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Weighted estimates for dyadic operators with complexity

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Dedication

To Gabi.

"As dificuldades são o aço estrututural que entra na construção do caráter."
" The difficulties are the structural steel that goes into the building of character."

– Carlos Drummond de Andrade

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Abstract

We extend the definitions of dyadic paraproduct, dual dyadic paraproduct and t-Haar multipliers to dyadic operators that depend on the complexity (m, n), for m and n positive integers. We will use the ideas developed by Nazarov and Volberg in [NV] to prove that the weighted $L^2(w)$-norm of a paraproduct with complexity (m, n) and the dual paraproduct associated to a function $b \in BMO$, depends linearly on the A_2-characteristic of the weight w, linearly on the BMO-norm of b, and polynomially in the complexity. Moreover we prove that the $L^2(w)$-norm of the composition of these operators depends linearly on the A_2-characteristic of the weight w, quadratic on the BMO-norm of b, and polynomially in the complexity. The argument for the paraproduct provides a new proof of the linear bound for the dyadic paraproduct [Be1] (the one with complexity $(0, 0)$). Paraproducts and their adjoints are examples of Haar shift multipliers of type 2 and 3. We adapt the Nazarov and Volberg method to show that for certain Haar shift multiplier of type 4 and complexity (m, n) the same type of bounds in $L^2(w)$ hold. Also we prove that the L^2-norm of a t-Haar multiplier for any t and weight w depends on the square root of the C_{2t}-characteristic of w times the square root of the A_q-characteristic of w^{2t} and polynomially in the complexity (m, n), recovering a result of Beznosova [Be] for the $(0, 0)$-complexity case. Last, we prove that for a pair of weights u and v and a class of locally integrable function b that satisfies certain conditions, the dyadic paraproduct π_b is bounded from $L^2(u)$ into $L^2(v)$ if and only if the weights satisfies the joint A_2 condition.

Contents

Chapter 1

Introduction

In the last four decades a number of mathematicians devoted their attention to study boundedness of operators in Lebesgue weighted spaces, $L^p(w)$. Many aspects of these theory have been studied in these years. In the 1970's their main concern was to find necessary and sufficient conditions for an operator to be bounded in $L^p(w)$. In these studies it was brought to attention the importance of the Muckenhoupt A_p-class. Recall that a weight w belongs to this class if and only if

$$[w]_{A_p} := \sup_I \left(\frac{1}{|I|} \int_I w(x)\, dx \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}}(x)\, dx \right)^{p-1} < \infty,$$

where $[w]_{A_p}$ denotes the A_p-characteristic of the weight. Muckenhoupt proved in 1972 that the maximal function is bounded in $L^p(w)$ for $p > 1$ if and only if $w \in A^p$. In 1973 Hunt-Muckenhoupt-Wheeden extended this result to the Hilbert transform. Also in 1973, Coiffman-Fefferman proved that

$$w \in A_p \Rightarrow \|Tf\|_{L^p(w)} \leq C([w]_{A_p}) \|f\|_{L^p(w)},$$

for all Calderón Zygmund operators T. Even though it was known that the constant C depended in the A_p - characteristic of w, it was not known how.

Only two decades later the mathematicians started to study how the norm of some operators in a weighted space depends on the so called A_p-characteristic of the weight. The first result of this type was due to Buckley in 1993, in [Bu] he proved that, for $p > 1$

$$\|Mf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)},$$

where M is the maximal function, w is a weight that is a locally integrable positive a.e. function, and $f \in L^p(w)$ if and only if $\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}} |f(x)|^p w(x)dx\right)^{1/p} < \infty$. Later on, many results of this type followed.

In 2000, Wittwer showed in [W] that the norm of the martingale transform in $L^2(w)$ depended linearly in the A_2-characteristic of the weight, $[w]_{A_2}$. Independently the same linear bound was shown to hold for the dyadic square function [W1], [HukTV]. This was then shown for the Ahlfors-Beurling transform [PetV], with important consequences in the theory of quasiconformal mappings. In 2007, Petermichl [Pet2] published the linear dependence for the Hilbert transform, and soon after for the Riesz transforms [Pet3]. Petermichl's work is based on her representation of the Hilbert transform as an average of dyadic shift operators of complexity $(0, 1)$, [Pet1]. In 2008, Beznosova [Be1] also proved linear dependence in the A_2-characteristic for the $L^2(w)$-norm of the dyadic paraproduct. More precisely, for T, any of the above mentioned operators, all $w \in A_2$ there is a $C_T > 0$ such that

$$\|Tf\|_{L^2(w)} \leq C_T[w]_{A_2}\|f\|_{L^2(w)}.$$

These linear estimates in $L^2(w)$ imply corresponding $L^p(w)$-bounds, by the sharp extrapolation theorem [DGPPet], i.e. it was enough to prove the linear bound in the A_2^d - characteristic in $L^2(w)$ to conclude that

$$\|Tf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

All these works, except the sharp extrapolation theorem, use the Bellman function technique. Those methods were used as well by Chung [Ch] to obtain quadratic bounds for the commutator of the Hilbert transform and a BMO function. This quadratic bound was later shown to be true for all operators for which the linear bound in $L^2(w)$ holds [ChPPz] with an argument that has nothing to do with Bellman functions. Bellman functions have impacted not only the theory of weights as described here, but also other areas in harmonic analysis, see [V] for more insights and references.

Many efforts were done to show a linear dependence on the A_2^d-characteristic of the $L^2(w)$ norm for a large class of operators. In particular for all Calderón-Zygmund operators, the so-called A_2-conjecture. Lacey-Petermichl-Reguera in [LPetR] proved the linear A_2-bounds for all Haar shift operators, and all operators that were averages of Haar shift operators with bounded complexity (including Hilbert, Riesz, and Beurling-Ahlfors transforms), this was the first class of operators to be shown to have the linear A_2-bounds. Their results depend exponentially in the complexity of the Haar shifts, so does an alternative proof presented soon after in [CrMPz]. Despite this fact, the argument in [CrMPz] is very flexible and can be adapted to obtain sharp bounds for paraproducts, square functions, vector-valued operators, and two-weight settings, as well as for fractional integrals and commutators [Le], [CrMoe], [Or]. Neither of these arguments uses Bellman functions, unlike all the previous work for individual operators.

Finally in the Summer 2010, Hytönen in [H] proved the A_2-conjecture, that is he showed that for *all* Calderón-Zygmund integral singular operators T in $\mathbb{R}^d$, weights $w \in A_p$, there is $C_{p,d,T} > 0$ such that,

$$\|Tf\|_{L^p(w)} \leq C_{p,d,T}[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

His result is based on results of Pérez, Treil, and Volberg in [PzTV], and in a very clever representation theorem for T in terms of Haar shift operators of *arbitrary complexity*, which generalizes Petermichl's representation theorem for the Hilbert transform [Pet1]. In [HPzTV] a more succinct proof of the A_2-conjecture was given.

See [L1] for a survey of the A_2-conjecture including a rather complete history of most results that appeared up to november 2010, and that contributed to the final resolution of this mathematical puzzle. An important and hard part of the proof was to obtain bounds for Haar shifts operators that depend linearly on the A_2-characteristic and at most polynomially on the complexity (m, n). In 2011, Nazarov and Volberg [NV] provided a beautiful new proof that still uses Bellman functions but minimally, and that can be transferred to geometric doubling metric spaces [NV1, NRezV]. Treil [T], independently [HLM+] are able to obtain linear dependence on the complexity. Crucial in both [NV] and [HLM+] is the use of some stopping time argument (it is called a corona decomposition in [LPetR, L1, HLM+]).

A *Haar shift operator of type 1 with complexity* (m, n), $m, n \in \mathbb{N}$, is defined by,

$$\big(T_1^{m,n} f\big)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^L \langle f, h_I\rangle h_J(x),$$

where $|c_{I,J}^L| \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$, $\mathcal{D}$ denotes the dyadic intervals, $|I|$ the length of interval I, $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L of length $2^{-m}|L|$, h_I are the Haar functions, and $\langle f, g\rangle$ denotes the L^2-inner product. Notice that the Haar shift operators are automatically uniformly bounded in $L^2(\mathbb{R})$, with operator norm less than or equal to one. The Haar shift of complexity $(0, 0)$ is the martingale transform. The Haar shift of complexity $(0, 1)$ corresponds to Petermichl's shift operator (Sha), introduced in [Pet1].

As the martingale transform was extended to the Haar shifts with complexity (m, n), it seems natural to attempt the same extension for other dyadic operators, and examine if we can recover the same dependence on the A_2-characteristic that we have for the original operator (the one with complexity $(0, 0)$) times a factor that depends at most polynomially in the complexity of these operators. The author and Pereyra started this analysis in [MoP], for the extension of the Haar multipliers and the the dyadic paraproduct and these results are part of this dissertation.

For $b \in BMO$, $m, n \in \mathbb{N}$, the *dyadic paraproduct of complexity* (m, n) is defined formally by, with $c^L_{I,J}$ as above,

$$\pi_b^{m,n} f(x) = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c^L_{I,J} \big\langle f, \frac{\chi_I}{|I|} \big\rangle \langle b, h_I\rangle h_J(x).$$

The *dual dyadic paraproduct of complexity* (m, n) is defined formally by, with $c^L_{I,J}$ as above,

$$\kappa_b^{m,n} f(x) = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c^L_{I,J} \langle b, h_I\rangle \langle f, h_I\rangle \frac{\chi_J(x)}{|J|}.$$

A formal calculation shows that $\kappa_b^{m,n} = (\pi_b^{n,m})^*$, where T^* is the formal adjoint of T: $\langle Tf, g\rangle = \langle f, T^* g\rangle$. In [NTV1, HPzTV], paraproducts of complexity $(0, r)$ depending on two weights (average is calculated with respect to one weight, Haar functions are with respect to the other weight, so is the inner product) were introduced and they have necessary an sufficient testing conditions for boundedness from one weighted space into the other with respect to the *same weights* that appear in the definition of the paraproduct. In our case there are no weights in the definition, and we are asking about boundedness in weighted space. One can check that the paraproduct of complexity (m, n) is the composition of a Haar shift operator of complexity (m, n) and the dyadic paraproduct of complexity $(0, 0)$, $\pi_b^{m,n} = T_1^{m,n}\pi_b$. since both the Haar shift operators [LPetR, CrMPz, H] and the dyadic paraproduct [Be1] obey linear bounds on $L^2(w)$ on the A_2-characteristic of the weight, these estimates immediately will provide a quadratic bound on the A_2-characteristic of the weight for the paraproduct of complexity (m, n), namely, $\|\pi_b^{m,n} f\|_{L^2(w)} \leq C_{m,n}\|b\|_{BMO^d}[w]^2_{A_2^d}\|f\|_{L^2(w)}$.

We prove in this dissertation, that in fact, the paraproduct of complexity (m, n) and the dual paraproduct of complexity (m, n) obey the same *linear bound* obtained by Beznosova for the dyadic paraproduct of complexity $(0, 0)$, multiplied by a polynomial factor that depends in the complexity.

Theorem 1.1. *For all $w \in A_2^d$, $b \in BMO^d$, then*

$$\|\pi_b^{m,n} f\|_{L^2(w)} \leq C(m+n+2)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)}.$$

Corollary 1.2. *For all $w \in A_2^d$, $b \in BMO^d$, then*

$$\|\kappa_b^{m,n} f\|_{L^2(w)} = \|(\pi_b^{n,m})^* f\|_{L^2(w^{-1})} \leq C(m+n+2)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)},$$

recall that $[w]_{A_2^d} = [w^{-1}]_{A_2^d}$.

The dyadic paraproduct and the dual dyadic paraproduct of complexity (m,n) are *generalized Haar shift operators*, defined as the Haar shift operators replacing the Haar functions h_I and h_J in the definition by characteristic functions, $\chi_I/|I|$ and $\chi_J/|J|$, and now one has to assume boundedness on $L^2(\mathbb{R})$, imposing size conditions on the coefficients is not enough. Generalized Haar shift operators where introduced in some preprints that have now been superseded by [HLM+]. For Hytönen's representation theorem (and hence for the resolution of the A_2-conjecture) one needs Haar shift operators of arbitrary complexity, and dyadic paraproducts of complexity $(0,0)$, and their adjoints, i.e. generalized Haar shift operators of arbitrary complexity are not needed.

We will also prove in this dissertation, similar estimates for a subclass of the generalized Haar shifts operators with complexity (m,n), these operators will be called *composition dual dyadic paraproduct with paraproduct of complexity* (m,n), and it is defined formally by, with $c_{I,J}^L$ as above,

$$\zeta_{b,d}^{m,n} f(x) = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^L \, \langle b, h_I\rangle \, \langle d, h_I\rangle \Big\langle f, \frac{\chi_I}{|I|}\Big\rangle \frac{\chi_J(x)}{|J|}.$$

for $b, d \in L^1_{loc}$, then we have the following result.

Theorem 1.3. *For all $w \in A_2^d$, $b, d \in BMO^d$, then*

$$\|\zeta_{b,d}^{m,n} f\|_{L^2(w)} \leq C(m+n+2)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|d\|_{BMO^d} \|f\|_{L^2(w)}.$$

These operators are a particularization of operators that we will call in this dissertation Haar shifts of type 4 with complexity (m,n) [1]. They are defined as

$$T_4^{m,n} f(x) = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,4} \left\langle f, \frac{\chi_I}{|I|}\right\rangle \frac{\chi_J(x)}{|J|}.$$

The linear dependence in the A_2^d characteristic for these operators is not a new result. This dependence was proved in [HLM+] for all generalized Haar shifts (Haar shifts of type 4 in our nomenclature) that are bounded on L^2, based on Sawyer two-weight testing conditions and a complicated corona or stopping time argument. For $b, d \in BMO^d$, these operators are bounded in L^2, since they are composition of a paraproduct and a dual paraproduct with a Haar shift of type 1 with complexity (m,n). This decomposition immediately yields a cubic dependence $[w]^3_{A_2^d}$ for the operator norm, we will recover the linear dependence $[w]_{A_2^d}$ and our result explicitly displays the dependence on $\|b\|_{BMO}$, the same for the paraproduct and dual paraproduct of complexity $(0,0)$ and the dependence on $\|b\|_{BMO}$ and $\|d\|_{BMO}$ for the composition of dual paraproduct and paraproduct. Also we present a new proof of these facts, bypassing the more complicated Sawyer two-weight testing conditions, providing a, from our point view, more transparent proof.

The operator $\zeta_{b,d}^{m,n}$ can be decomposed, formally, as $\pi_b^* T_1^{m,n} \pi_d$. We will prove that for the case $m = n = 0$ and $c_I^4 \geq 0$ for all dyadic intervals I, then if $T_4^{0,0}$ is bounded then it can be decomposed as $\pi_b^*\pi_b$ for some $b \in BMO^d$. Therefore our techniques allow us to prove linear bounds in the A_2-characteristic for all bounded positive operators $T_4^{0,0}$. We would like to extend this decomposition result for bounded positive operators of type 4 of arbitrary complexity. Such positive operators are often all one needs, after some reductions, to estimate all Haar shifts of type 4, see for example the recent work of Hytönen and Lacey [HL].

[1] The dyadic paraproduct and the dual dyadic paraproduct of complexity (m,n) are example of Haar shifts of type 2 and 3 respectively.

Also in the 1990's another question was raised, that was, to study the boundedness of operators that depend somehow on a weight in the Lebesgue space, L^p. In this problems the weight w is moved from the space into the operator. Independently Peréz in [Pz] and Pereyra in [P] proved that the weighted maximal function M_w is bounded in L^p if and only if the weight is in the RH_p class. Recall that a weight $w \in RH_p$ if $[w]_{RH_p} := \sup_I \big(\frac{1}{|I|}\int_I w^p(x)dx\big)^{1/p}\big(\frac{1}{|I|}\int_I w(x)dx\big)^{-1} < \infty$.

Another important example of these operators are the Haar multipliers introduced by Pereyra in [P1]. It was proved in [P2] that the L^2-norm for the Haar multiplier T_w depends on the square of the RH_2-characteristic of the weight w in the Haar multiplier's definition. The Haar multipliers and the dyadic paraproducts are closely related: the resolvent of the dyadic paraproduct is a cousin of T_w [P]. In her PhD dissertation, Beznosova showed that the L^2-norm of a t-Haar multipliers, T^t_w, defined in [KP], is bounded by a constant times the square root of the C_{2t}-characteristic of w times the square root of the A^d_q-characteristic of w^{2t}. For $t \in \mathbb{R}$, a weight $w \in C_{2t}$ if

$$[w]_{C_{2t}} := \sup_I \Big(\frac{1}{|I|}\int_I w^{2t}(x)dx\Big)\Big(\frac{1}{|I|}\int_I w(x)dx\Big)^{-2t} < \infty.$$

For $t \in \mathbb{R}$, $m, n \in \mathbb{N}$, and weight w, the *t-Haar multiplier of complexity* (m, n) is defined formally by

$$T^{t,w}_{m,n} f(x) = \sum_{I\in\mathcal{D}} \sum_{I\in\mathcal{D}_m(L), J\in\mathcal{D}_n(L)} \frac{\sqrt{|I|\,|J|}}{|L|} \frac{w^t(x)}{(m_I w)^t}\langle f, h_I\rangle h_J(x).$$

When $(m, n) = (0, 0)$ we denote the corresponding Haar multiplier by T^t_w. We will show in this dissertation that

Theorem 1.4. *For all $w \in C^d_{2t}$ such that $w^{2t} \in A^d_q$, for some $q > 1$, then*

$$\|T^{t,w}_{m,n} f\|_{L^2} \le C(m+n+2)^3 [w]^{\frac{1}{2}}_{C_{2t}} [w^{2t}]^{\frac{1}{2}}_{A^d_q} \|f\|_{L^2}.$$

Moreover, $w \in C^d_{2t}$ is a necessary condition for the boundedness of $T^{t,w}_{m,n}$.

This recovers results of Beznosova for T_w^t, the complexity $(0,0)$ case, [Be]. Observe that $T_{m,n}^{t,w}$ is different than both $T_1^{m,n}T_w^t$ and $T_w^tT_1^{m,n}$, where T_w^t denotes the t-Haar multiplier of complexity $(0,0)$. Notice that in this case, both $T_1^{m,n}T_w^t$ and $T_w^tT_1^{m,n}$ will obey exactly the same bound that T_w^t obeys in $L^2(\mathbb{R})$, because the Haar shift multipliers have L^2-norm less than or equal to one.

Another important problem that has been of concern to the harmonic analysts in the last 30 years is to find necessary and sufficient conditions for operators to be bounded from $L^2(u)$ into $L^2(v)$ where u and v are two weights. In fact one of the biggest open problems in the field nowadays is to find necessary and sufficient conditions for the boundedness of the Hilbert transform in the two weights setting.

Sawyer provided in 1982, [S] conditions that are necessary and sufficient for the boundedness of the maximal function from $L^p(u)$ into $L^p(v)$. His result states that it is enough to test the boundedness of the operators in the class of functions $u^{-1}\chi_I$, for an interval $I \subset \mathbb{R}$. Later in [S1], Sawyer proved a certain operator T_0 with positive kernel is bounded from $L^2(u)$ into $L^2(v)$ if and only if satisfies some type of similar conditions.

In 1999, Nazarov, Treil and Volberg presented necessary and sufficient conditions for the boundedness of the martingale transform and the square function from $L^2(u)$ into $L^2(v)$. For the martingale transform they proved the boundedness reducing the problem to analyze the boundedness of an operator T_0 with positive kernel, these conditions are Sawyer type conditions, i.e., we need to test the boundedness of the operator in a class of simple test functions.

We will prove in this dissertation that the dyadic paraproduct π_b is bounded from $L^2(u)$ into $L^2(v)$ for all b in a certain class, that we will call two weighted Carleson class u, v, and the weights u and v satisfiying certain condition if and only the pair of weights is in joint Muckenhoupt A_2^d. Let us be more precise.

A pair of weights, (u, v) belongs to the joint Muckenhoupt A_p-class if and only if

$$[u, v]_{A_p} := \sup_I \left(\frac{1}{|I|}\int_I v(x)\,dx\right)\left(\frac{1}{|I|}\int_I u^{-\frac{1}{p-1}}(x)\,dx\right)^{p-1} < \infty,$$

where $[u, v]_{A_p}$ denotes the A_p-characteristic of the weight. When $u = v = w$ this recovers the classical A_p-class of weights and $[w, w]_{A_p} = [w]_{A_p}$.

We say that a locally integrable function b belongs to two weighted Carleson class u, v, $Carl_{u,v}$ if there exists C such that for all dyadic intervals J,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{|b_I|^2}{m_I v} \le C\, m_J(u^{-1}),$$

where $b_I = \langle b, h_I\rangle$.

Theorem 1.5. *Let (u, v) be a pair of weights such that v is a regular weight and u^{-1} is also a regular weight and there exists B such that for all dyadic intervals J,*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \{|\Delta_I v|^2 |I| m_I(u^{-1})\} \le B m_J v.$$

Then π_b is bounded from $L^2(u)$ into $L^2(v)$ for all $b \in Carl_{u,v}$ if and only if $(u, v) \in A_2^d$.

By regularity of the weight we mean that the mass of the weight over both half lines should be infinity. One would like to find conditions for boundedness of the paraproduct from $L^2(u)$ into $L^2(v)$ with the minimal requirement possible on the weights, in our result regularity is the minimum that we can ask, this is a very mild condition, we will show in Chapter 2 that regularity condition is a weaker condition than the weight being doubling.

The last result that we will prove is a connection between the two weight boundedness of the maximal function and the dyadic square function. The dyadic maximal function is the operator defined as

$$S^d f(x) := \left(\sum_{I\in\mathcal{D}} |m_I f - m_{\hat{I}} f|^2 \chi_I(x)\right)^{\frac{1}{2}}$$

Theorem 1.6. *Let (u, v) be a pair of weight such that $v \in A^d_\infty$ and the Maximal function M is bounded from $L^2(u)$ to $L^2(v)$ then there exists $C > 0$, such that*

$$\|S^d f\|_{L^2(v)} \leq C\|f\|_{L^2(u)}.$$

This dissertation is organized as follows. In Chapter 2 we provide the basic definitions and basic results that will be used throughout this manuscript. In Chapter 3 we will prove the lemmas that are essential for the main results. In Chapter 4 we will discuss the different definitions of dyadic shift with complexity and some of the results known for these operators. We will also categorize the Generalized Haar shifts in four groups and show how these categories are related to each other. In Chapter 5 we will prove the main estimate for the dyadic paraproduct with complexity (m, n) (particular case of a dyadic shift of type 2) and we will provide a new proof of the linear bound for the dyadic paraproduct. In Chapter 6 we will prove the main estimate for the composition of a dual dyadic paraproduct and a dyadic paraproduct with complexity (m, n), particular case of a dyadic shift of type 4. In Chapter 6 we will also prove the main estimate for the t-Haar multipliers with complexity (m, n), we also provide necessary conditions for these operators to be bounded in $L^p(\mathbb{R})$, for $1 < p < \infty$. In Chapter 7 we will discuss some of the two weighted theory for dyadic operators and prove the main result of the dissertation, we will give conditions on u, v and b such that the dyadic paraproduct, π_b, is bounded from $L^2(u)$ into $L^2(v)$ if and only if $(u, v) \in A_2$.

Chapter 2

Preliminares

In this chapter we will review some basic definitions and introduce the notation that we will use throughout this dissertation. We will work on the Euclidean space $\mathbb{R}$, but most of the results presented here will also hold for the Euclidean space $\mathbb{R}^n$ and more generally for metric spaces with geometric doubling. All functions will be real valued $f : \mathbb{R} \to \mathbb{R}$. Given a measurable set E, $|E|$ will denote the Lebesgue measure of this set and the Lebesgue measure will be denoted by dx. For a bounded operator $T : X \to Y$,X, Y Banach spaces, the operator norm will be denoted by $\|T\|_{X\to Y}$, when $X = Y$ we may use the notation $\|T\|_X$.

Unless specified, p and q represent real numbers larger or equal than 1, $1 \leq p, q < \infty$, L^p will denote the Banach function space, $L^p(dx, \mathbb{R})$, with norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

A weight, w, is a locally integrable function in $\mathbb{R}$ that takes values in $(0, \infty)$ almost everywhere. The w-measure of a measurable set E, denoted by $w(E)$, is

$$w(E) = \int_E w(x) dx.$$

We also define $L^p(w)$ as the Banach function space, $L^p(d\mu, \mathbb{R})$ for $d\mu = wdx$, w is the Radon-Nikodym derivative of μ. The norm is defined by

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)dx \right)^{\frac{1}{p}}.$$

We denote

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx$$

the standard L^2- inner product on $\mathbb{R}$ and

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x)g(x)w(x)dx$$

the inner product in the weighted $L^2(w)$ space on $\mathbb{R}$.

For a measure σ and a set E, we define

$$\sigma(E) = \int_E d\sigma.$$

Let f be a locally integrable function, we define $m_E^\sigma f$ as the σ-average of f on E,

$$m_E^\sigma f := \frac{1}{\sigma(E)} \int_E f(x)d\sigma.$$

In the case that we are working with the Lebesgue measure the average will be denoted simply by $m_E f$,

$$m_E f := \frac{1}{|E|} \int_E f(x)dx.$$

2.1 Dyadic grid in $\mathbb{R}$

We will work in the dyadic setting $\mathcal{D}$, where $\mathcal{D}$ is the collection of dyadic intervals

$$\mathcal{D} := \{I \subset \mathbb{R} \ : \ I = [k2^{-j}, (k+1)2^{-j}), \ k, j \in \mathbb{Z}\},$$

for a dyadic interval L we define as $\mathcal{D}(L)$ the collection of dyadic intervals inside L

$$\mathcal{D}(L) := \{I \subset L \ : \ I \in \mathcal{D}\}$$

we also define the collection of dyadic intervals with length 2^{-j}, $\mathcal{D}_j$

$$\mathcal{D}_j := \{I \in \mathcal{D} \ : \ |I| = 2^{-j}\},$$

the cubes in $\mathcal{D}_j$ are called the *j-th generation*.

Combining the last two definitions we define as $\mathcal{D}_j(L)$ the collection of dyadic intervals inside the dyadic interval L with length equal to $2^{-j}|L|$

$$\mathcal{D}_j(L) := \{I \in \mathcal{D}(L) \ : \ |I| = 2^{-j}|L|\},$$

the intervals in $\mathcal{D}_j(L)$ are called the *j-th generation of L*.

Properties of the dyadic grids

- Any two dyadic interval $I, J \in \mathcal{D}$ are either disjoint or one is contained in the other. Any two distinct dyadic intervals $I, J \in \mathcal{D}_j(L)$ are disjoint.
- Each dyadic interval I is in an unique generation $\mathcal{D}_j$ and there are exactly 2 subsets of I in the next generation $\mathcal{D}_{j+1}$. Also, each dyadic interval $I \subset L$ is in an unique generation $\mathcal{D}_j(L)$ and there are exactly 2 subsets of I in the next generation $\mathcal{D}_{j+1}(L)$.
- The subsets of a dyadic intervals I that are in $\mathcal{D}_1(I)$ are called the children of I. We denote the children of the interval I by I_+ and I_-, where I_+ will always denote the right half part of I and I_- denotes the left half part of I. Since the children of I form a partition of I we have that $I = I_+ \bigcup I_-$.
- For every dyadic interval $I \in \mathcal{D}_j$ there is exactly one $\widehat{I} \in \mathcal{D}_{j-1}$, such that $I \subset \widehat{I}$, $\widehat{I}$ is the called the parent of I. Also, there is exactly 1 dyadic interval in $\mathcal{D}_1(\hat{I}) \setminus \{I\}$, this interval will be called the sibling of I, which is denoted by I^*, $I^* = \widehat{I} \setminus I$.

- For any j, the collection $\mathcal{D}_j$ forms a partition of the real line and for any dyadic cube L the collection $\mathcal{D}_j(L)$ forms a partition of L. In particular, the collection of children of I, $\mathcal{D}_1(I)$, is a partition of I.

A weight w is *dyadic doubling* if $\frac{w(\hat{I})}{w(I)} \leq C$ for all $I \in \mathcal{D}$. The smallest constant C is called the doubling constant of w and it is denoted by $D(w)$.

Remark 2.1. *Note that $D(w) \geq 2$, and that in fact the ratio between the length of a child and the length of its parent is comparable to one, more precisely, $D(w)^{-1} \leq \frac{w(I)}{w(\hat{I})} \leq 1 - D(w)^{-1}$.*

A weight w is *regular* if $w(\mathbb{R}^+) = w(\mathbb{R}^-) = \infty$. This condition is weaker than doubling in the sense that if a weight is doubling then the weight is regular, we can show this using the remark above. Consider the dyadic interval $I_n = [0, 2^n)$, then

$$\frac{w(I_0)}{w(\widehat{I_0})} = \frac{w(I_0)}{w(I_1)} \leq 1 - D(w)^{-1},$$

let $r^{-1} = 1 - D(w)^{-1}$, then $r > 1$ and $rw(I_0) < w(I_1)$. Iterating n times we obtain that $r^n w(I_0) < w(I_n)$. Since $r > 1$ and $w(I_0) > 0$, for any $M > 0$ we can choose n such that $r^n w(I_1) > M$. Now observe that $w(I_n) \leq w(\mathbb{R}^+)$ for all $n > 0$. Therefore $M < w(\mathbb{R}^+)$ for any M positive, i.e. $w(\mathbb{R}^+) = \infty$. Analogously we can show that $w(\mathbb{R}^-) = \infty$, so the weight is regular.

In order to show that regularity does not imply doubling consider the weight,

$$w(x) = \chi_{\mathbb{R}^-}(x) \sum_{n=0}^{\infty} 2^n \chi_{[2n, 2n+2-\frac{1}{2^n})}(x) + \chi_{[2n+2-\frac{1}{2^n}, 2n+2)}(x),$$

where $\chi_I(x) = 1$ if $x \in I$ and zero otherwise. The weight w is regular since clearly $v(\mathbb{R}^-) = v(\mathbb{R}^+) = \infty$. However this weight is not doubling, let $A_n = [2n + 2 - \frac{1}{2^n}, 2n + 2)$, then $\hat{A}_n = [2n + 2 - \frac{1}{2^{n-1}}, 2n + 2)$ and

$$\frac{w(\hat{A}_n)}{w(A_n)} = \frac{\frac{2^n+1}{2^n}}{\frac{1}{2^n}} = 2^n + 1,$$

which implies that w is not doubling.

This provides an "easy" divergence test to decide whether a weight is doubling or not.

2.2 Weighted Haar funtions

For a given weight v and an interval I define the *weighted Haar function* as

$$h_I^v(x) = \frac{1}{\sqrt{v(I)}} \left(\sqrt{\frac{v(I_-)}{v(I_+)}}\, \chi_{I_+}(x) - \sqrt{\frac{v(I_+)}{v(I_-)}}\, \chi_{I_-}(x) \right), \tag{2.1}$$

where χ_I is the characteristic function in the interval I, $h_I^v(x)$ can also be written as

$$h_I^v(x) = \begin{cases} \frac{1}{\sqrt{v(I)}} \sqrt{\frac{v(I_-)}{v(I_+)}}, & x \in I_+ \\ \frac{-1}{\sqrt{v(I)}} \sqrt{\frac{v(I_+)}{v(I_+)}}, & x \in I_- \\ 0, \quad \text{otherwise} \end{cases}$$

If v is the Lebesgue measure over $\mathbb{R}$, we will denote the Haar function simply by $h_I(x)$, and for any $I \in \mathcal{D}$

$$h_I(x) = \begin{cases} \frac{1}{\sqrt{|I|}}, & x \in I_+ \\ \frac{-1}{\sqrt{|I|}}, & x \in I_- \\ 0, \quad \text{otherwise} \end{cases}$$

It is an important fact that $\{h_I^v\}_{I\in\mathcal{D}}$ is an orthonormal system in $L^2(v)$ and $h_I^v(x)$ is constant in each each children of I. Also, h_I^v has v-mean zero over the line, $\int_{\mathbb{R}} h_I^v(x)v(x)dx = 0$. For any weight v any dyadic cube I we have that

$$\begin{aligned} \int_{\mathbb{R}} h_I^v(x)\, dv = \int_I h_I^v(x)\, dv &= \int_{I_+} \frac{1}{v(I)^{\frac{1}{2}}} \frac{v(I_-)^{\frac{1}{2}}}{v(I_+)^{\frac{1}{2}}} dv - \int_{I_-} \frac{1}{v(I)^{\frac{1}{2}}} \frac{v(I_+)^{\frac{1}{2}}}{v(I_-)^{\frac{1}{2}}} dv \\ &= \frac{\sqrt{v(I_-)v(I_+)} - \sqrt{v(I_+)v(I_-)}}{\sqrt{v(I)}} = 0. \end{aligned}$$

so, $h_I^v(x)$ has v-mean zero over the line, in fact, $h_I^v(x)$ has mean zero over any set that contains I. Moreover, $\|h_I^v\|_{L^2(v)} = 1$.

$$\|h_I^v\|_{L^2(v)} = \int_{\mathbb{R}} |h_I^v(x)|^2 \, dv = \int_{I_+} \frac{1}{v(I)} \frac{v(I_-)}{v(I_+)} dv + \int_{I_-} \frac{1}{v(I)} \frac{v(I_+)}{v(I_-)} dv$$
$$= \frac{v(I_-) + v(I_+)}{v(I)} = 1.$$

If the weight v is not regular, then $v([0,\infty))$ or $v((-\infty,0])$ is finite, maybe even both. If $v([0,\infty)) < \infty$ then $\frac{\chi_{[0,\infty)}}{\sqrt{v([0,\infty))}}$ will be orthonormal to h_I^v for all dyadic interval I, and normalized in $L^2(v)$, then in order to have a complete orthonormal system in $L^2(v)$ we will need to include it. The same occurs if $v((-\infty,0]) < \infty$, we would have to include $\frac{\chi_{(-\infty,0])}}{\sqrt{v((-\infty,0])}}$ in order to have an orthonormal basis in $L^2(v)$. Note that when $v = 1$, then $|\mathbb{R}^{\pm}| = \infty$ and we do not have this issue, likewise when v is doubling, $v(\mathbb{R}^{\pm}) = \infty$.

It is a simple exercise to verify that the weighted and unweighted Haar functions are related linearly as follows,

Proposition 2.2. *For any weight v, there are numbers α_I^v, β_I^v such that*

$$h_I(x) = \alpha_I^v \, h_I^v(x) + \beta_I^v \, \frac{\chi_I(x)}{\sqrt{|I|}} \tag{2.2}$$

where (i) $|\alpha_I^v| \le \sqrt{m_I v}$, (ii) $|\beta_I^v| \le \frac{|\Delta_I v|}{m_I v}$, and $\Delta_I v := m_{I_+} v - m_{I_-} v$.

