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Analysis of nonlinear Black Scholes models

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DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
Mathematics

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Albuquerque, New Mexico

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Dedication

To my family.

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ABSTRACT OF DISSERTATION

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Abstract

The Black Scholes equation is a fundamental model for derivative pricing. Modifying its assumptions will lead to more realistic but mathematically more complicated models. This dissertation consists of analytical and numerical studies about one particular type of nonlinear Black Scholes models, whose nonlinearity lies in the highest spatial derivative [1] with discontinuous coefficient function.

First we smooth out the discontinuous term and focus only on the nonlinearity. We consider the case where the volatility is a smooth function and present some basic existence and uniqueness results. To study the discontinuity we simplify the problem by discretizing the Partial Differential Equation PDE only in time and

[1] In physics, spatial derivative is the partial derivative with respect to space. Here the spatial derivative is with respect to the price of the underlying asset.

consider the evolution in a given tiny time step from initial data. We perform convergence and perturbation analysis to the Ordinary Differential Equation (ODE) with discontinuous coefficient and obtain some insight of how the curves, where the discontinuity occurs, evolve in the space-time plane for the PDE. Last we obtain numerical results for the nonlinear PDE in the setting of a moving boundary problem.

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Chapter 1

Introduction

The field of mathematical finance has gained significant attention since Black, Scholes, and Merton [1] published their Nobel Prize work in 1973. Using some simplified economic assumptions, they derived a linear partial differential equation (PDE) of convection–diffusion type which can be applied to the pricing of options. The solution to the linear PDE can be obtained analytically.

We are interested in nonlinear modifications of the Black-Scholes equation where the volatility σ is no longer constant, but depends on v_{ss}. Here v is the value of the option and s is the price of the underlying asset. The resulting PDEs become nonlinear in the highest derivative, and furthermore the nonlinear coefficients are discontinuous. Therefore the mathematical theory of the equation is by no means trivial. In this dissertation we treat the mathematical difficulties separately in the following chapters.

This chapter begins with a brief introduction to the classical Black-Scholes model in Section 1. The derivation of the modified Black-Scholes models is provided in Section 2.

1.1 Classical Black-Scholes Model

Option is an agreement that gives the holder a right, not obligation, to buy from, or sell to, the seller, or the buyer of the option certain amounts of underlying assets at a specified price (strike price) at a future time (expiration date). Clearly the value v of an option is a function of various parameters in the contract, written as

$$v(s, t; \mu, \sigma; E, T; r)$$

Here, s is the price of underlying asset; t is current time; μ is the drift of s; σ is the volatility of s; E is the strike price; T is the expiration date; r is the risk-free rate of interest. The assumptions for classic Black-Scholes model are the following:

1. The risk-free interest rate r is a known constant for the life of the option;
2. The price of underlying asset s follows log-normal random walk and the drift μ and volatility σ are constants known in advance;
3. Transaction costs associated with buying or selling underlying assets are not considered;
4. There are no dividends on the underlying asset;
5. Hedging can be done continuously;
6. The price of the underlying asset is divisible so that we can trade any share of the asset;
7. It is an arbitrage-free market.

Let Π denote the value of a portfolio with a long position in the option and a short position in some quantity Δ of the underlying asset,

$$\Pi = v(s, t) - \Delta s$$

By assumption, the price s of the underlying asset follows a log–normal random walk,

$$ds = \mu s dt + \sigma s dX \tag{1.1}$$

where X is standard Brownian motion.

As time changes from t to $t+dt$, the change in the value of the portfolio is due to the change in the value of the option and the change in the price of the underlying asset

$$d\Pi = dv - \Delta ds$$

By Itô's formula, we have

$$dv = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt + v_s ds$$

Combining the last two equations yields

$$d\Pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt + (v_s - \Delta)ds$$

Using a delta hedging strategy, we choose $\Delta = v_s$ and obtain

$$d\Pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt$$

By the assumption of an arbitrage–free market, the change $d\Pi$ equals the growth of Π in a risk–free interest–bearing account,

$$d\Pi = r\Pi dt = r(v - \Delta s)dt$$

Therefore,

$$r(v - \Delta s)dt = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt$$

Substituting $\Delta = v_s$, one arrives at the classic Black-Scholes equation,

$$v_t + rsv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} - rv = 0 \quad \text{for } 0 \le t \le T \tag{1.2}$$

The equation is supplemented by an end–condition at the expiration time T,

$$v(s,T) = \begin{cases} max(s-E,0), & \text{for call option,} \\ max(E-s,0), & \text{for put option,} \\ H(s-E), & \text{for binary call option.} \\ H(E-s), & \text{for binary put option.} \end{cases} \tag{1.3}$$

Here $H(x)$ is the Heaviside function. Equations (1.3) are examples of payoff functions for different options. Denote the right-hand side in formula (1.3) by $v_0(s)$.

An easy generalization can be made if we assume the asset receives a continuous and constant dividend yield D. The dividend rate can be viewed as risk-free rate for the underlying asset. After a time step dt each unit of the asset receives an amount $Dsdt$ of dividend. Thus, the change in the value of the portfolio after a time step dt becomes

$$d\Pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt + (v_s - \Delta)ds - D\Delta sdt \tag{1.4}$$

Under from delta hedging and the assumption of a non-arbitrage market, we get

$$v_t + (r-D)sv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} - rv = 0 \tag{1.5}$$

If one uses the transformation

$$\tau = T - t,\ x = ln(s) + (r - D - \frac{1}{2}\sigma^2)(T-t),\ w(x,\tau) = e^{r(T-t)}v(s,t)$$

Equation (1.5) transforms to the heat equation,

$$w_\tau = \frac{1}{2}\sigma^2 w_{xx}$$

and the end-condition becomes the initial condition,

$$w(x,0) = v(s,T) = v_0(s) = v_0(e^x) = w_0(x)$$

Therefore the classic Black Scholes model with a continuous constant dividend has the explicit solution

$$w(x,\tau) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{\frac{-(x-y)^2}{2\sigma^2\tau}} w_0(y)dy$$

or

$$v(s,t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{\frac{-(ln(s)+(r-D-\frac{1}{2}\sigma^2)(T-t)-y)^2}{2\sigma^2(T-t)}} v_0(e^y)dy \qquad (1.6)$$

For the Black-Scholes model, the "greeks" are important and useful for constructing option strategies. "Delta" (Δ) of an option defined as

$$\Delta = v_s$$

measures the sensitivity of the option or portfolio to the underlying asset. Call deltas are positive while put deltas are negative, reflecting the fact that the call option price is positively related to the underlying asset price while the put option price and the underlying asset price are inversely related. In fact, based on the put-call-parity, we have the put delta equals the call delta minus 1.

"Gamma" (Γ) measures how fast the delta changes for small changes in the underlying stock price. It is the second derivative of the option value with respect to the underlying asset.

$$\Gamma = v_{ss}$$

It shows by how much or how often a position should be rehedged in order to keep a delta neutral position. For hedging a portfolio with the delta-hedge strategy, then we want to keep gamma as small as possible, since the smaller it is the less often we will have to adjust the hedge to maintain a delta neutral position. The gammas are always positive for call options while negative for put options. However, gammas generally change signs for more complicated options such as binary options. This makes a big difference in the modified Black-Scholes model, which will be discussed in the next section.

"Theta"(Θ) is defined as

$$\Theta = v_t$$

It measures the sensitivity of the value of the option to the change of time, i.e: the "time decay". If the asset price does not move, then the option will change by theta with time.

"Vega" measures the sensitivity of the option price to the volatility of the underlying asset.

$$Vega = v_\sigma$$

It is an important but also confusing index, since volatility is not known with certainty in real market. Practically, Vega is expressed as the amount that the option's value will gain or lose as volatility rises or falls.

The last greek is "Rho" (ρ). It shows the rate of change of the option with respect to the interest rate.

$$\rho = v_r$$

Notice in the classic Black-Scholes we make the assumption of a constant interest rate. However, in practice one can use a time-dependent rate $r(t)$.

With the payoff functions for call, put and binary call/put options, we can calculate option values and "greeks" for these options explicitly. Tables 1.1- 1.3 list the formulas for values, Deltas and Gammas of options for the Black Scholes model (1.5). Given parameters as $E = 50$, $r = 0.01$, $D = 0$, $T = 1$ and $\sigma = 0.2$, Figures 1.1-1.6 show payoffs, option values at $t = 0$, deltas and gammas at $t = 0$ as functions of the underlying asset price.

The simple call and put options can be combined to construct advanced option strategies, such as butterfly options and bull options. The butterfly options consist of several simple call and put options. For example, the butterfly option could consist of two long calls at different strike prices and two short calls with the same strike price in between. The payoff function for a butterfly option, shown in Figure 1.7, can be expressed as

$$v(s,T) = max(s-90,0) + max(s-110,0) - 2max(s-100,0) \tag{1.7}$$

The bull options consists of either call or put options. For example, a bull call spread could be formed by buying a call option with a low exercise price, and selling another call option with a higher exercise price. A payoff function example in Figure 1.8 is shown as

$$v(s,T) = max(s-90,0) - max(s-100,0) \tag{1.8}$$

1.2 Modified Black-Scholes Model

We can modify the assumptions leading to the Black-Scholes model in different ways, which leads to different modified models. In this section we will focus on three different models.

1.2.1 Modified Black-Scholes Model with Variable Volatility

Volatility is the most fundamental input to an option pricing model. It is a measure of how much the underlying asset's price is likely to vary over time and is used to quantify the risk over the specified time period. The estimation of volatility is by no means an exact science, instead, it has many empirical features. Several approaches have been suggested such as using historical volatility or implied volatility, however neither of them is completely satisfactory. Volatility estimated from historical data is missing accurate indications of future volatility. While implied volatility from observed prices of traded options may differ for the same stock across strike prices and expiration date, like volatility smile [1] and volatility surface [2]. An alternative approach is to assume volatility is not known in advance as a constant but is an uncertain variable, which lies within a known range of values. This setting is very different from the classic Black Scholes model in the sense that the option price is no longer a unique value but lies in a range of possible prices (from worst case to best case). Work in this field was started by Avellaneda, Levy and Paras [13].

Assume the volatility σ lies within the range

$$0 < \sigma^- \leq \sigma \leq \sigma^+$$

where σ^+ and σ^- are estimates for the maximal and minimal values of σ. We then have

$$\min_{\sigma^- \leq \sigma \leq \sigma^+} \frac{1}{2}\sigma^2 s^2 v_{ss} = \begin{cases} \frac{1}{2}(\sigma^+)^2 s^2 v_{ss}, & \text{if } v_{ss} < 0, \\ \frac{1}{2}(\sigma^-)^2 s^2 v_{ss}, & \text{if } v_{ss} \geq 0. \end{cases}$$

This motivates to define the discontinuous function

$$\sigma_d(v_{ss}) = \begin{cases} \sigma^+, & \text{if } v_{ss} < 0, \\ \sigma^-, & \text{if } v_{ss} \geq 0. \end{cases} \quad (1.9)$$

As outlined in the previous section, under delta hedging, $\Delta = v_s$, we have

$$d\Pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt$$

[1]When implied volatility is plotted against strike price, the resulting graph typically turns up at either end. The shape of the curve is called the "smile".

[2]Implied volatility can be plotted against both maturity and strike price. The three-dimensional plot is called local volatility surface.

Assume the minimum return on the portfolio with volatility σ varying over the range $\sigma^- \le \sigma \le \sigma^+$ equals the risk-free return $r\Pi dt$. We then obtain

$$(v_t + \frac{1}{2}\sigma_d(v_{ss})^2 s^2 v_{ss})dt = r\Pi dt = r(v - sv_s)dt$$

with $\sigma_d(v_{ss})$ given by function (1.9). One obtains the non-linear PDE

$$v_t + rsv_s + \frac{1}{2}\sigma_d(v_{ss})^2 s^2 v_{ss} - rv = 0 \tag{1.10}$$

This is the model of Avellaneda, Levy, Paras for uncertain volatility.

We can also find the best option value by setting the maximum return on the portfolio to be the risk-free return. Then we will have a different discontinuous function of $\sigma_d(v_{ss})$ as

$$\sigma_d(v_{ss}) = \begin{cases} \sigma^+, & \text{if } v_{ss} \ge 0, \\ \sigma^-, & \text{if } v_{ss} < 0. \end{cases}. \tag{1.11}$$

In practice, we will not find much use for the best return model with (1.11), since it will be meaningless to assume the best outcome financially.

The nonlinearity in equation (1.10) has an important practical consequence. Individual options can not be calculated separately and aggregated. Any portfolio containing several options must be treated as a whole, such as binary options, butterfly spread and bull spread.

Because of the variability of $\sigma = \sigma_d(v_{ss})$, the transformation

$$x = ln(s) + (r - \frac{1}{2}\sigma^2)(T - t)$$

is not useful since it depends on the solution v. Instead, we apply the much simpler transformation

$$\tau = T - t,\ x = s,\ u(x, t) = v(s, \tau)$$

This transformation leads to

$$u_\tau = \frac{1}{2}\sigma_d^2(u_{xx})x^2 u_{xx} + rxu_x - ru \, , x > 0 \tag{1.12}$$

$$u(x,0) = v(s,T)$$

1.2.2 Modified Black Scholes Equation with Transaction Costs

Transaction costs are the costs appearing in the buying and selling of the underlying asset. The Black Scholes model requires the continuous rebalancing of a hedged portfolio and assumes no transaction costs in buying and selling. In reality, transaction costs do exist, of course. Depending on the underlying market, transaction costs may or may not be important. For example, transaction costs in emerging markets are more expensive and therefore it is not desirable to rehedge frequently. However, in a more liquid market, transaction costs may be very low and a portfolio can be easily rehedged to keep a delta neutral position. The classic Black-Scholes equation should be generalized to incorporate the effects of transaction costs in option pricing. Based on Leland's model [11], we assume that the transaction cost is proportional to the value of the underlying assets traded and the rate of proportion is a positive constant κ. Therefore, for buying (+) or selling (-) of $|\nu|$ shares at the price s, the transaction cost is

$$\kappa|\nu|s$$

It is possible to generalize the model by considering the components of transaction costs as a fixed cost for each transaction or a cost proportional to the number of shares of traded assets. Whalley and Wilmott [10] discussed the general models in greater detail. For completeness of model derivation, we quickly make a sketch of the Hoggard-Whalley-Wilmott model, which is based on Leland's assumption of transaction cost. Since transaction can only happen in discrete time step δt, we need to approximate the stochastic process (1.1) for the underlying asset by

$$\delta s = \mu s \delta t + \sigma s \phi \delta t^{\frac{1}{2}} \tag{1.13}$$

where ϕ is a standardized normal random variable. This approximation is based on the assumption that X is the standard Brownian motion. According to the normality of Brownian motion, we have

$$X(t + \delta t) - X(t) \sim N(0, \delta t)$$

and as $\delta t \to 0$, $X(t + \delta t) - X(t)$ approach to zero like $\sqrt{\delta t}$. We construct the same portfolio $\Pi = v(s,t) - \Delta s$. After a time step δt, the change in the value of the portfolio is given as

$$\delta \Pi = \sigma s(v_s - \Delta)\phi\sqrt{\delta t} + (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss}\phi^2 + \mu s v_s - \mu \Delta s)\delta t - \kappa s|\nu| \tag{1.14}$$

We use the same delta hedging strategy as in the classic Black-Scholes model and choose $\Delta = v_s$. The number of shares a trader holds is provided by Δ and hence the quantity ν is given by the change in the deltas

$$\nu = v_s(s + \delta s, t + \delta t) - v_s(s,t) \tag{1.15}$$

Using Taylor expansion, this can be approximated to the leading order as

$$\nu = v_{ss}\sigma s \phi \sqrt{\delta t} + O(\delta t) \tag{1.16}$$

Therefore, the change in portfolio can be approximated by

$$\delta \Pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss}\phi^2)\delta t - \kappa s^2 |v_{ss}|\sigma|\phi|\sqrt{\delta t} \tag{1.17}$$

And its expectation is

$$E[\delta \Pi] = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} - \kappa \sigma s^2 \sqrt{\frac{2}{\pi \delta t}}|v_{ss}|)\delta t \tag{1.18}$$

which follows from the facts

$$E[\phi] = 0, \quad E[\phi^2] = 1, \quad and \quad E[|\phi|] = \frac{2}{\pi}$$

Set the expected change of the portfolio value to be the amount that would have been earned by risk free deposit, namely $E[\delta\Pi] = r\Pi\delta t$. After dividing by δt and rearranging, we obtain

$$v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} - \kappa\sigma s^2 \sqrt{\frac{2}{\pi\delta t}}|v_{ss}| + rsv_s - rv = 0 \tag{1.19}$$

which is the Hoggard-Whalley-Wilmott model. Note that equation (1.10) with uncertain volatility model is exactly the same as the Hoggard-Whalley-Wilmott model (1.19). The nonlinear partial differential equations are essentially the same, however the reasons to form such equations are completely different.

1.2.3 Market Liquidity

The market price of the underlying asset is determined by supply and demand of the traded asset. Therefore the underlying asset price is affected by dynamic trading strategy. However, the classic Black Scholes model assumes that the market has perfect liquidity, meaning that investors can buy or sell a large amount of stock without affecting its price. In practice, there must be a feedback effect of trading strategies in any real market. Here we will take the market liquidity into account and introduce the modified model based on the analysis of Frey and Patie [14].

The price of the underlying asset is assumed to follow a stochastic process driven by some exogenous source of randomness such as a standard Brownian motion and also by the trading strategy of a representative trader. This leads to

$$ds = \sigma s dX + \rho\lambda(s) s d\alpha \tag{1.20}$$

where α is the stock trading strategy of a large trader, such as the amount of stock held by the trader. The variable ρ is a non-negative constant liquidity parameter. A small value of ρ means the market is more liquid and vise versa. Note if we set ρ to be zero then we are back to the Black Scholes assumption of perfect liquidity. The term $\frac{1}{\rho\lambda(s)s}$ measures the "depth of the market", which indicate the size of change in price caused by the change in one unit account of the large trader's stock position. The parameter λ describes the asymmetry of liquidity. In general, the market seems to be more liquid in bull market than in bear market range. Suppose the large trader uses a trading strategy of the form $\alpha = \phi(t, s)$. Then by Ito's Formula we will have

$$d\alpha = (\phi_t + \frac{1}{2}\phi_{ss}\sigma^2 s^2)dt + \phi_s ds \tag{1.21}$$

Inserting (1.21) into the right-hand-side of (1.20) and rearranging the terms, we obtain

$$ds = \nu(t, s)sdX + b(t, s)sdt \tag{1.22}$$

where

$$\nu(t, s) = \frac{\sigma}{1 - \rho\lambda(s)s\phi_s} \tag{1.23}$$

$$b(t, s) = \frac{\rho\lambda(s)(\phi_t + \frac{1}{2}\phi_{ss}\nu^2(t, s)s^2)}{1 - \rho\lambda(s)s\phi_s} \tag{1.24}$$

In Frey's model, the risk free interest rate is set to be zero for simplicity. Similar to the analysis of the portfolio in the Black Scholes model, we can derive the nonlinear Frey model as

$$v_t + \frac{1}{2}\frac{\sigma^2}{(1 - \rho\lambda(s)sv_{ss})^2}s^2 v_{ss} = 0 \tag{1.25}$$

with end condition at expiration date as payoff function.

$$v(s, T) = payoff(s) \tag{1.26}$$

while the trading strategy is

$$\phi(t, s) = v_s(t, s) \tag{1.27}$$

Note that the volatility ν is not a constant. Indeed, it depends on v_{ss}, the second spatial derivative of the solution.

