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DISSERTATION

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Dedication

To my wife, Willa, and our babies, Gus, Charlie, Othello, and Donovan, who have all endured so much over the years.

Also to the memory of my mother and father who did not live to see the end of this.

Acknowledgments

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Abstract

Links of isolated singularities defined by weighted homogeneous polynomials have a natural Sasakian structure. Since it is known that Sasaki-Einstein metrics have positive Ricci curvature, and since positive Sasakian structures give rise to Sasakian metrics with positive Ricci curvature, it is useful to determine which links have a positive Sasakian structure. This corresponds to the Fano index of the associated weighted projective variety being positive. Links of dimension $2n-1$ are $(n-2)$-connected. In dimension 5, there is a complete classification of simply connected spin manifolds due to Smale [28]. Hypersurface singularities yielding links of dimension 5 have been treated in [3] and [5]. This paper investigates isolated singularities of codimension 2 complete intersections with 5 dimensional links of positive index and provides a complete list up to degree 600, hence a complete (up to degree 600) list of types of links having positive Sasakian structures.

Contents

Chapter 1

Introduction

A singularity defined by weighted homogeneous polynomials in affine space invariant under a weighted $\mathbb{C}^*$ action is isolated if the projective variety defined by the same polynomials is quasismooth. Such quasismooth varieties have only orbifold (quotient) singularities. Links of isolated singularities defined by weighted homogeneous polynomials have a natural quasi-regular Sasakian structure. Since it is known that Sasaki-Einstein metrics have positive Ricci curvature, and since positive Sasakian structures give rise to Sasakian metrics with positive Ricci curvature, it is useful to determine which links have a positive Sasakian structure. Manifolds with positive Ricci curvature are of interest in their own right. Positivity of the Sasakian structure on a link corresponds to the index of the associated weighted projective variety being positive, hence Fano or log Fano. Furthermore, the existence of a Sasaki-Einstein metric on a link corresponds to the existence of a Kähler-Einstein metric on the corresponding log Fano variety. In particular, a 5-dimensional algebraic link that has a positive quasi-regular Sasakian structure is the link of an isolated singularity of an affine cone over a log del Pezzo surface. The classification problem for these links, therefore, is related to the classification problem for log del Pezzo surfaces. This is a subject of current widespread interest, see [6, 17, 13] for example. Del Pezzo surfaces and del Pezzo singularities are also of interest in high energy physics, see [15, 23, 11]. Links of dimension $2n-1$ are $(n-2)$-connected, so for dimension 5, links are simply connected. Also for dimension 5, there is a complete classification of simply connected spin manifolds due to Smale [28]. Hypersurface singularities yielding links of dimension 5 have been treated in [3] and [5]. This paper investigates isolated singularities of codimension 2 complete intersections with 5 dimensional links of positive index. It provides a start of a general classification and a complete list up to degrees $d_1 \le d_2 \le 600$.

Important definitions are given in Chapter 2. Weighted projective spaces and varieties are described in 2.1, Sasakian structures, especially the Sasakian structure on a weighted sphere, in 2.2, and links of isolated singularities of weighted complete intersections and their induced Sasakian structure in 2.3. Next, Chapter 3 gives numerical conditions for identifying which links possess positive Sasakian structures. Necessary and sufficient conditions on weights and degrees are given in 3.1 for quasismoothness. In 3.2, bounds on dimension and codimension as well as additional relations between weights and degrees are given relating to positivity. The main result is in 3.3: a complete listing of positive quasismooth complete intersections of codimension 2 in weighted $\mathbb{P}^4$ with degrees $d_1 \le d_2 \le 600$. Chapter 4 discusses some results about the topology of links. Simple formulas for the Alexander polynomial and Milnor number are given in special cases. General results for the middle Betti number are given as well. 4.3 applies these results to the 5 dimensional case, and includes a technique for computing the torsion of any Smale manifold admitting a Sasakian structure. Chapter 5 defines Sasaki-Einstein structures and gives some existence and obstruction results. In particular, some general properties of Sasaki-Einstein structures are in 5.1, a sufficient condition for existence of Sasaki-Einstein metrics is described in 5.2, and the Lichnerowicz obstruction is considered in 5.3. Appendix A contains a table of $(\mathbf{w}, \mathbf{d})$ types shown to possess Sasaki-Einstein metrics. Appendix B contains tables of families of types and sporadic types possessing positive Sasakian structures, up to $d_1 \leq d_2 \leq 600$. Appendix C contains tables of types that are well-formed, a concept of interest to those who study log del Pezzo surfaces. Details of a partial classification of types, based on the relations of the highest two weights to the degrees of the hypersurfaces are worked out in Appendix D.

This paper is in some senses an addendum to parts of Chapters 9, 10, and 11 of [3]. Future research will include an attempt to extend the classification approach here to arbitrary degree. In fact, I conjecture that the list of sporadic cases in B.3 is complete, and that the one and three parameter families of types included in B.1 and B.2 account for all higher degree types. Topology computations for the three parameter families will be completed. Moduli spaces should be examined as well.

Some work remains in the hypersurface case to complete the list of $(\mathbf{w}, \mathbf{d})$ types and compute the topology for each. The general approach used here will be much simpler in the hypersurface case, as there is an explicit formula for computation of the topology.

Finally, there is of course, the same question in higher dimensions.

Chapter 2

Weighted Projective Varieties, Sasakian structures, and Links

2.1 Weighted projective spaces and varieties

The main references for this section are [10] and [14]

Let $\mathbf{w} = (w_0, w_1, \ldots, w_n) \in \mathbb{Z}_+^{n+1}$, let $(x_0, x_1, \ldots, x_n)$ be affine coordinates on $\mathbb{C}^{n+1}$, and let $\mathbb{C}^*$ act by

$$\lambda(x_0, x_1, \ldots, x_n) = (\lambda^{w_0} x_0, \lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) \tag{2.1}$$

Then

$$\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, w_1, \ldots, w_n) := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

is *weighted projective space* of weight $\mathbf{w}$. $\mathbb{P}(\mathbf{w})$ is a rational n-dimensional projective variety. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}(\mathbf{w})$ be the canonical projection. $\mathbb{P}(\mathbf{w})$ has the structure of an orbifold (a complex variety possessing only quotient singularities). In particular, the affine pieces $U_i = (x_i \neq 0) \cong \mathbb{C}^n/(\mathbb{Z}/w_i\mathbb{Z})$ determine an orbifold atlas.

If ε is a primitive w_i^{th} root of unity, then the group acts via $z_j \mapsto \varepsilon^{w_i} z_j$ for $j \neq i$. Thus $z_j = (x_j/x_i)^{w_j/w_i}$. As varieties $\mathbb{P}(w_0, qw_1, \ldots, qw_n) \cong \mathbb{P}(w_0, w_1, \ldots, w_n)$ for $q \geq 1$, but they have different orbifold structures (see [10]).

Definition 1 *The expression* $\mathbb{P}(w_0, w_1, \ldots, w_n)$ *is* well-formed *if*

$$\gcd(w_0, \ldots, \hat{w}_1, \ldots, w_n) = 1$$

for each i.

A polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ is called a *weighted homogeneous polynomial* of degree d and weight $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ if for $\lambda \in \mathbb{C}^*$,

$$f(\lambda^{w_0} x_0, \lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^d f(x_0, x_1, \ldots, x_n)$$

A *weighted projective variety* is the zero-set of an ideal generated by weighted homogeneous polynomials all having the same weight.

If $V \subset \mathbb{C}^{n+1}$ is a variety defined by weighted homogeneous polynomials $f_1, \ldots, f_r$ all having the same weights $\mathbf{w}$, then V is invariant under the weighted $\mathbb{C}^*$ action (2.1). The converse is true as well [26]: if a variety V is invariant under the weighted $\mathbb{C}^*$ action (2.1), it can be defined by weighted homogeneous polynomials. Therefore, the quotient $V/\mathbb{C}^*$ is well-defined in $\mathbb{P}(\mathbf{w})$ and so is a weighted projective variety. Let X be any weighted projective variety and let $\mathcal{C}_X^* = \pi^{-1}(X)$ where π is the canonical projection. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_X^* & \rightarrow & \mathbb{C}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}(\mathbf{w}) \end{array}$$

Let $\mathcal{C}_X$ be the affine closure of $\mathcal{C}_X^*$ in $\mathbb{C}^{n+1}$. $\mathcal{C}_X$ is called the *affine cone* and $\mathcal{C}_X^*$ the *punctured affine cone* over X.

A variety V is a *complete intersection* if the minimal number of generators of its ideal is equal to its codimension.

Given weights $\mathbf{w}$ denote by $X_{d_1,\ldots,d_c} \subset \mathbb{P}(\mathbf{w})$ the family of all complete intersections of multidegree $\mathbf{d} = (d_1, \ldots, d_c)$. This notation will also sometimes denote a sufficiently general member of the family.

Definition 2 *A complete intersection* $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}(w_0, \ldots, w_n)$ *is* quasismooth *if its affine cone* C_X *is smooth outside its vertex.*

Conditions will be given in Section 3 for quasismoothness in terms of the weights and degrees.

Definition 3 *A variety* $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}(w_0, \ldots, w_n)$ *of codimension* c *is* well-formed *if the expression for* $\mathbb{P}$ *is well-formed (see Definition 1) and* X *contains no codimension* $c+1$ *singular stratum of* $\mathbb{P}$*.*

Conditions will be given in Section 3 for X to be well-formed in terms of the weights and degrees.

2.2 Sasakian manifolds and the Sasakian structure on the weighted sphere

The main reference for this section is [3]. A *contact manifold* is a $(2n+1)$-dimensional manifold with a contact form, that is, a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. The Reeb vector field ξ, is the unique vector field such that $\xi \lrcorner \eta = 1$ and $\xi \lrcorner d\eta = 0$. Let $\mathcal{D} = \ker \eta$. Then $(\mathcal{D}, d\eta)$ is a symplectic structure. An *almost contact structure* is a structure (ξ, η, Φ) where ξ is a vector field, η is a 1-form, and Φ is a $(1,1)$ tensor field such that

$\eta(\xi) = 1$ and $\Phi\circ\Phi = -\mathbb{1}+\xi\otimes\eta$. $\Phi\circ\xi = 0$ and $\eta\circ\Phi = 0$ follow. If (M,ξ,η,Φ) is an almost contact manifold and (M,g) is a Riemannian manifold then g is *compatible* with the almost contact structure if $g(\Phi X,\Phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all X, Y vector fields on M. If (M,η) is a contact manifold and $d\eta(\Phi X,\Phi Y) = d\eta(X,Y)$ for all X, Y vector fields on M and $d\eta(\Phi X, X) > 0$ for $X \neq 0$, then the almost contact structure is compatible with the contact structure. If, in addition, $g(X,\Phi Y) = d\eta(X,Y)$, (M,ξ,η,Φ,g) is a contact metric structure. The cone on (M,g_M), $C(M) = M\times\mathbb{R}^+$ has a Riemannian structure $(C(M),g)$ where $g = dr^2 + r^2 g_M$. Let $\Psi = r\frac{\partial}{\partial r}$. If M is almost contact, define I by $IY = \Phi Y + \eta(Y)\Psi$, $I\Psi = -\xi$. Then I is an almost complex structure. An almost contact structure (ξ,η,Φ) is *normal* if I is integrable. If it is contact as well, then it is *Sasakian*. That is, a structure is Sasakian if it is a normal contact metric structure.

Let $\mathbf{x} = (x_0,x_1,\dots,x_n)$ and $\mathbf{y} = (y_0,y_1,\dots,y_n)$. Consider the standard contact form on $\mathbb{R}^{2n+2} = \{(\mathbf{x},\mathbf{y})\}$ given by $\eta_{\mathbf{1}} = \sum_{i=0}^{n}(y_i dx_i - x_i dy_i)$ restricted to the sphere

$$S^{2n+1} = \{\mathbf{z} = (\mathbf{x},\mathbf{y}) \in \mathbb{C}^{n+1} | \sum_{i=o}^{n}|z_i|^2 = 1\}$$

The Reeb vector field is given by

$$\xi_{\mathbf{1}} = \sum_{i=0}^{n}(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i})$$

Let $\Phi_{\mathbf{1}}$ be the restriction of the standard complex structure on $\mathbb{R}^{2n+2}$ to $\mathcal{D} = \ker(\eta_{\mathbf{1}}|_{S^{2n+1}})$. Let $g_{\mathbf{1}}$ be the flat metric on $\mathbb{R}^{2n+2}$ restricted to S^{2n+1}. $g_{\mathbf{1}}$ has constant sectional curvature 1 and satisfies

$$g_{\mathbf{1}} = d\eta_{\mathbf{1}} \circ (\Phi_{\mathbf{1}} \otimes \mathbb{1}) + \eta_{\mathbf{1}} \otimes \eta_{\mathbf{1}}$$

so is compatible with the contact form. Since the structure is almost complex, the contact structure $(S^{2n+1},\eta_{\mathbf{1}},\Phi_{\mathbf{1}},g_{\mathbf{1}})$ is normal, hence Sasakian.

With the above notation let $H_i = y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}$ so $\xi_{\mathbf{1}} = \sum_{i=0}^{n} H_i$.

Let $\mathbf{w} = (w_0, w_1, \ldots, w_n) \in \mathbb{Z}_+^{n+1}$ and define $\xi_{\mathbf{w}} = \sum_{i=0}^{n} w_i H_i$ This determines a flow on S^{2n+1} given by

$$(z_0, z_1, \ldots, z_n) \mapsto (e^{2\pi i w_0 t}, e^{2\pi i w_1 t}, \ldots, e^{2\pi i w_n t})$$

Define $\eta_{\mathbf{w}} = \frac{\eta_{\mathbf{1}}}{\sum_{i=0}^{n} w_i |z_i|^2}$, $g_{\mathbf{w}} = d\eta_{\mathbf{w}} \circ (\Phi_{\mathbf{w}} \otimes \,) + \eta_{\mathbf{w}} \otimes \eta_{\mathbf{w}}$, and $\Phi_{\mathbf{w}} = \Phi_{\mathbf{1}} - \Phi(\xi_{\mathbf{w}} - \xi_{\mathbf{1}}) \otimes \eta_{\mathbf{w}}$ then $\mathcal{S}_{\mathbf{w}}^{2n+1} = (\eta_{\mathbf{w}}, \xi_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ is a weighted Sasakian structure on S^{2n+1}. [Note: the standard structure has $\mathbf{w} = (1, 1, \ldots, 1)$, whence the subscript $\mathbf{1}$.]

The leaves of the foliation $\mathcal{F}_{\xi_{\mathbf{w}}}$ generated by $\xi_{\mathbf{w}}$ are all circles, so $\mathcal{F}_{\xi_{\mathbf{w}}}$ is equivalent to a locally free circle action. The structure $\mathcal{S}_{\mathbf{w}}^{2n+1}$ is *quasiregular*, that is, there is a $k > 0$ such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_{\xi_{\mathbf{w}}}$ passes through U at most k times.

2.3 Links of weighted complete intersections and Sasakian structures on them

Let $V_f = V_{f_1,\ldots,f_c} = \{\mathbf{z} \in \mathbb{C}^{n+1} \big| f_1(\mathbf{z}) = \cdots = f_c(\mathbf{z})\}$. Suppose V_f has an isolated critical point at the origin. Then for ε sufficiently small the sphere $S_\varepsilon^{2n+1} = \{\mathbf{z} \in \mathbb{C}^{n+1} \big| \sum_{i=0}^{n} |z_i|^2 = \varepsilon\}$ only encloses one critical point. Let $L_f = V_f \cap S_\varepsilon^{2n+1}$. L_f is called the *link* of V_f at the origin. By scaling we can let $\varepsilon = 1$ so $L_f = V_f \cap S^{2n+1}$. L_f is a $(2(n+1-c)-1)$-dimensional manifold.

Suppose $f = (f_1, \ldots, f_c)$ and for each i, f_i is weighted homogeneous of degree d_i with weight $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ independent of i. If V_f has an isolated critical point at the origin and no other critical points, then $V_f = \mathcal{C}_{X_f}$ the affine cone over a quasismooth weighted projective variety X_f in the family $X_{d_1,\ldots,d_c} \subset \mathbb{P}(\mathbf{w})$. V_f is invariant under the weighted $\mathbb{C}^*$ action (2.1).

Given such V_f with weight $\mathbf{w}$, consider the Sasakian structure on $S_{\mathbf{w}}^{2n+1}$, $\mathcal{S}_{\mathbf{w}} =$

$(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$. It is invariant under the same weighted $\mathbb{C}^*$ action (2.1). For $P \in L_f$, $(\xi_{\mathbf{w}})_P \in \mathrm{T}_P L_f$ and $\Phi_{\mathbf{w}} \mathrm{T}_P L_f \subset \mathrm{T}_P L_f$ so $\mathcal{S}_{\mathbf{w}}$ restricts to a Sasakian structure on L_f, which is quasiregular as well.

Proposition 4 *([3, Proposition 9.2.4]) Given L_f and $\mathcal{S}_{\mathbf{w}}$ be its induced Sasakian structure. Then $\mathcal{S}_{\mathbf{w}}$ is*

(i) positive (anticanonical) if and only if $|\mathbf{w}| - |\mathbf{d}| > 0$,

(ii) null if and only if $|\mathbf{w}| - |\mathbf{d}| = 0$, and

(iii) negative (canonical) if and only if $|\mathbf{w}| - |\mathbf{d}| < 0$. □

The integer $I = |\mathbf{w}| - |\mathbf{d}|$ is called the *Fano index* of X_f when it is positive, and in general the *index* of V_f or L_f.

Theorem 5 *([3, Theorem 9.5.1]) If L_f is the link of an isolated complete intersection singularity of weighted homogeneous polynomials $f = (f_1, \dots, f_c)$, and $|\mathbf{w}| - |\mathbf{d}| > 0$, then L_f admits a Sasakian metric with positive Ricci curvature.* □

Chapter 3

Enumeration of Positive Sasakian Links

3.1 Numerical Conditions for Quasismoothness

3.1.1 Hypersurfaces

General conditions for quasismoothness of hypersurfaces are given by [14, Theorem 8.1]:

Theorem 6 *A general hypersurface, not a linear cone, $X_d \subset \mathbb{P}(w_0, \ldots, w_n)$ of degree d, where $n \geq 1$ is quasismooth if and only if for every nonempty subset $\mathcal{I} = \{i_0, \ldots, i_{k-1}\}$ of $\{0, \ldots, n\}$, either*

(a) there exists a monomial $x_{\mathcal{I}}^{M} = x_{i_0}^{m_0} \cdots x_{i_{k-1}}^{m_{k-1}}$ of degree d, or

(b) for $\nu = 1, \ldots, k$, there exist monomials $x_{\mathcal{I}}^{M_\nu} x_{e_\nu} = x_{i_0}^{m_{0,\nu}} \cdots x_{i_{k-1}}^{m_{k-1,\nu}} x_{e_\nu}$ of degree d and $\{e_\nu\}$ are k distinct elements. □

Precise conditions for curves in $\mathbb{P}(w_0, w_1, w_2)$ to be quasismooth are given by [14, Corollary 8.4]:

Corollary 7 *The curve $C_d \subset \mathbb{P}(w_0, w_1, w_2)$, with $d > w_i$, is quasismooth if and only if the following hold for all i:*

(1) there exists a monomial $x_i^n x_{e_i}$, for some e_i, of degree d.

(2) there exists a monomial of degree d which does not involve x_i. □

Precise conditions for surfaces in $\mathbb{P}(w_0, w_1, w_2, w_3)$ to be quasismooth are given by [14, Corollary 8.5]:

Corollary 8 *The surface $S_d \subset \mathbb{P}(w_0, w_1, w_2, w_3)$, with $d > w_i$, is quasismooth if and only if the following hold:*

(1) for all i there exists a monomial $x_i^n x_{e_i}$, for some e_i, of degree d.

(2) for all $i < j$ either

(a) there exists a monomial $x_i^m x_j^n$ of degree d, or

(b) there exist monomials $x_i^{m_1} x_j^{n_1} x_{e_1}$ $x_i^{m_2} x_j^{n2} x_{e_2}$ of degree d with $e_1 \neq e_2$.

(3) there exists a monomial of degree d which does not involve x_i. □

Precise conditions for 3-folds in $\mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ to be quasismooth are given by [14, Corollary 8.6]:

Corollary 9 *The 3-fold $X_d \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$, with $d > w_i$, is quasismooth if and only if the following hold:*

(1) for all i there exists a monomial $x_i^n x_{e_i}$, for some e_i, of degree d.

(2) for all $i < j$ either

(a) there exists a monomial $x_i^m x_j^n$ of degree d, or

(b) there exist monomials $x_i^{m_1} x_j^{n_1} x_{e_1}$ $x_i^{m_2} x_j^{n2} x_{e_2}$ of degree d with $e_1 \neq e_2$.

(3) for all $i < j$ there exists a monomial of degree d which does not involve either x_i or x_j. □

Remark 10 *A hypersurface variety satisfying condition (1) in any dimension is called* semiquasismooth *[2].*

Conditions for a hypersurface to be well-formed are given by ([14],6.10):

Remark 11 *A hypersurface* $X_d \subset \mathbb{P}(w_0, \ldots, w_n)$ *is well formed if and only if*

(i) $\gcd(w_0, \ldots, \hat{w}_i, \ldots, w_n) = 1$ *for each* i*, and*

(ii) $\gcd(w_0, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_n) \mid d$ *for each* $i < j$*.*

3.1.2 Codimension 2

General conditions for quasismoothness of codimension 2 weighted complete intersections are given by [14, Theorem 8.7]:

Theorem 12 *Suppose the general codimension 2 weighted complete intersection*

$$X_{d_1,d_2} \subset \mathbb{P}(w_0, \ldots, w_n)$$

of multidegree $\{d_1, d_2\}$*, where* $n \geq 2$*, is not the intersection of a linear cone with another hypersurface. Then* X_{d_1,d_2} *in* $\mathbb{P}$ *is quasismooth if and only if for each nonempty subset* $\mathcal{I} = \{i_0, \ldots, i_{k-1}\}$ *of* $\{0, \ldots, n\}$*, one of the following holds:*

(a) there exists a monomial $x_{\mathcal{I}}^{M_1}$ *of degree* d_1 *and there exists a monomial* $x_{\mathcal{I}}^{M_2}$ *of degree* d_2*, or*

(b) there exists a monomial $x_{\mathcal{I}}^{M}$ *of degree* d_1*, and for* $\nu = 1, \ldots, k-1$*, there exist monomials* $x_{\mathcal{I}}^{M_\nu} x_{e_\nu}$ *of degree* d_2*,where* $\{e_\nu\}$ *are* $k-1$ *distinct elements, or*

(c) there exists a monomial $x_{\mathcal{I}}^{M}$ *of degree* d_2*, and for* $\nu = 1, \ldots, k-1$*, there exist monomials* $x_{\mathcal{I}}^{M_\nu} x_{e_\nu}$ *of degree* d_1*,where* $\{e_\nu\}$ *are* $k-1$ *distinct elements, or*

(d) for $\nu = 1, \ldots, k$*, there exist monomials* $x_{\mathcal{I}}^{M_{\nu,1}} x_{e_{\nu,1}}$ *of degree* d_1*, and* $x_{\mathcal{I}}^{M_{\nu,2}} x_{e_{\nu,2}}$ *of degree* d_2*, such that* $\{e_{\nu,1}\}$ *are* k *distinct elements,* $\{e_{\nu,2}\}$ *are* k *distinct elements, and* $\{e_{\nu,1}, e_{\nu,2}\}$ *contains at least* $k+1$ *distinct elements.* □

Some general properties this condition requires are included in [14, Corollary 8.8]:

Corollary 13 *Suppose $X_{d_1,d_2} \subset \mathbb{P}(w_0,\ldots,w_n)$ is quasismooth and not the intersection of a linear cone with another hypersurface. Then the following hold:*

(i) Every variable x_i occurs in at least one of the defining equations.

(ii) All but at most one variable are in both equations.

(iii) If x_i does not appear in one defining equation then there exists a monomial x_i^m occurring in the other equation. □

Precise conditions for codimension 2 quasismooth complete intersections (curves) in $\mathbb{P}(w_0,w_1,w_2,w_3)$ are given by:

Corollary 14 *Suppose the general codimension 2 weighted complete intersection (curve) $X_{d_1,d_2} \subset \mathbb{P}(w_0,w_1,w_2,w_3)$ of multidegree $\{d_1,d_2\}$, is not the intersection of a linear cone with another hypersurface. Then X_{d_1,d_2} in $\mathbb{P}$ is quasismooth if and only if*

(1) for all i either

(b) there exists a monomial $x_i^{m_1}$ of degree d_1, or

(c) there exists a monomial $x_i^{m_2}$ of degree d_2, or

(d) there exist monomials $x_i^{n_1}x_{e_1}$ of degree d_1 for some e_1, and $x_i^{n_2}x_{e_2}$ of degree d_2 for some e_2, with $e_1 \neq e_2$

(2) for all $i < j$ either

(a) there exists a monomial $x_i^{m_1}x_j^{n_1}$ of degree d_1 and a monomial $x_i^{m_2}x_j^{n_2}$ of degree d_2, or

(b) there exists a monomial $x_i^{m_1}x_j^{n_1}$ of degree d_1 and a monomial $x_i^{m_2}x_j^{n_2}x_{e_2}$ of degree d_2, or

(c) there exists a monomial $x_i^{m_2}x_j^{n_2}$ of degree d_2 and a monomial $x_i^{m_1}x_j^{n_1}x_{e_1}$ of degree d_1

(3) for all $i < j < k$

(a) there exists monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ of degree d_1 and $x_i^{m_2}x_j^{n_2}x_k^{p_2}$ of degree d_2

Proof: Conditions (1), (2), and (3) come from applying the conditions of Theorem 12 for $|\mathcal{I}| = 1$, $|\mathcal{I}| = 2$, and $|\mathcal{I}| = 3$ respectively. □

Remark 15 *The condition (3) is equivalent to requiring for each $i = 0,1,2,3$ that there exists a monomial not involving x_i for each degree d_1, d_2.*

Precise conditions for codimension 2 quasismooth complete intersections (surfaces) in $\mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ are given by:

Corollary 16 *Suppose the general codimension 2 weighted complete intersection (surface) $X_{d_1,d_2} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ of multidegree $\{d_1, d_2\}$, is not the intersection of a linear cone with another hypersurface. Then X_{d_1,d_2} in $\mathbb{P}$ is quasismooth if and only if*

(1) for all i either

(b) there exists a monomial $x_i^{m_1}$ of degree d_1, or

(c) there exists a monomial $x_i^{m_2}$ of degree d_2, or

(d) there exist monomials $x_i^{n_1}x_{e_1}$ of degree d_1 for some e_1, and $x_i^{n_2}x_{e_2}$ of degree d_2 for some e_2, with $e_1 \neq e_2$

(2) for all $i < j$ either

(a) there exists a monomial $x_i^{m_1}x_j^{n_1}$ of degree d_1 and a monomial $x_i^{m_2}x_j^{n_2}$ of degree d_2, or

(b) there exists a monomial $x_i^{m_1}x_j^{n_1}$ of degree d_1 and a monomial $x_i^{m_2}x_j^{n_2}x_{e_2}$ of degree d_2, or

(c) there exists a monomial $x_i^{m_2}x_j^{n_2}$ of degree d_2 and a monomial $x_i^{m_1}x_j^{n_1}x_{e_1}$ of degree d_1, or

(d) there exist monomials $x_i^{m_{1,1}}x_j^{n_{1,1}}x_{e_{1,1}}$ and $x_i^{m_{1,2}}x_j^{n_{1,2}}x_{e_{1,2}}$ of degree d_1 and monomials $x_i^{m_{2,1}}x_j^{n_{2,1}}x_{e_{2,1}}$ and $x_i^{m_{2,2}}x_j^{n_{2,2}}x_{e_{2,2}}$ of degree d_2 such that $e_{1,1} \neq e_{1,2}$, $e_{2,1} \neq$

$e_{2,2}$, and $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$ contains 3 distinct elements.

(3) for all $i < j < k$ either

(a) there exists a monomial $x_i^{m_1} x_j^{n_1} x_k^{p_1}$ of degree d_1 and a monomial $x_i^{m_2} x_j^{n_2} x_k^{p_2}$ of degree d_2, or

(b) there exists a monomial $x_i^{m_1} x_j^{n_1} x_k^{p_1}$ of degree d_1 and monomials $x_i^{m_{2,1}} x_j^{n_{2,1}} x_k^{p_{2,1}} x_{e_{2,1}}$ and $x_i^{m_{2,2}} x_j^{n_{2,2}} x_k^{p_{2,2}} x_{e_{2,2}}$ of degree d_2, with $e_{2,1} \neq e_{2,2}$, or

(c) there exists a monomial $x_i^{m_2} x_j^{n_2} x_k^{p_2}$ of degree d_2 and monomials $x_i^{m_{1,1}} x_j^{n_{1,1}} x_k^{p_{1,1}} x_{e_{1,1}}$ and $x_i^{m_{1,2}} x_j^{n_{1,2}} x_k^{p_{1,2}} x_{e_{1,2}}$ of degree d_1, $e_{1,1} \neq e_{1,2}$

(4) for all $i < j < k < l$

(a) there exists monomials $x_i^{m_1} x_j^{n_1} x_k^{p_1} x_l^{q_1}$ of degree d_1 and $x_i^{m_2} x_j^{n_2} x_k^{p_2} x_l^{q_2}$ of degree d_2.

Proof: Conditions (1), (2), (3), and (4) come from applying the conditions of Theorem 12 for $|\mathcal{I}| = 1$, $|\mathcal{I}| = 2$, $|\mathcal{I}| = 3$, and $|\mathcal{I}| = 4$ respectively. □

Remark 17 *The condition (4) is equivalent to requiring for each $i = 0, 1, 2, 3, 4$ that there exists a monomial not involving x_i for each degree d_1, d_2.*

Conditions for a codimension 2 complete intersection to be well-formed are given by ([14],6.11):

Remark 18 *A complete intersection $X_{d_1,d_2} \subset \mathbb{P}(w_0, \ldots, w_n)$ is well formed if and only if*

(i) $\gcd(w_0, \ldots, \hat{w}_i, \ldots, w_n) = 1$ *for each i, and*

(ii) $\gcd(w_0, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_n) \mid d_m$ *for each m for each $i < j$, and*

(iii) $\gcd(w_0, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, \hat{w}_k, \ldots, w_n) \mid d_m$ *for some m for each $i < j < k$.*

3.1.3 Codimension 3

These results can be generalized to codimension 3 by a simple generalization of the proofs for the hypersurface [14, Theorem 8.1] and codimension 2 [14, Theorem 8.7] cases. First, the following, [14, Lemma 6.19], is used in the proof of 20 below.

Lemma 19 *Let Z be the affine variety of all points P which satisfy the determinantal condition:*

$$\operatorname{rank}\begin{pmatrix} g_1^1(P) & \cdots & g_1^m(P) \\ \vdots & & \vdots \\ g_c^1(P) & \cdots & g_c^m(P) \end{pmatrix} \leq k$$

where $\{g_i^j\}$ are general weighted homogeneous nonzero polynomials. If Z is nonempty then $\mathrm{codim} Z \leq (m-k)(c-k)$. □

Precise conditions for quasismoothness in codimension 3 are given by the following.

Theorem 20 *Suppose the general codimension 3 weighted complete intersection*

$$X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, \ldots, w_n)$$

of multidegree $\{d_1, d_2, d_3\}$, where $n \geq 3$, is not the intersection of a linear cone with a codimension 2 subvariety. Then X_{d_1,d_2,d_3} in $\mathbb{P}$ is quasismooth if and only if for each nonempty subset $\mathcal{I} = \{i_0, \ldots, i_{k-1}\}$ of $\{0, \ldots, n\}$, one of the following holds:

(a) there exist monomials $x_{\mathcal{I}}^{M_1}$ of degree d_1, $x_{\mathcal{I}}^{M_2}$ of degree d_2, and $x_{\mathcal{I}}^{M_3}$ of degree d_3, or

(b) there exist monomials $x_{\mathcal{I}}^{M_1}$ of degree d_1 and $x_{\mathcal{I}}^{M_2}$ of degree d_2, and for $\nu = 1, \ldots, k-2$, there exist monomials $x_{\mathcal{I}}^{M_\nu} x_{e_\nu}$ of degree d_3, where $\{e_\nu\}$ are $k-2$ distinct elements, or

(c) there exist monomials $x_{\mathcal{I}}^{M_1}$ *of degree* d_1 *and* $x_{\mathcal{I}}^{M_3}$ *of degree* d_3*, and for* $\nu = 1,\ldots,k-2$*, there exist monomials* $x_{\mathcal{I}}^{M_\nu} x_{e_\nu}$ *of degree* d_2*, where* $\{e_\nu\}$ *are* $k-2$ *distinct elements, or*

(d) there exist monomials $x_{\mathcal{I}}^{M_2}$ *of degree* d_2 *and* $x_{\mathcal{I}}^{M_3}$ *of degree* d_3*, and for* $\nu = 1,\ldots,k-2$*, there exist monomials* $x_{\mathcal{I}}^{M_\nu} x_{e_\nu}$ *of degree* d_1*, where* $\{e_\nu\}$ *are* $k-2$ *distinct elements, or*

(e) there exists a monomial $x_{\mathcal{I}}^{M}$ *of degree* d_1*, and for* $\nu = 1,\ldots,k-1$*, there exist monomials* $x_{\mathcal{I}}^{M_{\nu,2}} x_{e_{\nu,2}}$ *of degree* d_2 *and* $x_{\mathcal{I}}^{M_{\nu,3}} x_{e_{\nu,3}}$ *of degree* d_3*, where* $\{e_{\nu,2}\}$ *are* $k-1$ *distinct elements,* $\{e_{\nu,3}\}$ *are* $k-1$ *distinct elements, and* $\{e_{\nu,2}, e_{\nu,3}\}$ *contains at least* k *distinct elements, or*

(f) there exists a monomial $x_{\mathcal{I}}^{M}$ *of degree* d_2*, and for* $\nu = 1,\ldots,k-1$*, there exist monomials* $x_{\mathcal{I}}^{M_{\nu,1}} x_{e_{\nu,1}}$ *of degree* d_1 *and* $x_{\mathcal{I}}^{M_{\nu,3}} x_{e_{\nu,3}}$ *of degree* d_3*, where* $\{e_{\nu,1}\}$ *are* $k-1$ *distinct elements,* $\{e_{\nu,3}\}$ *are* $k-1$ *distinct elements, and* $\{e_{\nu,1}, e_{\nu,3}\}$ *contains at least* k *distinct elements, or*

(g) there exists a monomial $x_{\mathcal{I}}^{M}$ *of degree* d_3*, and for* $\nu = 1,\ldots,k-1$*, there exist monomials* $x_{\mathcal{I}}^{M_{\nu,1}} x_{e_{\nu,1}}$ *of degree* d_1 *and* $x_{\mathcal{I}}^{M_{\nu,2}} x_{e_{\nu,2}}$ *of degree* d_2*, where* $\{e_{\nu,1}\}$ *are* $k-1$ *distinct elements,* $\{e_{\nu,2}\}$ *are* $k-1$ *distinct elements, and* $\{e_{\nu,1}, e_{\nu,2}\}$ *contains at least* k *distinct elements, or*

(h) for $\nu = 1,\ldots,k$*, there exist monomials* $x_{\mathcal{I}}^{M_{\nu,1}} x_{e_{\nu,1}}$ *of degree* d_1*,* $x_{\mathcal{I}}^{M_{\nu,2}} x_{e_{\nu,2}}$ *of degree* d_2*, and* $x_{\mathcal{I}}^{M_{\nu,3}} x_{e_{\nu,3}}$ *of degree* d_3 *such that* $\{e_{\nu,1}\}$ *are* k *distinct elements,* $\{e_{\nu,2}\}$ *are* k *distinct elements,* $\{e_{\nu,3}\}$ *are* k *distinct elements, and* $\{e_{\nu,1}, e_{\nu,2}, e_{\nu,3}\}$ *contains at least* $k+2$ *distinct elements.*

Proof: Let F_1, F_2, F_3 be linear systems of all homogeneous polynomials of degrees d_1, d_2, and d_3, respectively, with respect to the weights $w_0,\ldots,w_n$. Let $f_1 \in F_1$, $f_2 \in F_2$, and $f_3 \in F_3$ be sufficiently general polynomials. Define

$$X = X_{d_1,d_2,d_3} \colon (f_1 = f_2 = f_3 = 0) \subset \mathbb{P}(\mathbf{w})$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_X^* & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}(\mathbf{w}) \end{array}$$

From Bertini's Theorem [12, Corollary III.10.9 and Remark III.10.9.2] the only singularities that can occur in the general $\mathcal{C}_X^*$ lie on the base loci of the linear systems F_1, F_2, and F_3. Any component of the base loci is a coordinate k-plane for some $k = 0, \dots, n$. So the complete intersection X_{d_1,d_2,d_3} is quasismooth if and only if its punctured affine cone $\mathcal{C}_X^*$ is nonsingular at each point of its intersection with every coordinate k-plane contained in the base loci. That is, X is quasismooth if and only if $\mathcal{C}_X^*$ is smooth along all the coordinate strata.