Proof. In order to find β_I^v we can multiply equation (2.2) by $\frac{\chi_I}{\sqrt{|I|}}$ and integrate with respect vdx we will obtain

$$\frac{m_{I_+} v - m_{I_-} v}{2} = \beta_I^v m_I v \Rightarrow \beta_I^v = \frac{1}{2} \frac{\Delta_I v}{m_I v}. \tag{2.3}$$

Therefore $|\beta_I| \le \frac{\Delta_I v}{m_I v}$. Also note that

$$\frac{|\Delta_I v|}{m_I v} = \frac{|m_{I_+} v - m_{I_-} v|}{m_I v} \le \frac{m_{I_+} v + m_{I_-} v}{m_I v} = \frac{2 m_I v}{m_I v} = 2.$$

Thus $\beta_I^v \leq 1$ for all $I \in \mathcal{D}$. Now in order to find α_I^v we will compute the $L^2(v)$ norm $h_I(x)$ in both sides of equation (2.2).

$$m_I v = \|h_I\|_{L^2(v)} = \left\| \alpha_I^v h_I^v + \beta_I^v \frac{\chi_I}{\sqrt{|I|}} \right\|_{L^2(v)} = (\alpha_I^v)^2 + (\beta_I^v)^2 m_I v \Rightarrow$$

$$\big(1 - (\beta_I^v)^2\big) m_I v = (\alpha^v)^2 \Rightarrow |\alpha_I^v| = \sqrt{\big(1 - (\beta_I^v)^2\big) m_I v} \tag{2.4}$$

Since $|\beta_I^v| \leq 1 \; \forall\, I \in \mathcal{D}$ then $1 - (\beta_I^v)^2 \leq 1$ which implies that $|\alpha^v| \leq \sqrt{m_I v}$. Note that if we plug equation (2.3) in (2.4) we have that

$$|\alpha_I^v| = \sqrt{m_I v - \frac{(m_{I_+} v - m_{I_-} v)^2}{m_I v}}. \tag{2.5}$$

□

2.3 The dyadic Muckenhoupt Class - A_p^d

Definition 2.3. *For $1 < p < \infty$, a weight w is in A_p^d, if*

$$[w]_{A_p^d} := \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I w(x)^{\frac{-1}{p-1}} dx \right)^{p-1} < \infty$$

Remark 2.4. *The characteristic of a weight in the A_2^d class is*

$$[w]_{A_2^d} = \sup_{I \in \mathcal{D}} m_I w \; m_I w^{-1}.$$

This class of weights is the dyadic analogue of the Muckenhoupt class A_p, where the supremum is taken over all intervals in $\mathbb{R}$. The constant $[w]_{A_p^d}$ is called the A_p^d characteristic of w.

It follows from Hölder's inequality that $1 \leq [w]_{A_p^d}$, for all $p > 1$, in order to let the reader be familiar with this kind of calculation we show it below, however later in the text, for similar calculations, we will just say that follows by Hölder's inequality.

For any $I \in \mathcal{D}$

$$\begin{aligned}
1 &= \left(\frac{|I|}{|I|}\right)^p = \left(\frac{1}{|I|}\int_I w^{\frac{1}{p}}(x)w^{\frac{-1}{p}}(x)dx\right)^p \\
&\leq \left(\left(\frac{1}{|I|}\int_I w^{\frac{p}{p}}(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|I|}\int_I w^{\frac{p'}{p}}(x)dx\right)^{\frac{1}{p'}}\right)^p \\
&= \left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{\frac{-1}{p-1}}dx\right)^{p-1} \leq [w]_{A_p^d}
\end{aligned}$$

Also follows by Hölder's inequality that if $w \in A_p^d$ then $w \in A_q^d$ for $1 < p \leq q < \infty$, i.e. $A_p^d \subseteq A_q^d$, moreover

$$[w]_{A_q^d} \leq [w]_{A_p^d}.$$

Proposition 2.5. *Let $w \in A_p^d$, for $p > 1$, then it follows that*

1) $w^{\frac{1}{p}} \in A^d_{2-\frac{1}{p}}$ and $[w]_{A^d_{2-\frac{1}{p}}} \leq [w]^{\frac{1}{p}}_{A_p^d}$.

2) w is a dyadic doubling weight and $D(w) \leq 2^p[w]_{A_p^d}$.

Proof. Note that,

$$\begin{aligned}
(m_I w^{\frac{1}{p}})\left(m_I (w^{\frac{1}{p}})^{\frac{-1}{2-\frac{1}{p}-1}}\right)^{2-\frac{1}{p}-1} &= (m_I w^{\frac{1}{p}})\left(m_I w^{\frac{-1}{p-1}}\right)^{1-\frac{1}{p}} \\
&= (m_I w^{\frac{1}{p}})\left(m_I w^{\frac{-1}{p-1}}\right)^{\frac{p-1}{p}} \leq (m_I w)^{\frac{1}{p}}\left(m_I w^{\frac{-1}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \left(m_I w \left(m_I w^{\frac{-1}{p-1}}\right)^{p-1}\right)^{\frac{1}{p}}
\end{aligned}$$

where the inequality in the last line follow by Hölder's inequality. Therefore if we assume that $w \in A_p^d$ then it follows that $w^{\frac{1}{p}} \in A_{2-\frac{1}{p}}$, since

$$\begin{aligned}
[w]_{A^d_{2-\frac{1}{p}}} &= \sup_I (m_I w^{\frac{1}{p}})\left(m_I (w^{\frac{1}{p}})^{\frac{-1}{2-\frac{1}{p}-1}}\right)^{2-\frac{1}{p}-1} \leq \sup_I (m_I w)^{\frac{1}{p}} m_I\left(w^{\frac{-1}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \sup_{I\in\mathcal{D}}\left(m_I w \left(m_I w^{\frac{-1}{p-1}}\right)^{p-1}\right)^{\frac{1}{p}} = [w]^{\frac{1}{p}}_{A_p^d}.
\end{aligned}$$

In order to prove the second part of the proposition, we will just use the definition of dyadic doubling constant.

$$D(w) = \sup_{I\in\mathcal{D}} \frac{w(\widehat{I})}{w(I)} \leq \sup_{I\in\mathcal{D}} \frac{|\widehat{I}|[w]_{A_p^d}\left(m_{\widehat{I}}\left(w^{-\frac{1}{p-1}}\right)\right)^{1-p}}{|I|\left(m_I\left(w^{-\frac{1}{p-1}}\right)\right)^{1-p}}$$
$$= [w]_{A_p^d} \sup_{I\in\mathcal{D}} \left(\frac{|\widehat{I}|}{|I|}\right)^p \left(\frac{\int_I w^{-\frac{1}{p-1}}}{\int_{\widehat{I}} w^{-\frac{1}{p-1}}}\right)^{p-1}.$$

However since w is positive a.e. and $I \subset \widehat{I}$, then $\dfrac{\int_I w^{-\frac{1}{p-1}}}{\int_{\widehat{I}} w^{-\frac{1}{p-1}}} \leq 1$, thus

$$D(w) \leq [w]_{A_p^d} \sup_{I\in\mathcal{D}} \left(\frac{|\widehat{I}|}{|I|}\right)^p \leq 2^p [w]_{A_p^d}.$$

□

In particular if w is in A_p^d then w is regular, because w is doubling.

Definition 2.6. *A weight w is in A_∞^d if*

$$[w]_{A_\infty^d} := \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|}\int_I w(x)dx\right) \exp\left(\frac{1}{|I|}\int_I \ln(w^{-1}(x))dx\right) \leq \infty. \tag{2.6}$$

The quantity defined above is called the A_∞^d-characteristic of w, it follows from Jensen's inequality that for any $1 \leq p < \infty$,

$$[w]_{A_\infty^d} \leq [w]_{A_p^d}.$$

The class A_∞ is defined in a similar fashion, with the supremum taken over all intervals I.

Definition 2.7. *A weight is in A_1^d if there exist $C > 0$ such that*

$$\frac{1}{|I|}\int_I w(y)dy \leq Cw(x)$$

a.e. *on I, for all dyadic intervals I. The smallest C that satisfies this condition is defined as the A_1^d-characteristic of w, denoted by $[w]_{A_1^d}$.*

The class A_1 is the analogue class where $\frac{1}{|I|}\int_I w(x)dx \leq Cw(x)$ *a.e.*, for all intervals I.

The classes of weights A_1 and A_∞ are considered the limit cases of the class A_p. This is because

$$\lim_{p\to\infty}\left(\frac{1}{|I|}\int_I w^{\frac{-1}{p-1}}(x)dx\right)^{p-1} = e^{-\int_I \ln w(x)\, dx},$$

which implies that if $w \in A_p$ then $w \in A_\infty$. It is also true that if $w \in A_\infty$ then $w \in A_p$ for some p, [CoFe]. Therefore

$$A_\infty = \bigcup_{p>1} A_p;$$

and

$$\lim_{p\to 1}\left(\frac{1}{|I|}\int_I w^{\frac{-1}{p-1}}(x)dx\right)^{p-1} = \|w^{-1}\|_{L^\infty},$$

which implies that $A_1 \subset A_p$ for all $p > 1$.

Remark 2.8. *Note that if $w \in A_\infty^d$ then $w \in A_p^d$ for some $p > 1$ and thus by Proposition 2.5, w is a dyadic doubling weight,and therefore the weight w is regular.*

2.4 The dyadic Reverse Hölder Class - RH_p^d

Definition 2.9. *A weight w is in RH_p^d , $1 < p < \infty$, if*

$$[w]_{RH_p^d} := \sup_{I\in\mathcal{D}}\left(\frac{1}{|I|}\int_I w(x)^p dx\right)^{\frac{1}{p}}\left(\frac{1}{|I|}\int_I w(x)dx\right)^{-1} < \infty$$

This class of weights is the dyadic analogous of the reverse Hölder RH_p, where the supremum is taken over all intervals in $\mathbb{R}$.

Remark 2.10. *The characteristic of a weight in the RH_2^d class is*

$$[w]_{RH_2^d}^2 = \sup_{I\in\mathcal{D}} \frac{m_I(w^2)}{(m_I w)^2}.$$

In the case of $p = 1$,

Definition 2.11. *A weight w belongs to the Reverse Hölder 1 class, RH_1^d, if*

$$[w]_{RH_1^d} := \sup_{I\in\mathcal{D}} m_I\left(\frac{w}{m_I w}\log\frac{w}{m_I w}\right)$$

Note if $w \in RH_p^d$ for some $p > 1$ then there exist C such that$(m_I w^p)^{\frac{1}{p}} \leq C m_I w$, for all $I \in \mathcal{D}$. Then for any $1 < q \leq p$, using Hölder's inequality we would have

$$(m_I w^q)^{\frac{1}{q}} \leq C m_I w.$$

Therefore if $1 < p \leq q$, then $RH_q^d \subseteq RH_p^d$ and

$$1 \leq [w]_{RH_p^d} \leq [w]_{RH_q^d}.$$

If we start with $w \in RH_p^d$ then $w \in RH_q^d$ for $1 < q \leq p$. A much deeper result, Gehring's theorem, is that there exists $\epsilon > 0$ such that $w \in RH_{p+\epsilon}^d$.

Theorem 2.12 (Gehring [Ge])**.** *If $w \in RH_p^d$ for some $1 < p < \infty$. Then there exists $\epsilon > 0$, depending only in p and RH_p^d characteristic if w such that*

$$w \in RH_{p+\epsilon}^d.$$

This result was proved in [Ge] for Lebesgue measure. For a proof of Gehring's Lemma for a general measure see [Pa].

The nondyadic version of the theorem was discovered by Gehring while studing quasiconformal mappings, see [Ge], the theorem states that RH_p classes are self-improving.

The next theorem relates the RH_p^d class with the A_∞^d class.

Theorem 2.13. *For $p > 1$ we have the following*

- *If $w \in A_\infty^d$ then $w^{\frac{1}{p}} \in RH_p^d$*
- *If $w \in RH_p^d$ and w is a doubling weight then $w \in A_\infty^d$.*

This theorem first appeared in [Bu1], the proof can be found in [KP]. These properties first appeared for the continuous Muckenhoupt and reverse Hölder classes in [CoFe].

Proposition 2.14. *Let w be a weight, then for any two real numbers p and r, such that $1 < p < \infty$ and $1 \leq r < \infty$, then*

1) $w^r \in A_p^d \quad \Longleftrightarrow \quad w \in RH_r^d \bigcap A^d_{\frac{p+r-1}{r}}$.

2) $[w^r]_{A_p^d} \leq [w]^r_{RH_r^d}[w]^r_{A^d_{\frac{p+r-1}{r}}}$,

3) $[w]_{RH_r^d} \leq [w^r]^{\frac{1}{r}}_{A_p^d}$,

4) $[w]^r_{A^d_{\frac{p+r-1}{r}}} \leq [w^r]^{\frac{1}{r}}_{A_p^d}$.

A proof of these statements can be found in [Be], Lemma 2.5.

Remark 2.15. *For $p = r$ we would have*

1') $w^p \in A_p^d \quad \Longleftrightarrow \quad w \in RH_p^d \bigcap A^d_{2-\frac{1}{p}}$,

2') $[w^p]_{A_p^d} \leq [w]^p_{RH_p^d}[w]^p_{A^d_{2-\frac{1}{p}}}$,

3') $[w]_{RH_p^d} \leq [w^p]^{\frac{1}{p}}_{A_p^d}$,

4') $[w]_{A^d_{2-\frac{1}{p}}} \leq [w^p]^{\frac{1}{p}}_{A_p^d}$.

2.5 C_s^d - Condition

The C_s^d was first defined in [KP]. Even though the condition is equivalent to a reverse Hölder condition for $s > 1$ and a Muckenhoupt type condition for $s < 0$, its definition is interesting because it simplifies notation when working with the type of Haar multiplier defined in [KP], which here we will call t-Haar multipliers.

Definition 2.16. *A weight w satisfies the C_s^d condition, for $s \in \mathbb{R}$, if*

$$[w]_{C_s^d} := \sup_{I \in \mathcal{D}} \frac{m_I(w^s)}{(m_I w)^s} < \infty.$$

The quantity defined above is called the C_s^d-characteristic of w. Let us analyze this definition.

For $0 \leq s \leq 1$, we have that any weight satisfies the condition and its characteristic is 1, this is just a consequence of Hölder's Inequality.

When $s > 1$, the condition is analogous to the reverse Hölder condition and

$$[w]_{C_s^d}^{\frac{1}{s}} = \sup_{I \in \mathcal{D}} \frac{m_I(w^s)^{\frac{1}{s}}}{(m_I w)} = [w]_{RH_s^d}.$$

For $s < 0$, we have that

$$[w]_{C_s^d}^{-\frac{1}{s}} = \sup_{I \in \mathcal{D}} (m_I w^s)^{-\frac{1}{s}} m_I w = [w]_{A^d_{1-\frac{1}{s}}}.$$

so, $w \in C_s^d \iff w \in A^d_{1-\frac{1}{s}}$. Moreover

$$[w]_{C_s^d} = [w]^d_{A_{1-\frac{1}{s}}}$$

or, alternatively

$$[w]_{A_s^d}^{\frac{1}{s-1}} = [w]^d_{C^d_{\frac{1}{1-p}}}.$$

2.6 Dyadic BMO

Definition 2.17. *A locally integrable function* b *has dyadic bounded mean oscillation,* $b \in BMO^d$*, if and only if*

$$\|b\|_{BMO_1^d} := \sup_{I\in\mathcal{D}} \frac{1}{|I|}\int_I |b(x) - m_I b|dx < \infty \tag{2.7}$$

Note that if $b(x)$ is constant almost everywhere then $\|b\|_{BMO_1^d} = 0$, therefore $\|\cdot\|_{BMO_1^d}$ is not a well defined norm. However if we consider this BMO^d the space of all locally integrable functions that satisfies (2.7) modulo constants then BMO^d is a Banach space and $||\cdot||_{BMO_1^d}$ is a norm.

The space BMO^d is a larger space than L^∞, $L^\infty \subsetneq BMO^d$. The classical example of a function in BMO^d that is not in L^∞ is $f(x) = \ln|x|$. In fact the famous John-Nirenberg inequality states that the only type of singularities that a function in BMO^d is allowed to have is of $\ln|x|$ type. Remember that the weight $w = |x|^\alpha$ is in A_∞^d if $-1 < \alpha$, now note that $\ln w = \alpha \ln|x|$ which belong to BMO^d. It is, in fact, true that if $w \in A_\infty^d$ then $\ln w$ is in BMO^d, for more detail we refer to [P1], Theorem 3.5.

Theorem 2.18 (John-Nirenberg Inequality)**.** *Given a function* b *in* BMO^d*, any dyadic interval* $I \in \mathcal{D}$ *and a positive number* $\lambda > 0$*, then there are positive constants* C_1*,* C_2 *that are independent of* b*,* I *and* λ*, such that*

$$|\{x \in I : |b(x) - m_I b| > \lambda\}| \leq C_1|I|e^{-\frac{C_2\lambda}{\|b\|_{BMO_1^d}}}.$$

A stopping time proof of this theorem and the next corollary can be found in [P], page 28.

Corollary 2.19 (Self-improvement)**.** *Given a function* b *in* BMO^d*, then for all*

$p > 1$ there exists a constant $C_p > 0$ such that for all dyadic intervals $I \in \mathcal{D}$

$$\left(\frac{1}{|I|}\int_I |b(x) - m_I b|^p dx\right)^{\frac{1}{p}} \leq C_p ||b||_{BMO_1^d}.$$

Note that using Hölder's inequality we also have that

$$\frac{1}{|I|}\int_I |b(x) - m_I b| dx \leq \left(\frac{1}{|I|}\int_I |b(x) - m_I b|^p dx\right)^{\frac{1}{p}}.$$

Therefore

$$\|b\|_{BMO_p^d} := \left(\frac{1}{|I|}\int_I |b(x) - m_I b|^p dx\right)^{\frac{1}{p}} \sim ||b||_{BMO_1^d}. \tag{2.8}$$

Therefore we have that for any $p > 1$, (2.8) provides an alternative definition for a norm in BMO^d. It will be convenient for us to define the BMO^d norm using this alternative definition $p = 2$, namely

$$\|b\|^2_{BMO^d} := \|b\|^2_{BMO_2^d} := \sup_{I\in\mathcal{D}} \frac{1}{|I|}\int_I |b(x) - m_I b|^2 dx \tag{2.9}$$

The reason that this definition for the BMO^d is preferred is because for any dyadic interval J

$$\int_J |b(x) - m_J b|^2 dx = \sum_{I\in\mathcal{D}(J)} |\langle b, h_I\rangle|^2 dx \tag{2.10}$$

and therefore, ultimately, we will have that

$$\|b\|_{BMO^d} = \left(\sup_{J\in\mathcal{D}} \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} |\langle b, h_I\rangle|^2\right)^{\frac{1}{2}} < \infty. \tag{2.11}$$

The equality in (2.10) follows just from the fact that $\{h_I\}_{I\in\mathcal{D}(J)}$ form a orthonormal basis for $\{f \in L_2(J) : m_I f = 0\}$.

Remark 2.20. *Note that if $b_I := \langle b, h_I\rangle$ then $\frac{|b_I|}{\sqrt{|I|}} \leq \|b\|_{BMO^d}$ $\forall\, I \in \mathcal{D}$.*

2.7 Carleson sequences

A positive sequence $\{\lambda_I\}_{I\in\mathcal{D}}$ is a v-Carleson sequence if there is a $C > 0$ such that for all dyadic intervals J

$$\sum_{I\in\mathcal{D}(J)} \lambda_I \leq Cv(J).$$

For the case that $v = 1$ almost everywhere we just say that the sequence is a Carleson sequence. Also note that if $\{\lambda_I\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity C, then if we divide both sides by $|J|$ then

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \lambda_I \leq Cm_J v.$$

We will more often use this definition of v-Carleson. The infimum among all C that satisfies this inequality is called the intensity of the v-Carleson sequence λ_I. Therefore if $b \in BMO^d$ then $\left\{|\langle b, h_I\rangle|^2\right\}_{I\in\mathcal{D}}$ is a Carleson sequence with intensity $\|b\|^2_{BMO^d}$.

Proposition 2.21. *Let v be a weight, $\{\lambda_I\}_{I\in\mathcal{D}}$ and $\{\gamma_I\}_{I\in\mathcal{D}}$ be two v-Carleson sequences with intensities A and B respectively then for any $c, d > 0$ we have that*

(i) $\{c\lambda_I + d\gamma_I\}_{I\in\mathcal{D}}$ *is a v-Carleson sequence with intensity at most $cA + dB$.*

(ii) $\{\sqrt{\lambda_I}\sqrt{\gamma_I}\}_{I\in\mathcal{D}}$ *is a v-Carleson sequence with intensity at most $\sqrt{AB}$.*

(iii) $\{(c\sqrt{\lambda_I} + d\sqrt{\gamma_I})^2\}_{I\in\mathcal{D}}$ *is a v-Carleson sequence with intensity at most $2c^2A + 2d^2B$.*

The proofs of these statements are quite simple. To prove the first one we just need properties of supremum, for the second one we just have to apply Cauchy-Schwarz and the third one is a consequence of the first two statements combined with the fact that $2cd\sqrt{A}\sqrt{B} \leq c^2A + d^2B$.

Remark 2.22. *Note that if b is in BMO^d and $\{\lambda_I\}_{I\in\mathcal{D}}$ is a Carleson sequence with intensity A, then $\{b_I\sqrt{\lambda_I}\}_{I\in\mathcal{D}}$ is a Carleson sequence with intensity $\|b\|_{BMO^d}\sqrt{A}$.*

Remark 2.23. *If v is not a regular weight then $v(\mathbb{R}^+) < \infty$ and/or $v(\mathbb{R}^-) < \infty$. in that case we can replace $\mathcal{D}$ by $\tilde{\mathcal{D}}$ where $\tilde{\mathcal{D}} = \mathcal{D}\bigcup\{\mathbb{R}^+\}$ if $v(\mathbb{R}^+) < \infty$, $\tilde{\mathcal{D}} = \mathcal{D}\bigcup\{\mathbb{R}^-\}$ if $v(\mathbb{R}^-) < \infty$ and $\tilde{\mathcal{D}} = \mathcal{D}\bigcup\{\mathbb{R}^+, \mathbb{R}^-\}$ if both are finite. We will say in this case that a sequence $\{\lambda_I\}_{I\in\mathcal{D}}$ is an extended v-Carleson sequence with intensity B if $\sum_{I\in\mathcal{D}(J)} \lambda_I \le B\, v(J)$ for all $J \in \tilde{\mathcal{D}}$.*

2.8 Maximal function

In this section we will define and state some important facts about the Hardy-Littlewood maximal function and its dyadic and weighted versions.

Definition 2.24. *The Hardy-Littlewood maximal function is defined as*

$$(Mf)(x) = \sup_{I\ni x} \frac{1}{|I|}\int_I f(x)dx \tag{2.12}$$

The next theorem was proved by Buckley in his PhD dissertation, [Bu2]. This was the first result showing the dependence in the A_p characteristic of a weight w of the $L^p(w)$ norm of an operator.

Theorem 2.25 (Buckley, [Bu2])**.** *Let $w \in A_p$ the the Hardy-Littlewood maximal function is bounded in $L^p(w)$ and satisfies the following estimate*

$$\|M\|_{L^p(w)} \le C_p[w]_{A_p}^{\frac{1}{p-1}} \tag{2.13}$$

Lerner later showed that $C_p = Cp^{\frac{1}{p'}}p'^{\frac{1}{p}}$, for some constant C where p' is the dual exponent of p.

In this dissertation will only be working with the dyadic maximal function, M^d, the definition is the same as the previous one, but the supremum is taken over dyadic intervals. Theorem 2.25 is also true if we change the maximal function by the dyadic maximal function and $[w]_{A_p}^{\frac{1}{p-1}}$ by $[w]_{A_p^d}^{\frac{1}{p-1}}$. Also we need to define the weighted dyadic maximal function.

Definition 2.26. *Let v be a weight, then we define the dyadic weighted maximal function M_v^d as follows*

$$(M_v^d f)(x) := \sup_{\substack{I \ni x \\ I \in \mathcal{D}}} \frac{1}{v(I)} \int_I f(x)v(x)dx \tag{2.14}$$

A very important fact about this operator is the following.

Lemma 2.27. *Let v be a locally integrable function such that $v > 0$ a.e.Then for all $1 < p < \infty$, M_v is bounded in $L^p(v)$. Moreover, for all $f \in L^p(v)$*

$$\|M_v f\|_{L^p(v)} \leq p' \|f\|_{L^p(v)}.$$

where p' is the dual exponent of p.

For a proof of this lemma see [CrMPz]. The important fact to note in the Lemma above is that the $L^p(v)$-norm of M_v is bounded just by p', there is no dependence on the weight v.

2.9 Dyadic Martingale transform

Definition 2.28. *Let $r(I)$ be a function from $\mathcal{D}$ into $\{-1, 1\}$, then we define the martingale transform as*

$$T_r f(x) = \sum_{I \in \mathcal{D}} r(I) \langle f, h_I \rangle h_I(x)$$

The next theorem was proved by Wittwer in [W]. This was the second result showing the dependence in the A_2 characteristic of a weight w of the $L^2(w)$ norm of an operator.

Theorem 2.29 (Wittwer, [W]). *For all $w \in A_2^d$ and all $f \in L^2(w)$, there exists a constant C, independent of r, such that*

$$\|T_r f\|_{L^2(w)} \leq C[w]_{A_2^d}\|f\|_{L^2(w)}$$

By Dragičević-Grafakos-Pereyra-Petermichl Sharp extrapolation Theorem 3.17, [DGPPet] we have the following corollary

Corollary 2.30. *For all $w \in A_p^d$ and all $f \in L^p(w)$, there exists a constant C, independent of r, such that*

$$\|T_r\|_{L^p(w)} \leq C[w]_{A_p^d}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}$$

2.10 Dyadic Paraproduct

One of the main operators that we will work with in this dissertation is dyadic paraproduct defined below.

Definition 2.31. *We define the dyadic paraproduct as the following operator*

$$(\pi_b f)(x) = \sum_{I\in\mathcal{D}} c_I \, m_I f \, \langle b, h_I\rangle h_I(x) \tag{2.15}$$

with $|c_I| \leq 1$.

The dyadic paraproduct is bounded L^p, [Fi]. It is also bounded in $L^2(w)$, for a proof see [C]. Beznosova proved in [Be1] that the bound of the $L^2(w)$ norm of the paraproduct depends linearly on $[w]_{A_2}$.

Theorem 2.32 (Beznosova, [Be1]). *There exists $C > 0$, such that for all $b \in BMO^d$ and for all $w \in A_2^d$*

$$\|\pi_b\|_{L^2(w)} \leq C[w]_{A_2^d}\|b\|_{BMO^d}$$

Beznosova result is for the field where the Hilbert Space $L^2(w)$ is defined over $\mathbb{R}$, the extension of this result to $L^2(w)$ over $\mathbb{R}^k$ is due to Chung, [Ch] and Cruz-Uribe, Martell and Pérez, [CrMPz]. Rubio de Francia Extrapolation theorem [Ru], give us boundedness in $L^p(w)$ for all $w \in A_p$ and Dragičevič *et al.* Sharp extrapolation Theorem 3.17, [DGPPet], will give us that, if $w \in A_p^d$ then $\|\pi_b\|_{L^p(w)\to L^p(w)} \leq C[w]_{A_p^d}^{\max\{1,\frac{1}{p-1}\}}\|b\|_{BMO^d}$, this is sharp by Chung's proof for the quadratic bound of the commutator of the Hilbert transform, more details in [P3].

Let us now compute the formal adjoint π_b^*

$$\begin{aligned}\left\langle \sum_{I\in\mathcal{D}} m_I f\langle b,h_I\rangle h_I \ , \ g\right\rangle &= \int_{\mathbb{R}} \sum_{I\in\mathcal{D}} m_I f\langle b,h_I\rangle h_I(x)g(x)dx \\ &= \sum_{I\in\mathcal{D}} m_I f\langle b,h_I\rangle\langle g,h_I\rangle \\ &= \sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle g,h_I\rangle \frac{1}{|I|}\int_I f(x)dx\end{aligned}$$

Now using the fact that $\int_{\mathbb{R}} f(x)\chi_I(x)dx = \int_I f(x)dx$ we have

$$\begin{aligned}\langle \pi_b f, g\rangle &= \int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle g,h_I\rangle \frac{\chi_I(x)}{|I|} f(x)dx \\ &= \left\langle f, \sum_{I\in\mathcal{D}} \langle g,h_I\rangle\langle b,h_I\rangle \frac{\chi_I}{|I|}\right\rangle\end{aligned}$$

Therefore

$$(\pi_b^* f)(x) = \sum_{I\in\mathcal{D}} \langle f,h_I\rangle\langle b,h_I\rangle \frac{\chi_I(x)}{|I|}. \tag{2.16}$$

2.11 Haar Multipliers

Definition 2.33. *Given a weight w, the Haar multiplier associated to it is defined as*

$$\big(T_w f\big)(x) = \sum_{I \in \mathcal{D}} \frac{w(x)}{m_I w} \langle f, h_I \rangle h_I(x) \tag{2.17}$$

Theorem 2.34 (Pereyra, [P]). *Given a dyadic doubling weight w, T_w is bounded in L^p, $1 < p < \infty$ iff $w \in RH_p^d$.*

Later Pereyra and Katz, in [KP], extended the definition of Haar multiplier to what it will be called here a t-Haar multiplier.

Definition 2.35. *Given a weight w and $t \in \mathbb{R}$, the t-Haar multiplier associated to it is defined as*

$$\big(T_w^t f\big)(x) = \sum_{I \in \mathcal{D}} \left(\frac{w(x)}{m_I w}\right)^t \langle f, h_I \rangle h_I(x) \tag{2.18}$$

They also proved that if $w \in A_\infty^d$, then the Haar multiplier operator T_w^t is bounded on $L^p(w)$ if and only if w satisfies the C_s^d condition for $s = tp$. Almost a decade later, Pereyra proved in [P2] sharp bounds in L^2 depending on the $[w]_{RH_2^d}^2$ for $t = 1$, $[w]_{A_2^d}^{\frac{1}{2}}$ for $t = \frac{1}{2}$ and $[w]_{A_2^d}$ for $t = \frac{-1}{2}$. In her PhD. dissertation, Beznosova attempted to extend these results for $t \in \mathbb{R}$, in fact she proved the following theorem.

Theorem 2.36 (Beznosova, [Be]). *Let t be a real number and w a weight in C_{2t}^d, such that $w^{2t} \in A_p$ for $1 < p < \infty$ and that satisfies the C_{2t}^d condition with constant $[w]_{C_{2t}^d}$. Then the Haar Multiplier is bounded in L_2. Moreover*

$$\|T_w^t\|_{L^2} \leq [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_p^d}^{\frac{1}{2}}$$

Unfortunately the dependence of the L^2-norm of the t-Haar multiplier on the C^d_{2t} - characteristic given above is not sharp. Since for $t = 1$, Theorem 2.36 will give that

$$\|T_w\|_{L^2} \leq C(p)[w]^2_{RH^d_2}[w^{2t}]^{\frac{1}{2}}_{A^d_{\frac{p+1}{2}}},$$

which is worse than the bound found by Pereyra in [P2], which is

$$\|T_w\|_{L^2} \leq CD(w)[w]^2_{RH^d_2},$$

recall that we proved in Proposition 2.5 that $D(w)$ is bounded by $2^p[w]_{A^d_p}$ for all $p > 1$.

Let us compute the formal adjoint $\left(T^t_w\right)^*$

$$\begin{aligned}\langle T^t_w f, g\rangle &= \left\langle \sum_{I\in\mathcal{D}} \frac{w^t(x)}{(m_I w)^t}\langle f, h_I\rangle h_I\ ,\ g\right\rangle \\ &= \sum_{I\in\mathcal{D}}\langle f, h_I\rangle\left\langle \frac{gw^t}{(m_I w)^t}, h_I\right\rangle \\ &= \left\langle f, \sum_{I\in\mathcal{D}} \frac{1}{(m_I w)^t}\langle gw^t\ ,\ h_I\rangle\right\rangle \\ &= \langle f\ ,\ (T^t_w)^* g\rangle.\end{aligned}$$

Thus

$$\big((T^t_w)^* f\big)(x) = \sum_{I\in\mathcal{D}} \frac{\langle fw^t\ ,\ h_I\rangle}{(m_I w)^t} h_I(x). \tag{2.19}$$

Chapter 3

Main Tools

In this section, we state and prove the lemmas and theorems necessary to obtain one and two weighted estimates that we will prove on the next chapters. The weighted Carleson Lemma 3.1 appears in all our main estimates, sharp weighted and two weighted estimates. Other lemmas, like α-Lemma 3.8 and Lift Lemma 3.13 are very important for sharp weighted estimates.

3.1 Carleson Lemmas

The weighted Carleson Lemma we present here is a variation in the same spirit of the Folklore Lemma [NV] of weighted Carleson embedding theorems that have appeared before in the literature, for example in [NTV1], in [LSU]. The Folklore Lemma 3.6 was introduced and used in [NV]. Here we obtain the Folklore Lemma as an immediate corollary of the weighted Carleson Lemma 3.1 and what we call the Little Lemma 3.3, introduced by Beznosova in her proof of the linear bound for the dyadic paraproduct.

3.1.1 Weighted Carleson Lemma

Lemma 3.1 (Weighted Carleson Lemma). *Let v be a regular weight, then $\{\alpha_J\}_{L\in\mathcal{D}}$ is a v-Carleson sequence with intensity B if and only if for all F non-negative v-measurable functions on the line,*

$$\sum_{L\in\mathcal{D}}(\inf_{x\in L} F(x))\alpha_L \leq B\int_{\mathbb{R}} F(x)\,v(x)\,dx; \tag{3.1}$$

Proof. ($\Rightarrow$) Assume that $F\in L^1(v)$ otherwise the first statement is automatically true. First we define $\gamma_L = \inf_{x\in L} F(x)$, we can write

$$\sum_{L\in\mathcal{D}}\gamma_L\alpha_L = \sum_{L\in\mathcal{D}}\int_0^\infty \chi(L,t)\,dt\,\alpha_L = \int_0^\infty \Big(\sum_{L\in\mathcal{D}}\chi(L,t)\,\alpha_L\Big)dt,$$

where $\chi(L,t)=1$ for $t<\gamma_L$ and zero otherwise, and the last equality by the monotone convergence theorem. Define $E_t=\{x\in\mathbb{R}\ :\ F(x)>t\}$. Since F is assumed a v-measurable function then E_t is a v-measurable set for every t. Moreover, since $F\in L^1(v)$ we have, by Chebychev's inequality, that the v-measure of E_t is finite for all real t, for a fixed t, there is an integer M_t such that $v(E_t)<\dfrac{1}{t}\displaystyle\int_{\mathbb{R}} F(x)v(x)dx := M_t$.

Also, there is a collection of maximal disjoint dyadic intervals $\mathcal{P}_t$ that will cover $E_t\setminus A$ where A is a set that has no interval inside of it. We will now describe a procedure to find such collection. Note that if E_t has no interval inside of it, then we have nothing to do. If E_t has an interval inside of it, then $v(E_t)>0$ then there exist an integer j_0 such that $2^{-j_0}\leq v(E_t)<2^{-j_0+1}$.

Define $\mathcal{D}^v_j := \{I\in\mathcal{D}\ :\ 2^{-j}\leq v(I)<2^{-j+1}\}$. We will say that a dyadic interval belongs to the level j with respect to the weight v if $I\in\mathcal{D}^v_j$. Note that is possible that for a given interval I in $\mathcal{D}^v_j$ its parent $\hat{I}$ is also in $\mathcal{D}^v_j$. In fact for any n positive is possible that I^n is also in $\mathcal{D}^v_j$, where I^n is the $n-th$ grandparent of I. However a given dyadic interval belongs to one and only one family $\mathcal{D}^v_j$, that is the collection of intervals $\{\mathcal{D}^v_j\}_{j\in\mathbb{Z}}$ are disjoint: $\mathcal{D}^v_j\cap\mathcal{D}^v_i=\emptyset$.