To summarize this section of modified Black Scholes models, we showed nonlinear pricing models followed from different assumptions. These three models exhibit common mathematical features. The coefficient functions of these PDEs are not smooth and they depend on the highest spatial derivatives of the solution. Based on model (1.10), analysis of the essential mathematical features are discussed in Chapter 2 and Chapter 3 while numerical results are shown in Chapter 4.

Option	Value
Call	$se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$
Put	$-se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$
Binary Call	$e^{-r(T-t)}N(d_2)$
Binary Put	$e^{-r(T-t)}(1 - N(d_2))$
	$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$
	$d_1 = \frac{ln(\frac{s}{E}) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$
	$d_2 = \frac{ln(\frac{s}{E}) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

Table 1.1: Option value

Option	Delta
Call	$e^{-D(T-t)}N(d_1)$
Put	$e^{-D(T-t)}(N(d_1)-1)$
Binary Call	$\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma s\sqrt{T-t}}$
Binary Put	$-\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma s\sqrt{T-t}}$
	$N'(x)=\dfrac{1}{2\pi}e^{-\frac{x^2}{2}}$

Table 1.2: Delta of option

Option	Gamma
Call	$\dfrac{e^{-D(T-t)}N'(d_1)}{\sigma s\sqrt{T-t}}$
Put	$\dfrac{e^{-D(T-t)}N'(d_1)}{\sigma s\sqrt{T-t}}$
Binary Call	$-\dfrac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 s^2\sqrt{T-t}}$
Binary Put	$-\dfrac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 s^2\sqrt{T-t}}$

Table 1.3: Gamma of option

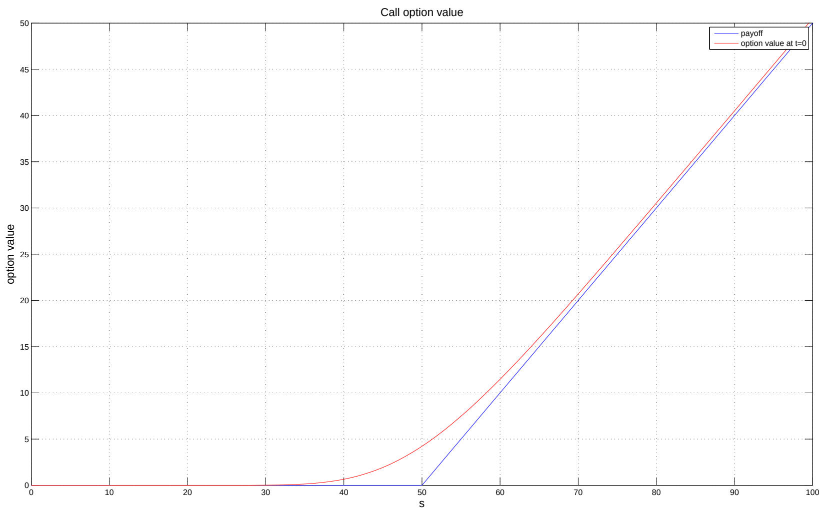

Figure 1.1: Call option payoff and value at t=0

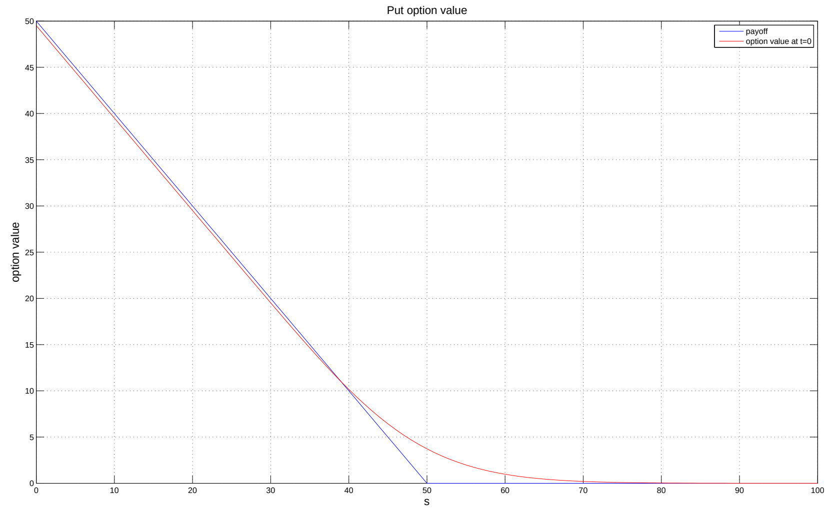

Figure 1.2: Put option payoff and value at t=0

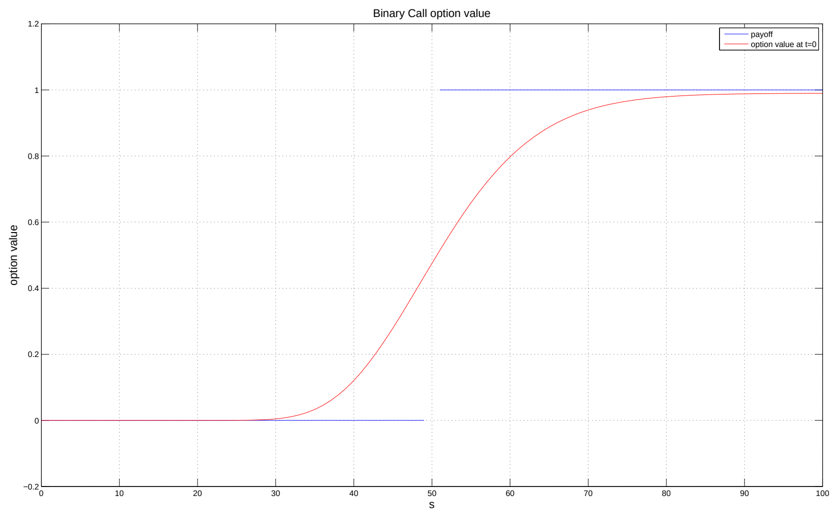

Figure 1.3: Binary call option payoff and value at t=0

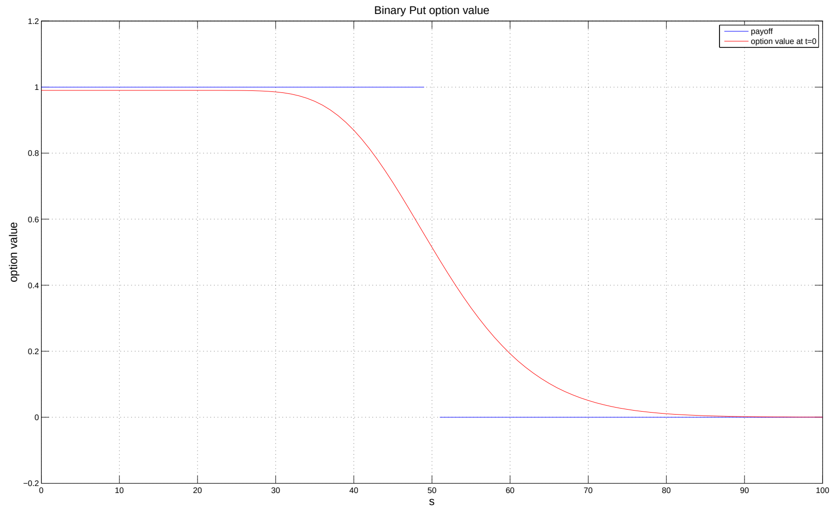

Figure 1.4: Binary put option payoff and value at t=0

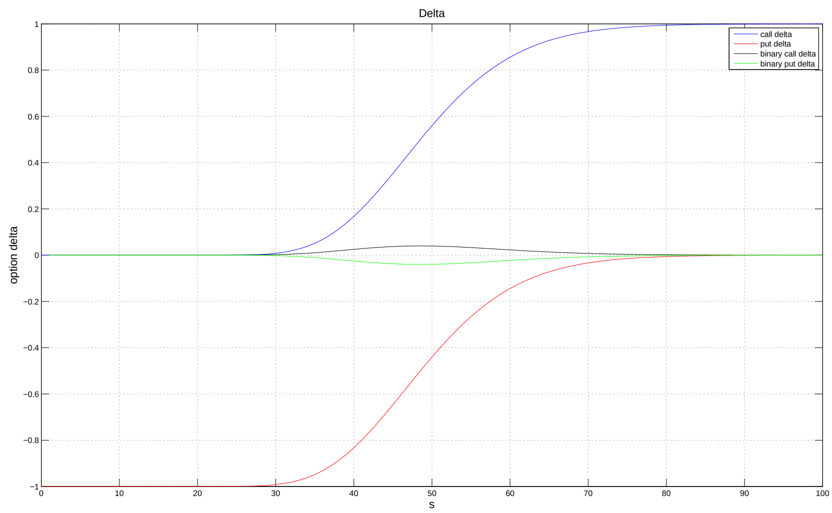

Figure 1.5: The deltas for call, put, binary call and binary put option at t=0

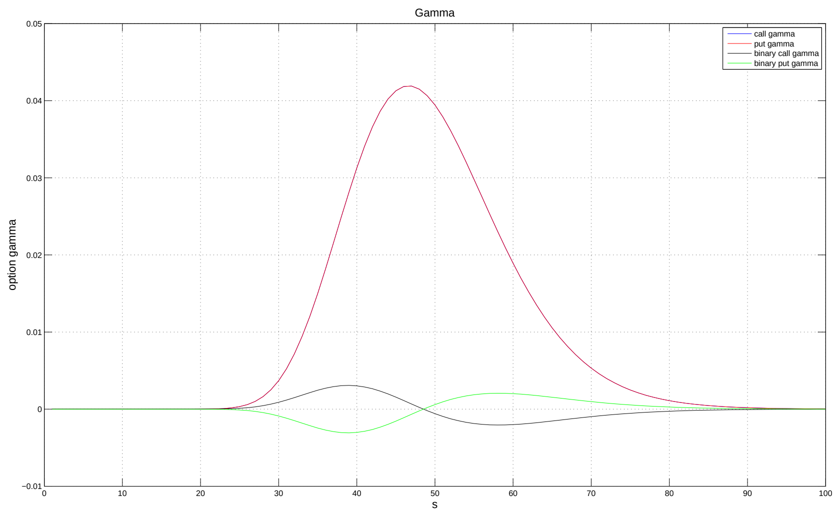

Figure 1.6: The gammas for call, put, binary call and binary put option at t=0

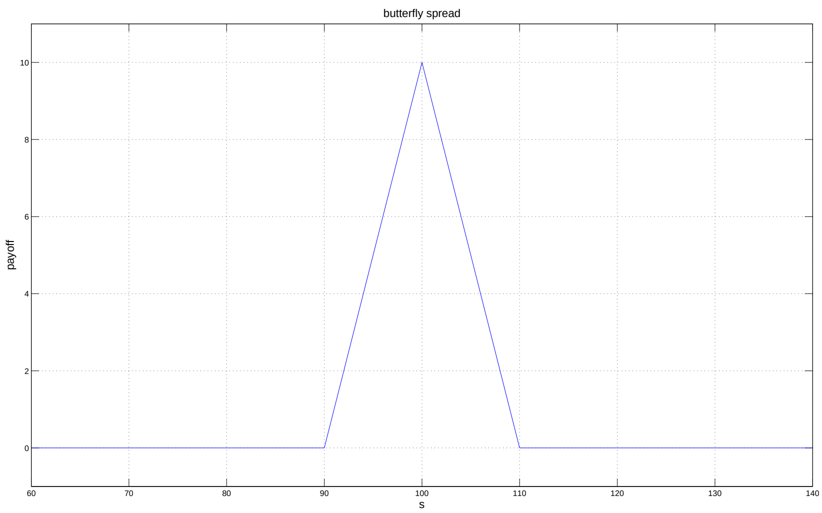

Figure 1.7: Payoff of butterfly spread

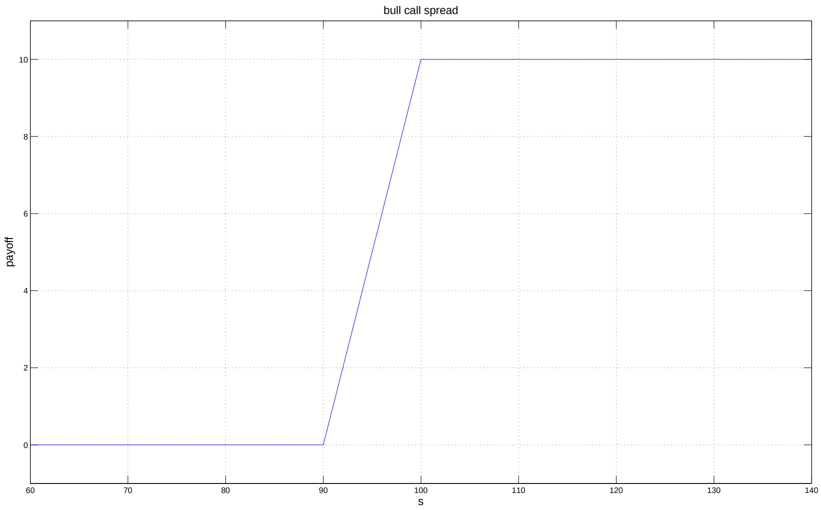

Figure 1.8: Payoff of bull call spread

Chapter 2

Existence and Uniqueness Analysis

The essential mathematical difficulty of equation (1.10) lies in the nonlinear term $\sigma_d^2(v_{ss})v_{ss}$. To address this difficulty, we consider an equation of the form

$$u_t = G(u_{xx})u_{xx}, \quad u(x,0) = u_0(x), \quad u(x+1,t) \equiv u(x,t) \tag{2.1}$$

where $G : \mathbb{R} \to (0, \infty)$ is a given smooth positive function, and $u_0(x)$ is a 1-periodic smooth function.

The function σ_d in (1.10) is not smooth, of course, but we can approximate σ_d by a smooth function like

$$\sigma_\epsilon(u_{xx}) = \frac{1}{2}(\sigma^+ + \sigma^-) - \frac{1}{2}(\sigma^+ - \sigma^-)tanh(\frac{1}{\epsilon}u_{xx}), \epsilon > 0 \tag{2.2}$$

An example of σ_ϵ is shown in Figure 2.1.

Differentiate equation (2.1) twice with respect to x and let $w(x,t) = u_{xx}(x,t)$ to get

$$w_t = D^2(G(w)w), \quad w(x,0) = u_0''(x), \quad w(x+1,t) \equiv w(x,t) \tag{2.3}$$

Here $D^2 = \frac{\partial^2}{\partial x^2}$. Therefore, we get

$$w_t = h(w)w_{xx} + h'(w)w_x^2 \tag{2.4}$$

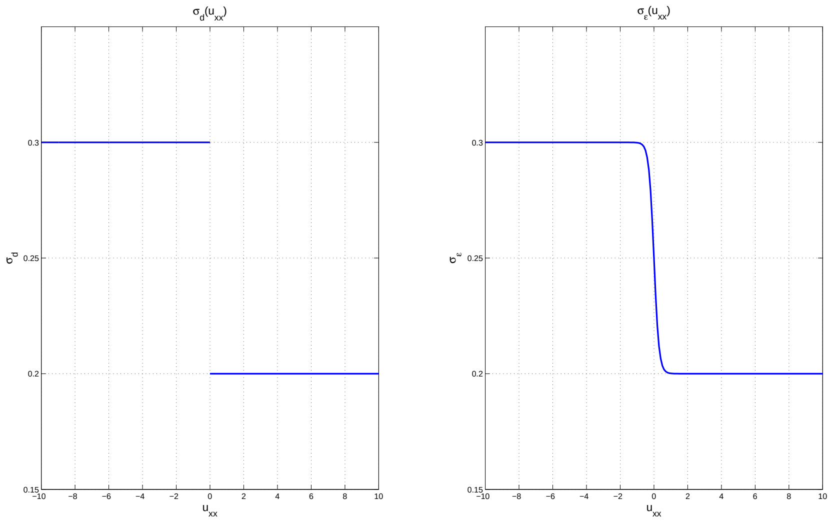

Figure 2.1: Plots of discontinuous function σ_d and corresponding smooth function σ_ϵ, where $\sigma^- = 0.2$, $\sigma^+ = 0.3$ and $\epsilon = 0.3$.

where $h(w) = G(w) + G^{'}(w)w$. It will be convenient to consider the slightly more general problem

$$w_t = h(w)w_{xx} + g(w, w_x),\ w(x, 0) = f(x), \quad w(x + 1, t) \equiv w(x, t) \tag{2.5}$$

where $h(w)$, $g(w, w_x)$ are C^∞ functions of their arguments, and $f(x) = u_0''(x)$ is 1-periodic smooth function. In addition, assume that [1]

$$h(w) \geq k > 0 \quad k\ is\ a\ constant$$

and that h, g and all their derivatives are bounded functions. If h , g or their derivatives are unbounded, we can use cut-off functions $\phi(w) \in C^\infty(\mathbb{R})$ for h and $\varphi(w, w_x) \in C^\infty(\mathbb{R}^2)$ for g with

$$\phi(w) = 1 \text{ for } |w| \leq R \quad , \quad \phi(w) = 0 \text{ for } |w| > 2R$$

$$\varphi(w, w_x) = 1 \text{ for } |w|^2 + |w_x|^2 \leq R^2 \quad , \quad \varphi(w, w_x) = 0 \text{ for } |w|^2 + |w_x|^2 > (2R)^2$$

[1]This assumption of $h(w)$ is plausible for any $\epsilon > 0$ with suitable σ^+ and σ^-.

for some real positive R. Then we can replace h by $\tilde{h}(w) = \phi(w)h(w)$ and replace g by $\tilde{g}(w, w_x) = \varphi(w, w_x)g(w, w_x)$. The functions $\tilde{h}(w)$ and $\tilde{g}(w, w_x)$ satisfy the assumptions of boundedness. Then for the original problem one obtains existence local in time.

After showing the uniqueness of a classical solution of equation (2.5) under the above assumptions, we prove a priori estimates in any finite time interval $0 \leq t < T < \infty$. To show existence of a solution in some time interval, we use the iteration approach as

$$w_t^{n+1} = h(w^n)w_{xx}^{n+1} + g(w^n, w_x^n) \tag{2.6}$$

$$w^n(x, 0) = f(x), \quad n = 1, 2, 3, ...$$

where $w^0(x, t) \equiv f(x)$. The sequence w^n will be shown to be convergent, and its limit is a solution to equation (2.5).

2.1 Uniqueness

A classical solution of equation (2.5) is a function $w \in C^1(t) \cap C^2(x)$ which satisfies (2.5) pointwise. Let $|.|_\infty$ be maximum norm, $||.||$ be L_2 norm, $(.,.)$ be L_2 inner product and $D^j = \frac{\partial^j}{\partial x^j}, \quad j = 0, 1, 2, ...$. We first show

Theorem 2.1.1. *Equation* (2.5) *has at most one classical solution.*

Proof. Let $u(x, t)$ and $v(x, t)$ be solutions to equation (2.5) in some time interval $0 \leq t < T$. This also assumes that u and v are 1-periodic in x. Their difference

$$\psi(x, t) = u(x, t) - v(x, t)$$

satisfies

$$\psi_t = h(u)u_{xx} + g(u, u_x) - h(v)v_{xx} - g(v, v_x), \quad \psi(x, 0) = 0$$

We can rewrite ψ_t as

$$\psi_t = h(u)(u_{xx} - v_{xx}) + (h(u) - h(v))v_{xx} + (g(u, u_x) - g(v, u_x)) + (g(v, u_x) - g(v, v_x))$$

By the Mean Value Theorem, we now get

$$\psi_t = h(u)\psi_{xx} + h^{'}(\xi)v_{xx}\psi + g_1(\zeta, u_x)\psi + g_2(v, \eta)\psi_x \tag{2.7}$$

where ξ and ζ lie between u and v, and η lies between u_x and v_x. Here g_1 is the partial derivative of g with respect to the first argument, while g_2 is the partial derivative of g with respect to the second argument. Applying energy estimation to ψ with (2.7), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel \psi(.,t) \parallel^2 &= (\psi, \psi_t) \\
&= -(D(h(u)\psi), \psi_x) + (\psi, (h^{'}(\xi)v_{xx} + g_1(\zeta, u_x))\psi) + (\psi, g_2(v, \eta)\psi_x) \\
&\leq -k \parallel \psi_x \parallel^2 + c_1 \parallel \psi \parallel \parallel \psi_x \parallel + c_2 \parallel \psi \parallel^2 \\
&\leq c_3 \parallel \psi \parallel^2
\end{aligned}$$

Therefore

$$\parallel \psi(.,t) \parallel^2 \leq e^{2c_3 t} \parallel \psi(.,0) \parallel^2 = 0$$

The initial condition $\psi(x, 0) = 0$ implies $\psi(x, t) \equiv 0$. □

2.2 A Priori Estimates

We show a priori estimates since the techniques to derive these will be useful for the existence argument in section 2.3. We first show the following lemmas which will be used for a priori estimates.