Let Π be a coordinate k-plane for some k. By renumbering, we can assume that Π is given by $x_k = \cdots = x_n = 0$, corresponding to the subset $I = \{0, \dots, k-1\}$. Let $\Pi^0 \subset \Pi$ be the open toric stratum where $x_0, \dots, x_{k-1}$ are all nonzero. Expand f_1, f_2, f_3 in terms of the coordinates $\{x_k, \dots, x_n\}$:

$$f_\lambda = h_\lambda(x_0, \dots, x_{k-1}) + \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \dots, x_{k-1}) + \text{h.o.t.}$$

for $\lambda = 1, 2, 3$.

Assume that one of the conditions (a), (b), (c), (d), (e), (f), (g), (h) holds for each non-empty subset I.

If (a) holds, then h_1, h_2, and h_3 are nonzero on Π^0. If any of h_1, h_2, and h_3 involve only two variables, then $\Pi^0 \cap \mathcal{C}_X^*$ is empty. This would include the cases $k = 1$ and $k = 2$, so without loss of generality, assume that h_1, h_2, and h_3 each involve at least three variables and hence $k \geq 3$. Π^0 is not part of the base locus of F_1, F_2,or F_3. By Bertini's Theorem, $(f_1 = 0)$, $(f_2 = 0)$, and $(f_3 = 0)$ are nonsingular on Π^0. Since $(h_1 = 0)$, $(h_2 = 0)$, and $(h_3 = 0)$ are free linear systems on Π^0, $(h_1 = 0)$, $(h_2 = 0)$, and $(h_3 = 0)$ intersect transversally. Thus, at each point of $(h_1 = h_2 = h_3 = 0) \cap \Pi^0$, there exist three distinct normals. Therefore $\mathcal{C}_X^*$ is nonsingular along Π^0.

Suppose (b) holds. Then h_1 and h_2 are nonzero and at least $k-2$ of the $\{g_3^i\}$ are nonzero. Π^0 is not part of the base locus for F_1 or F_2, so by Bertini's Theorem $(f_1 = f_2 = 0)$ is nonsingular on Π^0. Singular points occur exactly on the locus

$$Z = (h_1 = h_2 = 0) \bigcap_i (g_3^i = 0) \subset \Pi^0$$

which is an intersection of at least $k-2$ free linear systems on $(h_1 = h_2 = 0) \cap \Pi^0$. Thus $\dim Z \le 0$ and hence is at worst the origin. Therefore $\mathcal{C}_X^*$ is nonsingular along Π^0.

(c) and (d) are similar to (b).

Suppose (e) holds. Then h_1 is nonzero and at least $k-1$ of the $\{g_2^i\}$ are nonzero and at least $k-1$ of the $\{g_3^i\}$ are nonzero. Furthermore, the matrix

$$\begin{pmatrix} g_2^k & \cdots & g_2^n \\ g_3^k & \cdots & g_3^n \end{pmatrix}$$

has at least k nonzero columns. Π^0 is not part of the base locus for F_1, so by Bertini's Theorem $(f_1 = 0)$ is nonsingular on Π^0. Define the matrix M_P by

$$\begin{pmatrix} g_2^k(P) & \cdots & g_2^n(P) \\ g_3^k(P) & \cdots & g_3^n(P) \end{pmatrix}.$$

Singular points occur on the locus $Z = \{P | \operatorname{rank} M_P \le 2\}$.

As there are at least $k-1$ monomials of the form $x_I^M x_e$ of degree d_λ, $\lambda = 2,3$, at least $k-1$ of the $\{g_\lambda^i\}$ are nonzero. As these are free on Π^0, each row of the matrix M_P is nonzero for each $P \in \Pi^0$. Furthermore this matrix for any $P \in Z$ has at least k nonzero columns, since there are at least k distinct elements in $\{e_\nu^2, e_\nu^3\}$. By renumbering we can assume that the first k columns of M_P are not identically zero on Π^0. Fix $P \in \Pi^0$. Without loss of generality we can assume that $g_2^k(P) \neq 0$. If $g_3^k(P) = 0$ then $g_3^i(P) \neq 0$ for some $i > k$ so M_P has rank 2, and $P \in \mathcal{C}_X^*$ is nonsingular.

Suppose $g_3^k(P) \neq 0$. Define $a = g_2^k$, $b = g_3^k$, and

$$Z_P = \{Q \in \Pi^0 \big| \bigcap_{i>k} (a g_3^i(Q) - b g_2^i(Q) = 0)\}.$$

Then, $P \in Z_P$ if and only if $\operatorname{rank} M_P \leq 1$ in this case, which is equivalent to $P \in \mathcal{C}_X^*$ being singular. Since Z_P is the intersection of at least $k-1$ free linear systems on Π^0, $\dim Z_P \leq 0$ and so Z_P is at worst the origin. In particular, $P \notin Z_P$ and hence $P \in \mathcal{C}_X^*$ is nonsingular. Therefore, $\mathcal{C}_X^*$ is nonsingular along Π^0.

(f) and (g) are similar to (e).

Suppose only (h) holds. Then

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \dots, x_{k-1}) + \text{h.o.t.}$$

for $\lambda = 1, 2, 3$. The normal directions, perpendicular to the k-plane Π, to the hypersurfaces are $(g_1^k, \dots, g_1^n)$, $(g_2^k, \dots, g_2^n)$, and $(g_3^k, \dots, g_3^n)$. Define the matrix M_P by

$$M_P = \begin{pmatrix} g_1^k(P) & \cdots & g_1^n(P) \\ g_2^k(P) & \cdots & g_2^n(P) \\ g_3^k(P) & \cdots & g_3^n(P) \end{pmatrix}.$$

Singular points occur on the locus $Z = \{P \big| \operatorname{rank} M_P \leq 2\}$. As there are at least k monomials of the form $x_I^M x_e$ of degree d_λ, at least k of the $\{g_\lambda^i\}$ are nonzero. As these are free on Π^0, each row of the matrix M_P is nonzero for each $P \in \Pi^0$. Furthermore this matrix for any $P \in Z$ has at least $k+2$ nonzero columns, since there are at least $k+2$ distinct elements in $\{e_\nu^1, e_\nu^2, e_\nu^3\}$. By renumbering we can assume that the first $k+2$ columns of M_P are not identically zero on Π^0.

Fix $P \in \Pi^0$. Without loss of generality we can assume that $g_1^k(P) \neq 0$. If $g_2^k(P) = 0$ then $g_2^i(P) \neq 0$ for some $i > k$. Without loss of generality we can assume that $i = k+1$. If $g_3^k(P) = g_3^{k+1} = 0$, then $g_3^j(P) \neq 0$ for some $j > k+1$ so M_P has rank 3 and $P \in \mathcal{C}_X^*$

is nonsingular. Suppose $g_3^k(P) = 0$ and $g_3^{k+1}(P) \neq 0$. Define $b = g_2^{k+1}$, $c = g_3^{k+1}$, and

$$Z_P = \{Q \in \Pi^0 | \bigcap_{i>k+1} (bg_3^i(Q) - cg_2^i(Q) = 0)\}.$$

Then $P \in Z_P$ if and only if $\text{rank} M_P \leq 2$ in this case, which is equivalent to $P \in \mathcal{C}_X^*$ being singular. Since Z_P is the intersection of at least k free linear systems on Π^0, $\dim Z_P \leq 0$ and so Z_P is at worst the origin. In particular, $P \notin Z_P$ and hence $P \in \mathcal{C}_X^*$ is nonsingular. Now, suppose either $g_2^k(P) \neq 0$ or $g_3^k(P) \neq 0$. Then, define $a = g_1^k(P)$, $b = g_2^k(P)$, $c = g_2^k(P)$ and

$$\begin{aligned} Z_P \quad = \{Q \in \Pi^0 | \textstyle\bigcap_{j>i>k} \quad & (a(g_2^i(Q)g_3^j(Q) - g_2^j(Q)g_3^i(Q)) \\ & -b(g_1^i(Q)g_3^j(Q) - g_1^j(Q)g_3^i(Q)) \\ & +c(g_1^i(Q)g_2^j(Q) - g_1^j(Q)g_2^i(Q)) = 0)\}. \end{aligned}$$

Then, again, $P \in Z_P$ if and only if $\text{rank} M_P \leq 2$ in this case, which is equivalent to $P \in \mathcal{C}_X^*$ being singular. Since Z_P is the intersection of at least k free linear systems on Π^0, $\dim Z_P \leq 0$ and so Z_P is at worst the origin. In particular, $P \notin Z_P$ and hence $P \in \mathcal{C}_X^*$ is nonsingular. Therefore, $\mathcal{C}_X^*$ is nonsingular along Π^0.

As one of these eight conditions holds for every nonempty set I, $\mathcal{C}_X^*$ is nonsingular.

Conversely, assume that none of the conditions (a), (b), (c), (d), (e), (f), (g), (h) hold for some non-empty subset I. Without loss of generality we can assume that $I = \{0, \ldots, k-1\}$ for some k. Let Π be the corresponding coordinate plane $x_k = \cdots = x_n = 0$. There are several cases:

(i) $\Pi \not\subset \mathcal{C}_{X_{d_1}} \cup \mathcal{C}_{X_{d_2}}$. Then h_1 and h_2 are nonzero and since conditions (b) does not hold,there are at most $k-3$ of the $\{g_3^i\}$ which are nonzero. The singular points are exactly the locus

$$Z = (h_1 = h_2 = 0) \bigcap_i (g_3^i = 0)$$

so

$$\dim Z \geq k - (k-3) - 2 = 1.$$

Then Z contains more than the origin and $\mathcal{C}_X^*$ is singular along Π.

(ii) $\Pi \not\subset \mathcal{C}_{X_{d_1}} \cup \mathcal{C}_{X_{d_3}}$. As in case (i), $\mathcal{C}_X^*$ is singular along Π.

(iii) $\Pi \not\subset \mathcal{C}_{X_{d_2}} \cup \mathcal{C}_{X_{d_3}}$. As in case (i), $\mathcal{C}_X^*$ is singular along Π.

(iv) $\Pi \subset (\mathcal{C}_{X_{d_2}} \cap \mathcal{C}_{X_{d_3}}) \setminus \mathcal{C}_{X_{d_1}}$. Then h_1 is nonzero and since condition (e) does not hold, either

(1) there are at most $k-2$ of the $\{g_2^i\}$ which are nonzero,

(2) there are at most $k-2$ of the $\{g_3^i\}$ which are nonzero, or

(3) there are at most $k-1$ nonzero columns in the matrix

$$\begin{pmatrix} g_2^k & \cdots & g_2^n \\ g_3^k & \cdots & g_3^n \end{pmatrix}.$$

In case (1) the intersection $Z = \bigcap_i (g_2^i = 0)$ has dimension at least 1 and so the $\{g_2^i\}$ have a common solution and the matrix

$$M_P = \begin{pmatrix} g_2^k(P) & \cdots & g_2^n(P) \\ g_3^k(P) & \cdots & g_3^n(P) \end{pmatrix}$$

has rank less than 2 for some $P \in Z$ and hence $\mathcal{C}_X^*$ is singular along Π. Case (2) is similar to case (1). In case (3) let $Z = \{P | \text{rank} M_P \leq 1\}$. Then, by Lemma 19,

$$\dim Z \geq k - (k-2) - 1 = 1,$$

so Z contains more than just the origin. Therefore $\mathcal{C}_X^*$ is singular along Π.

(v) $\Pi \subset (\mathcal{C}_{X_{d_1}} \cap \mathcal{C}_{X_{d_3}}) \setminus \mathcal{C}_{X_{d_2}}$. As in case (iv), $\mathcal{C}_X^*$ is singular along Π.

(vi) $\Pi \subset (\mathcal{C}_{X_{d_1}} \cap \mathcal{C}_{X_{d_2}}) \smallsetminus \mathcal{C}_{X_{d_3}}$. As in case (iv), $\mathcal{C}_X^*$ is singular along Π.

(vii) $\Pi \subset \mathcal{C}_{X_{d_1}} \cap \mathcal{C}_{X_{d_3}} \cap \mathcal{C}_{X_{d_2}}$. In this case, h_1, h_2, and h_3 are all identically zero. Then

$$f_\lambda = \sum_{i=k}^{n} x_i g_\lambda^i(x_0, \dots, x_{k-1}) + \text{h.o.t.}$$

for $\lambda = 1, 2, 3$. As condition (h) does not hold, one of two cases occurs: (1) for some λ there are at most $k-1$ of the $\{g_\lambda^i\}$ which are nonzero, or (2) there are at most $k+1$ distinct elements in the set $\{e_\nu^1, e_\nu^2, e_\nu^3\}$. In case (1), the intersection $Z_\lambda = \bigcap_i (g_\lambda^i = 0)$ has dimension at least 1, so the matrix

$$M_P = \begin{pmatrix} g_1^k(P) & \cdots & g_1^n(P) \\ g_2^k(P) & \cdots & g_2^n(P) \\ g_3^k(P) & \cdots & g_3^n(P) \end{pmatrix}$$

has rank less than 3 for some $P \in Z_\lambda$ and hence $\mathcal{C}_X^*$ is singular along Π. In case (2) there are at most $k+1$ nonzero columns in M_P. Let $Z = \{P \big| \text{rank} M_P \leq 2\}$. Then

$$\dim Z \geq k - (k-1) = 1,$$

so Z contains more than just the origin. Therefore $\mathcal{C}_X^*$ is singular along Π. □

Corollary 13 has its counterpart in codimension 3:

Corollary 21 *Suppose $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, \dots, w_n)$ is quasismooth and not the intersection of a linear cone with another subvariety. Then the following hold:*

(i) Every variable x_i occurs in at least one of the defining equations.

(ii) If x_i does not appear in two of the defining equations then there exists a monomial x_i^m occurring in the other equation.

(iii) Every pair of variables x_i, x_j occurs in at least two of the defining equations.

(iv) If neither x_i and x_j appear in one of the defining equations, then both the other equations contain a monomial of the form $x_i^m x_j^n$.

(v) Each defining equation lacks at most two variables.

Proof: (i) and (ii) follow from Theorem 20 for $|I| = 1$. (iii) and (iv) follow from Theorem 20 for $|I| = 2$. (v) follows from Theorem 20 for $|I| = 3$. □

Precise conditions for codimension 3 quasismooth complete intersections (curves) in $\mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ are given by:

Corollary 22 *Suppose the general codimension 3 weighted complete intersection (curve) $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ of multidegree $\{d_1, d_2, d_3\}$ is not the intersection of a linear cone with a codimension 2 subvariety. Then X_{d_1,d_2,d_3} in $\mathbb{P}$ is quasismooth if and only if*

(1) for each i one of the following holds:

(e) there exists a monomial x_i^m of degree d_1, or

(f) there exists a monomial x_i^m of degree d_2, or

(g) there exists a monomial x_i^m of degree d_3, or

(h) there exist monomials $x_i^{m_1} x_{e_1}$ of degree d_1, $x_i^{m_2} x_{e_2}$ of degree d_2, and $x_i^{m_3} x_{e_3}$ of degree d_3 such that $\{e_1, e_2, e_3\}$ contains 3 distinct elements.

(2) for each $i < j$ one of the following holds:

(b) there exist monomials $x_i^{m_1} x_j^{n_1}$ of degree d_1 and $x_i^{m_2} x_j^{n_2}$ of degree d_2, or

(c) there exist monomials $x_i^{m_1} x_j^{n_1}$ of degree d_1 and $x_i^{m_3} x_j^{n_3}$ of degree d_3, or

(d) there exist monomials $x_i^{m_2} x_j^{n_2}$ of degree d_2 and $x_i^{m_3} x_j^{n_3}$ of degree d_3, or

(e) there exists a monomial $x_i^{m_1} x_j^{n_1}$ of degree d_1 and a monomial $x_i^{m_2} x_j^{n_2} x_{e_2}$ of degree d_2 and a monomial $x_i^{m_3} x_j^{n_3} x_{e_3}$ of degree d_3 where $e_2 \neq e_3$, or

(f) there exists a monomial $x_i^{m_2} x_j^{n_2}$ of degree d_2 and a monomial $x_i^{m_1} x_j^{n_1} x_{e_1}$ of degree d_1 and a monomial $x_i^{m_3} x_j^{n_3} x_{e_3}$ of degree d_3 where $e_1 \neq e_3$, or

(g) there exists a monomial $x_i^{m_3} x_j^{n_3}$ of degree d_3 and a monomial $x_i^{m_1} x_j^{n_1} x_{e_1}$ of degree d_1 and a monomial $x_i^{m_2} x_j^{n_2} x_{e_2}$ of degree d_2 where $e_1 \neq e_2$

(3) for each $i < j < k$ one of the following holds:

(a) there exist monomials $x_i^{m_1} x_j^{n_1} x_k^{p_1}$ of degree d_1, $x_i^{m_2} x_j^{n_2} x_k^{p_2}$ of degree d_2, and $x_i^{m_3} x_j^{n_3} x_k^{p_3}$ of degree d_3, or

(b) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}x_l$ *of degree* d_3*, or*

(c) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}x_l$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}$ *of degree* d_3*, or*

(d) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}x_l$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}$ *of degree* d_3

(4) for each $i < j < k < l$ *one of the following holds:*

(a) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}x_l^{q_1}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}x_l^{q_2}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}x_l^{q_3}$ *of degree* d_3

Proof: Direct application of Theorem 20 for $|\mathcal{I}| = 1$, $|\mathcal{I}| = 2$, $|\mathcal{I}| = 3$, and $|\mathcal{I}| = 4$. □

Remark 23 *The condition (4) is equivalent to requiring for each* $i = 0, 1, 2, 3, 4$ *that there exists a monomial not involving* x_i *for each degree* d_1*,* d_2*, and* d_3*.*

Precise conditions for quasismooth codimension 3 complete intersections (surfaces) in $\mathbb{P}(w_0, w_1, w_2, w_3, w_4, w_5)$ are given by:

Corollary 24 *Suppose the general codimension 3 weighted complete intersection (surface)* $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4, w_5)$ *of multidegree* $\{d_1, d_2, d_3\}$ *is not the intersection of a linear cone with a codimension 2 subvariety. Then* X_{d_1,d_2,d_3} *in* $\mathbb{P}$ *is quasismooth if and only if*

(1) for each i *one of the following holds:*

(e) there exists a monomial x_i^m *of degree* d_1*, or*

(f) there exists a monomial x_i^m *of degree* d_2*, or*

(g) there exists a monomial x_i^m *of degree* d_3*, or*

(h) there exist monomials $x_i^{m_1}x_{e_1}$ *of degree* d_1*,* $x_i^{m_2}x_{e_2}$ *of degree* d_2*, and* $x_i^{m_3}x_{e_3}$ *of degree* d_3 *such that* $\{e_1, e_2, e_3\}$ *contains* 3 *distinct elements.*

(2) for each $i < j$ *one of the following holds:*

(b) there exist monomials $x_i^{m_1}x_j^{n_1}$ *of degree* d_1 *and* $x_i^{m_2}x_j^{n_2}$ *of degree* d_2*, or*

(c) there exist monomials $x_i^{m_1}x_j^{n_1}$ *of degree* d_1 *and* $x_i^{m_3}x_j^{n_3}$ *of degree* d_3*, or*

(d) there exist monomials $x_i^{m_2}x_j^{n_2}$ *of degree* d_2 *and* $x_i^{m_3}x_j^{n_3}$ *of degree* d_3*, or*

(e) there exists a monomial $x_i^{m_1}x_j^{n_1}$ *of degree* d_1 *and a monomial* $x_i^{m_2}x_j^{n_2}x_{e_2}$ *of degree* d_2 *and a monomial* $x_i^{m_3}x_j^{n_3}x_{e_3}$ *of degree* d_3 *where* $e_2 \neq e_3$*, or*

(f) there exists a monomial $x_i^{m_2}x_j^{n_2}$ *of degree* d_2 *and a monomial* $x_i^{m_1}x_j^{n_1}x_{e_1}$ *of degree* d_1 *and a monomial* $x_i^{m_3}x_j^{n_3}x_{e_3}$ *of degree* d_3 *where* $e_1 \neq e_3$*, or*

(g) there exists a monomial $x_i^{m_3}x_j^{n_3}$ *of degree* d_3 *and a monomial* $x_i^{m_1}x_j^{n_1}x_{e_1}$ *of degree* d_1 *and a monomial* $x_i^{m_2}x_j^{n_2}x_{e_2}$ *of degree* d_2 *where* $e_1 \neq e_2$*, or*

(h)there exist monomials $x_i^{m_1}x_j^{n_1}x_{e_{1,1}}$ *and* $x_i^{m_1}x_j^{n_1}x_{e_{1,2}}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_{e_{2,1}}$ *and* $x_i^{m_2}x_j^{n_2}x_{e_{2,2}}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_{e_{3,1}}$ *and* $x_i^{m_3}x_j^{n_3}x_{e_{3,2}}$ *of degree* d_3 *such that* $\{e_{1,1}\} \neq \{e_{1,1}\}$*,* $\{e_{2,1}\} \neq \{e_{2,1}\}$*,* $\{e_{3,1}\} \neq \{e_{3,1}\}$*, and* $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, e_{3,1}, e_{3,2}\}$ *contains at least* 4 *distinct elements.*

(3) for each $i < j < k$ *one of the following holds:*

(a) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}$ *of degree* d_3*, or*

(b) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}x_l$ *of degree* d_3*, or*

(c) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}x_l$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}$ *of degree* d_3*, or*

(d) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}x_l$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}$ *of degree* d_2*, and* $x_i^{m_3}x_j^{n_3}x_k^{p_3}$ *of degree* d_3*, or*

(e) there exist monomials $x_i^{m_1}x_j^{n_1}x_k^{p_1}$ *of degree* d_1*,* $x_i^{m_{2,1}}x_j^{n_{2,1}}x_k^{p_{2,1}}x_{e_{2,1}}$ *and* $x_i^{m_{2,2}}x_j^{n_{2,2}}x_k^{p_{2,2}}x_{e_{2,2}}$ *of degree* d_2*, and* $x_i^{m_{3,1}}x_j^{n_{3,1}}x_k^{p_{3,1}}x_{e_{3,1}}$ *and* $x_i^{m_{3,2}}x_j^{n_{3,2}}x_k^{p_{3,2}}x_{e_{3,2}}$ *of degree* d_3*, where* $e_{2,1} \neq e_{2,2}$*,* $e_{3,1} \neq e_{3,2}$*, and* $\{e_{2,1}, e_{2,2}, e_{3,1}, e_{3,2}\}$ *contains at least* 3 *distinct elements, or*

(f) there exist monomials $x_i^{m_{1,1}}x_j^{n_{1,1}}x_k^{p_{1,1}}x_{e_{1,1}}$ *and* $x_i^{m_{1,2}}x_j^{n_{1,2}}x_k^{p_{1,2}}x_{e_{1,2}}$ *of degree* d_1*,* $x_i^{m_2}x_j^{n_2}x_k^{p_2}x_l$ *of degree* d_2*, and* $x_i^{m_{3,1}}x_j^{n_{3,1}}x_k^{p_{3,1}}x_{e_{3,1}}$ *and* $x_i^{m_{3,2}}x_j^{n_{3,2}}x_k^{p_{3,2}}x_{e_{3,2}}$ *of degree*

d_3, *where* $e_{1,1} \neq e_{1,2}$, $e_{3,1} \neq e_{3,2}$, *and* $\{e_{1,1}, e_{1,2}, e_{3,1}, e_{3,2}\}$ *contains at least* 3 *distinct elements, or*

(g) there exist monomials $x_i^{m_{1,1}} x_j^{n_{1,1}} x_k^{p_{1,1}} x_{e_{1,1}}$ *and* $x_i^{m_{1,2}} x_j^{n_{1,2}} x_k^{p_{1,2}} x_{e_{1,2}}$ *of degree* d_1 $x_i^{m_{2,1}} x_j^{n_{2,1}} x_k^{p_{2,1}} x_{e_{2,1}}$ *and* $x_i^{m_{2,2}} x_j^{n_{2,2}} x_k^{p_{2,2}} x_{e_{2,2}}$ *of degree* d_2, *and* $x_i^{m_3} x_j^{n_3} x_k^{p_3}$ *of degree* d_3, *where* $e_{1,1} \neq e_{1,2}$, $e_{2,1} \neq e_{2,2}$, *and* $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$ *contains at least* 3 *distinct elements*

(4) for each $i < j < k < l$ *one of the following holds:*

(a) there exist monomials $x_i^{m_1} x_j^{n_1} x_k^{p_1} x_l^{q_1}$ *of degree* d_1, $x_i^{m_2} x_j^{n_2} x_k^{p_2} x_l^{q_2}$ *of degree* d_2, *and* $x_i^{m_3} x_j^{n_3} x_k^{p_3} x_l^{q_3}$ *of degree* d_3, *or*

(b) there exist monomials $x_i^{m_1} x_j^{n_1} x_k^{p_1} x_l^{q_1}$ *of degree* d_1, $x_i^{m_2} x_j^{n_2} x_k^{p_2} x_l^{q_2}$ *of degree* d_2, *and* $x_i^{m_3} x_j^{n_3} x_k^{p_3} x_l^{q_3} x_{e_1}$ *and* $x_i^{m_3} x_j^{n_3} x_k^{p_3} x_l^{q_3} x_{e_2}$ *of degree* d_3, *where* $e_1 \neq e_2$, *or*

(c) there exist monomials $x_i^{m_1} x_j^{n_1} x_k^{p_1} x_l^{q_1}$ *of degree* d_1, $x_i^{m_2} x_j^{n_2} x_k^{p_2} x_l^{q_2} x_{e_1}$ *and* $x_i^{m_2} x_j^{n_2} x_k^{p_2} x_l^{q_2} x_{e_2}$ *of degree* d_2, *where* $e_1 \neq e_2$, *and* $x_i^{m_3} x_j^{n_3} x_k^{p_3} x_l^{q_3}$ *of degree* d_3, *or*

(d) there exist monomials $x_i^{m_1} x_j^{n_1} x_k^{p_1} x_l^{q_1} x_{e_1}$ *and* $x_i^{m_1} x_j^{n_1} x_k^{p_1} x_l^{q_1} x_{e_2}$ *of degree* d_1, *where* $e_1 \neq e_2$, $x_i^{m_2} x_j^{n_2} x_k^{p_2} x_l^{q_2}$ *of degree* d_2, *and* $x_i^{m_3} x_j^{n_3} x_k^{p_3} x_l^{q_3}$ *of degree* d_3

(5) for each $h < i < j < k < l$

(a) there exist monomials $x_h^{m_1} x_i^{n_1} x_j^{p_1} x_k^{q_1} x_l^{r_1}$ *of degree* d_1, $x_h^{m_2} x_i^{n_2} x_j^{p_2} x_k^{q_2} x_l^{r_2}$ *of degree* d_2, *and* $x_h^{m_3} x_i^{n_3} x_j^{p_3} x_k^{q_3} x_l^{r_3}$ *of degree* d_3.

Proof: Direct application of Theorem 20 for $|\mathcal{I}| = 1$, $|\mathcal{I}| = 2$, $|\mathcal{I}| = 3$, $|\mathcal{I}| = 4$, and $|\mathcal{I}| = 5$. □

Remark 25 *The condition (5) is equivalent to requiring for each* $i = 0, 1, 2, 3, 4, 5$ *that there exists a monomial not involving* x_i *for each degree* d_1, d_2, *and* d_3.

Conditions for a codimension 3 complete intersection to be well-formed are given by (see [14],6.12):

Remark 26 *A complete intersection* $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, \ldots, w_n)$ *is well formed if and only if* *(i)* $\gcd(w_0,\dots,\hat{w}_i,\dots,w_n) = 1$ *for each* i*, and*

(ii) $\gcd(w_0,\dots,\hat{w}_i,\dots,\hat{w}_j,\dots,w_n) \mid d_m$ *for each* m *for each* $i < j$*, and*

(iii) $\gcd(w_0,\dots,\hat{w}_i,\dots,\hat{w}_j,\dots,\hat{w}_k,\dots,w_n) \mid d_m$ *for at least two* m *for each* $i < j < k$*, and*

(iv) $\gcd(w_0,\dots,\hat{w}_i,\dots,\hat{w}_j,\dots,\hat{w}_k,\dots,\hat{w}_l,\dots,w_n) \mid d_m$ *for some* m *for each* $i < j < k < l$*.*

3.1.4 General Codimensions

The technique of proof given in [14] for Theorems 6 and 12, and extended to Theorem 20 above, clearly generalizes. The number of possible cases is 2^c, where c is the codimension.

Let Σ_c be the set of permutations of $\{1,\dots,c\}$.

Let $\Sigma_{c,i} = \{\sigma \in \Sigma_c | \sigma(1) < \cdots < \sigma(i), \sigma(i+1) < \cdots < \sigma(c)\}$

Theorem 27 *Suppose the general codimension* c *weighted complete intersection*

$$X_{d_1,\dots,d_c} \subset \mathbb{P}(w_0,\dots,w_n)$$

of multidegree $\{d_1,\dots,d_c\}$*, where* $n \geq 2$ *and* $n - c \geq 1$*, is not the intersection of a linear cone with a codimension* $c-1$ *subvariety. Then* $X_{d_1,\dots,d_c}$ *in* $\mathbb{P}$ *is quasismooth if and only if for each nonempty subset* $\mathcal{I} = \{i_0,\dots,i_{k-1}\}$ *of* $\{0,\dots,n\}$*, one of the following holds:*

(0) there exist monomials $x_{\mathcal{I}}^{M_1}$ *of degree* d_1*, …, and* $x_{\mathcal{I}}^{M_c}$ *of degree* d_c

(1) for some $\sigma \in \Sigma_{c,c-1}$*, there exist monomials* $x_{\mathcal{I}}^{M_{\sigma(1)}}$ *of degree* $d_{\sigma(1)}$*,…,* $x_{\mathcal{I}}^{M_{\sigma(c-1)}}$ *of degree* $d_{\sigma(c-1)}$*, and for* $\nu = 1,\dots,k-c+1$*, there exist monomials* $x_{\mathcal{I}}^{M_\nu} x_{e_\nu}$ *of degree* $d_{\sigma(c)}$*, where* $\{e_\nu\}$ *are* $k-c+1$ *distinct elements*

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

(j) for some $\sigma \in \Sigma_{c,c-j}$, there exist monomials $x_{\mathcal{I}}^{M_{\sigma(1)}}$ of degree $d_{\sigma(1)},\ldots,x_{\mathcal{I}}^{M_{\sigma(c-j)}}$ of degree $d_{\sigma(c-j)}$, and for $\nu = 1,\ldots,k-c+j$, there exist monomials $x_{\mathcal{I}}^{M_{c-j+1,\nu}} x_{e_{c-j+1,\nu}}$ of degree $d_{\sigma(c-j+1)}$, where $\{e_{c-j+1,\nu}\}$ are $k-c+j$ distinct elements,...,there exist monomials $x_{\mathcal{I}}^{M_{c,\nu}} x_{e_{c,\nu}}$ of degree $d_{\sigma(c)}$, where $\{e_{c,\nu}\}$ are $k-c+2j-1$ distinct elements

⋮ ⋮ ⋮ ⋮ ⋮

(n-1) for some $\sigma \in \Sigma_{c,1}$, there exists a monomial $x_{\mathcal{I}}^{M_{\sigma(1)}}$ of degree $d_{\sigma(1)}$, and for $\nu = 1,\ldots,k-1$, there exist monomials $x_{\mathcal{I}}^{M_{\sigma(2),\nu}} x_{e_{\sigma(2),\nu}}$ of degree $d_{\sigma(2)},\ldots,x_{\mathcal{I}}^{M_{\sigma(c),\nu}} x_{e_{\sigma(c),\nu}}$ of degree $d_{\sigma(c)}$, where $\{e_{\sigma(2),\nu}\}$ are $k-1$ distinct elements,...,$\{e_{\sigma(c),\nu}\}$ are $k-1$ distinct elements, and $\{e_{\sigma(2),\nu}, e_{\sigma(c)\nu}\}$ contains at least $k+c-3$ distinct elements

(n) for $\nu = 1,\ldots,k$, there exist monomials $x_{\mathcal{I}}^{M_{1,\nu}} x_{e_{1,\nu}}$ of degree d_1, ..., $x_{\mathcal{I}}^{M_{c,\nu}} x_{e_{c,\nu}}$ of degree d_c such that $\{e_{1,\nu}\}$ are k distinct elements, ..., $\{e_{c,\nu}\}$ are k distinct elements, and $\{e_{1,\nu},\ldots,e_{3,\nu}\}$ contains at least $k+c-1$ distinct elements.

Proof: As illustrated above in the case $c = 3$ (Theorem 20), using Bertini's Theorem, linear algebra, and dimensionality arguments. □

Corollaries 13 and 21 also generalize:

Corollary 28 *Suppose $X_{d_1,\ldots,d_c} \subset \mathbb{P}(w_0,\ldots,w_n)$ is quasismooth and not the intersection of a linear cone with another subvariety. Then the following hold for $k \in \{1,\ldots,c-1\}$:*

(i) Every k-tuple of variables $\{x_{i_0},\ldots,x_{i_{k-1}}\}$ occurs in at least k of the defining equations.

(ii) If none of a k-tuple of variables $\{x_{i_0},\ldots,x_{i_{k-1}}\}$ occur in $c-k$ of the defining equations then each of the other k equations contain a monomial of the form $x_{i_0}^{m_0}\cdots x_{i_{k-1}}^{m_{k-1}}$.

(iii) Each order k subset of defining equations lacks at most $c-k$ variables. □

[Note: Chen, Chen, and Chen [7, Proposition 3.1(1)] prove this for $k = 1$.]

Conditions for a general codimension complete intersection to be well-formed are given by ([14],6.12):

Remark 29 *A complete intersection $X_{d_1,\ldots,d_c} \subset \mathbb{P}(w_0,\ldots,w_n)$ is well formed if and only if*

(i) $\gcd(w_0,\ldots,\hat{w}_i,\ldots,w_n) = 1$ *for each i, and*

(ii) for each $\mu = 1,\ldots,c$, the greatest common divisor of any $(n-1-c+\mu)$ of the $\{w_i\}$ must divide at least μ of the $\{d_j\}$.

3.2 Additional constraints required for positivity

3.2.1 Codimension 2

Lemma 30 *Let $X_{d_1,d_2} \subset \mathbb{P}(w_0,\ldots,w_n)$ of multidegree $\{d_1,d_2\}$ be a codimension 2 weighted complete intersection. Suppose X_{d_1,d_2} is quasismooth and not the intersection of a linear cone with another hypersurface. Assume $w_0 \le \cdots \le w_n$ and $d_1 \le d_2$. Then (a) $d_2 \ge w_n + w_1$ and (b) $d_1 \ge w_{n-1} + w_0$.*

Proof: (a) Apply Theorem 12 to $I = \{n\}$. Then one of the following holds:

(i) $m_1 w_n = d_1$, or

(ii) $m_2 w_n = d_2$, or

(iii) $m_3 w_n + w_i = d_1$ and $m_4 w_n + w_j = d_2$ with $i,j \in \{0,\ldots,n-1\}$ and $i \ne j$.

(i)$\Rightarrow$ $d_2 \ge d_1 \ge 2w_n \ge w_n + w_1$. (ii)$\Rightarrow$ $d_2 \ge 2w_n \ge w_n + w_1$. (iii) If $j > 0$, $d_2 = m_4 w_n + w_j \ge w_n + w_j \ge w_n + w_1$. If $j = 0$, then $i > 0$, so $d_2 \ge d_1 = m_4 w_n + w_i \ge w_n + w_i \ge w_n + w_1$.

(b) Apply Theorem 12 to $I = \{n-1,n\}$. Then one of the following holds:

(i) $m w_{n-1} + p w_n = d_1$ or

(ii) $m_1 w_{n-1} + p_1 w_n + w_i = d_1$ with $i \in \{0,\ldots,n-2\}$.

(i)$\Rightarrow d_1 \geq 2w_{n-1} \geq w_{n-1} + w_0$. (ii)$\Rightarrow d_1 \geq w_{n-1} + w_i \geq w_{n-1} + w_0$. □

Lemma 31 *Let $X_{d_1,d_2} \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ of multidegree $\{d_1, d_2\}$ be a codimension 2 weighted complete intersection. Suppose X_{d_1,d_2} is quasismooth and not the intersection of a linear cone with another hypersurface. Assume $w_0 \leq w_1 \leq w_2 \leq w_3$ and $d_1 \leq d_2$. Then $|\mathbf{w}| \leq |\mathbf{d}|$. Furthermore, if $|\mathbf{w}| = |\mathbf{d}|$, then $w_3 < d_1$.*

Proof: First suppose $|\mathbf{w}| \geq |\mathbf{d}|$ and $d_1 < w_3$. From Lemma 30 we have $d_2 \geq w_3 + w_1$ and $d_1 \geq w_2 + w_0$, so under our assumption, $w_3 > w_2$. Then (iii) of Corollary 13 requires $d_2 \geq 2w_3$. Then

$$w_0 + w_1 + w_2 + w_3 \geq d_1 + d_2 \geq d_1 + 2w_3 > d_1 + w_2 + w_3 \Rightarrow w_0 + w_1 > d_1$$

which contradicts $d_1 \geq w_2 + w_0$.

Now suppose $|\mathbf{w}| > |\mathbf{d}|$. Then by Lemma 30

$$w_0 + w_1 + w_2 + w_3 > d_1 + d_2 \geq w_0 + w_1 + w_2 + w_3$$

which is a contradiction. Therefore $|\mathbf{w}| \leq |\mathbf{d}|$. □.