None dyadic interval with v-length bigger than 2^{-j_0+1} can be in E_t. Note that if J is in $\mathcal{D}^v_m$ for $m < j_0$ then $v(J) > 2^{-m} \geq 2^{-j_0+1} > v(E_t)$. Thus we start our search among all dyadic intervals in $\mathcal{D}^v_{j_0}$. If there is such I, then we have to make sure that it is the maximal dyadic interval in $\mathcal{D}^v_{j_0}$ that is in E_t, i.e., we want to capture the ancestor I^n such that I^n is in E_t and I^n is in $\mathcal{D}^v_{j_0}$, but $I_{n+1} = \hat{I}_n$ is not in E_t. We have to ask if $I^1 = \hat{I}$ is in E_t and is in $\mathcal{D}^v_{j_0}$. If it is, then we ask if I^2 is in E_t and is in $\mathcal{D}^v_{j_0}$. We keep repeating this process until we find some I^{n_0} that is not in E_t or in $\mathcal{D}^v_{j_0}$.

This process of looking for the maximal dyadic interval in a level m with respect to v is finite because of the regularity of the weight. Imagine that this process never stops, then we have that $\mathbb{R}^+$ or $\mathbb{R}^-$ is in E_t, depending if the starting interval I is a positive or a negative dyadic interval. Therefore

$$v(\mathbb{R}^-) < v(E_t) < 2^{-j_0+1} \quad \text{or} \quad v(\mathbb{R}^+) < v(E_t) < 2^{-j_0+1},$$

which is not possible by the regularity of the weight.

We allocate the maximal dyadic interval I^n in $\mathcal{P}^t_1$. Note that in this case $\mathcal{P}^t_1$ can have just one dyadic interval. Suppose that there exist another maximal dyadic interval J, by dyadic filtration we have that $J \cap I = \emptyset$, because if $I \subset J$ or $J \subset I$ then they could not be both maximal. However $J \cap I = \emptyset$ implies that

$$2^{-j_0+1} = 2^{-j_0} 2^{-j_0} \leq v(I \cup J) \leq v(E_t),$$

which contradicts the fact that $v(E_t) < 2^{-j_0+1}$.

If there is not an interval $I \in \mathcal{D}^v_{j_0}$ such that $I \in E_t$ then we move to the level $j_0 + 1$ and repeat the process. Observe that we can find at most 2 disjoint maximal intervals in E_t and in $\mathcal{D}^v_{j_0+1}$. If we do not find any in this level then we move to the next level $j_0 + 2$, the important thing is that if we find a dyadic interval that is in E_t for the first time in $\mathcal{D}^v_{j_0+n}$, then we can have at most 2^{n+1} maximal dyadic intervals that are in $\mathcal{D}^v_{j_0+n}$ are that are in E_t. Since $v(E_t) < 2^{-j_0+1}$ and each such I has $v(I) \geq 2^{-j_0-n}$, so if we can find $2^{n+1}+1$ such intervals, which by maximality are disjoint then, $2^{-j_0-n}(2^{n+1}+1) = 2^{-j_0+1} + 2^{-j_0-n} \leq v(E_t)$ which is a contradiction.

The collection $\mathcal{P}^1_t$ should contain all the "largest" maximal dyadic intervals that are completely inside E_t, since they will all belong to $\mathcal{D}^v_{j_0+k_1}$ for a fixed k_1.

Note that if $v(E_t) = \epsilon > 0$ then either there is a dyadic interval with v-length η_0, $0 < \eta_0 < \epsilon$ contained in E_t, or there is none and we stop. If there is such interval, then it must exists j, such that $2^{-j} \leq \eta_0 < 2^{-j+1}$, such that the collection $\mathcal{D}^v_j$ has at least one dyadic interval in E_t. After we find the collection of maximal dyadic intervals in E_t, $\mathcal{P}^1_t$, we repeat the same procedure in the set $E^1_t := E_t \setminus \left(\bigcup_{I\in\mathcal{P}^1_t} I\right)$, which means that we want to find the largest maximal dyadic intervals that are in $E_t \setminus \left(\bigcup_{I\in\mathcal{P}^1_t} I\right)$, we call this collection, $\mathcal{P}^2_t$. Again, if we are not able to find any dyadic intervals completely included in $E_t \setminus \left(\bigcup_{I\in\mathcal{P}^1_t} I\right)$, then we stop, we already accomplished our initial goal. If we find the maximal dyadic intervals inside $E_t \setminus \left(\bigcup_{I\in\mathcal{P}^1_t} I\right)$ then we will repeat the procedure in the set $E_t \setminus \left(\bigcup_{I\in\mathcal{P}^1_t \bigcup \mathcal{P}^2_t} I\right)$. We keep this process and at each stage we generate a finite collection of dyadic intervals $\mathcal{P}^l_t$, all in $\mathcal{D}^v_{j_0+k_l}$ where $k_{l+1} > k_l$, $l \geq 1$, or we stop, if we stop at stage l we say that $\mathcal{P}^n_t = \emptyset$ for all $n \geq l$. Then we write:

$$\mathcal{P}_t = \bigcup_{n=1}^{\infty} \mathcal{P}^n_t.$$

When $n \to \infty$, $\bigcup_{l=1}^n \bigcup_{I\in\mathcal{P}^l_t} I \to (E_t \setminus A)$ where $v(A) \geq 0$ and A does not contains any interval inside of it. Therefore

$$v\left(\bigcup_{l=1}^{n} \bigcup_{I\in\mathcal{P}^l_t} I\right) \to v(E_t \setminus A) \quad \text{when} \quad n \to \infty, \quad \text{i.e.}$$

$$v\left(\bigcup_{I\in\mathcal{P}_t} I\right) = \sum_{I\in\mathcal{P}_t} v(I) = v(E_t \setminus A) \leq v(E_t).$$

The last equality because by construction the families $\mathcal{P}_t^l \subset \mathcal{D}^v_{j_0+k_l}$ and are therefore disjoint families of intervals. Moreover, maximality implies that if $I, J \in \mathcal{P}_t^l$ then $I \cap J = \emptyset$, so on each family $\mathcal{P}_t^k$ the intervals themselves are disjoints. Hence $\mathcal{P}_t$ is a collection of disjoint intervals in E_t, hence $\sum_{L\in\mathcal{P}_t} v(L) \leq v(E_t)$.

Observe that $t \geq \gamma_L$ if and only if $\chi(L,t) = 0$, and that $t < \gamma_L$ if and only if $t < \inf_{x\in L} F(x)$. Together these imply that $L \subset E_t$ if and only if $\chi(L,t) = 1$. Then we can write that

$$\sum_{L\in\mathcal{D}} \chi(L,t)\alpha_L = \sum_{L\subset E_t} \alpha_L \leq \sum_{L\in\mathcal{P}_t} \sum_{I\in\mathcal{D}(L)} \alpha_I \leq B \sum_{L\in\mathcal{P}_t} v(L) \leq Bv(E_t), \tag{3.2}$$

where we used in the second inequality the fact that $\{\alpha_J\}_{I\in\mathcal{D}}$ is a Carleson sequence with intensity B.

$$\sum_{L\in\mathcal{D}} \gamma_L \alpha_L = \sum_{L\in\mathcal{D}} \int_0^{\gamma_L} dt\alpha_L = \sum_{L\in\mathcal{D}} \int_0^{\infty} \chi(L,t) dt \alpha_L = \int_0^{\infty} \sum_{L\in\mathcal{D}} \chi(L,t)\alpha_L dt$$

The last equality follows from Monotone Convergence Theorem, thus we can estimate

$$\sum_{L\in\mathcal{D}} \gamma_L\alpha_L \leq B \int_0^\infty v(E_t) dt = B \int_{\mathbb{R}} F(x)\, v(x)\, dx.$$

where the last equality follows from the layer cake representation.

($\Leftarrow$) Assume (3.1) is true, in particular it will hold for $F(x) = \frac{\chi_J(x)}{|J|}$, and since $\inf_{x\in I} F(x) = 0$ if $I \cap J = \emptyset$, $\inf_{x\in I} F(x) = \frac{1}{|J|}$ otherwise, then

$$\frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} \alpha_I \leq \sum_{I\in\mathcal{D}} \inf_{x\in I} F(x)\, \alpha_I \leq B \int_{\mathbb{R}} F(x)\, v(x)\, dx = B\, m_J v,$$

Therefore $\{\alpha_I\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity B. □

Let v be a regular weight, $\{\alpha_I\}_{I\in\mathcal{D}}$ a v-Carleson sequence with intensity B, $\{\lambda_I\}_{I\in\mathcal{D}}$ a sequence of positive numbers and we define the positive function $F(x) =$

$\lambda^*(x) = \sup_{I \ni x} \lambda_I$. Now apply Lemma 3.1 noting that $\lambda_L \leq \inf_{x \in L} F(x)$, to conclude that

$$\sum_{I \in \mathcal{D}} \lambda_I \alpha_I \leq B \int_{\mathbb{R}} \lambda^*(x) v(x)\, dx.$$

This is Lemma 6 in [P], but with the hypothesis that v is in A_∞ instead of regular.

Remark 3.2. *If we do not assume v is regular, and we assume instead that the sequence $\{\alpha_J\}_{J \in \tilde{\mathcal{D}}}$ is an extended v-Carleson sequence we will reach the same conclusion in Lemma 3.1 with $\mathcal{D}$ replaced by $\tilde{\mathcal{D}}$.*

3.1.2 Little Lemma

In order to prove Lemma 3.6 we need Lemma 3.3, which was proved by Beznosova in [Be1] using the Bellman function $B(u, v, l) = u - \dfrac{1}{v(1+l)}$.

Lemma 3.3 (Little Lemma, [Be1])**.** *Let v be a weight, such that v^{-1} is a weight as well, and let $\{\lambda_I\}_{I \in \mathcal{D}}$ be a Carleson sequence with intensity B then $\{\frac{\lambda_I}{m_I v^{-1}}\}_{I \in \mathcal{D}}$ is a v-Carleson sequence with intensity at most $4B$, that is for all $J \in \mathcal{D}$,*

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{m_I v^{-1}} \leq 4B\, m_J v.$$

For a proof of this result we refer [Be], Prop. 3.4 or [Be1], Prop. 2.1.

This lemma is to be compared to Lemma 4 in [P], that says if $\{\lambda_I\}_{I \in \mathcal{D}}$ is a Carleson sequence then $\{\lambda_I m_I v\}_{I \in \mathcal{D}}$ is a v-Carleson sequence Note that the assumption is $v \in A_\infty$, and there is no reference to v^{-1}, however if v^{-1} is a weight, then by Cauchy-Schwarz, $1 \leq m_I v\, m_I v^{-1}$, and we will deduce from that result that if $v \in A_\infty$ and v^{-1} is a weight, then $\{\frac{\lambda_I}{m_I v^{-1}}\}_{I \in \mathcal{D}}$ is a v-Carleson sequence. The Little Lemma provides the same result *without assuming* $v \in A_\infty$.

The next Lemma is a generalization of the Little Lemma, note that when $p = 2$, Lemma 3.4 will give us the same result as Lemma 3.3.

Lemma 3.4 (A_p^d Little Lemma). *Let $1 < p < \infty$, w a weight such that $w^{\frac{-1}{p-1}}$ is also a weight. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence of nonnegative numbers, i.e., there exists $B > 0$ s.t.*

$$\forall\ J \in \mathcal{D} \qquad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \lambda_I \le B,$$

then

$$\forall\ J \in \mathcal{D} \qquad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{\lambda_I}{\left(m_I w^{\frac{-1}{p-1}}\right)^{p-1}} \le 4Bm_J w.$$

Furthermore, if $w \in A_p^d$ then for any $J \in \mathcal{D}$

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} m_I w\ \lambda_I \le 4[w]_{A_p} m_J w.$$

Proof. We will show this inequality using a Bellman function type method. Consider $B(u,v,l) := u - \frac{1}{v^{p-1}(1+l)}$ defined on the domain $\mathbb{D} = \{(u,v,l) \in \mathbb{R}^3, u > 0, v > 0, uv^{p-1} > 1 \quad \text{and} \quad 0 \le l \le 1\}$. Note that $\mathbb{D}$ is convex. Note that

$$0 \le B(u,v,l) \le u \qquad \text{for all} \quad (u,v,l) \in \mathbb{D}$$

and

$$\frac{\partial B}{\partial l}(u,v,l) \ge \frac{1}{4v^{p-1}} \qquad \text{for all} \quad (u,v,l) \in \mathbb{D}. \tag{3.3}$$

and also

$$-(du,dv,dl)d^2B\begin{pmatrix} du \\ dv \\ dl \end{pmatrix} \tag{3.4}$$

$$= -(du,dv,dl)\begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p)\frac{v^{-p-1}}{1+l} & (1-p)\frac{v^{-p}}{(l+1)^2} \\ 0 & (1-p)\frac{v^{-p}}{(l+1)^2} & -2\frac{v^{1-p}}{(l+1)^3} \end{pmatrix}\begin{pmatrix} du \\ dv \\ dl \end{pmatrix}$$

$$= p(p-1)\frac{v^{-p-1}}{1+l}(du)^2 + 2(p-1)\frac{v^{-p}}{(l+1)^2}dudv + 2\frac{v^{1-p}}{(l+1)^3}(dv)^2 \ge 0, \tag{3.5}$$

since all terms are positive for $p > 1$.

Now let us show that if (u_-, v_-, l_-) and (u_+, v_+, l_+) are in $\mathbb{D}$ and we define (u_0, v_0, l) as $u_0 = \frac{u_- + u_+}{2}$, $v_0 = \frac{v_- + v_+}{2}$ and some l_0,

$$B(u_0, v_0, l_0) - \frac{B(u_-, v_-, l_-) + B(u_+, v_+, l_+)}{2} \geq \frac{C}{4v_0^{p-1}}$$

Consider for $-1 \leq t \leq 1$,

$$u(t) = \frac{(t+1)u_+ + (1-t)u_-}{2} \qquad v(t) = \frac{(t+1)v_+ + (1-t)v_-}{2}$$

and

$$l(t) = \frac{(t+1)l_+ + (1-t)l_-}{2}.$$

We define $b(t) := B(u(t), v(t), l(t))$, note that $b(0) = B(u_0, v_0, l_0)$, $b(1) = B(u_+, v_+, l_+)$, $b(-1) = B(u_-, v_-, l_-)$, $\frac{du}{dt} = \frac{u_+ - u_-}{2}$, $\frac{dv}{dt} = \frac{v_+ - v_-}{2}$ and $\frac{dl}{dt} = \frac{l_+ - l_-}{2}$. If (u_+, v_+, l_+) and (u_-, v_-, l_-) are in $\mathbb{D}$ then $(u(t), v(t), l(t))$ is also in $\mathbb{D}$ for all $|t| \leq 1$, since $\mathbb{D}$ is convex. It is a calculus exercise to show that

$$b(0) - \frac{b(1) + b(-1)}{2} = \frac{-1}{2} \int_{-1}^{1} (1 - |t|) b''(t) dt$$

Also it is easy to check that

$$-b''(t) = -\Big(\frac{du}{dt}, \frac{dv}{dt}, \frac{dl}{dt}\Big) d^2 B \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \\ \frac{dl}{dt} \end{pmatrix}$$

and

$$\begin{aligned} &B(u_0, v_0, l_0) - \frac{B(u_-, v_-, l_-) + B(u_+, v_+, l_+)}{2} = \\ &= \Big[B(u_0, v_0, l_0) - B\Big(\frac{u_- + u_+}{2}, \frac{v_- + v_+}{2}, \frac{l_- + l_+}{2}\Big)\Big] \\ &\quad + \Big[B\Big(\frac{u_- + u_+}{2}, \frac{v_- + v_+}{2}, \frac{l_- + l_+}{2}\Big) - \frac{B(u_-, v_-, l_-) + B(u_+, v_+, l_+)}{2}\Big] \end{aligned}$$

$$= C\frac{\partial B}{\partial l}(u_0, v_0, l') - \frac{1}{2}\int_{-1}^{1}(1-|t|)b''(t)dt \geq \frac{l}{4v_0^{p-1}}$$

where l' is a point between l_0 and $\frac{l_-+l_+}{2}$ and

$$\left[B(u_0, v_0, l_0) - B\Big(\frac{u_-+u_+}{2}, \frac{v_-+v_+}{2}, \frac{l_-+l_+}{2}\Big)\right] = C\frac{\partial B}{\partial l}(u_0, v_0, l') \qquad (3.6)$$

by the Mean Value Theorem.

Now we can use the Bellman function argument. Let $u_+ = m_{J_+}w$, $u_- = m_{J_-}w$, $v_+ = m_{J_+}w^{\frac{-1}{p-1}}$, $v_- = m_{J_-}v^{\frac{-1}{p-1}}$, $l_+ = \frac{1}{|J_+|B}\sum_{I\in\mathcal{D}(J_+)}\lambda_I$ and $l_- = \frac{1}{|J_-|B}\sum_{I\in\mathcal{D}(J_-)}\lambda_I$. Thus $(u_-, v_-, l_-), (u_+, v_+, l_+) \in \mathbb{D}$ and $u_0 = m_J w$, $v_0 = m_J w^{\frac{-1}{p-1}}$ and $l_0 = \frac{1}{|J|B}\sum_{I\in\mathcal{D}(J)}\lambda_I$. Then

$$(u_0, v_0, l_0) - \Big(\frac{u_-+u_+}{2}, \frac{v_-+v_+}{2}, \frac{l_-+l_+}{2}\Big) = \Big(0, 0, \frac{\lambda_J}{B|J|}\Big).$$

Then

$$\begin{aligned}
|J|m_J w &\geq |J|B(u_0, v_0, l_0) \\
&\geq |J|\frac{B(u_+, v_+, l_+)}{2} + |J|\frac{B(u_-, v_-, l_-)}{2} + \frac{1}{4B\big(m_J w^{\frac{-1}{p-1}}\big)^{p-1}}\lambda_J \\
&= |J_+|B(u_+, v_+, l_+) + |J_-|B(u_-, v_-, l_-) + \frac{1}{4B\big(m_J w^{\frac{-1}{p-1}}\big)^{p-1}}\lambda_J
\end{aligned}$$

Iterating, we get

$$m_J w \geq \frac{1}{4B|J|}\sum_{I\in\mathcal{D}(J)}\frac{\lambda_I}{(w^{\frac{-1}{p-1}})^{p-1}}$$

□

Similarly we can obtain the following result.

Lemma 3.5. *Let $1 < p < \infty$, w a weight such that $w^{\frac{-1}{p-1}}$ is also a weight. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence of nonnegative numbers, i.e., there exists $B > 0$ s.t.*

$$\forall\, J \in \mathcal{D} \qquad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\lambda_I \leq B,$$

then

$$\forall\, J \in \mathcal{D} \qquad \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{m_I w} \leq 4B \big(m_J w^{\frac{-1}{p-1}}\big)^{p-1}.$$

Furthermore, if $w \in A_p^d$ then for any $J \in \mathcal{D}$

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \big(m_J w^{\frac{-1}{p-1}}\big)^{p-1} \lambda_I \leq 4[w]_{A_p} \big(m_J w^{\frac{-1}{p-1}}\big)^{p-1}.$$

The proof is similar, however we use the Bellman function $B(u, v, l) = u^{p-1} - \frac{1}{v(1+l)}$.

The following lemma appeared in [NV] where they called it a folklore lemma, in their paper the lemma is stated without asking for v be regular. It is not clear for us if the regularity on the weight can be dropped. In all results that we will use the Folklore Lemma and for all purposes in [NV] paper, the weight is in A_2^d and therefore it was regular.

Corollary 3.6 (Folklore Lemma [NV])**.** *Let v be a regular weight such that v^{-1} is also a weight. Let $\{\lambda_J\}_{L \in \mathcal{D}}$ be a Carleson sequence with intensity B. Let F be a non-negative measurable function on the line. Then*

$$\sum_{L \in \mathcal{D}} \inf_{x \in L} F(x) \frac{\lambda_L}{m_L v^{-1}} \leq C\; B \int_{\mathbb{R}} F(x)\, v(x)\, dx.$$

The Folklore Lemma is a consequence of Lemma 3.3, and the weighted Carleson Lemma 3.1. Note that Lemma 3.3 can be deduced from the Folklore Lemma with $F(x) = \chi_J(x)$.

3.2 $\alpha\beta$-Lemma

The following lemma, for $v = w^{-1}$, for $\alpha = \beta = \frac{1}{4}$ appeared in the work of Beznosova, see [Be], and for $0 < \alpha = \beta < 1/2$ appeared in [NV]. With small modification in her proof, using the Bellman function $B(x,y)=x^\alpha y^\beta$ with domain of definition the first quadrant $x,y>0$ (a convex set), we can accomplish the result below, this was observed independently by Beznosova [Be2] and the author.

Lemma 3.7. ($\alpha\beta$-Lemma) *Let u,v be weights then for any $J\in\mathcal{D}$ and any $\alpha,\beta\in(0,\frac{1}{2})$*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{|\Delta_I u|^2}{(m_I u)^2}+\frac{|\Delta_I v|^2}{(m_I v)^2}\right)|I|(m_I u)^\alpha(m_I v)^\beta\le C_{\alpha,\beta}(m_J u)^\alpha(m_J v)^\beta. \qquad (3.7)$$

The constant $C_{\alpha,\beta}=\frac{36}{\min\{\alpha-2\alpha^2,\beta-2\beta^2\}}$.

Proof. We will show this inequality using a Bellman function type method. Consider $B(u,v)=u^\alpha v^\beta$ defined on the domain $\mathbb{D}=\{(u,v)\in\mathbb{R}^2, u>0, v>0\}$. Note that $\mathbb{D}$ is convex. Trivially we have that

$$0<B(u,v)\le u^\alpha v^\beta \qquad \text{for all} \quad (u,v)\in\mathbb{D}.$$

Our first goal is to show that

$$-(du,dv)d^2B\begin{pmatrix}du\\dv\end{pmatrix}\ge C_{\alpha,\beta}\big[u^{\alpha-2}v^\beta(du)^2+u^\alpha v^{\beta-2}(dv)^2\big], \qquad (3.8)$$

where $C_{\alpha,\beta}=\min\{\alpha-2\alpha^2,\beta-2\beta^2\}$, and d^2B is the Hessian matrix of $B(u,v)$.

$$\begin{aligned}
-(du,dv)d^2B\begin{pmatrix}du\\dv\end{pmatrix} &= \\
&= -(du,dv)\begin{bmatrix}\alpha(\alpha-1)u^{\alpha-2}v^\beta & \alpha\beta u^{\alpha-1}v^{\beta-1}\\ \alpha\beta u^{\alpha-1}v^{\beta-1} & \beta(\beta-1)u^\alpha v^{\beta-2}\end{bmatrix}\begin{pmatrix}du\\dv\end{pmatrix}\\
&= -\alpha(\alpha-1)u^{\alpha-2}v^\beta(du)^2-2\alpha\beta u^{\alpha-1}v^{\beta-1}dudv-\beta(\beta-1)u^\alpha v^{\beta-2}(dv)^2\\
&= \big[-\alpha(\alpha-1)-\alpha^2\big]u^{\alpha-2}v^\beta(du)^2+\big[-\beta(\beta-1)-\beta^2\big]u^\alpha v^{\beta-2}(dv)^2\\
&\quad+\alpha^2u^{\alpha-2}v^\beta(du)^2+\beta^2u^\alpha v^{\beta-2}(dv)^2-2\alpha\beta u^{\alpha-1}v^{\beta-1}dudv
\end{aligned}$$

$$= \big[-\alpha(\alpha-1)-\alpha^2\big]u^{\alpha-2}v^{\beta}(du)^2+\big[-\beta(\beta-1)-\beta^2\big]u^{\alpha}v^{\beta-2}(dv)^2+$$
$$\big(\alpha u^{\frac{\beta}{2}-1}v^{\frac{\alpha}{2}}du-\beta u^{\frac{\alpha}{2}}v^{\frac{\beta}{2}-1}du\big)^2$$
$$\geq\big[-\alpha(\alpha-1)-\alpha^2\big]u^{\alpha-2}v^{\beta}(du)^2+\big[-\beta(\beta-1)-\beta^2\big]u^{\alpha}v^{\beta-2}(dv)^2$$
$$=(\alpha-2\alpha^2)u^{\alpha-2}v^{\beta}(du)^2+(\beta-2\beta^2)u^{\alpha}v^{\beta-2}(dv)^2$$
$$\geq C_{\alpha,\beta}\big[u^{\alpha-2}v^{\beta}(du)^2+u^{\alpha}v^{\beta-2}(dv)^2\big],$$

where $C_{\alpha,\beta}=\min\{(\alpha-2\alpha^2),(\beta-2\beta^2)\}$, note $C_{\alpha,\beta}>0$ iff α and β are in $(0,1/2)$.

Now let us show that if (u_-,v_-) and (u_+,v_+) are in $\mathbb{D}$ and we define (u_0,v_0) as $u_0=\frac{u_-+u_+}{2}$ and $v_0=\frac{v_-+v_+}{2}$ then

$$B(u_0,v_0)-\frac{B(u_-,v_-)+B(u_+,v_+)}{2}\geq C_{\alpha,\beta}\left[\frac{v_0^{\beta}}{u_0^{2-\alpha}}|u_+-u_-|^2+\frac{u_0^{\alpha}}{v_0^{2-\beta}}|v_+-v_-|^2\right].$$

Consider for $-1\leq t\leq 1$,

$$u(t)=\frac{(t+1)u_++(1-t)u_-}{2}\qquad\text{and}\qquad v(t)=\frac{(t+1)v_++(1-t)v_-}{2}.$$

We define $b(t):=B(u(t),v(t))$, note that $b(0)=B(u_0,v_0)$, $b(1)=B(u_+,v_+)$, $b(-1)=B(u_-,v_-)$, $\frac{du}{dt}=\frac{u_+-u_-}{2}$ and $\frac{dv}{dt}=\frac{v_+-v_-}{2}$. If (u_+,v_+) and (u_-,v_-) are in $\mathbb{D}$ then $(u(t),v(t))$ is also in $\mathbb{D}$ for all $|t|\leq 1$, since $\mathbb{D}$ is convex. It is a calculus exercise to show that

$$b(0)-\frac{b(1)+b(-1)}{2}=\frac{-1}{2}\int_{-1}^{1}(1-|t|)b''(t)dt.$$

Also it is easy to check that

$$-b''(t)=-\big(\frac{du}{dt},\frac{dv}{dt}\big)d^2B\begin{pmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{pmatrix}.$$

Thus

$$-b''(t)\geq\frac{C_{\alpha,\beta}}{4}\left[\frac{(v(t))^{\beta}}{(u(t))^{2-\alpha}}|u_+-u_-|^2+\frac{(u(t))^{\alpha}}{(v(t))^{2-\beta}}|v_+-v_-|^2\right].$$

Since $u_0 = \frac{u_- + u_+}{2}$ then we can write $u(t) = u_0 + \frac{1}{2}t(u_+ - u_-)$ then for $t \in [\frac{-1}{2}, \frac{1}{2}]$

$$\frac{1}{2}u_0 \leq u(t) \leq \frac{3}{2}u_0.$$

By the same reasoning, for $t \in [\frac{-1}{2}, \frac{1}{2}]$

$$\frac{1}{2}v_0 \leq v(t) \leq \frac{3}{2}v_0.$$

Then, for $|t| \leq \frac{1}{2}$, and observing that because $0 < \alpha, \beta < 1/2$, then

$$-b''(t) \geq \frac{C_{\alpha,\beta}}{9}\left[\frac{v_0^\beta}{u_0^{2-\alpha}}|u_+ - u_-|^2 + \frac{u_0^\alpha}{v_0^{2-\beta}}|v_+ - v_-|^2\right].$$

Since $-b''(t) \geq 0$ for $\frac{1}{2} \leq |t| \leq 1$ we have that

$$\begin{aligned} b(0) - \frac{b(-1) + b(1)}{2} &\geq \frac{-1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}}(1 - |t|)b''(t)dt \\ &\geq \frac{C_{\alpha,\beta}}{18}\left(\frac{v_0^\beta}{u_0^{2-\alpha}}|u_+ - u_-|^2 + \frac{u_0^\alpha}{v_0^{2-\beta}}|v_+ - v_-|^2\right)\int_{\frac{-1}{2}}^{\frac{1}{2}}(1 - |t|)dt \\ &\geq \frac{C_{\alpha,\beta}}{36}\left(\frac{v_0^\beta}{u_0^{2-\alpha}}|u_+ - u_-|^2 + \frac{u_0^\alpha}{v_0^{2-\beta}}|v_+ - v_-|^2\right). \end{aligned}$$

Therefore we can conclude

$$\begin{aligned} B(u_0, v_0) &- \frac{B(u_-, v_-) + B(u_+, v_+)}{2} = b(0) - \frac{b(-1) + b(1)}{2} \\ &\geq \frac{C_{\alpha,\beta}}{36}\left(\frac{v_0^\beta}{u_0^{2-\alpha}}|u_+ - u_-|^2 + \frac{u_0^\alpha}{v_0^{2-\beta}}|v_+ - v_-|^2\right). \end{aligned} \tag{3.9}$$

Now we can use the Bellman function argument. Given weights u and v (we are abusing notation, u, v are also the variables in the Bellman function), let $u_+ = m_{J_+}u$, $u_- = m_{J_-}u$, $v_+ = m_{J_+}v$, $v_- = m_{J_-}v$. Thus $(u_-, v_-), (u_+, v_+) \in \mathbb{D}$ and $u = m_J u$ and $v = m_J v$.

$$|J|(m_J u)^\alpha (m_J v)^\beta = |J|u^\alpha v^\beta \geq |J|B(u, v) \geq$$

$$\begin{aligned}\geq & |J_+|B(u_+,v_+) + |J_-|B(u_-,v_-) \\ & \quad + |J|\frac{C_{\alpha,\beta}}{36}\left(\frac{(m_Jv)^\beta}{(m_Ju)^{2-\alpha}}|\Delta_J u|^2 + \frac{(m_Ju)^\alpha}{(m_Jv)^{2-\beta}}|\Delta_J v|^2\right) \\ = & |J_+|B(u_+,v_+) + |J_-|B(u_-,v_-) + \\ & \quad + |J|\frac{C_{\alpha,\beta}}{36}(m_Ju)^\alpha\,(m_Jv)^\beta\left(\frac{|\Delta_J u|^2}{(m_Ju)^2} + \frac{|\Delta_J v|^2}{(m_Jv)^2}\right).\end{aligned}$$

We can also estimate $B(u_+,v_+)$, $B(u_-,v_-)$ by (3.9), continuing this process we will have that

$$\frac{C_{\alpha,\beta}}{36}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(m_Iu)^\alpha\,(m_Iv)^\beta|I|\left(\frac{|\Delta_I u|^2}{(m_Iu)^2} + \frac{|\Delta_I v|^2}{(m_Iv)^2}\right) \leq B(u,v) \leq (m_Ju)^\alpha(m_Jv)^\beta.$$

□

We immediately deduce from the lemma the following,

Lemma 3.8 (α-Lemma, [Be1])**.** *Let $w \in A_2^d$, then for any $\alpha \in (0,\frac{1}{2})$, the sequence $\{\mu_I\}_{I\in\mathcal{D}}$, where*

$$\mu_I := (m_Iw)^\alpha(m_Iw^{-1})^\alpha|I|\left(\frac{|\Delta_I w|^2}{(m_Iw)^2} + \frac{|\Delta_I w^{-1}|^2}{(m_Iw^{-1})^2}\right),$$

is a Carleson sequence with intensity at most $2C_\alpha[w]^\alpha_{A_2}$, with $C_\alpha = \frac{36}{\alpha-2\alpha^2}$.

Proof. Apply Lemma 3.8 to the weights $u = w$, $v = w^{-1}$, $\beta = \alpha$, then

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{|\Delta_I w|^2}{(m_Iw)^2}+\frac{|\Delta_I w^{-1}|^2}{(m_Iw^{-1})^2}\right)|I|(m_Iw)^\alpha(m_Iw^{-1})^\alpha \leq C_\alpha(m_Jw)^\alpha(m_Jw^{-1})^\alpha. \quad (3.10)$$

Now in (3.10) use that $(m_Jw)^\alpha(m_Jw^{-1})^\alpha \leq [w]^\alpha_{A_2^d}$ to get

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\mu_I \leq C(m_Jw)^\alpha(m_Jw^{-1})^\alpha \leq C_\alpha[w]^\alpha_{A_2}.$$

□

A proof of this lemma that works on geometric doubling metric spaces can be found in [NV1, V]. In this paper $\alpha = 1/4$ can be used, and in that case the constant C_α can be replaced by 288.

Remark 3.9. *Throughout the proofs a constant C will be a numerical constant that may change from line to line, c_α will be a constant depending on $0 < \alpha < 1/2$, as a multiple of C_α or its square root that may change from line to line. Note that for that range of α, $e^\alpha \leq 2$.*

An interesting observation that we can make from the Lemma 3.7 is the fact that it tell us that if $w \in A_p^d$ then the sequence

$$\left\{ (m_I w)^\alpha (m_I w^{\frac{-1}{p-1}})^{\alpha(p-1)} |I| \left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{p-1}}|^2}{(m_I w^{\frac{-1}{p-1}})^2} \right) \right\}_{I \in \mathcal{D}}$$

is a Carleson sequence, Lemma 3.8 asserts this for $p = 2$.

Lemma 3.10. *Given $1 < p < \infty$, the sequence $\{\mu_I\}_{I \in \mathcal{D}}$, where*

$$\mu_I := (m_I w)^\alpha (m_I (w^{\frac{-1}{p-1}}))^{\alpha(p-1)} |I| \left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{p-1}}|^2}{(m_I w^{\frac{-1}{p-1}})^2} \right) \quad I \in \mathcal{D}$$

is a Carleson sequence with Carleson intensity at most $C_\alpha [w]_{A_p}^\alpha$ for any $\alpha \in \left(0, \max\{\frac{1}{2}, \frac{1}{2(p-1)}\}\right)$. Moreover, the sequence $\{\nu_I\}_{I \in \mathcal{D}}$, where

$$\nu_I := (m_I w)(m_I w^{\frac{-1}{p-1}})^{(p-1)} |I| \left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{p-1}}|^2}{(m_I w^{\frac{-1}{p-1}})^2} \right) \quad I \in \mathcal{D}$$

is a Carleson sequence with Carleson intensity at most $C[w]_{A_p}$.

Proof. Set $u = w$, $v = w^{-\frac{1}{p-1}}$, $\beta = \alpha(p-1)$. By hypothesis $0 < \alpha < \frac{1}{2}$ and also $0 < \alpha < \frac{1}{2(p-1)}$ which implies that $0 < \beta < \frac{1}{2}$, we can now use Lemma 3.7 to show that μ_I is a Carleson sequence with intensity at most $c_\alpha [w]_{A_p^d}^\alpha$. For the second statement suffices to notice that $\nu_I \leq \mu_I [w]_{A_p^d}^{1-\alpha}$ for all $I \in \mathcal{D}$. □

3.3 Buckley's Inequality

The following theorem was proved by Buckley in [Bu], for the purpose of this dissertation we just need Buckley's inequality for $p = 1$. However we will state the Buckley's inequality for the general case and also the sharp version of Beznosova and Reznikov for the case $p = 1$.