Lemma 2.2.1. *(Gronwall's Lemma): Suppose* $y \in C^1[0,T)$, $\psi \in C[0,T)$ *satisfy*

$$y^{'}(t) \leq cy(t) + \psi(t) + k, \quad 0 \leq t < T,$$

for some $c \geq 0$,$k \geq 0$. *Then*

$$y(t) \leq e^{ct}\{y(0) + \int_0^t |\psi(s)|ds + kt\}, \quad 0 \leq t < T.$$

Proof. For the function $z(t) = e^{-ct}y(t)$ it holds that

$$z^{'}(t) = -ce^{-ct}y(t) + e^{-ct}y'(t) \leq e^{-ct}(\psi(t) + k).$$

Thus integration of both sides yields

$$z(t) \leq \int_0^t |\psi(s)|ds + kt + z(0),$$

and the lemma follows. □

Lemma 2.2.2. *Suppose* $u \in C^1[0,1]$, *then*

$$|u|_\infty^2 \leq ||u||^2 + 2||u||||Du||, \ D = \frac{d}{dx}$$

Proof. There exist x_0 and x_1 with

$$min|u(x)| : 0 \leq x \leq 1 = |u(x_0)|$$

$$max|u(x)| : 0 \leq x \leq 1 = |u(x_1)| = |u|_\infty$$

Let $x_0 < x_1$ for definiteness. Then we have

$$\begin{aligned} |u(x_1)|^2 - |u(x_0)|^2 &= \int_{x_0}^{x_1} [\frac{d}{dx}|u(x)|^2]dx \\ &= \int_{x_0}^{x_1} 2(u, u_x)dx \\ &\leq 2||u||||u_x||. \end{aligned}$$

Since $|u(x_0)| \leq ||u||$, the lemma follows. □

Lemma 2.2.2 is an example of Sobolev Inequality which gives bounds of the maximum norm of a smooth function by L_2-norms of its derivatives.

Let w denote a C^∞ solution of (2.5) defined for $0 \leq t < T$, any finite T, with periodic boundary conditions. We can show estimations for spatial derivatives of w. In the following, let c_i, $i = 1, 2, ...$ be suitable positive constants. First note that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \parallel w \parallel^2 &= (w,\ w_t) \\ &= (w,\ h(w)w_{xx} + g(w, w_x)) \\ &\leq c_1 \parallel w \parallel \parallel D^2 w \parallel + c_2 \parallel w \parallel + c_3 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \parallel Dw \parallel^2 &= (Dw,\ Dw_t) \\ &= -(D^2 w,\ w_t) \\ &\leq -k \parallel D^2 w \parallel^2 + c_4 \parallel D^2 w \parallel + c_5 \\ &\leq -\frac{k}{2} \parallel D^2 w \parallel^2 + c_6 \end{aligned} \tag{2.9}$$

Adding (2.8) to (2.9), we obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}(\parallel w \parallel^2 + \parallel Dw \parallel^2) \leq c_7(\parallel w \parallel^2 + \parallel Dw \parallel^2) + c_8 \tag{2.10}$$

As follows from Gronwall's Lemma **??**, inequality (2.10) implies that $\parallel w \parallel$ and $\parallel Dw \parallel$ are bounded for $0 \leq t < T < \infty$. By Lemma 2.2.2 , we obtain the boundedness for $|w|_\infty$ for $0 \leq t < T$. Furthermore, integrating both sides of equation (2.9), we have $\int_0^t ||D^2 w(., \tau)||^2 d\tau$ is bounded for $0 \leq t < T$. The function Dw_t satisfies

$$\begin{aligned} Dw_t &= D(h(w)w_{xx}) + Dg(w, w_x) \\ &= hD^3 w + h^{'} D^2 w Dw + g_1 Dw + g_2 D^2 w \end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel D^2 w \parallel^2 &= -(D^3 w,\ Dw_t) \\
&= -(D^3 w,\ hD^3 w + h^{'} D^2 w Dw + g_1 Dw + g_2 D^2 w) \\
&\leq -k \parallel D^3 w \parallel^2 + c_9 |Dw|_\infty \parallel D^2 w \parallel\parallel D^3 w \parallel \\
&\quad + c_{10} \parallel D^3 w \parallel\parallel Dw \parallel + c_{11} \parallel D^3 w \parallel\parallel D^2 w \parallel
\end{aligned}$$

The function $||Dw||$ has been shown to be bounded, and $|Dw|_\infty$ is bounded by $||D^2w||$ from Lemma 2.2.2. We obtain

$$\frac{1}{2}\frac{d}{dt} \parallel D^2 w \parallel^2 \leq c_{12} \parallel D^2 w \parallel^4 + c_{13} ||D^2 w||^2 + c_{14}$$

Using the abbreviation $\alpha(t) = ||D^2w(.,t)||^2$ and $\beta(t) = ||D^2w(.,t)||^2 + 1$, we have shown the differential inequality with some positive constants a and c

$$\alpha'(t) \leq a\alpha(t)\beta(t) + c \tag{2.11}$$

Since $\int_0^t \beta(s)ds$ is bounded for $0 \leq t < T$, we can rewrite the differential inequality (2.11) as

$$(e^{-a\int_0^t \beta(s)ds}\alpha(t))' \leq e^{-a\int_0^t \beta(s)ds} c$$

Then integration of both sides yields

$$\alpha(t) \leq e^{a\int_0^t \beta(s)ds}[\int_0^t e^{-a\int_0^s \beta(\tau)d\tau} cds + \alpha(0)]$$

Therefore, we obtain $\alpha(t) = ||D^2w(.,t)||^2$ is bounded for $0 \leq t < T$. Furthermore, $|Dw|_\infty$ is bounded by Lemma 2.2.2. Next we consider

$$\begin{aligned}
D^2 w_t &= D^2(h(w)w_{xx}) + D^2 g(w, w_x) \\
&= hD^4 w + h^{''}(Dw)^2 D^2 w + 2h^{'} D^3 w Dw + h^{'}(D^2 w)^2 \\
&\quad + g_{11}(Dw)^2 + 2g_{12} Dw D^2 w + g_{22}(D^2 w)^2
\end{aligned}$$

where g_{ii}, $i = 1, 2$, is the second partial of g with respect to the ith argument, while g_{12} is the mixed second partial of g with respect to both arguments. We get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel D^3 w \parallel^2 &= -(D^4 w, D^2 w_t) \\
&\leq -k \parallel D^4 w \parallel^2 + c_{15} \parallel D^4 w \parallel\parallel D^3 w \parallel + c_{16} \parallel D^4 w \parallel\parallel D^2 w \parallel |D^2 w|_\infty \\
&\quad + c_{17} \parallel D^4 w \parallel\parallel D^2 w \parallel + c_{18} \parallel D^4 w \parallel \\
&\leq c_{19} \parallel D^3 w \parallel^2 + c_{20}
\end{aligned}$$

Again, as follows from Gronwall's lemma 2.2.1, we have $\parallel D^3 w \parallel$ is bounded for $0 \leq t < T$. Lemma 2.2.2 gives a bound for $|D^2 w|_\infty$.

Lemma 2.2.3. *Suppose w is a C^∞ solution of equation* (2.5) *defined for $0 \leq t < T$, T is any finite time. Assume $w(x, t)$ is 1-periodic in x for each t. Then $\parallel D^j w \parallel$ are bounded for $j = 0, 1, 2 \cdots$ for $0 \leq t < T$.*

Proof. We use induction on j.

The cases when $j = 0, 1, 2, 3$ have been treated above; thus let $j \geq 4$, we only need to show $\parallel D^j w||$ is bounded, given $\parallel D^l w||$ are bounded for $l = 0, 1, \cdots, j-1$. We have

$$\frac{1}{2}\frac{d}{dt} \parallel D^j w \parallel^2 = -(D^{j+1} w, D^{j-1} w_t)$$

For term $D^{j-1} w_t$, the leading order terms give

$$\begin{aligned}
D^{j-1} w_t &= \alpha_1 h(w) D^{j+1} w + (\alpha_2 h' Dw + \alpha_3 g_2) D^j w \\
&\quad + (\alpha_4 h' D^2 w + \alpha_5 g_{22} D^2 w + \alpha_6 g_1) D^{j-1} w + o(D^{j-2} w)
\end{aligned}$$

where α_i, $i = 1, 2, ..6$ are positive constant coefficients. The terms $|D^{l-1} w|_\infty$ are bounded by Lemma 2.2.2 for $1 \leq l \leq j-1$, $j \geq 4$, therefore

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel D^j w \parallel^2 &\leq -k \parallel D^{j+1} w \parallel^2 + c_{21} \parallel D^{j+1} w \parallel \parallel D^j w \parallel \\
&\quad + c_{21} \parallel D^{j+1} w \parallel \parallel D^{j-1} w \parallel + c_{22} \parallel D^{j+1} w \parallel \\
&\leq c_{23} \parallel D^j w \parallel^2 + c_{24}
\end{aligned}$$

From Gronwall's lemma 2.2.1, the induction step is completed, and the lemma follows. □

Since we have bounded space derivatives of w in $0 \leq t < T < \infty$, we can use the differential equation (2.5) to bound all time derivatives and mixed derivatives, ie: each term

$$\frac{\partial^{p+q}}{\partial x^p \partial t^q} w(x,t)$$

can be written as a sum of products of space derivatives and hence it is bounded in $0 \leq t < T$.

2.3 Existence via Iteration

First, notice $w^n(x,t)$ in iteration scheme (2.6) are defined as solutions to linear equations. They exist for $0 \leq t < \infty$. Fix $T > 0$, to prove existence via the iteration scheme (2.6). We then estimate the function w^n independently of the index n. We start with the following estimates to show uniform smoothness of the sequence w^n. Let $c_i, i = 1, 2, ...,$ be suitable positive constants. First note that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel w^n \parallel^2 &= (w^n,\ w^n_t) \\
&= (w^n,\ h(w^{n-1}) w^n_{xx} + g(w^{n-1}, w^{n-1}_x)) \\
&\leq c_1 \parallel w^n \parallel \parallel D^2 w^n \parallel + c_2 \parallel w^n \parallel + c_3 \qquad (2.12)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel Dw^n \parallel^2 &= (Dw^n, Dw_t^n) \\
&= -(D^2w^n, w_t^n) \\
&\leq -k \parallel D^2w^n \parallel^2 + c_4 \parallel D^2w^n \parallel + c_5 \\
&\leq -\frac{k}{2} \parallel D^2w^n \parallel^2 + c_6 \qquad (2.13)
\end{aligned}$$

We then obtain the following differential inequality by adding inequalities (2.12) to (2.13)

$$\frac{1}{2}\frac{d}{dt}(\parallel w^n \parallel^2 + ||Dw^n||^2) \leq c_7(\parallel w^n \parallel^2 + ||Dw^n||^2) + c_8$$

Therefore, as follows from Lemma 2.2.1, $\parallel w^n \parallel$ and $\parallel Dw^n \parallel$ are bounded independently of n for $0 \leq t < T$. We also obtain that $|w^n|_\infty$ is bounded independently of n by Lemma 2.2.2. Integrating both sides of inequality (2.13), we have $\int_0^t \parallel D^2w^n(,.\tau) \parallel d\tau$ is bounded uniformly for $0 \leq t < T$. Notice Dw_t^n satisfies

$$\begin{aligned}
Dw_t^n &= D(h(w^{n-1})w_{xx}^n) + Dg(w^{n-1}, w_x^{n-1}) \\
&= h' Dw^{n-1}D^2w^n + hD^3w^n + g_1Dw^{n-1} + g_2D^2w^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel D^2w^n \parallel^2 &= -(D^3w^n, Dw_t^n) \\
&\leq -k \parallel D^3w^n \parallel^2 + c_9|Dw^{n-1}|_\infty \parallel D^2w^n \parallel\parallel D^3w^n \parallel \\
&\quad + c_{10} \parallel D^3w^n \parallel\parallel Dw^{n-1} \parallel + c_{11} \parallel D^3w^n \parallel\parallel D^2w^{n-1} \parallel
\end{aligned}$$

Since $|Dw^{n-1}|_\infty$ is bounded by $\parallel D^2w^{n-1}||$ by Lemma 2.2.2, we get

$$\frac{1}{2}\frac{d}{dt} \parallel D^2w^n \parallel^2 \leq c_{12} \parallel D^2w^n \parallel^2\parallel D^2w^{n-1} \parallel^2 + c_{13} \parallel D^2w^{n-1} \parallel^2 + c_{14}$$

Using the abbreviations $\alpha(t) = \parallel D^2 w^n(.,t) \parallel^2$ and $\beta(t) = \parallel D^2 w^{n-1}(.,t) \parallel^2$, we have shown the differential inequality with positive constants a,b and c

$$\alpha^{'}(t) \leq a\alpha(t)\beta(t) + b\beta(t) + c \tag{2.14}$$

Since $\int_0^t \beta(\tau)d\tau$ has been shown to be bounded for $0 \leq t < T$, we can rewrite inequality (2.14) as

$$(e^{-a\int_0^t \beta(s)ds}\alpha(t))' \leq e^{-a\int_0^t \beta(s)ds}(b\beta(t) + c)$$

Integration of both sides gives

$$\alpha(t) \leq e^{a\int_0^t \beta(s)ds}[\int_0^t e^{-a\int_0^s \beta(\tau)d\tau}(b\beta(s) + c)ds + \alpha(0)]$$

Thus $\alpha(t) = \parallel D^2 w^n \parallel$ is bounded uniformly for $0 \leq t < T$. It follows that $|Dw^n|$ is also bounded independently of n by Lemma 2.2.2.

$$\begin{aligned}
D^2 w_t^n &= D^2(h(w^{n-1})w_{xx}^n) + D^2 g(w^{n-1}, w_x^{n-1}) \\
&= hD^4 w + h^{'} Dw^{n-1} D^3 w + \\
&\quad h^{''}(Dw^{n-1})^2 D^2 w^n + h^{'} D^2 w^{n-1} D^2 w^n + h^{'} Dw^{n-1} D^3 w^n + \\
&\quad g_{11}(Dw^{n-1})^2 + 2g_{12} D^2 w^{n-1} Dw^{n-1} + g_1 Dw^{n-1} \\
&\quad + g_{22}(D^2 w^{n-1})^2 + g_2 D^3 w^{n-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel D^3 w^n \parallel^2 &= -(D^4 w^n, D^2 w_t^n) \\
&\leq c_{15} \parallel D^3 w^n \parallel^2 + c_{16} \parallel D^3 w^{n-1} \parallel^2 + c_{17}
\end{aligned}$$

Using the abbreviation $y_n(t) = \parallel D^3 w^n(.,t) \parallel^2$, we have shown the following differential inequality

$$y_n'(t) \leq 2c_{15} y_n(t) + 2c_{16} y_{n-1}(t) + 2c_{17}$$

To estimate $y_n(t)$, we can use Gronwall's lemma 2.2.1 and Picard's lemma stated as follows:

Lemma 2.3.1. *(Picard's Lemma): Let $y^k(t)$, $k = 0, 1, \cdots$, denote a sequence of nonnegative continuous functions which satisfy the inequalities:*

$$y^{k+1}(t) \leq a + b\int_0^t y^k(s)ds, \quad 0 \leq t \leq T,$$

with nonnegative constants a, b. Then

$$y^k(t) \leq a\sum_{n=0}^{k-1} \frac{b^n t^n}{n!} + \frac{b^k t^k}{k!} \max_{0\leq s\leq t} y^0(s)$$

for $0 \leq t \leq T$ and $k = 0, 1, \cdots$. In particular, the sequence $y^k(t)$, $0 \leq t \leq T$, is uniformly bounded. If $a = 0$, then the sequence converges uniformly to zero.

Proof. For $k = 0$ the estimate is true. Assume it holds up to the index k. Then

$$y^{k+1}(t) \leq a + a\sum_{n=1}^{k} \frac{b^n t^n}{n!} + \frac{b^{k+1} t^{k+1}}{(k+1)!} \max_{0\leq s\leq t} y^0(s)$$

and the lemma is proved. □

By Gronwall's lemma 2.2.1 and Picard's lemma 2.3.1, we obtain that the term $\| D^3 w^n(.,t) \|$ is bounded independently of n for $0 \leq t < T$.

We now show that all spatial derivatives of w^n can be estimated for any finite time interval $0 \leq t < T$.

Lemma 2.3.2. *For $j = 0, 1, 2, \cdots$, $\| D^j w^n \|$ are bounded independently of n for $0 \leq t < T$,*

Proof. We use induction on j.
The cases with $j = 0, 1, 2, 3$ have been shown above. Thus let $j \geq 4$, we now show

that $\| D^j w^n \|$ is bounded independently of n , given $\| D^l w^n \|$ are bounded for $0 \leq l \leq j-1$. We have

$$\frac{1}{2}\frac{d}{dt} \| D^j w^n \|^2 = -(D^{j+1}w^n, D^{j-1}w_t^n)$$

For function $D^{j-1}w_t^n$, the leading term analysis leads to

$$\begin{aligned} D^{j-1}w_t^n &= \alpha_1 h(w^{n-1})D^{j+1}w^n \\ &\quad +\alpha_2 h' Dw^{n-1}D^j w^n + \alpha_3 g_2 D^j w^{n-1} \\ &\quad +\alpha_4 h' D^2 w^{n-1} D^{j-1}w^n + (\alpha_5 g_{22} D^2 w^{n-1} + \alpha_6 g_1 + \alpha_7 h' D^2 w^{n-1})D^{j-1}w^{n-1} \\ &\quad +o(D^{j-2}w^n) + o(D^{j-2}w^{n-1}) \end{aligned}$$

where α_i, $i = 1, 2, .., 7$ are positive constant coefficients. We have $|D^{l-1}w^n|_\infty$ are bounded independently of n for $1 \leq l \leq j-1$, $j \geq 4$ by lemma 2.2.2. Thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| D^j w^n \|^2 &\leq -k \| D^{j+1}w^n \|^2 + c_{18} \| D^{j+1}w^n \| \| D^j w^n \| \\ &\quad + c_{19} \| D^{j+1}w^n \| \| D^j w^{n-1} \| + c_{20} \| D^{j+1}w^n \| \\ &\leq c_{21} \| D^j w^n \|^2 + c_{22} \| D^j w^{n-1} \|^2 + c_{23} \end{aligned}$$

Followed from Gronwall's lemma 2.2.1 and Picard's lemma 2.3.1, we obtain that $\| D^j w^n \|$ is bounded independently of n for $0 \leq t < T$. Thus the induction step is completed and the lemma follows. □

So far, we have estimated the $L_2 - norm$ of all spatial derivatives of the sequence w^n in $0 \leq t < T$, any finite T. By the Sobolev inequality stated in Lemma 2.2.2, the functions $D^j w^n$ are also bounded in maximum norm, for any $j \geq 0$. Since we can always replace time derivatives by spatial derivatives using the differential equation (2.6), it follows the uniform smoothness:

$$|\frac{\partial^{p+q} w^n}{\partial^{x^p t^q}}(x,t)| \leq C(p,q)$$

for $0 \leq t < T$. Here $C(p,q)$ are some constants depending only on p and q, but independent of n.