Example 32 *$X_{2,2} \subset \mathbb{P}(1,1,1,1)$ is quasismooth and has $|\mathbf{w}| = |\mathbf{d}|$.*

Example 33 *$X_{2,6} \subset \mathbb{P}(1,1,1,3)$ is quasismooth and has $|\mathbf{w}| < |\mathbf{d}|$ and $w_3 < d_1$*

Lemma 34 *Let $X_{d_1,d_2} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ of multidegree $\{d_1, d_2\}$ be a codimension 2 weighted complete intersection. Suppose X_{d_1,d_2} is quasismooth and not the intersection of a linear cone with another hypersurface. Assume $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4$ and $d_1 \leq d_2$. If $|\mathbf{w}| \geq |\mathbf{d}|$, then $w_4 < d_1$.*

Proof: Suppose, on the contrary, that $d_1 < w_4$. By Lemma 30 we have $d_2 \geq w_4 + w_1$ and $d_1 \geq w_3 + w_0$ respectively, so under our assumption, $w_4 > w_3$. From (iii) of Corollary 13 we have that $d_2 = kw_4 \geq 2w_4$. Then

$$w_0 + w_1 + w_2 + w_3 + w_4 \geq d_1 + d_2 \geq w_0 + w_3 + 2w_4$$

so

$$w_1 + w_2 \geq w_4 > d_1.$$

From (3) of Corollary 16 applied to $\{2,3,4\}$, we must have either (i) $mw_2+nw_3+pw_4 = d_1$ or both (ii) $m_0w_2 + n_0w_3 + p_0w_4 + w_0 = d_1$ and (iii) $m_1w_2 + n_1w_3 + p_1w_4 + w_1 = d_1$. (i) and (iii) are impossible since $w_1 + w_2 > d_1$. □

Example 35 $X_{2,6} \subset \mathbb{P}(1,1,1,1,3)$ *is quasismooth and has both* $d_1 < w_4$ *and* $|\mathbf{w}| < |\mathbf{d}|$.

3.2.2 Codimension 3

Lemma 36 *Let* $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0,\ldots,w_n)$ *of multidegree* $\{d_1,d_2,d_3\}$ *be a codimension 3 weighted complete intersection. Suppose* X_{d_1,d_2,d_3} *is quasismooth and not the intersection of a linear cone with another hypersurface. Assume* $w_0 \leq \cdots \leq w_n$ *and* $d_1 \leq d_2$. *Then (a)* $d_3 \geq w_n + w_2$, *(b)* $d_2 \geq w_{n-1} + w_1$, *and (c)* $d_1 \geq w_{n-2} + w_0$.

Proof: (a) Apply Theorem 20 to $I = \{n\}$. Then one of the following holds:

(i) $m_1w_n = d_1$, or

(ii) $m_2w_n = d_2$, or

(ii) $m_3w_n = d_3$, or

(iv) $m_4w_n+w_i = d_1$, $m_5w_n+w_j = d_2$, and $m_6w_n+w_k = d_3$ with $i,j,k \in \{0,\ldots,n-1\}$, $\{i,j,k\}$ distinct.

(i)⇒ $d_3 \geq d_2 \geq d_1 \geq 2w_n \geq w_n + w_2$. (ii)⇒ $d_3 \geq d_2 \geq 2w_n \geq w_n + w_2$. (iii)⇒ $d_3 \geq 2w_n \geq w_n + w_2$. (iv) If $k \geq 2$, $d_3 = m_6w_n + w_k \geq w_n + w_k \geq w_n + w_2$. If $k < 2$, then either

$i \geq 2$ in which case $d_3 \geq d_2 \geq d_1 = m_4 w_n + w_i \geq w_n + w_i \geq w_n + w_2$ or $j \geq 2$ in which case $d_3 \geq d_2 = m_5 w_n + w_j \geq w_n + w_j \geq w_n + w_2$.

(b) Apply Theorem 20 to $I = \{n-1, n\}$. Then at least one of the following holds:

(i) $m_1 w_{n-1} + p_1 w_n = d_1$, or

(ii) $m_2 w_{n-1} + p_2 w_n = d_2$ or

(iii) $m_3 w_{n-1} + p_3 w_n + w_i = d_1$ and $m_4 w_{n-1} + p_4 w_n + w_j = d_2$ with $i, j \in \{0, \dots, n-2\}$ and $i \neq j$.

(i)$\Rightarrow$ $d_2 \geq d_1 \geq 2w_{n-1} \geq w_{n-1} + w_1$. (i)$\Rightarrow$ $d_2 \geq 2w_{n-1} \geq w_{n-1} + w_1$. (iii) If $j > 0$, $d_2 \geq w_{n-1} + w_j \geq w_{n-1} + w_1$. If $j = 0$, then $i > 0$ so $d_2 \geq d_1 \geq w_{n-1} + w_i \geq w_{n-1} + w_1$.

(c) Apply Theorem 20 to $I = \{n-2, n-1, n\}$. Then at least one of the following holds:

(i) $m w_{n-2} + p w_{n-1} + q w_n = d_1$ or

(ii) $m_1 w_{n-2} + p_1 w_{n-1} + q_1 w_n + w_i = d_1$ with $i \in \{0, \dots, n-3\}$.

(i)$\Rightarrow$ $d_1 \geq 2w_{n-2} \geq w_{n-2} + w_0$. (ii)$\Rightarrow$ $d_1 \geq w_{n-2} + w_i \geq w_{n-2} + w_0$. □

Lemma 37 *Let $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ of multidegree $\{d_1, d_2, d_3\}$ be a codimension 3 weighted complete intersection. Suppose X_{d_1,d_2,d_3} is quasismooth and not the intersection of a linear cone with another subvariety. Assume $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4$ and $d_1 \leq d_2 \leq d_3$. Then $|\mathbf{w}| < |\mathbf{d}|$.*

Proof: Suppose on the contrary that $|\mathbf{w}| \geq |\mathbf{d}|$. Lemma 36 implies that $d_3 \geq w_4 + w_2$, $d_2 \geq w_3 + w_1$, and $d_1 \geq w_2 + w_0$. Then

$$w_0 + w_1 + w_2 + w_3 + w_4 \geq d_1 + d_2 + d_3 \geq (w_0 + w_2) + (w_1 + w_3) + (w_2 + w_4)$$

which is a contradiction. Therefore $|\mathbf{w}| < |\mathbf{d}|$. □

Lemma 38 *Let $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4, w_5)$ of multidegree $\{d_1, d_2, d_3\}$ be a codimension 3 weighted complete intersection. Suppose X_{d_1,d_2,d_3} is quasismooth and* *not the intersection of a linear cone with another subvariety. Then* $|\mathbf{w}| \leq |\mathbf{d}|$*. Suppose* $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4 \leq w_5$ *and* $d_1 \leq d_2 \leq d_3$*. If* $|\mathbf{w}| = |\mathbf{d}|$*, then* $w_5 < d_1$*.*

Proof: Suppose $|\mathbf{w}| \geq |\mathbf{d}|$. Lemma 36 implies that $d_3 \geq w_5 + w_2$, $d_2 \geq w_4 + w_1$, and $d_1 \geq w_3 + w_0$. We have

$$w_0 + w_1 + w_2 + w_3 + w_4 + w_5 \geq d_1 + d_2 + d_3 \geq (w_0 + w_3) + (w_1 + w_4) + (w_2 + w_5)$$

so $|\mathbf{w}| = |\mathbf{d}|$ Now suppose further that $w_5 > d_1$. Then, since $d_1 \geq w_3 + w_0$, $w_5 > d_1$. From (1) of Corollary 24, we have that $w_5 \mid d_2$ or $w_5 \mid d_3$. Either way $d_3 \geq 2w_5$ which would imply

$$\begin{aligned} w_0 + w_1 + w_2 + w_3 + w_4 + w_5 &= d_1 + d_2 + d_3 \\ &\geq (w_0 + w_3) + (w_1 + w_4) + 2w_5 \\ &> (w_0 + w_3) + (w_1 + w_4) + (w_2 + w_5) \end{aligned}$$

which is a contradiction. □

Example 39 $X_{2,2,2} \subset \mathbb{P}(1,1,1,1,1,1)$ *is quasismooth and has* $|\mathbf{w}| = |\mathbf{d}|$*.*

Example 40 $X_{2,2,6} \subset \mathbb{P}(1,1,1,1,1,3)$ *is quasismooth and has both* $w_5 > d_2$ *and* $|\mathbf{w}| < |\mathbf{d}|$*.*

Example 41 $X_{2,6,6} \subset \mathbb{P}(1,1,1,1,1,3)$ *is quasismooth and has both* $d_2 > w_5 > d_1$ *and* $|\mathbf{w}| < |\mathbf{d}|$*.*

Proposition 42 *1. Let* $X_{d_1,\ldots,d_{n-1}} \subset \mathbb{P}(w_0,\ldots,w_n)$ *be a weighted complete intersection curve.*

(a) If $|\mathbf{w}| > |\mathbf{d}|$ *then* $n = 2$*.*

(b) If $|\mathbf{w}| = |\mathbf{d}|$ *then* $n = 2$ *or* $n = 3$*.*

2. Let $X_{d_1,\ldots,d_{n-2}} \subset \mathbb{P}(w_0,\ldots,w_n)$ *be a weighted complete intersection surface.*

(a) If $|\mathbf{w}| > |\mathbf{d}|$ *then* $n = 3$ *or* $n = 4$*.*

(b) If $|\mathbf{w}| = |\mathbf{d}|$ *then* $n = 3$*,* $n = 4$*, or* $n = 5$*.*

Proof: These are implied by Lemmas 31, 34, 37, and 38 above. □

3.2.3 General Codimension

Lemma 43 *Let* $X_{d_1,\ldots,d_c} \subset \mathbb{P}(w_0,\ldots,w_n)$ *be a codimension c weighted complete intersection. Suppose* $X_{d_1,\ldots,d_c}$ *is quasismooth and is not the intersection of a linear cone with a codimension* $c-1$ *subvariety. Assume* $w_0 \le \cdots \le w_n$ *and* $d_1 \le \cdots \le d_c$*. Then* $d_c \ge w_n + w_{c-1}, \ldots, d_1 \ge w_{n-c+1} + w_0$*.*

Proof: Apply Theorem 27 here as Theorem 12 and Theorem 20 were used in the proofs of Lemma 30 and Lemma 36 respectively. □

Proposition 44 *Assume* $w_0 \le \cdots \le w_n$ *and* $d_1 \le \cdots \le d_c$*. Let* $p = n - c$*. 1. If a weighted complete intersection p-fold* $X_{d_1,\ldots,d_{n-p}} \subset \mathbb{P}(w_0,\ldots,w_n)$ *has* $|\mathbf{w}| > |\mathbf{d}|$ *then* $p + 1 \le n < 2p + 1$*.*

2. If a weighted complete intersection p-fold $X_{d_1,\ldots,d_{n-p}} \subset \mathbb{P}(w_0,\ldots,w_n)$ *has* $|\mathbf{w}| = |\mathbf{d}|$ *then* $p + 1 \le n < 2p + 2$*.*

[Note: Chen, Chen, and Chen prove a stronger result with a slightly different approach in [7, Theorem 1.3]]

Proof: By Lemma 43 we have $d_{n-p} \ge w_n + w_{n-p-1}, \ldots, d_1 \ge w_{p+1} + w_0$ so

$$d_1 + \cdots + d_{n-p} \ge (w_n + w_{n-p-1}) + \cdots + (w_{p+1} + w_0).$$

Then $p + 1 \leq n - p - 1 \Rightarrow |\mathbf{w}| < |\mathbf{d}|$ and $p < n - p - 1 \Rightarrow |\mathbf{w}| \leq |\mathbf{d}|$ □

3.3 Enumeration of 5-dimensional links of codimension 2 complete intersection singularities

3-dimensional positive links of hypersurface singularities were enumerated in [26] (with corrections in [2]). From Proposition 42 above, there are no positive 3-dimensional links of higher codimension. 5-dimensional positive links of hypersurface singularities were enumerated in [29, 30].

Based on the conditions in Corollary 16 and Lemma 34, along with positivity, we have the following result:

Lemma 45 *If* $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ *with* $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4$ *and* $\mathbf{d} = (d_1, d_2)$ *with* $d_1 \leq d_2$*, then if* $X_\mathbf{d} \subset \mathbb{P}(\mathbf{w})$ *is quasismooth and* $|\mathbf{w}| > |\mathbf{d}|$*,* $X_\mathbf{d}$ *belongs to one of 41 classes when categorized by how condition (1) of Corollary 16 is satisfied with respect to* $I = \{3\}$ *and* $I = \{4\}$ *and how condition (2) is satisfied with respect to* $I = \{3, 4\}$*. These classes are listed in Appendix D.*

Proof: Details are worked out in Appendix D. □

Based on these classes, a program was written in Mathematica 9 (see documentation at [1]) to implement the conditions of Corollary 16.

Theorem 46 *Let* $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ *and* $\mathbf{d} = (d_1, d_2)$ *with* $d_1 \leq d_2 \leq 600$*. If* $X_\mathbf{d} \subset \mathbb{P}(\mathbf{w})$ *is quasismooth and* $|\mathbf{w}| > |\mathbf{d}|$*, then* $(\mathbf{w}, \mathbf{d})$

(i) belongs to one of the one parameter families in Table B.1 of the Appendix,

(ii) belongs to one of the three parameter families in Table B.2, or

(iii) is one of the sporadic cases in Table B.3. □

Proposition 42 implies that positive quasismooth weighted projective surfaces (which yield 5 dimensional links) must be either codimension 1 or 2. Therefore, if this list were completed in the codimension 2 case, all 5 dimensional Sasakian algebraic links would be known.

Appendix C lists all the well-formed types amongst the list above.

Chapter 4

Topology of Links

4.1 Hypersurface singularities

In the hypersurface case, we have [24]:

Theorem 47 *(Milnor fibration theorem for hypersurface singularities) Let* $z_0 \in V_f$*, a hypersurface in* $\mathbb{C}^{n+1}$*. Then, for* $\varepsilon > 0$ *sufficiently small, the map*

$$\phi: S_\varepsilon^{2n+1}(z_0) \setminus L_f \to S^1$$

defined by

$$\phi(z) = \frac{f(z)}{|f(z)|}$$

is the projection map of a smooth fiber bundle, with smooth parallizable fiber. If z_0 *is an isolated singular point of* f*, then each fiber* F *has the homotopy type of a bouquet of n-spheres:* $S^n \vee \cdots \vee S^n$*.* $\overline{F}$ *is a compact manifold with boundary* L_f*. Furthermore,* L_f *is a smooth* $(n-2)$*-connected manifold of dimension* $2n-1$*, and* F *is a* $(n-1)$*-connected manifold of dimension* $2n$*.* □

The number of S^n is $\mu = \mu_f$, the *Milnor number* of the singularity. It is strictly positive and can be computed in general as the degree of the Gauss map

$$z \mapsto \frac{df(z)}{\|df(z)\|}.$$

Alternately,

$$\mu_f = \dim_{\mathbb{C}} \frac{\mathbb{C}[z_0, \dots, z_n]}{\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right)}.$$

$\mathrm{H}_n(F, \mathbb{Z})$ is free abelian of rank μ.

For a weighted homogeneous polynomial with an isolated critical point at the origin,

$$\mu = (q_0 - 1) \cdots (q_n - 1)$$

where $\{q_0, \dots, q_i\}$ are the rational weights $q_i = d/w_i$. μ must be an integer, even though the q_i may not be. This puts a constraint on the q_i, which is satisfied by the conditions for quasismoothness.

Consider the covering homotopy:

$$F_0 \times [0, 2\pi] \xrightarrow{h_t} S_\varepsilon^{2n+1} \smallsetminus L_f.$$

h_0 is the identity on F_0 and $h = h_{2\pi} : F_0 \to F_{2\pi} \cong F_0$ is the *characteristic homeomorphism* or *monodromy map* of the fiber. This induces an exact sequence:

$$0 \to \mathrm{H}_n(L_f, \mathbb{Z}) \to \mathrm{H}_n(F, \mathbb{Z}) \xrightarrow{\mathbb{1} - h_*} \mathrm{H}_n(F, \mathbb{Z}) \to \mathrm{H}_{n-1}(L_f, \mathbb{Z}) \to 0.$$

Thus, $\mathrm{H}_n(L_f, \mathbb{Z}) = \ker(\mathbb{1} - h_*)$ is free abelian. $\mathrm{H}_{n-1}(L_f, \mathbb{Z}) = \operatorname{coker}(\mathbb{1} - h_*)$ may have torsion, but the free part is $\mathrm{H}_{n-1}(L_f, \mathbb{Z}) \otimes \mathbb{Q} = \mathrm{H}_n(L_f, \mathbb{Z})$ by duality. Since L_f is $(n-2)$-connected, the only non-trivial homology is in dimensions 0, $n-1$, n, and $2n-1$. Let

$$\Delta(t) = \det(t\mathbb{1}_* - h_*).$$

$\Delta(t)$ is the *characteristic polynomial* of the monodromy map or *Alexander polynomial* of the link. $\Delta(1) \neq 0$ implies $\mathbb{1}_* - h_*$ is nonsingular so $\mathrm{H}_n(L_f, \mathbb{Z}) = 0$ and L_f is a rational homology sphere. If $|\Delta(1)| = 1$ then L_f is a homology sphere. More generally, $\Delta(1) \neq 0 \Rightarrow |\Delta(1)| = |\mathrm{H}_{n-1}(L_f, \mathbb{Z})|$.

Let

$$\Lambda_n = \mathrm{div}(t^n - 1) = \langle 1 \rangle + \langle \zeta_n \rangle + \cdots + \langle \zeta_n^{n-1} \rangle$$

where ζ_n is a primitive n^{th} root of unity. Rewrite $q_i = d/w_i = u_i/v_i$ with $\gcd(u_i, v_i) = 1$. Then

$$\mathrm{div}\Delta = \prod_i \left(\frac{\Lambda_{u_i}}{v_i} - 1 \right)$$

and $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $t-1$ in $\Delta(t)$, that is, the order of vanishing of $\Delta(t)$ at $t = 1$.

This can be calculated explicitly [25]:

Corollary 48 *Given the above situation:*

$$b_n(L_f) = \sum (-1)^{n+1-s} \frac{u_{i_1} \cdots u_{i_s}}{v_{i_1} \cdots v_{i_s} \mathrm{lcm}(u_{i_1}, \ldots, u_{i_s})}$$

where the sum is taken over all the 2^{n+1} *subsets* $\{i_1, \ldots, i_s\}$ *of* $\{0, \ldots, n\}$. □

4.2 Higher codimension singularities

In higher codimension, we have [22]:

Theorem 49 *(Generalized Milnor Fibration Theorem) Let* $f = (f_1, \ldots, f_c)$, $V_f = \{\mathbf{z} \in \mathbb{C}^{n+1} \big| f_1(\mathbf{z}) = \cdots = f_c(\mathbf{z}) = 0\}$. *Suppose* V_f *has an isolated singularity at the origin* *and let* $L_f = V_f \cap S_\varepsilon^{2n+1}$ *for* $\varepsilon > 0$ *sufficiently small that* S_ε^{2n+1} *only encloses the one singularity. Consider* f *as a map* $f : \mathbb{C}^{n+1} \to \mathbb{C}^c$*. Then the map*

$$\phi \colon S_\varepsilon^{2n+1} \smallsetminus L_f \to S^{2c-1}$$

defined by

$$\phi_j(z) = \frac{f_j(z)}{|f(z)|}$$

is the projection map of a smooth fiber bundle, with smooth parallizable fiber. The general fiber F *has the homotopy type of a bouquet of* $(n+1-c)$*-spheres.* $\overline{F}$ *is a compact manifold with boundary* L_f*. Furthermore,* L_f *is a smooth* $(n-c-1)$*-connected* $(2(n-c)+1)$ *manifold, and* F *is a* $(n-c)$*-connected* $(2(n+1-c))$*-manifold.* □

Again, the number of S^{n+1-c} is $\mu = \mu_f$, the Milnor number of the singularity. The Milnor number can be calculated as follows [21, Corollary 3.7.2]:

Theorem 50 *If for* $1 \le j \le c$ *the equations* $f_1 = 0, \ldots, f_j = 0$ *define a complete intersection with isolated singularity at 0 (that is,* $X_{d_1,\ldots,d_j} \subset \mathbb{P}^n$ *is quasismooth), then*

$$\mu(L_{d_1,\ldots,d_c}) = \sum_{j=1}^{c} (-1)^{c-j} \dim_{\mathbb{C}} A_j$$

where

$$A_j = \mathcal{O}_{\mathbb{C}^{n+1},0} / (\{\frac{\partial(f_1,\ldots,f_j)}{\partial(z_{\nu_1},\ldots,z_{\nu_j})} : 1 \le \nu_1 \le \cdots \le \nu_j \le n+1\}, f_1, \ldots, f_{j-1}) \mathcal{O}_{\mathbb{C}^{n+1},0}. \qquad \square$$

In particular, we have [27, Theorem 1]:

Corollary 51 *If* $\deg f_1 = \cdots = \deg f_c$ *then* $q_i = d_j / w_i$ *is independent of* j*. Let*

$$p(t) = \prod_{i=1}^{n+1} (q_i t + (q_i - 1)) = \beta_{n+1} t^{n+1} + \cdots + \beta_1 t + \beta_0.$$

Then

$$\mu(V_f) = \beta_{c-1} - \beta_{c-2} + \cdots + (-1)^{c-1} \beta_0. \qquad \square$$

Again, since L_f is $(n-c-1)$-connected, the only non-trivial homology is in dimensions 0, $n-c$, $n+1-c$, and $(2(n-c)+1)$.

The Alexander polynomial also generalizes easily [27, Theorem 2] if $\deg f_1 = \cdots = \deg f_c$ so $q_i = d_j/w_i$ is independent of j. Again, rewrite $q_i = d/w_i = u_i/v_i$ with $\gcd(u_i, v_i) = 1$, and let Λ_{u_i} be as above. Then

$$\mathrm{div}\Delta(t) = \sum_{r=0}^{c-1}\sum_{I_r}(-1)^{r-c+1}\frac{\Lambda_{u_{\sigma_0}}}{v_{\sigma_1}}\cdots\frac{\Lambda_{u_{\sigma_{r-1}}}}{v_{\sigma_{r-1}}}\left(\frac{\Lambda_{u_{\sigma_r}}}{v_{\sigma_r}}-1\right)\cdots\left(\frac{\Lambda_{u_{\sigma_n}}}{v_{\sigma_n}}-1\right)$$

where I_r runs over partitions of $\{0,\ldots,n\}$ into sets $\{\sigma_0,\ldots,\sigma_{r-1}\}$ and $\{\sigma_r,\ldots,\sigma_n\}$.

Dimca [9] provides another approach to finding the middle Betti numbers of links. Let $f_a = (f_1,\ldots,f_p,f)$ be an ordered set of weighted homogeneous polynomials of the same weights $\mathbf{w} = (w_0,\ldots,w_n)$, and degrees $\mathbf{d} = (d_1,\ldots,d_p,d)$, and suppose $f_0 = (f_1,\ldots,f_p)$. Suppose V_{f_0} and V_{f_a} have isolated singularities at the origin. If f is considered as a map $f : V_{f_0} \to \mathbb{C}$, then $V_{f_a} = f^{-1}(0)$. Let $L_{f_0} = V_{f_0} \cap S^{2n+1}$ and $L_{f_a} = V_{f_a} \cap S^{2n+1}$ and let X_{f_0} and X_{f_a} be the projective quasi-smooth weighted complete intersections defined by $f_0 = (f_1,\ldots,f_p)$ and $f_a = (f_1,\ldots,f_p,f)$. Let $\mathcal{O}_{n+1}$ be the $\mathbb{C}$-algebra of germs of holomorphic functions at the origin of $\mathbb{C}^{n+1}$, I_X the ideal generated by $f_1,\ldots,f_p$ in $\mathcal{O}_{n+1}$. Let Ω^k be the $\mathcal{O}_{n+1}$-module of germs of holomorphic k-forms at the origin of $\mathbb{C}^{n+1}$. If $a = (a_0,\ldots,a_n)$, let $x^a = x_0^{a_0}\cdots x_n^{a_n}$. The weights $\mathbf{w}$ induce a filtration on Ω^k such that a monomial form $\phi \in \Omega^k$,

$$\phi = x^a \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

has degree

$$\deg(\phi) = \deg(x^a) + w_{i_1} + \cdots + w_{i_k},$$

where

$$\deg(x^a) = a_0 w_0 + \cdots + a_n w_n.$$

In turn, this induces a filtration on the stalk at the origin of the sheaf of holomorphic k-forms relative to f,

$$\Omega_f^k = \Omega^k / I_X \cdot \Omega^k + \mathrm{d}f_1 \wedge \Omega^{k-1} + \cdots + \mathrm{d}f_p \wedge \Omega^{k-1} + \mathrm{d}f \wedge \Omega^{k-1}.$$

Then $\Omega_f^{n-p}/\mathrm{d}\Omega_f^{n-p-1}$ is a free $\mathcal{O}_1$-module of rank $\mu = \mu(X_{f_a})$, the Milnor number of X_{f_a} at the origin. Let

$$A := \Omega_f^{n-p}/\mathrm{d}\Omega_f^{n-p-1} + (f)\Omega_f^{n-p} = \Omega_{X_{f_a}}^{n-p}/\mathrm{d}\Omega_{X_{f_a}}^{n-p-1}$$

Then A is a μ-dimensional vector space over $\mathbb{C}$ with a natural grading $A = \bigoplus_{k\geq 0} A_k$ coming from the above filtration. Let

$$P(s) = \sum_{k\geq 0} (\dim A_k) s^k$$

be the Poincaré series of A. Then

$$P(s) = \mathrm{res}_{t=0} \frac{t^{-n-1+p}}{1+t} \left[\prod_{i=1}^{n+1} \frac{1+ts^{w_i}}{1-s^{w_i}} \prod_{j=1}^{p+1} \frac{1-s^{d_j}}{1+ts^{d_j}} \right]$$

where $d_{p+1} = d$.

Theorem 52 *([9, Theorem 1]) The complex monodromy operator h is diagonalizable and its eigenvalues are d-roots of unity. The multiplicity of the root $e^{-2\pi k i/d}$ is*

$$\sum_{j\equiv k \mod d} \dim A_j = d^{-1} \sum_{s^d=1} P(s) s^{-k}.$$

□

Then the following proposition [9, Proposition 6] allows computation of the Betti numbers of L_f, X_f, L_{f_0}, and X_{f_0}.

Proposition 53 *Given the notation above,*

(i) $b_k(X_f) = b_k(\mathbb{P}^n)$ for $k \neq n$ and $b_n(X_f) = b_n(\mathbb{P}^n) + b_n(L_f)$, where $\mathbb{P}^n$ is the

usual projective n-space.

(ii) For $n \geq 2$,

$$b_n(L_f) + b_{n-1}(L_{f_0}) = \dim\ker(h - \mathbb{1})$$

□

Remark 54 *As a special case, for $f_a = (f_1)$ ($p = 0$), $b_{n-1}(L_{f_0}) = \dim\ker(h - \mathbb{1})$, where h is the complex monodromy operator associated with the map $f_1 : \mathbb{C}^{n+1} \to \mathbb{C}$.*

Proposition 55 *([9, Proposition 3]) If f, f' both have same type $(\mathbf{w}, \mathbf{d})$, then L_f and $L_{f'}$ are homeomorphic.* □

4.3 Topology of Sasakian Structures in Dimension 5

Smale [28] classified all closed simply connected 5-manifolds admitting spin structures. Any such manifold M has the form

$$M = kM_\infty \# M_{m_1} \# \cdots \# M_{m_n}$$

where $M_\infty = S^2 \times S^3$, kM_∞ is the k-fold connected sum of M_∞,$k \in \mathbb{N}$, $\{m_i\}$ are positive integers with $1 \leq m_1 \mid \cdots \mid m_n$ and M_m is a rational homology sphere with $H_2(M_m, \mathbb{Z}) = \mathbb{Z}/m \otimes \mathbb{Z}/m$ if $m > 1$, and $M_1 = S^5$. For convenience, let $0M_\infty = S^5$. In other words, closed simply-connected 5-manifolds with vanishing second Stiefel-Whitney class are characterized completely by their homology.

If $X_{d_1,d_2} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ and $X_{d_1} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ are both quasi-smooth, the Milnor number of L_{f_1,f_2} can be computed by applying Theorem 50:

$$\mu(L_{f_1,f_2}) + \mu(L_{f_1}) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^5,0} / (\{\frac{\partial(f_1, f_2)}{\partial(z_i, , z_j)} : 0 \leq i \leq j \leq 4\}, f_1)\mathcal{O}_{\mathbb{C}^5,0}.$$

In particular, from Corollary 51, if $d_1 = d_2 = d$, we have:

$$\mu(L_{f_1,f_2}) = \left(-1 + \sum_{j=0}^{4} \frac{d}{d - w_j}\right)\left(\prod_{i=0}^{4} \frac{d - w_i}{w_i}\right).$$

The Dimca technique [9] may be applied in two different ways to compute the Betti numbers of 5-dimensional links of codimension 2 complete intersection singularities in $\mathbb{P}^4$, say $X_{d_1,d_2} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$. Let f_1, f_2 be sufficiently general weighted homogeneous polynomials of degrees d_1, d_2 respectively. First, suppose $X_{d_1} \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ is quasismooth, and h_1, h_2 the complex monodromy operators associated with the maps $f_1 : \mathbb{C}^5 \to \mathbb{C}$, $f_2 : X_{d_1} \to \mathbb{C}$, respectively. Then

$$b_2(L_{f_1,f_2}) = \dim\ker(h_2 - \mathbb{1}) - b_3(L_{f_1}) = \dim\ker(h_2 - \mathbb{1}) - \dim\ker(h_1 - \mathbb{1})$$

Example 56 *Consider* $\mathbf{w} = (1, 1, 2, 2, 2)$, $\mathbf{d} = (3, 4)$. $X_{\mathbf{w},\mathbf{d}}$ *is quasismooth.* $X_{(\mathbf{w},3)}$ *is not quasismooth ((3) of Corollary 9 is not satisfied for* $i = 0$, $j = 1$*).* $X_{(\mathbf{w},4)}$ *is quasismooth, however. Let* $d/w_i = u_i/v_i$ *with* $\gcd(u_i, v_i) = 1$*. Then* $u_0 = u_1 = 4$, $u_2 = u_3 = u_4 = 2$*, and* $v_0 = v_1 = v_2 = v_3 = v_4 = 1$*.* $b_3(L_{(\mathbf{w},4)})$ = *the order of vanishing of* $\Delta(t)$ *at* $t = 1$*. From Corollary 48:*

$$\begin{aligned} b_3 \;=\; & (-1)^5(1) \\ & + (-1)^4\left(2\left(\frac{4}{4}\right) + 3\left(\frac{2}{2}\right)\right) \\ & + (-1)^3\left(\frac{4\cdot 4}{4} + 6\left(\frac{4\cdot 2}{4}\right) + 3\left(\frac{2\cdot 2}{2}\right)\right) \\ & + (-1)^2\left(3\left(\frac{4\cdot 4\cdot 2}{4}\right) + 6\left(\frac{4\cdot 2\cdot 2}{4}\right) + \frac{2\cdot 2\cdot 2}{2}\right) \\ & + (-1)\left(3\left(\frac{4\cdot 4\cdot 2\cdot 2}{4}\right) + 2\left(\frac{4\cdot 2\cdot 2\cdot 2}{4}\right)\right) \\ & + \frac{4\cdot 4\cdot 2\cdot 2\cdot 2}{4} \\ =\; & 2 \end{aligned}$$

$$\begin{aligned} P(s) &= \mathrm{res}_{t=0}\frac{t^{-4}}{1+t}\left[{\frac{1+ts}{1-s}}^{2}\cdot{\frac{1+ts^2}{1-s^2}}^{3}\cdot\frac{1-s^3}{1+ts^3}\cdot\frac{1-s^4}{1+ts^4}\right] \\ &= 3s^6+5s^7+7s^8+6s^9+3s^{10}+s^{11} \end{aligned}$$

Then $\dim\ker(h-\mathbb{1}) = \sum_{j\equiv 0(\mod (3))} a_j = 3+6 = 9$.

Therefore, $b_2(L_{\mathbf{w},\mathbf{d}}) = 9-2 = 7$.

Another technique is necessary if neither $X_{d_1} \subset \mathbb{P}(\mathbf{w})$ or $X_{d_2} \subset \mathbb{P}(\mathbf{w})$ is quasismooth. If $X_{d_1,d_2,d_3} \subset \mathbb{P}(\mathbf{w})$ is quasismooth for some $f : X_{d_1,d_2} \to \mathbb{C}$, f of degree d_3, then in fact, X_{d_1,d_2,d_3} is a smooth curve and

$$b_1(L_{f_1,f_2,f_3}) = b_1(X_{d_1,d_2,d_3}) = 2p_g(X_{d_1,d_2,d_3})$$

where $p_g(Y)$ is the geometric genus. Then

$$b_2(L_{f_1,f_2}) + 2p_g(X_{d_1,d_2,d_3}) = \dim\ker(h-\mathbb{1})$$

A general formula for the genus $p_g(Y)$ in terms of $(\mathbf{w},\mathbf{d})$ is given below in Corollary 60.

Such an f always exists, as, in particular, $d_3 = 2\cdot\mathrm{lcm}(w_0,w_1,w_2,w_3,w_4)$ will work ($d_3 = \mathrm{lcm}(w_0,w_1,w_2,w_3,w_4)$ if $\mathrm{lcm}(w_0,w_1,w_2,w_3,w_4) > w_4$). That is, if $X_{d_1,d_2} \subset \mathbb{P}(w_0,w_1,w_2,w_3,w_4)$ is quasismooth, then $X_{d_1,d_2,d_3} \subset \mathbb{P}(w_0,w_1,w_2,w_3,w_4)$ is as well.

Example 57 *Consider* $\mathbf{w} = (2,3,3,4,4)$, $\mathbf{d} = (6,8)$. $X_{\mathbf{w},\mathbf{d}}$ *is quasismooth. On the other hand,* $X_{(\mathbf{w},6)}$ *is not quasismooth (Corollary 9 is not satisfied for* $\{x_3,x_4\}$*) and* $X_{(\mathbf{w},8)}$ *is not quasismooth (Corollary 9 is not satisfied for* $\{x_1,x_2\}$*).* $X_{(\mathbf{w},(6,8,7))}$ *is quasismooth as well, with* $|\mathbf{d}|-|\mathbf{w}| = 5$.

$$P(s) = \mathrm{res}_{t=0}\frac{t^3}{1+t}\left[\frac{1+ts^2}{1-s^2}\frac{1+ts^3}{1-s^3}\frac{1+ts^3}{1-s^3}\frac{1+ts^4}{1-s^4}\frac{1+ts^4}{1-s^4}\frac{1-s^6}{1+ts^6}\frac{1-s^8}{1+ts^8}\frac{1-s^7}{1+ts^7}\right]$$

$$P(s) = s^8+4s^9+4s^{10}+6s^{11}+7s^{12}+9s^{13}+8s^{14}+9s^{15}+6s^{16}+6s^{17}+4s^{18}+3s^{19}+s^{20}+s^{21}$$

Then $\dim\ker(h - \mathbb{1}) = 8 + 1 = 9$. *Then (see Corollary 60 below):* $p_g(X_{6,7,8}) = a_{|\mathbf{d}|-|\mathbf{w}|}$ *in the series*

$$\sum_{i=0}^{\infty} a_i t^i = \frac{(1-t^6)(1-t^7)(1-t^8)}{(1-t^2)(1-t^3)(1-t^3)(1-t^4)(1-t^4)} = 1 + t^2 + 2s^3 + 3s^4 + 2s^5 + O(s^6)$$

$b_2(L_{\mathbf{w},\mathbf{d}}) = 9 - 2(2) = 5.$

Kollár has shown in [19] that if a Smale manifold M admits a Sasakian structure, then

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \sum_i (\mathbb{Z}_{m_i})^{2g(D_i)}$$

where $g(D_i)$ is the genus, and m_i the ramification index, of the branch divisor D_i. Here k is the second Betti number of M which we showed earlier how to calculate. $2g(D) \neq 0$ precisely when D is non-rational.

Kollár has also shown in [19]:

Theorem 58 *If M is a 5-dimensional simply connected positive Sasakian manifold, then the torsion subgroup of* $\mathrm{H}_2(M, \mathbb{Z})$ *is one of the following:* $(\mathbb{Z}_m)^2$ *for any* $m \in \mathbb{Z}^+$, $(\mathbb{Z}_5)^4$, $(\mathbb{Z}_4)^4$, $(\mathbb{Z}_3)^4$, $(\mathbb{Z}_3)^6$, $(\mathbb{Z}_3)^8$, *or* $(\mathbb{Z}_2)^{2n}$ *for any* $n \in \mathbb{Z}^+$ *(where* $\mathbb{Z}_1$ *denotes trivial torsion).*□.