Theorem 3.11 (Buckley, [Bu]). *Suppose w is a weight, then for $p \neq 0$, $p \neq 1$ then $w \in C_p^d$ if and only if for all $J \in \mathcal{D}$*

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} (m_I w)^p \left(\frac{\Delta_I w}{m_I w} \right)^2 |I| \leq C[w]_{C_p^d} (m_J w)^p, \tag{3.11}$$

where C is constant that only depends on p.

For $p = 1$, Buckley showed that the weight w is A_∞^d if and only if the sequence $\{\mu_I\}_{I \in \mathcal{D}}$, $\mu_I := m_I w \dfrac{|\Delta_I w|^2}{(m_I w)^2} |I|$, is a w-Carleson sequence with intensity that depends on the $[w]_{A_\infty^d}$, however this dependence was not provided. Later, in [W], Wittwer proved that if the $w \in A_2^d$ then the sequence $\{\mu_I\}_{\mathcal{D}}$ is a w-Carleson sequence with intensity at most $C[w]_{A_2^d}$, where C does not depend on the weight w. However the sharp dependence was proved by Beznosova and Reznikov recentely in [BeRez].

Theorem 3.12 (Beznosova-Reznikov, [BeRez]). *Let $w \in RH_1^d$, then for any dyadic interval $J \in \mathcal{D}$*

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} (m_I w) \left(\frac{\Delta_I w}{m_I w} \right)^2 |I| \leq C[w]_{RH_1^d} m_J w$$

3.4 Lift Lemma

Given a dyadic interval L, and weights u, v, we introduce a family of stoping time intervals $\mathcal{ST}_L^m$ such that the averages of the weights over any stopping time interval

$K \in \mathcal{ST}_L^m$ are comparable to the averages on L. This construction was introduced in [NV] for the case $u = w$, $v = w^{-1}$ in their proof of the A_2-conjecture for Haar shift operators with complexity (m, n) with polynomial dependence in the complexity. We also present a lemma that lifts w-Carleson sequences on intervals to w-Carleson sequences on "stopping intervals", this was used in [NV] for the very specific stoping time intervals $\mathcal{ST}_L^m$. We present the proofs for the convenience of the reader.

Lemma 3.13 (Lift Lemma [NV])**.** *Let u and v be weights, L be a dyadic interval and m, n be fixed positive integers. Let $\mathcal{ST}_L^m$ be the collection of maximal stopping time intervals $K \in \mathcal{D}(L)$, where the stopping criteria are either* (i) $\frac{|\Delta_K u|}{m_K u} + \frac{|\Delta_K v|}{m_K v} \geq \frac{1}{m+n+2}$, *or* (ii) $|K| = 2^{-m}|L|$. *Then for any stopping interval $K \in \mathcal{ST}_L^m$, $\frac{e^{-1}}{2} m_L u \leq m_K u \leq e\, m_L u$, and hence also $\frac{e^{-1}}{2} m_L v \leq m_K v \leq e\, m_L v$.*

Note that the roles of m and n can be interchanged and we get the family $\mathcal{ST}_L^n$ using the same stopping condition (i) and condition (ii) replaced by $|K| = 2^{-n}|L|$. Notice that $\mathcal{ST}_L^m$ is a partition of L in dyadic subintervals of length at least $2^{-m}|L|$. Any collection of subintervals of L with this property will be an m-stopping time for L.

Proof. Let K be a maximal stopping time interval, no dyadic interval strictly bigger than K can satisfy any of the stopping criteria. If F is a dyadic interval strictly bigger than K and contained in L then necessarily

$$\frac{|\Delta_F u|}{m_F u} \leq \frac{1}{m+n+2} \qquad \text{and} \qquad \frac{|\Delta_F v|}{m_F v} \leq \frac{1}{m+n+2}. \tag{3.12}$$

In particular this is true for the parent of K. Let us denote $\widehat{K}$ the parent of K and K^* its sibling, and

$$\begin{aligned} |m_K u - m_{\widehat{K}} u| &= \big|m_K w - \frac{m_K w + m_{K^*} w}{2}\big| \leq \big|\frac{m_K u - m_{K^*} u}{2}\big| \\ &= \frac{|\Delta_{\widehat{K}} u|}{2} \leq \frac{m_{\widehat{K}} u}{2(m+n+2)}. \end{aligned}$$

So, $m_{\widehat{K}}u\big(1-\frac{1}{2(m+n+2)}\big) \leq m_K u \leq m_{\widehat{K}}u\big(1+\frac{1}{2(m+n+2)}\big)$. Iterating this process until we reach L, we will get that

$$m_L u\left(1-\frac{1}{2(m+n+2)}\right)^m \leq m_K u \leq m_L u\left(1+\frac{1}{2(m+n+2)}\right)^m$$

remember that $|K| = 2^{-j}|L|$ where $0 \leq j \leq m$ so we will iterate at most m times. Now observe that

$$\left(1-\frac{1}{m}\right)^m < \left(1-\frac{1}{2(m+n+2)}\right)^m,$$

and

$$\left(1+\frac{1}{2(m+n+2)}\right)^m < \left(1+\frac{1}{2(m+n+2)}\right)^{2(m+n+2)}.$$

It is a calculus exercise to show that $\left(1-\frac{1}{m}\right)^m$ is bounded below by $\frac{e^{-1}}{2}$ and $\left(1+\frac{1}{m}\right)^m$ is an increasing sequence that goes to e. We prove these in the appendix Lemmas 8.1 and 8.3. Therefore

$$\frac{e^{-1}}{2} < m_L u\left(1-\frac{1}{m}\right)^m \leq m_K u \leq m_L u\left(1+\frac{1}{2(m+n+2)}\right)^{2(m+n+2)} < e.$$

□

The following lemma lifts a w-Carleson sequence to m-stopping time intervals with comparable intensity, it was spelled for the particular stopping time $\mathcal{ST}_L^m$ and $w = 1$ in [NV]. This is a property of any stopping time that stops once the m^{th}-generation is reached.

Lemma 3.14. *For each $L \in \mathcal{D}$ let $\mathcal{ST}_L^m$ be a partition of L in dyadic subintervals of length at least $2^{-m}|L|$ (in particular it could be the stopping time intervals defined in Lemma 3.13). Assume $\{\nu_I\}_{I\in\mathcal{D}}$ is a w-Carleson sequence with intensity at most A, let $\nu_L^m := \sum_{K\in\mathcal{ST}_L^m} \nu_I$, then $\{\nu_L^m\}_{L\in\mathcal{D}}$ is a w-Carleson sequence with intensity at most $(m+1)A$.*

Proof. In order to show that $\{\nu_L^m\}_{L\in\mathcal{D}}$ is a w-Carleson sequence with intensity at most $(m+1)A$, is enough to show that for any $J \in \mathcal{D}$

$$\frac{1}{|J|}\sum_{L\in\mathcal{D}(J)}\nu_L^m < (m+1)A\, m_J w.$$

Observe that for each dyadic interval K inside a fixed dyadic interval J there exist at most $m+1$ dyadic intervals L such that $K \in \mathcal{ST}_L^m$. Let us denote K^i the dyadic interval that contains K and such that $|K^i| = 2^i|K|$. If $K \in \mathcal{D}(J)$ then L must be $K^0, K^1, \dots$ or K^m. We just have to notice that if $L = K^i$, for $i > m$ then K cannot be in $\mathcal{ST}_L^m$ because $|K| < 2^{-m}|L|$. Therefore

$$\begin{aligned}\frac{1}{|J|}\sum_{L\in\mathcal{D}(J)}\nu_L^m =& \frac{1}{|J|}\sum_{L\in\mathcal{D}(J)}\sum_{K\in\mathcal{ST}_L^m}\nu_K = \frac{1}{|J|}\sum_{K\in\mathcal{D}(J)}\sum_{\substack{L\in\mathcal{D}(J)\, s.t.\\ K\in\mathcal{ST}_L^m}}\nu_K\\ &\leq \frac{1}{|J|}\sum_{K\in\mathcal{D}(J)}(m+1)\nu_K \leq (m+1)A\, m_J w.\end{aligned}$$

The last inequality follows by the definition of w-Carleson sequence with intensity A. The lemma is proved. □

Corollary 3.15 (Nazarov-Volverg, [NV]). *Let L be a dyadic interval and let $\mathcal{ST}_L^m$ be the collection of maximal stopping time intervals $K \in \mathcal{D}(L)$, where the stopping criteria are either* (i) $\frac{|\Delta_K u|}{m_K u} + \frac{|\Delta_K v|}{m_K v} \geq \frac{1}{m+n+2}$, *or* (ii) $|K| = 2^{-m}|L|$, *where n is a fixed positive integer. Then for any stopping cube $K \in \mathcal{ST}_L^m$*

$$\begin{aligned}\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} m_I(|f|w)\frac{|\Delta_I w|}{m_I w}\frac{|I|}{\sqrt{|L|}} \leq\\ \leq 2e^{\alpha}(m+n+2)m_K(|f|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}\sqrt{\mu_K}(m_L w\; m_L w^{-1})^{\frac{-\alpha}{2}}\end{aligned}$$

Proof. For any stopping interval K in $\mathcal{ST}_L^m$ we have that

$$\frac{1}{4}e^{-2}m_L w\; m_L w^{-1} \leq m_K w\; m_K w^{-1} \leq e^2 m_L w\; m_L w^{-1}$$

and if K is stopping interval by the first criteria then

$$1 \leq (m+n+2)\left(\frac{|\Delta_K w|}{m_K w} + \frac{|\Delta_K w^{-1}|}{m_K w^{-1}}\right) \leq (m+n+2)\sqrt{2}\sqrt{\frac{|\Delta_K w|^2}{(m_K w)^2} + \frac{|\Delta_K w^{-1}|^2}{(m_K w^{-1})^2}}.$$

If K is a stopping interval by the first criteria we will have that, where in the first inequality we just use that $\frac{|\Delta_I w|}{m_I w} \leq 2$,

$$\begin{aligned}
\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} m_I(|f|w)\frac{|\Delta_I w|}{m_I w}\frac{|I|}{\sqrt{|L|}} &\leq 2\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} m_I(|f|w)\frac{|I|}{\sqrt{|L|}}\\
&=2\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} \frac{1}{\sqrt{|L|}}\int_I |f(x)|w(x)dx\\
&=\frac{1}{\sqrt{|L|}}\int_K |f(x)|w(x)dx = m_K(|f|w)\frac{|K|}{\sqrt{|L|}}\\
&\leq(m+n+2)\sqrt{2}\sqrt{\frac{|\Delta_K w|^2}{(m_K w)^2} + \frac{|\Delta_K w^{-1}|^2}{(m_K w^{-1})^2}}\, m_K(|f|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}\\
&=(m+n+2)\sqrt{2}\sqrt{\mu_K}\,(m_K w)^{\frac{-\alpha}{2}}(m_K w^{-1})^{\frac{-\alpha}{2}}\, m_K(|f|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}\\
&\leq(m+n+2)\sqrt{2}e^{-\alpha}\sqrt{\mu_K}\,(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}\, m_K(|f|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}.
\end{aligned}$$

Now if K is a stopping interval by the second criteria then $K \in \mathcal{D}_m$

$$\begin{aligned}
\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} m_I(|f|w)\frac{|\Delta_I w|}{m_I w}\frac{|I|}{\sqrt{|L|}} &= m_K(|f|w)\frac{|\Delta_K w|}{m_K w}\frac{|K|}{\sqrt{|L|}}\\
&\leq\sqrt{2}\sqrt{\mu_K}\,(m_K w)^{\frac{-\alpha}{2}}(m_K w^{-1})^{\frac{-\alpha}{2}}\, m_K(|f|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}\\
&\leq(m+n+2)\sqrt{2}e^{-\alpha}\sqrt{\mu_K}\,(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}\, m_K(|f|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}
\end{aligned}$$

□

3.5 Sharp extrapolation theorem

In this section we will present one of the most important tools in the study of weighted inequalities, the extrapolation theorem of Rubio de Francia. We also present its sharp version due to Dragicevic *et al.*

Theorem 3.16 (Rubio de Francia's extrapolation theorem [Ru]). *Given an operator T, suppose that for some r, $1 \le r < \infty$ and every $w \in A_r$ there exists a constant C depending only on $[w]_{A_r}$, such that*

$$\|Tf\|_{L^r(w)} \le \|f\|_{L^r(w)}$$

Then for every p, $1 < p < \infty$ and every $w \in A_p$ there exists a constant depending only on $[w]_{A_p}$ such that

$$\|Tf\|_{L^p(w)} \le \|f\|_{L^p(w)}$$

For a proof see [GC-RF].

Dragičevič, Grafakos, Pereyra and Petermichl proved a sharp version of this result. They trace the dependence of the $L^p(w)$ norm on the the $L^r(w)$ norm of the operator.

Theorem 3.17 (Sharp extrapolation theorem [DGPPet]). *Given an operator T, suppose there is r, $1 \le r < \infty$, such that the operator T is bounded on $L^r(w)$ for all weights $w \in A_r$. Then the operator T is bounded on $L^p(w)$ for all $1 < p < \infty$ and weights $w \in A_p$. More precisely, suppose for each $B > 1$ there is a constant $N_r(B) > 0$, such that*

$$\|T\|_{L^r(w)} \le N_r(B) \quad \forall\, w \in A_r \quad \textit{with} \quad [w]_{A_r} \le B,$$

then for any $1 < p < \infty$ and $B > 1$ there exists $N_p(B) > 0$ such that for all weights $w \in A_p$ with $[w]_{A_p} \le N_p(B)$, $\|T\|_{L^p(w)} \le N_p(B)$, where

$$N_p(B) \le \begin{cases} 2^{\frac{1}{r}} N_r(2C(p')^{\frac{p-r}{p-1}}B), & \textit{if} \quad p > r; \\ 2^{\frac{r-1}{r}} N_r(2^{r-1}(C(p')^{p-r})^{\frac{p-r}{p-1}}B), & \textit{if} \quad p < r. \end{cases} \tag{3.13}$$

where $C(p) = Cp^{\frac{1}{p'}}(p')^{\frac{1}{p}}$*, this is the constant coming from the boundedness of maximal function in* $L^p(w)$*.*

Remark 3.18. *Note that if* $N_r(B) = B$*, then* $N_p(B) = C_p B^{\max\{1, \frac{r-1}{p-1}\}}$*, more specifically if* $r = 2$ *then* $N_p(B) = C_p B^{\max\{1 \frac{1}{p-1}\}}$ *for some constant* C_p*. Since for all Calderón-Zygmund operators it was proved by Hytönen in [H] that the dependence of the* $L^2(w)$ *norm on the* A_2 *characteristic is linear, by the extrapolation we have that the dependence of the* $L^p(w)$ *norm on the* A_p *is given by* $[w]_{A_p}^{\max\{1 \frac{1}{p-1}\}}$*.*

Cruz-Uribe and Pérez observed in [CrPz] that Theorem 3.16 holds for any pair of of function (f, g) and the same idea can be applied for Theorem 3.17 i.e., if

$$\|g\|_{L^r(w)} \leq N_r(B)\|f\|_{L_r(w)} \qquad \text{for all} \quad w \in A_r$$

with $[w]_{A_r} \leq B$ then

$$\|g\|_{L^p(w)} \leq N_p(B)\|f\|_{L_p(w)} \qquad \text{for all} \quad w \in A_p$$

and Theorem 3.17 is a particular case of this result for $g = Tf$.

Chapter 4

Haar shift operators with complexity (m, n)

Given a nonnegative integer number $\tau > 0$, the generalized Haar shift operator of index τ is an operator action on a locally integrable functions, and defined as follows.

Definition 4.1. *The generalized Haar shift operator of index τ, G^τ is defined as*

$$(G^\tau f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I,J\in\mathcal{D}(L)\\ 2^{-\tau}|L|\leq|I|,|J|}} C^L_{I,J}\langle f, P_I\rangle Q_J(x) \qquad f\in L^1_{loc} \tag{4.1}$$

where $P_I, Q_I \in \{h_I, \chi_I/|I|\}$

We will decompose this operator in four types of operators, where in each of them for every dyadic interval I, P_I and Q_I are always the Haar functions h_I or the characteristic function $\frac{\chi_I}{|I|}$. The table 4 summarizes how we are going to construct these operators. Some authors define Q_I normalized in L^2 , i.e. $\frac{\chi_I}{\sqrt{|I|}}$, in our case Q_I is normalized in L^1.

Generalized Haar shift	P_I	Q_J
type 1	h_I	h_J
type 2	$\chi_I/\lvert I\rvert$	h_J
type 3	h_I	$\chi_J/\lvert J\rvert$
type 4	$\chi_I/\lvert I\rvert$	$\chi_J/\lvert J\rvert$

A Haar shift operator of index τ can be decomposed as sum of operators of type $1-4$.

$$G^\tau f = \sum_{i=1}^{4} G_i^\tau f$$

$$(G_1^\tau f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I,J\in\mathcal{D}(L)\\ 2^{-\tau}|L|\leq |I|,|J|}} C_{I,J}^{L,1}\langle f, h_I\rangle h_J(x); \tag{4.2}$$

$$(G_2^\tau f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I,J\in\mathcal{D}(L)\\ 2^{-\tau}|L|\leq |I|,|J|}} C_{I,J}^{L,2}\langle f, \frac{\chi_I}{|I|}\rangle h_J(x); \tag{4.3}$$

$$(G_3^\tau f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I,J\in\mathcal{D}(L)\\ 2^{-\tau}|L|\leq |I|,|J|}} C_{I,J}^{L,3}\langle f, h_I\rangle \frac{\chi_J(x)}{|J|}; \tag{4.4}$$

$$(G_4^\tau f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I,J\in\mathcal{D}(L)\\ 2^{-\tau}|L|\leq |I|,|J|}} C_{I,J}^{L,4}\langle f, \frac{\chi_I}{|I|}\rangle \frac{\chi_J(x)}{|J|}. \tag{4.5}$$

We will name the operator G_i^τ generalized Haar shift operator of type i and index τ.

The generalized Haar shift operators of type 1 were defined by Lacey, Petermichl and Reguera, in [LPetR], where they proved that the $L^2(w)$ norm of any operator of this type is bounded linearly by the A_2^d characteristic of the weight w, i.e., for a weight $w\in A_2^d$

$$\|G_1^\tau\|_{L^2(w)} \leq C[w]_{A_2^d},$$

provided $C_{I,J}^{L,1} \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ for all $I,J\in\mathcal{D}$. This was the first result where the linear dependence in the A_2^d characteristic was obtained for a whole class of operators. This class of operators played an important role in the solution of the A_2-conjecture. The down side of their method of proof was that the dependence of the constant C on the index τ was exponential. In order to prove A_2-conjecture better dependence had to be proven. In fact it was proved later by [H, HPzTV, NV, T, L] that this dependence could be improved to be polynomial. However instead of working with these generalized Haar shifts operators, most of these authors preferred to work with what are called in [HPzTV] elementary Haar shifts, which we will call here elementary Haar shifts of type 1. We will discuss in the next section that if we can show that these operators are bounded on $L^2(w)$ by $C[w]_{A_2}$ where C depends polynomially on the complexity (m,n) then operators of the form G_1^{τ} will also be bounded by a constant that depends polynomially in the index τ and linearly in the A_2^d characteristic.

4.1 Elementary Haar shifts of type 1 with complexity (m,n)

We will define the elementary Haar shifts of type i for $i \in \{1,2,3,4\}$ with complexity (m,n) in a similar fashion that we defined the generalized Haar shifts of type i, however in their definition instead of having I and J dyadic intervals such that $I, J \in \mathcal{D}(L)$ with $2^{-\tau}|L| \leq |I|, |J|$ we will have $I, J \in \mathcal{D}(L)$ with $2^{-n}|L| = |I|$ and $2^{-m}|L| = |J|$. Let us be more precise and define these operators. We will often omit the world elementary and call these operators just dyadic shifts of type 1 with complexity (m,n).

Definition 4.2. *An operator is elementary Haar Shifts of type 1 with complexity*

(m, n) *if the operator has the following form*

$$(T_1^{m,n} f)(x) := \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^{L,1} \langle f, h_I \rangle h_J(x). \tag{4.6}$$

where $c_{I,J}^{L,1} \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ *for all dyadic intervals* I, J, L.

We impose the size condition in order for the operator to be bounded in L^2.

$$\begin{aligned} \|T_1^{m,n} f\|_{L^2}^2 &= \left\| \sum_{J \in \mathcal{D}} \Big(\sum_{I \in \mathcal{D}_m(J^n)} c_{I,J}^{J^n,1} \langle f, h_I \rangle \Big) h_I \right\|_{L^2}^2 = \sum_{J \in \mathcal{D}} \left| \sum_{I \in \mathcal{D}_m(J^n)} c_{I,J}^{J^n,1} \langle f, h_I \rangle \right|^2 \\ &\leq 2^m \sum_{J \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(J^n)} \frac{|I||J|}{|J^n|^2} |\langle f, h_I \rangle|^2. \end{aligned}$$

In the last inequality we used the fact that $\left| \sum_{i=1}^N a_i \right|^2 \leq N \sum_{i=1}^N a_i^2$ where N in this case is 2^m, which is the amount of dyadic intervals in $\mathcal{D}_m(I^n)$. There are 2^n interval J whose ancestor is J^n. Also note that for each $I \in \mathcal{D}_m(J^n)$, $|I| = 2^{-m}|J^n|$ and since $|J| = 2^{-n}|J^n|$ so $\frac{|I||J|}{|J^n|^2} = 2^{-m-n}$, thus

$$\|T_1^{m,n} f\|_{L^2}^2 \leq 2^m \cdot 2^n \sum_{I \in \mathcal{D}} 2^{-n-m} |\langle f, h_I \rangle|^2 = \|f\|_{L^2}^2,$$

we can conclude $\|T_1^{m,n} f\|_{L^2} \leq \|f\|_{L^2}$, so $\|T_1^{m,n}\|_{L^2} \leq 1$.

In the case $m = n = 0$ and $c_{I,I}^{I,1} \in \{-1, 1\}$ this operator is the martingale transform. Another important example of elementary Haar shift is Petermichl's Sha operator, which is defined as

$$(T_1^{0,1} f)(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \big(h_{I_+}(x) - h_{I_-}(x) \big) \tag{4.7}$$

The operator Sha is an an elementary Haar shift operator of complexity $(0, 1)$. A breakthrough result from Petermichl says that the Hilbert transform can be written as an average of dilations and translations of the Sha operator. In [Pet2], Petermichl makes use of this representation and Bellman function techniques to prove that the $L^2(w)$ norm of the Hilbert transform depends linearly on the A_2 characteristic. It should be clear that a Haar shift of index τ is the sum of the elementary shifts of complexity (m,n) where $0 \le m,n \le \tau$.

An important and hard part of the proof of the A_2 conjecture was to obtain bounds for Haar shifts operators of type 1 that depend linearly on the A_2- characteristic and at most polynomially on the complexity (m,n). In 2011, Nazarov and Volberg [NV] provided a beautiful new proof that still uses Bellman functions but minimally, and that can be transferred to geometric doubling metric spaces [NV1, NRezV]. Treil [T], independently [HLM+] are able to obtain linear dependence in the complexity. Crucial in both [NV] and [HLM+] is the use of some stopping time argument (it is called a corona decomposition in [LPetR, L1, HLM+]).

Theorem 4.3 (Hytönen-Nazarov-Pérez-Lacey-Volberg-Treil, [HPzTV, NV, L, T]). *Let* (m,n) *be nonnegative integer numbers and* w *a weight in* A_2^d *then*

$$\|T_1^{m,n}\|_{L^2(w)} \le C(m+n+1)^k [w]_{A_2^d} \tag{4.8}$$

This result was proved by Hytönen,Pérez,Volberg and Treil in [HPzTV] with $k=3$, later Nazarov and Volberg gave a new and very interesting prove with $k=4$ and Lacey in [L] and Treil in [T] proved with $k=1$. For the proof of the A_2 conjecture estimate 4.8 for any finite k is enough.

Let us now prove that composition of Haar shits of type 1 is also a Haar shift of type 1.

Theorem 4.4 (Composition of Haar shifts). *Let* m,n,r,s *be nonnegative numbers and* $T_1^{m,n}$ *and* $T_1^{r,s}$ *Haar shifts operators of type 1, then the composition* $T_1^{m,n}T_1^{r,s}$ *is a Haar shift of type 1 with complexity* $\big(\max\{m-s+r,r\},\max\{n,n+s-m\}\big)$.

Proof. Let

$$(T_1^{m,n} f)(x) = \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^{L,1} \langle f, h_I \rangle h_J(x). \tag{4.9}$$

$$(T_1^{r,s} f)(x) = \sum_{U \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_r(U) \\ Y \in \mathcal{D}_s(U)}} c_{X,Y}^{U,1} \langle f, h_I \rangle h_J(x). \tag{4.10}$$

where

$$|c_{I,J}^{L,1}| \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|} \quad \text{and} \quad |c_{X,Y}^{U,1}| \leq \frac{\sqrt{|X|}\sqrt{|Y|}}{|U|}.$$

Then for a given f

$$\begin{aligned}
T_1^{m,n} T_1^{r,s} f(x) &= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^{L,1} \left\langle \sum_{U \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_r(U) \\ Y \in \mathcal{D}_s(U)}} c_{X,Y}^{U,1} \langle f\,,\, h_X \rangle h_Y \,,\, h_I \right\rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} \sum_{U \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_r(U) \\ Y \in \mathcal{D}_s(U)}} c_{I,J}^{L,1} c_{X,Y}^{U,1} \langle f\,,\, h_X \rangle \langle h_Y\,,\, h_I \rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{U \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_r(U) \\ J \in \mathcal{D}_n(L)}} \left(\sum_{\substack{I \in \mathcal{D}_m(L) \\ Y \in \mathcal{D}_s(U)}} c_{I,J}^{L,1} c_{X,Y}^{U,1} \langle h_Y\,,\, h_I \rangle \right) \langle f\,,\, h_X \rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{U \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_r(U) \\ J \in \mathcal{D}_n(L)}} c_{X,J}^{LU,1} \langle f\,,\, h_X \rangle h_J(x)
\end{aligned}$$

where

$$c_{X,J}^{LU,1} := \sum_{\substack{I \in \mathcal{D}_m(L) \\ Y \in \mathcal{D}_s(U)}} c_{I,J}^{L,1} c_{X,Y}^{U,1} \langle h_Y\,,\, h_I \rangle.$$

Since $Y \in \mathcal{D}_s(U)$ and $I \in \mathcal{D}_m(L)$ and $\langle h_Y\,,\, h_I \rangle \neq 0$ if and only if $Y = I$ we have that if $U \bigcap L = \emptyset$ then $\langle h_Y\,,\, h_I \rangle = 0$. Moreover, if $Y = I$ we have that $|U| 2^{-s} = 2^{-m} |L|$, which implies that $U \in \mathcal{D}_{m-s}(L)$ if $m > s$ or $L \in \mathcal{D}_{s-m}(U)$ if $s > m$ and $U = L$ if $s = m$.

If $m \geq s$ then

$$T_1^{m,n} T_1^{r,s} f(x) = \sum_{L \in \mathcal{D}} \sum_{U \in \mathcal{D}_{m-s}(L)} \sum_{\substack{X \in \mathcal{D}_r(U) \\ J \in \mathcal{D}_n(L)}} c_{X,J}^{LU,1} \langle f\,,\, h_X \rangle h_J(x)$$
$$= \sum_{L \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_{m+r-s}(L) \\ J \in \mathcal{D}_n(L)}} c_{X,J}^{LU,1} \langle f\,,\, h_X \rangle h_J(x),$$

because if $U \in \mathcal{D}_{m-s}(L)$ and $X \in \mathcal{D}_r(U)$ then $X \in \mathcal{D}_{m+r-s}(L)$. Therefore if $c_{X,J}^{LU,1} \leq \dfrac{\sqrt{|X|}\sqrt{|J|}}{|L|}$, then $T_1^{m,n} T_1^{r,s}$ is Haar shift of type 1 with complexity $(m + r - s, n)$. Let us show now that $c_{X,J}^{LU,1}$ satisfies the size condition, where U is the unique interval in $\mathcal{D}_{m-s}$ containing $X \in \mathcal{D}_{m+s}$.

$$|c_{X,J}^{LU,1}| = \left| \sum_{\substack{I \in \mathcal{D}_m(L) \\ Y \in \mathcal{D}_s(U)}} c_{I,J}^{L,1} c_{X,Y}^{U,1} \langle h_Y\,,\, h_I \rangle \right|$$
$$\leq \sum_{Y = I \in \mathcal{D}_s(U)} |c_{I,J}^{L,1} c_{X,Y}^{U,1}| \leq \sum_{Y \in \mathcal{D}_s(U)} \frac{\sqrt{|Y|}\sqrt{|J|}}{|L|} \frac{\sqrt{|X|}\sqrt{|Y|}}{|U|}$$
$$= 2^s \frac{2^{-\frac{s}{2}}\sqrt{|U|}\sqrt{|J|}}{|L|} \frac{\sqrt{|X|} 2^{-\frac{s}{2}}\sqrt{|U|}}{|U|} = \frac{\sqrt{|X|}\sqrt{|J|}}{|L|}.$$

Note that $I = Y$ should be in U and in L, but since $U \subset L$ then the first sum just collapses to a sum for $Y \in \mathcal{D}_s(U)$.

If $s > m$ then

$$T_1^{m,n} T_1^{r,s} f(x) = \sum_{U \in \mathcal{D}} \sum_{L \in \mathcal{D}_{s-m}(U)} \sum_{\substack{X \in \mathcal{D}_r(U) \\ J \in \mathcal{D}_n(L)}} c_{X,J}^{LU,1} \langle f\,,\, h_X \rangle h_J(x)$$
$$= \sum_{U \in \mathcal{D}} \sum_{\substack{X \in \mathcal{D}_r(U) \\ J \in \mathcal{D}_{s+n-m}(U)}} c_{X,J}^{LU,1} \langle f\,,\, h_X \rangle h_J(x)$$

Note if $L \in \mathcal{D}_{s-m}(U)$ and $J \in \mathcal{D}_n(L)$ then $J \in \mathcal{D}_{s+n-m}(U)$. Therefore we just have to show now that $c_{X,J}^{LU,1} \leq \dfrac{\sqrt{|X|}\sqrt{|J|}}{|U|}$ in order to $T_1^{m,n} T_1^{r,s}$ be Haar shift of type 1 with complexity $(n, n + s - m)$. Lets us show now that if $c_{X,J}^{LU,1}$ satisfies the size condition.

$$|c_{X,J}^{LU,1}| = \left| \sum_{\substack{I \in \mathcal{D}_m(L) \\ Y \in \mathcal{D}_s(U)}} c_{I,J}^{L,1} c_{X,Y}^{U,1} \langle h_Y \, , \, h_I \rangle \right|$$

$$\leq \sum_{Y = I \in \mathcal{D}_m(L)} |c_{I,J}^{L,1} c_{X,Y}^{U,1}| \leq \sum_{I \in \mathcal{D}_m(U)} \frac{\sqrt{|I|}\sqrt{|J|}}{|L|} \frac{\sqrt{|X|}\sqrt{|I|}}{|U|}$$

$$= 2^m \frac{2^{-\frac{m}{2}}\sqrt{|L|}\sqrt{|J|}}{|L|} \frac{\sqrt{|X|}2^{-\frac{m}{2}}\sqrt{|L|}}{|U|} = \frac{\sqrt{|X|}\sqrt{|J|}}{|U|}.$$

Again note that $I = Y$ should be in U and in L, but since $L \subset U$ then the first sum just collapses into a sum for $I \in \mathcal{D}_m(U)$. Thus,

$$T_1^{m,n} T_1^{r,s} = T_1^{\left(\max\{m-s+r,r\},\max\{n,n+s-m\}\right)}.$$

□

4.2 Elementary Haar shifts of type 2, 3 and 4 with complexity (m, n)

Analogously we define the elementary Haar shifts of type 2, 3 and 4. For these operators the size condition is not enough to guarantee boundedness on L^2, so boundedness on L^2 is assumed in order to prove the linear dependence on the A_2^d characteristic on weighted spaces.

Definition 4.5. *We define an elementary Haar Shifts of type 2 with complexity (m, n) as*

$$(T_2^{m,n} f)(x) := \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^{L,2} \langle f, \frac{\chi_I(x)}{|I|} \rangle h_J(x). \tag{4.11}$$

For $m = n = 0$ and $c_{I,I}^{I,2} = \langle b, h_I \rangle$for some function $b \in L^1_{loc}$ this operator is the dyadic paraproduct, π_b, known to be bounded in L^2 if and only if $b \in BMO^d$.

Definition 4.6. *An operator is an elementary Haar shifts of type 3 with complexity* (m,n) *if the operator has the following form*

$$(T_3^{m,n} f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,3} \langle f, h_I\rangle \frac{\chi_J(x)}{|J|}. \tag{4.12}$$

Remark 4.7. *For* $m = n = 0$ *and* $c_{I,I}^{I,3} = \langle d, h_I\rangle$, *for some function* $d \in L^1_{loc}$, *this operator is an adjoint of a dyadic paraproduct,* π_d^*, *known to be bounded in* L^2 *if and only if* $b \in BMO^d$.

Definition 4.8. *We define an elementary Haar Shifts of type 4 with complexity* (m,n) *as*

$$(T_4^{m,n} f)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,4} \langle f, \frac{\chi_I}{|I|}\rangle \frac{\chi_J(x)}{|J|}. \tag{4.13}$$

Remark 4.9. *For* $m = n = 0$ *and* $c_{I,I}^{I,4} = \langle b, h_I\rangle \langle d, h_I\rangle$ *for some function* $b, d \in L^1_{loc}$ *this operator is formally the composition of an adjoint of a dyadic paraproduct with a dyadic paraproduct,* $\pi_d^*\pi_b$.

Note that for $i \in \{1, 2, 3, 4\}$ we can estimate the norm of G_i^τ by the norm of an elementary Haar shift of type i. For all m, n such that $0 \le m, n \le \tau$ let $T_i^{m,n}$ be the elementary Haar shift of type i where the $c_{I,J}^{L,i} = C_{I,J}^{L,i}$ and $C_{I,J}^{L,i}$ are the coefficients of G^τ. Then for w a weight

$$\|T_i^\tau f\|_{L^2(w)} \le (\tau+1)^2 \max_{1\le m,n\le\tau} \|T_i^{m,n}\|_{L^2(w)} \tag{4.14}$$

Therefore if for $i \in \{1, 2, 3, 4\}$ we can estimate $T_i^{m,n}$ for an arbitrary m, n by

$$\|T_i^{m,n}\|_{L^2(w)} \le C(m+n+2)^a [w]_{A_d^2},$$

for some positive integer a. Note if $\tau = \max\{m, n\}$, then $(m+n+2)^a \le 2^a(\tau+1)^a$, then

$$\|T_i^\tau\|_{L^2(w)} \le C(\tau+1)^{a+2} [w]_{A_d^2}.$$

It is easy to see that Haar shifts of type 1, 2, 3 can be written as Haar shifts of type 4, we just use the fact that a Haar function in an interval I can be written as a linear combination of characteristic functions supported in the children of this interval. We summarize these facts in the next remark.