We finally show

Theorem 2.3.3. *For any finite time T, equation (2.5) has a C^∞ solution, which is 1-periodic in x, defined for $0 \leq t < T$.*

Proof. Consider the sequence w^n defined by the iteration (2.6) and let

$$\delta^n = w^{n+1} - w^n$$

$$\delta^n(x,0) = 0$$

Then

$$\begin{aligned}
\delta^n_t &= h(w^n)w^{n+1}_{xx} + g(w^n, w^n_x) - h(w^{n-1})w^n_{xx} - g(w^{n-1}, w^{n-1}_x) \\
&= (h(w^n)w^{n+1}_{xx} - h(w^n)w^n_{xx}) + (h(w^n)w^n_{xx} - h(w^{n-1})w^n_{xx}) + \\
&\quad (g(w^n, w^n_x) - g(w^{n-1}, w^n_x)) + (g(w^{n-1}, w^n_x) - g(w^{n-1}, w^{n-1}_x))
\end{aligned}$$

We apply the Mean Value Theorem and get

$$\delta^n_t = h(w^n)\delta^n_{xx} + h'(\xi)w^n_{xx}\delta^{n-1} + g_1(\zeta, w^n_x)\delta^{n-1} + g_2(w^{n-1}, \eta)\delta^{n-1}_x$$

where ξ and ζ lie between $w^{n-1}$and w^n, and η lies between w^{n-1}_x and w^n_x. Then we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel \delta^n \parallel^2 &= (\delta^n, \delta^n_t) \\
&= -(D(\delta^n h(w^n)), \delta^n_x) + (\delta^n, (h'(\xi)w^n_{xx} + g_1(\zeta, w^{n-1}_x))\delta^{n-1}) \\
&\quad -(D(\delta^n g_2(w^n, \eta)), \delta^{n-1}) \\
&\leq -k \parallel \delta^n_x \parallel^2 + c_1 \parallel \delta^n_x \parallel\parallel \delta^n \parallel + c_2 \parallel \delta^n_x \parallel\parallel \delta^{n-1} \parallel + c_3 \parallel \delta^n \parallel\parallel \delta^{n-1} \parallel \\
&\leq c_4 \parallel \delta^n \parallel^2 + c_5 \parallel \delta^{n-1} \parallel^2
\end{aligned}$$

As follows from Gronwall's Lemma 2.2.1, we obtain

$$\| \delta^n(.,t) \|^2 \leq c_5 e^{2c_4 t} \int_0^t \| \delta^{n-1}(.,\tau) \|^2 \, d\tau$$

Let $C = c_5 e^{2c_4 T}$. For $0 \leq t < T$, we now have

$$\| \delta^n(.,t) \|^2 \leq C \int_0^t \| \delta^{n-1}(.,\tau) \|^2 \, d\tau$$

where C is independent of n. Therefore, by Picard's lemma 2.3.1, the sequence $w^n(\cdot,t)$ converges to a L_2 function $w(\cdot,t)$. The smoothness estimates (Lemma 2.3.2) imply that $w \in C^\infty$ and the convergence holds pointwise and also for all derivatives. (see Arzela-Ascoli theorem in [2] for details). Then the equation (2.6) implies that w solves the equation (2.5). $\square$

2.4 Construction of Solution

So far we have shown, for problem (2.3):

$$w_t = D^2(G(w)w), \quad w(x,0) = u_0''(x), \quad w(x+1,t) \equiv w(x,t)$$

if the function G satisfies the assumption that $h(w) = G(w) + G'(w)w$ is bounded from below by some positive constant, we then obtain the local existence of a unique solution $w(x,t)$. The Fourier expansion of $w(x,t)$ is written as

$$w(x,t) = \sum_{k=-\infty}^{\infty} \hat{w}(k,t) e^{2\pi ikx} \tag{2.15}$$

Notice that

$$\frac{d}{dt}(1,w) = (1,w_t) = (1, D^2(G(w)w)) = 0$$

We then have

$$\int_0^1 w(x,t)dx \equiv 0, \quad t \geq 0$$

Therefore, the term $\hat{w}(0,t) \equiv 0$ in the equation (2.15), and

$$w(x,t) = \sum_{k \neq 0} \hat{w}(k,t) e^{2\pi i k x} \tag{2.16}$$

We can integrate (2.16) twice in x and obtain a 1-periodic function $u(x,t)$

$$u(x,t) = \alpha(t) + \sum_{k \neq 0} \frac{1}{(2\pi i k)^2} \hat{w}(k,t) e^{2\pi i k x} \tag{2.17}$$

The term $\alpha(t)$ needs to be determined.

We are now to show that $u(x,t)$ defined in equation (2.17) with suitable $\alpha(t)$ solves problem (2.1)

$$u_t = G(u_{xx})u_{xx}, \quad u(x,0) = u_0(x), \quad u(x+1,t) \equiv u(x,t)$$

First we show that $u(x,t)$, constructed in equation (2.17), satisfies the initial condition $u(x,0) = u_0(x)$, if $\alpha(0) = \hat{u}_0(0)$. Here $\hat{u_0}(k)$, $k \in \mathcal{Z}$, are the Fourier coefficients of $u_0(x)$. Express $u_0(x)$ in Fourier expansion as following

$$u_0(x) = \hat{u}_0(0) + \sum_{k \neq 0} \hat{u_0}(k) e^{2\pi i k x}$$

Then we have

$$u_0''(x) = \sum_{k \neq 0} (2\pi i k)^2 \hat{u}_0(k) e^{2\pi i k x}$$

Since $w(x,0) = u_0''(x)$, we get

$$\begin{aligned} w(x,0) &= \sum_{k \neq 0} \hat{w}(k,0) e^{2\pi i k x} \\ &= \sum_{k \neq 0} (2\pi i k)^2 \hat{u_0}(k) e^{2\pi i k x} \end{aligned}$$

This leads to

$$\hat{w}(k,0) = (2\pi i k)^2 \hat{u_0}(k), \quad for\ each\ k \neq 0$$

Then we can show that

$$\begin{aligned}
u(x,0) &= \alpha(0) + \sum_{k\neq 0} \frac{1}{(2\pi ik)^2}\hat{w}(k,0)e^{2\pi ikx} \\
&= \hat{u}_0(0) + \sum_{k\neq 0} \frac{1}{(2\pi ik)^2}(2\pi ik)^2\hat{u}_0(k)e^{2\pi ikx} \\
&= u_0(x)
\end{aligned}$$

Next, we will derive the conditions for $\alpha(t)$ such that $u(x,t)$ satisfies the PDE $u_t = G(u_{xx})u_{xx}$. Note that the term $G(u_{xx})u_{xx}$ can be written as $G(w)w$, which is a 1-periodic function in x. Let $H(x,t) = G(w)w$. The Fourier expansion of $H(x,t)$ leads to

$$H(x,t) = \sum_{-\infty}^{+\infty} \hat{H}(k,t)e^{2\pi ikx}$$

and

$$D^2H(x,t) = \sum_{k\neq 0}(2\pi ik)^2\hat{H}(k,t)e^{2\pi ikx}$$

As follows from $w_t = D^2H(x,t)$, we have

$$\hat{w}_t(k,t) = (2\pi ik)^2\hat{H}(k,t), \quad for\ each\ k \neq 0$$

Therefore, we can show that

$$\begin{aligned}
u_t &= \alpha'(t) + \sum_{k\neq 0} \frac{1}{(2\pi ik)^2}\hat{w}_t(k,t)e^{2\pi ikx} \\
&= \alpha'(t) + \sum_{k\neq 0} \frac{1}{(2\pi ik)^2}(2\pi ik)^2\hat{H}(k,t)e^{2\pi ikx} \\
&= \alpha'(t) + \sum_{k\neq 0} \hat{H}(k,t)e^{2\pi ikx}
\end{aligned}$$

Choose $\alpha'(t) = \hat{H}(0,t)$, then we have

$$u_t = G(u_{xx})u_{xx}$$

Let u_1 and u_2 solve problem (2.1). Set $w_1 = u_{1xx}$, $w_2 = u_{2xx}$. As follows from the uniqueness argument of equation (2.3), we then have $w_1 = w_2 := w$. This leads to $u_{1xx} = u_{2xx}$. The PDE $u_t = G(w)w$ implies that $u_{1t} = u_{2t}$. Therefore we obtain $u_1 = u_2$. Then the uniqueness of solution u to (2.1) follows.

Finally, we summarize this section as following:
Based on the solution $w(x,t)$ to the problem (2.3), we can construct a function $u(x,t)$ by equation (2.17), where $\alpha(t)$ satisfies $\alpha'(t) = \hat{H}(0,t)$, $\alpha(0) = \hat{u}_0(0)$, here $H = G(w)w$. Such u is the unique solution to problem (2.1).

We have obtain the uniqueness and existence results for PDE $u_t = G(u_{xx})u_{xx}$ with periodic boundary conditions. Although we have not carried out the analysis in details, it is reasonable to expect the similar but more complicated argument will lead to the same results for u with the Dirichlet boundary conditions.

Chapter 3

Analysis of Discontinuity

In this chapter we drop the assumption of smoothness for $G(u_{xx})$ in equation (2.1) and allow it to be a discontinuous function. As motivated by our application, we choose $G(u_{xx})$ as a positive piecewise constant function. We consider equation (2.1) for $0 \leq x \leq l$, with smooth initial condition $u(x,0) = f(x)$ and Dirichlet boundary conditions. To study the discontinuity in the coefficient function, we discretize only in time and consider the evolution of equation (2.1) in a given tiny time step dt from initial data. For simplicity we use $u_{xx}(x,0)$ instead of $u_{xx}(x,dt)$ in the function $G(u_{xx})$, which is similar to using a semi-implicit difference scheme in numerical computations. We obtain

$$\frac{u(x,dt) - u(x,0)}{dt} = G(u_{xx}(x,0))u_{xx}(x,dt) \tag{3.1}$$

Note that $u_{xx}(x,0) = f^{''}(x)$ is a given function. We assume, for $f(x)$ there is $\bar{x} \in (0,l)$ such as

$$\begin{aligned} f^{''}(x) &< 0, \quad for\ x \in [0,\bar{x}) \\ f^{''}(x) &> 0, \quad for\ x \in (\bar{x},l] \end{aligned} \tag{3.2}$$

A typical function $f(x)$ is shown in Figure 3.1

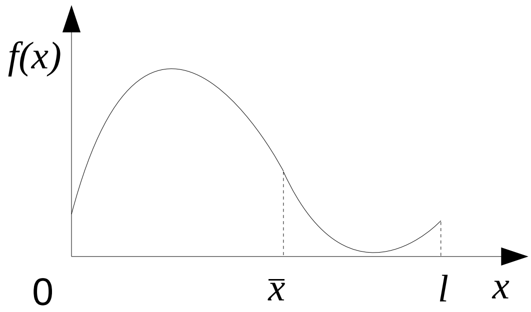

Figure 3.1: $f(x)$ satisfies condition (3.2).

Therefore, $G(u_{xx}(x,0))$ is a positive piecewise constant function defined in $[0,\bar{x})$ and $(\bar{x},l]$. Denote $u(x,dt)=u(x)$, then equation(3.1) can be written as

$$u(x)-g(x)u''(x)=f(x), \quad u(0)=u(l)=0 \tag{3.3}$$

where

$$g(x)=\begin{cases} K, & for\, 0\leq x<\bar{x} \\ D, & for\, \bar{x}<x\leq l \end{cases} \tag{3.4}$$

where K and D are both positive constants, $K=\sigma^{+}dt$ and $D=\sigma^{-}dt$.

3.1 Existence of Solutions to Smoothed ODEs

There are at least two obvious formulations to smooth out the ODE (3.3) with Dirichlet boundary conditions:

$$u_\epsilon(x)-g_\epsilon(x)u_\epsilon''(x)=f(x) \quad , \; u_\epsilon(0)=u_\epsilon(l)=0 \tag{3.5}$$

$$u_\epsilon(x)-(g_\epsilon(x)u_\epsilon'(x))'=f(x) \quad , \; u_\epsilon(0)=u_\epsilon(l)=0 \tag{3.6}$$

Here $g_\epsilon(x)$ is positive smooth function and $g_\epsilon(x)\geq c_0>0$, c_0 is a constant. Furthermore, we assume that $g_\epsilon(x)$ converges to $g(x)$ defined in (3.4) in L_1 norm for equation (3.5), while in L_2 norm for equation (3.6), as $\epsilon \to 0$. The Maximum Principle guarantees that there is only the trivial solution to the homogenous systems of (3.5) and (3.6). Therefore, as follows from Fredholm's Alternative (see [7] for details), we conclude

Theorem 3.1.1. *The boundary value problem* (3.5) *has a unique solution.*

Theorem 3.1.2. *The boundary value problem* (3.6) *has a unique solution.*

3.2 Existence of Solutions to Discontinuous ODEs

Typically, no C^2 solution exists to equation (3.3) with discontinuous function $g(x)$. We need to impose conditions at $x = \bar{x}$ in order to define what we mean by a solution of equation (3.3). Let $u(\bar{x}^{\pm}) = \lim_{x \to \bar{x}^{\pm}} u(x)$ and $u'(\bar{x}^{\pm}) = \lim_{x \to \bar{x}^{\pm}} u'(x)$

Theorem 3.2.1. *Let $f \in C^{\infty}[0, l]$. The boundary value problem* (3.3) *with $g(x)$ defined in* (3.4) *has a unique solution, if we require the solution $u(x)$ to satisfy $u(\bar{x}^-) = u(\bar{x}^+)$ and $Ku^{'}(\bar{x}^-) = Du^{'}(\bar{x}^+)$.*

Proof. Equation (3.3) with $g(x)$ defined in (3.4) can be viewed as two initial value problems as

$$\begin{aligned} u(x) - Ku^{''}(x) &= f(x)\,, 0 \leq x < \bar{x} \\ u(0) &= 0 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} u(x) - Du^{''}(x) &= f(x)\,, \bar{x} < x \leq l \\ u(l) &= 0 \end{aligned} \tag{3.8}$$

Let u_L and u_R be solutions to (3.7) and (3.8), respectively. We have

$$u_L = u_L^p + u_L^g$$

where u_L^p is the unique solution to

$$u(x) - Ku^{''}(x) = f(x)$$

$$u(0) = 0,\ u(\bar{x}) = 0$$

while u_L^g satisfies

$$u(x) - Ku^{''}(x) = 0$$

$$u(0) = 0$$

We have

$$u_L^g = a_1(e^{\frac{1}{\sqrt{K}}x} - e^{-\frac{1}{\sqrt{K}}x})$$

and a_1 is a constant to be determined. Similarly we have

$$u_R = u_R^p + u_R^g$$

where u_R^p is the unique solution to

$$u(x) - Du^{''}(x) = f(x)$$

$$u(\bar{x}) = 0,\ u(l) = 0$$

while u_R^g satisfies

$$u(x) - Du^{''}(x) = 0$$

$$u(l) = 0$$

We have

$$u_R^g = a_2(e^{\frac{1}{\sqrt{D}}x} - e^{\frac{2}{\sqrt{D}}l - \frac{1}{\sqrt{D}}x})$$

and a_2 is a constant to be determined. The assumptions of $u_L(\bar{x}) = u_R(\bar{x})$ and $Ku_L'(\bar{x}) = Du_R'(\bar{x})$ lead to a linear system for a_1 and a_2:

$$\begin{aligned} a_1(e^{\frac{1}{\sqrt{K}}\bar{x}} - e^{-\frac{1}{\sqrt{K}}\bar{x}}) &\quad - \quad a_2(e^{\frac{1}{\sqrt{D}}\bar{x}} - e^{\frac{2}{\sqrt{D}}l - \frac{1}{\sqrt{D}}\bar{x}}) = 0 \\ a_1\sqrt{K}(e^{\frac{1}{\sqrt{K}}\bar{x}} + e^{-\frac{1}{\sqrt{K}}\bar{x}}) &\quad - \quad a_2\sqrt{D}(e^{\frac{1}{\sqrt{D}}\bar{x}} + e^{\frac{2}{\sqrt{D}}l - \frac{1}{\sqrt{D}}\bar{x}}) = Du_R^{p\,\prime}(\bar{x}) - Ku_L^{p\,\prime}(\bar{x}) \end{aligned}$$

To show the existence of a unique solution of (3.3), we only need to show $det(A) \neq 0$, where

$$A = \begin{pmatrix} (e^{\frac{1}{\sqrt{K}}\bar{x}} - e^{-\frac{1}{\sqrt{K}}\bar{x}}) & -(e^{\frac{1}{\sqrt{D}}\bar{x}} - e^{\frac{2}{\sqrt{D}}l - \frac{1}{\sqrt{D}}\bar{x}}) \\ \sqrt{K}(e^{\frac{1}{\sqrt{K}}\bar{x}} + e^{-\frac{1}{\sqrt{K}}\bar{x}}) & -\sqrt{D}(e^{\frac{1}{\sqrt{D}}\bar{x}} + e^{\frac{2}{\sqrt{D}}l - \frac{1}{\sqrt{D}}\bar{x}}) \end{pmatrix}$$

The term $det(A)$ is a smooth function of $\bar{x}$, and we call $d(\bar{x}) = det(A(\bar{x}))$. Let $s_1 = \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{D}}$ and $s_2 = \frac{1}{\sqrt{K}} - \frac{1}{\sqrt{D}}$. We have

$$det(A) = d(\bar{x}) = (\sqrt{K} - \sqrt{D})(e^{s_1\bar{x}} - e^{\frac{2}{\sqrt{D}}l - s_1\bar{x}}) + (\sqrt{K} + \sqrt{D})(e^{-s_2\bar{x}} - e^{\frac{2}{\sqrt{D}}l + s_2\bar{x}})$$

and

$$d(0) = (1 - e^{\frac{2}{\sqrt{D}}l})2\sqrt{K} < 0$$

$$d(l) = (e^{-s_2 l} - e^{s_1 l})2\sqrt{D} < 0$$

while

$$d^{'}(\bar{x}) = (\sqrt{\frac{K}{D}} - \sqrt{\frac{D}{K}})(e^{s_1\bar{x}} + e^{\frac{2}{\sqrt{D}}l - s_1\bar{x}} + e^{-s_2\bar{x}} + e^{\frac{2}{\sqrt{D}}l + s_2\bar{x}})$$

$d(\bar{x})$ is monotonic since $d^{'}(\bar{x}) > 0$ if $K > D$ while $d^{'}(\bar{x}) < 0$ if $K < D$. Therefore we obtain

$$d(\bar{x}) < 0 \; for \; all \; \bar{x} \in (0, l)$$

Then the theorem follows. $\square$

Theorem 3.2.2. *Let* $f \in C^\infty[0, l]$. *The boundary value problem* (3.3) *with* $g(x)$ *defined in* (3.4) *has a unique solution, if we require the solution* $u(x)$ *to satisfy* $u(\bar{x}^-) = u(\bar{x}^+)$ *and* $u^{'}(\bar{x}^-) = u^{'}(\bar{x}^+)$.