The genus of the branch divisor can be computed by a method given by Dolgachev in [10]: For X a quasismooth weighted complete intersection, define the Poincaré series of X by

$$P_X(t) = \sum_{m=0}^{\infty} a_m t^m = \sum_{m=0}^{\infty} (dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(m)))t^m.$$

Theorem 59 *If X is a quasismooth weighted complete intersection with weights* $\mathbf{w} = (w_0, \ldots, w_n)$ *and multidegree* $\mathbf{d} = (d_1, \ldots, d_c)$, *then*

$$P_X(t) = \frac{\prod_{i=1}^{c}(1-t^{d_i})}{\prod_{j=0}^{n}(1-t^{w_j})}. \qquad \square$$

Corollary 60 *The genus, $p_g(X) = \dim_{\mathbb{C}} H^{\dim X}(X, \mathcal{O}_X)$, is given by*

$$p_g(X) = a_{|\mathbf{d}|-|\mathbf{w}|} \qquad \square$$

Example 61 *1. The formula of Corollary 60 reduces to the standard formula in the case of a curve X_d in the standard $\mathbb{P}^2$:*

$$P_{X_d}(t) = \frac{(1-t^d)}{(1-t)^3} = (1-t^d)\sum_{m=0}^{\infty}\binom{m+2}{2} t^m$$

and $|\mathbf{d}| - |\mathbf{w}| = d-3$. So the coefficient of t^{d-3} in $P_{X_d}(t)$ is

$$\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$$

2. The formula of Corollary 60 also reduces to the standard formula in the case of a curve X_{d_1,d_2} in the standard $\mathbb{P}^3$:

$$P_{X_{d_1,d_2}}(t) = \frac{(1-t^{d_1})(1-t^{d_2})}{(1-t)^4} = (1-t^{d_1}-t^{d_2}+t^{d_1+d_2})\sum_{m=0}^{\infty}\binom{m+3}{3} t^m$$

and $|\mathbf{d}| - |\mathbf{w}| = d_1+d_2-4$. So the coefficient of $t^{d_1+d_2-4}$ in $P_{X_{d_1,d_2}}(t)$ is

$$a_{d_1+d_2-4} = \binom{d_1+d_2-1}{3} - \binom{d_2-1}{3} - \binom{d_1-1}{3}$$

$$a_{d_1+d_2-4} = \frac{d_1d_2(d_1+d_2-4)}{2} + 1$$

3. Consider $(\mathbf{w}, \mathbf{d}) = ((3,4,4,6,6),(10,12))$. Since $\gcd(w_1,w_2,w_3,w_4) = 2$, there is a branch divisor D_0 of ramification index 2. $D_0 \cong X_{((2,2,3,3),(5,6))}$. The genus of D_0 is the coefficient of t^1 in

$$\frac{(1-t^5)(1-t^6)}{(1-t^2)(1-t^2)(1-t^3)(1-t^3)} = 1 + 2t^2 + 2t^3 + O(t^4).$$

Thus $g_p(D_0) = 0$ and thus D_0 does not contribute to torsion.

4. Consider $(\mathbf{w},\mathbf{d}) = ((6,8,8,10,15),(16,30))$. *Since* $\gcd(w_0,w_1,w_2,w_3) = 2$, *there is a branch divisor* D_4 *of ramification index* 2. *We must compute the genus of the complete intersection* $X_{((3,4,4,5),(8,15))}$ *which is the coefficient of* t^7 *in*

$$\frac{(1-t^8)(1-t^{15})}{(1-t^3)(1-t^4)(1-t^4)(1-t^5)} = 1 + t^3 + 2t^4 + t^5 + t^6 + 2t^7 + O(t^8).$$

Then the torsion component of $\mathrm{H}_2(L,\mathbb{Z})$ *is* $\mathbb{Z}_2^4$.

Programs in Mathematica 9 (see documentation at [1]) were written to compute $\mathrm{H}_2(L,\mathbb{Z})$ based on the above techniques for each of the entries of Tables B.1 and B.3 and selected entries in Table B.2 in the Appendix, and are listed in Tables B.1, B.2.1, and B.3 of the Appendix. This shows:

Corollary 62 *There exist positive Sasakian structures on links of weighted complete intersection singularities of the following topological types:*

(i) $k\#(S^2 \times S^3)$, *for all* $k \geq 0$,

(ii) kM_2, *for all* $k \geq 1$,

(iii) M_3, $2M_3$,

(iv) M_4,

(v) $M_\infty \# M_{2k+1}$, *for all* $k \geq 1$,

(vi) $5M_\infty \# M_k$, *for all* $k \geq 2$,

(vii) $M_\infty \# kM_2$, *for all* $k \geq 1$,

(viii) $2M_\infty \# M_2$, $2M_\infty \# 3M_2$,

(ix) $M_\infty \# M_3$, $2M_\infty \# M_3$, $M_\infty \# 2M_3$,

(x) $M_\infty \# M_4$,

(xi) $M_\infty \# M_5$

Furthermore, there exist countably many $(\mathbf{w},\mathbf{d})$ *types on:*

(i) $k\#(S^2 \times S^3)$, *for all* $k \geq 0$,

(ii) $M_\infty \# kM_2$, *for all* $k \geq 1$,

(iii) $M_\infty \# M_3$, $2M_\infty \# M_2$, $5M_\infty \# M_2$. □

Structures on all of these topologies were previously exhibited in the hypersurface singularity case, for example, in [3] and [5].

Chapter 5

Sasaki-Einstein Structures on Links

5.1 Einstein metrics

Recall, a Riemannian metric g is called an *Einstein* metric if there is a constant λ such that $\mathrm{Ric}_g = \lambda g$. If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure and g is Einstein, then $\mathcal{S}$ is *Sasaki-Einstein*.

If $\mathcal{S}$ is a quasi-regular Sasaki manifold, then [3, Theorem 7.1.3] then the space of leaves $M/\mathcal{F}_\xi$ is an almost Kähler orbifold $(\mathcal{Z}, h)$. Then [3, Theorem 11.1.3, Corollary 11.1.4]:

Theorem 63 *Let M be a compact manifold of dimension $2n+1$ with a quasi-regular K-contact structure (ξ, η, Φ, g). Then g is Sasaki-Einstein if and only if h is Kähler-Einstein with scalar curvature $4n(n+1)$. In this case, g has Einstein constant $2n$.* □

5.2 Existence Results

Demailly and Kollár [8], give sufficient conditions for a log del Pezzo surface to possess a Kähler-Einstein metric. The link of the singularity at the origin of the affine cone over a log del Pezzo surface has a positive Sasakian structure, which is Einstein if the log del Pezzo surface has a Kähler-Einstein orbifold metric. Johnson and Kollár [16] obtain a sufficient bound on weights and degrees of weighted projective hypersurfaces to guarantee the existence of Kähler-Einstein metrics. This bound was extended to the existence of Sasaki-Einstein metrics on 5-manifolds given as links of hypersurface singularities by Boyer, Galicki, and Nakamaye [4].

Definition 64 *Let $\mathcal{Z}$ be a log del Pezzo surface and D a $\mathbb{Q}$-divisor on $\mathcal{Z}$. Then the pair $(\mathcal{Z}, D)$ is* klt *or* Kawamata log-terminal *if for each local uniformizing neighborhood $\tilde{U}$ there exists a log resolution of singularities $\mu : X \to \tilde{U}$ and a $\mathbb{Q}$-divisor $D_X = \sum a_i E_i$ on X such that*

$$K_X \equiv_n \mu^*(K_{\tilde{U}}^{orb} + D) + D_X$$

with $a_i > -1$ for all i.

Then [8]:

Theorem 65 *Let X be an n dimensional Fano variety (possibly with quotient singularities). Assume there is an $\epsilon > 0$ such that*

$$(X, \frac{n+\epsilon}{n+1} D)$$

is klt for every effective $\mathbb{Q}$-divisor $D \equiv -K_X$. Then X has a Kähler-Einstein metric.
□

Johnson and Kollár give sufficient conditions for (X, D) to be klt given a surface X and a $\mathbb{Q}$-divisor D on X. Let X be a surface with quotient singularities $P_i \in X$, and write these locally analytically as

$$p_i : (\mathbb{C}^2, Q_i) \to (\mathbb{C}^2/G_i, P_i) \cong (X, P_i),$$

where $G_i \subset GL(2, \mathbb{C})$ is a finite subgroup (see [18] and [20]). Let D be an effective $\mathbb{Q}$-divisor on X. Then (X, D) is klt if the following three conditions hold:

(i) D does not contain an irreducible component with coefficient ≥ 1.

(ii) $\mathrm{mult}_P D \leq 1$ at every smooth point $P \in X$.

(iii) $\mathrm{mult}_{Q_i} D_i \leq 1$ for every i where $D_i := p_i^* D$.

They give the following estimate for multiplicity of points [16, Proposition 11]:

Proposition 66 *Let $X \subset \mathbb{P}(w_0, \ldots, w_n)$ be a d-dimensional subvariety of weighted projective space. Assume that X is not contained in the singular locus and $w_0 \leq \cdots \leq w_n$. Let $X_i \subset \mathbb{C}^n$ denote the preimage of X in the orbifold chart*

$$\mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z} \cong \mathbb{P}(w_0, \ldots, w_n) \setminus (x_i = 0).$$

Then for every i and every $p \in X_i$,

$$\mathrm{mult}_p X_i \leq (w_n \cdots w_{n-d})(X \cdot \mathcal{O}(1)^d).$$

Moreover, if $Z \neq (z_0 = \cdots z_{n-d-1} = 0)$ then a stronger inequality holds:

$$\mathrm{mult}_p X_i \leq (w_n \cdots w_{n-d+1} w_{n-d-1})(X \cdot \mathcal{O}(1)^d). \qquad \square$$

The following is the Sasakian equivalent of [16, Corollary 13] (see also [5, Lemma 2.3]).

Lemma 67 *Let $L(\mathbf{w}, d)$ be a link of a weighted homogeneous hypersurface with weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3)$ ordered as $w_0 \leq w_1 \leq w_2 \leq w_3$. Let $\mathcal{Z}_{\mathbf{w}}$ denote* *the corresponding projective algebraic orbifold. Furthermore, let $I = |\mathbf{w}| - d$ denote the Fano index. Then*

(1) The 5-manifold $L(\mathbf{w}, d)$ admits a Sasaki-Einstein metric if $2Id < 3w_0w_1$.

(2) If the line $z_0 = z_1 = 0$ does not lie in $\mathcal{Z}_{\mathbf{w}}$ and the weaker condition $2Id < 3w_0w_2$ holds, then $L(\mathbf{w}, d)$ admits a Sasaki-Einstein metric.

Proof: Let $D \equiv -\frac{2+\epsilon}{3}K_X$ be $\mathbb{Q}$-effective. (1) Then we have, from Proposition 66,

$$\text{mult}_P D_i \leq (w_3w_2)(D \cdot \mathcal{O}(1)) \leq (w_3w_2)dI(\frac{2+\epsilon}{3w_0w_1w_2w_3})$$

so $\text{mult}_P D_i \leq 1$ if $\frac{2dI}{3w_0w_1} < 1$

(2) If the line $\{z_0 = z_1 = 0\}$ does not lie in $\mathcal{Z}_{\mathbf{w}}$, then $\{z_0 = z_1 = 0\}$ does not lie in D either, and from Proposition 66,

$$\text{mult}_P D_i \leq (w_3w_1)(D \cdot \mathcal{O}(1)) \leq (w_3w_1)dI(\frac{2+\epsilon}{3w_0w_1w_2w_3})$$

so $\text{mult}_P D_i \leq 1$ if $\frac{2dI}{3w_0w_2} < 1$.

□

This easily extends to the codimension 2 case since we can still use Proposition 66.

Lemma 68 *Let $L(\mathbf{w}, \mathbf{d})$ be a link of a weighted complete intersection with weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ ordered as $w_0 \leq w_1 \leq w_2 \leq w_3 \leq w_4$ and multidegree $\mathbf{d} = (d_1, d_2)$. Let $\mathcal{Z}_{\mathbf{w}}$ denote the corresponding projective algebraic orbifold. Furthermore, let $I = |\mathbf{w}| - |\mathbf{d}|$ denote the Fano index. Then*

(1) The 5-manifold $L(\mathbf{w}, \mathbf{d})$ admits a Sasaki-Einstein metric if $2Id_1d_2 < 3w_0w_1w_2$.

(2) If the line $z_0 = z_1 = z_2 = 0$ does not lie in $\mathcal{Z}_{\mathbf{w}}$ and the weaker condition $2Id_1d_2 < 3w_0w_1w_3$ holds, then $L(\mathbf{w}, \mathbf{d})$ admits a Sasaki-Einstein metric.

Proof: Let $D \equiv -\frac{2+\epsilon}{3}K_X$ be $\mathbb{Q}$-effective. (1) Then we have, from Proposition 66,

$$\text{mult}_P D_i \leq (w_4w_3)(D \cdot \mathcal{O}(1)) \leq (w_4w_3)d_1d_2I(\frac{2+\epsilon}{3w_0w_1w_2w_3w_4})$$

so $\text{mult}_P D_i \leq 1$ if $\frac{2d_1d_2I}{3w_0w_1w_2} < 1$

(2) If $\{z_0 = z_1 = z_2 = 0\}$ does not lie in $\mathcal{Z}_{\mathbf{w}}$, then $\{z_0 = z_1 = z_2 = 0\}$ does not lie in D either, and from Proposition 66,

$$\text{mult}_P D_i \leq (w_4w_2)(D \cdot \mathcal{O}(1)) \leq (w_4w_2)d_1d_2I(\frac{2+\epsilon}{3w_0w_1w_2w_3w_4})$$

so $\text{mult}_P D_i \leq 1$ if $\frac{2d_1d_2I}{3w_0w_1w_3} < 1$.

□

This is consistent with Lemma 67 above. If $w_0 = 1$, and $d_2 = 1$, and $\mathcal{Z}_{\mathbf{w}} = \mathcal{Z}'_{\mathbf{w}} \cap Z_0$ where Z_0 is the hyperplane $z_0 = 0$ and $\mathcal{Z}'_{\mathbf{w}}$ has a defining equation not containing z_0, then the conjecture applied to $\mathcal{Z}_{\mathbf{w}}$ reduces to Lemma 67 applied to $\mathcal{Z}'_{\mathbf{w}}$.

The third result in [5, Lemma 2.3] requires a slightly different argument (but see the proof of Proposition 66 in [16, Proposition 11]). It also can be extended to the complete intersection case.

Lemma 69 *(1) In the hypersurface case, if the point* $(0,0,0,1)$ *does not lie in* $\mathcal{Z}_{\mathbf{w}}$ *and the weaker condition* $2Id < 3w_0w_3$ *holds, then* $L(\mathbf{w}, d)$ *admits a Sasaki-Einstein metric.*

(2) In the codimension 2 case, if the point $(0,0,0,0,1)$ *does not lie in* $\mathcal{Z}_{\mathbf{w}}$ *and the weaker condition* $2Id_1d_2 < 3w_0w_1w_4$ *holds, then* $L(\mathbf{w}, \mathbf{d})$ *admits a Sasaki-Einstein metric.*

Proof: (1) Again, let $D \equiv -\frac{2+\epsilon}{3}K_X$ be $\mathbb{Q}$-effective, and let $P \in D$. Let $C(P) \subset \mathbb{C}^4$ be the cone over P with vertex 0. Let $D_i = D \cap \{z_i = 1\}$. $\text{mult}_0 C(P) = \text{mult}_P D \leq w_3$. $Q = (0,0,0,1)$ is the only point in $\mathcal{Z}_{\mathbf{w}}$ with $\text{mult}_Q \mathcal{Z}_{\mathbf{w}} = w_3$, all other points having multiplicity $\leq w_2$, so in fact, $\text{mult}_0 C(P) = \text{mult}_P D \leq w_2$. Let $D_i = \sum a_j V_j$ where each V_j is irreducible. Now, $(0,0,0,1) \notin D_i$ and $(0,0,0,1) \in \{z_0 = z_1 = 0\}$ so $\{z_0 = z_1 = 0\} \notin D_i$. Then for each j, either $\{z_0 = 0\}$ meets V_j properly or $\{z_1 = 0\}$ meets V_j properly, and in either case, with multiplicity $w_j \leq w_1$ at any point.

Therefore,

$$\begin{aligned}\text{mult}_0 C(D_i) &= \sum_j \text{mult}_0 C(V_j) \\ &\leq \sum_j w_j V_j \\ &\leq w_1 \text{mult}_0 C(\mathcal{Z}_{\mathbf{w}}) \\ &\leq w_2 w_1 (D \cdot \mathcal{O}(1)) \\ &\leq (w_2 w_1) dI(\frac{2+\epsilon}{3w_0 w_1 w_2 w_3})\end{aligned}$$

so $\text{mult}_P D_i \leq 1$ if $\frac{2dI}{3w_0 w_3} < 1$.

(2) Again, let $D \equiv -\frac{2+\epsilon}{3} K_X$ be $\mathbb{Q}$-effective, and let $P \in D$. Let $C(P) \subset \mathbb{C}^5$ be the cone over P with vertex 0. $\text{mult}_0 C(P) = \text{mult}_P D \leq w_4$. $Q = (0,0,0,0,1)$ is the only point with $\text{mult}_Q \mathcal{Z}_{\mathbf{w}} = w_4$, all other points having multiplicity $\leq w_3$, so in fact, $\text{mult}_0 C(P) = \text{mult}_P D \leq w_3$. Let $D = \sum a_j V_j$ where each V_j is irreducible. Now, $(0,0,0,0,1) \notin D$ and $(0,0,0,0,1) \in \{z_0 = z_1 = z_2 = 0\}$ so $\{z_0 = z_1 = z_2 = 0\} \notin D$. Then for each j, at least one of $\{z_0 = 0\}$, $\{z_1 = 0\}$, or $\{z_2 = 0\}$ meets V_j properly, and in any case, with multiplicity $\leq w_2$ at any point. Therefore,

$$\begin{aligned}\text{mult}_0 C(D_i) &= \sum_j \text{mult}_0 C(V_j) \\ &\leq \sum_j w_j V_j \\ &\leq w_2 \text{mult}_0 C(\mathcal{Z}_{\mathbf{w}}) \\ &\leq w_3 w_2 (D \cdot \mathcal{O}(1)) \\ &\leq (w_3 w_2) d_1 d_2 I(\frac{2+\epsilon}{3w_0 w_1 w_2 w_3 w_4})\end{aligned}$$

so $\text{mult}_P D_i \leq 1$ if $\frac{2d_1 d_2 I}{3w_0 w_1 w_4} < 1$. □

Table A lists the 154 cases which meet the bounds of Lemmas 68 or 69 with $d_1 \leq d_2 \leq 600$, along with their Smale type. 36 families are structures on S^5. 71 are structures on $S^2 \times S^3$. 20 are structures on $2\#(S^2 \times S^3)$. 21 are structures on

$3\#(S^2 \times S^3)$. There is one family on the rational homology sphere M_2 and two on M_3. There are 2 families on the connected sum $M_\infty \# M_2$ and one on $M_\infty \# 2M_2$. Sasaki-Einstein structures on all of these topologies were previously exhibited in the hypersurface singularity case (see [3] and [5]).

5.3 An obstruction

One important obstruction to the existence of a Sasaki-Einstein metric is the Lichnerowicz obstruction, [3, Corollary 11.3.11ff.]:

Theorem 70 *Let M^{2n-1} be a compact manifold with a Sasaki-Einstein structure $\mathcal{S} = (\xi, \eta, \Phi, g)$. Then the first non-zero eigenvalue λ_1 of the Laplace operator Δ_g is bounded: $\lambda_1 \geq 2n-1$ and $\lambda_1 = 2n-1$ if and only if $\mathcal{S}$ is the standard Sasaki-Einstein structure on S^{2n-1}.* □

Let $Y = C(M)$ be the associated cone with the induced Kähler structure. Let f be a holomorphic function on Y with $\mathcal{L}_\xi f = cif$ where $c > 0$ is a real constant called the *charge* of f with respect to ξ. We have $\Delta_Y f = 0$, so at $r = 1$,

$$\Delta_Y = \frac{1}{r^2}\Delta_M - \frac{1}{r^{2n-1}}\frac{\partial}{\partial r}\left(r^{2n-1}\frac{\partial}{\partial r}\right)$$

so $\Delta_M \tilde{f} = \lambda \tilde{f}$, where $\lambda = c[c + (2n-2)]$ and $f = r^c \tilde{f}$. Then, if (M, g_M) is Sasaki-Einstein, by Theorem 70 $\lambda_1 \geq 2n-1$, so $c \geq 1$.

The following is from [11]. Now consider a link $L(\mathbf{w}, d)$ of a hypersurface in weighted projective space with isolated singularity, defined by a weighted homogeneous polynomial F of weights $\mathbf{w} = (w_0, \dots, w_n)$, with $w_0 \leq \dots \leq w_n$. Let X be the affine cone $X = C(L)$. Let $\{U_j\}$ be a cover of X given by $U_j = \{z \in X | \frac{\partial F}{\partial z_j} \neq 0\}$. Then on each U_j we can define a nowhere zero holomorphic $(n,0)$-form

$$\Omega = \frac{dz_0 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n}{\partial F / \partial z_j}$$

If ζ is the holomorphic vector field on $X \smallsetminus \{0\}$ with $\mathcal{L}_\zeta z_j = w_j i z_j$ for each $j = 0, \dots, n$, then $\mathcal{L}_\zeta \Omega = (|w| - d) i\Omega$. If there is a Ricci-flat Kähler metric on X then ζ normalizes to $\xi = \frac{n}{|w|-d}\zeta$.

Proposition 71 *If $L(\mathbf{w}, d)$ is a smooth link, then if the index $I = |w| - d > nw_0$, $L(\mathbf{w}, d)$ cannot admit any Sasaki-Einstein structure.*

Proof: In fact, z_0 has charge $c = \frac{nw_0}{|w|-d}$ with respect to ξ. □

Corollary 72 *In particular, in the 5-dimensional link case, $n = 3$, so $L(\mathbf{w}, d)$ cannot admit any Sasaki-Einstein structure if $I > 3w_0$.* □

This generalizes to the codimension 2 complete intersection case in the following way. Let $\mathbf{w} = (w_0, \dots, w_{n+1})$. Suppose $X_{d_1,d_2} \subset \mathbb{P}(\mathbf{w})$ is quasismooth, $X = C(X_{d_1,d_2})$ and suppose (f_1, f_2) generate I_X. Then the sets $\{U_{j,k}\}$ cover $X \smallsetminus \{0\}$ where $U_{j,k} = \{z \in X | \frac{\partial f_1}{\partial z_j}\frac{\partial f_2}{\partial z_k} - \frac{\partial f_1}{\partial z_k}\frac{\partial f_2}{\partial z_j} \neq 0\}$. Then on $\{U_{j,k}\}$

$$\Omega = \frac{dz_0 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge \widehat{dz_k} \wedge \dots \wedge dz_{n+1}}{\frac{\partial f_1}{\partial z_j}\frac{\partial f_2}{\partial z_k} - \frac{\partial f_1}{\partial z_k}\frac{\partial f_2}{\partial z_j}}$$

defines a nowhere zero $(n, 0)$-form on X. Again, as above, if ζ is the holomorphic vector field on $X \smallsetminus \{0\}$ with $\mathcal{L}_\zeta z_j = w_j i z_j$ for each $j = 0, \dots, n+1$, then $\mathcal{L}_\zeta \Omega = (|w| - |d|) i\Omega$. If there is a Ricci-flat Kähler metric on X then ζ normalizes to $\xi = \frac{n}{|w|-|d|}\zeta$.

Then we have:

Proposition 73 *If $L(\mathbf{w}, \mathbf{d})$ is a smooth link, then if the index $I = |w| - |d| > nw_0$, $L(\mathbf{w}, \mathbf{d})$ cannot admit any Sasaki-Einstein structure.*

Proof: In fact, z_0 has charge $c = \frac{nw_0}{|w|-|d|}$ with respect to ξ. □

Corollary 74 *Again, in the* 5*-dimensional link case,* $n = 3$*, so* $L(\mathbf{w}, d)$ *cannot admit any Sasaki-Einstein structure if* $I > 3w_0$*.* □

A similar argument will generalize this to any codimension complete intersection link.

Tables B.1, B.2, and B.3 of the Appendix indicate when this obstruction occurs in the 23438 types with $d_1 \leq d_2 \leq 600$.

Appendices

Appendix A

Types satisfying the bounds of Lemmas 68 or 69

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(5, 16, 24, 28, 32)$	$(48, 56)$	1	$2M_\infty$
$(6, 6, 10, 10, 15)$	$(16, 30)$	1	M_∞
$(6, 7, 9, 11, 14)$	$(18, 28)$	1	$3M_\infty$
$(6, 8, 8, 10, 15)$	$(16, 30)$	1	$M_\infty \# 2M_2$
$(6, 8, 9, 11, 13)$	$(22, 24)$	1	$2M_\infty$
$(6, 9, 10, 13, 18)$	$(19, 36)$	1	$3M_\infty$
$(6, 9, 14, 14, 22)$	$(28, 36)$	1	$3M_\infty$
$(6, 10, 10, 15, 15)$	$(25, 30)$	1	M_∞
$(6, 10, 10, 15, 20)$	$(30, 30)$	1	M_1
$(6, 10, 14, 18, 23)$	$(24, 46)$	1	$M_\infty \# M_2$
$(6, 10, 15, 15, 15)$	$(30, 30)$	1	M_1
$(6, 10, 15, 20, 20)$	$(30, 40)$	1	M_∞
$(6, 12, 14, 17, 22)$	$(34, 36)$	1	$2M_\infty$
$(6, 12, 16, 21, 27)$	$(33, 48)$	1	M_∞
$(6, 12, 22, 27, 33)$	$(33, 66)$	1	M_1
$(6, 14, 18, 19, 23)$	$(37, 42)$	1	$3M_\infty$

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(6,14,18,23,28)$	$(42,46)$	1	$2M_\infty$
$(6,14,19,24,29)$	$(43,48)$	1	$3M_\infty$
$(6,20,25,30,35)$	$(55,60)$	1	$2M_\infty$
$(6,20,30,35,40)$	$(60,70)$	1	M_∞
$(7,12,18,18,24)$	$(36,42)$	1	M_∞
$(7,12,18,24,30)$	$(42,48)$	1	M_∞
$(8,10,16,17,23)$	$(33,40)$	1	$3M_\infty$
$(8,10,16,23,30)$	$(40,46)$	1	$2M_\infty$
$(8,10,17,24,31)$	$(41,48)$	1	$3M_\infty$
$(8,12,18,19,29)$	$(37,48)$	1	$3M_\infty$
$(8,13,20,20,32)$	$(40,52)$	1	M_∞
$(8,14,21,28,35)$	$(49,56)$	1	M_∞
$(8,14,26,32,39)$	$(40,78)$	1	$M_\infty\#M_2$
$(8,14,28,35,42)$	$(56,70)$	1	M_1
$(8,18,24,31,41)$	$(49,72)$	1	$3M_\infty$
$(8,20,23,26,30)$	$(46,60)$	1	M_∞
$(8,20,27,34,46)$	$(54,80)$	1	M_∞
$(8,26,32,39,46)$	$(72,78)$	1	$2M_\infty$
$(8,26,32,39,70)$	$(78,96)$	1	$2M_\infty$
$(8,34,48,55,62)$	$(96,110)$	1	M_∞
$(8,42,56,63,70)$	$(112,126)$	1	M_∞
$(9,10,12,15,21)$	$(30,36)$	1	M_3
$(9,12,13,16,24)$	$(25,48)$	1	$2M_\infty$
$(9,13,15,18,21)$	$(36,39)$	1	M_∞
$(9,14,21,29,34)$	$(43,63)$	1	$3M_\infty$
$(9,15,22,30,36)$	$(45,66)$	1	M_∞
$(9,15,22,30,51)$	$(60,66)$	1	M_∞
$(9,15,23,23,31)$	$(46,54)$	1	$3M_\infty$
$(9,15,23,23,37)$	$(46,60)$	1	$3M_\infty$
$(9,21,28,28,35)$	$(56,63)$	2	M_∞

w	**d**	I	Smale type
$(9, 23, 30, 38, 67)$	$(76, 90)$	1	M_∞
$(9, 24, 32, 32, 40)$	$(64, 72)$	1	M_∞
$(10, 12, 20, 29, 31)$	$(41, 60)$	1	$3M_\infty$
$(10, 12, 21, 30, 39)$	$(51, 60)$	1	$2M_\infty$
$(10, 12, 30, 39, 48)$	$(60, 78)$	1	M_∞
$(10, 16, 25, 40, 55)$	$(65, 80)$	1	M_∞
$(10, 16, 40, 55, 70)$	$(80, 110)$	1	M_1
$(10, 17, 25, 26, 34)$	$(51, 60)$	1	$3M_\infty$
$(10, 17, 25, 34, 41)$	$(51, 75)$	1	$3M_\infty$
$(10, 17, 25, 34, 43)$	$(60, 68)$	1	$3M_\infty$
$(10, 17, 25, 34, 58)$	$(68, 75)$	1	$3M_\infty$
$(10, 22, 40, 49, 58)$	$(80, 98)$	1	M_∞
$(10, 24, 32, 55, 86)$	$(96, 110)$	1	M_2
$(10, 27, 36, 45, 54)$	$(81, 90)$	1	M_∞
$(10, 27, 45, 54, 63)$	$(90, 108)$	1	M_∞
$(11, 18, 27, 28, 44)$	$(55, 72)$	1	$3M_\infty$
$(11, 18, 27, 37, 44)$	$(55, 81)$	1	$3M_\infty$
$(11, 18, 27, 44, 61)$	$(72, 88)$	1	$3M_\infty$
$(11, 18, 27, 44, 70)$	$(81, 88)$	1	$3M_\infty$
$(11, 25, 32, 34, 41)$	$(66, 75)$	2	M_∞
$(11, 25, 34, 43, 52)$	$(77, 86)$	2	M_∞
$(11, 25, 34, 43, 57)$	$(68, 100)$	2	M_∞
$(11, 27, 36, 62, 97)$	$(108, 124)$	1	M_∞
$(11, 29, 38, 39, 48)$	$(77, 87)$	1	M_∞
$(11, 29, 38, 48, 85)$	$(96, 114)$	1	M_∞
$(11, 29, 39, 49, 59)$	$(88, 98)$	1	M_∞
$(11, 29, 39, 49, 67)$	$(78, 116)$	1	M_∞
$(11, 36, 45, 54, 63)$	$(99, 108)$	2	M_1
$(11, 40, 50, 60, 70)$	$(110, 120)$	1	M_1
$(12, 14, 15, 18, 21)$	$(36, 42)$	2	M_3

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(12, 14, 24, 35, 46)$	$(60, 70)$	1	$2M_\infty$
$(12, 14, 24, 35, 58)$	$(70, 72)$	1	$2M_\infty$
$(12, 15, 20, 26, 34)$	$(46, 60)$	1	$2M_\infty$
$(12, 15, 25, 25, 35)$	$(50, 60)$	2	M_∞
$(12, 18, 22, 27, 33)$	$(45, 66)$	1	M_∞
$(12, 21, 32, 32, 52)$	$(64, 84)$	1	M_∞
$(12, 30, 40, 51, 69)$	$(81, 120)$	1	M_∞
$(12, 32, 42, 43, 53)$	$(85, 96)$	1	$2M_\infty$
$(12, 32, 43, 54, 65)$	$(97, 108)$	1	$2M_\infty$
$(12, 42, 52, 63, 114)$	$(126, 156)$	1	M_1
$(12, 44, 55, 66, 77)$	$(121, 132)$	1	M_∞
$(13, 20, 29, 31, 47)$	$(60, 78)$	2	M_∞
$(13, 20, 31, 42, 49)$	$(62, 91)$	2	M_∞
$(13, 22, 55, 76, 97)$	$(110, 152)$	1	M_∞
$(13, 23, 34, 35, 56)$	$(69, 91)$	1	M_∞
$(13, 23, 34, 56, 89)$	$(102, 112)$	1	M_∞
$(13, 23, 35, 47, 57)$	$(70, 104)$	1	M_∞
$(13, 23, 35, 57, 79)$	$(92, 114)$	1	M_∞
$(14, 16, 42, 55, 68)$	$(84, 110)$	1	M_∞
$(14, 17, 27, 29, 39)$	$(56, 68)$	2	M_∞
$(14, 17, 29, 41, 44)$	$(58, 85)$	2	M_∞
$(14, 19, 25, 32, 43)$	$(57, 75)$	1	M_∞
$(14, 19, 25, 32, 45)$	$(64, 70)$	1	M_∞
$(15, 24, 35, 48, 57)$	$(72, 105)$	2	M_∞
$(15, 26, 40, 65, 90)$	$(105, 130)$	1	M_∞
$(15, 26, 65, 90, 115)$	$(130, 180)$	1	M_∞
$(15, 27, 40, 54, 66)$	$(81, 120)$	1	M_∞
$(15, 27, 40, 54, 93)$	$(108, 120)$	1	M_∞
$(15, 33, 44, 57, 75)$	$(90, 132)$	2	M_∞
$(15, 39, 52, 66, 90)$	$(105, 156)$	1	M_∞

w	**d**	I	Smale type
$(15, 39, 52, 90, 141)$	$(156, 180)$	1	M_∞
$(15, 48, 59, 72, 129)$	$(144, 177)$	2	M_1
$(15, 54, 67, 81, 147)$	$(162, 201)$	1	M_1
$(16, 18, 48, 63, 78)$	$(96, 126)$	1	M_∞
$(16, 21, 28, 36, 48)$	$(64, 84)$	1	M_∞
$(16, 21, 28, 48, 68)$	$(84, 96)$	1	M_∞
$(16, 28, 42, 43, 69)$	$(85, 112)$	1	$2M_\infty$
$(16, 28, 43, 70, 97)$	$(113, 140)$	1	$2M_\infty$
$(16, 29, 44, 72, 100)$	$(116, 144)$	1	M_1
$(16, 42, 56, 71, 97)$	$(113, 168)$	1	$2M_\infty$
$(16, 44, 59, 74, 102)$	$(118, 176)$	1	M_1
$(16, 46, 56, 69, 82)$	$(128, 138)$	3	M_∞
$(16, 58, 72, 87, 102)$	$(160, 174)$	1	M_∞
$(16, 58, 72, 87, 158)$	$(174, 216)$	1	M_∞
$(16, 62, 88, 101, 114)$	$(176, 202)$	3	M_1
$(16, 74, 104, 119, 134)$	$(208, 238)$	1	M_1
$(16, 78, 104, 117, 130)$	$(208, 234)$	3	M_1
$(16, 90, 120, 135, 150)$	$(240, 270)$	1	M_1
$(17, 20, 35, 50, 65)$	$(85, 100)$	2	M_1
$(18, 21, 35, 51, 54)$	$(72, 105)$	2	M_∞
$(18, 22, 27, 33, 39)$	$(66, 72)$	1	M_∞
$(18, 22, 27, 33, 48)$	$(66, 81)$	1	M_∞
$(18, 23, 30, 39, 51)$	$(69, 90)$	2	M_1
$(18, 24, 32, 41, 55)$	$(73, 96)$	1	$2M_\infty$
$(18, 24, 40, 63, 102)$	$(120, 126)$	1	M_1
$(18, 32, 48, 65, 79)$	$(97, 144)$	1	$2M_\infty$
$(18, 33, 49, 81, 129)$	$(147, 162)$	1	M_1
$(18, 42, 50, 59, 76)$	$(118, 126)$	1	M_1
$(18, 50, 66, 83, 148)$	$(166, 198)$	1	M_1
$(20, 28, 47, 66, 74)$	$(94, 140)$	1	M_1

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(20, 36, 55, 90, 125)$	$(145, 180)$	1	M_∞
$(21, 24, 29, 36, 51)$	$(72, 87)$	2	M_1
$(21, 24, 41, 60, 99)$	$(120, 123)$	2	M_1
$(22, 40, 60, 99, 138)$	$(160, 198)$	1	M_∞
$(22, 40, 60, 99, 158)$	$(180, 198)$	1	M_∞
$(22, 60, 80, 139, 218)$	$(240, 278)$	1	M_1
$(24, 30, 38, 53, 82)$	$(106, 120)$	1	M_1
$(24, 34, 40, 63, 86)$	$(120, 126)$	1	M_1
$(24, 34, 56, 79, 134)$	$(158, 168)$	1	M_1
$(24, 38, 84, 107, 130)$	$(168, 214)$	1	M_1
$(24, 66, 88, 111, 153)$	$(177, 264)$	1	M_∞
$(24, 90, 112, 135, 246)$	$(270, 336)$	1	M_1
$(26, 36, 48, 83, 118)$	$(144, 166)$	1	M_1
$(26, 36, 60, 95, 154)$	$(180, 190)$	1	M_1
$(26, 48, 120, 167, 214)$	$(240, 334)$	1	M_1
$(30, 42, 70, 99, 111)$	$(141, 210)$	1	M_∞
$(30, 56, 140, 195, 250)$	$(280, 390)$	1	M_1
$(30, 84, 112, 195, 306)$	$(336, 390)$	1	M_1

Appendix B

Lists of types

B.1 One parameter families of types

$\mathbf{w}$ $\mathbf{d}$	t constraint I	Smale type Lichnerowicz obstruction
$(1,1,1,t,t)$ $(t+1,t+1)$	$1 \le t$ 1	$(2t+3)M_\infty$ none
$(1,1,t,t,t)$ $(1+t,2t)$	$1 \le t$ 1	$(2t+3)M_\infty$ none
$(1,1,t+1,t+1,2t+1)$ $(2t+2,2t+2)$	$0 \le t$ 1	$(2t+5)M_\infty$ none
$(1,2,t+2,t+2,2t+3)$ $(2t+4,2t+5)$	$0 \le t$ 1	$(t+7)M_\infty$ none
$(1,t,t,t,t)$ $(2t,2t)$	$1 \le t$ 1	$5M_\infty \# M_t$ none
$(1,t+1,2t+1,2t+1,3t+1)$ $(3t+2,4t+2)$	$0 \le t$ $t+1$	$5M_\infty$ $t > 2$
$(1,2t+1,2t+1,3t+1,4t+1)$ $(4t+2,6t+2)$	$0 \le t$ $t+1$	$5M_\infty$ $t > 2$