Remark 4.10. *Given m,n two nonnegative integer numbers, let $T_i^{m,n}$ for $i \in \{1,2,3\}$ be elementary operators of type i with complexity (m,n) as we defined with coefficients $c_{I,J}^{L,i}$. Then*

(i) $T_1^{m,n} = T_4^{m+1,n+1}$ *where* $c_{I_+,J_+}^{L,4} = c_{I_-,J_-}^{L,4} = \frac{\sqrt{|I|}\sqrt{|J|}}{4} c_{I,J}^{L,1}$ *and* $c_{I_-,J_+}^{L,4} = c_{I_+J_-}^{L,4} = -\frac{\sqrt{|I|}\sqrt{|J|}}{4} c_{I,J}^{L,1}$

(ii) $T_2^{m,n} = T_4^{m,n+1}$ *where* $c_{I,J_+}^{L,4} = \frac{\sqrt{|J|}}{2} c_{I,J}^{L,1}$ *and* $c_{I,J_-}^{L,4} = -\frac{\sqrt{|J|}}{2} c_{I,J}^{L,1}$

(iii) $T_3^{m,n} = T_4^{m+1,n}$ *where* $c_{I,J_+}^{L,4} = \frac{\sqrt{|I|}}{2} c_{I,J}^{L,1}$ *and* $c_{I,J_-}^{L,4} = -\frac{\sqrt{|I|}}{2} c_{I,J}^{L,1}$

As we already discussed, Haar shifts of type 1 are bounded in L^2 if we impose size conditions on the coefficients, but in order to get boundedness of operators of type 4 we need to impose further conditions, see Theorem 3.4 in [HPzTV]. The advantage to separate the operators of type 1, 2 and 3 from the operators of type 4, is that we can impose less conditions on them in order to be bounded in $L^2(w)$ for a weight $w \in A_2$. Also instead of testing condition we will impose conditions on the coefficients of the operator.

Operator	Complexity as an operator of type 4
Martingale transform	$(1,1)$
Paraproduct	$(0,1)$
Dual paraproduct	$(1,0)$
Sha	$(1,2)$

Next theorem relates a Haar shift of type i with complexity (m, n) with its formal adjoint. The formal adjoint of Haar shift of type 1 is also a Haar shift of type 1. The same phenomena occurs with Haar shifts of type 4. Haar shifts of type 2 have Haar shifts of type 3 as adjoints.

Proposition 4.11. *Given nonnegative integers m and n then the formal adjoint of the operator $T_i^{m,n}$ are given by the following*

(i) $(T_1^{m,n})^* = T_1^{n,m}$ *and* $(T_4^{m,n})^* = T_4^{n,m}$

(ii) $(T_2^{m,n})^* = T_3^{n,m}$ *and* $(T_3^{m,n})^* = T_2^{n,m}$

Proof. *(i)*

$$\begin{aligned}\langle T_1^{m,n} f, g\rangle &= \Bigg\langle \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,1}\langle f, h_I\rangle h_J, g\Bigg\rangle \\ &= \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,1}\langle f, h_I\rangle\langle h_J, g\rangle \\ &= \Bigg\langle f, \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,1}\langle g, h_J\rangle h_I\Bigg\rangle = \langle f, T_1^{n,m} g\rangle,\end{aligned}$$

where $T_1^{n,m} g := \sum_{L\in\mathcal{D}} \sum_{\substack{J\in\mathcal{D}_m(L)\\ I\in\mathcal{D}_n(L)}} d_{J,I}^{L,1}\langle g, h_J\rangle h_I$ and $d_{J,I}^{L,1} = c_{I,J}^{L,1}$ obey size condition. And

$$\begin{aligned}\langle T_4^{m,n} f, g\rangle &= \Bigg\langle \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,4}\langle f, \chi_I/|I|\rangle \chi_J/|J|, g\Bigg\rangle \\ &= \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,4}\langle f, \chi_I/|I|\rangle\langle \chi_J/|J|, g\rangle \\ &= \Bigg\langle f, \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,4}\langle g, \chi_J/|J|\rangle \chi_I/|I|\Bigg\rangle = \langle f, T_4^{n,m} g\rangle,\end{aligned}$$

where $T_4^{n,m} g := \sum_{L\in\mathcal{D}} \sum_{\substack{J\in\mathcal{D}_m(L)\\ I\in\mathcal{D}_n(L)}} d_{J,I}^{L,4}\langle g, h_J\rangle h_I$ and $d_{J,I}^{L,4} = c_{I,J}^{L,4}$ obey size condition.

(ii)

$$\begin{aligned}\langle T_2^{m,n} f, g\rangle &= \left\langle \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,2}\langle f, \chi_I/|I|\rangle h_J, g\right\rangle \\ &= \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,2}\langle f, \chi_I/|I|\rangle \langle h_J, g\rangle \\ &= \left\langle f, \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L,2}\langle g, h_J\rangle \chi_I/|I|\right\rangle = \langle f, T_3^{n,m} g\rangle,\end{aligned}$$

where $T_3^{n,m} g := \sum_{L\in\mathcal{D}} \sum_{\substack{J\in\mathcal{D}_m(L)\\ I\in\mathcal{D}_n(L)}} d_{J,I}^{L,3}\langle g, h_J\rangle h_I$ and $d_{J,I}^{L,3} = c_{I,J}^{L,2}$ obey size condition. By what we just proved we have that $(T_3^{m,n})^* = (T_2^{n,m})^{**} = T_2^{n,m}$. □

Remark 4.12. *For any nonnegative integer m, $T_1^{m,m}$ and $T_4^{m,m}$ are self adjoint operators.*

As a corollary, if $T_i^{m,n}$ is bounded in $L^2(w)$ with a bound that depends on $[w]_{A_2^d}$ and on $(m+n)$ then $(T_i^{m,n})^*$ is bounded on $L^2(w^{-1})$ with the same bound depending on $[w^{-1}]_{A_2^d} = [w]_{A_2^d}$ and $m + n = n + m$.

4.3 A further particularization

We will now define the particular cases of operators of type 2, 3 and 4 that we will work on this dissertation. A *paraproduct of complexity* (m, n) is the operator defined formally by

$$\left(\pi_b^{m,n} f\right)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^{L} b_I m_I f h_J(x), \tag{4.15}$$

where $|c^L_{I,J}| \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ for all dyadic intervals I, J and L and $b_I = \langle b, h_I \rangle$ for a locally integrable function b.

A paraproduct of complexity $(0, 0)$ is the dyadic paraproduct π_b, known to be bounded in $L^p(\mathbb{R})$ if and only if $b \in BMO^d$. Similarly $\pi_b^{m,n}$ will be bounded in $L^p(\mathbb{R})$ if and only if $b \in BMO^d$, furthermore it will be bounded in $L^p(w)$ whenever $w \in A_p$, and we will trace the dependence of the operator bound in the A_p-characteristic of the weight, the BMO^d norm of b and the complexity (m, n) in the next chapter.

The definition of paraproduct of complexity (m, n) is inspired by the definition of $T_1^{m,n}$, the Haar Shift operators with complexity (m, n), in [H], [HPzTV] [NV].

One can observe that the paraproduct of complexity (m, n) is the composition of the Haar shift operator of type 1 with complexity (m, n) and the dyadic paraproduct of complexity $(0, 0)$.

Proposition 4.13. *Consider a paraproduct of complexity* (m, n), $\pi_b^{m,n}$, *with coefficients* $c^L_{I,J}$ *then*

$$\pi_b^{m,n} = T_1^{m,n} \pi_b,$$

where $T_1^{m,n}$ *has coefficients* $c^{L,1}_{I,J} = c^L_{I,J}$.

Proof. For any f

$$\begin{aligned}
(T_1^{m,n} \pi_b f)(x) &= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c^{L,1}_{I,J} \langle \pi_b f, h_I \rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c^L_{I,J} \Big\langle \sum_{K \in \mathcal{D}} b_K \, m_K f \, h_k, h_I \Big\rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c^L_{I,J} \sum_{K \in \mathcal{D}} b_K \, m_K f \langle h_K, h_I \rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c^L_{I,J} \, b_I \left\langle f, \frac{\chi_I(x)}{|I|} \right\rangle h_J(x) = \pi_b^{m,n} f.
\end{aligned}$$

□

A paraproduct of complexity (m,n) is a type 2 Haar shift operator and therefore by proposition 4.11 its adjoint should be a Haar shift operator of type 3. For a function $b \in L^1_{loc}$, we define the *dual paraproduct operator of complexity* (m,n), $\kappa_b^{m,n}$ by

$$\left(\kappa_b^{m,n} f\right)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_n(L)\\ J\in\mathcal{D}_m(L)}} c^L_{I,J}\, b_I \,\langle f, h_I\rangle \frac{\chi_J(x)}{|J|}, \tag{4.16}$$

where $|c^L_{I,J}| \le \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ for all dyadic intervals I, J and L and $b_I = \langle b, h_I\rangle$ for all $I \in \mathcal{D}$.

The dual paraproduct with complexity (m,n) is a type 3 Haar shift operator, moreover

$$(\pi_b^{m,n})^* = \kappa_b^{n,m} \tag{4.17}$$

Using identity 4.17 and Propositions 4.14 and 4.11 we have that

$$\kappa_b^{m,n} = (\pi_b^{n,m})^* = (T_1^{n,m}\pi_b)^* = \pi_b^*(T_1^{n,m})^* = \pi_b^* T_1^{m,n}. \tag{4.18}$$

Given $b, d \in L^1_{loc}$ then composition of a paraproduct with dual paraproduct operator of complexity (m,n),

$$\left(\zeta_{b,d}^{m,n} f\right)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_n(L)\\ J\in\mathcal{D}_m(L)}} c^L_{I,J}\, b_I\, d_I\, m_I f\, \frac{\chi_J(x)}{|J|}, \tag{4.19}$$

where $|c^L_{I,J}| \le \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ for all dyadic intervals I, J and L and $b_I = \langle b, h_I\rangle$ and $d_I = \langle d, h_I\rangle$. These operators are particular case of a dyadic shift of type 4 with complexity (m,n).

Proposition 4.14. *Given* $b, d \in L^1_{loc}$ *and composition of a paraproduct with dual paraproduct operator of complexity* (m, n)*,* $\zeta^{m,n}_{b,d}$ *with coefficients* $\{c^L_{I,J}\}$ *then formally*

$$\zeta^{m,n}_{b,d} = \pi^*_b T^{m,n}_1 \pi_d,$$

where $T^{m,n}_1$ *have coefficients* $c^{L,1}_{I,J} = c^L_{I,J}$ *for all* $I \in \mathcal{D}$*.*

Proof. By proposition 4.14,

$$(T^{m,n}_1 \pi_d) = \pi^{m,n}_d$$

Then

$$\begin{aligned}(\pi^*_b T^{m,n}_1 \pi_d f)(x) &= \sum_{K\in\mathcal{D}} b_K \Big\langle \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c^L_{I,J}\, d_I \Big\langle f, \frac{\chi_I(x)}{|I|} \Big\rangle h_J(x), h_K \Big\rangle \frac{\chi_J}{|J|}(x)\\
&= \sum_{K\in\mathcal{D}} \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} b_K d_I\, c^L_{I,J} \Big\langle f, \frac{\chi_I(x)}{|I|} \Big\rangle \langle h_I, h_K\rangle \frac{\chi_J}{|J|}(x)\\
&= \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} b_I d_I\, c^L_{I,J} \Big\langle f, \frac{\chi_I(x)}{|I|} \Big\rangle \frac{\chi_J}{|J|}(x)\\
&= \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} b_I\, d_I\, c^L_{I,J} \Big\langle f, \frac{\chi_I(x)}{|I|} \Big\rangle \frac{\chi_J}{|J|}(x)\end{aligned}$$

Remark 4.15. *For* m, n, r, s *nonnegative integers we have,*

$$\begin{aligned}\kappa^{m,n}_b \pi^{r,s}_d &= \pi^*_b (T^{m,n}_1)^* T^{r,s}_1 \pi_d = \pi^*_b T^{n,m}_1 T^{r,s}_1 \pi_d\\
&= \pi^*_b T_1^{\big(\max\{n-s+r,r\},\max\{m,m+s-n\}\big)} \pi_d = \zeta_{b,d}^{\big(\max\{n-s+r,r\},\max\{m,m+s-n\}\big)}\end{aligned}$$

□

Chapter 5

Bounds for Operators type 2 and 3

A *paraproduct of complexity* (m, n) is the operator defined formally for $b \in L^1_{loc}$ by

$$\big(\pi_b^{m,n} f\big)(x) := \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c^L_{I,J} m_I f \langle b, h_I\rangle h_J(x), \tag{5.1}$$

where $|c^L_{I,J}| \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$ for all dyadic intervals I, J and L.

In Chapter 4 we showed that the paraproduct of complexity (m, n) is the composition of the Haar shift operator of complexity (m, n) and the dyadic paraproduct of complexity $(0, 0)$, $\pi_b^{m,n} = T_1^{m,n}\pi_b$. It is well known that both the Haar shift operators [LPetR, CrMPz, H, T, L] and the dyadic paraproduct [Be1] obey linear bounds in $L^2(w)$ on the A_2-characteristic of the weight, these estimates immediately will provide a quadratic bound on the A_2-characteristic of the weight for the paraproduct of complexity (m, n), namely, $\|\pi_b^{m,n} f\|_{L^2(w)} \leq C_{n,m}[w]^2_{A_2^d}\|f\|_{L^2(w)}$, where $C_{n,m}$ depends polynomially (even linearly) on $n+m$. We will show that in fact, the paraproduct of complexity (m, n) obeys the same *linear bound* obtained by Beznosova for the dyadic paraproduct of complexity $(0, 0)$, multiplied by a polynomial factor that depends on the complexity.

The paper [LPetR] was the first to introduce the class of Haar shifts operators, it also proved the linear A_2-bound for this class, however their bound depends exponentially in the complexity of the operator. The proof given by Nazarov and Volberg, in [NV], to show that Haar shifts operators with complexity (m,n) are bounded in $L^2(w)$ linearly by the A_2-characteristic of w and polynomially in the complexity, with appropriate modifications would also work for generalized Haar shifts operators with complexity (m,n), which includes paraproduct of complexity (m,n). The modifications that are needed to cover the class of generalized Haar shift multipliers, for the particular case that we called composition of dual paraproduct with paraproduct will be addressed in the next chapter. In this Chapter we describe those modifications for the paraproduct, and in our proof we trace the linear dependence in the BMO-norm of b as well. But before, we will present this new and conceptually simpler (in our opinion) proof for the linear bound in the A_2-characteristic for the dyadic paraproduct, which will allow us to highlight certain elements of the general proof without dwelling with the complexity.

5.1 Complexity $(0,0)$

For complexity $(0,0)$ the operator is

$$(\pi_b f)(x) = \sum_{I\in\mathcal{D}} c_I\, m_I f\ \langle b, h_I\rangle h_I(x) \tag{5.2}$$

with $|c_I| \leq 1$.

Beznosova proved in [Be1] that the dyadic paraproduct π_b obeys a linear bound in $L^2(w)$ both in terms of the A_2-characteristic of the weight w and the BMO-norm of b.

Theorem 5.1 (Beznosova, [Be1])**.** *There exists $C>0$, such that for all $b \in BMO^d$*

and for all $w \in A_2^d$,

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2}\|b\|_{BMO^d}\|f\|_{L^2(w)}.$$

Beznosova's proof is based on the α-Lemma, the Little Lemma, which were the new Bellman function ingredients that she introduced, and on Nazarov-Treil-Volberg's two-weight Carleson embedding theorem, which can be found on [NTV]. Next we give another proof of this result, this proof is still based in the α-Lemma 3.8, however it does not make use of the two-weight Carleson embedding theorem, instead we will use properties of Carleson sequences such as the Little Lemma 3.3, and the Weighted Carleson Lemma 3.1, following [NV] argument for Haar Shifts of complexity (m, n). We are using the same Bellman function ingredients that Beznosova introduced in her proof, but in a more direct way.

Proof of Theorem 5.1. Fix $f \in L^2(w)$ and $g \in L^2(w^{-1})$ and define $b_I = \langle b, h_I\rangle$, b_I is a Carleson sequence with intensity $\|b\|^2_{BMO^d}$.

By duality, suffices to prove:

$$|\langle \pi_b(fw), gw^{-1}\rangle| \leq C\|b\|_{BMO^d}[w]_{A_2^d}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}. \tag{5.3}$$

Note that

$$|\langle \pi_b(fw), gw^{-1}\rangle| = \left|\left\langle \sum_{I\in\mathcal{D}} c_I b_I m_I(fw)h_I(x), gw^{-1}\right\rangle\right|. \tag{5.4}$$

Replace $h_I = \alpha_I h_I^{w^{-1}} + \beta_I \frac{\chi_I}{\sqrt{|I|}}$ where $\alpha_I = \alpha_I^{w^{-1}}$ and $\beta_I = \beta_I^{w^{-1}}$ as described in Proposition 2.2. Use the triangle inequality to break the sum in (5.4) into two summands to be estimated separately.

$$\left|\left\langle \sum_{I\in\mathcal{D}} c_I b_I m_I(fw)h_I(x), gw^{-1}\right\rangle\right| \leq \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w)|\langle gw^{-1}, h_I\rangle|$$

$$\leq \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w) \big| \langle gw^{-1}, \alpha_I h_I^{w^{-1}} + \beta_I \frac{\chi_I}{\sqrt{|I|}} \rangle|$$

Using the estimates $\alpha_I \leq \sqrt{m_I w^{-1}}$, and $\beta_I \leq \frac{|\Delta_I w^{-1}|}{m_I w^{-1}}$, we have that,

$$\left| \left\langle \sum_{I\in\mathcal{D}} c_I b_I m_I(fw) h_I(x), gw^{-1} \right\rangle \right| \leq \Sigma_1 + \Sigma_2$$

where

$$\begin{aligned} \Sigma_1 &:= \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w) |\langle gw^{-1}, h_I^{w^{-1}} \rangle| \sqrt{m_I w^{-1}} \\ \Sigma_2 &:= \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w) |\langle gw^{-1}, \chi_I \rangle| \frac{|\Delta_I w^{-1}|}{m_I w^{-1}} \frac{1}{\sqrt{|I|}}. \end{aligned}$$

Estimating Σ_1: First using that $\dfrac{m_I(|f|w)}{m_I w} \leq M_w f(x)$ for all $x \in I$, and that $\langle gv, f \rangle = \langle g, f \rangle_v$; second using the Cauchy-Schwarz inequality and $m_I w\, m_I w^{-1} \leq [w]_{A_2}$, we get

$$\begin{aligned} \Sigma_1 &\leq \sum_{I\in\mathcal{D}} |b_I| \frac{m_I^w |f|}{\sqrt{m_I w^{-1}}} \big| \langle g, h_I^{w^{-1}} \rangle_{w^{-1}} \big| \; m_I w^{-1} \; m_I w \\ &\leq \sum_{I\in\mathcal{D}} |b_I| \frac{\inf_{x\in I} M_w f(x)}{\sqrt{m_I w^{-1}}} \big| \langle g, h_I^{w^{-1}} \rangle_{w^{-1}} \big| \; m_I w^{-1} \; m_I w \\ &\leq [w]_{A_2} \left(\sum_{I\in\mathcal{D}} |b_I|^2 \frac{\inf_{x\in I} M_w^2 f(x)}{m_I w^{-1}} \right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} |\langle g, h_I^{w^{-1}} \rangle_{w^{-1}}|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Using Lemma 3.6 with $F(x) = M_w^2 f(x)$ and $v = w^{-1}$, then Using Weighted Carleson Lemma 3.1, with $F(x) = M_w^2 f(x)$, $v = w$, and $\alpha_I = \frac{|b_I|^2}{m_I w^{-1}}$ (which is an w-Carleson sequence with intensity $4\|b\|^2_{BMO}$, according to Lemma 3.3), then, together with the fact that $\{h_I^{w^{-1}}\}_{I\in\mathcal{D}}$ is an orthonormal system in $L^2(w^{-1})$, we get

$$\Sigma_1 \;\; \leq \;\; [w]_{A_2} \|b\|_{BMO^d} \left(\int_{\mathbb{R}} M_w^2 f(x) w(x) dx \right)^{\frac{1}{2}} \|g\|_{L^2(w^{-1})}$$

$$\leq \ C[w]_{A_2}\|b\|_{BMO^d}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.$$

In the last inequality we used the fact that M_w is bounded in $L^2(w)$ with operator norm independent of w.

Estimating Σ_2: Let $\alpha \in (0, \frac{1}{2})$, using similar arguments than the ones used for Σ_1, we conclude that,

$$\begin{aligned}\Sigma_2 \ &\leq \ \sum_{I\in\mathcal{D}} |b_I| m_I^w|f| \, m_I^{w^{-1}}|g| \sqrt{\frac{|\Delta_I w^{-1}|^2}{(m_I w^{-1})^2}|I|(m_I w \, m_I w^{-1})^{\alpha}} \, (m_I w \, m_I w^{-1})^{1-\frac{\alpha}{2}} \\ &\leq \ [w]_{A_2}^{1-\frac{\alpha}{2}} \sum_{I\in\mathcal{D}} |b_I|\sqrt{\mu_I} \inf_{x\in I} M_w f(x) M_{w^{-1}} g(x),\end{aligned}$$

where μ_I is defined in Lemma 3.8, and in the last inequality we used the fact that for any $I \in \mathcal{D}$ for all $x \in I$,

$$m_I^w|f| \, m_I^{w^{-1}}|g| \leq M_w f(x) M_{w^{-1}} g(x).$$

Since $\{|b_I|^2\}_{I\in\mathcal{D}}$ and $\{\mu_I\}_{I\in\mathcal{D}}$ are Carleson sequences with intensities $\|b\|^2_{BMO^d}$ and $[w]^{\alpha}_{A_2}$, respectively then, by Proposition 2.21, the sequence $\{|b_I|\sqrt{\mu_I}\}_{I\in\mathcal{D}}$ is a Carleson sequence with intensity $\|b\|_{BMO^d}[w]^{\frac{\alpha}{2}}_{A_2}$. Thus, by Lemma 3.1 with $F(x) = M_w f(x) M_{w^{-1}} g(x)$, $\alpha_I = |b_I|\sqrt{\mu_I}$, and $v = 1$,

$$\Sigma_2 \ \leq \ [w]_{A_2}^{1-\frac{\alpha}{2}} \|b\|_{BMO^d} [w]_{A_2}^{\frac{\alpha}{2}} \int_{\mathbb{R}} M_w f(x) M_{w^{-1}} g(x) \, dx,$$

finally using Cauchy-Schwarz and the fact that $w^{\frac{1}{2}} w^{\frac{-1}{2}} = 1$ we get

$$\begin{aligned}\Sigma_2 \ &\leq \ [w]_{A_2}\|b\|_{BMO^d}\left(\int_{\mathbb{R}} M_w^2 f(x) w(x) dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} M_{w^{-1}}^2 g(x) w^{-1}(x) dx\right)^{\frac{1}{2}} \\ &= \ [w]_{A_2}\|b\|_{BMO^d}\|M_w f\|_{L^2(w)}\|M_{w^{-1}} g\|_{L^2(w^{-1})} \\ &\leq \ C[w]_{A_2}\|b\|_{BMO^d}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.\end{aligned}$$

These estimates together give (5.3), and the theorem is proved. □

5.2 Complexity (m, n)

In this section we prove the linear in A_2-characteristic, polynomial in complexity estimate for the paraproducts of complexity (m, n). The proof will follow the general lines of the argument presented in Section 5.1 for the complexity $(0, 0)$ case, with the added refinements devised by Nazarov and Volberg [NV], adapted to our setting, to handle the general complexity.

Theorem 5.2. *For all $b \in BMO^d$ and $w \in A_2^d$, there is $c > 0$ such that*

$$\|\pi_b^{m,n} f\|_{L^2(w)} \leq c(n+m+2)^4 [w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)}.$$

Proof. Fix $f \in L^2(w)$ and $g \in L^2(w^{-1})$ and define $b_I = \langle b, h_I \rangle$ and let $C_m^n := C(m+n+2)$. By duality, it is enough to show that

$$|\langle \pi_b^{m,n}(fw), gw^{-1} \rangle| \leq c(C_m^n)^4 [w]_{A_2} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \|f\|_{L^2(w)}.$$

We can write the left-hand-side as a double sum that we will estimate,

$$\begin{aligned} &\Big| \Big\langle \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}\, b_I m_I(fw)\, h_J \,,\, gw^{-1} \Big\rangle \Big| \\ &\qquad \leq \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} |b_I|\, \frac{\sqrt{|I|}\sqrt{|J|}}{|L|} m_I(|f|w)\, |\langle gw^{-1}, h_J \rangle|. \end{aligned}$$

As before we will replace $h_I = \alpha_I h_I^{w^{-1}} + \beta_I \frac{\chi_I}{\sqrt{|I|}}$ and break into two terms to be estimated separately.

$$\begin{aligned} &\Big| \Big\langle \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}\, b_I m_I(fw)\, h_J(x) \,,\, gw^{-1} \Big\rangle \Big| \\ &\qquad \leq \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} |b_I|\, \frac{\sqrt{|I|}\sqrt{|J|}}{|L|} m_I(|f|w)\, |\langle gw^{-1}, h_J \rangle| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} |b_I| \,\frac{\sqrt{|I|}\sqrt{|J|}}{|L|}\, m_I(|f|w)\, \big|\langle gw^{-1}, \alpha_J h_J^{w^{-1}} + \beta_J \frac{\chi_J}{\sqrt{|J|}}\rangle\big| \\
&\leq \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} |b_I| \,\frac{\sqrt{|I|}\sqrt{|J|}}{|L|} m_I(|f|w)|\langle gw^{-1}, h_J^{w^{-1}}\rangle|\sqrt{m_J w^{-1}} + \\
&\quad + \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} |b_I| \,\frac{\sqrt{|I|}}{|L|} m_I(|f|w)|\langle gw^{-1}, \chi_J\rangle|\frac{|\Delta_J w^{-1}|}{m_J w^{-1}} \\
&= \Sigma_1^{m,n} + \Sigma_2^{m,n}
\end{aligned}
$$

We define for each weight v, and ϕ a locally integrable function the quantities,

$$
S_L^{v,n}\phi := \sum_{J\in\mathcal{D}_n(L)} |\langle \phi, h_J^v\rangle_v| \sqrt{m_J v}\frac{\sqrt{|J|}}{\sqrt{|L|}}, \tag{5.5}
$$

$$
R_L^{v,n}\phi := \sum_{J\in\mathcal{D}_n(L)} \frac{|\Delta_J v|}{m_J v} m_J(|\phi|v)\, \frac{|J|}{\sqrt{|L|}}, \tag{5.6}
$$

$$
Pb_L^{v,m}\phi := \sum_{I\in\mathcal{D}_m(L)} |b_I|\, m_I(|\phi|v)\frac{\sqrt{|I|}}{\sqrt{|L|}}. \tag{5.7}
$$

We also define the following Carleson sequences, see Corollaries 3.14 and 3.8,

$$
\mu_K := (m_K w)^\alpha (m_K w^{-1})^\alpha \left(\frac{|\Delta_K w^{-1}|^2}{(m_K w^{-1})^2} + \frac{|\Delta_K w|^2}{(m_K w)^2}\right)|K|, \text{ intensity } C[w]_{A_2}^\alpha,
$$

$$
\mu_L^n := \sum_{K\in\mathcal{ST}_L^n} \mu_K, \text{ intensity } C(n+1)[w]_{A_2}^\alpha,
$$

$$
\mu_K^b := \frac{|b_K|^2}{|K|}(m_K w\, m_K w^{-1})^\alpha, \text{ intensity } C\|b\|^2_{BMO^d}[w]_{A_2}^\alpha,
$$

$$
\mu_L^{b,n} := \sum_{K\in\mathcal{ST}_L^n} \mu_K^b, \text{ intensity } C(n+1)\|b\|^2_{BMO^d}[w]_{A_2}^\alpha,
$$

where $\mathcal{ST}_L^n$ is the stopping time defined in Lemma 3.13. Note that

$$
\Sigma_1^{m,n} \leq \sum_{L\in\mathcal{D}} Pb_L^{w,m} f\; S_L^{w^{-1},n} g, \quad \Sigma_2^{m,n} \leq \sum_{L\in\mathcal{D}} Pb_L^{w,m} f\; R_L^{w^{-1},n} g,
$$

thus in order to estimate $\Sigma_1^{m,n}$ and $\Sigma_2^{m,n}$ we will use the following estimates for $Pb_L^{w,n}f$, $S_L^{w^{-1},m}g$ and $R_L^{w^{-1},m}g$, where $0<\alpha<1/2$ so we can use the α-Lemma 3.8,

$$S_L^{w^{-1}}g \leq \Big(\sum_{J\in\mathcal{D}_n}|\langle g,h_J^{w^{-1}}\rangle_{w^{-1}}|^2\Big)^{\frac{1}{2}}\sqrt{m_L w^{-1}}, \tag{5.8}$$

$$R_L^{w^{-1}}g \leq e^{\alpha}C_m^n(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{1-\frac{\alpha}{2}}\inf_{x\in L}M_{w^{-1}}(|g|^p)^{\frac{1}{p}}(x)\sqrt{\mu_L^n}, \tag{5.9}$$

$$Pb_L^w f \leq e^{\alpha}C_m^n(m_L w)^{1-\frac{\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}\inf_{x\in L}M_w(|f|^p)^{\frac{1}{p}}(x)\big(\|b\|_{BMO^d}\sqrt{\mu_L^m}+\sqrt{\mu_L^{b,m}}\big). \tag{5.10}$$

Estimate (5.8) is easy to show, we just need to use Cauchy-Schwarz inequality and the fact that $\{J\in\mathcal{D}_m(L)\}$ is a partition of L.

$$\begin{aligned} S_L^{w^{-1}}g &= \sum_{J\in\mathcal{D}_n(L)}|\langle g,h_J^{w^{-1}}\rangle_{w^{-1}}|\sqrt{m_J w^{-1}}\frac{\sqrt{|J|}}{\sqrt{|L|}} \\ &\leq \Big(\sum_{J\in\mathcal{D}_n(L)}|\langle g,h_J^{w^{-1}}\rangle_{w^{-1}}|^2\Big)^{\frac{1}{2}}\Big(\sum_{J\subset L,|J|=2^{-n}|L|}m_J w^{-1}\frac{|J|}{|L|}\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{J\in\mathcal{D}_n(L)}|\langle g,h_J^{w^{-1}}\rangle_{w^{-1}}|^2\Big)^{\frac{1}{2}}\big(m_L w^{-1}\big)^{\frac{1}{2}} \end{aligned}$$

Estimate (5.9) was obtained in [NV], we include their proof for completeness in Lemma 5.5. With a variation over their argument we prove estimate (5.10) in Lemma 5.4, both lemmas we prove in Section 5.2.2. Let us first use estimates (5.8), (5.9) and (5.10) to estimate $\Sigma_1^{m,n}$ and $\Sigma_2^{m,n}$.

Estimate for $\Sigma_1^{m,n}$: Estimating the first term we get, after using Cauchy-Schwarz inequality and the fact that $\{h_J^{w^{-1}}\}_{J\in\mathcal{D}}$ is an orthonormal system in $L^2(w^{-1})$ and

$\mathcal{D} = \cup_{L\in\mathcal{D}}\mathcal{D}_m(L)$,

$$\begin{aligned}
\Sigma_1^{m,n} \leq & \sum_{L\in\mathcal{D}} Pb_L^{w,m} f\; S_L^{w^{-1},n} g \\
\leq & C_m^n \sum_{L\in\mathcal{D}} (m_L w\; m_L w^{-1})^{1-\frac{\alpha}{2}} \inf_{x\in L}\Big(M_w(|f|^p)(x)\Big)^{\frac{1}{p}} \nu_L^m \frac{1}{\sqrt{m_L w^{-1}}} \|g\|_{L^2(w^{-1})} \\
\leq & \, e^\alpha C_m^n [w]_{A_2}^{1-\frac{\alpha}{2}} \sum_{L\in\mathcal{D}} \frac{\nu_L^m}{\sqrt{m_L w^{-1}}} \inf_{x\in L}\Big(M_w(|f|^p)(x)\Big)^{\frac{1}{p}} \|g\|_{L^2(w^{-1})} \\
\leq & \, e^\alpha C_m^n [w]_{A_2}^{1-\frac{\alpha}{2}} \left(\sum_{L\in\mathcal{D}} \frac{(\nu_L^m)^2}{m_L w^{-1}} \inf_{x\in L}\Big(M_w(|f|^p)(x)\Big)^{\frac{2}{p}}\right)^{\frac{1}{2}} \|g\|_{L^2(w^{-1})}.
\end{aligned}$$

Since by Bessel's inequality

$$\left(\sum_{J\in\mathcal{D}_n(L)} |\langle g, h_J^{w^{-1}}\rangle_{w^{-1}}|^2\right)^{\frac{1}{2}} \leq \|g\|_{L^2(w^{-1})}.$$

Using the Weighted Carleson Lemma 3.1 with $F(x) = \big(M_w(|f|^p)(x)\big)^{2/p}$, $v = w$, and $\alpha_L = \frac{(\nu_L^m)^2}{m_L w^{-1}}$. Recall that $\nu_L^m := \big(\|b\|_{BMO^d}\sqrt{\mu_L^m} + \sqrt{\mu_L^{b,m}}\big)$, by Proposition 2.21, $(\nu_L^m)^2$ is a Carleson measure with intensity at most $C_m^n\|b\|^2_{BMO^d}[w]^\alpha_{A_2^d}$. By Lemma 3.3, $\frac{(\nu_L^m)^2}{m_L w^{-1}}$ is an w-Carleson sequence with comparable intensity, thus we will have that

$$\begin{aligned}
\Sigma_1^{m,n} \leq & (C_m^n)^{\frac{3}{2}} \|b\|_{BMO^d} [w]_{A_2}^{1-\frac{\alpha}{2}} [w]_{A_2}^{\frac{\alpha}{2}} \|g\|_{L^2(w^{-1})} \left(\int_{\mathbb{R}} M_w(|f|^p)(x)^{\frac{2}{p}} w(x)dx\right)^{\frac{1}{2}} \\
= & \, e^\alpha (C_m^n)^{\frac{3}{2}} [w]_{A_2} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \Big\|M_w(|f|^p)\Big\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(w)} \\
\leq & \, e^\alpha \left[\left(\frac{2}{p}\right)'\right]^{\frac{1}{p}} (C_m^n)^{\frac{3}{2}} [w]_{A_2} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \big\|\,|f|^p\big\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(w)} \\
= & \, e^\alpha (C_m^n)^{\frac{5}{2}} [w]_{A_2} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \|f\|_{L^2(w)}.
\end{aligned}$$

Here we are using the fact that M_w is bounded in $L^q(w)$ for all $q > 1$ and furthermore

$$\|M_w f\|_{L^q(w)} \leq Cq' \|f\|_{L^q(w)}$$

In our case $q = \dfrac{2}{p}$ and $q' = \left(\dfrac{2}{p}\right)' = \dfrac{2}{2-p} = 2(m+n+2)$.

