dominant and β is surjective, hence β is a rational dominant morphism. For any $b \in \Delta$, let $(W/T)_b, V_b$ be the fibers of the corresponding families at b. Fix a point $b_0 \in \Delta$, let $u_b : V_{b_0} \rightarrow (W/T)_b$ be the composition of morphisms $V_{b_0} \xrightarrow{\sigma_b} V_b$ with $V_b \xrightarrow{\gamma_b} (W/T)_b$. Since $W_b, (W/T)_b$ are of general type, by the Big Picard Theorem we get that u_b is rational; therefore we have that $Alb(V_{b_0}) \rightarrow Alb((W/T)_b)$ is a rational dominant map at fibers levels, hence a surjective one. Because there exist only countable many quotients of an abelian variety and by Lemma 56, we get that

$W = (Z_K + Q) \times W/T$ for some k-subvariety Z_K of A and some $Q \in A(K)$ such that $Alb(W/T)$ descends to k. Denote W/T by $W^{'}$, hence $Alb(W^{'})$ descends to k.

□

We now recall Buium's Lemma (see [Bu2]) for isotrivial abelian varieties:

Lemma 60 (Buium's Lemma)**.** *Let W be a projective variety of general type over K. Assume W is a closed subvariety of A_K, where A is an abelian k-variety (here k denotes the field of complex numbers). Let G be any algebraic D-group, $V \subset G$ an absolutely irreducible, reduced, D-scheme and $u : V \to W$ be a dominant morphism of K-schemes.Then, after replacing K by a finite extension of it, one may find a closed k-subvariety $Z \subset A$ and a point $Q \in A(K)$ such that $W = Z_K + Q$ in A_K. Moreover, if we view W as a D-scheme by trivially lifting δ from K to $W \simeq Z_K$, then $u : V \to W$ is necessarily a morphism of D-schemes.*

Let us state our following result:

Lemma 61. *Let W be a projective variety of general type over K. Asume W is a closed subvariety of A_K, where A is an abelian k-variety, $A = B \times C$ with B isotrivial and C with trace zero. Let G be any algebraic D-group, $V \subset G$ an absolutely irreducible, reduced D-scheme and $u : V \to W$ be a dominant morphism of K-schemes. Then after replacing K by a finite extension of it, one may find a closed k-subvariety $Z \subset A$and a point $Q \in A(K)$ such that $W = (Z_K + Q)$.*

Proof. By using some notations and arguments of the proof of Buium' s Theorem 2 in the Introduction, for some fixed $b_0 \in \Delta$ we have that $V_{b_0} \to W_b \subset A_b = B_b \times C_{b_0} \to B_b$, for all $b \in \Delta$. Let $\pi_b : Alb(W_b) \to B_b$ and $\rho_b : Alb(W_b) \to A_b$. Because $Alb(W_b) \to \pi_b(Alb(W_b))$ is a surjective morphism and $Alb(W_b)$ is isotrivial then $\pi_b(Alb(W_b))$ is isotrivial, too. Because the abelian variety $Alb(V_{b_0})$ has only contable many quotients then $Alb(W_b) = Alb(W_{b_0})$. Also $\pi_b(Alb(W_b)) \subset B_b$ and B_b is with trace 0, therefore if $\pi_b(Alb(W_b)) \neq 0$ then we would get a contradiction, since $\pi_b(Alb(W_b))$ is isotrivial. Hence $\pi_b(Alb(W_b)) = 0$. Also Im $\rho_p \subset 0 \times C_{b_0}$ so we get that $W_b \subset 0 \times C_{b_0}$ with W_b of general type and C_{b_0} isotrivial, so by the above Buium's Lemma, after replacing K by a finite extension of it, one may find a closed k-subvariety Z_{bK} of A and a point $Q_b \in A(K)$ such that $W_b = (Z_{bK} + Q_b)$. Therefore, we conclude that after replacing K by a finite extension of it, one may find a closed k-subvariety $Z \subset A$ and a point $Q \in A(K)$ such that $W = (Z_K + Q)$.