$\mathbf{w}$ $\mathbf{d}$	t constraint I	Smale type Lichnerowicz obstruction
$(1, 3t+2, 4t+2, 6t+3, 9t+4)$ $(9t+5, 12t+6)$	$0 \le t$ $t+1$	$7M_\infty$ $t > 2$
$(1, 4t+2, 6t+3, 9t+4, 12t+5)$ $(12t+6, 18t+8)$	$0 \le t$ $t+1$	$7M_\infty$ $t > 2$
$(2, 2, 2t+1, 2t+1, 2t+1)$ $(2t+3, 4t+2)$	$1 \le t$ 2	$(2t+2)M_\infty$ none
$(2, 2, 2t+1, 2t+1, 4t)$ $(4t+2, 4t+2)$	$1 \le t$ 2	$(2t+2)M_\infty$ none
$(2, 3, t+1, t+2, t+2)$ $(t+4, 2t+4)$	$1 \le t,\ t \ne 0 \mod 3$ 2	$\begin{cases} (\frac{t-1}{3}+4)M_\infty & \text{if } t = 1 \mod 3 \\ (\frac{t-2}{3}+5)M_\infty & \text{if } t = 2 \mod 3 \end{cases}$ none
$(2, 3, t+1, t+2, 2t+1)$ $(2t+3, 2t+4)$	$1 \le t,\ t \ne 2 \mod 3$ 2	$\begin{cases} (\frac{t}{3}+3)M_\infty & \text{if } t = 0 \mod 3 \\ (\frac{t-1}{3}+4)M_\infty & \text{if } t = 1 \mod 3 \end{cases}$ none
$(2, 4, t+1, t+2, t+3)$ $(t+5, 2t+4)$	$1 \le t,\ t \ne 1 \mod 4$ 3	$\begin{cases} 3M_\infty & \text{if } t = 2 \mod 4 \\ M_\infty \# M_2 & \text{if } t = 3 \mod 4 \\ 2M_\infty & \text{if } t = 0 \mod 4 \end{cases}$ none
$(2, 4, 2t+3, 2t+3, 4t+4)$ $(4t+6, 4t+8)$	$0 \le t$ 2	$(t+5)M_\infty$ none
$(2, t+1, t+1, t+2, 2t+1)$ $(2t+3, 3t+3)$	$0 \le t$ 1	$\begin{cases} 7M_\infty & \text{if } t = 0 \mod 2 \\ 5M_\infty & \text{if } t = 1 \mod 2 \end{cases}$ none
$(2, t+1, t+1, 2t+1, 3t+1)$ $(3t+3, 4t+2)$	$1 \le t$ 1	$\begin{cases} 6M_\infty & \text{if } t = 0 \mod 2 \\ 2M_\infty \# 3M_2 & \text{if } t = 1 \mod 2 \end{cases}$ none
$(2, t+1, t+2, 2t+2, 3t+2)$ $(3t+4, 4t+4)$	$0 \le t$ 1	$\begin{cases} 5M_\infty \# M_2 & \text{if } t = 0 \mod 2 \\ 5M_\infty & \text{if } t = 1 \mod 2 \end{cases}$ none

w **d**	t constraint I	Smale type Lichnerowicz obstruction
$(2, t+1, 2t+2, 3t+2, 4t+2)$ $(4t+4, 6t+4)$	$0 \le t$ 1	$\begin{cases} 5M_\infty \# M_2 & \text{if } t = 0 \mod 2 \\ M_\infty \# 4M_2 & \text{if } t = 1 \mod 2 \end{cases}$ none
$(2, 2t+1, 2t+1, 2t+1, 2t+1)$ $(4t+2, 4t+2)$	$1 \le t$ 2	$M_\infty \# M_{2t+1}$ none
$(2, 2t+3, 4t+4, 4t+4, 6t+5)$ $(6t+7, 8t+8)$	$0 \le t$ $2t+3$	$3M_\infty$ $t > 1$
$(2, 4t+4, 4t+4, 6t+5, 8t+6)$ $(8t+8, 12t+10)$	$0 \le t$ $2t+3$	$M_\infty \# M_2$ $t > 1$
$(2, 6t+7, 8t+8, 12t+12, 18t+17)$ $(18t+19, 24t+24))$	$0 \le t$ $2t+3$	$3M_\infty$ $t > 1$
$(2, 8t+8, 12t+12, 18t+17, 24t+22)$ $(24t+24, 36t+34)$	$0 \le t$ $2t+3$	$3M_2$ $t > 1$
$(3, t+1, t+2, t+2, 2t+1)$ $(2t+4, 3t+3)$	$1 \le t$ 2	$\begin{cases} 3M_\infty & \text{if } t = 0 \mod 3 \\ M_\infty \# M_3 & \text{if } t = 1 \mod 3 \\ 5M_\infty & \text{if } t = 2 \mod 3 \end{cases}$ none
$(3, t+1, t+2, 2t+1, 3t)$ $(3t+3, 4t+2)$	$1 \le t,\ t \ne 2 \mod 3$ 2	$\begin{cases} 3M_\infty & \text{if } t = 0 \mod 3 \\ M_\infty \# M_3 & \text{if } t = 1 \mod 3 \end{cases}$ none
$(3, t+2, 2t+1, 2t+1, 3t)$ $(3t+3, 4t+2)$	$1 \le t,\ t \ne 1 \mod 3$ $t+2$	$3M_\infty$ $t > 7$
$(3, 2t+5, 2t+5, 3t+6, 4t+7)$ $(4t+10, 6t+12)$	$0 \le t,\ t \ne 2 \mod 3$ $3t+4$	$3M_\infty$ $t > 1$
$(3, 3t, 3t+1, 3t+1, 3t+2)$ $(6t+2, 6t+3)$	$1 \le t$ 2	$5M_\infty$ none
$(3, 3t+3, 4t+2, 6t+3, 9t+3)$ $(9t+6, 12t+6)$	$0 \le t,\ t \ne 1 \mod 3$ $t+2$	$M_\infty \# M_3$ $t > 7$

$\mathbf{w}$ $\mathbf{d}$	t constraint I	Smale type Lichnerowicz obstruction
$(3, 4t+2, 6t+3, 9t+3, 12t+3)$ $(12t+6, 18t+6)$	$0 \leq t,\ t \neq 1 \mod 3$ $t+2$	$M_\infty \# M_3$ $t > 7$
$(4, 6, 2t+1, 2t+3, 2t+3)$ $(2t+7, 4t+6)$	$1 \leq t,\ t \neq 2 \mod 3$ 4	$\begin{cases} (\frac{t}{3}+2)M_\infty & \text{if } t = 0 \mod 3 \\ (\frac{t-1}{3}+3)M_\infty & \text{if } t = 1 \mod 3 \end{cases}$ none
$(4, 6, 2t+1, 2t+3, 4t)$ $(4t+4, 4t+6)$	$1 \leq t,\ t \neq 1 \mod 3$ 4	$\begin{cases} (\frac{t}{3}+2)M_\infty & \text{if } t = 0 \mod 3 \\ (\frac{t-2}{3}+2)M_\infty & \text{if } t = 2 \mod 3 \end{cases}$ none
$(4, t+1, t+2, t+3, t+3)$ $(2t+4, 2t+6)$	$1 \leq t,\ t = 0, 3 \mod 4$ 3	$2M_\infty$ none
$(4, t+1, t+2, t+3, 2t)$ $(2t+4, 3t+3)$	$1 \leq t,\ t = 0, 3 \mod 4$ 3	$2M_\infty$ none
$(4, 2t+2, 2t+3, 2t+4, 2t+5)$ $(4t+7, 4t+8)$	$0 \leq t$ 3	$3M_\infty$ none
$(4, 2t+1, 2t+1, 2t+3, 4t)$ $(4t+4, 6t+3)$	$1 \leq t$ 2	$5M_\infty$ none
$(4, 2t+1, 2t+3, 4t+2, 6t+1)$ $(6t+5, 8t+4)$	$1 \leq t$ 2	$4M_\infty$ none
$(4, 2t+3, 2t+3, 4t+4, 6t+5)$ $(6t+9, 8t+8)$	$0 \leq t$ 2	$5M_\infty$ none
$(4, 2t+3, 4t+2, 4t+2, 6t+1)$ $(6t+5, 8t+4)$	$1 \leq t$ $2t+3$	$2M_\infty$ $t > 4$
$(4, 2t+1, 4t+2, 6t+1, 8t)$ $(8t+4, 12t+2)$	$1 \leq t$ 2	$4M_\infty$ none
$(4, 4t+2, 4t+2, 6t+1, 8t)$ $(8t+4, 12t+2)$	$1 \leq t$ $2t+3$	M_∞ $t > 4$
$(4, 6t+5, 8t+4, 12t+6, 18t+7)$ $(18t+11, 24t+12)$	$0 \leq t$ $2t+3$	$3M_\infty$ $t > 4$

$\mathbf{w}$ $\mathbf{d}$	t constraint I	Smale type Lichnerowicz obstruction
$(6, t+1, t+4, 2t+2, 3t)$ $(3t+6, 4t+4)$	$1 \le t,\ t \neq 2 \mod 3$ 3	$3M_\infty$ none
$(6, t+2, 2t+4, 3t+3, 4t+2)$ $(4t+8, 6t+6)$	$1 \le t,\ t \neq 1 \mod 3$ 3	$\begin{cases} M_\infty \# M_2 & \text{if } t = 0, 2 \mod 6 \\ 3M_\infty & \text{if } t = 3, 5 \mod 6 \end{cases}$ none
$(6, 2t+1, 2t+3, 2t+3, 4t)$ $(4t+6, 6t+2)$	$1 \le t,\ t \neq 2 \mod 3$ 4	$\begin{cases} M_\infty & \text{if } t = 0 \mod 3 \\ 3M_\infty & \text{if } t = 1 \mod 3 \end{cases}$ none
$(6, 2t+2, 2t+2, 2t+5, 4t+1)$ $(4t+7, 6t+6)$	$1 \le t,\ t \neq 2 \mod 3$ 3	$3M_\infty$ none
$(6, 2t+2, 2t+2, 4t+1, 6t)$ $(6t+6, 8t+2)$	$1 \le t,\ t \neq 2 \mod 3$ 3	$2M_\infty$ none
$(6, 3t+1, 3t+3, 3t+4, 3t+5)$ $(6t+6, 6t+8)$	$1 \le t$ 5	M_∞ none
$(6, 4t+4, 4t+4, 6t+3, 8t+2)$ $(8t+8, 12t+6)$	$1 \le t,\ t \neq 2 \mod 3$ $2t+5$	M_∞ $t > 1$
$(6, 6t+3, 6t+5, 6t+5, 6t+7)$ $(12t+10, 12t+12)$	$0 \le t$ 4	$3M_\infty$ none
$(6, 6t+7, 6t+9, 12t+12, 18t+15)$ $(18t+21, 24t+24)$	$0 \le t$ 4	M_∞ none
$(6, 6t+9, 8t+8, 12t+12, 18t+15)$ $(18t+21, 24t+24)$	$0 \le t,\ t \neq 2 \mod 3$ $2t+5$	M_∞ $t > 1$
$(6, 8t+8, 12t+12, 18t+15, 24t+18)$ $(24t+24, 36t+30))$	$0 \le t,\ t \neq 2 \mod 3$ $2t+5$	M_1 $t > 6$

$\mathbf{w}$ $\mathbf{d}$	t constraint I	Smale type Lichnerowicz obstruction
$(7, 4t+6, 6t+9, 9t+10, 12t+11)$ $(12t+18, 18t+20)$	$0 \leq t,\ t \neq 2 \mod 7$ $t+5$	M_∞ $t > 16$
$(8, 4t+1, 4t+3, 4t+5, 4t+7)$ $(8t+8, 8t+10)$	$1 \leq t$ 6	M_∞ none
$(8, 4t+5, 4t+7, 4t+9, 8t+6)$ $(8t+14, 12t+15)$	$0 \leq t$ 6	M_∞ none
$(8, 6t+7, 8t+4, 12t+6, 18t+5)$ $(18t+13, 24t+12)$	$0 \leq t$ $2t+5$	M_∞ $t > 14$
$(9, 3t+2, 3t+5, 3t+8, 6t+1)$ $(6t+10, 9t+9)$	$1 \leq t$ 6	M_∞ none
$(9, 3t+5, 3t+8, 6t+7, 9t+6)$ $(9t+15, 12t+14)$	$0 \leq t$ 6	M_∞ none
$(9, 3t+6, 4t+2, 6t+3, 9t)$ $(9t+9, 12t+6)$	$1 \leq t,\ t \neq 1 \mod 3$ $t+5$	M_∞ $t > 22$
$(9, 4t+6, 6t+9, 9t+9, 12t+9)$ $(12t+18, 18t+18)$	$0 \leq t,\ t \neq 0 \mod 3$ $t+6$	M_∞ $t > 21$
$(10, 2t+4, 4t+8, 6t+7, 8t+6)$ $(8t+16, 12t+14)$	$0 \leq t,\ t \neq 3 \mod 5$ 5	M_∞ none
$(12, 4t+4, 4t+7, 4t+10, 8t+2)$ $(8t+14, 12t+12)$	$1 \leq t,\ t \neq 2 \mod 3$ 9	M_1 none
$(12, 6t+5, 6t+9, 12t+6, 18t+3)$ $(18t+15, 24t+12)$	$1 \leq t$ 8	M_1 none
$(12, 6t+9, 8t+4, 12t+6, 18t+3)$ $(18t+15, 24t+12)$	$1 \leq t,\ t \neq 1 \mod 3$ $2t+7$	M_∞ $t > 21$
$(14, 8t+8, 12t+12, 18t+11, 24t+10)$ $(24t+24, 36t+22)$	$0 \leq t,\ t \neq 6 \mod 7$ 9	M_1 none
$(18, 8t+8, 12t+12, 18t+9, 24t+6)$ $(24t+24, 36t+18)$	$1 \leq t,\ t \neq 2 \mod 3$ $2t+11$	M_1 $t > 21$

B.2 Three parameter families of types

1. $\mathbf{w} = (u, u+2s, t(u+2s), t(u+2s)+s, 2t(u+2s)-u)$

$\mathbf{d} = (2t(u+2s), 2t(u+2s)+2s)$

$I = u + s$

Lichnerowicz obstruction when $s > 2u$

$u \geq 1$,

$$s \geq 1, \left\{ \begin{array}{ll} \gcd(s,u) = 1 & \text{if } u = 2v+1 \\ \gcd(2s,u) = 2 & \text{if } u = 4v \\ \gcd(s,2v+1) = 1 & \text{if } u = 4v+2 \end{array} \right\}$$

$$t \geq 1, \left\{ \begin{array}{ll} t = v, 2v \mod (2v+1) & \text{if } u = 2v+1 \\ 2t = (2v-1), (4v-2), (4v-1) \mod (4v) & \text{if } u = 4v \\ \left\{ \begin{array}{ll} 2t = (2v-1), (2v) \mod (2v+1) & \text{if } s = 1 \mod (2) \\ 2t = (4v+1) \mod (4v+2) & \text{if } s = 0 \mod (2) \end{array} \right\} & \text{if } u = 4v+2 \end{array} \right\}$$

(t can be half integer)

2. $\mathbf{w} = (u, u+2s, t(u+2s)+s, t(u+2s)+2s, 2t(u+2s)+2s-u)$

$\mathbf{d} = (2t(u+2s)+2s, 2t(u+2s)+4s)$

$I = u + s$

Lichnerowicz obstruction when $s > 2u$

$u \geq 1$

$s \geq 1$, $\gcd(s,u) = 1$

$$t \geq 1,\ t = \left\{ \begin{array}{ll} (v-1), 2v \mod (2v+1) & \text{if } u = 2v+1 \\ 2v \mod (2v+1) & \text{if } u = 4v+2 \\ (v-1), (2v-1) \mod (2v) & \text{if } u = 4v \end{array} \right.$$

3. $\mathbf{w} = (u, u+2s, t(u+2s), t(u+2s)+s, t(u+2s)+2s)$

$\mathbf{d} = (t(u+2s)+u+2s, 2t(u+2s)+2s)$

$I = u+s$

Lichnerowicz obstruction when $s > 2u$

$u \geq 1,$

$s \geq 1,\ \gcd(s,u) = 1,$

$$t \geq 1,\ t = \begin{cases} v, 2v \mod (2v+1) & \text{if } u = 2v+1 \text{ or } u = 4v+2 \\ (2v-1) \mod (2v) & \text{if } u = 4v \end{cases}$$

4. $\mathbf{w} = (u, u+2s, t(u+2s)-s, t(u+2s), t(u+2s)+s)$

$\mathbf{d} = (t(u+2s)+s+u, 2t(u+2s))$

$I = u+s$

Lichnerowicz obstruction when $s > 2u$

$u \geq 1,$

$$s \geq 1, \begin{cases} \gcd(s,u) = 1 & \text{if } u = 2v+1 \\ \gcd(s,u) = 1 & \text{if } u = 4v \\ \text{any } s \geq 1 & \text{if } u = 2 \\ \gcd(s, 2v+1) = 1 & \text{if } u = 4v+2,\ v \geq 1 \end{cases}$$

$$2t \geq 3, \begin{cases} t = 0, v \mod (2v+1) & \text{if } u = 2v+1 \\ 2t = 0, (2v-1), 2v \mod (4v) & \text{if } u = 4v \\ 2t = \left\{ \begin{array}{ll} r(2v+1) & \text{if } s = 1 \mod (2) \\ (2q+1)(2v+1) & \text{if } s = 0 \mod (2) \end{array} \right\} & \text{if } u = 4v+2 \end{cases}$$

(t can be half integer)

5. $\mathbf{w} = (u, u+s, u+2s, t(u+2s) - u - s, t(u+2s) - u)$

$\mathbf{d} = (t(u+2s), t(u+2s) + s)$

$I = u + s$

Lichnerowicz obstruction when $s > 2u$

$u \geq 1$

$s \geq 1,\ \gcd(s,u) = 1$

$$t \geq 2,\ t = \begin{cases} 0, v \mod (2v+1) & \text{if } u = 2v+1 \\ rv & \text{if } u = 2v \end{cases}$$

6. $\mathbf{w} = (u, u+s, u+2s, (t-1)(u+2s), (t-1)(u+2s) + s)$

$\mathbf{d} = (t(u+2s) - s, t(u+2s))$

$I = u + s$

Lichnerowicz obstruction when $s > 2u$

$u \geq 1$

$s \geq 1,\ \gcd(s,u) = 1$

$$t \geq 2,\ t = \begin{cases} 0, (v+1) \mod (2v+1) & \text{if } u = 2v+1 \\ rv & \text{if } u = 2v \end{cases}$$

B.2.1 Selected topology

type	$(\mathbf{w},\mathbf{d})$	Smale type
1	$((1,3,3t,3t+1,6t-1),(6t,6t+2))$	$(2t+3)M_\infty$
1	$((1,5,5t,5t+2,10t-1),(10t,10t+4))$	$(2t+3)M_\infty$
1	$((1,7,7t,7t+3,14t-1),(14t,14t+6))$	$(2t+3)M_\infty$
1	$((2,4,4t,4t+1,8t-2),(8t,8t+2))$	$M_\infty \# tM_2$
1	$((2,4,4t+2,4t+3,8t+2),(8t+4,8t+6))$	$(t+1)M_2$
1	$((2,6,6t+3,6t+5,12t+4),(12t+6,12t+10))$	$(2t+2)M_\infty$
1	$((2,8,8t,8t+3,16t-2),(16t,16t+6))$	$M_\infty \# tM_2$
1	$((2,8,8t+4,8t+7,16t+6),(16t+8,16t+14))$	$(t+1)M_2$
1	$((2,10,10t+5,10t+9,20t+8),(20t+10,20t+18))$	$(2t+2)M_\infty$
1	$((3,5,5t,5t+1,10t-3),(10t,10t+2)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+3)M_\infty$
1	$((3,5,5t,5t+1,10t-3),(10t,10t+2)),\ t=2 \mod 3$	$(2\left(\frac{t-2}{3}\right)+3)M_\infty$
1	$((3,7,7t,7t+2,14t-3),(14t,14t+4)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+3)M_\infty$
1	$((3,7,7t,7t+2,14t-3),(14t,14t+4)),\ t=2 \mod 3$	$(2\left(\frac{t-2}{3}\right)+3)M_\infty$
2	$((1,3,3t+1,3t+2,6t+1),(6t+2,6t+4))$	$(2t+3)M_\infty$
2	$((1,5,5t+2,3t+4,6t+3),(6t+4,6t+8))$	$(2t+3)M_\infty$
2	$((1,7,7t+3,3t+6,6t+5),(6t+6,6t+12))$	$(2t+3)M_\infty$
2	$((2,4,4t+1,4t+2,8t),(8t+2,8t+4))$	$M_\infty \# tM_2$
2	$((2,8,8t+3,8t+6,16t+4),(16t+6,16t+12))$	$M_\infty \# tM_2$
2	$((2,12,12t+5,12t+10,24t+8),(24t+10,24t+20))$	$M_\infty \# tM_2$
2	$((3,5,5t+1,5t+2,10t-1),(10t+2,10t+4)),\ t=2 \mod 3$	$(\frac{t-2}{3}+3)M_\infty$
2	$((3,5,5t+1,5t+2,10t-1),(10t+2,10t+4)),\ t=0 \mod 3$	$(\frac{t}{3}+2)M_\infty$
2	$((3,7,7t+2,7t+4,14t+1),(14t+4,14t+8)),\ t=2 \mod 3$	$(\frac{t-2}{3}+3)M_\infty$
2	$((3,7,7t+2,7t+4,14t+1),(14t+4,14t+8)),\ t=0 \mod 3$	$(\frac{t}{3}+2)M_\infty$

type	$(\mathbf{w},\mathbf{d})$	Smale type
3	$((1,3,3t,3t+1,3t+2),(3t+3,6t+2))$	$(2t+3)M_\infty$
3	$((1,5,5t,5t+2,5t+4),(5t+5,10t+4))$	$(2t+3)M_\infty$
3	$((1,7,7t,7t+3,7t+6),(7t+7,14t+6))$	$(2t+3)M_\infty$
3	$((2,4,4t,4t+1,4t+2),(4t+4,8t+2))$	$M_\infty \# tM_2$
3	$((2,8,8t,8t+3,8t+6),(8t+8,16t+6))$	$M_\infty \# tM_2$
3	$((2,12,12t,12t+5,12t+10),(12t+12,24t+10))$	$M_\infty \# tM_2$
3	$((3,5,5t,5t+1,5t+2),(5t+5,10t+2)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+3)M_\infty$
3	$((3,5,5t,5t+1,5t+2),(5t+5,10t+2)),\ t=2 \mod 3$	$(2\left(\frac{t-2}{3}\right)+3)M_\infty$
3	$((3,7,7t,7t+2,7t+4),(7t+7,14t+4)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+3)M_\infty$
3	$((3,7,7t,7t+2,7t+4),(7t+7,14t+4)),\ t=2 \mod 3$	$(2\left(\frac{t-2}{3}\right)+3)M_\infty$
4	$((1,3,3t-1,3t,3t+1),(3t+2,6t))$	$(2t+3)M_\infty$
4	$((1,5,5t-2,5t,5t+2),(5t+3,10t))$	$(2t+3)M_\infty$
4	$((1,7,7t-3,7t,7t+3),(7t+4,14t))$	$(2t+3)M_\infty$
4	$((2,4,4t+1,4t+2,4t+3),(4t+5,8t+4))$	$2M_\infty$
4	$((2,4,4t+3,4t+4,4t+5),(4t+7,8t+8))$	$3M_\infty$
4	$((2,6,6t+1,6t+3,6t+5),(6t+7,12t+6))$	$(2t+2)M_\infty$
4	$((2,8,8t+1,8t+4,8t+7),(8t+9,16t+8))$	$2M_\infty$
4	$((2,8,8t+5,8t+8,8t+11),(8t+13,16t+16))$	$3M_\infty$
4	$((2,10,10t+1,10t+5,10t+9),(10t+11,20t+10))$	$(2t+2)M_\infty$
4	$((3,5,5t-1,5t,5t+1),(5t+4,10t)),\ t=0 \mod 3$	$(2\left(\frac{t}{3}\right)+5)M_\infty$
4	$((3,5,5t-1,5t,5t+1),(5t+4,10t)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+5)M_\infty$
4	$((3,7,7t-2,7t,7t+2),(7t+4,14t)),\ t=0 \mod 3$	$(2\left(\frac{t}{3}\right)+5)M_\infty$
4	$((3,7,7t-2,7t,7t+2),(7t+4,14t)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+5)M_\infty$

type	$(\mathbf{w},\mathbf{d})$	Smale type
5	$((1,2,3,3t-2,3t-1),(3t,3t+1))$	$(2t+1)M_\infty$
5	$((1,3,5,5t-3,5t-1),(5t,5t+2))$	$(2t+1)M_\infty$
5	$((1,4,7,7t-4,7t-1),(7t,7t+3))$	$(2t+1)M_\infty$
5	$((2,3,4,4t-3,4t-2),(4t,4t+1))$	$(t+1)M_\infty$
5	$((2,5,8,8t-5,8t-2),(8t,8t+3))$	$(t+1)M_\infty$
6	$((1,2,3,3t,3t+1),(3t+2,3t+3))$	$(2t+3)M_\infty$
6	$((1,3,5,5t,5t+2),(5t+3,5t+5))$	$(2t+3)M_\infty$
6	$((1,4,7,7t,7t+3),(7t+4,7t+7))$	$(2t+3)M_\infty$
6	$((2,3,4,4t,4t+1),(4t+3,4t+4))$	$(t+2)M_\infty$
6	$((2,5,8,8t,8t+3),(8t+5,8t+8))$	$(t+2)M_\infty$
6	$((3,4,5,5t,5t+1),(5t+4,5t+5)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+3)M_\infty$
6	$((3,4,5,5t,5t+1),(5t+4,5t+5)),\ t=2 \mod 3$	$(2\left(\frac{t-2}{3}\right)+3)M_\infty$
6	$((3,5,7,7t,7t+2),(7t+5,7t+7)),\ t=1 \mod 3$	$(2\left(\frac{t-1}{3}\right)+3)M_\infty$
6	$((3,5,7,7t,7t+2),(7t,5,7t+7)),\ t=2 \mod 3$	$(2\left(\frac{t-2}{3}\right)+3)M_\infty$
6	$((4,5,6,6t,6t+1),(6t+5,6t+6)),\ t=1 \mod 2$	tM_∞

B.3 Sporadic types

None of these instances meet the Lichnerowicz obstruction.

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(1,2,2,3,3)$	$(4,6)$	1	$7M_\infty$
$(1,3,3,4,6)$	$(7,9)$	1	$8M_\infty$
$(1,3,3,5,5)$	$(6,10)$	1	$9M_\infty$
$(1,3,4,6,6)$	$(7,12)$	1	$8M_\infty$
$(1,3,4,6,8)$	$(9,12)$	1	$8M_\infty$
$(1,3,4,6,9)$	$(10,12)$	1	$8M_\infty$
$(1,4,5,7,11)$	$(12,15)$	1	$8M_\infty$
$(1,4,5,8,8)$	$(9,16)$	1	$9M_\infty$
$(1,4,5,8,12)$	$(13,16)$	1	$9M_\infty$
$(1,4,6,8,11)$	$(12,17)$	1	$9M_\infty$
$(1,4,7,10,13)$	$(14,20)$	1	$8M_\infty$
$(1,5,6,9,14)$	$(15,19)$	1	$9M_\infty$
$(1,5,7,10,14)$	$(15,21)$	1	$10M_\infty$
$(1,5,8,12,19)$	$(20,24)$	1	$8M_\infty$
$(1,5,9,13,17)$	$(18,26)$	1	$9M_\infty$
$(1,6,10,15,15)$	$(16,30)$	1	$9M_\infty$
$(1,6,10,15,20)$	$(21,30)$	1	$9M_\infty$
$(1,6,10,15,24)$	$(25,30)$	1	$9M_\infty$
$(1,7,11,17,27)$	$(28,34)$	1	$9M_\infty$
$(1,7,12,17,23)$	$(24,35)$	1	$9M_\infty$
$(1,7,12,18,18)$	$(19,36)$	1	$10M_\infty$
$(1,7,12,18,24)$	$(25,36)$	1	$10M_\infty$
$(1,8,13,19,31)$	$(32,39)$	1	$9M_\infty$
$(1,8,13,20,20)$	$(21,40)$	1	$10M_\infty$
$(1,8,13,20,32)$	$(33,40)$	1	$10M_\infty$

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(1, 8, 14, 20, 27)$	$(28, 41)$	1	$10M_\infty$
$(1, 9, 15, 22, 30)$	$(31, 45)$	1	$10M_\infty$
$(1, 9, 15, 22, 36)$	$(37, 45)$	1	$10M_\infty$
$(1, 9, 15, 23, 23)$	$(24, 46)$	1	$11M_\infty$
$(1, 10, 16, 24, 39)$	$(40, 49)$	1	$10M_\infty$
$(1, 10, 17, 25, 34)$	$(35, 51)$	1	$11M_\infty$
$(1, 11, 18, 27, 44)$	$(45, 55)$	1	$11M_\infty$
$(2, 2, 2, 2, 3)$	$(4, 6)$	1	$M_\infty \# 4M_2$
$(2, 2, 3, 3, 3)$	$(6, 6)$	1	$6M_\infty$
$(2, 2, 3, 3, 4)$	$(6, 7)$	1	$6M_\infty$
$(2, 2, 3, 4, 4)$	$(6, 8)$	1	$3M_\infty \# 2M_2$
$(2, 3, 4, 4, 5)$	$(8, 9)$	1	$5M_\infty$
$(2, 3, 4, 5, 5)$	$(8, 10)$	1	$5M_\infty$
$(2, 3, 4, 5, 6)$	$(9, 10)$	1	$5M_\infty$
$(2, 3, 4, 6, 8)$	$(10, 12)$	1	$5M_\infty \# M_2$
$(2, 3, 5, 6, 7)$	$(10, 12)$	1	$5M_\infty$
$(2, 4, 5, 6, 6)$	$(10, 12)$	1	$2M_\infty \# 2M_2$
$(2, 4, 5, 6, 7)$	$(11, 12)$	1	$4M_\infty$
$(2, 4, 6, 7, 8)$	$(12, 14)$	1	$M_\infty \# 3M_2$
$(2, 4, 6, 9, 14)$	$(16, 18))$	1	$M_\infty \# 4M_2$
$(2, 4, 8, 11, 14)$	$(16, 22))$	1	$M_\infty \# 4M_2$
$(2, 5, 6, 9, 13)$	$(15, 18)$	2	$4M_\infty$
$(2, 5, 6, 10, 14)$	$(16, 20)$	1	$5M_\infty \# M_2$
$(2, 5, 8, 11, 14)$	$(16, 22)$	2	$5M_\infty$
$(2, 6, 8, 9, 10)$	$(16, 18)$	1	$M_\infty \# 3M_2$
$(2, 7, 8, 13, 19)$	$(21, 26)$	1	$5M_\infty$
$(2, 7, 10, 13, 18)$	$(20, 28)$	2	$5M_\infty$
$(2, 7, 10, 15, 15)$	$(17, 30)$	2	$6M_\infty$
$(2, 7, 10, 15, 20)$	$(22, 30)$	2	$6M_\infty$
$(2, 9, 12, 17, 24)$	$(26, 36)$	2	$6M_\infty$

w	**d**	I	Smale type
$(2,9,12,17,27)$	$(29,36)$	2	$6M_\infty$
$(2,9,12,19,19)$	$(21,38)$	2	$7M_\infty$
$(2,11,14,21,33)$	$(35,44)$	2	$7M_\infty$
$(3,3,4,4,6)$	$(7,12)$	1	$4M_\infty$
$(3,3,4,6,6)$	$(9,12)$	1	$M_\infty\#2M_3$
$(3,3,4,6,9)$	$(12,12)$	1	$M_\infty\#2M_3$
$(3,3,5,5,7)$	$(10,12)$	1	$7M_\infty$
$(3,4,4,6,6)$	$(10,12)$	1	$4M_\infty$
$(3,4,4,6,8)$	$(12,12)$	1	$4M_\infty$
$(3,4,4,6,9)$	$(12,13)$	1	$5M_\infty$
$(3,4,5,6,7)$	$(10,12)$	1	$2M_\infty$
$(3,4,5,6,8)$	$(9,16)$	1	$5M_\infty$
$(3,4,5,7,9)$	$(12,14)$	2	$3M_\infty$
$(3,4,6,6,6)$	$(12,12)$	1	$M_\infty\#M_3$
$(3,4,6,6,9)$	$(12,15)$	1	$2M_\infty\#M_3$
$(3,4,6,8,8)$	$(12,16)$	1	$5M_\infty$
$(3,4,6,9,9)$	$(12,18)$	1	$2M_\infty\#M_3$
$(3,5,6,7,8)$	$(13,15)$	1	$5M_\infty$
$(3,5,6,8,10)$	$(15,16)$	1	$5M_\infty$
$(3,5,6,8,13)$	$(16,18)$	1	$5M_\infty$
$(3,5,6,9,12)$	$(15,18)$	2	$M_\infty\#M_3$
$(3,5,7,9,11)$	$(14,18)$	3	$3M_\infty$
$(3,5,7,9,11)$	$(16,18)$	1	$5M_\infty$
$(3,6,7,9,15)$	$(18,21)$	1	$2M_3$
$(3,6,8,8,10)$	$(16,18)$	1	$5M_\infty$
$(3,7,8,9,13)$	$(16,21)$	3	$3M_\infty$
$(3,7,8,12,12)$	$(15,24)$	3	$4M_\infty$
$(3,7,8,12,16)$	$(19,24)$	3	$4M_\infty$
$(3,8,9,15,21)$	$(24,30)$	2	M_∞
$(3,8,10,12,14)$	$(22,24)$	1	$4M_\infty$

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(3,8,10,12,17)$	$(20,27)$	3	$4M_\infty$
$(3,8,12,14,16)$	$(24,28)$	1	$4M_\infty$
$(3,10,11,15,22)$	$(25,33)$	3	$5M_\infty$
$(4,4,6,7,9)$	$(13,16)$	1	$5M_\infty$
$(4,4,7,10,10)$	$(14,20)$	1	$4M_\infty$
$(4,4,7,10,13)$	$(17,20)$	1	$5M_\infty$
$(4,5,7,10,11)$	$(15,21)$	1	$4M_\infty$
$(4,5,7,10,13)$	$(18,20)$	1	$4M_\infty$
$(4,5,7,10,16)$	$(20,21)$	1	$4M_\infty$
$(4,5,8,8,12)$	$(16,20)$	1	$M_\infty \# M_4$
$(4,5,8,12,16)$	$(20,24)$	1	$M_\infty \# M_4$
$(4,6,6,6,9)$	$(12,18)$	1	M_2
$(4,6,6,7,9)$	$(13,18)$	1	$3M_\infty$
$(4,6,6,8,11)$	$(12,22)$	1	$M_\infty \# 2M_2$
$(4,6,7,9,9)$	$(16,18)$	1	$3M_\infty$
$(4,6,7,9,14)$	$(18,21)$	1	$3M_\infty$
$(4,6,8,11,13)$	$(17,24)$	1	$3M_\infty$
$(4,6,8,11,14)$	$(20,22)$	1	$2M_\infty \# M_2$
$(4,6,8,11,18)$	$(22,24)$	1	$2M_\infty \# M_2$
$(4,6,9,12,15)$	$(21,24)$	1	$3M_\infty$
$(4,6,9,14,14)$	$(18,28)$	1	$2M_\infty \# M_2$
$(4,6,10,12,15)$	$(16,30)$	1	$M_\infty \# 2M_2$
$(4,6,12,15,18)$	$(24,30)$	1	$M_\infty \# M_2$
$(4,7,8,10,13)$	$(20,21)$	1	$3M_\infty$
$(4,8,10,11,13)$	$(21,24)$	1	$4M_\infty$
$(4,8,11,14,14)$	$(22,28)$	1	$3M_\infty$
$(4,8,11,14,17)$	$(25,28)$	1	$4M_\infty$
$(4,8,11,14,18)$	$(22,32)$	1	$3M_\infty$
$(4,9,15,18,21)$	$(30,36)$	1	$2M_\infty$
$(4,10,12,15,18)$	$(28,30)$	1	$2M_\infty \# M_2$