Estimate for $\Sigma_2^{m,n}$: Using the fact that $(m_I w\, m_I w^{-1})^{1-\alpha} \leq [w]_{A_2}^{1-\alpha}$,

$$\begin{aligned}\Sigma_2^{m,n} \leq & \sum_{L\in\mathcal{D}} Pb_L^{w,n} f\; R_L^{w^{-1},m} g \\ \leq & e^{2\alpha}(C_m^n)^2 \sum_{L\in\mathcal{D}} [w]_{A_2}^{1-\alpha} \inf_{x\in L}\big(M_w(|f|^p)(x)\big)^{\frac{1}{p}} \inf_{x\in L}\big(M_{w^{-1}}(|g|^p)(x)\big)^{\frac{1}{p}} \nu_L^m \sqrt{\mu_L^n}.\end{aligned} \tag{5.11}$$

Since $(\nu_L^m)^2$ and μ_L^n have intensity at most $C(m+1)[w]_{A_2}^{\alpha}\|b\|_{BMO}^2$ and $C(n+1)[w]_{A_2^d}^{\alpha}$ respectively then, by Proposition 2.21, we have that $\nu_L^m\sqrt{\mu_L^n}$ is a Carleson measure with intensity at most $C(m+n+2)\|b\|_{BMO^d}[w]_{A_2}^{\alpha}$. If we now apply Lemma 3.1 in (5.11), with $F^p(x) = M_w(|f|^p)(x)M_{w^{-1}}(|g|^p)(x)$, $\alpha_L = \nu_L^m\sqrt{\mu_L^n}$, and $v=1$, we will have, by Cauchy-Schwarz and the boundedness of M_v in $L^q(v)$ for $q = p/2 > 1$,

$$\begin{aligned}\Sigma_2^{m,n} &\leq (C_m^n)^2[w]_{A_2}^{1-\alpha} \sum_{L\in\mathcal{D}} \inf_{x\in L}\big(M_w(|f|^p)(x)M_{w^{-1}}(|g|^p)(x)\big)^{\frac{1}{p}} \nu_L^m\sqrt{\mu_L^n} \\ &\leq e^{2\alpha}(C_m^n)^3[w]_{A_2}\|b\|_{BMO^d} \int_{\mathbb{R}} \big(M_w(|f|^p)(x)\big)^{\frac{1}{p}}\big(M_{w^{-1}}(|g|^p)(x)\big)^{\frac{1}{p}} dx\end{aligned}$$

Using Cauchy-Schwarz and the fact that

$$\left(\int_{\mathbb{R}} \big(M_w(|f|^p)(x)\big)^{\frac{2}{p}} w(x)dx\right)^{\frac{1}{2}} = \big\|M_w(|f|^p)\big\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}}$$

and

$$\left(\int_{\mathbb{R}} \big(M_{w^{-1}}(|g|^p)(x)\big)^{\frac{2}{p}} w^{-1}dx\right)^{\frac{1}{2}} = \big\|M_{w^{-1}}(|g|^p)\big\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}},$$

we have that

$$\begin{aligned}\Sigma_2^{m,n} &\leq e^{2\alpha}(C_m^n)^3[w]_{A_2}\|b\|_{BMO^d}\big\|M_w(|f|^p)\big\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}}\big\|M_{w^{-1}}(|g|^p)\big\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}} \\ &\leq e^{2\alpha}\left[\left(\frac{2}{p}\right)'\right]^{\frac{2}{p}}(C_m^n)^3[w]_{A_2}\|b\|_{BMO^d}\big\||f|^p\big\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}}\big\||g|^p\big\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}}\end{aligned}$$

$$= e^{2\alpha}(C_m^n)^5[w]_{A_2}\|b\|_{BMO^d}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.$$

Together these estimates prove the theorem with $c \leq 6$, under the assumption that estimate (5.10) holds. □

5.2.1 Bounds for $\kappa_b^{m.n}$

As an immediate corollary of the estimate for the paraproduct of complexity (m,n) we get similar bounds for the adjoint.

Corollary 5.3. *For all $b \in BMO^d$ and $w \in A_2$, there is $c > 0$ such that*

$$\|\kappa_b^{m,n} f\|_{L^2(w)} \leq c(n+m+2)^4[w]_{A_2}\|b\|_{BMO^d}\|f\|_{L^2(w)}.$$

Proof. By Proposition 4.11 $\kappa_b^{m,n} = (\pi_b^{n,m})^*$. Therefore

$$\|\kappa_b^{m,n}\|_{L^2(w)\to L^2(w)} = \|\pi_b^{n,m}\|_{L^2(w^{-1})\to L^2(w^{-1})}$$

Using Theorem 5.2 we have that

$$\|\pi_b^{m,n}\|_{L^2(w^{-1})\to L^2(w^{-1})} \leq c(m+n+2)^4\|b\|_{BMO^d}[w^{-1}]_{A_2^d};$$

and since $[w]_{A_2^d} = [w^{-1}]_{A_2^d}$ we have that

$$\|\kappa_b^{m,n}\|_{L^2(w)\to L^2(w)} \leq (n+m+2)^4\|b\|_{BMO^d}[w]_{A_2^d}.$$

□

5.2.2 Bounds for $Pb_L^{v,m}$ and $R_L^{v,n}$

The missing step in the previous proof is estimate (5.10) and (5.9), which we now prove. Inequality (5.9) was proved by Nazarov and Volberg in [NV], with an adaptation of their argument we will prove (5.10).

Lemma 5.4. *Let $b \in BMO$, ϕ a locally integrable function, then*

$$Pb_L^{w,m}\phi \leq e^{\alpha} C_m^n (m_L w)^{1-\frac{\alpha}{2}} (m_L w^{-1})^{\frac{-\alpha}{2}} \inf_{x\in L} \left(M_w(|\phi|^p)(x)\right)^{\frac{1}{p}} \nu_L^m,$$

where $\nu_L^m = \left(\|b\|_{BMO^d}\sqrt{\mu_L^m} + \sqrt{\mu_L^{b,m}}\right)$, *and* $p = 2 - \frac{1}{m+n+2}$.

Proof. Let $\mathcal{ST}_L^m$ be the collection of stopping time intervals defined in Lemma 3.13, then

$$\begin{aligned} Pb_L^w\phi &= \sum_{I\in\mathcal{D}_m(L)} |b_I|\, m_I(|\phi|w)\frac{\sqrt{|I|}}{\sqrt{|L|}} \\ &= \sum_{K\in\mathcal{ST}_L^m} \sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} \frac{|b_I|}{\sqrt{|I|}}\, m_I(|\phi|w)\frac{|I|}{\sqrt{|L|}} \end{aligned} \tag{5.12}$$

Now note that if K is a stopping time interval by the first criterium then

$$\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} \frac{|b_I|}{\sqrt{|I|}}\, m_I(|\phi|w)\frac{|I|}{\sqrt{|L|}} \leq \|b\|_{BMO^d} m_K(|\phi|w)\frac{|K|}{\sqrt{|L|}}$$
$$\leq \sqrt{2}(m+n+2)\|b\|_{BMO^d} m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K}(m_K w)^{\frac{-\alpha}{2}}(m_K w^{-1})^{\frac{-\alpha}{2}}$$

The first inequality because $\frac{|b_I|}{\sqrt{|I|}} \leq \|b\|_{BMO^d}$ and the second inequality because $1 \leq \sqrt{2}(m+n+2)\sqrt{\mu_K^m}(m_K w\; m_K w^{-1})^{\frac{-\alpha}{2}}$.

Now we use the fact, proved in Lemma 3.13, that we can compare the averages of the weights in the stopping interval with their averages in L, paying a price of a constant e^2, then

$$\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} \frac{|b_I|}{\sqrt{|I|}}\, m_I(|\phi|w)\frac{|I|}{\sqrt{|L|}} \leq$$
$$C_n^m e^{\alpha}\|b\|_{BMO^d} m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K}(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}$$

If K is a stopping time interval by the second criteria then the sum collapses to just one term

$$\begin{aligned}\sum_{I\in\mathcal{D}(K)\bigcap\mathcal{D}_m(L)} |b_I|\ m_I(|\phi|w)\frac{\sqrt{|I|}}{\sqrt{|L|}} &= \frac{|b_K|}{\sqrt{|K|}}\ m_K(|\phi|w)\frac{|K|}{\sqrt{|L|}}\\ &\leq C_n^m\ m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K^b}(m_K w)^{\frac{-\alpha}{2}}(m_K w^{-1})^{\frac{-\alpha}{2}}\\ &\leq C_n^m e^{\alpha} m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K^b}(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}},\end{aligned}$$

where in the last inequality we used Lemma 3.13 again. Then plugging 5.2.2 into we will have

$$\begin{aligned}Pb_L^w f \leq C_n^m e^{\alpha}\ \|b\|_{BMO^d} \sum_{K\in\Xi_1} m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K}(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}\\ C_n^m e^{\alpha} \sum_{K\in\Xi_2} m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K^b}(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}\end{aligned}$$

where

$$\Xi_1(L) = \{K \in \mathcal{ST}_L^n : K \text{ is a stopping time interval by criteria } 1\}$$

and

$$\Xi_2(L) = \{K \in \mathcal{ST}_L^n : K \text{ is a stopping time interval by criteria } 2\},$$

note that $\Xi_1 \bigcup \Xi_2$ is a partition of L.

Let $\Sigma_{Pb}^1 := \sum_{K\in\Xi_1} m_K(|\phi|w)\frac{\sqrt{|K|}}{\sqrt{|L|}}\sqrt{\mu_K}$
and
$\Sigma_{Pb}^2 := (m_L w^{-1})^{\frac{-\alpha}{2}} \sum_{K\in\Xi_2} m_K(|\phi|w)\frac{\sqrt{|K|}}{|L|}\sqrt{\mu_K^b}$, so

$$Pb_L^w \leq C_n^m e^{\alpha}\ (m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}}\Big(\|b\|_{BMO^d}\Sigma_{Pb}^1 + \Sigma_{Pb}^2\Big).$$

$$\Sigma_{Pb}^1 \ \leq\ \left(\sum_{K\in\Xi_1}(m_K(|\phi|w))^2\frac{|K|}{|L|}\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{ST}_L^m}\mu_K\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{K \in \Xi_1(L)} (m_K(|\phi|w))^p \left(\frac{|K|}{|L|} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \sqrt{\mu_L^m}. \tag{5.13}$$

Note that $\mu_L^m := \sum_{I \in \mathcal{D}_m(L)} \mu_K \geq \sum_{I \in \Xi_1} \mu_K$ and that the last inequality follows because $\frac{p}{2} < 1$. Then

$$\left(\sum_{K \in \Xi_1} (m_K(|f|w))^2 \frac{|K|}{|L|} \right)^{\frac{p}{2}} \leq \sum_{K \in \mathcal{ST}_L^m} (m_K(|\phi|w))^p \left(\frac{|K|}{|L|} \right)^{\frac{p}{2}}.$$

Also, by the second stop criteria we have that $\frac{|K|}{|L|} = 2^{-j}$ for $0 \leq j \leq m$, then

$$\begin{aligned} \left(\frac{|K|}{|L|} \right)^{\frac{p}{2}} &= \left(\frac{2^{-j}|L|}{|L|} \right)^{\frac{p}{2}} = 2^{-j(1-\frac{1}{2(m+n+2)})} \\ &\leq 2^{-j+\frac{j}{2(m+n+2)}} < 2.2^{-j} = 2\frac{|K|}{|L|}. \end{aligned} \tag{5.14}$$

Plug (5.14) in (5.13) we will have:

$$\begin{aligned} \Sigma_{Pb}^1 &\leq \left(2 \sum_{K \in \Xi_1} (m_K(|\phi|w))^p \frac{|K|}{|L|} \right)^{\frac{1}{p}} \sqrt{\mu_L^m} \\ &\leq \left(2 \sum_{K \in \Xi_1} (m_K(|\phi|^p w))(m_K w)^{p-1} \frac{|K|}{|L|} \right)^{\frac{1}{p}} \sqrt{\mu_L^m}. \end{aligned}$$

One more time using Lemma 3.13, we have that

$$\begin{aligned} \Sigma_{Pb}^1 &\leq 2^{\frac{1}{p}} e^{\alpha(1-\frac{1}{p})} (m_L w)^{1-1/p} \left(\sum_{K \in \Xi_1} (m_K(|\phi|^p w)) \frac{|K|}{|L|} \right)^{\frac{1}{p}} \sqrt{\mu_L^m} \\ &\leq e^\alpha m_L w (m_L w)^{\frac{-1}{p}} \Big(m_L(|\phi|^p w) \Big)^{\frac{1}{p}} \sqrt{\mu_L^m} \\ &= e^\alpha m_L w \left(\frac{m_L(|\phi|^p w)}{m_L w} \right)^{\frac{1}{p}} = e^\alpha m_L w \Big(m_L^w(|\phi|^p) \Big)^{\frac{1}{p}} \sqrt{\mu_L^m} \\ &\leq e^\alpha m_L w \inf_{x \in L} \Big(M_w(|\phi|^p)(x) \Big)^{\frac{1}{p}} \sqrt{\mu_L^m} \end{aligned}$$

and

$$\Sigma^2_{Pb} \leq \left(\sum_{K \in \Xi_2} (m_K(|f|w))^2 \frac{|K|}{|L|} \right)^{\frac{1}{2}} \left(\sum_{K \in \Xi_2} \mu^b_K \right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{K \in \mathcal{ST}^m_L} (m_K(|\phi|w))^p \left(\frac{|K|}{|L|} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \sqrt{\mu^{b,m}_L}.$$

Following now the same steps as we did in the estimative of Σ^1_{Pb} we will have

$$\Sigma^2_{Pb} \leq e^\alpha m_L w \inf_{x \in L} \Big(M_w(|\phi|^p)(x) \Big)^{\frac{1}{p}} \sqrt{\mu^{b,m}_L}.$$

Thus,

$$Pb^w_L \leq C^m_n e^\alpha \, (m_L w)^{1-\frac{\alpha}{2}} (m_L w^{-1})^{\frac{-\alpha}{2}} \inf_{x \in L} \Big(M_w(|\phi|^p)(x) \Big)^{\frac{1}{p}} \big(\|b\|_{BMO^d} \sqrt{\mu^m_L} + \sqrt{\mu^{b,m}_L} \big).$$

□

For completeness we also present Nazarov and Volberg proof's of estimative 5.9.

Lemma 5.5. *Let ϕ a locally integrable function, then*

$$R^{w^{-1},n}_L \phi \leq C(m+n+2)(m_L w)^{\frac{-\alpha}{2}} (m_L w^{-1})^{1-\frac{\alpha}{2}} \inf_{x \in L} M_{w^{-1}}(|\phi|^p)^{\frac{1}{p}}(x) \sqrt{\mu^n_L}.$$

where $\mu_L = (m_L w)^\alpha (m_L w^{-1})^\alpha \left(\frac{|\Delta_L w^{-1}|^2}{(m_L w^{-1})^2} + \frac{|\Delta_L w|^2}{(m_L w)^2} \right) |L|$ *and* $p = 2 - \frac{1}{m+n+2}$.

Proof. Let $\mathcal{ST}^n_L$ be the collection of stopping time intervals from Lemma 3.13, then

$$R^{w^{-1},n}_L \phi = \sum_{J \in \mathcal{D}_n(L)} \frac{|\Delta_K w^{-1}|}{m_K w^{-1}} m_J(|\phi|w^{-1}) \frac{|J|}{\sqrt{|L|}} \tag{5.15}$$
$$= \sum_{K \in \mathcal{ST}^n_L} \sum_{J \in \mathcal{D}(K) \bigcap \mathcal{D}_n(L)} \frac{|\Delta_K w^{-1}|}{m_K w^{-1}} m_J(|\phi|w^{-1}) \frac{|J|}{\sqrt{|L|}}.$$

By Lemma 3.13 we have that for any stopping time cube K

$$\sum_{J \in \mathcal{D}(K) \bigcap \mathcal{D}_n(L)} \frac{|\Delta_K w^{-1}|}{m_K w^{-1}} m_J(|\phi|w^{-1}) \frac{|J|}{\sqrt{|L|}} \leq \tag{5.16}$$

$$\leq 2C_m^n.e^{\alpha}(m_L w \; m_L w^{-1})^{\frac{-\alpha}{2}} m_K(|\phi| w^{-1})\sqrt{\mu_K}\frac{\sqrt{|K|}}{\sqrt{|L|}}.$$

Thus, plugging 5.16 into 5.15

$$R_L^{w^{-1},n}\phi \leq 2C_m^n.e^{\alpha}(m_L w \; m_L w^{-1})^{\frac{-\alpha}{2}} \sum_{K\in\mathcal{ST}_L^n} m_K(|\phi| w^{-1})\sqrt{\mu_K^n}\frac{\sqrt{|K|}}{\sqrt{|L|}}.$$

Following the same steps as in the proof of Lemma 5.4 we can bound

$$\sum_{K\in\mathcal{ST}_L^n} m_K(|\phi| w^{-1})\sqrt{\mu_K}\frac{\sqrt{|K|}}{\sqrt{|L|}} \leq e^{\alpha} m_L w^{-1} \inf_{x\in L}\Big(M_{w^{-1}}(|\phi|^p)(x)\Big)^{\frac{1}{p}}\sqrt{\mu_L^n}.$$

and then

$$R_L^{w^{-1},n}\phi \leq 2C_m^n e^{\alpha}(m_L w)^{\frac{-\alpha}{2}}(m_L w^{-1})^{1-\frac{\alpha}{2}} \inf_{x\in L}\Big(M_{w^{-1}}(|f|^p)(x)\Big)^{\frac{1}{p}}\sqrt{\mu_L^n}. \tag{5.17}$$

□

Chapter 6

Bounds for Haar shift operators type 4 and Haar Multipliers

In this chapter we will prove sharp bounds for Haar shifts of type 4 and for for t-Haar multipliers of complexity (m, n). We will prove the linear bound in the A_2 characteristic for $T_4^{(0,0)}$ and later adapt Nazarov-Volberg method, [NV] one more time time to deal with the complexity (m, n) for Haar shifts of type 4, the particular case composition of dual paraproduct and paraproduct, $\zeta^{m,n}$. We will also extend the bounds proved by Beznosova, [Be], for the t-Haar multipliers of complexity $(0, 0)$ to complexity (m, n). For the $(0, 0)$ we will present a new proof of her result based in a Bellman function argument. However it is important to say that for the complexity $(0, 0)$ the best dependence is given by Pereyra, in [P], and the proof is also based in a Bellman function techniques.

6.1 Haar shifts operators of type 4

We prove in this section a result similar to Theorem 5.2 and Corollary 5.3 for the composition of dual dyadic paraproduct and dyadic paraproduct. Before proving for the complexity (m, n), let us discuss the complexity $(0, 0)$.

6.1.1 Bounds for $\zeta_{b,d}^{0,0}$

The composition of a dyadic dual paraproduct and dyadic paraproduct is a Haar shift of type 4 with complexity $(0, 0)$. Let $b(x)$ and $d(x)$ be two functions in BMO^d; calculating $\pi_b^* \pi_d f$.

$$\begin{aligned}
\pi_b^* \pi_d f &= \sum_{I \in \mathcal{D}} b_I \Big\langle \sum_{J \in \mathcal{D}} d_J \; m_J f \; h_J \; , \; h_I \Big\rangle \frac{\chi_I}{|I|} \\
&= \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} b_I \; d_I \; m_J f \langle h_J, h_I \rangle \frac{\chi_I}{|I|} \\
&= \sum_{I \in \mathcal{D}} d_I \; b_I \; m_I f \; \frac{\chi_I}{|I|} \\
&= \sum_{I \in \mathcal{D}} d_I \; b_I \langle f, \frac{\chi_I}{|I|} \rangle \frac{\chi_I}{|I|} = \zeta_{b,d}^{(0,0)}
\end{aligned}$$

Estimating the $L^2(w)$-norm for $w \in A_2^d$, by Theorem 5.1 we trivially have that

$$\|\pi_b^* \pi_d\|_{L^2(w)} \leq C \, \|b\|_{BMO^d} \|d\|_{BMO^d} [w]_{A_2}^2$$

However we can improve that bound for

$$\|\pi_b^* \pi_d\|_{L^2(w)} \leq C \, \|b\|_{BMO^d} \|d\|_{BMO^d} [w]_{A_2}.$$

In the proofs of the sharp linear dependence in the A_2 characteristics for paraproducts, martingale transform and Sha we see that the term that looks like a Haar shift of type 4 (the one that has characteristic function in both linear products) is the hardest to prove the sharp bound, in all cases we have to use a very powerful lemma, more commonly the α-Lemma 3.8. Surprisingly to show the sharp dependence of these operators in the A_2 characteristic is not that hard, if we can show that we are dealing with a Haar shift of type 4 given as composition $\pi_b^*\pi_d$ of the adjoint of a paraproduct and a paraproduct. This will be the case for positive operators of type 4 and complexity $(0,0)$ if we know that they are bounded in L^2.

Theorem 6.1. *There exists $C > 0$, such that for all $b, d \in BMO^d$ and for all $w \in A_2^d$,*

$$\|\pi_b^*\pi_d f\|_{L^2(w)} \leq C[w]_{A_2}\|b\|_{BMO^d}\|d\|_{BMO^d}\|f\|_{L^2(w)}.$$

Proof. By duality,

$$\begin{aligned}
\left|\left\langle \pi_b^*\pi_d(wf), w^{-1}g\right\rangle\right| &\leq \sum_{I\in\mathcal{D}} |b_I||d_I| m_I(wf) m_I(w^{-1}g) \\
&= \sum_{I\in\mathcal{D}} |b_I||d_I| \frac{m_I(wf)}{m_I w}\frac{m_I(w^{-1}g)}{m_I w^{-1}} m_I w m_I w^{-1} \\
&\leq [w]_{A_2} \sum_{I\in\mathcal{D}} |b_I||d_I| m_I^w f m_I^{w^{-1}} g \\
&\leq [w]_{A_2} \sum_{I\in\mathcal{D}} |b_I||d_I| \inf_{x\in I}(M_w f)(x)(M_{w^{-1}}g)(x).
\end{aligned}$$

Using Lemma 3.6 with Carleson sequence $\{|b_I||d_I|\}$ and intensity $\|b\|_{BMO^d}\|d\|_{BMO^d}$, positive function $F(x) = (M_w f)(x)(M_{w^{-1}})(x)$, and Lebesgue measure $dv = dx$,

$$\begin{aligned}
\left|\left\langle \pi_b^*\pi_d(wf), w^{-1}g\right\rangle\right| &\leq [w]_{A_2}\|b\|_{BMO^d}\|d\|_{BMO^d}\int_{\mathbb{R}}(M_w f)(x)(M_{w^{-1}}g)(x)dx \\
&\leq [w]_{A_2}\|b\|_{BMO^d}\|d\|_{BMO^d}\left(\int_{\mathbb{R}} M_w^2 f\, w(x)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} M_{w^{-1}}^2 g\, w^{-1}dx\right)^{\frac{1}{2}} \\
&= [w]_{A_2}\|b\|_{BMO^d}\|d\|_{BMO^d}\|M_w f\|_{L^2(w)}\|M_{w^{-1}}g\|_{L^2(w^{-1})} \\
&\leq [w]_{A_2}\|b\|_{BMO^d}\|d\|_{BMO^d}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.
\end{aligned}$$

□

Brett Wick, [Wi], called us the attention to the fact that if a positive operator of type 4 with complexity $(0,0)$ is bounded in L^2 then it can be decomposed as $\pi_b^*\pi_b$ for some $b \in BMO^d$. Assume $(T_4^{0,0}f)(x) = \sum_{I\in\mathcal{D}} c_I \big\langle f, \frac{\chi_I}{|I|}\big\rangle \frac{\chi_J(x)}{|J|}$ where $c_I \geq 0$ for all dyadic intervals I. Then there exist $C > 0$ such that for any $f, g \in L^2$,

$$\left|\big\langle T_4^{0,0} f, g\big\rangle\right| \leq C\|f\|_{L^2}\|g\|_{L^2}.$$

In particular for $f = g = \chi_K$ for a fixed interval K, $\|\chi_K\|_{L^2}^2 = |K|$ and

$$\begin{aligned}\left|\big\langle T_4^{0,0} h_K, h_K\big\rangle\right| &= \left|\Big\langle \sum_{I\in\mathcal{D}} c_I \big\langle \chi_K, \frac{\chi_I}{|I|}\big\rangle \frac{\chi_I}{|I|}, \chi_K\Big\rangle\right| \\ &\leq \sum_{I\in\mathcal{D}(K)} c_I \frac{|\langle\chi_I,\chi_K\rangle|^2}{|I|^2} + \sum_{\substack{I\in\mathcal{D} s.t\\ K\subset I}} c_I \frac{|\langle\chi_I,\chi_K\rangle|^2}{|I|^2} \\ &\leq \sum_{I\in\mathcal{D}(K)} c_I + \sum_{\substack{I\in\mathcal{D} s.t\\ K\subset I}} c_I \frac{|K|^2}{|I|^2}\end{aligned}$$

Since both sums are positive, then

$$\sum_{I\in\mathcal{D}(K)} c_I \leq |\langle T_4^{(0,0)}\chi_K, \chi_K\rangle| \leq \|\chi_K\|_{L^2}^2 \leq C\,|K|.$$

Since K was arbitrary, we have that $\{c_I\}_{I\in\mathcal{D}}$ is Carleson and therefore $\sqrt{c}(x) := \sum_{I\in\mathcal{D}} \sqrt{c_I} h_I(x)$ is in BMO^d, remember that $c_I \geq 0$ for dyadic interval I. Hence, $T_4^{(0,0)} f = \pi^*_{\sqrt{c}}\pi_{\sqrt{c}}$. Notice that if $c_I \geq 0$ and $\{c_I\}_{I\in\mathcal{D}}$ is Carleson then $b(x) = \sqrt{c}(x)$ is in BMO^d and $T_4^{(0,0)}$ is bounded in L^2.

6.1.2 Bounds for $\zeta_{b,d}^{m,n}$

Now we prove that the $L^2(w)$, for $w \in A_2$, norm of composition of dual dyadic paraproduct and dyadic paraproduct of complexity (m,n) depends linearly in the A_2^d characteristic and polynomially on the complexity (m,n).

Theorem 6.2. *For all $a, d \in BMO^d$ and $w \in A_2$ and $c_{I,J}^L \leq \frac{\sqrt{|I|}\sqrt{|J|}}{|L|}$, there is $c > 0$ such that*

$$\|\zeta_b^{m,n} f\|_{L^2(w)} \leq c\|a\|_{BMO^d}\|d\|_{BMO^d}(n+m+2)^4[w]_{A_2}\|f\|_{L^2(w)};$$

where

$$\zeta_{a,d}^{m,n} = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}^L\, a_I\, d_J \langle f, \frac{\chi_I}{|I|}\rangle \frac{\chi_J}{|J|}$$

$a_I = \langle a, h_I\rangle$ *and* $d_J = \langle d, h_J\rangle$.

Proof. Fix $f \in L^2(w)$ and $g \in L^2(w^{-1})$ and let $C_m^n := C(m+n+2)$. By duality, it is enough to show that

$$|\langle \zeta_{a,d}^{m,n}(fw), gw^{-1}\rangle B| \leq c(C_m^n)^4\|a\|_{BMO^d}\|d\|_{BMO^d}[w]_{A_2}\|g\|_{L^2(w^{-1})}\|f\|_{L^2(w)}.$$

We can write the left-hand-side as a double sum that we will estimate,

$$\begin{aligned}
&\Big|\Big\langle \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} c_{I,J}\, a_I\, d_J m_I(fw)\, \frac{\chi_J}{|J|}\ ,\ gw^{-1}\Big\rangle\Big| \\
&\qquad \leq \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} |a_I|\,|d_J|\ \frac{\sqrt{|I|}\sqrt{|J|}}{|L|} m_I(|f|w)\ \frac{1}{|J|}|\langle |g|w^{-1}, \chi_J\rangle| \\
&\quad = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} |a_I|\,|d_J|\ \frac{\sqrt{|I|}\sqrt{|J|}}{|L|} m_I(|f|w)\ m_J(gw^{-1}) := \Sigma_0^{m,n}
\end{aligned}$$

We define for each weight v, and ϕ a locally integrable function the quantities,

$$Pa_L^{v,n}\phi := \sum_{I\in\mathcal{D}_m(L)} |a_I|\ m_I(|\phi|v)\frac{\sqrt{|I|}}{\sqrt{|L|}}. \tag{6.1}$$

$$Pd_L^{v,n}\phi := \sum_{J\in\mathcal{D}_m(L)} |d_J|\ m_I(|\phi|v)\frac{\sqrt{|I|}}{\sqrt{|L|}}. \tag{6.2}$$

This is the same quantity defined in (5.7) with b instead of a or d. We also define the following Carleson sequences, see Corollaries 3.14 and 3.8,

$$\mu_K := (m_K w)^{\alpha}(m_K w^{-1})^{\alpha}\left(\frac{|\Delta_K w^{-1}|^2}{(m_K w^{-1})^2} + \frac{|\Delta_K w|^2}{(m_K w)^2}\right)|K|, \text{ intensity } C[w]^{\alpha}_{A_2},$$

$$\mu^n_L := \sum_{K\in\mathcal{ST}^n_L} \mu_K, \text{ intensity } C(n+1)[w]^{\alpha}_{A_2},$$

$$\mu^a_K := \frac{|a_K|^2}{|K|}(m_K w\ m_K w^{-1})^{\alpha}, \text{ intensity } C\|a\|_{BMO^d}[w]^{\alpha}_{A^d_2},$$

$$\mu^d_K := \frac{|d_K|^2}{|K|}(m_K w\ m_K w^{-1})^{\alpha}, \text{ intensity } C\|d\|_{BMO^d}[w]^{\alpha}_{A^d_2},$$

$$\mu^{a,m}_L := \sum_{K\in\mathcal{ST}^m_L} \mu^a_K, \text{ intensity } C(m+1)\|a\|_{BMO^d}[w]^{\alpha}_{A^d_2}.$$

$$\mu^{d,m}_L := \sum_{K\in\mathcal{ST}^n_L} \mu^d_K, \text{ intensity } C(n+1)\|d\|_{BMO^d}[w]^{\alpha}_{A^d_2}.$$

Note that

$$\Sigma^{m,n}_0 \leq \sum_{L\in\mathcal{D}} Pa^{w,m}_L f\ Pd^{w^{-1},n}_L g,$$

thus in order to estimate $\Sigma^{m,n}_0$ we will use the following estimates for $Pa^{w,m}_L f$ and $Pd^{w^{-1},n}_L g$, where $0 < \alpha < 1/2$ so we can use the α-Lemma 3.8,

$$Pa^{w,m}_L f \leq e^{\alpha} C^n_m (m_L w)^{1-\frac{\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}} \inf_{x\in L} M_w(|f|^p)^{\frac{1}{p}}(x)\nu^{a,n}_L. \tag{6.3}$$

where $\left(\nu^{a,n}_L := \|a\|_{BMO^d}\sqrt{\mu^n_L} + \sqrt{\mu^{a,n}_L}\right)$

$$Pd^{w^{-1}}_L g \leq e^{\alpha} C^n_m (m_L w)^{1-\frac{\alpha}{2}}(m_L w^{-1})^{\frac{-\alpha}{2}} \inf_{x\in L} M_{w^{-1}}(|g|^p)^{\frac{1}{p}}(x)\nu^{d,m}_L. \tag{6.4}$$

where $\nu^{d,m}_L := \|d\|_{BMO^d}\sqrt{\mu^n_L} + \sqrt{\mu^{d,n}_L}$. These estimates are proved in Lemma 5.4, just have to interchange the roles of b by a or d.

Estimate for $\Sigma_0^{m,n}$: Using the fact that $(m_I w\, m_I w^{-1})^{1-\alpha} \leq [w]_{A_2}^{1-\alpha}$,

$$\begin{aligned}
\Sigma_0^{m,n} &\leq \sum_{L\in\mathcal{D}} Pa_L^{w,m} f \; Pd_L^{w^{-1},m} g \\
&\leq e^{2\alpha}(C_m^n)^2 \sum_{L\in\mathcal{D}} [w]_{A_2^d}^{1-\alpha} \inf_{x\in L} \left(M_w(|f|^p)(x)\right)^{\frac{1}{p}} \inf_{x\in L} \left(M_{w^{-1}}(|g|^p)(x)\right)^{\frac{1}{p}} \nu_L^{a,n}\nu_L^{d,m} \\
&\leq e^{2\alpha}(C_m^n)^2 [w]_{A_2^d}^{1-\alpha} \sum_{L\in\mathcal{D}} \inf_{x\in L} \left(M_w(|f|^p)(x) M_{w^{-1}}(|g|^p)(x)\right)^{\frac{1}{p}} \nu_L^{a,n}\nu_L^{d,m} \qquad (6.5)
\end{aligned}$$

Since $(\nu_L^{a,n})^2$ and $(\nu_L^{d,m})^2$ have intensity at most $C(n+1)\|a\|_{BMO^d}^2 [w]_{A_2}^{\alpha}$ and $C(m+1)\|b\|_{BMO^d}^2 [w]_{A_2}^{\alpha}$ respectively then, by Proposition 2.21, we have that $\nu_L^{a,n}\nu_L^{d,m}$ is a Carleson measure with intensity at most $C(m+n+2)\|a\|_{BMO^d}\|d\|_{BMO^d}[w]_{A_2}^{\alpha}$. If we now apply Lemma 3.1 in (6.5), with $F^p(x) = M_w(|f|^p)(x) M_{w^{-1}}(|g|^p)(x)$, $p = 2-(m+n+2)^{-1}$, so that $q' = \left(\frac{2}{p}\right)' \sim C_m^n$, $\alpha_L = \nu_L^{a,n}\nu_L^{d,m}$, and $v = 1$, we will have, by Cauchy-Schwarz and the boundedness of M_v in $L^q(v)$ for $q = p/2 > 1$,

$$\begin{aligned}
\Sigma_0^{m,n} &\leq (C_m^n)^2 [w]_{A_2}^{1-\alpha} \sum_{L\in\mathcal{D}} \inf_{x\in L} \left(M_w(|f|^p)(x) M_{w^{-1}}(|g|^p)(x)\right)^{\frac{1}{p}} \nu_L^{a,n}\nu_L^{d,m} \\
&\leq e^{2\alpha}(C_m^n)^3 [w]_{A_2}\|a\|_{BMO^d}\|d\|_{BMO^d} \int_{\mathbb{R}} \left(M_w(|f|^p)(x)\right)^{\frac{1}{p}} \left(M_{w^{-1}}(|g|^p)(x)\right)^{\frac{1}{p}} dx \\
&\leq e^{2\alpha}(C_m^n)^3 [w]_{A_2^d} \|a\|_{BMO^d}\|d\|_{BMO^d} \left\|M_w(|f|^p)\right\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}} \left\|M_{w^{-1}}(|g|^p)\right\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}} \\
&\leq e^{2\alpha}\left[\left(\frac{2}{p}\right)'\right]^{\frac{2}{p}} (C_m^n)^3 [w]_{A_2} \|a\|_{BMO^d}\|d\|_{BMO^d} \left\||f|^p\right\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}} \left\||g|^p\right\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}} \\
&= e^{2\alpha}(C_m^n)^5 [w]_{A_2^d} \|a\|_{BMO^d}\|d\|_{BMO^d} \|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}.
\end{aligned}$$

Together these estimates prove the theorem with $c \leq 6$. □

6.2 Haar Multipliers

For a weight w, $t \in \mathbb{R}$, $m, n \in \mathbb{N}$, a *t-Haar multiplier of complexity* (m, n) is the operator defined as

$$T_{t,w}^{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} \frac{\sqrt{|I|\,|J|}}{|L|} \left(\frac{w(x)}{m_L w} \right)^t \langle f, h_I \rangle h_J(x). \tag{6.6}$$

For complexity $(0, 0)$ these operators are the Haar multipliers introduced for $t = 1$ in [P] and denoted by T_w, and for other real numbers t introduced in [KP], denoted by T_w^t. Note that these operators have symbols, namely $\frac{\sqrt{|I|}\sqrt{|J|}}{|L|}\left(\frac{w(x)}{m_L w}\right)^t$, that depend on: the space variable x, the frequency encoded on the dyadic interval L, and the complexity encoded on the subintervals $I \in \mathcal{D}_n(L)$ and $J \in \mathcal{D}_m(L)$. These makes these operators more akin to pseudodifferential operators where the trigonometric functions have been replaced by the Haar functions. Let us formally calculate $T_1^{m,n} T_w^t$ and $T_w^t T_1^{m,n}$ to check that these compositions are not $T_{t,w}^{m,n}$.

$$\begin{aligned}
(T_1^{m,n} T_w^t f)(x) &= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^L \langle T_w^t, h_I \rangle h_J(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} \frac{c_{I,J}^L}{(m_I w)^t} \langle f w^t, h_I \rangle h_J(x) \neq T_{t,w}^{m,n},
\end{aligned}$$

however for $m = 0$, $T_1^{m,n} T_w^t = (T_{t,w}^{0,n})^*$.

$$\begin{aligned}
(T_w^t T_1^{m,n} f)(x) &= \sum_{K \in \mathcal{D}} \left(\frac{w(x)}{m_K w} \right)^t \langle T_1^{m,n} f, h_K \rangle h_K(x) \\
&= \sum_{K \in \mathcal{D}} \left(\frac{w(x)}{m_K w} \right)^t \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^L \langle f, h_I \rangle \langle h_J, h_K \rangle h_K(x) \\
&= \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} c_{I,J}^L \left(\frac{w(x)}{m_J w} \right)^t \langle f, h_I \rangle h_J(x) \neq T_{t,w}^{m,n}.
\end{aligned}$$

but for $n = 0$, $T_w^t T_1^{m,0} = T_{t,w}^{m,0}$.