Proof. Follow the argument in Theorem 3.2.1 and get the same setting for u_L and u_R. The assumptions of $u_L(\bar{x}) = u_R(\bar{x})$ and $u_L'(\bar{x}) = u_R'(\bar{x})$ lead to a linear system for a_1 and a_2:

$$\begin{aligned}
a_1(e^{\frac{1}{\sqrt{K}}\bar{x}} - e^{-\frac{1}{\sqrt{K}}\bar{x}}) \quad &- \quad a_2(e^{\frac{1}{\sqrt{D}}\bar{x}} - e^{\frac{2}{\sqrt{D}}l-\frac{1}{\sqrt{D}}\bar{x}}) = 0 \\
a_1\frac{1}{\sqrt{K}}(e^{\frac{1}{\sqrt{K}}\bar{x}} + e^{-\frac{1}{\sqrt{K}}\bar{x}}) \quad &- \quad a_2\frac{1}{\sqrt{D}}(e^{\frac{1}{\sqrt{D}}\bar{x}} + e^{\frac{2}{\sqrt{D}}l-\frac{1}{\sqrt{D}}\bar{x}}) = u_R^{p\,\prime}(\bar{x}) - u_L^{p\,\prime}(\bar{x})
\end{aligned}$$

Therefore, we have matrix A as

$$A = \begin{pmatrix} (e^{\frac{1}{\sqrt{K}}\bar{x}} - e^{-\frac{1}{\sqrt{K}}\bar{x}}) & -(e^{\frac{1}{\sqrt{D}}\bar{x}} - e^{\frac{2}{\sqrt{D}}l-\frac{1}{\sqrt{D}}\bar{x}}) \\ \frac{1}{\sqrt{K}}(e^{\frac{1}{\sqrt{K}}\bar{x}} + e^{-\frac{1}{\sqrt{K}}\bar{x}}) & -\frac{1}{\sqrt{D}}(e^{\frac{1}{\sqrt{D}}\bar{x}} + e^{\frac{2}{\sqrt{D}}l-\frac{1}{\sqrt{D}}\bar{x}}) \end{pmatrix}$$

Let $s_1 = \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{D}}$ and $s_2 = \frac{1}{\sqrt{K}} - \frac{1}{\sqrt{D}}$, we have

$$det(A) = d(\bar{x}) = s_2(e^{s_1\bar{x}} - e^{\frac{2}{\sqrt{D}}l-s_1\bar{x}}) + s_1(e^{-s_2\bar{x}} - e^{\frac{2}{\sqrt{D}}l+s_2\bar{x}})$$

and

$$d(0) = (1 - e^{\frac{2}{\sqrt{D}}l})\frac{2}{\sqrt{K}} < 0$$

$$d(l) = (e^{-s_2 l} - e^{s_1 l})\frac{2}{\sqrt{D}} < 0$$

while

$$\begin{aligned}
d^{'}(\bar{x}) \quad &= \quad s_1 s_2(e^{s_1\bar{x}} + e^{\frac{2}{\sqrt{D}}l-s_1\bar{x}} - e^{-s_2\bar{x}} - e^{\frac{2}{\sqrt{D}}l+s_2\bar{x}}) \\
&= \quad s_1 s_2[(e^{s_1\bar{x}} - e^{-s_2\bar{x}}) + e^{\frac{2}{\sqrt{D}}l}(e^{-s_1\bar{x}} - e^{-s_2\bar{x}})] \\
&= \quad s_1 s_2(1 - e^{(s_1+s_2)\bar{x}})\frac{e^{s_1\bar{x}} - e^{s_2\bar{x}+\frac{2}{\sqrt{D}}l}}{e^{(s_1+s_2)\bar{x}}}
\end{aligned}$$

Now we can show $d^{'}(\bar{x}) < 0$ if $K > D$, while $d^{'}(\bar{x}) > 0$ if $K < D$. Therefore we obtain the monotonicity for $d(\bar{x})$ and then show

$$d(\bar{x}) < 0 \; for \; all \; \bar{x} \in (0, l)$$

The theorem follows. $\square$

3.3 Convergence Results

We now connect the smooth boundary value problems (3.5) and (3.6) with the discontinuous boundary value problem (3.3) to show some convergence results. First of all, we consider

$$\begin{aligned} u(x) - g(x)u^{''}(x) &= f(x) \\ u(0) = u(l) &= 0 \\ u(\bar{x}^-) &= u(\bar{x}^+) \\ u'(\bar{x}^-) &= u'(\bar{x}^+) \end{aligned} \tag{3.9}$$

where $g(x)$ is defined in (3.4) , and prove the following lemma.

Lemma 3.3.1. *Let $u(x)$ be the solution to* (3.9)*. Then $u(x)$ is bounded in maximum norm by the L_1 norm of $f(x)$, i.e: $|u(x)|_\infty \leq c||f||_{L_1}$ with some positive constant c, here c is independent of f.*

Proof. We use Green's functions to solve the system (3.9). For $0 \leq x < \bar{x}$, we have

$$\begin{aligned} Ku^{''}(x) - u(x) &= -f(x) \\ u(0) = 0 & \end{aligned} \tag{3.10}$$

The Green's function $G(x)$ satisfies

$$\begin{aligned} KG^{''}(x) - G(x) &= \delta(x - \xi) \\ G(0) = 0 & \end{aligned}$$

Therefore, $G(x)$ can be solved as

$$\begin{cases} c_1(e^{\frac{1}{\sqrt{K}}x} - e^{-\frac{1}{\sqrt{K}}x}), & 0 \leq x < \xi \\ c_1(e^{\frac{1}{\sqrt{K}}x} - e^{-\frac{1}{\sqrt{K}}x}) + \frac{1}{2\sqrt{K}}(e^{\frac{1}{\sqrt{K}}(x-\xi)} - e^{-\frac{1}{\sqrt{K}}(x-\xi)}), & \xi < x < \bar{x} \end{cases}$$

Let $u_L(x) = u(x)\ for\ 0 \leq x < \bar{x}$. Then

$$u_L(x) = -c_1(e^{\frac{1}{\sqrt{K}}x} - e^{-\frac{1}{\sqrt{K}}x})\int_0^{\bar{x}} f(\xi)d\xi - \frac{1}{2\sqrt{K}}\int_0^x (e^{\frac{1}{\sqrt{K}}(x-\xi)} - e^{-\frac{1}{\sqrt{K}}(x-\xi)})f(\xi)d\xi$$

and

$$u_L'(x) = -c_1\frac{1}{\sqrt{K}}(e^{\frac{1}{\sqrt{K}}x} + e^{-\frac{1}{\sqrt{K}}x})\int_0^{\bar{x}} f(\xi)d\xi - \frac{1}{2K}\int_0^x (e^{\frac{1}{\sqrt{K}}(x-\xi)} + e^{-\frac{1}{\sqrt{K}}(x-\xi)})f(\xi)d\xi$$

For $\bar{x} < x \leq l$, we have

$$\begin{aligned} Du''(x) - u(x) &= -f(x) \\ u(l) = 0 & \end{aligned} \tag{3.11}$$

The Green's function $G(x)$ satisfies

$$\begin{aligned} DG''(x) - G(x) &= \delta(x-\xi) \\ G(l) = 0 & \end{aligned}$$

And $G(x)$ can be written as

$$\begin{cases} c_3(e^{\frac{1}{\sqrt{D}}x} - e^{\frac{1}{\sqrt{D}}(2l-x)}) - \frac{1}{2\sqrt{D}}(e^{\frac{1}{\sqrt{D}}(x-\xi)} - e^{-\frac{1}{\sqrt{D}}(x-\xi)}), & \bar{x} < x < \xi \\ c_3(e^{\frac{1}{\sqrt{D}}x} - e^{\frac{1}{\sqrt{D}}(2l-x)}), & \xi < x \leq l \end{cases}$$

Let $u_R(x) = u(x)\ for\ \bar{x} < x \leq l$, we have

$$u_R(x) = -c_3(e^{\frac{1}{\sqrt{D}}x} - e^{\frac{1}{\sqrt{D}}(2l-x)})\int_{\bar{x}}^l f(\xi)d\xi + \frac{1}{2\sqrt{D}}\int_x^l (e^{\frac{1}{\sqrt{D}}(x-\xi)} - e^{-\frac{1}{\sqrt{D}}(x-\xi)})f(\xi)d\xi$$

and

$$u_R'(x) = -c_3\frac{1}{\sqrt{D}}(e^{\frac{1}{\sqrt{D}}x} + e^{\frac{1}{\sqrt{D}}(2l-x)})\int_{\bar{x}}^l f(\xi)d\xi + \frac{1}{2D}\int_x^l (e^{\frac{1}{\sqrt{D}}(x-\xi)} + e^{-\frac{1}{\sqrt{D}}(x-\xi)})f(\xi)d\xi$$

Let

$$a_1(x) = -(e^{\frac{1}{\sqrt{K}}x} - e^{-\frac{1}{\sqrt{K}}x})$$

$$a_2(x) = -(e^{\frac{1}{\sqrt{D}}x} - e^{\frac{1}{\sqrt{D}}(2l-x)})$$

$$a_3(x) = -\frac{1}{\sqrt{K}}(e^{\frac{1}{\sqrt{K}}x} + e^{-\frac{1}{\sqrt{K}}x})$$

$$a_4(x) = -\frac{1}{\sqrt{D}}(e^{\frac{1}{\sqrt{D}}x} + e^{\frac{1}{\sqrt{D}}(2l-x)})$$

and

$$B_1(x;f) = -\frac{1}{2\sqrt{K}}\int_0^x h_1(x,\xi)f(\xi)d\xi$$

$$B_2(x;f) = \frac{1}{2\sqrt{D}}\int_x^l h_2(x,\xi)f(\xi)d\xi$$

$$B_3(x;f) = -\frac{1}{2K}\int_0^x h_3(x,\xi)f(\xi)d\xi$$

$$B_4(x;f) = \frac{1}{2D}\int_x^l h_4(x,\xi)f(\xi)d\xi$$

with

$$h_{1,3}(x,\xi) = e^{\frac{1}{\sqrt{K}}(x-\xi)} \mp e^{-\frac{1}{\sqrt{K}}(x-\xi)},\ for\ 0 < \xi < \bar{x}$$

$$h_{2,4}(x,\xi) = e^{\frac{1}{\sqrt{D}}(x-\xi)} \mp e^{-\frac{1}{\sqrt{D}}(x-\xi)},\ for\ \bar{x} < \xi < l$$

and

$$p = \int_0^{\bar{x}} f(\xi)d\xi$$

$$q = \int_{\bar{x}}^l f(\xi)d\xi$$

The conditions at $\bar{x}$ lead to two linear equations for c_1 and c_3, and we get

$$\begin{aligned} c_1 &= \frac{1}{a(\bar{x})p}[a_2(\bar{x})(B_4(\bar{x};f) - B_3(\bar{x};f)) - a_4(\bar{x})(B_2(\bar{x};f) - B_1(\bar{x};f))] \\ c_3 &= \frac{1}{a(\bar{x})q}[a_1(\bar{x})(B_4(\bar{x};f) - B_3(\bar{x};f)) - a_3(\bar{x})(B_2(\bar{x};f) - B_1(\bar{x};f))] \end{aligned}$$

where $a(x) = a_2(x)a_3(x) - a_1(x)a_4(x)$. Theorem 3.2.2 shows $a(\bar{x}) \neq 0$. Therefore, the solution to (3.9) is

$$\begin{aligned} u_L(x) &= \tfrac{a_1(x)}{a(\bar{x})}[a_2(\bar{x})(B_4(\bar{x};f) - B_3(\bar{x};f)) - a_4(\bar{x})(B_2(\bar{x};f) - B_1(\bar{x};f))] + B_1(x;f) \\ u_R(x) &= \tfrac{a_2(x)}{a(\bar{x})}[a_1(\bar{x})(B_4(\bar{x};f) - B_3(\bar{x};f)) - a_3(\bar{x})(B_2(\bar{x};f) - B_1(\bar{x};f))] + B_2(x;f) \end{aligned} \tag{3.12}$$

Notice

$$|B_i(x;f)|_\infty \leq b_i||f||_{L_1}$$

where b_i are positive constants ,$i = 1, 2, 3, 4$. From (3.12), we can bound $u_L(x)$ and $u_R(x)$ in $|.|_\infty$ by the L_1 norm of $f(x)$. □

Lemma 3.3.2. *Let u_ϵ be the solution to* (3.5)*. There exists a positive constant c independent of ϵ, such that $|u''_\epsilon(x)|_\infty \leq c|f(x)|_\infty$.*

Proof. By the maximum principle, we get

$$|u_\epsilon(x)|_\infty \leq c_1|f(x)|_\infty$$

where $c_1 > 0$ is independent of ϵ. Therefore for

$$u''_\epsilon(x) = \frac{u_\epsilon(x) - f(x)}{g_\epsilon(x)}$$

with assumption

$$g_\epsilon(x) \geq c_0 > 0$$

we have

$$|u''_\epsilon(x)|_\infty \leq \frac{c_1 + 1}{c_0}|f(x)|_\infty$$

Let $c = \frac{c_1 = 1}{c_0}$, c is independent of ϵ. The lemma is proved. □

Now we are ready to show the convergence result for the first case.

Theorem 3.3.3. *Let $u_\epsilon(x)$ be the solution to equation (3.5), and let $u(x)$ be the solution to (3.9). Then $u_\epsilon(x)$ converges to $u(x)$ in $|.|_\infty$ as $\epsilon \to 0$, if $g_\epsilon(x)$ converges to $g(x)$ in L_1 norm as $\epsilon \to 0$.*

Proof. Let the linear operator L_0 be defined as

$$L_0 u = u - g u''$$

while

$$L_\epsilon u_\epsilon = u_\epsilon - g_\epsilon u_\epsilon''$$

and set

$$\varphi_\epsilon(x) = g_\epsilon(x) - g(x)$$

Consider

$$\begin{aligned} L_\epsilon u_\epsilon &= u_\epsilon - (g + \varphi_\epsilon) u_\epsilon'' \\ &= u_\epsilon - g u_\epsilon'' - \varphi_\epsilon u_\epsilon'' \\ &= L_0 u_\epsilon - \varphi_\epsilon u_\epsilon'' \end{aligned} \tag{3.13}$$

Meanwhile we have

$$L_\epsilon u_\epsilon = f = L_0 u$$

Then (3.13) can be rewritten as

$$L_0(u_\epsilon - u) = \varphi_\epsilon u_\epsilon''$$

Lemma **??** and lemma 3.3.2 imply

$$\begin{aligned} |u_\epsilon - u|_\infty &\le c_1 \quad ||\varphi_\epsilon u_\epsilon''||_{L_1} \\ &\le c_1 \quad ||\varphi_\epsilon||_{L_1} |u_\epsilon''|_\infty \\ &\le c_2 \quad ||\varphi_\epsilon||_{L_1} |f|_\infty \end{aligned}$$

with positive constants c_1 and c_2. Therefore $|u_\epsilon - u|_\infty \to 0$ as $\epsilon \to 0$ follows from the assumption $||\varphi_\epsilon(x)||_{L_1} \to 0$. $\square$

Now consider the ODE system

$$
\begin{aligned}
u(x) - g(x)u''(x) &= f(x) \\
u(0) = u(l) &= 0 \\
u(\bar{x}^-) &= u(\bar{x}^+) \\
Ku'(\bar{x}^-) &= Du'(\bar{x}^+)
\end{aligned}
\tag{3.14}
$$

and equation (3.6). We show the convergence result for the second case.

Theorem 3.3.4. *Let $u_\epsilon(x)$ be solution to equation* (3.6)*, while $u(x)$ denotes the solution to equation* (3.14)*. Then u_ϵ converges to $u(x)$ in L_2 as $\epsilon \to 0$, if $g_\epsilon(x)$ converges to $g(x)$ in L_2 norm as $\epsilon \to 0$.*

Proof. Let

$$\varphi_\epsilon(x) = g_\epsilon(x) - g(x)$$

and

$$\delta_\epsilon(x) = u_\epsilon(x) - u(x)$$

The Dirichlet boundary conditions imply

$$\delta_\epsilon(0) = \delta_\epsilon(l) = 0$$

The solution $u(x)$ is a smooth function in $[0, \bar{x})$ and in $(\bar{x}, l]$. The difference of equation (3.6) and (3.14) gives, for $x \neq \bar{x}$

$$\delta_\epsilon - [(g_\epsilon u'_\epsilon)' - (g_\epsilon u')'] - [(g_\epsilon u')' - gu''] = 0$$

Since $g(x)$ is a piecewise constant function away from $\bar{x}$, we have $gu'' = (gu')'$ for $x \neq \bar{x}$. Therefore,

$$\delta_\epsilon - (g_\epsilon \delta_\epsilon')' = (\varphi_\epsilon u')' \quad , \ x \neq \bar{x}.$$

Multiplying $\delta_\epsilon(x)$ to both sides above and integrating, we get

$$||\delta_\epsilon||_{L_2}^2 - \int_0^l (g_\epsilon \delta_\epsilon')' \delta_\epsilon dx = \int_0^l (\varphi_\epsilon u')' \delta_\epsilon dx \tag{3.15}$$

And

$$\begin{aligned} -\int_0^l (g_\epsilon \delta_\epsilon')' \delta_\epsilon dx &= \int_0^{\bar{x}} (g_\epsilon \delta_\epsilon') \delta_\epsilon' dx - g_\epsilon \delta_\epsilon' \delta_\epsilon |_{x=0}^{x=\bar{x}^-} + \int_{\bar{x}}^l (g_\epsilon \delta_\epsilon') \delta_\epsilon' dx - g_\epsilon \delta_\epsilon' \delta_\epsilon |_{x=\bar{x}^+}^{x=l} \\ &= \int_0^l g_\epsilon (\delta_\epsilon')^2 dx + g_\epsilon \delta_\epsilon' \delta_\epsilon |_{x=\bar{x}^-}^{x=\bar{x}^+} \end{aligned}$$

Note that, for $x \neq \bar{x}$

$$g_\epsilon \delta_\epsilon' \delta_\epsilon = (g_\epsilon u_\epsilon' - gu') \delta_\epsilon - \varphi_\epsilon u' \delta_\epsilon$$

where g_ϵ, u_ϵ' and δ_ϵ are continuous functions. The conditions at $\bar{x}$ for ODE system(3.14) show that $gu' \in C$. Therefore

$$g_\epsilon \delta_\epsilon' \delta_\epsilon |_{x=\bar{x}^-}^{x=\bar{x}^+} = -\varphi_\epsilon u' \delta_\epsilon |_{x=\bar{x}^-}^{x=\bar{x}^+}$$

For the right hand side of (3.15), we also have

$$\int_0^l (\varphi_\epsilon u')' \delta_\epsilon dx = -\int_0^l (\varphi_\epsilon u') \delta_\epsilon' dx - \varphi_\epsilon u' \delta_\epsilon |_{x=\bar{x}^-}^{x=\bar{x}^+}$$

Thus

$$\begin{aligned} ||\delta_\epsilon||_{L_2}^2 + \int_0^l g_\epsilon (\delta_\epsilon')^2 dx &= -\int_0^l (\varphi_\epsilon u') \delta_\epsilon' dx \\ &\leq ||\varphi_\epsilon u'||_{L_2} ||\delta_\epsilon'||_{L_2} \end{aligned}$$

With the assumption $g_\epsilon \geq c_0 > 0$, we have

$$\int_0^l g_\epsilon (\delta_\epsilon')^2 dx \geq c_0 ||\delta_\epsilon'||_{L_2}^2$$

Therefore, we obtain

$$\begin{aligned} ||\delta_\epsilon||_{L_2}^2 + c_0||\delta_\epsilon'||_{L_2}^2 &\leq ||\varphi_\epsilon u'||_{L_2}||\delta_\epsilon'||_{L_2} \\ &\leq \frac{1}{2c_0}||\varphi_\epsilon u'||_{L_2}^2 + c_0||\delta_\epsilon'||_{L_2}^2 \end{aligned}$$

Thus

$$||\delta_\epsilon||_{L_2}^2 \leq \frac{1}{2c_0}||\varphi_\epsilon u'||_{L_2}^2 \leq \frac{1}{2c_0}|u'|_\infty^2||\varphi_\epsilon||_{L_2}^2$$

Here $|u'|_\infty$ is finite by construction. Therefore, we get

$$||\delta_\epsilon||_{L_2}^2 \leq c||\varphi_\epsilon||_{L_2}^2$$

and the theorem follows. □

3.4 Perturbation Result

In this section, we consider equation (3.9) and note that $g(x) = g(x;\bar{x})$ is also a function of $\bar{x}$, see equation (3.4). Recall that for the solution of the PDE $u_t = G(u_{xx})u_{xx}$, the function u_{xx} changes sign at the discontinuity of $G(u_{xx})$. This motivate the following considerations.