□

We now state and prove our main result, which is stronger than the previous lemma:

Proposition 62. *Let W be a projective variety of general type over K. Asume W is a closed subvariety of A_K, where A is an abelian k-variety, $A = B \times C$ with B isotrivial and C with trace zero. Let G be any algebraic D-group, $V \subset G$ an absolutely irreducible, reduced D-scheme and $u : V \to W$ be a dominant morphism of K-schemes. Then after replacing K by a finite extension of it, one may find an isotrivial k-subvariety $Z \subset B$ and an abelian k-subvariety $T \subset C$ with trace zero and points $Q \in B(K)$, $P \in C(K)$ such that $W = (Z_K + Q) \times (T + P)$.*

Proof. Let $A = B \times C$ where B is an isotrivial abelian variety and C is an abelian variety with trace 0. By Ueno's Theorem there exists an abelian subvariety S of A such that W, is up to isogeny, $W/S \times S$ with $W/S \subset A$ subvariety of general type. We apply the previous lemma to W/S, so after replacing K by a finite extension of it, one may find an isotrivial k-subvariety $Z \subset B$ and a point $Q \in B(K)$ such that $W/S = (Z_K + Q)$. Also $S = S + 0$, therefore one may find an abelian k-subvariety $T \subset C$ with trace zero and a point $P \in C(K)$ and conclude that $W = (Z_K + Q) \times (T + P)$. □

Now let X be a smooth projective curve, A an abelian variety, and Y an effective divisor on A. Assume that Y contains no translate of a non zero abelian subvariety.

Let A_K, Y_K denote the K-schemes, obtained from A, Y by tensorization with K. So, let's consider for each n, the n-th jet spaces (along a non-zero k-derivation on

K) A^n_K, Y^n_K assotiated to A_K, Y_K. Let

$$\bigtriangledown_n : W(K) \to W^n(K)$$

and for $P \in W(K)$ write $P^n = \bigtriangledown_n(P)$.

By composing $A(K) \xrightarrow{\bigtriangledown_2} A^2(K)$ with $A^2(K) \to K^N$ we get the Buium-Manin map $\phi : A(K) \to K^N$ (where $g = dimA$ and by Theorem 1.1 in [Bu4]: $g \leq N \leq 2g$). We will next apply Theorem 2 in [Bu1], which is the following:

Theorem 63. *Let A be an abelian K-variety and $\Gamma \subset A(K)$ a finite-rank subgroup. Then there exists a horizontal, irreducible, closed subgroup scheme H/K of finite type of $jet_\infty(A/K)$ such that $\Gamma \subset A_H(K)$.*

Let us mention that a closed subsheme of $jet_\infty(A/K)$ is called horizontal, if its ideal is preserved by the derivation on $jet_\infty(A/K)$. By this theorem, there exists $H \in (K^N)^\infty$ an algebraic D-group such that for any $Q \in \phi(A(K))$ we have $Q^\infty \in H(K)$. Let $G \in A^\infty$ be the pull back of H via $A^\infty \to (K^N)^\infty$. It is easy to show that G is an algebraic D-group. Let $G_n \in A^n$ be the projection of G via $\pi : A^\infty \to A^n$. Let $V_n = Y^n \bigcap G_n$.

Assume $A = B \times C$, where B is an isotrivial abelian variety and C is an abelian variety with trace zero. Let us denote the Zariski closure of $\pi(V)$ by W where $W \subset A$. By our previous Proposition applied to $V := lim_{\leftarrow} V_n$ and the Zariski closure of $\pi(V)$, after replacing K by a finite extension of it, one may find an isotrivial k-subvariety $Z \subset B$ and an abelian k-subvariety $T \subset C$ with trace zero and points $Q \in B(K)$, $P \in C(K)$ such that $W = (Z_K + Q) \times (T + P)$.

Remark 64. *We hope to be able to use these above results in order to prove a version of an ABC Theorem for any abelian variety over a function field.*

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