6.2.1 Necessary conditions

Let us first show a necessary condition on the weight w so that $T^{m,n}_{w,t}$ is bounded in $L^p(\mathbb{R})$. This necessary C^d_{tp}-condition is the same condition found in [KP] for the t-Haar multiplier of complexity $(0,0)$, see also [P1]

Theorem 6.3. *Let m, n be positive integers and let t be a real number then if $T^{m,n}_{t,w}$ is a bounded operator on $L^p(\mathbb{R})$ then w is a weight in C^d_{tp}.*

Proof. Assume that $T^{m,n}_{t,w}$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, there exists C such that for any $f \in L^p(\mathbb{R})$ we have that $\|T^{m,n}_{t,w} f\|_p \leq C\|f\|_p$. Thus for any $I_0 \in \mathcal{D}$ we should have that

$$\|T^{m,n}_{t,w} h_{I_0}\|^p_{L^p} \leq C^p \|h_{I_0}\|^p_{L^p}. \tag{6.7}$$

Let us compute then the norm on the left-hand-side of (6.7). First observe that,

$$T^{m,n}_{t,w} h_{I_0}(x) = \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} \frac{\sqrt{|I|\,|J|}}{|L|} \left(\frac{w(x)}{m_L w}\right)^t \langle h_{I_0}, h_I\rangle h_J(x). \tag{6.8}$$

By the properties of dyadic filtration $\langle h_{I_0}, h_I\rangle = 1$ if $I_0 = I$ and $\langle h_{I_0}, h_I\rangle = 0$ otherwise. Also there exists just one dyadic interval L_0 such that $I_0 \subset L_0$ and $|I_0| = 2^{-m}|L_0|$. Therefore we can collapse the sums in (6.8) in just one sum, and calculate the L^p-norm as follows,

$$\|T^{m,n}_{t,w} h_{I_0}\|^p_{L^p} = \int_{\mathbb{R}} \Bigg| \sum_{J\in\mathcal{D}_n(L_0)} \frac{\sqrt{|I_0|}\sqrt{|J|}}{|L_0|} \Big(\frac{w(x)}{m_{L_0} w}\Big)^t h_J(x)\Bigg|^p dx.$$

Furthermore, since $\mathcal{D}_n(L_0)$ is a partition of L_0, on the functions h_J are supported on $J \in \mathcal{D}_n(L_0)$, the power p can travel inside the sum, and we get,

$$\|T^{m,n}_{t,w} h_{I_0}\|^p_p = \frac{|I_0|^{\frac{p}{2}}}{|L_0|^{p-1}} \frac{m_{L_0}(w^{tp})}{(m_{L_0} w)^{pt}}. \tag{6.9}$$

Inserting $\|h_{I_0}\|_p^p = |I_0|^{1-\frac{p}{2}}$ and (6.9) in (6.7), we will have that for any dyadic interval I_0 there exists C such that

$$\frac{|I_0|^{\frac{p}{2}}}{|L_0|^{p-1}} \frac{m_{L_0}(w^{tp})}{(m_{L_0}w)^{pt}} \leq C^p |I_0|^{1-\frac{p}{2}}$$

which implies $\frac{m_{L_0}(w^{tp})}{(m_{L_0}w)^{pt}} \leq C^p|I_0|^{1-p}|L_0|^{p-1} = C^p 2^{m(p-1)} =: C_{m,p}$. Now observe that this inequality should hold for any $L_0 \in \mathcal{D}$, we just have to choose as I_0 any of the descendants of L_0 in the m-th generation, also note that m is a fixed value. Therefore

$$[w]_{C^d_{2t}} = \sup_{L\in\mathcal{D}} \big(m_L(w^{tp})\big)\big(m_L w\big)^{-pt} \leq C_{m,p}.$$

We conclude that $w \in C^d_{tp}$, moreover $[w]_{C^d_{tp}} \leq 2^{n(p-1)} \|T^{m,n}_{t,w}\|^p_{L^p}$. □

6.2.2 Sufficient condition

The C^d_{2t}-condition is not only necessary but also sufficient for most t for a t-Haar multiplier of complexity (m,n) to be bounded in $L^2(\mathbb{R})$, this was proved in [KP] for the case $m = n = 0$. Here we are concerned not only with the boundedness but also with the dependence on the C^d_{2t}-constant of the operator norm. For the case $m = n = 0$ and $t = 1, \pm 1/2$ this was studied in [P2]. Beznosova [Be] was able to obtain estimates, under the additional condition on the weight $w^{2t} \in A_q$ for some $q > 1$, for the case of complexity $(0,0)$ and for all $t \in \mathbb{R}$. We recover her results and we will extended it for complexity (m,n). Our proof differs from hers in that we are adapting the methods of Nazarov and Volberg [NV] to this setting as well. Both proofs rely on the $A^d_p - \alpha$-Lemma 3.10 and on the A^d_p-Little Lemma 3.4.

Theorem 6.4. *Let t be a real number and w a weight in C^d_{2t}, such that $w^{2t} \in A^d_q$, for $q > 1$ and that satisfies the C^d_{2t} condition with constant $[w]_{C^d_{2t}}$. Then the Haar Multiplier with depth (m,n) is bounded in $L_2(\mathbb{R})$. Moreover*

$$\|T^{m,n}_{t,w} f\|_{L^2} \leq C(m+n+2)^3 [w]^{\frac{1}{2}}_{C^d_{2t}} [w^{2t}]^{\frac{1}{2}}_{A^d_q} \|f\|_{L^2}.$$

Proof. Fix $f,\ g \in L^2(\mathbb{R})$. By duality, it is enough to show that

$$|\langle T^{m,n}_{t,w} f, g\rangle| \leq C(m+n+2)^3[w]^{\frac{1}{2}}_{C^d_{2t}}[w^{2t}]^{\frac{1}{2}}_{A^d_2}\|f\|_{L^2}\|g\|_{L^2}.$$

The inner product on the left-hand-side can be expanded into a double sum, that we now estimate,

$$\begin{aligned}\Big|\Big\langle \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} \frac{\sqrt{|I|\,|J|}}{|L|}\,\frac{w^t}{(m_L w)^t}\langle f, h_I\rangle h_J, g\Big\rangle\Big| \\ \leq \sum_{L\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}} \frac{\sqrt{|I|\,|J|}}{|L|}\,\frac{|\langle f, h_I\rangle|}{(m_L w)^t}\,|\langle g w^t, h_J\rangle|.\end{aligned}$$

Once again, we will replace h_J by a linear combination of a weighted Haar function and a characteristic function, $h_J = \alpha_J h^{w^{2t}}_J + \beta_J \frac{\chi_J}{\sqrt{|J|}}$, where $\alpha_J = \alpha^{w^{2t}}_J$, $\beta_J = \beta^{w^{2t}}_J$, $|\alpha_J| \leq \sqrt{m_J w^{2t}}$, and $|\beta_J| \leq \frac{|\Delta_J(w^{2t})|}{m_J w^{2t}}$. Now break into two terms to be estimated separately so that,

$$|\langle T^{m,n}_{t,w} f, g\rangle| \leq \Sigma^{m,n}_1 + \Sigma^{m,n}_2,$$

where

$$\begin{aligned}\Sigma^{m,n}_1 &= \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_n(L);J\in\mathcal{D}_m(L)} \frac{\sqrt{|I|\,|J|}}{|L|}\,\frac{\sqrt{m_J(w^{2t})}}{(m_L w)^t}|\langle f, h_I\rangle|\;|\langle g w^t, h^{w^{2t}}_J\rangle|,\\ \Sigma^{m,n}_2 &= \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_n(L);J\in\mathcal{D}_m(L)} \frac{|J|\sqrt{|I|}}{|L|(m_L w)^t}\,\frac{|\Delta_J(w^{2t})|}{m_J(w^{2t})}|\langle f, h_I\rangle|\; m_J(|g|w^t).\end{aligned}$$

Again let $p = 2 - (C^m_n)^{-1}$, and define as in (5.5) and (5.6), the quantities $S^{v,n}_L\phi$ and $R^{v,n}_L\phi$, we will use here the case $v = w^{2t}$, and corresponding estimates. Define a new quantity

$$P^m_L\phi := \sum_{I\in\mathcal{D}_m(L)} |\langle f, h_I\rangle|\frac{\sqrt{|I|}}{\sqrt{|L|}}.$$

We also define the following sequences for $0<\alpha<\dfrac{1}{2(q-1)}$,

$$\eta_I := (m_I(w^{2t}))^{\alpha}\,(m_I w^{\frac{-2t}{q-1}})^{(q-1)\alpha}\left(\frac{|\Delta_I(w^{2t})|^2}{|m_I w^{2t}|^2}+\frac{|\Delta_I(w^{-2t})|^2}{|m_I w^{-2t}|^2}\right)|I|,$$

by Lemma 3.10, a Carleson sequence with intensity $C_\alpha[w^{2t}]^{\alpha}_{A^d_q}$, and

$$\eta^m_L := \sum_{I\in\mathcal{ST}^m_L}\eta_I,$$

where the stopping time $\mathcal{ST}^m_L$ is defined as in Lemma 3.13 but with respect to the weight w^{2t}, and by Lemma 3.14, it is a Carleson sequence with intensity $C_\alpha(m+1)[w^{2t}]^{\alpha}_{A^d_q}$.

Observe that on the one hand $\langle gw^t, h^{w^{2t}}_J\rangle = \langle gw^{-t}, h^{w^{2t}}_J\rangle_{w^{2t}}$, and on the other $m_J(|g|w^t)=m_J(|gw^{-t}|w^{2t})$. Therefore,

$$\Sigma^{m,n}_1 = \sum_{L\in\mathcal{D}}\frac{1}{(m_Lw)^t}S^{w^{2t},n}_L(gw^{-t})\;P^m_L f,$$

$$\Sigma^{m,n}_2 = \sum_{L\in\mathcal{D}}\frac{1}{(m_Lw)^t}R^{w^{2t},n}_L(gw^{-t})\;P^m_L f.$$

Estimates (5.8) and (5.9) hold for $S^{w^{2t},m}_L(gw^{-t})$ and $R^{w^{2t},m}_L(gw^{-t})$ with w^{-1} and g replaced by w^{2t} and gw^{-t}:

$$S^{w^{2t},n}_L(gw^{-t}) \le (m_Lw^{2t})^{\frac12}\Big(\sum_{J\in\mathcal{D}_m(L)}|\langle gw^{-t},h^{w^{2t}}_J\rangle_{w^{2t}}|^2\Big)^{\frac12},$$
$$R^{w^{2t},n}_L(gw^{-t}) \le e^{\alpha}C^n_m(m_Lw^{2t})^{1-\frac{\alpha}{2}}(m_Lw^{\frac{2t}{q-1}})^{\frac{-(q-1)\alpha}{2}}\inf_{x\in L}F^{\frac12}(x)\sqrt{\eta^m_L},$$

where $F(x)=\big(M_{w^{2t}}(|gw^{-t}|^p)(x)\big)^{\frac{2}{p}}$. Estimating $P^n_L f$ is very simple:

$$P^m_L f = \sum_{I\in\mathcal{D}_m(L)}|\langle f,h_I\rangle|\frac{\sqrt{|I|}}{\sqrt{|L|}} \le \Big(\sum_{I\in\mathcal{D}_m(L)}\frac{|I|}{|L|}\Big)^{\frac12}\Big(\sum_{I\in\mathcal{D}_m(L)}|\langle f,h_I\rangle|^2\Big)^{\frac12}$$
$$=\Big(\sum_{I\in\mathcal{D}_m(L)}|\langle f,h_I\rangle|^2\Big)^{\frac12}.$$

Estimating $\Sigma_1^{m,n}$: Plug in the estimates for $S_L^{w^{2t},n}(gw^{-t})$ and $P_L^m f$, observe that $\frac{(m_L w^{2t})^{\frac{1}{2}}}{(m_L w)^t} \leq [w]_{C_{2t}^d}^{\frac{1}{2}}$, use Cauchy-Schwarz, and we get,

$$\begin{aligned}
\Sigma_1^{m,n} &\leq \sum_{L\in\mathcal{D}} [w]_{C_{2t}^d}^{\frac{1}{2}} \Bigg(\sum_{J\in\mathcal{D}_n(L)} |\langle gw^{-t}, h_J^{w^{2t}}\rangle_{w^{2t}}|^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{I\in\mathcal{D}_m(L)} |\langle f, h_I\rangle|^2 \Bigg)^{\frac{1}{2}} \\
&\leq [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \Bigg(\sum_{L\in\mathcal{D}} \sum_{J\in\mathcal{D}_n(L)} |\langle gw^{-t}, h_J^{w^{2t}}\rangle_{w^{2t}}|^2 \Bigg)^{\frac{1}{2}} \\
&\leq [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_{L^2} \|gw^{-t}\|_{L^2(w^{2t})} = [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}.
\end{aligned}$$

Estimating $\Sigma_2^{m,n}$: Plug in the estimates for $R_L^{w^{2t},n}(gw^{-t})$ and $P_L^m f$, where $F(x) = \big(M_{w^{2t}}(|gw^{-t}|^p)(x)\big)^{2/p}$, use Cauchy-Schwarz observing that

$$\frac{(m_L w^{\frac{-2t}{q-1}})^{\frac{-(q-1)\alpha}{2}} (m_L w^{2t})^{1-\frac{\alpha}{2}}}{(m_L w)^t} \leq [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1-\alpha}{2}} (m_L w^{\frac{-2t}{q-1}})^{-\frac{q-1}{2}},$$

and get

$$\Sigma_2^{m,n} \leq 4e^{\alpha} C_m^n [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1-\alpha}{2}} \|f\|_2 \Bigg(\sum_{L\in\mathcal{D}} \frac{\eta_L^m}{(m_L w^{\frac{-2t}{q-1}})^{q-1}} \inf_{x\in L} F(x) \Bigg)^{\frac{1}{2}}.$$

Now using Lemma 3.1 with $\alpha_L = \dfrac{\eta_L^m}{(m_L w^{\frac{-2t}{q-1}})^{q-1}}$ (which by Lemma 3.4 is a w^{2t}-Carleson sequence with intensity $c_\alpha(m+1)[w]_{A_q^d}^{\alpha}$), $F(x) = \big(M_{w^{2t}}|gw^{-t}|^p(x)\big)^{2/p}$, and $v = w^{2t}$,

$$\Sigma_2^{m,n} \leq c_\alpha (C_m^n)^2 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \Big\| M_{w^{2t}}(|gw^{-t}|^p) \Big\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{p}}.$$

Using Lemma 2.27, that is the boundedness of $M_{w^{2t}}$ in $L^{\frac{2}{p}}(w^{2t})$ for $2/p > 1$, $(\dfrac{2}{p})' \sim c_m^n$

$$\begin{aligned}
\Sigma_2^{m,n} &\leq c_\alpha (C_m^n)^2 (2/p)' [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \Big\| |gw^{-t}|^p \Big\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{p}} \\
&\leq c_\alpha (C_m^n)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}.
\end{aligned}$$

The theorem is proved. □

The following observations were made by Beznosova in the complexity $(0,0)$ case, see her dissertation [Be].

Note that when $t = -\frac{1}{2}$, we have that

$$\|T^{m,n}_{-\frac{1}{2},w} f\|_{L^2} \leq C(m+n+2)^3 [w]^{\frac{1}{2}}_{C^d_{-1}} [w^{-1}]^{\frac{1}{2}}_{A^d_2} \|f\|_{L^2},$$

by definition, $[w]_{C^d_{-1}} = [w]_{A^d_2} = [w^{-1}]_{A^d_2}$, therefore

$$\|T^{m,n}_{-\frac{1}{2},w} f\|_2 \leq C(m+n+2)^3 [w]_{A^d_2} \|f\|_2. \tag{6.10}$$

For $t = \frac{1}{2}$, we have that

$$\|T^{m,n}_{\frac{1}{2},w} f\|_{L^2} \leq C(m+n+2)^3 [w]^{\frac{1}{2}}_{C^d_1} [w]^{\frac{1}{2}}_{A^d_2} \|f\|_{L^2},$$

since $[w]_{C^d_1} \leq 1$ by Hölder's inequality then

$$\|T^{m,n}_{\frac{1}{2},w} f\|_{L^2} \leq C(m+n+2)^3 [w]^{\frac{1}{2}}_{A^d_2} \|f\|_{L^2}, \tag{6.11}$$

Both estimates (6.10) and (6.11) are sharp, because the same dependence on the A_2 -characteristic for the operators with complexity $(0,0)$ are known to be sharp, [P2].

For $t = 1$ we unfortunately cannot recover the sharp dependence found in [P2], in this case we will have

$$\|T^{m,n}_{w} f\|_{L^2} \leq C(m+n+2)^3 [w]^{\frac{1}{2}}_{C^d_2} [w^2]^{\frac{1}{2}}_{A^d_2} \|f\|_{L^2},$$

by definition $[w]^{\frac{1}{2}}_{C^d_2} = [w]_{RH^d_2}$, and by a result of Beznosova [Be], $w^2 \in A^d_q$ if and only if $w \in RH^d_2 \bigcap A^d_{\frac{q+1}{2}}$, moreover $[w^2]^{\frac{1}{2}}_{A^d_q} \leq [w]_{RH^d_2} [w]_{A^d_{\frac{q+1}{2}}}$. Therefore

$$\|T^{m,n}_{w} f\|_{L^2} \leq C(m+n+2)^3 [w]^2_{RH^d_2} [w]_{A^d_{\frac{q+1}{2}}} \|f\|_{L^2}.$$

which for complexity $(0,0)$ is a little worse than the bound from [P2]

$$\|T_w f\|_{L^2} \leq C D(w) [w]^2_{RH^d_2} \|f\|_{L^2}.$$

because we have that $D(w) \leq [w]_{A^d_{\frac{q+1}{2}}}$, see [Be].

Chapter 7

Two weighted estimates

In this chapter we will prove sufficient conditions for the boundedness of the dyadic paraproduct π_b from a weighted Lebesgue space to another, possibly different, weighted Lebesgue space. Our conditions consist of some type of test conditions on the pair of weights and the function b. This problem has been addressed for other dyadic operator such as the martingale transform, the dyadic maximal function and the dyadic square function. In fact, for all these three operators necessary and sufficient conditions over the weights u and v are known in order for them to be bounded from $L^2(u)$ into $L^2(v)$. Also many authors proved boundedness of specific paraproducts under certain hypothesis from a weighted Lebesgue space to another weighted Lebesgue space. In this case b comes from a known operator T with specific properties, and b, as well as the paraproduct depend on u and v, the paraproduct is based on weighted Haar functions and weighted averages, this adapted paraproduct appears for instance in [HPzTV, NTV1, HLM+]. Our conditions are the first type of conditions that assure that the rigid form of the dyadic paraproduct, i.e. the paraproduct based on the Haar functions $\{h_I\}_{I\in\mathcal{D}}$ and Lebesgue averages $\{m_I f\}_{I\in\mathcal{D}}$, is bounded from $L^2(u)$ to $L^2(v)$. First let us define a two weight condition that is necessary for all dyadic operators described above, the Muckenhoupt condition, also called A_p^d condition or joint A_p^d condition.

Definition 7.1. *A pair of weights* $(u,v) \in A_p^d$ *if* $u^{\frac{-1}{p-1}}$, $v \in L^1_{loc}$ *and*

$$[u,v]_{A_p^d} := \sup_{I\in\mathcal{D}} \left(m_I u^{\frac{-1}{p-1}}\right)^{p-1} m_I v < \infty.$$

Remark 7.2. *Note that* $(u,v) \in A_p^d$ *if and only if there exists* $C > 0$ *such that* $\left(m_I u^{\frac{-1}{p-1}}\right)^{p-1} m_I v < C$ *for all dyadic intervals* $I \in \mathcal{D}$*. Moreover, Lebesgue Differentiation Theorem implies that*

$$v(x) \leq Cu(x) \qquad a.e.$$

Remark 7.3. *Note that if* u *and* v *are weights such that* $u^{\frac{-1}{p-1}}$ *is in* L^1_{loc} *and* u^{-1} *and* v *are bounded then there is* $C > 0$

$$\forall\, I \in \mathcal{D} \quad m_I v < C \text{ and } m_I u^{\frac{-1}{p-1}} < C^{\frac{1}{p-1}} \quad \Rightarrow \quad m_I v\, m_I u^{\frac{-1}{p-1}} < CC^{\frac{1}{p-1}} \quad \forall\, I \in \mathcal{D}$$

which implies that $(u,v) \in A_p^d$

7.1 The issue of reduction to the one weight theory

In the study of two weight inequalities, we frequently want to find conditions on the weights u and v, i.e. a class of weights (u,v) such that a given operator is bounded from $L^p(u)$ into $L^p(v)$. One should be careful because some conditions imposed on the pair of weights u and v might reduce the problem to a one weight theory problem, i.e. these type of conditions would imply that an operator is bounded from $L^p(u)$ into $L^p(v)$ if and only if the operator is bounded from $L^p(u)$ into $L^p(u)$ or $L^p(v)$ into $L^p(v)$. The first type of condition that reduce the two weight problem to a one weight problem is the comparability of the weights. Consider $T : L^p(u) \to L^p(v)$ an operator, and u and v weights such that there is $C > 1$ where

$$\frac{1}{C} v(x) \leq u(x) \leq Cv(x) \quad a.e.$$

Then for any $g \in L^p(v)$

$$\int_{\mathbb{R}} |g(x)|^p v(x)dx \leq C \int_{\mathbb{R}} |g(x)|^p u(x)dx \leq C^2 \int_{\mathbb{R}} |g(x)|^p v(x)dx.$$

Thus

$$\|g\|_{L^p(v)} \leq C\|g\|_{L^p(u)} \leq C^2\|g\|_{L^p(v)},$$

and this implies that

$$\|Tf\|_{L^p(v)} \leq B\|f\|_{L^p(u)} \Rightarrow \|Tf\|_{L^p(v)} \leq CB\|f\|_{L^p(v)}.$$

and

$$\|Tf\|_{L^p(v)} \leq B\|f\|_{L^p(v)} \Rightarrow \|Tf\|_{L^p(v)} \leq CB\|f\|_{L^p(u)}.$$

Therefore

$$\|Tf\|_{L^p(v)} \leq B\|f\|_{L^p(u)} \Leftrightarrow \|Tf\|_{L^p(v)} \leq CB\|f\|_{L^p(v)}.$$

As we easily showed in the previous lines, comparability between the weights reduces a two weight problem to a one one weight problem. Since this is kind of obvious we never assumed comparability "per se" as a hypothesis for the results we were trying to get. However comparability between the weights is sometimes disguised in other conditions, i.e. conditions over the weights that imply that the weights are comparable. We collect in the next proposition some conditions over the weights u and v that imply they are comparable, we faced them at some point working in the results of this chapter of the dissertation.

Proposition 7.4. *If the pair of weights u and v, where $u^{\frac{-1}{p-1}}$ is also a weight, satisfies any of the properties below then the weights are comparable.*

(i) $(u, v) \in A_p$ *and* $(v, u) \in A_p$*;*

(ii) If there is $C > 0$ *such that* $C < \left(m_I u^{\frac{-1}{p-1}}\right)^{p-1} m_I v$ *for all dyadic intervals* I *and* $(u, v) \in A_p$.

(iii) $(u,v) \in A_p$*, and* v^{-1} *and* u *are bounded;*

Proof. *(i)* Assume $(u,v) \in A_p$ and $(v,u) \in A_p$; then

$$\big(m_I u^{\frac{-1}{p-1}}\big)^{p-1} m_I v < C_2 \quad \forall I \in \mathcal{D} \tag{7.1}$$

and

$$\big(m_I v^{\frac{-1}{p-1}}\big)^{p-1} m_I u < C_1 \quad \forall I \in \mathcal{D} \tag{7.2}$$

Note that, 7.1, is equivalent to

$$m_I v < C_2 \big(m_I u^{\frac{-1}{p-1}}\big)^{1-p} \quad \forall I \in \mathcal{D} \tag{7.3}$$

By Lebesgue Differentiation Theorem, equation 7.1 implies that $v(x) < C_2 u(x)$ *a.e.* Analogously we can show that 7.2 implies that $u(x) \leq C_1 v(x)$, thus the weights are comparable.

(ii) Assume there is $C > 0$ such that $C < \big(m_I u^{\frac{-1}{p-1}}\big)^{p-1} m_I v$ for all dyadic intervals I. Then by Lebesgue Differentiation Theorem $C < u^{-1}(x)v(x)\, a.e$ which implies that $Cu(x) < v(x)\, a.e$, and $(u,v) \in A_p$ implies that there exists $D > 0$ such that $v(x) < Du(x)\, a.e.$ Therefore the weights are comparable.

(iii) Assume $(u,v) \in A_p$ for $p > 1$, v^{-1} and u bounded . Then by Remark 7.3 $(v,u) \in A_p$, therefore by part (i) the weights are comparable.

□

7.1.1 Power weights

The classical example in the one weighted theory of a weight w in the A_p class are the power weights $w(x) = |x|^\alpha$ for $-1 < \alpha < p-1$. Let us study power weights in the two weights setting, i.e, let us answer the following question. For which β and γ $(|x|^\beta, |x|^\gamma)$ is in A_p^d?

Note that if $(|x|^\beta, |x|^\gamma)$ is in A_p^d for any $p > 1$ there is $C > 0$ such that $|x|^\gamma \leq C|x|^\beta$ *a.e*, which implies that $|x|^{\gamma-\beta} \leq C$ *a.e.* $\gamma = \beta$.

Thus in order for $(|x|^\beta, |x|^\gamma)$ be A_p we need $-1 < \beta = \gamma < p - 1$.

7.1.2 A_2^d and joint A_2^d do not imply comparability

We will show, by providing an example, that it is not true that if the two weights u, v are in A_2^d and the pair (u, v) is in joint A_2 then the weights are comparable. This also implies that both weights in A_∞^d and in joint A_2^d is not a sufficient condition for the weights to be comparable.

Example 7.5. *Let* $v(x) = 1$ *and*

$$u(x) = \chi_{\mathbb{R}\setminus(0,1)}(x) + \sum_{n=0}^{\infty} 2^{\frac{n}{2}} \chi_{[2^{-n-1}, 2^{-n})}(x)$$

Note that for $x \in (0,1)$, $0 \leq u(x) \leq \frac{1}{\sqrt{x}}$ *and for* $x \in \mathbb{R} \setminus (0,1)$ $u(x) = 1$ *. Therefore* $u \in L^1_{loc}$*. Also*

$$\forall I \in \mathcal{D} \quad m_I v \, m_I u^{-1} \leq 1 \Rightarrow (u, v) \in A_2^d \quad \textit{with} \quad [u, v]_{A_2^d} = 1$$

By definition of the weights we have $v(x) \leq u(x)$*, which is always true if* $(u, v) \in A_2^d$*. Note that*

$$u^{-1}(x) = \chi_{\mathbb{R}\setminus(0,1)}(x) + \sum_{n=0}^{\infty} 2^{\frac{-n}{2}} \chi_{[2^{-n-1}, 2^{-n})}(x).$$

Note that $u^{-1} \in L^1_{loc}$ *since* $u^{-1}(x) \leq 1$ *for all* $x \in \mathbb{R}$*. Suppose that there exists* $C > 0$ *such that* $u(x) \leq Cv(x)$ *a.e then* $u(x) \leq C$ *a.e, i.e* u *is bounded almost everywhere, this cannot happen because if* $n > 2\log_2 C$ *then* $u(x) > C$ *for all* $x \in [2^{-n-1}, 2^{-n})$*, therefore the weights are not comparable.*

The second type of condition that reduces the two weights problem to a one weight problem is if $(u, v) \in A_q^d$ for $q > 1$, u or v is in A_r for some $r > 1$ and the operator that we are trying to analyze is bounded in $L^p(w)$ if $w \in A_r$. Let us use the dyadic paraproduct π_b to exemplify this, it is known that if $b \in BMO$ then π_b is bounded in $L^2(w)$ if and only if $w \in A_2$. Therefore for any pair of weights u and v, such that (u, v) is in joint A_2^d and one of them is in A_2^d we have that π_b is bounded from $L^2(u)$ to $L^2(v)$, one can conclude that just using the one weight theory. This fact is enounced and proved in the next proposition.

Proposition 7.6. *Given a pair of weights $(u, v) \in A_q$ for $q > 1$. Suppose T is an operator that is bounded on $L^p(w)$ if $w \in A_r$ for some $r > 1$. Then if $u \in A_r$ or if $v \in A_r$ the operator T is bounded from $L^p(u)$ to $L^p(v)$.*

Proof. Assume that $(u, v) \in A_q^d$, then there exist $C > 0$ such that $v(x) < Cu(x) a.e.$ This implies that

$$\|Tf\|_{L^p(v)} \le \|Tf\|_{L^p(u)} \quad \text{and} \quad \|f\|_{L^p(v)} \le \|f\|_{L^p(u)} \tag{7.4}$$

Therefore if $v \in A_r$, then by hypothesis, $\|Tf\|_{L^p(v)} \le C\|f\|_{L^p(v)}$, hence by (7.4)

$$\|Tf\|_{L^p(v)} \le C\|f\|_{L^p(v)} \le C\|f\|_{L^p(u)} \tag{7.5}$$

Analogous, if $u \in A_r$ then by hypothesis, $\|Tf\|_{L^p(u)} \le C\|f\|_{L^p(u)}$, by 7.4

$$\|Tf\|_{L^p(v)} \le \|Tf\|_{L^p(u)} \le C\|f\|_{L^p(u)} \tag{7.6}$$

□

7.2 Two weighted results for dyadic operators

In this section we present dyadic operators for which conditions for the two weight norm inequality are known. The first result is for the Maximal function, due to E. Sawyer, [S], this result is also valid for the dyadic Maximal function.

Theorem 7.7 (Sawyer, [S]). *Let u and v be weights and $1 < p \leq q \leq \infty$ with $p < \infty$, then there exists $C > 0$ such that*

$$\|Mf\|_{L^q(v)} \leq C\|f\|_{L^p(u)}$$

for all $f \in L^p(u)$ if and only if

$$\left(\int_I [M(\chi_I u^{\frac{-1}{p-1}})(x)]^q v(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_I u^{\frac{-1}{p-1}}(x)dx\right)^{\frac{1}{p}}.$$

for all dyadic intervals I.

Remark 7.8. *The theorem above in fact works for weighted spaces defined over $\mathbb{R}^k$, we stated for weighted spaces over $\mathbb{R}$, because we are working over $\mathbb{R}$ throughout this dissertation and we had not set the proper definition of dyadic cubes $\mathbb{R}^k$.*

The next result gives necessary and sufficient condition over the weights to obtain two weight inequalities for the square function. The necessary and sufficient condition was proved by Nazarov Treil and Volberg in [NTV].

Theorem 7.9 (Nazarov-Treil-Volberg [NTV]). *Let a couple of weights,$(u, v) \in A_2$. The dyadic Square function, defined by*

$$S^d f(x) := \left(\sum_{I\in\mathcal{D}} |m_I f - m_{\hat{I}} f|^2 \chi_I(x)\right)^{\frac{1}{2}} = \left(\frac{1}{2}\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 \frac{\chi_I(x)}{|I|}\right)^{\frac{1}{2}}$$

is bounded from $L^2(u)$ to $L^2(v)$ if and only if u^{-1} and v are weights and

$\{\Delta_I u^{-1} m_I v |I|\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence.

Beznonova recently proved the sharp bound for the norm of the dyadic square form $L^2(u)$ into $L^2(v)$, she proved the following

$$\|S^d\|_{L^2 u \to L^2(v)} \leq [u, v]^{\frac{1}{2}}_{A_2^d} [w]^{\frac{1}{2}}_{RH_1^d}.$$

The next and last theorem is from Nazarov-Treil-Volberg and it gives necessary and sufficient conditions for the Martingale Transform

$$T_\sigma f(x) = \sum_{I \in \mathcal{D}} \sigma_I \langle f, h_I \rangle h_I(x)$$

to be uniformly bounded from $L^2(u)$ into $L^2(v)$ with respect to all possible choices of σ, where $\sigma = \{\sigma_I\}_{I\in\mathcal{D}}$ and $\sigma_I \in \{+1, -1\}$, $\forall I \in \mathcal{D}$. Before we state the theorem we need to introduce some notation. Let

$$\alpha_I = \frac{|\Delta_I v|}{m_I v} \frac{|\Delta_I u^{-1}|}{m_I u^{-1}}.$$

Also consider the operator

$$T_0 f(x) = \sum_{I \in \mathcal{D}} \frac{\alpha_I}{|I|} m_I f \, \chi_I(x);$$

T_0 is an positive dyadic shift operator of type 4.

Theorem 7.10 (Nazarov-Treil-Volberg, [NTV]). *The martingale transform T_σ is bounded from $L^2(u)$ to $L^2(v)$ if and only if the following four assertions hold simultaneously:*

(i) $(u, v) \in A_2$

(ii) $\{\Delta_I |u^{-1}|^2 \, m_I v |I|\}_{I\in\mathcal{D}}$ *is a u^{-1}-Carleson sequence.*

(iii) $\{|\Delta_I v|^2 \, m_I u^{-1} |I|\}_{I\in\mathcal{D}}$ *is a v-Carleson sequence.*

(iv) The operator T_0 is bounded from $L^2(u)$ into $L^2(v)$.