For equation (3.9) we have defined a solution $u(x) \in C^1[0,l]$. Let $u_L(x) = u(x)$ for $0 \leq x \leq \bar{x}$ and $u_R(x) = u(x)$ for $\bar{x} \leq x \leq l$. Then the following three statements are equivalent.

Statement1: $u(x) \in C^2$

Statement2: $u_L''(\bar{x}) = u_R''(\bar{x}) = 0$

Statement3: $u_L(\bar{x}) = u_R(\bar{x}) = f(\bar{x})$.

Assume that for some given $f(x)$ there exists $\bar{x}$ so that

$$u_L(\bar{x}) = u_R(\bar{x}) = f(\bar{x}).$$

We then perturb $f(x)$ to become $f(x) + \epsilon h(x)$ and ask if we can find $\tilde{x} = \bar{x} + \epsilon a$ so that the solution $\tilde{u}(x; \epsilon)$ of the problem

$$\tilde{u} - g(x; \tilde{x})\tilde{u}'' = f(x) + \epsilon h(x)$$

$$\tilde{u}(0) = \tilde{u}(l) = 0$$

$$\tilde{u}(\tilde{x}^-) = \tilde{u}(\tilde{x}^+)$$

$$\tilde{u}'(\tilde{x}^-) = \tilde{u}'(\tilde{x}^+)$$

will satisfy

$$\tilde{u}_L(\tilde{x}) = \tilde{u}_R(\tilde{x}) = f(\tilde{x}) + \epsilon h(\tilde{x}).$$

Here $\epsilon > 0$, $h(x) \in C^\infty$ and a is a constant. We carry out the analysis to first order in ϵ and show that if non-degeneracy condition is satisfied, then the perturbed problem will again have a solution $\tilde{u} \in C^2[0, l]$, if $\tilde{x} = x + \epsilon a$ is properly chosen.

Note $u(x)$ depends on parameter $\bar{x}$ and data $f(x)$. To separate the perturbation effects of $\bar{x}$ from the effects of $f(x)$, different from the construction in Theorem 3.3.3, we set $u_L(x)$ as $u_L(x) = u_{L1}(x) + u_{L2}(x)$, where $u_{L1}(x)$ is the unique solution to initial value problem

$$\begin{aligned} u(x) - Ku''(x) &= f(x) \\ u(0) &= 0 \\ u'(0) &= 0 \end{aligned}$$

Thus $u_{L1}(x)$ only depends on $f(x)$ smoothly, denote as $u_{L1}(x; f)$. While $u_{L2}(x)$ is solution to

$$\begin{aligned} u(x) - Ku''(x) &= 0 \\ u(0) &= 0 \end{aligned}$$

therefore

$$u_{L2}(x) = \alpha w(x)$$

where $w(x) = e^{\lambda_1 x} - e^{-\lambda_1 x}$, $\lambda_1 = \sqrt{\frac{1}{K}}$ and α is a free parameter. Similarly, we have

$$u_R(x) = u_{R1}(x) + u_{R2}(x)$$

where $u_{R1}(x)$ is the unique solution to the initial value problem

$$\begin{aligned} u(x) - Du''(x) &= f(x) \\ u(l) &= 0 \\ u'(l) &= 0 \end{aligned}$$

The function $u_{R1}(x)$ also only depends on $f(x)$ smoothly, denote as $u_{R1}(x; f)$. And $u_{R2}(x)$ is solution to

$$\begin{aligned} u(x) - Du''(x) &= 0 \\ u(l) &= 0 \end{aligned}$$

therefore

$$u_{R2}(x) = \beta(e^{\lambda_2(x-l)} - e^{-\lambda_2(x-l)})$$

with $\lambda_2 = \sqrt{\frac{1}{D}}$ and β is a free parameter.
The assumption of $u(x) \in C^1$ leads to a linear system for α and β. We can show α and β are smooth functions of $\bar{x}$ and also smoothly depend on $f(x)$. Therefore, denote $u_{L2}(x) = u_{L2}(x, \bar{x}; f) = \alpha(\bar{x}; f)w(x)$.
When $f(x)$ gets perturbed to be

$$\tilde{f}(x) = f(x) + \epsilon h(x)$$

As follows from Taylor expansion, we have, to the leading order

$$u_L(x;\bar{x},\tilde{f}) = u_{L1}(x;f) + \alpha(\bar{x};f)w(x) + \epsilon v(x;\bar{x},h) + O(\epsilon^2)$$

where $v(x)$ is some smooth function. With $\bar{x}$ being perturbed to be

$$\tilde{x} = \bar{x} + \epsilon a$$

we have

$$\begin{aligned} u_L(x;\tilde{x},\tilde{f}) &= u_{L1}(x;f) + \alpha(\bar{x};f)w(x) + \alpha'(\bar{x};f)\epsilon a w(x) \\ &\quad + \epsilon v(x;\bar{x},h) + O(\epsilon^2) \end{aligned}$$

Then for the new solution $\tilde{u}_L$, we obtain

$$\begin{aligned} \tilde{u}_L(\tilde{x};\tilde{x},\tilde{f}) &= u_{L1}(\bar{x};f) + \epsilon a\, u'_{L1}(\bar{x};f) + \alpha(\bar{x};f)w(\bar{x}) + \\ &\quad \epsilon a\, \alpha(\bar{x};f)w'(\bar{x}) + \epsilon a\, \alpha'(\bar{x};f)w(\bar{x}) + \epsilon v(\bar{x};\bar{x},h) + O(\epsilon^2) \\ &= u_L(\bar{x}) + \epsilon[a\, u'_{L1}(\bar{x}) + a\, \alpha(\bar{x})w'(\bar{x}) + a\, \alpha'(\bar{x})w(\bar{x}) + v(\bar{x};\bar{x},h)] + O(\epsilon^2) \end{aligned}$$

The Taylor expansion for $\tilde{f}(\tilde{x})$ gives

$$\tilde{f}(\tilde{x}) = f(\bar{x}) + \epsilon[af'(\bar{x}) + h(\bar{x})] + O(\epsilon^2)$$

As follows from the assumption $u_L(\bar{x}) = f(\bar{x})$, we have $\tilde{u}_L(\tilde{x}) = \tilde{f}(\tilde{x})$ to the leading order if and only if

$$a\, u'_{L1}(\bar{x}) + a\, \alpha(\bar{x})w'(\bar{x}) + a\, \alpha'(\bar{x})w(\bar{x}) + v(\bar{x};\bar{x},h) = af'(\bar{x}) + h(\bar{x})$$

which can be arranged as

$$a[u'_{L1}(\bar{x}) + \alpha(\bar{x})w'(\bar{x}) + \alpha'(\bar{x})w(\bar{x}) - f'(\bar{x})] = h(\bar{x}) - v(\bar{x};\bar{x},h) \tag{3.16}$$

Note

$$u'_{L1}(\bar{x}) + \alpha(\bar{x})w'(\bar{x}) + \alpha'(\bar{x})w(\bar{x}) = [\frac{\partial}{\partial x}u_L(x,\bar{x}) + \frac{\partial}{\partial \bar{x}}u_L(x,\bar{x})]|_{x=\bar{x}}$$

Following from (3.16), we conclude

Theorem 3.4.1. *Assume for equation* (3.9)*. There is a solution* $u(x) \in C^2$*. Let* u_L *be defined as above. If* $[\frac{\partial}{\partial x}u_L(x;\bar{x}) + \frac{\partial}{\partial \bar{x}}u_L(x;\bar{x})]|_{x=\bar{x}} - f'(\bar{x}) \neq 0$*, then there exists constant* a*, such that with perturbed* $\tilde{f}(x) = f(x) + \epsilon h(x)$ *and* $\tilde{x} = \bar{x} + \epsilon a$*, where* $\epsilon > 0$ *and* $h(x) \in C^\infty$*, the new solution* $\tilde{u}(x) \in C^2$ *.*

This result provides an insight of how the curve, where discontinuity of coefficient function occurs, evolves in space-time plane for the corresponding PDE.

3.5 Local Convergence of Simplified Newton Iteration

Consider an iteration in function space for the following boundary value problem

$$\begin{aligned} u - g(u'')u'' &= f \\ u(0) = u(l) &= 0 \end{aligned} \tag{3.17}$$

We assume here that $g(w)$ is a smooth positive function, $D^n g(w)$ is bounded for $n = 0, 1, 2\cdots$, and $g(w) + Dg(w)w \geq c_0 > 0$, where $D^n g(w) = \frac{d^n}{dw^n}g(w)$. We define the simplified Newton iteration as

$$u_{n+1}(x) = u_n(x) + h_n(x)\ ,\ n = 0, 1, 2, \cdots$$

where $h_n(x)$ satisfies

$$\begin{aligned} h_n - [g(u_0'') + Dg(u_0'')u_0'']h_n'' &= f - [u_n - g(u_n'')u_n''] \\ h_n(0) = h_n(l) &= 0 \end{aligned} \tag{3.18}$$

We first use induction to show

Lemma 3.5.1. *Consider the sequence $h_n(x)$, $n = 0, 1, 2, ...$ defined in (3.18). If $|h_0''|_\infty$ is small enough, then there exists k, $0 < k < 1$, with the following property: $|h_i''|_\infty \le k|h_{i-1}''|_\infty$ for $1 \le i \le n-1$, implies $|h_n''|_\infty \le k|h_{n-1}''|_\infty$.*

Proof. Let RHS denote the right-hand-side of (3.18). Substituting u_n by $u_{n-1}+h_{n-1}$ and rewriting in Taylor expansion, we get

$$\begin{aligned} RHS &= f - [u_{n-1} - g(u''_{n-1})u''_{n-1}] + [g(u''_{n-1}) + Dg(u''_{n-1})u''_{n-1}]h''_{n-1} - h_{n-1} \\ &\quad +[\frac{1}{2}D^2g(u''_{n-1})u''_{n-1} + Dg(u''_{n-1})](h''_{n-1})^2 + O((h''_{n-1})^3) \end{aligned}$$

Note the first two parts on right of above equation are the right-hand-side of equation (3.18) at $n-1$. Thus

$$\begin{aligned} RHS &= -[g(u''_0) + Dg(u''_0)u''_0]h''_{n-1} + [g(u''_{n-1}) + Dg(u''_{n-1})u''_{n-1}]h''_{n-1} \\ &\quad +[\frac{1}{2}D^2g(u''_{n-1})u''_{n-1} + Dg(u''_{n-1})](h''_{n-1})^2 + O((h''_{n-1})^3) \\ &= [g(u''_{n-1}) - g(u''_0)]h''_{n-1} \\ &\quad +[Dg(u''_{n-1})u''_{n-1} - Dg(u''_{n-1})u''_0 + Dg(u''_{n-1})u''_0 - Dg(u''_0)u''_0]h''_{n-1} \\ &\quad +[\frac{1}{2}D^2g(u''_{n-1})u''_{n-1} + Dg(u''_{n-1})](h''_{n-1})^2 \\ &\quad +O((h''_{n-1})^3) \\ &= [Dg(\xi) + Dg(u''_{n-1}) + D^2g(\eta)](u''_{n-1} - u''_0)h''_{n-1} \\ &\quad +[\frac{1}{2}D^2g(u''_{n-1})u''_{n-1} + Dg(u''_{n-1})](h''_{n-1})^2 + O((h''_{n-1})^3) \end{aligned} \tag{3.19}$$

where ξ and η lie between u''_0 and u''_{n-1}. Notice $D^n g$ are bounded by assumption and

$$u''_{n-1} - u''_0 = h''_0 + h''_1 + \cdots + h''_{n-1}$$

Also followed from assumption, we have

$$|h''_0|_\infty + |h''_1|_\infty + \cdots + |h''_{n-1}|_\infty \le \frac{|h''_0|_\infty}{1-k}$$

Therefore we can bound RHS as

$$|RHS|_\infty \leq c_1 \frac{|h_0^{''}|_\infty}{1-k} |h_{n-1}^{''}|_\infty + (c_2 \frac{|h_0^{''}|_\infty}{1-k} + c_3)|h_{n-1}^{''}|_\infty^2$$

where $c_{1,2,3}$ are positive constants. Note $|h_{n-1}^{''}|_\infty \leq |h_0^{''}|_\infty$, we have

$$|RHS|_\infty \leq c_4 (\frac{|h_0^{''}|_\infty}{1-k} + \frac{1}{1-k} + 1)|h_0^{''}|_\infty |h_{n-1}^{''}|_\infty$$

where $c_4 \geq max(c_1, c_2, c_3)$. For equation (3.18), by the maximum principle one obtains

$$|h_n|_\infty \leq |RHS|_\infty$$

Rewrite equation (3.18) as

$$h_n^{''} = \frac{h_n - RHS}{g(u_0^{''}) + Dg(u_0^{''})u_0^{''}}$$

With the assumption $g(w) + Dg(w)w \geq c_0 > 0$, we can get

$$|h_n^{''}|_\infty \leq \frac{2}{c_0} |RHS|_\infty = \frac{2c_4}{c_0} (\frac{|h_0^{''}|_\infty}{1-k} + \frac{1}{1-k} + 1)|h_0^{''}|_\infty |h_{n-1}^{''}|_\infty$$

For any chosen k, $0 < k < 1$, start from u_0 very close to real solution u such that $|h_0^{''}|_\infty$ is small enough to make

$$\frac{2c_4}{c_0} (\frac{|h_0^{''}|_\infty}{1-k} + \frac{1}{1-k} + 1)|h_0^{''}|_\infty \leq k$$

Then we have

$$|h_0^{''}|_\infty \leq \frac{2}{c_0} |RHS|_\infty \leq k |h_{n-1}^{''}|_\infty$$

and the lemma follows. □

It is easy to check that $|h_1^{''}|_\infty \leq k |h_0^{''}|_\infty$ by using equation (3.19) for $n = 1$. Therefore we have shown the contraction for $|h_n^{''}|_\infty$ for all n . This leads to the fact $|h_n^{''}|_\infty \to 0$ as $n \to \infty$. Now rewrite (3.18) as

$$h_n = RHS + [g(u_0^{''}) + Dg(u_0^{''})u_0^{''}]h_n^{''}$$

Then

$$\begin{aligned} |h_n|_\infty &\leq |RHS|_\infty + c_1|h_n''| \\ &\leq \frac{kc_0}{2}|h_{n-1}''| + c_1|h_n''| \end{aligned}$$

Here c_1 is a positive constant independent of n. Thus we have shown

Theorem 3.5.2. *The simplified Newton iteration defined in* (3.18) *is locally convergent.*

Chapter 4

Moving Boundary Problem

4.1 Formulation

We continue the analysis of the discontinuity occurring for the partial differential problem:

$$\begin{aligned}
&u_t = g(x,t)u_{xx}, \quad l_1 \le x \le l_2,\ t \ge 0\\
&u(l_1,t) = 0\\
&u(l_2,t) = 0\\
&u(x,0) = f(x) \qquad (4.1)
\end{aligned}$$

with smooth initial data $f(x)$ satisfying assumption (3.2) on $l_1 \le x \le l_2$.

Let $s : [0, \infty) \to (l_1, l_2)$ denote a smooth function. We first consider $s(t)$ as a given function, but below we will derive conditions to determine $s(t)$. Assume that $g(x,t)$ is a piecewise constant function defined in the $x - t$ plane of the following form: for $t \geq 0$

$$g(x,t) = \begin{cases} k_1, & for \quad l_1 \leq x < s(t) \\ k_2, & for \quad s(t) < x \leq l_2 \end{cases} \tag{4.2}$$

Here we assume $k_{1,2} > 0$ and $k_1 \neq k_2$. Let u be a solution of problem (4.1), u_1 be the solution in region R_1: $l_1 \leq x < s(t)$ and u_2 be the solution in region R_2: $s(t) < x \leq l_2$. We have a PDE problem, shown in Figure 4.1, as follows:

$$\frac{\partial u_1}{\partial t} = k_1 \frac{\partial^2 u_1}{\partial x^2}, \quad l_1 \leq x < s(t), \tag{4.3}$$

$$\frac{\partial u_2}{\partial t} = k_2 \frac{\partial^2 u_2}{\partial x^2}, \quad s(t) < x \leq l_2, \tag{4.4}$$

$$u(l_1,t) = 0 \tag{4.5}$$

$$u(l_2,t) = 0 \tag{4.6}$$

$$u(x,0) = f(x) \tag{4.7}$$

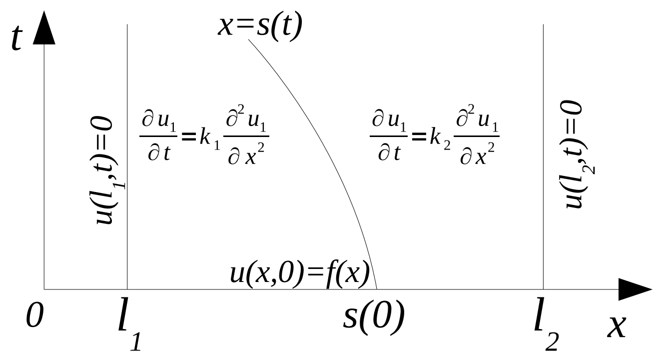

Figure 4.1: Moving boundary problem.

One assumes that u_1 and u_2 have C^2 extensions to the closure of the regions R_1 and R_2. The position of the moving boundary $x = s(t)$ has to be determined as well as the unknown solution $u(x,t)$.

Usually two conditions are needed on the moving boundary, one to determine the boundary itself and the other to complete the definition of the solution to the differential equation. For example, the non-dimensional two phase Stefan problem [8], recall:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= k_1 \frac{\partial^2 u_1}{\partial x^2}, \quad 0 \le x < s(t), && (4.8)\\ \frac{\partial u_2}{\partial t} &= k_2 \frac{\partial^2 u_2}{\partial x^2}, \quad x > s(t), && (4.9)\\ u_1 &= U_1, \quad x = 0, \quad t \ge 0, && (4.10)\\ u_2 &= -U_2, \quad x \to \infty, \quad t \ge 0, && (4.11)\\ on\ x = s(t) \quad &, \quad \begin{cases} u_1 = u_2 = 0, \\ \gamma_2 \frac{\partial u_2}{\partial x} - \gamma_1 \frac{\partial u_1}{\partial x} = \frac{ds}{dt} \end{cases} && (4.12) \end{aligned}$$

where $k_{1,2}$ and $\gamma_{1,2}$ are non-dimensional parameters, which relate to specific heat capacity, density, heat conductivity and latent heat. The terms $U_{1,2}$ are given constants. The boundary condition (4.12) is called Stefan condition, which can be derived from energy conservation.