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(4, 10, 12, 15, 26)$	$(30, 36)$	1	$2M_\infty \# M_2$
$(4, 12, 15, 18, 18)$	$(30, 36)$	1	$2M_\infty$
$(4, 12, 15, 18, 21)$	$(33, 36)$	1	$3M_\infty$
$(4, 14, 20, 23, 26)$	$(40, 46)$	1	$M_\infty \# M_2$
$(4, 18, 24, 27, 30)$	$(48, 54)$	1	$M_\infty \# M_2$
$(5, 6, 6, 9, 9)$	$(15, 18)$	2	$2M_\infty$
$(5, 6, 8, 10, 12)$	$(18, 20)$	3	$3M_\infty$
$(5, 6, 8, 12, 19)$	$(24, 25)$	1	$3M_\infty$
$(5, 6, 9, 13, 13)$	$(18, 26)$	2	$3M_\infty$
$(5, 6, 10, 12, 14)$	$(20, 24)$	3	$3M_\infty$
$(5, 6, 10, 15, 20)$	$(25, 30)$	1	$M_\infty \# M_5$
$(5, 6, 14, 18, 22)$	$(28, 36)$	1	$3M_\infty$
$(5, 6, 15, 20, 25)$	$(30, 40)$	1	$M_\infty \# M_5$
$(5, 7, 8, 11, 14)$	$(21, 22)$	2	$3M_\infty$
$(5, 7, 10, 11, 14)$	$(21, 25)$	1	$5M_\infty$
$(5, 7, 10, 14, 16)$	$(21, 30)$	1	$5M_\infty$
$(5, 7, 10, 14, 18)$	$(25, 28)$	1	$5M_\infty$
$(5, 7, 10, 14, 23)$	$(28, 30)$	1	$5M_\infty$
$(5, 8, 8, 12, 12)$	$(20, 24)$	1	$2M_\infty$
$(5, 8, 9, 12, 19)$	$(24, 27)$	2	$2M_\infty$
$(5, 8, 12, 14, 16)$	$(24, 28)$	3	$2M_\infty$
$(5, 9, 12, 15, 18)$	$(27, 30)$	2	$3M_\infty$
$(5, 9, 12, 16, 20)$	$(25, 36)$	1	$3M_\infty$
$(5, 9, 12, 20, 31)$	$(36, 40)$	1	$3M_\infty$
$(5, 9, 15, 18, 21)$	$(30, 36)$	2	$3M_\infty$
$(5, 11, 14, 18, 22)$	$(33, 36)$	1	$3M_\infty$
$(5, 12, 16, 20, 24)$	$(36, 40)$	1	$3M_\infty$
$(5, 12, 18, 21, 24)$	$(36, 42)$	2	$2M_\infty$
$(5, 12, 20, 24, 28)$	$(40, 48)$	1	$3M_\infty$
$(5, 14, 17, 21, 37)$	$(42, 51)$	1	$2M_\infty$

w	**d**	I	Smale type
$(5, 16, 24, 28, 32)$	$(48, 56)$	1	$2M_\infty$
$(6, 6, 8, 11, 13)$	$(19, 24)$	1	$4M_\infty$
$(6, 6, 8, 11, 16)$	$(22, 24)$	1	$3M_\infty$
$(6, 6, 10, 10, 15)$	$(16, 30)$	1	M_∞
$(6, 6, 10, 15, 15)$	$(21, 30)$	1	M_∞
$(6, 6, 10, 15, 24)$	$(30, 30)$	1	M_1
$(6, 7, 9, 11, 14)$	$(18, 28)$	1	$3M_\infty$
$(6, 7, 9, 12, 15)$	$(21, 24)$	4	$1M_\infty$
$(6, 8, 8, 10, 15)$	$(16, 30)$	1	$M_\infty\#2M_2$
$(6, 8, 9, 9, 12)$	$(18, 24)$	2	$M_\infty\#M_3$
$(6, 8, 9, 11, 13)$	$(22, 24)$	1	$2M_\infty$
$(6, 8, 12, 17, 19)$	$(25, 36)$	1	$4M_\infty$
$(6, 8, 18, 23, 28)$	$(36, 46)$	1	$M_\infty\#M_2$
$(6, 8, 20, 27, 34)$	$(40, 54)$	1	$2M_2$
$(6, 9, 10, 13, 18)$	$(19, 36)$	1	$3M_\infty$
$(6, 9, 13, 21, 33)$	$(39, 42)$	1	M_3
$(6, 9, 14, 14, 22)$	$(28, 36)$	1	$3M_\infty$
$(6, 10, 10, 15, 15)$	$(25, 30)$	1	M_∞
$(6, 10, 10, 15, 20)$	$(30, 30)$	1	M_1
$(6, 10, 12, 15, 24)$	$(30, 36)$	1	M_∞
$(6, 10, 14, 18, 23)$	$(24, 46)$	1	$M_\infty\#M_2$
$(6, 10, 15, 15, 15)$	$(30, 30)$	1	M_1
$(6, 10, 15, 20, 20)$	$(30, 40)$	1	M_∞
$(6, 12, 14, 17, 22)$	$(34, 36)$	1	$2M_\infty$
$(6, 12, 16, 21, 27)$	$(33, 48)$	1	M_∞
$(6, 12, 16, 27, 42)$	$(48, 54)$	1	M_1
$(6, 14, 18, 19, 23)$	$(37, 42)$	1	$3M_\infty$
$(6, 14, 18, 23, 28)$	$(42, 46)$	1	$2M_\infty$
$(6, 14, 18, 23, 40)$	$(46, 54)$	1	$2M_\infty$
$(6, 14, 19, 24, 29)$	$(43, 48)$	1	$3M_\infty$

w	**d**	I	Smale type
$(6,18,22,27,48)$	$(54,66)$	1	M_1
$(6,20,25,30,35)$	$(55,60)$	1	$2M_\infty$
$(6,20,30,35,40)$	$(60,70)$	1	M_∞
$(7,8,12,12,16)$	$(24,28)$	3	M_∞
$(7,8,12,16,20)$	$(28,32)$	3	M_∞
$(7,9,15,21,27)$	$(36,42)$	1	$3M_\infty$
$(7,9,21,27,33)$	$(42,54)$	1	$3M_\infty$
$(7,10,15,15,20)$	$(30,35)$	2	M_∞
$(7,10,15,20,25)$	$(35,40)$	2	M_∞
$(7,12,18,18,24)$	$(36,42)$	1	M_∞
$(7,12,18,24,30)$	$(42,48)$	1	M_∞
$(8,8,10,15,22)$	$(30,32)$	1	$3M_\infty$
$(8,9,9,12,15)$	$(24,27)$	2	$2M_\infty$
$(8,9,12,20,28)$	$(36,40)$	1	M_4
$(8,10,15,20,25)$	$(35,40)$	3	M_∞
$(8,10,16,17,23)$	$(33,40)$	1	$3M_\infty$
$(8,10,16,19,22)$	$(32,38)$	5	M_∞
$(8,10,16,23,30)$	$(40,46)$	1	$2M_\infty$
$(8,10,16,23,38)$	$(46,48)$	1	$2M_\infty$
$(8,10,17,24,31)$	$(41,48)$	1	$3M_\infty$
$(8,10,20,25,30)$	$(40,50)$	3	M_1
$(8,12,13,14,18)$	$(26,36)$	3	M_∞
$(8,12,13,18,23)$	$(31,36)$	7	$2M_\infty$
$(8,12,17,22,26)$	$(34,48)$	3	M_∞
$(8,12,18,19,29)$	$(37,48)$	1	$3M_\infty$
$(8,12,19,30,41)$	$(49,60)$	1	$3M_\infty$
$(8,13,20,20,32)$	$(40,52)$	1	M_∞
$(8,13,20,32,44)$	$(52,64)$	1	M_∞
$(8,14,16,21,26)$	$(40,42)$	3	$2M_\infty$
$(8,14,16,21,34)$	$(42,48)$	3	$2M_\infty$

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(8, 14, 21, 28, 35)$	$(49, 56)$	1	M_∞
$(8, 14, 26, 32, 39)$	$(40, 78)$	1	$M_\infty \# M_2$
$(8, 14, 28, 35, 42)$	$(56, 70)$	1	M_1
$(8, 18, 24, 27, 30)$	$(48, 54)$	5	M_∞
$(8, 18, 24, 31, 41)$	$(49, 72)$	1	$3M_\infty$
$(8, 20, 23, 26, 30)$	$(46, 60)$	1	M_∞
$(8, 20, 27, 34, 46)$	$(54, 80)$	1	M_∞
$(8, 22, 32, 37, 42)$	$(64, 74)$	3	M_∞
$(8, 26, 32, 39, 46)$	$(72, 78)$	1	$2M_\infty$
$(8, 26, 32, 39, 70)$	$(78, 96)$	1	$2M_\infty$
$(8, 30, 40, 45, 50)$	$(80, 90)$	3	M_∞
$(8, 34, 48, 55, 62)$	$(96, 110)$	1	M_∞
$(8, 42, 56, 63, 70)$	$(112, 126)$	1	M_∞
$(9, 10, 12, 15, 18)$	$(27, 30)$	7	M_∞
$(9, 10, 12, 15, 21)$	$(30, 36)$	1	M_3
$(9, 10, 15, 22, 23)$	$(32, 45)$	2	$3M_\infty$
$(9, 11, 12, 17, 25)$	$(34, 36)$	4	M_∞
$(9, 12, 13, 16, 24)$	$(25, 48)$	1	$2M_\infty$
$(9, 12, 16, 16, 20)$	$(32, 36)$	5	M_∞
$(9, 12, 17, 24, 27)$	$(36, 51)$	2	M_∞
$(9, 12, 17, 24, 39)$	$(48, 51)$	2	M_∞
$(9, 12, 17, 27, 42)$	$(51, 54)$	2	M_∞
$(9, 12, 19, 19, 26)$	$(38, 45)$	2	$3M_\infty$
$(9, 12, 19, 19, 29)$	$(38, 48)$	2	$3M_\infty$
$(9, 13, 15, 18, 21)$	$(36, 39)$	1	M_∞
$(9, 14, 15, 21, 27)$	$(36, 42)$	8	M_∞
$(9, 14, 21, 29, 34)$	$(43, 63)$	1	$3M_\infty$
$(9, 15, 20, 20, 25)$	$(40, 45)$	4	M_∞
$(9, 15, 22, 30, 36)$	$(45, 66)$	1	M_∞
$(9, 15, 22, 30, 51)$	$(60, 66)$	1	M_∞

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(9,15,22,36,57)$	$(66,72)$	1	M_∞
$(9,15,23,23,31)$	$(46,54)$	1	$3M_\infty$
$(9,15,23,23,37)$	$(46,60)$	1	$3M_\infty$
$(9,19,24,31,53)$	$(62,72)$	2	M_∞
$(9,21,28,28,35)$	$(56,63)$	2	M_∞
$(9,23,30,38,67)$	$(76,90)$	1	M_∞
$(9,24,32,32,40)$	$(64,72)$	1	M_∞
$(10,11,15,18,22)$	$(33,40)$	3	$3M_\infty$
$(10,11,15,22,23)$	$(33,45)$	3	$3M_\infty$
$(10,11,15,22,29)$	$(40,44)$	3	$3M_\infty$
$(10,11,15,22,34)$	$(44,45)$	3	$3M_\infty$
$(10,12,16,25,38)$	$(48,50)$	3	M_2
$(10,12,20,29,31)$	$(41,60)$	1	$3M_\infty$
$(10,12,21,30,39)$	$(51,60)$	1	$2M_\infty$
$(10,12,30,39,48)$	$(60,78)$	1	M_∞
$(10,13,25,31,37)$	$(50,62)$	4	M_∞
$(10,16,25,40,55)$	$(65,80)$	1	M_∞
$(10,16,30,37,44)$	$(60,74)$	3	M_∞
$(10,16,40,55,70)$	$(80,110)$	1	M_1
$(10,17,25,26,34)$	$(51,60)$	1	$3M_\infty$
$(10,17,25,34,41)$	$(51,75)$	1	$3M_\infty$
$(10,17,25,34,43)$	$(60,68)$	1	$3M_\infty$
$(10,17,25,34,58)$	$(68,75)$	1	$3M_\infty$
$(10,19,35,43,51)$	$(70,86)$	2	M_∞
$(10,21,28,35,42)$	$(63,70)$	3	M_∞
$(10,21,35,42,49)$	$(70,84)$	3	M_∞
$(10,22,40,49,58)$	$(80,98)$	1	M_∞
$(10,24,32,55,86)$	$(96,110)$	1	M_2
$(10,27,36,45,54)$	$(81,90)$	1	M_∞
$(10,27,45,54,63)$	$(90,108)$	1	M_∞

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(11, 12, 15, 18, 21)$	$(33, 36)$	8	M_1
$(11, 13, 14, 19, 20)$	$(33, 39)$	5	M_∞
$(11, 13, 14, 20, 29)$	$(40, 42)$	5	M_∞
$(11, 13, 19, 25, 27)$	$(38, 52)$	5	M_∞
$(11, 13, 19, 25, 31)$	$(44, 50)$	5	M_∞
$(11, 14, 21, 23, 33)$	$(44, 56)$	2	$3M_\infty$
$(11, 14, 21, 30, 33)$	$(44, 63)$	2	$3M_\infty$
$(11, 14, 21, 33, 45)$	$(56, 66)$	2	$3M_\infty$
$(11, 14, 21, 33, 52)$	$(63, 66)$	2	$3M_\infty$
$(11, 15, 20, 32, 49)$	$(60, 64)$	3	M_∞
$(11, 16, 20, 24, 28)$	$(44, 48)$	7	M_1
$(11, 17, 20, 24, 27)$	$(44, 51)$	4	M_∞
$(11, 17, 20, 27, 43)$	$(54, 60)$	4	M_∞
$(11, 17, 24, 31, 37)$	$(48, 68)$	4	M_∞
$(11, 17, 24, 31, 38)$	$(55, 62)$	4	M_∞
$(11, 18, 27, 28, 44)$	$(55, 72)$	1	$3M_\infty$
$(11, 18, 27, 37, 44)$	$(55, 81)$	1	$3M_\infty$
$(11, 18, 27, 44, 61)$	$(72, 88)$	1	$3M_\infty$
$(11, 18, 27, 44, 70)$	$(81, 88)$	1	$3M_\infty$
$(11, 20, 25, 30, 35)$	$(55, 60)$	6	M_1
$(11, 21, 26, 29, 34)$	$(55, 63)$	3	M_∞
$(11, 21, 26, 34, 57)$	$(68, 78)$	3	M_∞
$(11, 21, 28, 47, 73)$	$(84, 94)$	2	M_∞
$(11, 21, 29, 37, 45)$	$(66, 74)$	3	M_∞
$(11, 21, 29, 37, 47)$	$(58, 84)$	3	M_∞
$(11, 24, 30, 36, 42)$	$(66, 72)$	5	M_1
$(11, 25, 32, 34, 41)$	$(66, 75)$	2	M_∞
$(11, 25, 32, 41, 71)$	$(82, 96)$	2	M_∞
$(11, 25, 34, 43, 52)$	$(77, 86)$	2	M_∞
$(11, 25, 34, 43, 57)$	$(68, 100)$	2	M_∞

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(11, 27, 36, 62, 97)$	$(108, 124)$	1	M_∞
$(11, 28, 35, 42, 49)$	$(77, 84)$	4	M_1
$(11, 29, 38, 39, 48)$	$(77, 87)$	1	M_∞
$(11, 29, 38, 48, 85)$	$(96, 114)$	1	M_∞
$(11, 29, 39, 49, 59)$	$(88, 98)$	1	M_∞
$(11, 29, 39, 49, 67)$	$(78, 116)$	1	M_∞
$(11, 32, 40, 48, 56)$	$(88, 96)$	3	M_1
$(11, 36, 45, 54, 63)$	$(99, 108)$	2	M_1
$(11, 40, 50, 60, 70)$	$(110, 120)$	1	M_1
$(12, 14, 15, 18, 21)$	$(36, 42)$	2	M_3
$(12, 14, 18, 20, 27)$	$(32, 54)$	5	M_∞
$(12, 14, 24, 35, 46)$	$(60, 70)$	1	$2M_\infty$
$(12, 14, 24, 35, 58)$	$(70, 72)$	1	$2M_\infty$
$(12, 15, 20, 26, 34)$	$(46, 60)$	1	$2M_\infty$
$(12, 15, 25, 25, 35)$	$(50, 60)$	2	$1M_\infty$
$(12, 16, 18, 23, 25)$	$(41, 48)$	5	$2M_\infty$
$(12, 16, 23, 30, 37)$	$(53, 60)$	5	$2M_\infty$
$(12, 18, 20, 27, 42)$	$(54, 60)$	5	M_1
$(12, 18, 22, 27, 33)$	$(45, 66)$	1	M_∞
$(12, 20, 21, 30, 39)$	$(51, 60)$	11	M_∞
$(12, 20, 25, 30, 35)$	$(55, 60)$	7	M_∞
$(12, 21, 32, 32, 52)$	$(64, 84)$	1	M_∞
$(12, 28, 35, 42, 49)$	$(77, 84)$	5	M_∞
$(12, 30, 40, 51, 69)$	$(81, 120)$	1	M_∞
$(12, 32, 42, 43, 53)$	$(85, 96)$	1	$2M_\infty$
$(12, 32, 43, 54, 65)$	$(97, 108)$	1	$2M_\infty$
$(12, 42, 52, 63, 114)$	$(126, 156)$	1	M_1
$(12, 44, 55, 66, 77)$	$(121, 132)$	1	M_∞
$(13, 14, 19, 23, 29)$	$(42, 52)$	4	M_∞
$(13, 14, 19, 29, 44)$	$(57, 58)$	4	M_∞

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(13, 14, 23, 32, 33)$	$(46, 65)$	4	M_∞
$(13, 14, 35, 46, 57)$	$(70, 92)$	3	M_∞
$(13, 17, 24, 27, 38)$	$(51, 65)$	3	M_∞
$(13, 17, 24, 38, 59)$	$(72, 76)$	3	M_∞
$(13, 17, 27, 37, 41)$	$(54, 78)$	3	M_∞
$(13, 18, 45, 61, 77)$	$(90, 122)$	2	M_∞
$(14, 15, 19, 26, 31)$	$(45, 57)$	3	M_∞
$(14, 15, 19, 26, 37)$	$(52, 56)$	3	M_∞
$(14, 15, 25, 35, 45)$	$(60, 70)$	4	M_∞
$(14, 15, 35, 45, 55)$	$(70, 90)$	4	M_∞
$(14, 16, 42, 55, 68)$	$(84, 110)$	1	M_∞
$(14, 17, 27, 29, 39)$	$(56, 68)$	2	M_∞
$(14, 17, 27, 39, 64)$	$(78, 81)$	2	M_∞
$(14, 17, 29, 41, 44)$	$(58, 85)$	2	M_∞
$(14, 17, 29, 41, 53)$	$(70, 82)$	2	M_∞
$(14, 19, 25, 32, 43)$	$(57, 75)$	1	M_∞
$(14, 19, 25, 32, 45)$	$(64, 70)$	1	M_∞
$(15, 16, 20, 28, 32)$	$(48, 60)$	3	M_∞
$(15, 16, 20, 32, 44)$	$(60, 64)$	3	M_∞
$(15, 18, 19, 27, 39)$	$(54, 57)$	7	M_1
$(15, 18, 25, 36, 39)$	$(54, 75)$	4	M_∞
$(15, 18, 25, 36, 57)$	$(72, 75)$	4	M_∞
$(15, 21, 28, 39, 45)$	$(60, 84)$	4	M_∞
$(15, 21, 28, 45, 69)$	$(84, 90)$	4	M_∞
$(15, 22, 55, 75, 95)$	$(110, 150)$	2	M_∞
$(15, 24, 35, 48, 57)$	$(72, 105)$	2	M_∞
$(15, 24, 35, 48, 81)$	$(96, 105)$	2	M_∞
$(15, 26, 65, 90, 115)$	$(130, 180)$	1	M_∞
$(15, 27, 40, 54, 66)$	$(81, 120)$	1	M_∞
$(15, 27, 40, 54, 93)$	$(108, 120)$	1	M_∞

w	**d**	I	Smale type
$(15, 33, 44, 57, 75)$	$(90, 132)$	2	M_∞
$(15, 33, 44, 75, 117)$	$(132, 150)$	2	M_∞
$(15, 36, 43, 54, 93)$	$(108, 129)$	4	M_1
$(15, 39, 52, 66, 90)$	$(105, 156)$	1	M_∞
$(15, 39, 52, 90, 141)$	$(156, 180)$	1	M_∞
$(15, 48, 59, 72, 129)$	$(144, 177)$	2	M_1
$(15, 54, 67, 81, 147)$	$(162, 201)$	1	M_1
$(16, 18, 24, 27, 30)$	$(48, 54)$	13	M_1
$(16, 18, 24, 35, 37)$	$(53, 72)$	5	$2M_\infty$
$(16, 18, 48, 63, 78)$	$(96, 126)$	1	M_∞
$(16, 20, 29, 38, 42)$	$(58, 80)$	7	M_1
$(16, 20, 30, 33, 47)$	$(63, 80)$	3	$2M_\infty$
$(16, 21, 28, 36, 48)$	$(64, 84)$	1	M_∞
$(16, 21, 28, 48, 68)$	$(84, 96)$	1	M_∞
$(16, 22, 24, 33, 42)$	$(64, 66)$	7	M_∞
$(16, 22, 24, 33, 50)$	$(66, 72)$	7	M_∞
$(16, 26, 40, 47, 54)$	$(80, 94)$	9	M_1
$(16, 28, 39, 50, 62)$	$(78, 112)$	5	M_1
$(16, 28, 42, 43, 69)$	$(85, 112)$	1	$2M_\infty$
$(16, 30, 40, 45, 50)$	$(80, 90)$	11	M_1
$(16, 30, 40, 53, 67)$	$(83, 120)$	3	$2M_\infty$
$(16, 34, 40, 51, 62)$	$(96, 102)$	5	M_∞
$(16, 34, 40, 51, 86)$	$(102, 120)$	5	M_∞
$(16, 36, 49, 62, 82)$	$(98, 144)$	3	M_∞
$(16, 38, 56, 65, 74)$	$(112, 130)$	7	M_∞
$(16, 42, 56, 63, 70)$	$(112, 126)$	9	M_1
$(16, 42, 56, 71, 97)$	$(113, 168)$	1	$2M_\infty$
$(16, 44, 59, 74, 102)$	$(118, 176)$	1	M_1
$(16, 46, 56, 69, 82)$	$(128, 138)$	3	M_∞
$(16, 46, 56, 69, 122)$	$(138, 168)$	3	M_∞

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(16, 50, 72, 83, 94)$	$(144, 166)$	5	M_1
$(16, 54, 72, 81, 90)$	$(144, 162)$	7	M_1
$(16, 58, 72, 87, 102)$	$(160, 174)$	1	M_∞
$(16, 58, 72, 87, 158)$	$(174, 216)$	1	M_∞
$(16, 62, 88, 101, 114)$	$(176, 202)$	3	M_1
$(16, 66, 88, 99, 110)$	$(176, 198)$	5	M_1
$(16, 74, 104, 119, 134)$	$(208, 238)$	1	M_1
$(16, 78, 104, 117, 130)$	$(208, 234)$	3	M_1
$(16, 90, 120, 135, 150)$	$(240, 270)$	1	M_1
$(17, 20, 35, 50, 65)$	$(85, 100)$	2	M_1
$(18, 19, 24, 33, 39)$	$(57, 72)$	4	M_1
$(18, 20, 21, 27, 33)$	$(54, 60)$	5	M_1
$(18, 21, 29, 45, 69)$	$(87, 90)$	5	M_1
$(18, 21, 35, 51, 54)$	$(72, 105)$	2	M_∞
$(18, 21, 35, 54, 87)$	$(105, 108)$	2	M_∞
$(18, 22, 27, 33, 39)$	$(66, 72)$	1	M_∞
$(18, 22, 27, 33, 48)$	$(66, 81)$	1	M_∞
$(18, 23, 30, 39, 51)$	$(69, 90)$	2	M_1
$(18, 24, 26, 35, 46)$	$(70, 72)$	7	M_1
$(18, 24, 32, 41, 55)$	$(73, 96)$	1	$2M_\infty$
$(18, 24, 40, 63, 102)$	$(120, 126)$	1	M_1
$(18, 26, 30, 41, 64)$	$(82, 90)$	7	M_1
$(18, 30, 34, 43, 56)$	$(86, 90)$	5	M_1
$(18, 32, 48, 65, 79)$	$(97, 144)$	1	$2M_\infty$
$(18, 33, 49, 81, 129)$	$(147, 162)$	1	M_1
$(18, 34, 42, 55, 92)$	$(110, 126)$	5	M_1
$(18, 42, 50, 59, 76)$	$(118, 126)$	1	M_1
$(18, 50, 66, 83, 148)$	$(166, 198)$	1	M_1
$(19, 20, 24, 36, 52)$	$(72, 76)$	3	M_1
$(20, 24, 41, 58, 62)$	$(82, 120)$	3	M_1

w	**d**	I	Smale type
$(20, 28, 47, 66, 74)$	$(94, 140)$	1	M_1
$(21, 24, 29, 36, 51)$	$(72, 87)$	2	M_1
$(21, 24, 41, 60, 99)$	$(120, 123)$	2	M_1
$(22, 24, 32, 49, 74)$	$(96, 98)$	7	M_1
$(22, 24, 36, 55, 74)$	$(96, 110)$	5	M_∞
$(22, 24, 36, 55, 86)$	$(108, 110)$	5	M_∞
$(22, 32, 48, 77, 106)$	$(128, 154)$	3	M_∞
$(22, 32, 48, 77, 122)$	$(144, 154)$	3	M_∞
$(22, 36, 48, 79, 122)$	$(144, 158)$	5	M_1
$(22, 40, 60, 99, 138)$	$(160, 198)$	1	M_∞
$(22, 40, 60, 99, 158)$	$(180, 198)$	1	M_∞
$(22, 48, 64, 109, 170)$	$(192, 218)$	3	M_1
$(22, 60, 80, 139, 218)$	$(240, 278)$	1	M_1
$(24, 26, 32, 51, 70)$	$(96, 102)$	5	M_1
$(24, 26, 40, 59, 94)$	$(118, 120)$	5	M_1
$(24, 26, 60, 77, 94)$	$(120, 154)$	7	M_1
$(24, 30, 32, 45, 66)$	$(90, 96)$	11	M_1
$(24, 30, 38, 53, 82)$	$(106, 120)$	1	M_1
$(24, 30, 40, 57, 63)$	$(87, 120)$	7	M_∞
$(24, 34, 40, 63, 86)$	$(120, 126)$	1	M_1
$(24, 34, 56, 79, 134)$	$(158, 168)$	1	M_1
$(24, 38, 84, 107, 130)$	$(168, 214)$	1	M_1
$(24, 42, 56, 75, 93)$	$(117, 168)$	5	M_∞
$(24, 54, 64, 81, 138)$	$(162, 192)$	7	M_1
$(24, 66, 80, 99, 174)$	$(198, 240)$	5	M_1
$(24, 66, 88, 111, 153)$	$(177, 264)$	1	M_∞
$(24, 90, 112, 135, 246)$	$(270, 336)$	1	M_1
$(26, 30, 40, 67, 94)$	$(120, 134)$	3	M_1
$(26, 30, 50, 77, 124)$	$(150, 154)$	3	M_1
$(26, 32, 80, 107, 134)$	$(160, 214)$	5	M_1

$\mathbf{w}$	$\mathbf{d}$	I	Smale type
$(26, 36, 48, 83, 118)$	$(144, 166)$	1	M_1
$(26, 36, 60, 95, 154)$	$(180, 190)$	1	M_1
$(26, 40, 100, 137, 174)$	$(200, 274)$	3	M_1
$(26, 48, 120, 167, 214)$	$(240, 334)$	1	M_1
$(30, 32, 80, 105, 130)$	$(160, 210)$	7	M_1
$(30, 42, 70, 99, 111)$	$(141, 210)$	1	M_∞
$(30, 48, 64, 105, 162)$	$(192, 210)$	7	M_1
$(30, 56, 140, 195, 250)$	$(280, 390)$	1	M_1
$(30, 84, 112, 195, 306)$	$(336, 390)$	1	M_1

Appendix C

Well-formed types

C.1 One parameter families of well-formed types

$\mathbf{w}$	$\mathbf{d}$	I
$(1, 1, t+1, t+1, 2t+1),\ 0 \le t$	$(2t+2, 2t+2)$	1
$(1, 2, t+2, t+2, 2t+3),\ 0 \le t,\ t$ odd	$(2t+4, 2t+5)$	1
$(2, t+1, t+1, 2t+1, 3t+1),\ 1 \le t,\ t$ even	$(3t+3, 4t+2)$	1
$(2, 2, 2t+1, 2t+1, 4t),\ 1 \le t$	$(4t+2, 4t+2)$	2
$(2, 3, t+1, t+2, t+2),\ 1 \le t,\ t = 2 \mod 3$	$(t+4, 2t+4)$	2
$(2, 3, t+1, t+2, 2t+1),\ 1 \le t,\ t = 0 \mod 3$	$(2t+3, 2t+4)$	2
$(2, 4, 2t+3, 2t+3, 4t+4),\ 0 \le t$	$(4t+6, 4t+8)$	2
$(3, t+1, t+2, t+2, 2t+1),\ t \ne 1 \mod 3$	$(2t+4, 3t+3)$	2
$(3, 3t, 3t+1, 3t+1, 3t+2),\ 1 \le t$	$(6t+2, 6t+3)$	2
$(3, 3t+1, 3t+2, 6t+1, 6t+3),\ 1 \le t$	$(9t+3, 12t+2)$	2
$(4, 2t+1, 2t+1, 2t+3, 4t)$,$1 \le t$	$(4t+4, 6t+3)$	2
$(4, 2t+3, 2t+3, 4t+4, 6t+5),\ 0 \le t$	$(6t+9, 8t+8)$	2
$(4, 2t+1, 2t+3, 4t+2, 6t+1),\ 1 \le t$	$(6t+5, 8t+4)$	2
$(4, 2t+1, 4t+2, 6t+1, 8t),\ 1 \le t$	$(8t+4, 12t+2)$	2
$(4, 4t+1, 4t+2, 4t+3, 4t+3),\ 1 \le t$	$(8t+4, 8t+6))$	3
$(4, 6, 6t+3, 6t+5, 6t+5),\ 1 \le t$	$(6t+9, 12t+10)$	4
$(4, 6, 6t+5, 6t+7, 12t+8),\ 1 \le t$	$(12t+12, 12t+14)$	4
$(6, 6t+3, 6t+5, 6t+5, 6t+7),\ 0 \le t$	$(12t+10, 12t+12)$	4
$(6, 6t+3, 6t+5, 6t+5, 12t+4),\ 1 \le t$	$(12t+10, 18t+9)$	4
$(6, 6t+1, 6t+3, 6t+4, 6t+5),\ 1 \le t$	$(12t+6, 12t+8)$	5
$(8, 4t+1, 4t+3, 4t+5, 4t+7),\ 1 \le t$	$(8t+8, 8t+10)$	6
$(8, 4t+5, 4t+7, 4t+9, 8t+6),\ 0 \le t$	$(8t+14, 12t+15)$	6
$(9, 3t+2, 3t+5, 3t+8, 6t+1),\ 1 \le t$	$(6t+10, 9t+9)$	6
$(9, 3t+5, 3t+8, 6t+7, 9t+6),\ 0 \le t$	$(9t+15, 12t+14)$	6
$(1, 4t-2, 6t-3, 9t-5, 12t-7)$	$(12t-6, 18t-10)$	t
$(1, 3t-1, 4t-2, 6t-3, 9t-5),\ t$ even	$(9t-4, 12t-6)$	t
$(7, 4t+6, 6t+9, 9t+10, 12t+11)$ $0 \le t,\ t \ne 2 \mod 7$	$(12t+18, 18t+20)$	$t+5$

C.2 Three parameter families of well-formed types

1. For index $I \geq 2$:

$\mathbf{w} = (u, 2I - u, t(2I - u), t(2I - u) + I - u, 2t(2I - u) - u)$

$\mathbf{d} = (2t(2I - u), 2t(2I - u) + 2I - 2u)$

Either:

a. $1 \leq u < I$, u odd, $\gcd(u, I) = 1$

$t \geq 1$, $t = \frac{u-1}{2}, u - 1 \mod (u)$

or

b. $u = 4v + 2$ $I - u$ even, $\gcd(I - u, 2v + 1) = 1$

$2t = (4v + 1) \mod (4v + 2)$

2. For index $I \geq 2$:

$\mathbf{w} = (u, 2I - u, t(2I - u) + I - u, t(2I - u) + 2I - 2u, 2t(2I - u) + 2I - 3u)$

$\mathbf{d} = (2t(2I - u) + 2I - 2u, 2t(2I - u) + 4I - 4u)$

$1 \leq u < I$, $\gcd(u, 2) = 1$ $\gcd(u, I) = 1$

$t \geq 1$, $t = \frac{u-3}{2}, u - 1 \mod (u)$

3. For index $I \geq 2$:

$\mathbf{w} = (u, 2I - u, t(2I - u), t(2I - u) + I - u, t(2I - u) + 2I - 2u)$

$\mathbf{d} = (t(2I - u) + 2I - u, 2t(2I - u) + 2I - 2u)$

$1 \leq u < I$, $\gcd(u, 2) = 1$ $\gcd(u, I) = 1$

$t \geq 1$, $t = \frac{u-1}{2}, u - 1 \mod (u)$

4. For index $I \geq 2$:

$\mathbf{w} = (u, 2I - u, t(2I - u) + u - I, t(2I - u), t(2I - u) + I - u)$

$\mathbf{d} = (t(2I - u) + I, 2t(2I - u))$

Either:

a. $1 \leq u < I$, u odd, $\gcd(u, I) = 1$

$t \geq 1$, $t = 0, \frac{u-1}{2} \mod (u)$

or

b. $u = 4v + 2$ $I - u$ even, $\gcd(I - u, 2v + 1) = 1$

$2t = (2q + 1)(2v + 1)$ for some $q \geq 1$, $\gcd(2t, I - u) = 1$ t can be half integer.

5. For index $I \geq 2$:

$$\mathbf{w} = (u, I, 2I - u, t(2I - u) - I, t(2I - u) - u)$$

$$\mathbf{d} = (t(2I - u), t(2I - u) + I - u)$$

$$u \equiv 1 \mod 2,\ \gcd(u, I) = 1$$

$$t \geq 2,\ t \equiv 0, \tfrac{u-1}{2} \mod (u)\ \gcd(I - u, t - 1) = 1$$

6. For index $I \geq 2$:

$$\mathbf{w} = (u, I, 2I - u, (t-1)(2I - u), (t-1)(2I - u) + I - u)$$

$$\mathbf{d} = (t(2I - u) + u - I, t(2I - u))$$

$$u \equiv 1 \mod 2,\ \gcd(u, I) = 1$$

$$t \geq 2,\ t \equiv 0, \tfrac{u+1}{2} \mod (u)\ \gcd(I - u, t - 1) = 1$$

C.3 Sporadic well-formed types

$\mathbf{w}$	$\mathbf{d}$	I	KE
$(1, 2, 2, 3, 3)$	$(4, 6)$	1	?
$(1, 3, 3, 5, 5)$	$(6, 10)$	1	?
$(1, 4, 5, 7, 11)$	$(12, 15)$	1	?
$(1, 4, 7, 10, 13)$	$(14, 20)$	1	?
$(1, 5, 8, 12, 19)$	$(20, 24)$	1	?
$(1, 5, 9, 13, 17)$	$(18, 26)$	1	?
$(1, 7, 11, 17, 27)$	$(28, 34)$	1	?
$(1, 7, 12, 17, 23)$	$(24, 35)$	1	?
$(1, 8, 13, 19, 31)$	$(32, 39)$	1	?
$(1, 9, 15, 23, 23)$	$(24, 46)$	1	?
$(2, 2, 3, 3, 3)$	$(6, 6)$	1	?
$(2, 3, 4, 5, 5)$	$(8, 10)$	1	?
$(2, 3, 5, 6, 7)$	$(10, 12)$	1	?
$(3, 3, 5, 5, 7)$	$(10, 12)$	1	?
$(3, 5, 6, 8, 13)$	$(16, 18)$	1	?
$(3, 5, 7, 9, 11)$	$(16, 18)$	1	?
$(4, 5, 7, 10, 13)$	$(18, 20)$	1	?
$(5, 7, 10, 14, 23)$	$(28, 30)$	1	?
$(5, 9, 12, 20, 31)$	$(36, 40)$	1	?
$(5, 14, 17, 21, 37)$	$(42, 51)$	1	?
$(6, 7, 9, 11, 14)$	$(18, 28)$	1	Y
$(6, 8, 9, 11, 13)$	$(22, 24)$	1	Y
$(9, 15, 23, 23, 31)$	$(46, 54)$	1	Y
$(9, 15, 23, 23, 37)$	$(46, 60)$	1	Y
$(9, 23, 30, 38, 67)$	$(76, 90)$	1	Y

$\mathbf{w}$	$\mathbf{d}$	I	KE
$(10,17,25,34,43)$	$(60,68)$	1	Y
$(11,18,27,44,61)$	$(72,88)$	1	Y
$(11,27,36,62,97)$	$(108,124)$	1	Y
$(11,29,38,48,85)$	$(96,114)$	1	Y
$(11,29,39,49,59)$	$(88,98)$	1	Y
$(11,29,39,49,67)$	$(78,116)$	1	Y
$(13,22,55,76,97)$	$(110,152)$	1	Y
$(13,23,34,56,89)$	$(102,112)$	1	Y
$(13,23,35,47,57)$	$(70,104)$	1	Y
$(13,23,35,57,79)$	$(92,114)$	1	Y
$(14,19,25,32,45)$	$(64,70)$	1	Y
$(2,5,6,9,13)$	$(15,18)$	2	?
$(2,5,8,11,14)$	$(16,22)$	2	?
$(2,7,8,13,19)$	$(21,26)$	2	?
$(2,7,10,13,18)$	$(20,28)$	2	?
$(2,9,12,19,19)$	$(21,38)$	2	?
$(3,4,5,6,7)$	$(11,12)$	2	?
$(3,4,6,7,8)$	$(12,14)$	2	?
$(5,6,9,13,13)$	$(18,26)$	2	?
$(5,7,8,11,14)$	$(21,22)$	2	?
$(5,8,9,12,19)$	$(24,27)$	2	?
$(9,10,15,22,23)$	$(32,45)$	2	?
$(9,12,19,19,26)$	$(38,45)$	2	?
$(9,12,19,19,29)$	$(38,48)$	2	?
$(9,19,24,31,53)$	$(62,72)$	2	?
$(10,19,35,43,51)$	$(70,86)$	2	?
$(11,14,21,33,52)$	$(63,66)$	2	?
$(11,21,28,47,73)$	$(84,94)$	2	?

w	**d**	I	KE
$(11, 25, 32, 34, 41)$	$(66, 75)$	2	Y
$(11, 25, 32, 41, 71)$	$(82, 96)$	2	?
$(11, 25, 34, 43, 52)$	$(77, 86)$	2	Y
$(11, 25, 34, 43, 57)$	$(68, 100)$	2	Y
$(13, 18, 45, 61, 77)$	$(90, 122)$	2	?
$(13, 20, 29, 31, 47)$	$(60, 78)$	2	Y
$(13, 20, 29, 47, 74)$	$(87, 94)$	2	?
$(13, 20, 31, 42, 49)$	$(62, 91)$	2	Y
$(13, 20, 31, 49, 67)$	$(80, 98)$	2	?
$(14, 17, 27, 39, 64)$	$(78, 81)$	2	?
$(14, 17, 29, 41, 44)$	$(58, 85)$	2	Y
$(14, 17, 29, 41, 53)$	$(70, 82)$	2	?
$(3, 4, 5, 6, 7)$	$(10, 12)$	3	?
$(3, 5, 7, 9, 11)$	$(14, 18)$	3	?
$(3, 7, 8, 9, 13)$	$(16, 21)$	3	?
$(1, 10, 15, 22, 29)$	$(30, 44)$	3	?
$(10, 11, 15, 22, 29)$	$(40, 44)$	3	?
$(11, 15, 20, 32, 49)$	$(60, 64)$	3	?
$(11, 21, 26, 34, 57)$	$(68, 78)$	3	?
$(11, 21, 29, 37, 45)$	$(66, 74)$	3	?
$(11, 21, 29, 37, 47)$	$(58, 84)$	3	?
$(13, 14, 35, 46, 57)$	$(70, 92)$	3	?
$(13, 17, 24, 38, 59)$	$(72, 76)$	3	?
$(13, 17, 27, 37, 41)$	$(54, 78)$	3	?
$(13, 17, 27, 41, 55)$	$(68, 82)$	3	?
$(14, 15, 19, 26, 37)$	$(52, 56)$	3	?

w	**d**	I	KE
$(4,5,6,7,8)$	$(12,14)$	4	?
$(9,11,12,17,25)$	$(34,36)$	4	?
$(10,13,25,31,37)$	$(50,62)$	4	?
$(11,17,20,24,27)$	$(44,51)$	4	?
$(11,17,20,27,43)$	$(54,60)$	4	?
$(11,17,24,31,37)$	$(48,68)$	4	?
$(11,17,24,31,38)$	$(55,62)$	4	?
$(13,14,19,23,29)$	$(42,52)$	4	?
$(13,14,19,29,44)$	$(57,58)$	4	?
$(13,14,23,32,33)$	$(46,65)$	4	?
$(13,14,23,33,43)$	$(56,66)$	4	?
$(11,13,14,20,29)$	$(40,42)$	5	?
$(11,13,19,25,27)$	$(38,52)$	5	?
$(11,13,19,25,31)$	$(44,50)$	5	?