Note that a pair of weights satisfies condition *(i)* and *(ii)* if and only if the dyadic square function S^d is bounded from $L^2(u)$ into $L^2(v)$, and satisfies condition *(i)* and *(iii)* if and only if the dyadic square function S^d is bounded from $L^2(v^{-1})$ into $L^2(u^{-1})$, note that $(u, v) \in A_2^d$ if and only if $(v^{-1}, u^{-1}) \in A_2^d$.

7.3 Main Result

Before stating and proving our main result, we need to define a class of objects that will take the place of the BMO^d class in the one weighted theory, we will call this class the two weighted Carleson class.

Definition 7.11. *Given two weights u and v, such that v is a regular weight and u^{-1} is also a regular weight, then we say that a locally integrable function b belongs to the two weighted Carleson class u, v, $Carl_{u,v}$ if $\left\{\frac{|b_I|^2}{m_I v}\right\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence, where $b_I = \langle b, h_I\rangle$.*

Note that if $u = v$, then we have that that $b \in Carl_{v,v}$ if $\left\{\frac{|b_I|^2}{m_I v}\right\}_{I\in\mathcal{D}}$ is a v^{-1}-Carleson sequence, which is true by Lemma 3.3 if $\{b_I\}_{I\in\mathcal{D}}$ is Carleson sequence. This is equivalent to say that $b \in BMO^d$. Therefore for any weight v, such that v^{-1} is also a weight, we have that

$$BMO^d \subset Carl_{v,v}.$$

Theorem 7.12. *Let (u, v) be a pair of weights such that v is a regular weight and u^{-1} is also a regular weight and $\{|\Delta_I v|^2 |I| m_I u^{-1}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity B_2. Then π_b is bounded from $L^2(u)$ into $L^2(v)$ for all $b \in Carl_{u,v}$ if and only if $(u, v) \in A_2^d$. Moreover, if B_1 is the intensity of the u^{-1}-Carleson sequence $\left\{\frac{|b_I|^2}{m_I v}\right\}_{I\in\mathcal{D}}$ then*

$$\|\pi_b f\|_{L^2(v)} \le CB[u, v]_{A_2}\|f\|_{L^2(u)};$$

for some $C > 0$ and $B = \max\{B_1, B_2\}$.

Proof. Sufficiency of A_2^d:

Fix $f \in L^2(u^{-1})$ and $g \in L^2(v)$. Note that $fu^{-1} \in L^2(u)$ and $\|fu^{-1}\|_{L^2(u)} = \|f\|_{L^2(u^{-1})}$, $gv \in L^2(v^{-1})$ and $\|gv\|_{L^2(v^{-1})} = \|g\|_{L^2(v)}$, $\pi_b(fu^{-1})$ is expected to be in

$L^2(v)$, then $gv \in L^2(v^{-1})$ is in the right space for the pairing. Then, by duality, suffices to prove:

$$|\langle \pi_b(fu^{-1}), gv\rangle| \le CB[u,v]_{A_2}\|f\|_{L^2(u^{-1})}\|g\|_{L^2(v)}. \tag{7.7}$$

Note that

$$|\langle \pi_b(fu^{-1}), gv\rangle| = \Big|\Big\langle \sum_{I\in\mathcal{D}} c_I b_I m_I(fu^{-1})h_I, gv\Big\rangle\Big|.$$

Replace $h_I = \alpha_I h_I^v + \beta_I \frac{\chi_I}{\sqrt{|I|}}$ where $\alpha_I = \alpha_I^v$ and $\beta_I = \beta_I^v$ as described in Proposition 2.2, and get

$$|\langle \pi_b(fu^{-1}), gv\rangle| \;\le\; \sum_{I\in\mathcal{D}} |b_I| m_I(|f|u^{-1})\Big|\Big\langle gv, \alpha_I h_I^v + \beta_I \frac{\chi_I}{\sqrt{|I|}}\Big\rangle\Big|. \tag{7.8}$$

Use the triangle inequality to break the sum in (7.8) into two summands to be estimated separately,

$$\begin{aligned}|\langle \pi_b(fu^{-1}), gv\rangle| \;\le\; & \sum_{I\in\mathcal{D}} |b_I||\alpha_I| m_I(|f|u^{-1})\big|\big\langle gv, h_I^v\big\rangle\big| \\ & + \sum_{I\in\mathcal{D}} |b_I|\,\frac{|\beta_I|}{\sqrt{|I|}} m_I(|f|u^{-1})\big|\big\langle gv, \chi_I\big\rangle\big|.\end{aligned}$$

Using the estimates $|\alpha_I| \le \sqrt{m_I v}$, and $|\beta_I| \le \dfrac{1}{\sqrt{|I|}}\dfrac{|\Delta_I v|}{m_I v}$, we have that,

$$|\langle \pi_b(fu^{-1}), gv\rangle| \le \;\Sigma_1 + \Sigma_2,$$

where

$$\begin{aligned}\Sigma_1 \;&:=\; \sum_{I\in\mathcal{D}} |b_I| m_I(|f|u^{-1})|\langle gv, h_I^v\rangle|\sqrt{m_I v} \\ \Sigma_2 \;&:=\; \sum_{I\in\mathcal{D}} |b_I| m_I(|f|u^{-1})|\langle gv, \chi_I\rangle|\frac{|\Delta_I v|}{m_I v}\frac{1}{\sqrt{|I|}}.\end{aligned}$$

Estimating Σ_1: First using that $\dfrac{m_I(|f|u^{-1})}{m_I u^{-1}} \le M_{u^{-1}}f(x)$ for all $x \in I$, and that $\langle gv, f\rangle = \langle g, f\rangle_v$; second using the Cauchy-Schwarz inequality and $m_I(u^{-1})\, m_I v \le$

$[u,v]_{A_2^d}$, we get

$$\begin{aligned}\Sigma_1 &\leq \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{m_I v}} m_I^{u^{-1}}(|f|)\big|\langle g, h_I^v\rangle_v\big|\; m_I(u^{-1})\, m_I v \\ &\leq [u,v]_{A_2^d} \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{m_I v}} \inf_{x\in I} M_{u^{-1}} f(x)\big|\langle g, h_I^v\rangle_v\big| \\ &\leq [u,v]_{A_2^d}\Bigg(\sum_{I\in\mathcal{D}} \frac{|b_I|^2}{m_I v} \inf_{x\in I} M_{u^{-1}}^2 f(x)\Bigg)^{\frac12}\Bigg(\sum_{I\in\mathcal{D}}\big|\langle g, h_I^v\rangle_v\big|^2\Bigg)^{\frac12}.\end{aligned}$$

Using Weighted Carleson Lemma 3.1, with $F(x) = M_{u^{-1}}^2 f(x)$, and $\alpha_I = \frac{|b_I|^2}{m_I v}$, which is a u^{-1}-Carleson sequence with intensity B_1, by condition (i), then, together with the fact that $\{h_I^v\}_{I\in\mathcal{D}}$ is an orthonormal system in $L^2(v)$, we get

$$\begin{aligned}\Sigma_1 &\leq [u,v]_{A_2^d} B_1\Bigg(\int_{\mathbb{R}} M_{u^{-1}}^2 f(x) u^{-1}(x)dx\Bigg)^{\frac12}\|g\|_{L^2(v)} \\ &\leq CB_1[u,v]_{A_2^d}\|f\|_{L^2(u^{-1})}\|g\|_{L^2(v)} \\ &\leq CB[u,v]_{A_2^d}\|f\|_{L^2(u^{-1})}\|g\|_{L^2(v)}.\end{aligned}$$

In the second inequality we used the fact that $M_{u^{-1}}$ is bounded in $L^2(u^{-1})$ with operator norm independent of u^{-1}, Lemma 2.27

Estimating Σ_2:

Using similar arguments than the ones used for Σ_1, we conclude that,

$$\begin{aligned}\Sigma_2 &\leq \sum_{I\in\mathcal{D}} |b_I| m_I^{u^{-1}}(|f|)\; m_I^v(|g|)\frac{|\Delta_I v|}{m_I v}\;\sqrt{|I|}\; m_I u^{-1} m_I v \\ &= \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{m_I v}} m_I^{u^{-1}}(|f|)\; m_I^v(|g|)|\Delta_I v|\;\sqrt{|I|}\; m_I u^{-1}\sqrt{m_I v} \\ &\leq [u,v]_{A_2^d}^{\frac12}\sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{m_I v}}|\Delta_I v|\sqrt{|I|}\;(m_I u^{-1})^{\frac12} \inf_{x\in I} M_{u^{-1}} f(x) \inf_{x\in I} M_v g(x)\end{aligned}$$

$$\leq \quad [u,v]_{A_2^d}^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} \frac{|b_I|^2}{m_I v} \inf_{x\in I} M_{u^{-1}}^2 f(x) \right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} |\Delta_I v|^2 \, m_I u^{-1} \, |I| \inf_{x\in I} M_v^2 g(x) \right)^{\frac{1}{2}}$$

Since $\{\frac{|b_I|^2}{m_I v}\}_{I\in\mathcal{D}}$ is a u^{-1} - Carleson sequence and $\{|\Delta_I v|\,|I| m_I u^{-1}\}_{I\in\mathcal{D}}$ is a v - Carleson sequence with intensity B_1 and B_2 respectively. Thus, by Lemma 3.1

$$\begin{aligned}
\Sigma_2 \quad &\leq \quad [u,v]_{A_2^d} \sqrt{B_1\, B_2} \left(\int_{\mathbb{R}} M_{u^{-1}}^2 f(x) u^{-1}(x) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} M_v^2 g(x) v(x) dx \right)^{\frac{1}{2}} \\
&= \quad [u,v]_{A_2^d} \sqrt{B_1\, B_2} \|M_{u^{-1}} f\|_{L^2(u^{-1})} \|M_v g\|_{L^2(v)} \\
&\leq \quad C[u,v]_{A_2^d} \sqrt{B_1\, B_2} \|f\|_{L^2(u^{-1})} \|g\|_{L^2(v)} \\
&\leq \quad CB[u,v]_{A_2^d} \|f\|_{L^2(u^{-1})} \|g\|_{L^2(v)}.
\end{aligned}$$

These estimates together give (7.7), and the sufficiency of the joint A_2^d condition is proved.

In order to prove the necessity A_2^d we use a trick that is applied very often when checking necessity of A_d^2, which is particularize the inequality to $f = h_K$ for some dyadic K. In our case we also have to make the right choice of b in $Carl_{u,v}$, which is $b(x) = \sqrt{|K|} h_K(x)$, this was observed by D. Chung, [Ch3].

Necessity of A_2^d:

Let us assume $\pi_b : L^2(u) \to L^2(v)$ is bounded, i.e. there exist $C > 0$ such that for all $f \in L^2(u)$,

$$\|\pi_b f\|_{L^2(v)} \leq C \|f\|_{L^2(u)}\,, \tag{7.9}$$

for all $b \in Carl_{u,v}$. Fix a dyadic interval K, let us consider $f(x) = \chi_K u^{-1}(x)$ and $b(x) = \sqrt{|K|} h_K(x)$. Then

$$\|f\|_{L^2(u)} = \left(\int |\chi_K(x) u^{-1}(x)|^2 u(x) dx \right)^{1/2}$$

$$= \left(\int_K u^{-1}(x)dx \right)^{1/2} = \sqrt{u^{-1}(K)}\,.$$

$$\|\pi_b f\|_{L^2(v)} = \left(\int \left| \sum_{I \in \mathcal{D}} m_I(\chi_K u^{-1}) \langle \sqrt{|K|}\, h_K, h_I \rangle h_I(x) \right|^2 v(x)dx \right)^{1/2}$$

$$= m_K u^{-1} \left(\int_K |\sqrt{|K|} h_K(x)|^2 v(x)dx \right)^{1/2} = m_K u^{-1} \sqrt{v(K)}\,.$$

By the assumption (7.9) we have

$$m_K u^{-1} \sqrt{v(K)} \leq C \sqrt{u^{-1}(K)}\,,$$

and

$$\frac{\sqrt{u^{-1}(K) v(K)}}{|K|} = \sqrt{m_K u^{-1} m_K v} < C\,. \tag{7.10}$$

Thus, the joint A_2 condition for the pair of weights (u, v) , is a necessary condition for the two weights estimate of the family of dyadic paraproducts indexed by a function $b \in Carl_{u,v}$. □

Corollary 7.13. *Given $b \in L^1_{loc}$ and (u, v) a pair of weights that satisfies the condition $\{\frac{|b_I|^2}{m_I v}\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence with intensity B_1. If the dyadic square function S^d is bounded from $L_2(v^{-1})$ into $L_2(u^{-1})$ then paraproducts π_b is bounded from $L_2(u)$ into $L_2(v)$.*

Proof. Assume S^d is bounded from $L_2(v^{-1})$ into $L_2(u^{-1})$, then Theorem 7.10 implies that $(u, v) \in A_2$ and $\{|\Delta_I v|^2 |I| m_I u^{-1}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence. These two facts with the hypothesis that $\{\frac{|b_I|^2}{m_I v}\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence implies, by Theorem 7.12, that π_b is bounded from $L^2(u)$ to $L^2(v)$. □

When we apply our theorem for the one weight theory we see that we are asking no additional hypothesis to the ones that we have for boundedness of the paraproduct in a weighted Lebesgue space. If $u = v$ the sequence, the joint A^d_2 condition is nothing more than the A^d_2 condition, which implies that $\{|\Delta_I v|^2 |I| m_I v^{-1}\}_{I\in\mathcal{D}}$ is a v-Carleson by Buckley's Theorem 3.11 inequality, also as we argued before if $b \in BMO^d$ then $b \in Carl_{u,v}$ for $u = v$.

It is interesting to register that the conditions on Theorem 7.12 were not the first ones that we obtained to prove two weighted estimates for the paraproduct. Let us state the theorem that gave origin to the main result of the dissertation.

Theorem 7.14. *Let u and v be a pair of weights in joint A_2^d such that for any Carleson sequences $\{\alpha_I\}_{I\in\mathcal{D}}$, $\{\beta_I\}_{I\in\mathcal{D}}$ the sequence $\{\frac{\alpha_I}{m_I u^{-1}}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence and $\{\frac{\beta_I}{m_I v}\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence then for all $b \in BMO^d$ there exist $C > 0$ such that for all $f \in L^2(u)$*

$$\|\pi_b f\|_{L^2(u)\to L^2(v)} \leq C\|f\|_{L^2(u)}$$

The proof is very similar to the proof of Theorem 7.12. The hypothesis that for any Carleson sequence $\{\alpha_I\}_{I\in\mathcal{D}}$, $\{\beta_I\}_{I\in\mathcal{D}}$ the sequence $\{\frac{\alpha_I}{m_I u^{-1}}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence and $\{\frac{\alpha_I}{m_I v}\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence are stronger than the one in Theorem 7.12, since in theorem we are asking the same thing for specific Carleson sequences $\{\beta_I = b_I\}_{I\in\mathcal{D}}$ and $\{\alpha_I = |\Delta_I|^2|I|\}_{I\in\mathcal{D}}$. Also we thought that such strong hypothesis would imply that the weights are comparable, this is certainly true if we ask for some certain kind of uniformity in the v and u^{-1}-Carleson sequences.

Proposition 7.15. *If the a pair of weights u and v, $(u, v) \in A_p$, where $u^{\frac{-1}{p-1}}$ is also a weight, satisfies any of the properties below then the weights are comparable.*

(i) There is $C > 0$ such that for all Carleson sequences $\{\alpha_I\}_{I\in\mathcal{D}}$,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{\alpha_I}{m_I v} \leq C m_J u^{-1} \quad \forall\, J \in \mathcal{D}.$$

(ii) There is $C > 0$ such that for all Carleson sequences $\{\beta_I\}_{I\in\mathcal{D}}$,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{\beta_I}{m_I u^{-1}} \leq C m_J v \quad \forall\, J \in \mathcal{D}.$$

Proof. *(i)* Fix K a dyadic interval, let $\{\lambda_I\}_{I\in\mathcal{D}}$ be the Carleson sequence where $\lambda_K = |K|$ and $\lambda_I = 0$ if $I \neq K$, then for every dyadic interval J, such that $K \subset J$

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\frac{\lambda_I}{m_I u^{-1}} = \frac{|K|}{|J|}\frac{1}{m_K u^{-1}} \leq C\, m_J v.$$

Now take $J = K$ then we have

$$\frac{1}{C} \leq m_K u^{-1} m_K v.$$

Since K was arbitrary, we would have $\frac{1}{C} \leq m_I u^{-1} m_I v$. for all dyadic intervals I, then by *(ii)* the weights must be comparable.

(ii) Similar to the proof of *(i)*. □

Let us state the result for the adjoint of the paraproduct.

Theorem 7.16. *Let (u, v) be a pair of weight such that u^{-1} is also a weight and $\{|\Delta_I u^{-1}|^2 |I| m_I v^{-1}\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence with intensity B_2. Then π_b^* is bounded from $L^2(u)$ into $L^2(v)$ for all $b \in Carl_{v^{-1},u^{-1}}$ if and only if $(u, v) \in A_2^d$. Moreover, if B_1 is the intensity of the v-Carleson sequence $\left\{\frac{|b_I|^2}{m_I u^{-1}}\right\}_{I\in\mathcal{D}}$ then*

$$\|\pi_b^* f\|_{L^2(v)} \leq CB[u, v]_{A_2}\|f\|_{L^2(u)};$$

for some $C > 0$ and $B = \max\{B_1, B_2\}$.

Proof. Fix $f \in L^2(u^{-1})$ and $g \in L^2(v)$.Since $(u, v) \in A_2^d$ and the weights are such that $\{|\Delta_I u^{-1}|^2 |I| m_I v^{-1}\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson, then by Theorem 7.12 π_b is bounded from $L^2(v^{-1})$ into $L^2(u^{-1})$ for all b in $Carl_{v^{-1},u^{-1}}$ which implies that the adjoint of the paraproduct is bounded from $L^2(u)$ into $L^2(v)$.

□

Corollary 7.17. *Given $b \in L^1_{loc}$ and (u, v) be a pair of weight that satisfies the condition $\{\frac{|b_I|^2}{m_I u^{-1}}\}_{I \in \mathcal{D}}$ is a v-Carleson sequence. If the dyadic square function S^d is bounded from $L^2(u)$ into $L^2(v)$ then paraproduct adjoint π_b^* is also bounded from $L^2(u)$ into $L^2(v)$.*

Remark 7.18. *Both paraproduct and adjoint are bounded from $L^2(u)$ to $L^2(v)$ if*

(i) $\{\frac{|b_I|^2}{m_I v}\}_{I \in \mathcal{D}}$ *is a u^{-1}-Carleson sequence.*

(ii) $\{|\Delta_I v|^2 |I| m_I u^{-1}\}_{I \in \mathcal{D}}$ *is a v-Carleson sequence.*

(iii) $\{\frac{|b_I|^2}{m_I u^{-1}}\}_{I \in \mathcal{D}}$ *is a v-Carleson sequence.*

(iv) $\{|\Delta_I u^{-1}|^2 |I| m_I v\}_{I \in \mathcal{D}}$ *is a u^{-1}-Carleson sequence.*

(v) $(u, v) \in A_2$.

Remember that $(u, v) \in A_2^d \iff (v^{-1}, u^{-1}) \in A_2^d$, so we are not introducing extra joint A_2^d condition.

7.4 The maximal and the square functions

The next theorem relates the boundedness of the square function with the boundedness of the Maximal function from $L^2(u)$ into $L^2(v)$. If the weight v is in A_∞^d and the Maximal function is bounded then the square function is also bounded. This result is an adaptation of Buckley's proof, in [Bu], for the fact that if $w \in A_2^d$ then S_d is bounded in $L^2(w)$. Pereyra, in [P], proved a similar result for the weighted maximal function and the weighted square function in L^q.

Theorem 7.19. *Let (u, v) be a pair of weight such that $v \in A_\infty$ and the Maximal function M is bounded from $L^2(u)$ to $L^2(v)$ then there exists $C > 0$, such that*

$$\|S^d f\|_{L^2(v)} \leq C \|f\|_{L^2(u)}.$$

First note that

$$\|S^d f\|^2_{L^2(v)} = \sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 m_I v.$$

It is enough to check that for all $f \in L^2(u)$

$$2\Sigma_1 := \sum_{I\in\mathcal{D}} |m_I f - m_{\tilde{I}} f|^2 v(\tilde{I}) \leq C\|f\|^2_{L^2(u)}$$

Pairing the terms with the same parent, we have that

$$\Sigma_1 = \sum_{I\in\mathcal{D}} (m_I^2 f - m_{\tilde{I}}^2 f) v(\tilde{I}).$$

Adding and subtracting $2v(I)m_I^2 f$, then we get

$$\Sigma_1 = \sum_{I\in\mathcal{D}} \big(2v(I)m_I^2 f - v(\tilde{I})m_{\tilde{I}}^2 f\big) + \sum_{I\in\mathcal{D}} \big(v(\tilde{I}) - 2v(I)\big)m_I^2 f = \Sigma_2 + \Sigma_3$$

Estimating Σ_2:

$$\Sigma_2 = \sum_{I\in\mathcal{D}} \big(2v(I)m_I^2 f - v(\tilde{I})m_{\tilde{I}}^2 f\big) = \sum_{m=-\infty}^{\infty} (a_m - a_{m-1})$$

where $a_m = \sum_{I\in\mathcal{D}_m} 2v(I)m_I^2 f = 2\int (E_m f(x))^2 v(x)dx$, $E_m f(x) := m_I f \quad x \in I \in \mathcal{D}_m$ thus

$$|a_m| \leq 2\int_{\mathbb{R}} |Mf(x)|^2 v(x)dx = 2\|Mf\|^2_{L^2(v)} \leq \|f\|^2_{L^2(u)}$$

The last inequality follows since M is assumed to be bounded from $L^2(u)$ to $L^2(v)$.

Estimating Σ_3: First note that paring the terms with the parents

$$\sum_{I\in\mathcal{D}} \big(v(\tilde{I}) - 2v(I)\big)m_{\tilde{I}}^2 f = 0\,, \quad \text{hence}$$

$$\Sigma_3 = \sum_{I\in\mathcal{D}} (v(\tilde{I}) - 2v(I))(m_I^2 f - m_{\tilde{I}}^2 f)$$

Thus, by Cauchy-Schwartz inequality

$$\begin{aligned}\Sigma_3 &\leq \left(\sum_{I\in\mathcal{D}} \frac{v(\tilde{I})-2v(I))^2}{v(\tilde{I})}(m_I f + m_{\tilde{I}} f)^2\right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} v(\tilde{I})(m_I f - m_{\tilde{I}} f)^2\right)^{\frac{1}{2}} \\ &= \Sigma_2^{\frac{1}{2}}\Sigma_3^{\frac{1}{2}} \leq \frac{\Sigma_2+\Sigma_3}{2}\end{aligned}$$

so

$$\Sigma_1 \leq \Sigma_2 + \Sigma_3 \leq C\|f\|^2_{L^2(u)} + \frac{\Sigma_4+\Sigma_1}{2}$$

which implies that

$$\frac{1}{2}\Sigma_5 \leq C\|f\|^2_{L^2(u)} + \frac{\Sigma_4}{2}$$

now suffices to show that $\Sigma_4 \leq C\|f\|^2_{L^2(u)}$.

Estimating Σ_4:

$$\begin{aligned}\Sigma_4 &\leq 4\sum_{I\in\mathcal{D}} \frac{|\Delta_I v|^2}{m_I v}|I| m_I f \\ &\leq \sum_{I\in\mathcal{D}} \frac{|\Delta_I v|^2}{m_I v}|I| \inf_{x\in I} M^2 f(x) \\ &\leq C\int_{\mathbb{R}} M^2 f(x) v(x) dx = \|Mf\|^2_{L^2(v)} \leq C\,\|f\|^2_{L^2(u)}\end{aligned}$$

Note that in the last inequality we use the fact that if $v \in A^d_\infty$ then by Buckley's inequality for $p = 1$, or Theorem 3.12, $\{\frac{|\Delta_I v|^2}{m_I v}|I|\}_{I\in\mathcal{D}}$ is a v-Carleson sequence which implies that the last inequality follows from 3.1.

Remark 7.20. *Even though not explicitly we are still assuming that $(u, v) \in A^d_2$, since we assumed that $M : L^2(u) \to L^2(v)$ which implies $(u, v) \in A^d_2$ ([GC-RF], Theorem 1.12, page 392),*

Chapter 8

Future research

In this chapter we will point some directions for future research. We will keep this discussion restricted to the problems related to those presented in the previous chapters.

The first problem for future concerns is to study if the estimates for t-Haar multipliers can be improved. Recall that for a weight w, $t \in \mathbb{R}$, $m, n \in \mathbb{N}$, a t-Haar multiplier of complexity (m, n) is the operator defined as

$$T_{t,w}^{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_m(L) \\ J \in \mathcal{D}_n(L)}} \frac{\sqrt{|I|\,|J|}}{|L|} \left(\frac{w(x)}{m_L w} \right)^t \langle f, h_I \rangle h_J(x). \tag{8.1}$$

We proved in Theorem 6.4 that if t is a real number and w a weight in C_{2t}^d, such that $w^{2t} \in A_q^d$, for $q > 1$ and that satisfies the C_{2t}^d condition with constant $[w]_{C_{2t}^d}$. Then

$$\|T_{t,w}^{m,n} f\|_{L^2} \leq C(m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_{L^2}.$$

We argued in Chapter 6 that the dependence in the C_{2t}^d -characteristic is not sharp, since for the case $m = n = 0$, Pereyra in [P2] proved

$$\|T_{0,w}^{0,0} f\|_{L^2} \leq C\, D(w)\, [w]_{RH_2^d}^2 \|f\|_{L^2},$$

where $D(w)$ is the doubling constant of w. This is a better dependence for $T^{0,0}_{0,w}$ that the one that we obtained for all complexity (m,n), which after some observations we conclude that

$$\|T^{0,0}_{0,w}f\|_{L^2} \le C\ [w]^2_{RH^d_2}[w]_{A^d_{\frac{q+1}{2}}}\|f\|_{L^2}.$$

However is not known even if the bounds proved by Pereyra are the best possible. Pérez and Hytönen improved the sharp dependence in the A_2 characteristic for Calderón-Zygmund operators to bounds that involve the A_2 and the RH^d_1 characteristic of the weight. The Haar multipliers $T^{0,0}_{0,w}$, were not in the scope of their analysis and using their ideas the bounds for this operators might be improved to some sort of mixed type bound in the RH_2 and RH_1 characteristic of the weight, i.e., we would hope to prove that

$$\|T^{0,0}_{0,w}f\|_{L^2} \le C\,[w]_{RH^d_2}[w]_{RH^d_1}\|f\|_{L^2},$$

where C might depend on the doubling constant. Later we shall study if similar bounds would also hold for t-Haar multipliers and for t-Haar multipliers with complexity (m,n). Also it is not clear for us if one can use Nazarov-Volberg techniques to obtain the mixed type bounds proved by Pérez and Hytönen in [HPz] for Haar shifts, and dyadic paraproduct.

Another interesting problem, at least from the theoretical perspective, is to analyze if the $L^p(w)$ norm of a square function with complexity (m,n) obeys the same sharp dependence on the A_p-characteristic that we have for the original original square function (complexity $(0,0)$, times a factor that depends at most polynomially in the complexity of these operators. Given $f \in L^1_{loc}$ we define the dyadic square function of complexity (m,n) as the

$$S^{m,n}f := \Bigg(\sum_{L\in\mathcal{D}}\sum_{\substack{I\in\mathcal{D}_m(L)\\ J\in\mathcal{D}_n(L)}}|m_I f - m_{\hat{I}}f|^2\chi_J\Bigg)^{\frac{1}{2}}.$$

Note that for $m=n=0$, this is exactly the dyadic square function, S^d. Cruz-Uribe, Martell and Pérez proved in [CrMPz] that for $p > 1$ and $w \in A_p^d$

$$\|S^d f\|_{L^p(w)} \leq C_p[w]_{A_p^d}^{\max\{\frac{1}{2},\frac{1}{p-1}\}} \|f\|_{L^p(w)} \quad \forall\, f \in L^p(w).$$

This result is obtained by sharp extrapolation from $p = 3$, they prove that

$$\|S^d f\|_{L^3(w)} \leq C_p[w]_{A_3^d}^{\frac{1}{2}} \|f\|_{L^3(w)} \quad \forall f \in L^3(w)$$

and then use Sharp Extrapolation Theorem 3.17. Analogous to that, in order to prove

$$\|S^{m,n}\|_{L^p(w)} \leq C_{m,n,p}[w]_{A_p^d}^{\max\{\frac{1}{2},\frac{1}{p-1}\}} \|f\|_{L^p(w)} \quad \forall\, f \in L^p(w),$$

it would be enough to prove that

$$\|S^{m,n}\|_{L^3(w)} \leq C_{m,n}[w]_{A_3^d}^{\frac{1}{2}} \|f\|_{L^3(w)} \quad \forall\, f \in L^p(w).$$

In chapter 7 we proved that if (u, v) is a pair of weights such that v is a regular weight and u^{-1} is also a regular weight and $\{|\Delta_I v|^2 |I| m_I u^{-1}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence then π_b is bounded from $L^2(u)$ into $L^2(v)$ for all $b \in Carl_{u,v}$ if and only if $(u, v) \in A_2^d$. A locally integrable function b belongs to the space $Carl_{u,v}$ if $\left\{\frac{|b_I|^2}{m_I v}\right\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence, where $b_I = \langle b, h_I\rangle$, this space is clearly a vector space. One might define the following norm in this space,

$$\|b\|_{Carl_{u,v}} := \left(\sup_{J\in\mathcal{D}} \frac{1}{m_J u^{-1}} \sum_{I\in\mathcal{D}(J)} \frac{|\langle b, h_I\rangle|^2}{m_I v} \right)^{\frac{1}{2}}.$$

Note that if $\|b\|_{Carl_{u,v}} = 0$ if and only if b is a constant function, therefore as for the BMO^d norm, $\|\cdot\|_{Carl_{u,v}}$ defines a norm in the the quotient space $Carl_{u,v}$ modulo constant functions. Also if $u = v = 1$ then $\|\cdot\|_{Carl_{u,v}}$ is exactly the BMO^d norm. Two questions are in order here, the first one is what are the conditions, if any, on the weights u, and v such that the vector space $Carl_{u,v}$ with norm $\|\cdot\|_{Carl_{u,v}}$ is a Banach space. The second question is if there is any class of weights w such that $Carl_{w,w} = BMO^d$. If $Carl_{w,w} \neq BMO^d$ for $w \in A_2^d$ then Theorem 7.12 guarantee the boundedness of the paraproduct on $L^2(w)$ for functions b other that BMO^d functions, recall we know that $BMO^d \subset Carl_{w,w}$ for any weight w.

The last question for future research is about decomposition of Haar shifts of type 4 with complexity (m, n) as the composition of a dyadic dual paraproduct with Haar shift operator of type 1 with complexity (m, n) and a dyadic paraproduct. We proved in Chapter 6 that if a positive operator of type 4 with complexity $(0, 0)$ is bounded in L^2 then it can be decomposed as $\pi_b^* \pi_b$ for some $b \in BMO^d$, can this be generalized to complexity (m, n), i.e., given a positive operator of type 4 with complexity (m, n), we want to find $b, d \in BMO^d$ such that $T_4^{m,n} = \pi_d^* T_1^{m,n} \pi_b$. Can one find a bounded Haar shift operator type 4 that cannot be written as a composition $\pi_b^* \pi_d$ for b and d in BMO^d? In the recent paper by Hytönen and Lacey, [HL], on mixed $A_p - A_\infty$ estimates, they reduce their estimate to studying a positive dyadic shift operator of type 4 with complexity $(i, 0)$ given by

$$(S_4^{i,0} f)(x) = \sum_{L \in \mathcal{D}} m_L f \Bigg(\sum_{\substack{J : J \in \mathcal{L} \\ J^i = L}} \chi_J \Bigg),$$

where $\mathcal{L}$ are the Lerner cubes obtained from Lerner's Calderón Zygmund decomposition with respect to the mean oscillation. It will be interesting if we can realize this positive type 4 Haar shift operator as a composition of the paraproduct and dual paraproduct with some complexity. If possible this will be based on geometric considerations dictated by the Lerner's cubes.

Appendix

Lemma 8.1. *The sequence* $a_n = \big(1+\dfrac{1}{n}\big)^n$*, for* n *a positive integer, is an increasing sequence.*

Proof. First let us show that the function $f(x) = \big(1+\dfrac{1}{x}\big)^x$ define for all real numbers $x > 1$ is a increasing function. We can achieve that just using calculus tools. Note that $\ln(f(x)) = x\ln\big(1+\frac{1}{x}\big)$, implicit differentiating we have

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \ln\big(1+\frac{1}{x}\big) + x\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)\\ &= \ln\big(1+\frac{1}{x}\big) - \frac{1}{x+1}\end{aligned}$$

Since $f(x) > 0$ for all $x > 1$, then $f'(x)$ is positive in $(1,\infty)$ if and only if

$$\ln\big(1+\frac{1}{x}\big) > \frac{1}{x+1} \tag{8.2}$$

Since $x > 1$ then $0 < \dfrac{1}{x} < 1$, so we can expand $\ln\big(1+\frac{1}{x}\big)$ in its Taylor expansion for all $x > 1$.

$$\ln\big(1+\frac{1}{x}\big) = \frac{1}{x} - \frac{1}{2}\left(\frac{1}{x}\right)^2 + \frac{1}{3}\left(\frac{1}{x}\right)^3 - \ldots$$

Clearly, for any fixed $x \geq 2$ we have that the sequence $\dfrac{1}{n}\left(\dfrac{2}{x}\right)^n$ is a decreasing sequence, then by the Alternating Test Series (AST), we have that

$$\frac{1}{x} - \frac{1}{2}\frac{1}{x^2} < \ln\big(1+\frac{1}{x}\big) < \frac{1}{x} \tag{8.3}$$

It is easy to check that if $x > 1$ then $\frac{1}{x} - \frac{1}{2}\frac{1}{x^2} > \frac{1}{x+1}$, plug this in (8.3) we would have that for $x > 1$

$$\frac{1}{x+1} \leq \frac{1}{x} - \frac{1}{2}\frac{1}{x^2} < \ln\left(1 + \frac{1}{x}\right) \tag{8.4}$$

Therefore (8.2) is satisfied for all $x > 1$, so $f'(x) > 0$ for $x > 1$, which implies that $f(x)$ is increasing. Note that this certainly imply that a_n is an increasing sequence for $n \geq 2$. Now observe that $a_1 = 2 < \frac{9}{4} = a_2$, therefore a_n is an increasing sequence for all n positive integer. □

Remark 8.2. *It is an immediate consequence of this that $a_n < e^2$ for all $n \geq 1$.*

Lemma 8.3. *For all positive integers $n \geq 2$, $\frac{e^{-1}}{2} < \left(1 - \frac{1}{n}\right)^n$.*

Let $a_n = \left(1 - \frac{1}{n}\right)^n$ for all $n > 0$, note that if $n > m$, then

$$\frac{1}{m} > \frac{1}{n} \Rightarrow 1 - \frac{1}{n} > 1 - \frac{1}{m} \Rightarrow a_n = \left(1 - \frac{1}{n}\right)^n > \left(1 - \frac{1}{m}\right)^m = a_m.$$

Thus a_n is a increasing sequence, which implies that

$$\frac{e^{-1}}{2} < \frac{1}{4} = a_2 \leq a_n \qquad \forall \quad n \leq 2.$$

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