Since our problem is not based on physics, but is derived from the financial world, different moving boundary conditions are appropriate. In this case, we assume $u(.,t) \in C^2$ across the phase-change curve $x = s(t)$. First notice $u(.,t) \in C^1$ implies

$$u_1 = u_2 = h(t) \quad on \quad x = s(t) \tag{4.13}$$

where $h(t)$ is some unknown function to be determined, and

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \quad on \quad x = s(t). \tag{4.14}$$

Then $u(.,t) \in C^2$ is equivalent to

$$\frac{\partial^2 u_i}{\partial x^2} = 0 \quad on\ x = s(t), \quad i = 1, 2 \tag{4.15}$$

and

$$\frac{\partial u_i}{\partial t} = 0 \quad on\ x = s(t), \quad i = 1, 2 \tag{4.16}$$

due to the change of coefficients in equations (4.3) and (4.4).Differentiate $u_i(s(t), t)$, $i = 1, 2$ with respect to t, to obtain

$$\frac{du_i(s(t), t)}{dt} = \frac{\partial u_i}{\partial t}|_{x=s(t)} + \frac{\partial u_i}{\partial x}|_{x=s(t)}\dot{s} = \dot{h} \tag{4.17}$$

where $\dot{} = \frac{d}{dt}$. By using (4.16) we obtain

$$\frac{\partial u_i}{\partial x}\dot{s} = \dot{h}, \quad on\ x = s(t) \tag{4.18}$$

Therefore, we have a moving-boundary problem defined by (4.3)-(4.7) together with the moving boundary conditions (4.13), (4.14) and (4.18). For this problem, the unknowns are $u_1(x, t)$, $u_2(x, t)$, $h(t)$ and $s(t)$. Notice $u_{1,2}(x, t)$ are determined if $s(t)$ and $h(t)$ are given. It is reasonable to expect that $h(t)$ and $s(t)$ can be solved from conditions (4.14) and (4.18).

4.2 Numerical Method

Very few analytical solutions are available in closed form for moving boundary problems. With appropriate boundary conditions and initial conditions, the exact solutions are usually known as similarity solutions, which, for the problem under consideration, take the form of functions of the single variable $x/t^{\frac{1}{2}}$. Numerical methods have been developed for the solution of moving boundary problems. Front-tracking methods compute the position of the moving boundary at each step in time using a finite-difference scheme or a finite-element scheme. An alternative way to track the moving boundary is to fix it by a suitable choice of new space coordinates. Of course, the partial differential equations need to be transformed correspondingly.

In this section we will discuss numerical methods for problem (4.3)-(4.7) with moving boundary conditions (4.13), (4.14) and (4.18).

4.2.1 Front Tracking Method with Fixed Grid

Suppose the partial differential equation is to be solved by using finite-difference replacements for the derivatives to compute values of $u_{i,j}$, at discrete points ($i\delta x$, $j\delta t$) on a fixed grid in the (x, t) plane, $1 \le i \le M$ and $1 \le j \le N$. At any time $j\delta t$, the phase-change boundary $x = s(t)$ will usually be located between two neighbouring grid points, say $i\delta x$ and $(i + 1)\delta x$. The value of u at the moving boundary $x = s(t)$ is h_j at time $t = j\delta t$. The value h_j is also unknown.

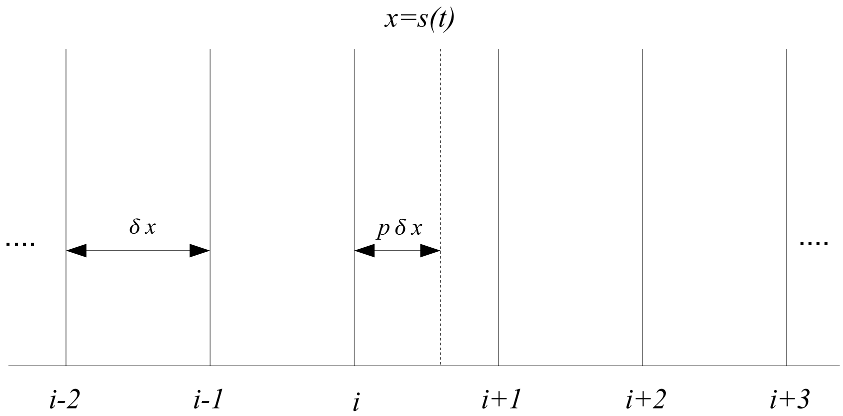

Figure 4.2: Fixed grid for front-tracking method

Figure 4.2 shows the moving boundary at time $t = j\delta t$, which is at a fractional distance $p\delta x$ between the grid lines $i\delta x$ and $(i+1)\delta x$. Lagrangian interpolation can be used to approximate u_{xx} at $x = i\delta x$, $x = (i+1)\delta x$ and $x = s(t)$, and u_x at $x = s(t)$ [8]. For $l_1 < x < s(t)$ we have

$$\frac{\partial u}{\partial x} \simeq \frac{1}{\delta x}\left(\frac{pu_{i-1}}{p+1} - \frac{(p+1)u_i}{p} + \frac{(2p+1)h}{p(p+1)}\right), \quad x = s(t) \tag{4.19}$$

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{2}{(\delta x)^2}\left(\frac{u_{i-1}}{p+1} - \frac{u_i}{p} + \frac{h}{p(p+1)}\right), \quad x = i\delta x \tag{4.20}$$

Similarly for $s(t) < x < l_2$ we have

$$\frac{\partial u}{\partial x} \simeq \frac{1}{\delta x}\left(\frac{(2p-3)h}{(1-p)(2-p)} + \frac{(2-p)u_{i+1}}{1-p} - \frac{(1-p)u_{i+2}}{2-p}\right), \quad x = s(t) \tag{4.21}$$

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{2}{(\delta x)^2}\left(\frac{h}{(1-p)(2-p)} - \frac{u_{i+1}}{1-p} + \frac{u_{i+2}}{2-p}\right), \quad x = (i+1)\delta x \tag{4.22}$$

These formulas for the space derivatives are used with usual explicit replacement of time derivatives in partial differential equations and conditions on the phase-change boundary. For points not next to the moving boundary, the standard central finite-difference formulas for equal space intervals are used. Then we obtain

$$u_{m,j+1} = u_{m,j} + k_1\frac{\delta t}{(\delta x)^2}(u_{m-1,j} - 2u_{m,j} + u_{m+1,j}), \quad m = 2, ..., i-1, \tag{4.23}$$

$$u_{m,j+1} = u_{m,j} + k_2\frac{\delta t}{(\delta x)^2}(u_{m-1,j} - 2u_{m,j} + u_{m+1,j}), \quad m = i+2, ..., M-1 \tag{4.24}$$

From boundary conditions (4.5) and (4.6), we have $u_{1,j} = u_{M,j} = 0$. At points $i\delta x$ and $(i+1)\delta x$, we use (4.20) and (4.22) instead, and get

$$u_{i,j+1} = u_{i,j} + k_1\frac{2\delta t}{(\delta x)^2}\left(\frac{u_{i-1,j}}{p_j+1} - \frac{u_{i,j}}{p_j} + \frac{h_j}{p_j(p_j+1)}\right) \tag{4.25}$$

$$u_{i+1,j+1} = u_{i+1,j} + k_2\frac{2\delta t}{(\delta x)^2}\left(\frac{h_j}{(1-p_j)(2-p_j)} - \frac{u_{i+1,j}}{1-p_j} + \frac{u_{i+2,j}}{2-p_j}\right) \tag{4.26}$$

Write $s_j = (i+p)\delta x$, the conditions at moving boundary (4.14) and (4.18) lead to equations for h_{j+1} and p_{j+1} at time $t = (j+1)\delta t$.

$$\begin{aligned}
&\frac{p_{j+1}u_{i-1,j+1}}{p_{j+1}+1} - \frac{(p_{j+1}+1)u_{i,j+1}}{p_{j+1}} + \frac{(2p_{j+1}+1)h_{j+1}}{p_{j+1}(p_{j+1}+1)} = \\
&\frac{(2p_{j+1}-3)h_{j+1}}{(1-p_{j+1})(2-p_{j+1})} + \frac{(2-p_{j+1})u_{i+1,j+1}}{1-p_{j+1}} - \frac{(1-p_{j+1})u_{i+2,j+1}}{2-p_{j+1}} = \\
&\frac{h_{j+1}-h_j}{p_{j+1}-p_j}
\end{aligned} \tag{4.27}$$

The steps in the numerical solution, starting from known values of all variables at each grid point at time $j\delta t$, are:

(i) Calculate $u_{m,j+1}$, $m = 2, ..., i-1$, from (4.23) and $u_{m,j+1}$, $m = i+2, ..., M-1$, from (4.24).

(ii) Calculate $u_{i,j+1}$ from (4.25) and $u_{i+1,j+1}$ from (4.26).

(iii) Calculate p_{j+1} and h_{j+1} from (4.27).

(iv) Repeat the steps (i)-(iii). When p exceeds unity or p becomes negative, the special equations (4.25),(4.26) and (4.27) are applied to the points $(i+1)\delta x$ and $(i+2)\delta x$ or $(i-1)\delta x$ and $i\delta x$.

There are two problems for the method above. When the moving boundary gets close to a grid point, singularities will cause loss of accuracy. For example, the Lagrangian interpolation formulas (4.19) and (4.20) will lead to big coefficient for u_i, if p is very close to 0. Another issue for the method is that it is a nonlinear system (4.27) for p_{j+1} and h_{j+1}. To avoid these two issues, we can use Lagrangian interpolation at points $(i-2)\delta x$, $(i-1)\delta x$ and $i\delta x$ to compute u_{xx} at $i\delta x$ and similarly points $(i+1)\delta x$, $(i+2)\delta x$ and $(i+3)\delta x$ to compute u_{xx} at $(i+1)\delta x$. As for u_x at $x = s(t)$, we use one-side difference of points $(i-1)\delta x$ and $x = s(t)$ for $x < s(t)$

$$\frac{\partial u}{\partial x} = \frac{h - u_{i-1}}{(1+p)\delta x}, \quad x = s(t) \tag{4.28}$$

while use $(i+2)\delta x$ and $x = s(t)$ for $x > s(t)$

$$\frac{\partial u}{\partial x} = \frac{u_{i+2} - h}{(2-p)\delta x}, \quad x = s(t) \tag{4.29}$$

Therefore the moving boundary conditions (4.14) and (4.18) yield

$$\frac{h_{j+1} - u_{i-1,j+1}}{1 + p_{j+1}} = \frac{u_{i+2,j+1} - h_{j+1}}{2 - p_{j+1}} = \frac{h_{j+1} - h_j}{p_{j+1} - p_j} \tag{4.30}$$

Notice (4.30) can be simplified into two linear equations about $p(j+1)$ and $h(j+1)$. However no solution exists for $p(j+1)$ and $h(j+1)$. Therefore this alternative finite-difference scheme does not work. So far we have not succeeded in applying the explicit finite-difference front tracking method with fixed grid to our moving boundary problem (4.3)-(4.7).

4.2.2 Front-Fixing Methods with Coordinate Transforms

In the moving boundary problem specified by equations (4.3)-(4.7), let phase 1 be the region $l_1 \leq x \leq s(t)$. Then the moving boundary can be fixed by the coordinate transformation

$$z_1 = \frac{x - l_1}{s(t) - l_1} \tag{4.31}$$

so that $x = l_1$ becomes $z_1 = 0$ and $x = s(t)$ becomes $z_1 = 1$. Similarly, transformation for phase 2, $s(t) \leq x \leq l_2$, is

$$z_2 = \frac{x - l_2}{s(t) - l_2} \tag{4.32}$$

and $x = l_2$ becomes $z_2 = 0$ and $x = s(t)$ becomes $z_2 = 1$. Let

$$u_i(x,t) = v_i(z_i,t),\ i = 1,2.$$

We apply the chain rule for partial derivatives and get, for $i = 1,2$

$$\frac{\partial u_i}{\partial x} = \frac{\partial v_i}{\partial z_i}\frac{1}{s(t) - l_i} \tag{4.33}$$

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 v_i}{\partial z_i^2}\frac{1}{(s(t) - l_i)^2} \tag{4.34}$$

and

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \frac{\partial v_i}{\partial z_i}(-\frac{x - l_i}{(s(t) - l_i)^2})\dot{s} + \frac{\partial v_i}{\partial t} \\ &= \frac{\partial v_i}{\partial z_i}(-\frac{z_i}{s(t) - l_i})\dot{s} + \frac{\partial v_i}{\partial t} \end{aligned} \tag{4.35}$$

Correspondingly, equations (4.3)-(4.4) become

$$k_i\frac{\partial^2 v_i}{\partial z_i^2} + z_i(s - l_i)\frac{\partial v_i}{\partial z_i}\dot{s} - (s - l_i)^2\frac{\partial v_i}{\partial t} = 0, \quad i = 1,2 \tag{4.36}$$

after multiplying through by $(s - l_i)^2$ for numerical convenience. The moving boundary conditions (4.14) and (4.18) become

$$\frac{\partial v_1}{\partial z_1}|_{z_1=1}\frac{\dot{s}}{s - l_1} = \frac{\partial v_2}{\partial z_2}|_{z_2=1}\frac{\dot{s}}{s - l_2} = \dot{h} \tag{4.37}$$

Let ξ_i, $1 \le i \le N+1$ be equally spaced grid points for z_1. While let ξ_i, $N+1 \le i \le 2N+1$ be equally spaced grid points for z_2. Let $\delta\xi = \frac{1}{N}$, we then have

$$\xi_i = (i-1)\delta\xi, \quad i = 1, 2, .., N+1$$

and

$$\xi_i = 1 - (i-N-1)\delta\xi, \quad i = N+1, N+2, .., 2N+1.$$

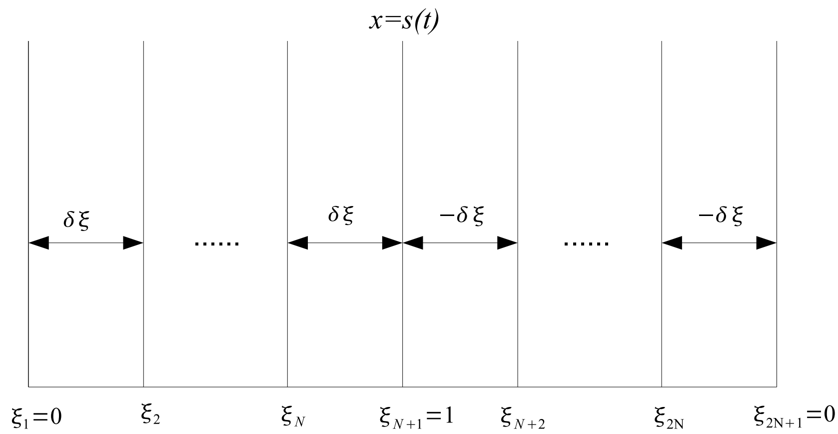

Figure 4.3: Fixed grid for front-fixing method with coordinate transform

Figure 4.3 shows the fixed grid mesh of z and the moving boundary $x = s(t)$ is transformed to the straight line $\xi_{N+1} = 1$. The usual finite-difference discretization in time at $t^{j+1} = (j+1)\delta t$, a fully implicit scheme, is combined with central differences for space derivatives to give the system for $i \neq N+1$

$$\begin{aligned} C_m v(i,j) = \quad & v(i-1,j+1)[-B_m + \xi_i A_m] + v(i,j+1)[2B_m + C_m)] \\ & +v(i+1,j+1)[-B_m - \xi_i A_m], \quad m = 1, 2 \end{aligned} \tag{4.38}$$

where

$$B_m = \frac{\delta t}{(\delta\xi)^2} k_m$$

$$C_m = (s(j+1) - l_m)^2$$
$$A_1 = C_1 \frac{s(j+1) - s(j)}{2\delta\xi}$$

and

$$A_2 = -C_2 \frac{s(j+1) - s(j)}{2\delta\xi}$$

Notice that the negative sign in A_2 occurs since z_2 changes form 1 to 0 decreasingly. We use two points backwards derivatives for approximating $\frac{\partial v}{\partial z_1}$ at ξ_{N+1}

$$\frac{\partial v}{\partial z_1} = \frac{v_{N+1} - v_N}{\delta\xi} \tag{4.39}$$

while two points forwards for $\frac{\partial u}{\partial z_2}$ at ξ_{N+1}

$$\frac{\partial v}{\partial z_2} = -\frac{v_{N+2} - v_{N+1}}{\delta\xi} \tag{4.40}$$

From moving boundary condition (4.37), one obtains

$$v(N,j+1)(l_2-s(j+1))+v(N+1,j+1)(l_1-l_2)+v(N+2,j+1)(s(j+1)-l_1) = 0 \tag{4.41}$$

and

$$s(j+1) = \frac{(v(N+1,j+1) - v(N,j+1))s(j) - \delta\xi l_1(v(N+1,j+1) - v(N+1,j))}{v(N+1,j+1) - v(N,j+1) - \delta\xi(v(N+1,j+1) - v(N+1,j))} \tag{4.42}$$

Notice that equation (4.41) exhibits linearity between $v(N, j+1)$, $v(N+2, j+1)$ and $v(N+1, j+1)$ at each time step if $s(j+1)$ is known. Therefore we can combine equation (4.38) with (4.41) together to form a linear system for $u(:, j+1)$ with a given $s(j+1)$. Let P denotes the matrix defined as

$$
\begin{pmatrix}
1 & & & & & & & & \\
\xi_2 A_1 - B_1 & 2B_1 + C_1 & -\xi_2 A_1 - B_1 & & & & & & \\
 & . & . & . & & & & & \\
 & & \xi_N A_1 - B_1 & 2B_1 + C_1 & -\xi_N A_1 - B_1 & & & & \\
 & & & l_2 - s(j+1) & l_1 - l_2 & s(j+1) - l_1 & & & \\
 & & & & \xi_{N+1} A_2 - B_2 & 2B_2 + C_2 & -\xi_{N+1} A_2 - B_2 & & \\
 & & & & & . & . & . & \\
 & & & & & & \xi_{2N} A_2 - B_2 & 2B_2 + C_2 & -\xi_{2N} A_2 - B_2 \\
 & & & & & & & & 1
\end{pmatrix}
\tag{4.43}
$$

where $A_{1,2}$, $B_{1,2}$ and $C_{1,2}$ are described as for equation (4.38). Let

$$V = \begin{pmatrix} v(1, j+1) \\ v(2, j+1) \\ . \\ . \\ v(N, j+1) \\ v(N+1, j+1) \\ v(N+2, j+1) \\ . \\ . \\ v(2N, j+1) \\ v(2N+1, j+1) \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 0 \\ C_1 v(2, j) \\ . \\ . \\ C_1 v(N, j) \\ 0 \\ C_2 v(N+2, j) \\ . \\ . \\ C_2 v(2N, j) \\ 0 \end{pmatrix}$$

We then derive the matrix system $PV = Q$ to solve for $v(:, j+1)$. An iteration scheme can be used with known $v(:, j)$ and $s(j)$ as following:

(i) Make an initial guess for $s^{(0)}(j+1)$, for example, let $s^{(0)}(j+1) = s(j)$.

(ii) Solve for $v^{(0)}(:, j+1)$ from the matrix system $PV = Q$.

(iii) Compute next $s^{(1)}(j+1)$ from equation (4.42).

(iv) Repeat the steps (ii)-(iii) until the error $|s^{(n)}(j+1) - s^{(n+1)}(j+1)|$ is less than a prescribed tolerance, say $tol = 10^{-6}$.

At all interior ξ_i, except for $i = N+1$, we use central difference formulas for both the first and second space derivatives. Therefore, we expect the order of method for space discretization to be 2. While the time derivative is always approximated by two point backward formula, the order of method for time discretization is 1. These results are confirmed in next section for test problems. Of course, rather than using two-points approximation for $\frac{\partial v}{\partial \xi_{1,2}}$ as in (4.39) and (4.40), we have other choices for approximation, such as the three-point backward/forward formula

$$(3v_{N+1} - 4v_N + v_{N-1})/2\delta\xi \tag{4.44}$$

However, for the sake of numerical efficiency, we want the matrix P to be tridiagonal.