Appendix D

Cases broken down by highest two weights

D.1 General requirements

Suppose, as usual, that $w_0 \le w_1 \le w_2 \le w_3 \le w_4 < d_1 \le d_2$ and $w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$. We have $d_1 \ge w_3 + w_0$, $d_1 \ge w_4$, and $d_2 \ge w_4 + w_1$ from Lemmas 30 and 34.

D.1.1 Possibilities for w_3

From Corollary 16 applied to $\{3\}$, we need one of the following: (i) $d_1 = m_1 w_3$, (ii) $d_2 = m_2 w_3$, or (iii) $d_1 = m_3 w_3 + w_i$ and $d_2 = m_4 w_3 + w_j$ with $i, j \in \{0, 1, 2, 4\}$ and $i \ne j$. (i) $2 \le m_1 < 3$ else $d_1 + d_2 \ge 3w_3 + w_1 + w_4 \ge w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. (ii) $2 \le m_2 < 4$ else $d_1 + d_2 \ge w_4 + 4w_3 \ge w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. (iii) $d_1 < 2w_3 + w_0$ else $d_1 + d_2 \ge 2w_3 + w_0 + w_1 + w_4 \ge w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. Therefore $d_1 = w_3 + w_0$, $d_1 = w_3 + w_1$, $d_1 = w_3 + w_2$, or $d_1 = w_3 + w_4$ in this case. $d_2 < 3w_3 + w_0$ else $d_1 + d_2 >$

$w_4 + 3w_3 + w_0 \geq w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. Therefore $d_2 = w_3 + w_0$, $d_2 = w_3 + w_1$, $d_2 = w_3 + w_2$, $d_2 = w_3 + w_4$, $d_2 = 2w_3 + w_0$, $d_2 = 2w_3 + w_1$, or $d_2 = 2w_3 + w_2$. Without loss of generality we can assume that if $d_1 = w_3 + w_i$ and $d_2 = w_3 + w_j$ then $i < j$, since, if not, then $d_1 \leq d_2$ would imply $w_j = w_i$.

Tabling these possibilities:

d_1	d_2	
$2w_3$		
	$2w_3$	
	$3w_3$	
$w_0 + w_3$	$w_1 + w_3$	
$w_0 + w_3$	$w_2 + w_3$	
$w_0 + w_3$	$w_3 + w_4$	
$w_0 + w_3$	$2w_3 + w_1$	
$w_0 + w_3$	$2w_3 + w_2$	
$w_1 + w_3$	$w_2 + w_3$	
$w_1 + w_3$	$w_3 + w_4$	
$w_1 + w_3$	$2w_3 + w_0$	
$w_1 + w_3$	$2w_3 + w_2$	
$w_2 + w_3$	$w_3 + w_4$	
$w_2 + w_3$	$2w_3 + w_0$	
$w_2 + w_3$	$2w_3 + w_1$	
$w_3 + w_4$	$2w_3 + w_0$	too great
$w_3 + w_4$	$2w_3 + w_1$	too great
$w_3 + w_4$	$2w_3 + w_2$	too great

D.1.2 Possibilities for w_4

From the Corollary applied to $\{4\}$, we need one of the following: (i) $d_1 = n_1 w_4$, (ii) $d_2 = n_2 w_4$, or (iii) $d_1 = n_3 w_4 + w_i$ and $d_2 = n_4 w_4 + w_j$ with $i, j \in \{0, 1, 2, 4\}$ and $i \neq j$. (i) $2 \leq n_1 < 3$ else $d_1 + d_2 \geq w_0 + w_3 + 3w_4 \geq w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. (ii) $2 \leq n_2 < 3$ else $d_1 + d_2 \geq 3w_4 + w_3 + w_0 \geq w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. (iii) $d_1 < 2w_4 + w_0$ else $d_1 + d_2 \geq 2w_4 + w_0 + w_1 + w_4 \geq w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. Therefore $d_1 = w_4 + w_0$, $d_1 = w_4 + w_1$, $d_1 = w_4 + w_2$, or $d_1 = w_4 + w_3$ in this case. $d_2 < 2w_4 + w_1$ else $d_1 + d_2 > w_4 + 3w_3 + w_0 \geq w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2$ which is a contradiction. Therefore $d_2 = w_4 + w_0$, $d_2 = w_4 + w_1$, $d_2 = w_4 + w_2$, $d_2 = w_4 + w_3$, or $d_2 = 2w_4 + w_0$. Without loss of generality we can assume that if $d_1 = w_4 + w_i$ and $d_2 = w_4 + w_j$ then $i < j$, since, if not, then $d_1 \leq d_2$ would imply $w_j = w_i$.

Tabling these possibilities:

d_1	d_2	
$2w_4$		
	$2w_4$	
$w_0 + w_4$	$w_1 + w_4$	
$w_0 + w_4$	$w_2 + w_4$	
$w_0 + w_4$	$w_3 + w_4$	
$w_1 + w_4$	$w_2 + w_4$	
$w_1 + w_4$	$w_3 + w_4$	
$w_1 + w_4$	$2w_4 + w_0$	too great
$w_2 + w_4$	$w_3 + w_4$	
$w_3 + w_4$	$2w_4 + w_0$	too great

D.1.3 Possible pairs for w_3 and w_4

Combining the previous conditions and noting those ruled out those by immediately violating positivity:

d_1	d_2	
$2w_3 = 2w_4$		see below
$2w_3$	$2w_4$	
$2w_3 = w_0 + w_4$	$w_1 + w_4$	
$2w_3 = w_0 + w_4$	$w_2 + w_4$	
$2w_3 = w_0 + w_4$	$w_3 + w_4$	
$2w_3 = w_1 + w_4$	$w_2 + w_4$	
$2w_3 = w_1 + w_4$	$w_3 + w_4$	
$2w_3 = w_2 + w_4$	$w_3 + w_4$	
$2w_4$	$2w_3$	
	$2w_3 = 2w_4$	see below
$w_0 + w_4$	$2w_3 = w_1 + w_4$	
$w_0 + w_4$	$2w_3 = w_2 + w_4$	
$w_0 + w_4$	$2w_3 = w_3 + w_4$	
$w_1 + w_4$	$2w_3 = w_2 + w_4$	
$w_1 + w_4$	$2w_3 = w_3 + w_4$	
$w_2 + w_4$	$2w_3 = w_3 + w_4$	

$2w_4$	$3w_3$	too great
	$3w_3 = 2w_4$	see below
$w_0 + w_4$	$3w_3 = w_1 + w_4$	too great
$w_0 + w_4$	$3w_3 = w_2 + w_4$	too great
$w_0 + w_4$	$3w_3 = w_3 + w_4$	too great
$w_1 + w_4$	$3w_3 = w_2 + w_4$	too great
$w_1 + w_4$	$3w_3 = w_3 + w_4$	too great
$w_2 + w_4$	$3w_3 = w_3 + w_4$	too great
d_1	d_2	
$w_0 + w_3 = 2w_4$	$w_1 + w_3$	
$w_0 + w_3$	$w_1 + w_3 = 2w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_1 + w_4$	see below
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_2 + w_4$	see below
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_3 + w_4$	
$w_0 + w_3 = w_1 + w_4$	$w_1 + w_3 = w_2 + w_4$	
$w_0 + w_3 = w_1 + w_4$	$w_1 + w_3 = w_3 + w_4$	
$w_0 + w_3 = w_2 + w_4$	$w_1 + w_3 = w_3 + w_4$	
$w_0 + w_3 = 2w_4$	$w_2 + w_3$	
$w_0 + w_3$	$w_2 + w_3 = 2w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_1 + w_4$	see below
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_2 + w_4$	see below
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_3 + w_4$	
$w_0 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_2 + w_4$	see below
$w_0 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_3 + w_4$	
$w_0 + w_3 = w_2 + w_4$	$w_2 + w_3 = w_3 + w_4$	

$w_0 + w_3 = 2w_4$	$w_3 + w_4$	
$w_0 + w_3$	$w_3 + w_4 = 2w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_0 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	
$w_0 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_0 + w_3 = w_2 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_0 + w_3 = 2w_4$	$w_1 + 2w_3$	too great
$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_1 + 2w_3 = w_1 + w_4$	too great
$w_0 + w_3 = w_0 + w_4$	$w_1 + 2w_3 = w_2 + w_4$	too great
$w_0 + w_3 = w_0 + w_4$	$w_1 + 2w_3 = w_3 + w_4$	too great
$w_0 + w_3 = w_1 + w_4$	$w_1 + 2w_3 = w_2 + w_4$	too great
$w_0 + w_3 = w_1 + w_4$	$w_1 + 2w_3 = w_3 + w_4$	too great
$w_0 + w_3 = w_2 + w_4$	$w_1 + 2w_3 = w_3 + w_4$	too great
$w_0 + w_3 = 2w_4$	$w_2 + 2w_3$	too great
$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	
$w_0 + w_3 = w_0 + w_4$	$w_2 + 2w_3 = w_1 + w_4$	too great
$w_0 + w_3 = w_0 + w_4$	$w_2 + 2w_3 = w_2 + w_4$	too great
$w_0 + w_3 = w_0 + w_4$	$w_2 + 2w_3 = w_3 + w_4$	too great
$w_0 + w_3 = w_1 + w_4$	$w_2 + 2w_3 = w_2 + w_4$	too great
$w_0 + w_3 = w_1 + w_4$	$w_2 + 2w_3 = w_3 + w_4$	too great
$w_0 + w_3 = w_2 + w_4$	$w_2 + 2w_3 = w_3 + w_4$	too great

d_1	d_2	
$w_1 + w_3 = 2w_4$	$w_2 + w_3$	
$w_1 + w_3$	$w_2 + w_3 = 2w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_1 + w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_2 + w_4$	see below
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_3 + w_4$	
$w_1 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_2 + w_4$	see below
$w_1 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_3 + w_4$	
$w_1 + w_3 = w_2 + w_4$	$w_2 + w_3 = w_3 + w_4$	
$w_1 + w_3 = 2w_4$	$w_3 + w_4$	
$w_1 + w_3$	$w_3 + w_4 = 2w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_1 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	
$w_1 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_1 + w_3 = w_2 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_1 + w_3 = 2w_4$	$w_0 + 2w_3$	too great
$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4$	see below
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	
$w_1 + w_3 = w_1 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	too great
$w_1 + w_3 = w_1 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	too great
$w_1 + w_3 = w_2 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	too great

$w_1 + w_3 = 2w_4$	$w_2 + 2w_3$	too great
$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_2 + 2w_3 = w_1 + w_4$	too great
$w_1 + w_3 = w_0 + w_4$	$w_2 + 2w_3 = w_2 + w_4$	too great
$w_1 + w_3 = w_0 + w_4$	$w_2 + 2w_3 = w_3 + w_4$	too great
$w_1 + w_3 = w_1 + w_4$	$w_2 + 2w_3 = w_2 + w_4$	too great
$w_1 + w_3 = w_1 + w_4$	$w_2 + 2w_3 = w_3 + w_4$	too great
$w_1 + w_3 = w_2 + w_4$	$w_2 + 2w_3 = w_3 + w_4$	too great
$w_2 + w_3 = 2w_4$	$w_3 + w_4$	
$w_2 + w_3$	$w_3 + w_4 = 2w_4$	
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	
$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_2 + w_3 = w_2 + w_4$	$w_3 + w_4 = w_3 + w_4$	
$w_2 + w_3 = 2w_4$	$w_0 + 2w_3$	too great
$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4$	
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	see below
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	
$w_2 + w_3 = w_1 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	too great
$w_2 + w_3 = w_1 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	too great
$w_2 + w_3 = w_2 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	too great

d_1	d_2	
$w_2 + w_3 = 2w_4$	$w_1 + 2w_3$	too great
$w_2 + w_3$	$w_1 + 2w_3 = 2w_4$	
$w_2 + w_3 = w_0 + w_4$	$w_1 + 2w_3 = w_1 + w_4$	too great
$w_2 + w_3 = w_0 + w_4$	$w_1 + 2w_3 = w_2 + w_4$	too great
$w_2 + w_3 = w_0 + w_4$	$w_1 + 2w_3 = w_3 + w_4$	too great
$w_2 + w_3 = w_1 + w_4$	$w_1 + 2w_3 = w_2 + w_4$	too great
$w_2 + w_3 = w_1 + w_4$	$w_1 + 2w_3 = w_3 + w_4$	too great
$w_2 + w_3 = w_2 + w_4$	$w_1 + 2w_3 = w_3 + w_4$	too great

D.1.4 Restrictions because of $\{3, 4\}$

We also need to consider Corollary 16 applied to $\{3, 4\}$. All of the above satisfy it immediately except the following:

d_1	d_2
$2w_3 = 2w_4$	
	$2w_3 = 2w_4$
	$3w_3 = 2w_4$
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_1 + w_4$
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_2 + w_4$
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_1 + w_4$
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_2 + w_4$
$w_0 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_2 + w_4$
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_2 + w_4$
$w_1 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_2 + w_4$
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4$
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$

Take each of these in turn. If $d_1 = 2w_3 = 2w_4$, positivity requires $d_1 + d_2 < w_0 + w_1 + w_2 + w_3 + w_4 \leq w_0 + 4w_4$, so $d_2 < w_0 + 2w_4 = w_0 + 2w_3$. But Corollary 14 applied to $\{3,4\}$ implies that d_2 must involve at least one of w_3 or w_4, so $d_2 = 2w_3$, $d_2 = w_3 + w_0$, $d_2 = w_3 + w_1$, or $d_2 = w_3 + w_2$. In any case, Therefore $2w_3 \geq d_2 \geq d_1 = 2w_3$.

If $d_2 = 2w_3 = 2w_4$, Corollary 14 applied to $\{3,4\}$ implies that $d_1 = 2w_3$ or $d_1 = w_3 + w_i$ for some $i = 0,1,2$.

If $d_2 = 3w_3 = 2w_4$, Corollary 14 applied to $\{3,4\}$ gives nine possibilities for d_1: $2w_4$, $2w_3$, $w_0 + w_3$, $w_0 + w_4$, $w_1 + w_3$, $w_1 + w_4$, $w_2 + w_3$, $w_2 + w_4$, $w_3 + w_4$. Positivity eliminates all of these that contain w_4 since $d_2 = 3w_3$. This leaves $d_1 = 2w_3$, $d_1 = w_0 + w_3$, $d_1 = w_1 + w_3$, $d_1 = w_2 + w_3$.

If $d_1 = w_0 + w_3 = w_0 + w_4$ and $d_2 = w_1 + w_3 = w_1 + w_4$ then $w_3 = w_4$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_2 = 2w_3$, (iii) $d_1 = w_2 + w_3$ and $d_2 = w_0 + w_3$, (iv) $d_1 = w_1 + w_3$ and $d_2 = w_2 + w_3$, or (v) $d_1 = w_2 + w_3$ and $d_2 = w_2 + w_3$. (i) implies $w_0 = w_3$, hence $w_0 = w_4$, (ii) implies $w_1 = w_3$, hence $w_1 = w_4$, and (iii),(iv), or (v) imply $w_0 = w_2$.

If either $d_1 = w_0 + w_3 = w_0 + w_4$ and $d_2 = w_1 + w_3 = w_2 + w_4$, or $d_1 = w_0 + w_3 = w_0 + w_4$ and $d_2 = w_2 + w_3 = w_1 + w_4$, then $w_1 = w_2$ and $w_3 = w_4$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_2 = 2w_3$, (iii) $d_1 = w_1 + w_3$, or (iv) $d_1 = w_2 + w_3$. (i) implies $w_0 = w_3$, hence $w_0 = w_4$, (ii) implies $w_1 = w_3$, hence $w_1 = w_4$, and (iii) or (iv) imply $w_0 = w_2$.

If $d_1 = w_0 + w_3 = w_0 + w_4$ and $d_2 = w_2 + w_3 = w_2 + w_4$ then $w_3 = w_4$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_2 = 2w_3$, (iii) $d_1 = w_1 + w_3$ and $d_2 = w_0 + w_3$, (iv) $d_1 = w_2 + w_3$ and $d_2 = w_1 + w_3$, or (v) $d_1 = w_1 + w_3$ and $d_2 = w_1 + w_3$. (i) implies $w_0 = w_3$, hence $w_0 = w_4$, (ii) implies $w_2 = w_3$, hence $w_2 = w_4$, and (iii),(iv), or (v) imply $w_0 = w_2$.

If either $d_1 = w_0 + w_3 = w_1 + w_4$ and $d_2 = w_2 + w_3 = w_2 + w_4$, or $d_1 = w_1 + w_3 = w_0 + w_4$

and $d_2 = w_2 + w_3 = w_2 + w_4$, then $w_0 = w_1$ and $w_3 = w_4$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_2 = 2w_3$, (iii) $d_2 = w_0 + w_3$, or (iv) $d_2 = w_1 + w_3$. (i) implies $w_0 = w_3$, hence $w_0 = w_4$, (ii) implies $w_2 = w_3$, hence $w_2 = w_4$, and (iii) or (iv) imply $w_0 = w_2$.

If $d_1 = w_1 + w_3 = w_1 + w_4$ and $d_2 = w_2 + w_3 = w_2 + w_4$ then $w_3 = w_4$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_2 = 2w_3$, (iii) $d_1 = w_0 + w_3$ and $d_2 = w_0 + w_3$, (iv) $d_1 = w_0 + w_3$ and $d_2 = w_1 + w_3$, or (v) $d_1 = w_2 + w_3$ and $d_2 = w_0 + w_3$. (i) implies $w_1 = w_3$, hence $w_1 = w_4$, (ii) implies $w_2 = w_3$, hence $w_2 = w_4$, and (iii),(iv), or (v) imply $w_0 = w_2$.

If $d_1 = w_1 + w_3 = w_0 + w_4$ and $d_2 = w_0 + 2w_3 = w_1 + w_4$, then $w_0 < w_1$, $w_3 < w_4$, and $d_1 < d_2$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_1 = w_3 + w_4$, (iii) $d_1 = 2w_4$, (iv) $d_2 = 2w_3$, (v) $d_2 = w_3 + w_4$, (vi) $d_2 = 2w_4$, (vii) $d_1 = w_2 + w_3$, (viii) $d_1 = w_2 + w_4$, (ix) $d_2 = w_2 + w_3$, (x) $d_2 = w_2 + 2w_3$, or (xi) $d_2 = w_2 + w_4$. (i) or (v) imply $w_1 = w_3$, (ii) implies $w_0 = w_3$ which is a contradiction, (iii) implies $w_0 = w_4$ which is a contradiction, (iv) is impossible, (vi) implies $w_1 = w_4$ which is a contradiction, (vii) or (xi) imply $w_1 = w_2$, (viii) or (x) imply $w_0 = w_2$ which is a contradiction, and (ix) is impossible.

If $d_1 = w_2 + w_3 = w_0 + w_4$ and $d_2 = w_0 + 2w_3 = w_2 + w_4$, then $w_0 < w_1$, $w_3 < w_4$, and $d_1 < d_2$. Consider Corollary 14 applied to $\{3,4\}$. We need either (i) $d_1 = 2w_3$, (ii) $d_1 = w_3 + w_4$, (iii) $d_1 = 2w_4$, (iv) $d_2 = 2w_3$, (v) $d_2 = w_3 + w_4$, (vi) $d_2 = 2w_4$, (vii) $d_1 = w_1 + w_3$, (viii) $d_1 = w_1 + w_4$, (ix) $d_2 = w_1 + w_3$, (x) $d_2 = w_1 + 2w_3$, or (xi) $d_2 = w_1 + w_4$. (i) or (v) imply $w_2 = w_3$, (ii) implies $w_0 = w_3$ which is a contradiction, (iii) implies $w_0 = w_4$ which is a contradiction, (iv) is impossible, (vi) implies $w_2 = w_4$ which is a contradiction, (vii) or (xi) imply $w_1 = w_2$, (viii) or (x) imply $w_0 = w_1$ and is hence too great, and (ix) is impossible.

In sum, replace the above twelve cases with the following:

d_1	d_2
$2w_3 = 2w_4$	$2w_3 = 2w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$
$w_1 + w_3 = w_1 + w_4$	$2w_3 = 2w_4$
$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$
$2w_3$	$3w_3 = 2w_4$
$w_0 + w_3$	$3w_3 = 2w_4$
$w_1 + w_3$	$3w_3 = 2w_4$
$w_2 + w_3$	$3w_3 = 2w_4$
$2w_0 = 2w_4$	$2w_0 = 2w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_1 = 2w_4$
$w_0 + w_3 = w_2 + w_4$	$w_0 + w_3 = w_2 + w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_2 = 2w_4$
$2w_1 = 2w_4$	$2w_1 = 2w_4$
$w_1 + w_3 = w_1 + w_4$	$2w_2 = 2w_4$
$w_1 + w_3 = 2w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4 = w_3 + w_4$
$w_1 + w_3 = w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4 = w_2 + w_4$
$w_2 + w_3 = 2w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4 = w_3 + w_4$

D.1.5 Order constraints

Next, we note any additional constraints put on the weights by the relations.

d_1	d_2	constraints
$2w_3 = 2w_4$	$2w_3 = 2w_4$	$w_3 = w_4$, $d_1 = d_2$
$2w_3$	$2w_4$	
$2w_3 = w_0 + w_4$	$w_1 + w_4$	
$2w_3 = w_0 + w_4$	$w_2 + w_4$	
$2w_3 = w_0 + w_4$	$w_3 + w_4$	
$2w_3 = w_1 + w_4$	$w_2 + w_4$	
$2w_3 = w_1 + w_4$	$w_3 + w_4$	
$2w_3 = w_2 + w_4$	$w_3 + w_4$	
$2w_4$	$2w_3$	$w_3 = w_4$, $d_1 = d_2$
$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$	$w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$2w_3 = 2w_4$	$w_3 = w_4$
$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_3 = w_4$
$w_0 + w_4$	$2w_3 = w_1 + w_4$	
$w_0 + w_4$	$2w_3 = w_2 + w_4$	
$w_0 + w_4$	$2w_3 = w_3 + w_4$	$w_3 = w_4$
$w_1 + w_4$	$2w_3 = w_2 + w_4$	
$w_1 + w_4$	$2w_3 = w_3 + w_4$	$w_3 = w_4$
$w_2 + w_4$	$2w_3 = w_3 + w_4$	$w_3 = w_4$
$2w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, w_3 even
$w_0 + w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, w_3 even
$w_1 + w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, w_3 even
$w_2 + w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, w_3 even
$w_0 + w_3 = 2w_4$	$w_1 + w_3$	$w_0 = w_1 = w_2 = w_3 = w_4$
$w_0 + w_3$	$w_1 + w_3 = 2w_4$	$w_1 = w_2 = w_3 = w_4$

d_1	d_2	constraints
$2w_0 = 2w_4$	$2w_0 = 2w_4$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3 = w_0 + w_4$	$2w_1 = 2w_4$	$w_1 = w_2 = w_3 = w_4$
$w_0 + w_3 = w_2 + w_4$	$w_0 + w_3 = w_2 + w_4$	$w_0 = w_1 = w_2,\ w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_3 + w_4$	$w_1 = w_2 = w_3 = w_4$
$w_0 + w_3 = w_1 + w_4$	$w_1 + w_3 = w_2 + w_4$	$w_0 = w_1 = w_2,\ w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3 = w_1 + w_4$	$w_1 + w_3 = w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3 = w_2 + w_4$	$w_1 + w_3 = w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3 = 2w_4$	$w_2 + w_3$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3$	$w_2 + w_3 = 2w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_2 = 2w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_0 = w_1,\ w_2 = w_3 = w_4$
$w_0 + w_3 = w_2 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3 = 2w_4$	$w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_0 + w_3$	$w_3 + w_4 = 2w_4$	$w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	$w_1 = w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_3 = w_4$
$w_0 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_0 = w_1,\ w_2 = w_3 = w_4$
$w_0 + w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 = w_1,\ w_3 = w_4$
$w_0 + w_3 = w_2 + w_4$	$w_3 + w_4$	$w_0 = w_1 = w_2,\ w_3 = w_4$

d_1	d_2	constraints
$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_3 < w_4$, w_1 even $d_1 < d_2$
$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_3 < w_4$, w_2 even $d_1 < d_2$
$w_1 + w_3 = 2w_4$	$w_2 + w_3$	$w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_1 + w_3$	$w_2 + w_3 = 2w_4$	$w_2 = w_3 = w_4$
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_1 + w_4$	
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_0 = w_1$, $w_2 = w_3 = w_4$
$2w_1 = 2w_4$	$2w_1 = 2w_4$	
$w_1 + w_3 = w_1 + w_4$	$2w_2 = 2w_4$	
$w_1 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_3 = w_4$
$w_1 + w_3 = w_2 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_3 = w_4$ $d_1 = d_2$
$w_1 + w_3 = 2w_4$	$w_3 + w_4$	$w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
$w_1 + w_3$	$w_3 + w_4 = 2w_4$	$w_3 = w_4$
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	$w_1 = w_2 = w_3$
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3$
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4$	
$w_1 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$w_3 + w_4$	$w_3 = w_4$
$w_1 + w_3 = w_2 + w_4$	$w_3 + w_4$	$w_1 = w_2$, $w_3 = w_4$
$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_3 < w_4$, w_0 even $d_1 < d_2$
$2w_3 = w_0 + w_4$ $w_1 + w_3$	$w_0 + 2w_3 = w_3 + w_4$ $w_1 + w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$
$w_2 + w_3 = w_0 + w_4$ $w_1 + w_3$	$w_0 + 2w_3 = w_2 + w_4$ $w_1 + w_4$	$w_0 < w_1 = w_2$, $w_3 < w_4$ $d_1 < d_2$

d_1	d_2	constraints
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	$w_0 < w_1,\ w_3 < w_4$ $d_1 < d_2$
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	$w_0 < w_1,\ w_3 < w_4$ $d_1 < d_2$
$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_3 < w_4,\ w_2$ even $d_1 < d_2$
$w_2 + w_3 = 2w_4$	$w_3 + w_4$	$w_2 = w_3 = w_4$ $d_1 = d_2$
$w_2 + w_3$	$w_3 + w_4 = 2w_4$	$w_3 = w_4$
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	$w_1 = w_2 = w_3$
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3$
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4$	
$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3$
$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4$	
$w_2 + w_3 = w_2 + w_4$	$w_3 + w_4$	$w_3 = w_4$
$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_3 < w_4,\ w_0$ even $d_1 < d_2$
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4$	$w_0 < w_1,\ w_3 < w_4$ $d_1 < d_2$
$= 2w_3 = w_0 + w_4$ $w_2 + w_3$	$w_0 + 2w_3 = w_3 + w_4$ $w_2 + w_4$	$w_0 < w_1,\ w_2 = w_3 < w_4$ $d_1 < d_2$
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	$w_0 < w_1,\ w_3 < w_4$ $d_1 < d_2$
$w_2 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_3 < w_4,\ w_1$ even $d_1 < d_2$

D.2 Cases by constraints

Now regroup the above by constraints.

D.2.1 Cases with at most 3 distinct weights

1.

d_1	d_2	constraints
$w_0 + w_3 = 2w_4$	$w_1 + w_3$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_0 + w_3 = w_1 + w_4$	$w_1 + w_3 = w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_0 + w_3 = w_2 + w_4$	$w_1 + w_3 = w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_0 + w_3 = 2w_4$	$w_2 + w_3$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_0 + w_3 = w_2 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_0 + w_3 = 2w_4$	$w_3 + w_4$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$2w_0 = 2w_4$	$2w_0 = 2w_4$	$w_0 = w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$

Given the constraint, these are all identical.

2.

d_1	d_2	constraints
$w_1 + w_3 = 2w_4$	$w_2 + w_3$	$w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_1 + w_3 = 2w_4$	$w_3 + w_4$	$w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$w_1 + w_3 = w_2 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$
$2w_1 = 2w_4$	$2w_1 = 2w_4$	$w_1 = w_2 = w_3 = w_4,\ d_1 = d_2$

Given the constraint, these are identical. Assume that $w_0 < w_1$ else this reduces to case 1.

3.

d_1	d_2	constraint
$w_0 + w_3$	$w_1 + w_3 = 2w_4$	$w_1 = w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_1 + w_3 = w_3 + w_4$	$w_1 = w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	$w_1 = w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_1 = 2w_4$	$w_1 = w_2 = w_3 = w_4$

Given the constraint, these are identical. Assume that $w_0 < w_1$ (and hence also $d_1 < d_2$) else this reduces to case 1.

We must have $\gcd(w_0, w_1) = 1$. Thus this form is $\mathbf{w} = (r, s, s, s, s)$, $\mathbf{d} = (r+s, 2s)$, with $\gcd(r, s) = 1$ and $r < s$. The Corollary for $\{0\}$ requires either $r \mid 2s$, or $r \mid (r+s)$, or $r \mid (2s-s)$ and $r \mid (r+s-s)$. In any case, $r \mid 2s$, so $r = 1, 2$. Consider the Corollary for $\{2,3,4\}$. Since $s \nmid (r+s)$ unless $r = s$, this rules out cases (a) and (b), leaving case (c). This requires $s \mid (r+s) - w_i$ for two different weights w_i, but $r = w_0$ is the only weight satisfying the relation, so case (c) cannot be satisfied either. Therefore, there are no possibilities.

4.

d_1	d_2	constraints
$w_0 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_0 = w_1$, $w_2 = w_3 = w_4$
$w_0 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_0 = w_1$, $w_2 = w_3 = w_4$
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_0 = w_1$, $w_2 = w_3 = w_4$

Given the constraints, these are identical. Assume $w_1 < w_2$ else this reduces to case 1.

5.

d_1	d_2	constraints
$w_0 + w_3 = w_1 + w_4$	$w_1 + w_3 = w_2 + w_4$	$w_0 = w_1 = w_2$, $w_3 = w_4$, $d_1 = d_2$
$w_0 + w_3 = w_2 + w_4$	$w_0 + w_3 = w_2 + w_4$	$w_0 = w_1 = w_2$, $w_3 = w_4$, $d_1 = d_2$

Given the constraints, these are identical. Assume $w_2 < w_3$ else this reduces to case 1.

6.

d_1	d_2	constraints
$w_0 + w_3 = w_2 + w_4$	$w_3 + w_4$	$w_0 = w_1 = w_2$, $w_3 = w_4$

Assume $w_2 < w_3$ else this reduces to case 1.

7.

d_1	d_2	constraints
$w_2 + w_3 = 2w_4$	$w_3 + w_4$	$w_2 = w_3 = w_4$, $d_1 = d_2$

Assume $w_1 < w_2$ else this reduces to case 2. This case can be split up into two:

case	d_1	d_2	constraints
7a	$w_2 + w_3 = 2w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 = w_3 = w_4,\ d_1 = d_2$
7b	$w_2 + w_3 = 2w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 = w_3 = w_4,\ d_1 = d_2$

8.

d_1	d_2	constraints
$w_0 + w_3$	$w_2 + w_3 = 2w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_2 = 2w_4$	$w_2 = w_3 = w_4$

Given the constraint, these are identical. Assume $w_0 < w_1$ else this reduces to case 4. Assume $w_1 < w_2$ else this reduces to case 3.

There are no instances. We must have $\gcd(w_0, w_1, w_2) = 1$. Thus this form is $\mathbf{w} = (r, s, t, t, t)$, $\mathbf{d} = (r + t, 2t)$, with $\gcd(r, s, t) = 1$ and $r < s < t$.

Consider the Corollary for $\{2, 3, 4\}$. Then since $d_1 = r + t < 2t$, (a) and (b) cannot hold. (c) requires $r + t = d_1 = mt + r$ and $r + t = d_1 = nt + s$. The first is satisfied with $m = 1$. The second requires $r + t = nt + s$ so $r = (n-1)t + s > (n-1)r + s > (n-1)r$ so $n - 1 \leq 0$. On the other hand $s + t > r + t$ so this is a contradiction.

9.

d_1	d_2	constraints
$w_1 + w_3$	$w_2 + w_3 = 2w_4$	$w_2 = w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$w_2 + w_3 = w_3 + w_4$	$w_2 = w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$2w_2 = 2w_4$	$w_2 = w_3 = w_4$

Given the constraint, these are identical. Assume $w_0 < w_1$ else this reduces to case 4. Assume $w_1 < w_2$ else this reduces to case 2.

There are no instances. We must have $\gcd(w_0, w_1, w_2) = 1$. This form is $\mathbf{w} = (r, s, t, t, t)$, $\mathbf{d} = (s + t, 2t)$, with $\gcd(r, s, t) = 1$ and $r < s < t$.

Consider the Corollary for $\{2,3,4\}$. Then since $d_1 = s+t < 2t$, (a) and (b) cannot hold. (c) requires $s+t = d_1 = mt+r$ and $s+t = d_1 = nt+s$. The second is satisfied with $n = 1$. The first requires $s+t = mt+r$ so $s = (m-1)t+r > (m-1)s+r > (m-1)s$ so $m-1 \leq 0$. On the other hand $s+t > r+t$ so this is a contradiction.

10.

d_1	d_2	constraints
$w_1 + w_3 = w_2 + w_4$	$w_3 + w_4$	$w_1 = w_2,\ w_3 = w_4$

Assume $w_0 < w_1$ else this reduces to case 6. Assume $w_2 < w_3$ else this reduces to case 2.

11.

d_1	d_2	constraints
$w_0 + w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 = w_1,\ w_3 = w_4$

Assume $w_1 < w_2$ else this reduces to case 6. Assume $w_2 < w_3$ else this reduces to case 4.

12.

d_1	d_2	constraints
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	$w_1 = w_2 = w_3$
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_1 + w_4$	$w_1 = w_2 = w_3$

Given the constraint, these are identical. Assume $w_0 < w_1$ and $w_3 < w_4$ else this reduces to case 1.