An alternative way of treating equation (4.36) suggested in [9] is the Method of Lines in space, i.e: discretize only the space derivatives and then integrate the resulting ordinary differential equations in time along all constant $\xi = \xi_i$ lines. For example, equation (4.36) is approximated by

$$\frac{\partial v(i)}{\partial t} = k_m \frac{1}{(s-l_m)^2} \frac{v(i+1) + v(i-1) - 2v(i)}{(\delta\xi)^2} + \frac{1}{s-l_m} \dot{s}\xi_i \frac{v(i+1) - v(i-1)}{2\delta\xi} \tag{4.45}$$

where $i = 2, ..., N$ for $m = 1$, while $i = N+2, ..., 2N$ for $m = 2$. The moving boundary conditions (4.37) can then be approximated by

$$v(N+1) = \frac{v(N)(l_2 - s) + v(N+2)(s - l_1)}{l_2 - l_1} \tag{4.46}$$

$$\dot{s} = \frac{\dot{v}(N+1)(s - l_1)\delta\xi}{v(N+1) - v(N)} \tag{4.47}$$

for example. Well-established ODE solvers can automatically integrate the resulting ODE systems to the required accuracy and will produce the solution vector $(v(2), ..., v(2N), s)$ at required time intervals. With Dirichlet boundary conditions, we then have the full solution vector $(v(1), ..., v(2N+1), s)$.

4.3 Numerical Results for Front Fixing Method

4.3.1 Test Examples

We have no exact solution for the moving boundary problem (4.3)-(4.7) to compare with the numerical solution. Therefore some special examples are used to test the method.

A. Base Case

Assume for problem (4.3)-(4.7), we have $l_1 = -l_2$ and a smooth initial function $f(x)$ satisfying the following conditions:

$$f(-x) = -f(x)$$

and

$$f''(x) > 0 \quad for \quad x < 0 \quad and \quad f''(x) > 0 \quad for \quad x > 0.$$

Since $f(x)$ changes concavity at $x = 0$, we have $s(0) = 0$. Let $(u(x,t),\ s(t))$ be the solution to the base case.

B. Case I

Consider a solution pair $(\tilde{u}(x,t),\ s(t))$, where $\tilde{u}(x,t) = -u(x,t)$. It is trivial to show $\tilde{u}(x,t)$ satisfies PDE (4.3) and (4.4) with Dirichilet boundary condition, while $\tilde{u}(x,0) = -f(x)$. On the moving boundary $x = s(t)$, we have

$$\tilde{u}_1(x,t) = \tilde{u}_2(x,t) = \tilde{h}(t) = -h(t)$$

and

$$\frac{\partial \tilde{u}_{1,2}}{\partial x} = \frac{\partial u_{1,2}}{\partial x}$$

Therefore, the moving boundary condition (4.18) still holds for $(\tilde{u}(x,t),\, s(t))$

$$\frac{\partial \tilde{u}_1}{\partial x} = \frac{\partial \tilde{u}_2}{\partial x} = \frac{\dot{\tilde{h}}}{\dot{s}}$$

Hence with the initial condition in Base Case changed to $-f(x)$, the solution will become $(-u(x,t),\, s(t))$.

C. Case II

Now take a solution pair $(\tilde{u}(x,t),\, \tilde{s}(t))$, where $\tilde{u}(x,t) = u(-x,t)$ and $\tilde{s}(t) = -s(t)$. It is easy to see that $\tilde{u}$ satisfies Dirichilet boundary condition and $\tilde{u}(x,0) = f(-x)$.

$$\frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial t}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial \tilde{u}}{\partial x} \quad \textit{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \tilde{u}}{\partial x^2}$$

Then we obtain

$$\frac{\partial \tilde{u}_1}{\partial t} = k_1 \frac{\partial^2 \tilde{u}_1}{\partial x^2}, \quad l_1 < -x < s(t)$$

and

$$\frac{\partial \tilde{u}_2}{\partial t} = k_2 \frac{\partial^2 \tilde{u}_2}{\partial x^2}, \quad s(t) < -x < l_2$$

Given the assumption of $l_1 = -l_2$, we can rewrite the regions for $\tilde{u}_{1,2}(x,t)$ as $\tilde{s}(t) < x \le l_2$ and $l_1 \le x < \tilde{s}(t)$, respectively. The moving boundary condition (4.18) is satisfied, since on $x = \tilde{s}(t)$

$$\tilde{u}_1(\tilde{s}(t),t) = \tilde{u}_2(\tilde{s}(t),t) = h(t)$$

and

$$\frac{\partial \tilde{u}_1}{\partial x} = \frac{\partial \tilde{u}_2}{\partial x} = -\frac{\dot{h}}{\dot{s}} = \frac{\dot{h}}{\dot{\tilde{s}}}$$

Thus, if we change the initial condition to $f(-x)$ and exchange coefficients k_1 and k_2 in PDEs, the solution $(u(-x,t),\ -s(t))$ should be expected.

D. Case III

Now consider an extreme case where $k_1 = k_2$. We have shown from Case I, a solution pair $(-u(x,t),\ s(t))$ exists with initial condition $-f(x)$. While in Case II, there is a solution pair $(u(-x,t),\ -s(t))$ with initial condition $f(-x)$ and exchanged k_1 and k_2. Under assumption $k_1 = k_2$ together with $f(-x) = -f(x)$, Case I is the same as Case II. Therefore we conclude the solution should have the following properties:

$$-u(x,t) = u(-x,t)$$

and

$$s(t) \equiv 0.$$

A complete specification of the test problem for Base Case, including the PDE parameter, is given in Table 4.3.1. The numerical comparison results, shown in Figures 4.8-4.9, verify the symmetry properties for the cases discussed above.

4.3.2 Rate of Convergence

Test problem of Base Case was solved by front fixing method with coordinate transformation. Solutions were computed on a sequence of uniformly refined grids, initial spacestep dz is 0.01 and timestep dt is 0.01. At each grid refinement, the spacestep and timestep were halved. The convergence tolerance for $s(t)$ iteration is 10^{-6}. Since there exists no exact solution for comparison, we take the numerical solution with very fine mesh, where $dz = 0.01/32$ and $dt = 0.01/64$, as a proxy for exact solution. Let u_{ex} be "exact solution" and u be numerical solution. Define $error$ as

$$error = |u(:,T) - u_{ex}(:,T)|_\infty$$

Convergent results are given in Table 4.3.2 and 4.3.2. Figure 4.10 and 4.11 are the log-log plots of errors versus the number of grid points. We showed the front fixing method described above converges at quadratic rate in space and at linear rate in time. These results are consistent with the order of the discretization formulas, which are central difference in space except for points on the moving boundary and backward difference in time everywhere.

Initial Condition	$f(x) = -x^3 + x$
Initial Moving Boundary Position	$s(0) = 0$
Boundary	$l_1 = -1$, $l_2 = 1$
Final Time	$T = 0.1$
PDE Parameter	$k_1 = 0.1$, $k_2 = 0.5$

Table 4.1: Description of test problem for Base Case

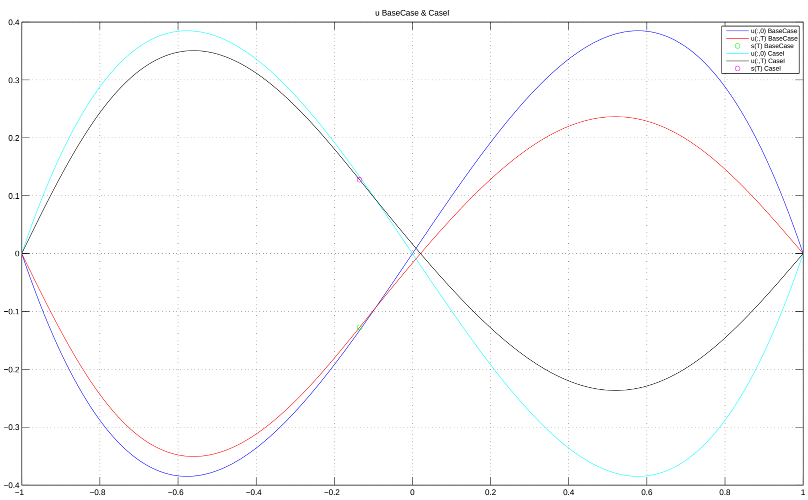

Figure 4.4: Solution $u(x,t)$ at $t = 0$ and $t = T$, final position of moving boundary $x = s(T)$ for Base Case and Case I.

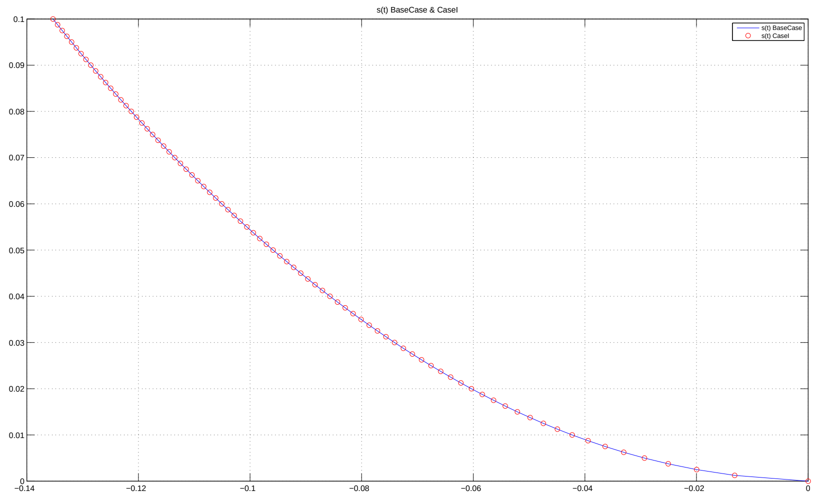

Figure 4.5: Moving boundary $x = s(t)$ for Base Case and Case I.

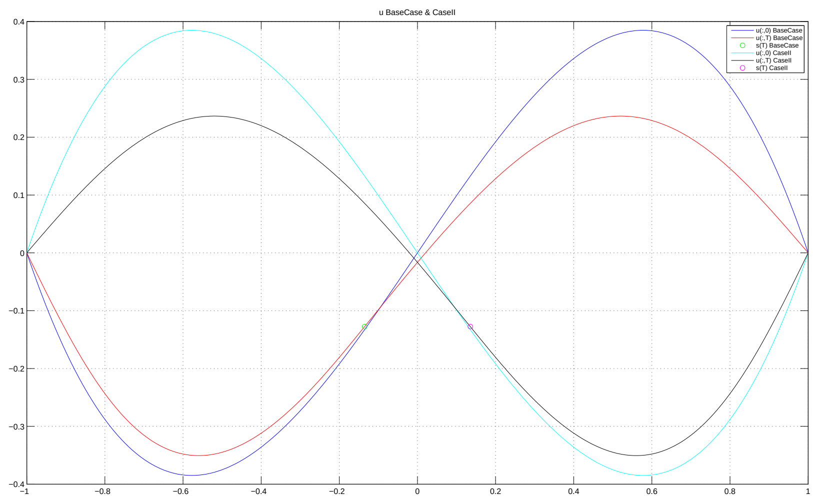

Figure 4.6: Solution $u(x,t)$ at $t = 0$ and $t = T$, final position of moving boundary $x = s(T)$ for Base Case and Case II.

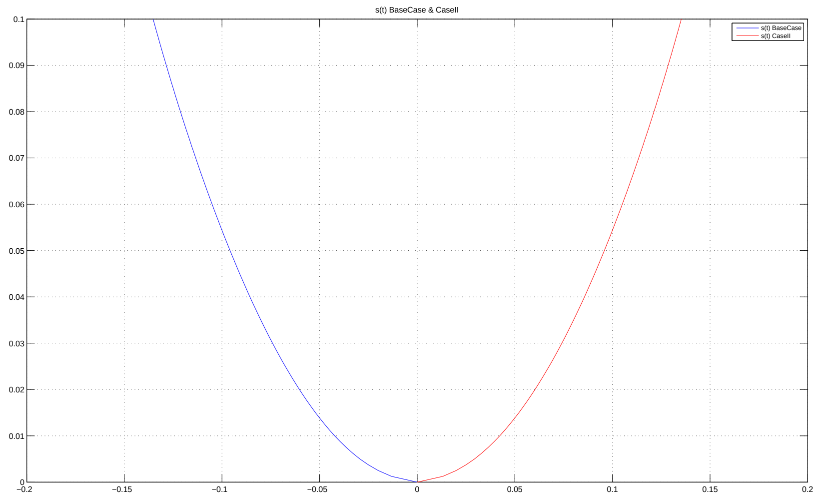

Figure 4.7: Moving boundary $x = s(t)$ for Base Case and Case II.

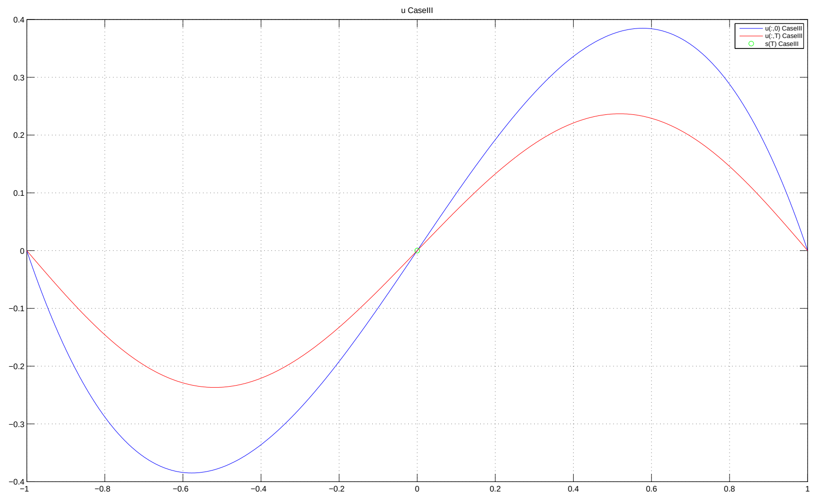

Figure 4.8: Solution $u(x, t)$ at $t = 0$ and $t = T$, final position of moving boundary $x = s(T)$ for Case III.

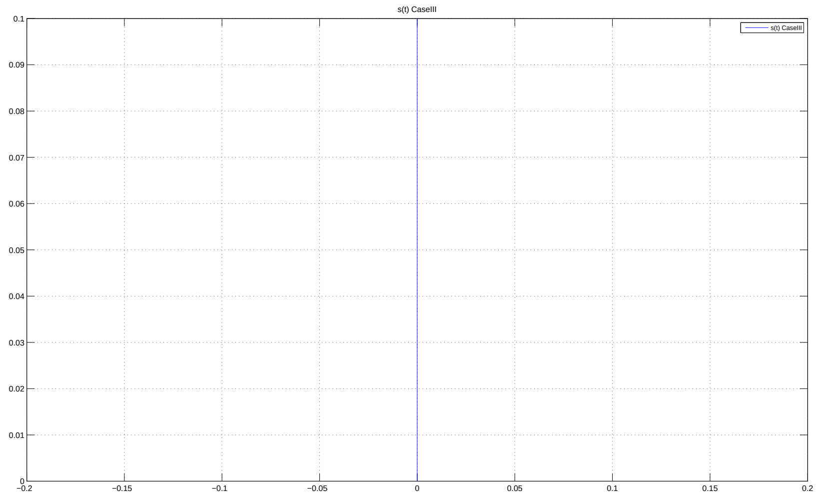

Figure 4.9: Moving boundary $x = s(t)$ for Case III.

Spacial Nodes	Error (10^{-4})	Ratio
201	0.1798	
401	0.0454	3.96
801	0.0113	4.01
1601	0.0027	4.18

Table 4.2: Convergence results for test problem defined in Table 4.3.1. The time step is fixed at 0.01/64 at each grid refinement. "Ratio" is the ratio of successive error

Time Nodes	Error	Ratio
11	0.0034	
21	0.0017	2
41	0.0008	2.12
81	0.0004	2.00

Table 4.3: Convergence results for test problem defined in Table 4.3.1. The space step is fixed at 0.01/32 at each grid refinement. "Ratio" is the ratio of successive error

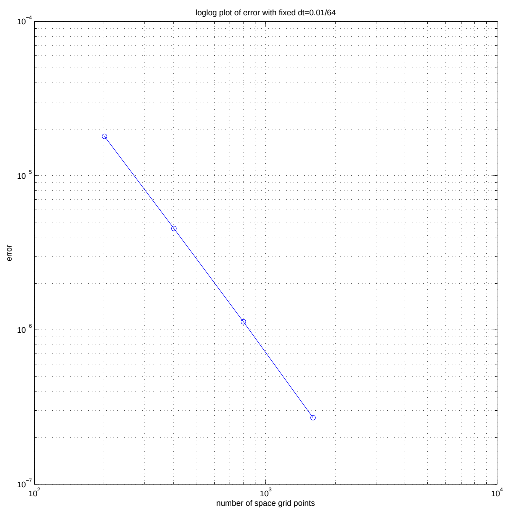

Figure 4.10: Log-log plot of error versus number of space grid points when time step is fixed at 0.01/64

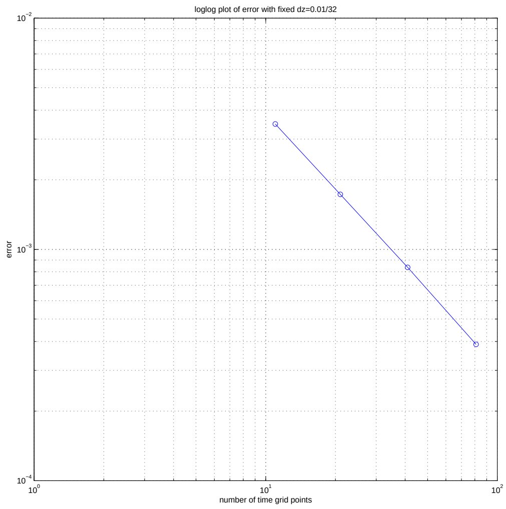

Figure 4.11: Log-log plot of error versus number of space grid points when space step is fixed at 0.01/32

Chapter 5

Conclusion and Future Work

Although the classic Black Scholes model is the most important step in the development of derivative pricing analysis, the assumptions of constant volatility of the underlying asset, no transaction costs and the market being perfectly liquid are clearly not plausible in real markets. We have derived nonlinear models corresponding to revised assumptions. Despite substantial differences in the financial framework, these nonlinear Black-Scholes models have a very similar mathematical structure. This dissertation focuses on the properties of the resulting nonlinear PDEs.

The main results consist of three parts. Using energy estimates, we have shown existence and uniqueness of a solution to the smoothed nonlinear PDE with periodic boundary conditions. To study the discontinuity in the coefficient function, we discretize the nonlinear PDE into ODE systems. The analysis focuses on the connection between the ODE systems with discontinuous coefficient function and the ones with smoothed coefficient function. Numerical results are based on the framework of moving boundary problems. We first formulate the nonlinear PDE model as a moving boundary problem with appropriate moving boundary conditions. Two general methods are applied to our model. The front tracking method with a fixed grid does not work well, due to the nonlinearity of the discretized system of moving boundary conditions. The front fixing method with coordinate transform has been validated for some test problems. The order of the method is shown to be quadratic in space while linear in time.

One part of our future work is to connect the PDE with discontinuous coefficient function to the one with smooth coefficient function through some convergence analysis. The problem can be formulated as follows: consider a PDE with smooth coefficient function, as such coefficient function approaches some discontinuous function in certain norm, will the solution also converge and if yes, will it converge to the solution of the discontinuous PDE? Another part of our interest is the analysis of moving boundary problems. We would like to establish analytical validation of moving boundary conditions of our model. Leading order analysis has been tried for validation, however it does not seem to be helpful. For numerical analysis, we have seen the iteration scheme works for the front fixing method. Therefore, the proof of convergence of the iteration scheme will be of interest.

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