D.2.2 Cases with $d_2 = 2w_3 = 2w_4$ cases

13.

d_1	d_2	constraint
$2w_3 = 2w_4$	$2w_3 = 2w_4$	$w_3 = w_4,\ d_1 = d_2$
$2w_4$	$2w_3$	$w_3 = w_4,\ d_1 = d_2$

Given the constraint, these are identical.

If $w_0 = w_1 = w_2 = w_3$ this reduces to case 1. If $w_0 < w_1 = w_2 = w_3$ this reduces to case 2. If $w_0 = w_1 < w_2 = w_3$ this reduces to case 7a. If $w_0 < w_1 < w_2 = w_3$ this reduces to case 7b. So, assume $w_2 < w_3$. Positivity requires $w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2 = 3w_3 + w_4$, so $w_0 + w_1 + w_2 > 2w_3$ and $w_3 < (w_0 + w_1 + w_2)/2$

Consider the corollary for $\{2\}$. Then we must have either (i) $mw_2 = 2w_4$ for some $m > 1$, or (ii) $n_1w_2 + w_i = 2w_4$ and $n_2w_2 + w_j = 2w_4$ for some $i \neq j$ and some $n_1, n_2 > 0$. (i) $mw_2 = 2w_4 < w_0 + w_1 + w_2 \leq 3w_2$ so $m = 2$. But then $2w_2 = d_1 = 2w_3 > 2w_2$ which is a contradiction. This leaves: (ii) $n_1w_2 + w_i = 2w_4 < w_0 + w_1 + w_2 \leq w_0 + 2w_2 \leq w_1 + 2w_2 < 2w_2 + w_3 = 2w_2 + w_4$. Thus at most, $n_1 = 1$. Then $w_2 + w_i = d_1 = 2w_4 > w_2 + w_4 \geq w_2 + w_j$ for any $j = 0, 1, 3, 4$, which is a contradiction. Therefore, there are no instances.

14.

d_1	d_2	constraint
$w_0 + w_4$	$2w_3 = w_3 + w_4$	$w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_3 = w_4$
$w_0 + w_3$	$w_3 + w_4 = 2w_4$	$w_3 = w_4$
$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$	$w_3 = w_4$

Given the constraint, these are identical. If $w_0 = w_1 = w_2 = w_3$ this reduces to case 1. If $w_0 < w_1 = w_2 = w_3$ this reduces to case 3. If $w_0 = w_1 < w_2 = w_3$ this reduces to case 4. If $w_0 < w_1 < w_2 = w_3$ this reduces to case 8. So, assume $w_2 < w_3$. If $w_0 = w_1 = w_2$ this reduces to case 6. If $w_0 = w_1 < w_2$ this reduces to case 11. So, assume $w_0 < w_1$ and split up as follows:

case	d_1	d_2	constraints
14a	$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 = w_4$
14b	$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 = w_4$

15.

d_1	d_2	constraint
$w_1 + w_4$	$2w_3 = w_3 + w_4$	$w_3 = w_4$
$w_1 + w_3$	$w_3 + w_4 = 2w_4$	$w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$w_3 + w_4$	$w_3 = w_4$
$w_1 + w_3 = w_1 + w_4$	$2w_3 = 2w_4$	$w_3 = w_4$

Given the constraint, these are identical. If $w_0 = w_1 = w_2 = w_3$ this reduces to case 1. If $w_0 < w_1 = w_2 = w_3$ this reduces to case 2. If $w_0 = w_1 < w_2 = w_3$ this reduces to case 4. If $w_0 < w_1 < w_2 = w_3$ this reduces to case 9. So, assume $w_2 < w_3$. If $w_0 = w_1 = w_2$ this reduces to case 6. If $w_0 < w_1 = w_2$ this reduces to case 10. If $w_0 = w_1 < w_2$ this reduces to case 11. So we can assume $w_0 < w_1 < w_2 < w_3$.

16.

d_1	d_2	constraint
$w_2 + w_4$	$2w_3 = w_3 + w_4$	$w_3 = w_4$
$w_2 + w_3$	$w_3 + w_4 = 2w_4$	$w_3 = w_4$
$w_2 + w_3 = w_2 + w_4$	$w_3 + w_4$	$w_3 = w_4$
$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_3 = w_4$

Given the constraint, these are identical. If $w_0 = w_1 = w_2 = w_3$ this reduces to case 1. If $w_0 < w_1 = w_2 = w_3$ this reduces to case 2. If $w_0 = w_1 < w_2 = w_3$ this reduces to case 7a. If $w_0 < w_1 < w_2 = w_3$ this reduces to case 7b. So, assume $w_2 < w_3$. If $w_0 = w_1 = w_2$ this reduces to case 6. If $w_0 < w_1 = w_2$ this reduces to case 10. So assume $w_1 < w_2$ and split up as follows:

case	d_1	d_2	constraints
16a	$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 = w_4$
16b	$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 = w_4$

D.2.3 Cases with $w_2 = w_3 < w_4$

17.

d_1	d_2	constraint
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3$

Assume $w_3 < w_4$ else $w_0 = w_1$ and this reduces to case 4. Assume $w_0 < w_1$ else $w_3 = w_4$ and this reduces to case 4. Assume $w_1 < w_2$ else this reduces to case 12.

18.

d_1	d_2	constraint
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3$

Assume $w_3 < w_4$ else $w_0 = w_1 = w_2$ and this reduces to case 1. Assume $w_1 < w_2$ else this reduces to case 12. Split up as follows:

case	d_1	d_2	constraints
18a	$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$
18b	$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$

19.

d_1	d_2	constraint
$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4 = w_2 + w_4$	$w_2 = w_3$

Assume $w_3 < w_4$ else $w_1 = w_2$ and this reduces to case 2. Assume $w_1 < w_2$ else $w_3 = w_4$ and this reduces to case 2. Assume $w_0 < w_1$ else this reduces to case 18a.

D.2.4 Cases with $d_1 = 2w_3 < 2w_4 = d_2$

20.

d_1	d_2
$2w_3$	$2w_4$

We can assume $w_3 < w_4$ else this reduces to Case 13. Split up as follows:

case	d_1	d_2	constraint
20a	$2w_3$	$2w_4$	$w_0 = w_1 = w_2 = w_3 < w_4,\ d_1 < d_2$
20b	$2w_3$	$2w_4$	$w_0 < w_1 = w_2 = w_3 < w_4,\ d_1 < d_2$
20c	$2w_3$	$2w_4$	$w_0 = w_1 < w_2 = w_3 < w_4,\ d_1 < d_2$
20d	$2w_3$	$2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4,\ d_1 < d_2$
20e	$2w_3$	$2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4,\ d_1 < d_2$
20f	$2w_3$	$2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4,\ d_1 < d_2$
20g	$2w_3$	$2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4,\ d_1 < d_2$
20h	$2w_3$	$2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4,\ d_1 < d_2$

D.2.5 Cases with $d_1 = 2w_3 = w_i + w_4$

21.

d_1	d_2
$2w_3 = w_0 + w_4$	$w_1 + w_4$

Assume $w_3 < w_4$ else this reduces to case 1. Assume $w_0 < w_3$ else this reduces to case 1. If $w_1 = w_2 = w_3$ then this reduces to case 12. Split up as follows:

case	d_1	d_2	constraint
21a	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$
21b	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$
21c	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$
21d	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$
21e	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$
21f	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

22.

d_1	d_2
$2w_3 = w_0 + w_4$	$w_2 + w_4$

Assume $w_3 < w_4$ else this reduces to case 1. Assume $w_2 < w_3$ else this reduces to case 18. Assume $w_1 < w_2$ else this reduces to case 21. Split up as follows:

case	d_1	d_2	constraint
22a	$2w_3 = w_0 + w_4$	$w_2 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$
22b	$2w_3 = w_0 + w_4$	$w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

23.

d_1	d_2
$2w_3 = w_0 + w_4$	$w_3 + w_4$

Assume $w_3 < w_4$ else this reduces to case 1. Assume $w_2 < w_3$ else this reduces to case 18. Split up as follows:

case	d_1	d_2	constraint
23a	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$
23b	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$
23c	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$
23d	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

24.

d_1	d_2
$2w_3 = w_1 + w_4$	$w_2 + w_4$

Assume $w_3 < w_4$ else this reduces to case 2. Assume $w_2 < w_3$ else this reduces to case 19. If $w_0 = w_1 = w_2$ this reduces to case 21c. If $w_0 = w_1 < w_2$ this reduces to case 22a. Assume $w_0 < w_1$ and split up as follows:

case	d_1	d_2	constraint
24a	$2w_3 = w_1 + w_4$	$w_2 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$
24b	$2w_3 = w_1 + w_4$	$w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

25.

d_1	d_2
$2w_3 = w_1 + w_4$	$w_3 + w_4$

Assume $w_3 < w_4$ else this reduces to case 2. Assume $w_2 < w_3$ else this reduces to case 19. If $w_0 = w_1 = w_2$ this reduces to case 23a. If $w_0 = w_1 < w_2$ this reduces to case 23c. Assume $w_0 < w_1$ and split up as follows:

case	d_1	d_2	constraint
25a	$2w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$
25b	$2w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

26.

d_1	d_2
$2w_3 = w_2 + w_4$	$w_3 + w_4$

Assume $w_3 < w_4$ else this reduces to case 7. Assume $w_2 < w_3$ else this reduces to case 7. If $w_0 = w_1 = w_2$ this reduces to case 23a. If $w_0 < w_1 = w_2$ this reduces to case

25a. Assume $w_1 < w_2$ and split up as follows:

case	d_1	d_2	constraint
26a	$2w_3 = w_2 + w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$
26b	$2w_3 = w_2 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

D.2.6 Cases with $d_2 = 2w_3 = w_i + w_4$

27.

d_1	d_2
$w_0 + w_4$	$2w_3 = w_1 + w_4$

Assume $w_3 < w_4$ else $w_1 = w_3$ and this reduces to case 3. Assume $w_1 < w_3$ else $w_3 = w_4$ and this reduces to case 3. Assume $w_0 < w_1$ else $d_1 = d_2$ and this reduces to case 21. Split up as follows:

case	d_1	d_2	constraint
27a	$w_0 + w_4$	$2w_3 = w_1 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$
27b	$w_0 + w_4$	$2w_3 = w_1 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$
27c	$w_0 + w_4$	$2w_3 = w_1 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$

28.

d_1	d_2
$w_0 + w_4$	$2w_3 = w_2 + w_4$

Assume $w_3 < w_4$ else $w_2 = w_3$ and this reduces to case 8. Assume $w_2 < w_3$ else $w_3 = w_4$ and this reduces to case 8. Assume $w_1 < w_2$ else this reduces to case 27a. Split up as follows:

case	d_1	d_2	constraint
28a	$w_0 + w_4$	$2w_3 = w_2 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$
28b	$w_0 + w_4$	$2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$

29.

d_1	d_2
$w_1 + w_4$	$2w_3 = w_2 + w_4$

Assume $w_3 < w_4$ else $w_2 = w_3$ and this reduces to case 9. Assume $w_2 < w_3$ else $w_3 = w_4$ and this reduces to case 9. Assume $w_1 < w_2$ else $d_1 = d_2$ and this reduces to case 24a. Assume $w_0 < w_1$ else this reduces to case 28a.

D.2.7 A case involving all five weights

30.

d_1	d_2
$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_1 + w_4$

Assume $w_3 < w_4$ else $w_0 = w_1 = w_2$ and this reduces to case 5. Assume $w_0 < w_1$ else $w_3 = w_4$ and this reduces to case 5. Assume $w_1 < w_2$ else $w_3 = w_4$ and this reduces to case 5. Assume $w_2 < w_3$ else this reduces to case 27b.

D.2.8 $d_1 = w_j + w_3 = w_i + w_4$, $w_2 < w_3$

31.

d_1	d_2
$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4$

Assume $w_3 < w_4$ else $w_0 = w_1$ and this reduces to case 11. Assume $w_2 < w_3$ else this reduces to case 17. Assume $w_0 < w_1$ else $w_3 = w_4$ and this reduces to case 11. Split up as follows:

case	d_1	d_2	
31a	$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$
31b	$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

32.

d_1	d_2
$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4$

Assume $w_3 < w_4$ else $w_0 = w_1 = w_2$ and this reduces to case 6. Assume $w_2 < w_3$ else this reduces to case 18. Assume $w_1 < w_2$ else this reduces to case 31a. Split up as follows:

case	d_1	d_2	
32a	$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$
32b	$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$

33.

d_1	d_2
$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4$

Assume $w_3 < w_4$ else $w_1 = w_2$ and this reduces to case 10. Assume $w_2 < w_3$ else this reduces to Case 19. Assume $w_1 < w_2$ else $w_3 = w_4$ and this reduces to case 10. Assume $w_0 < w_1$ else this reduces to case 32a.

D.2.9 $d_2 = 2w_3 + w_i$

34.

d_1	d_2	constraints
$2w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_3 even

This is a special case of case 20.

35.

d_1	d_2	
$w_0 + w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_3 even

Assume $w_0 < w_3$ else this is a special case of 20.

case	d_1	d_2	
35a	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$, $d_1 < d_2$, w_3 even
35b	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$, w_3 even
35c	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$, w_3 even
35d	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even
35e	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even
35f	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even
35g	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even

36.

d_1	d_2	
$w_1 + w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_3 even

Assume $w_1 < w_3$ else this reduces to a special case of 20. Assume $w_0 < w_1$ else this reduces to 35.

case	d_1	d_2	
36a	$w_1 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$, w_3 even
36b	$w_1 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even
36c	$w_1 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even

37.

d_1	d_2	
$w_2 + w_3$	$3w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_3 even

Assume $w_2 < w_3$ else this reduces to a special case of 20. Assume $w_1 < w_2$ else this reduces to 36. Split up as follows:

case	d_1	d_2	
37a	$w_2 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even
37b	$w_2 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_3 even

38.

d_1	d_2	
$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_1 even

Assume $w_1 < w_3$ else this is a special case of case 35. Split up as follows:

case	d_1	d_2	
38a	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$, w_1 even
38b	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$, w_1 even
38c	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$, w_1 even
38d	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$, w_1 even
38e	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_1 even
38f	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_1 even

39.

d_1	d_2	
$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_2 even

Assume $w_2 < w_3$ else this is a special case of case 35. Assume $w_1 < w_2$ else this is reduces to case 38c or 38d. Split up as follows:

case	d_1	d_2	
39a	$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_2 even
39b	$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_2 even

40.

d_1	d_2	
$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_0 even

Assume $w_0 < w_3$ else this is a special case of case 36. Assume $w_0 < w_1$ else this reduces to case 38.

case	d_1	d_2	
40a	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$, $d_1 < d_2$, w_0 even
40b	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$, $d_1 < d_2$, w_0 even
40c	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$, $d_1 < d_2$, w_0 even
40d	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$, $d_1 < d_2$, w_0 even

41.

d_1	d_2	
$2w_3 = w_0 + w_4$ $w_1 + w_3$	$w_0 + 2w_3 = w_3 + w_4$ $w_1 + w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$

This is a special case of 12.

42.

d_1	d_2	
$w_2 + w_3 = w_0 + w_4$ $w_1 + w_3$	$w_0 + 2w_3 = w_2 + w_4$ $w_1 + w_4$	$w_0 < w_1 = w_2$, $w_3 < w_4$ $d_1 < d_2$

Assume $w_2 < w_3$ else this reduces to 41.

43.

d_1	d_2	
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	$w_3 < w_4,\ d_1 < d_2,\ w_0 < w_1$

Assume $w_1 < w_2$ else this reduces to case 41 or 42. Split up as follows:

case	d_1	d_2	
43a	$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4,\ d_1 < d_2$
43b	$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4,\ d_1 < d_2$

44.

d_1	d_2	
$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	$w_3 < w_4,\ d_1 < d_2,\ w_0 < w_1$

This is a special case of 31.

45.

d_1	d_2	
$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_3 < w_4,\ d_1 < d_2,\ w_2$ even

Assume $w_2 < w_3$ else this is a special case of case 36. Split up as follows:

case	d_1	d_2	
45a	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4,\ d_1 < d_2,\ w_2$ even
45b	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4,\ d_1 < d_2,\ w_2$ even
45c	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4,\ d_1 < d_2,\ w_2$ even
45d	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4,\ d_1 < d_2,\ w_2$ even

46.

d_1	d_2	
$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_3 < w_4,\ d_1 < d_2,\ w_0$ even

Assume $w_0 < w_2$ else this is reduces to case 45. Assume $w_2 < w_3$ else this is a special case of 20. Split up as follows:

case	d_1	d_2	
46a	$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4,\ d_1 < d_2,\ w_0$ even
46b	$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4,\ d_1 < d_2,\ w_0$ even
46c	$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4,\ d_1 < d_2,\ w_0$ even

47.

d_1	d_2	
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4$	$w_3 < w_4$, $d_1 < d_2$, $w_0 < w_1$

Assume $w_1 < w_2$ else this is a special case of 42. Assume $w_2 < w_3$ else this is a special case of 21.

48.

d_1	d_2	constraints
$2w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	$w_0 < w_1$, $w_2 = w_3 < w_4$
$w_2 + w_3$	$w_2 + w_4$	$d_1 < d_2$

Assume $w_1 < w_2$ else this is a special case of case 47. This is a special case of 18.

49.

d_1	d_2	constraints
$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_3 + w_4$	$w_3 < w_4$, $d_1 < d_2$, $w_0 < w_2$

This is a special case of 32.

50.

d_1	d_2	constraints
$w_2 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_3 < w_4$, $d_1 < d_2$, w_1 even

Assume $w_1 < w_2$ else this reduces to case 45. Assume $w_2 < w_3$ else this is a special case of 20. Assume $w_0 < w_1$ else this is a special case of case 49.

Positivity requires $w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2 = w_2 + w_3 + 2w_4$ so $w_0 + w_1 > w_4 = w_3 + w_1/2 > w_2 + w_1/2 > 3w_1/2$, thus $w_1 < 2w_0$ and $w_2 < w_3 < w_0 + w_1/2$.

Consider the Corollary applied to $\{1\}$. At least one of the following must hold: (i) $w_1 \mid d_1$, (ii) $w_1 \mid d_2$, (iii) $w_1 \mid (d_1 - w_0)$, (iv) $w_1 \mid (d_1 - w_2)$, (v) $w_1 \mid (d_1 - w_3)$, or (vi) $w_1 \mid (d_1 - w_4)$,

(i) $d_1 = kw_1$. But $2w_1 < 2w_2 < 2w_0 + w_1 < 3w_1$ so no such k exists.

(ii) $d_2 = kw_1$. But $3w_1 < 2w_3 + w_1 < 2w_0 + 2w_1 < 4w_1$ so no such k exists.

(iii) $d_1 = kw_1 + w_0$. $kw_1 = d_1 - w_0 = w_2 + w_3 - w_0 < 2w_0 + w_1 - w_0 = w_0 + w_1 < 2w_1$ so $k = 1$ and $d_1 = w_0 + w_1 < w_2 + w_3$ which is a contradiction.

(iv) $d_1 = kw_1 + w_2$. $kw_1 = d_1 - w_2 = w_3$, so $w_1 < w_3 < w_0 + w_1/2 < 2w_1$ so no such k exists.

(v) $d_1 = kw_1 + w_3$. $kw_1 = d_1 - w_3 = w_2$, so $w_1 < w_2 < w_0 + w_1/2 < 2w_1$ so no such k exists.

(vi) $d_1 = kw_1 + w_4$. $kw_1 = d_1 - w_4 = w_2 + w_3 - w_4 = w_2 + w_3 - (w_3 + w_1/2) = w_2 - w_1/2$. $kw_0 + w_1/2 < kw_1 + w_1/2 = w_2 < w_0 + w_1/2$ so $k < 1$ which is impossible. Therefore, there are no instances.

D.3 Summary

case	d_1	d_2	constraints
1	$2w_0 = 2w_4$	$2w_0 = 2w_4$	$w_0 = w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
2	$2w_1 = 2w_4$	$2w_1 = 2w_4$	$w_0 < w_1 = w_2 = w_3 = w_4$ $d_1 = d_2$
4	$w_0 + w_2 = w_1 + w_4$	$2w_2 = 2w_4$	$w_0 = w_1 < w_2 = w_3 = w_4$ $d_1 < d_2$
5	$w_0 + w_3 = w_2 + w_4$	$w_0 + w_3 = w_2 + w_4$	$w_0 = w_1 = w_2 < w_3 = w_4$ $d_1 = d_2$
6	$w_0 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 = w_4$ $d_1 < d_2$
7a	$2w_2 = 2w_4$	$2w_2 = 2w_4$	$w_0 = w_1 < w_2 = w_3 = w_4$ $d_1 = d_2$
7b	$2w_2 = 2w_4$	$2w_2 = 2w_4$	$w_0 < w_1 < w_2 = w_3 = w_4$ $d_1 = d_2$
10	$w_1 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 = w_4$ $d_1 < d_2$
11	$w_0 + w_3 = w_1 + w_4$	$2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 = w_4$ $d_1 < d_2$
12	$w_1 + w_3 = 2w_3 = w_0 + w_4$	$w_1 + w_4 = w_3 + w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$
14a	$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 = w_4$ $d_1 < d_2$
14b	$w_0 + w_3 = w_0 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 = w_4$ $d_1 < d_2$
15	$w_1 + w_3 = w_1 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 = w_4$ $d_1 < d_2$

case	d_1	d_2	constraints
16a	$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 = w_4$ $d_1 < d_2$
16b	$w_2 + w_3 = w_2 + w_4$	$2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 = w_4$ $d_1 < d_2$
17	$w_1 + w_3 = w_0 + w_4$	$w_2 + w_4 = w_3 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
18a	$2w_2 = 2w_3 = w_0 + w_4$	$w_2 + w_4 = w_3 + w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
18b	$2w_2 = 2w_3 = w_0 + w_4$	$w_2 + w_4 = w_3 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
19	$2w_2 = 2w_3 = w_1 + w_4$	$w_2 + w_4 = w_3 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
20a	$2w_3$	$2w_4$	$w_0 = w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$
20b	$2w_3$	$2w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$
20c	$2w_3$	$2w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
20d	$2w_3$	$2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
20e	$2w_3$	$2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
20f	$2w_3$	$2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
20g	$2w_3$	$2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
20h	$2w_3$	$2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$

case	d_1	d_2	constraints
21a	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$ $d_1 = d_2$
21b	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
21c	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$ $d_1 = d_2$
21d	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
21e	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 = d_2$
21f	$2w_3 = w_0 + w_4$	$w_1 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
22a	$2w_3 = w_0 + w_4$	$w_2 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
22b	$2w_3 = w_0 + w_4$	$w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
23a	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
23b	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
23c	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
23d	$2w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
24a	$2w_3 = w_1 + w_4$	$w_2 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 = d_2$
24b	$2w_3 = w_1 + w_4$	$w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$

case	d_1	d_2	constraints
25a	$2w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
25b	$2w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
26a	$2w_3 = w_2 + w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
26b	$2w_3 = w_2 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
27a	$w_0 + w_4$	$2w_3 = w_1 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
27b	$w_0 + w_4$	$2w_3 = w_1 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
27c	$w_0 + w_4$	$2w_3 = w_1 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
28a	$w_0 + w_4$	$2w_3 = w_2 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
28b	$w_0 + w_4$	$2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
29	$w_1 + w_4$	$2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
30	$w_1 + w_3 = w_0 + w_4$	$w_2 + w_3 = w_1 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
31a	$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$
31b	$w_1 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$

case	d_1	d_2	constraints
32a	$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
32b	$w_2 + w_3 = w_0 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
33	$w_2 + w_3 = w_1 + w_4$	$w_3 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
35a	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$, w_3 even
35b	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$, w_3 even
35c	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$, w_3 even
35d	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
35e	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
35f	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
35g	$w_0 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
36a	$w_1 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$, w_3 even
36b	$w_1 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
36c	$w_1 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even

case	d_1	d_2	constraints
37a	$w_2 + w_3$	$3w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
37b	$w_2 + w_3$	$3w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_3 even
38a	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$, w_1 even
38b	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$, w_1 even
38c	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_1 even
38d	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_1 even
38e	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_1 even
38f	$w_0 + w_3$	$w_1 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_1 even
39a	$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_2 even
39b	$w_0 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_2 even
40a	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$, w_0 even
40b	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$, w_0 even
40c	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_0 even
40d	$w_1 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_0 even

case	d_1	d_2	constraints
41	$2w_3 = w_0 + w_4$ $w_1 + w_3$	$w_0 + 2w_3 = w_3 + w_4$ $w_1 + w_4$	$w_0 < w_1 = w_2 = w_3 < w_4$ $d_1 < d_2$
42	$w_2 + w_3 = w_0 + w_4$ $w_1 + w_3$	$w_0 + 2w_3 = w_2 + w_4$ $w_1 + w_4$	$w_0 < w_1 = w_2$, $w_3 < w_4$ $d_1 < d_2$
43a	$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 = w_3 < w_4$ $d_1 < d_2$
43b	$w_1 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_2 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$
45a	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 = w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_2 even
45b	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_2 even
45c	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_2 even
45d	$w_1 + w_3$	$w_2 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_2 even
46a	$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 = w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_0 even
46b	$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 = w_2 < w_3 < w_4$ $d_1 < d_2$, w_0 even
46c	$w_2 + w_3$	$w_0 + 2w_3 = 2w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$, w_0 even
47	$w_2 + w_3 = w_0 + w_4$	$w_0 + 2w_3 = w_1 + w_4$	$w_0 < w_1 < w_2 < w_3 < w_4$ $d_1 < d_2$

D.4 Details of cases

1. Given that $\gcd(w_0, w_1, w_2, w_3, w_4) = 1$, these reduce to the single instance $\mathbf{w} = (1,1,1,1,1)$. By positivity, the only choice for degree is $d_1 = d_2 = 2$. Thus, the only instance is

$\mathbf{w} = (1,1,1,1,1)$, $\mathbf{d} = (2,2)$.

2. We must have $\gcd(w_0, w_1) = 1$. Thus this form is $\mathbf{w} = (r, s, s, s, s)$, $\mathbf{d} = (2s, 2s)$, with $\gcd(r, s) = 1$ and $r < s$. The Corollary for $\{0\}$ requires either $r \mid 2s$, or $r \mid (2s-s)$. Thus $r = 1, 2$. Possibilities are:

$\mathbf{w} = (1, s, s, s, s)$, $\mathbf{d} = (2s, 2s)$, $2 \le s$ and

$\mathbf{w} = (2, 2t+1, 2t+1, 2t+1, 2t+1)$, $\mathbf{d} = (4t+2, 4t+2)$, $1 \le t$.

For $s = 1$, the first is case 1.

4. We must have $\gcd(w_0, w_2) = 1$. Thus this form is $\mathbf{w} = (r, r, s, s, s)$, $\mathbf{d} = (r+s, 2s)$, with $\gcd(r, s) = 1$ and $r < s$. The Corollary for $\{0\}$ requires either $r \mid 2s$, or $r \mid (r+s)$, or $r \mid (2s - s)$ and $r \mid (r + s - s)$, or $r \mid (2s - r)$ and $r \mid (r + s - s)$, or $r \mid (2s - s)$ and $r \mid (r + s - r)$. In any case, $r \mid 2s$, so $r = 1, 2$. This leaves

$\mathbf{w} = (1, 1, s, s, s)$, $\mathbf{d} = (1 + s, 2s)$ $2 \le s$ and

$\mathbf{w} = (2, 2, 2t+1, 2t+1, 2t+1)$, $\mathbf{d} = (2t+3, 4t+2)$, $1 \le t$.

For $t = 1$ this is the same as 17b for $t = 1$.

For $s = 1$, the first is case 1.

5. We must have $\gcd(w_0, w_3) = 1$. Thus this form is $\mathbf{w} = (r, r, r, s, s)$, $\mathbf{d} = (r + s, r + s)$, with $\gcd(r, s) = 1$ and $r < s$. Consider the Corollary for $\{0, 1\}$. If (a), (b), or (c) hold, then $r \mid (r + s) \Rightarrow r \mid s \Rightarrow r = 1$. If (d) holds, then $r \mid (r + s - s)$ and $r \mid (r + s - r)$. The latter implies $r \mid s$, so again $r = 1$. Thus the only case is

$\mathbf{w} = (1,1,1,s,s)$, $\mathbf{d} = (s+1,s+1)$, $2 \le s$.

For $s = 1$, this is case 1.

6. We must have $\gcd(w_0, w_3) = 1$. Thus this form is $\mathbf{w} = (r,r,r,s,s)$, $\mathbf{d} = (r+s, 2s)$, with $\gcd(r,s) = 1$ and $r < s$. Consider the Corollary for $\{0\}$. Then either $r \mid (r+s)$, or $r \mid 2s$, or $r \mid (r+s-s)$ and $r \mid (2s-s)$, or $r \mid (r+s-s)$ and $r \mid (2s-r)$, or $r \mid (r+s-r)$ and $r \mid (2s-s)$, or $r \mid (r+s-r)$ and $r \mid (2s-r)$. In any case, $r \mid s$ or $r \mid 2s$, so $r = 1,2$. If $r = 1$, then for positivity, we need $3 + 2s > 3s + 1$, so $1 < s < 2$, which is a contradiction. If $r = 2$, then for positivity, we need $6 + 2s > 3s + 2$, so $2 < s < 4$. The only instance is:

$\mathbf{w} = (2,2,2,3,3)$, $\mathbf{d} = (5,6)$.

7a.

We must have $\gcd(w_0, w_2) = 1$. Thus this form is $\mathbf{w} = (r,r,s,s,s)$, $\mathbf{d} = (2s, 2s)$, with $\gcd(r,s) = 1$ and $r < s$. Positivity implies $2r + 3s > 4s \Rightarrow 2r > s$. The Corollary for $\{0\}$ implies $r \mid 2s$, or $r \mid (2s-s) = s$, or $r \mid (2s-r)$ and $r \mid (2s-s) = s$. In any case, $r \mid 2s$, so $r = 1,2$. $r = 1 \Rightarrow 2 > s > 1$ which is a contradiction. $r = 2 \Rightarrow 4 > s > 2 \Rightarrow s = 3$. The only case is:

$(\mathbf{w} = (2,2,3,3,3), \mathbf{d} = (6,6))$.

7b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3,4,6,6,6),(12,12))$

$(\mathbf{w}, \mathbf{d}) = ((6,10,15,15,15),(30,30))$

10. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3,4,4,6,6),(10,12))$

$(\mathbf{w}, \mathbf{d}) = ((5,6,6,9,9),(15,18))$

$(\mathbf{w}, \mathbf{d}) = ((5, 8, 8, 12, 12), (20, 24))$

$(\mathbf{w}, \mathbf{d}) = ((6, 10, 10, 15, 15), (25, 30))$

11. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((2, 2, 3, 4, 4), (6, 8))$

$(\mathbf{w}, \mathbf{d}) = ((3, 3, 4, 6, 6), (9, 12))$

$(\mathbf{w}, \mathbf{d}) = ((4, 4, 5, 6, 6), (10, 12))$

$(\mathbf{w}, \mathbf{d}) = ((4, 4, 7, 10, 10), (14, 20))$

$(\mathbf{w}, \mathbf{d}) = ((6, 6, 10, 15, 15), (21, 30))$

12. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((1, 2, 2, 2, 3), (4, 5))$

14a. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((1, 2, 2, 3, 3), (4, 6))$

$(\mathbf{w}, \mathbf{d}) = ((1, 3, 3, 5, 5), (6, 10))$

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 3, 4, 4), (6, 8))$

14b.

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 3t + 2, 3t + 3, 3t + 3), (3t + 5, 6t + 6)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 3t + 3, 3t + 4, 3t + 4), (3t + 6, 6t + 8)),\ 0 \leq t$

or $(\mathbf{w}, \mathbf{d}) = ((2, 3, t + 1, t + 2, t + 2), (t + 4, 2t + 4)),\ 1 \leq t,\ t \neq 0 \mod (3)$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 6t + 1, 6t + 3, 6t + 3), (6t + 7, 12t + 6)),\ 1 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 6t + 3, 6t + 5, 6t + 5), (6t + 9, 12t + 10)),\ 0 \leq t$

or $(\mathbf{w},\mathbf{d}) = ((4,6,2t+1,2t+3,2t+3),(2t+7,4t+6)),\ 1 \le t,\ t \ne 2 \mod (3)$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((1,3,4,6,6),(7,12))$

$(\mathbf{w},\mathbf{d}) = ((1,4,5,8,8),(9,16))$

$(\mathbf{w},\mathbf{d}) = ((1,6,10,15,15),(16,30))$

$(\mathbf{w},\mathbf{d}) = ((1,7,12,18,18),(19,36))$

$(\mathbf{w},\mathbf{d}) = ((1,8,13,20,20),(21,40))$

$(\mathbf{w},\mathbf{d}) = ((1,9,15,23,23),(24,46))$

$(\mathbf{w},\mathbf{d}) = ((2,7,10,15,15),(17,30))$

$(\mathbf{w},\mathbf{d}) = ((2,9,12,19,19),(21,38))$

$(\mathbf{w},\mathbf{d}) = ((3,4,6,9,9),(12,18))$

$(\mathbf{w},\mathbf{d}) = ((3,7,8,12,12),(15,24))$

$(\mathbf{w},\mathbf{d}) = ((4,6,9,14,14),(18,28))$

$(\mathbf{w},\mathbf{d}) = ((5,6,9,13,13),(18,26))$

15.

$(\mathbf{w},\mathbf{d}) = ((4,4t+1,4t+2,4t+3,4t+3),(8t+4,8t+6)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((4,4t+4,4t+5,4t+6,4t+6),(8t+10,8t+12)),\ 1 \le t$

or $(\mathbf{w},\mathbf{d}) = ((4,t+1,t+2,t+3,t+3),(2t+4,2t+6)),\ 1 \le t,\ t = 0,3 \mod (4)$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((2,3,4,5,5),(8,10))$

$(\mathbf{w}, \mathbf{d}) = ((2, 4, 5, 6, 6), (10, 12))$

$(\mathbf{w}, \mathbf{d}) = ((3, 4, 6, 8, 8), (12, 16))$

$(\mathbf{w}, \mathbf{d}) = ((4, 8, 11, 14, 14), (22, 28))$

$(\mathbf{w}, \mathbf{d}) = ((4, 12, 15, 18, 18), (30, 36))$

$(\mathbf{w}, \mathbf{d}) = ((6, 10, 15, 20, 20), (30, 40))$

16a. There are no instances. We have $w_0 = w_1 < w_2 < w_3 = w_4$, $d_1 = w_2 + w_4$, $d_2 = 2w_3 = 2w_4$.

$$w_0 + w_1 + w_2 + w_3 + w_4 > d_1 + d_2 = w_2 + 2w_3 + w_4 \Rightarrow w_0 + w_1 > w_3$$

Consider the corollary for $\{0, 1, 2\}$. We must have at least one of (i) $m_1w_0 + n_1w_1 + p_1w_2 = w_2 + w_4$ for some $m_1 + n_1 + p_1 \geq 2$, or (ii) $m_2w_0 + n_2w_1 + p_2w_2 = 2w_3$ for some $m_2 + n_2 + p_2 \geq 2$. (i) $(m_1 + n_1)w_0 + (p_1 - 1)w_2 = w_4 = w_3 < w_0 + w_1 = 2w_0 < w_0 + w_2$. This is only satisfied by $m_1 + n_1 = 2$ and $p_1 = 0$, $m_1 + n_1 = 1$ and $p_1 = 1$, or $m_1 + n_1 = 0$ and $p_1 = 2$. At most, then $d_1 = 2w_2 < w_2 + w_4 = d_1$ which is a contradiction.

(ii) $(m_2 + n_2)w_0 + p_2w_2 = 2w_4 < 2w_0 + 2w_1 = 4w_0 < 4w_2$. This is only satisfied if $m_2 + n_2 + p_2 < 4$. Since (i) does not hold, in addition we require $d_1 = m_3w_0 + n_3w_1 + p_3w_2 + w_4$ with $m_3 + n_3 + p_3 \geq 1$. Then $0 = (m_4 + n_4)w_0 + (p_4 - 1)w_2$. This can only be true if $p_4 = 0$, in which case it would be $(m_4 + n_4)w_0 = w_2 < w_3 < w_0 + w_1 = 2w_0$, and so $m_4 + n_4 = 1$. This would imply $w_0 = w_2$ which is a contradiction.

16b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 7, 9, 9), (16, 18))$

17. No instances were found by computer search.

18a. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((2, 2, 3, 3, 4), (6, 7))$

$(\mathbf{w},\mathbf{d}) = ((3,3,4,4,5),(8,9))$

$(\mathbf{w},\mathbf{d}) = ((3,3,5,5,7),(10,12))$

18b.

$(\mathbf{w},\mathbf{d}) = ((3,t+1,t+2,t+2,2t+1),(2t+4,3t+3)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((6,6t+1,6t+3,6t+3,12t),(12t+6,18t+3)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((6,6t+3,6t+5,6t+5,12t+4),(12t+10,18t+9)),\ 1 \le t$

or $(\mathbf{w},\mathbf{d}) = ((6,2t+1,2t+3,2t+3,4t),(4t+6,6t+2)),\ 1 \le t,\ t \ne 2 \mod (3)$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((3,4,6,6,9),(12,15))$

$(\mathbf{w},\mathbf{d}) = ((4,5,8,8,12),(16,20))$

$(\mathbf{w},\mathbf{d}) = ((6,9,14,14,22),(28,36))$

$(\mathbf{w},\mathbf{d}) = ((8,13,20,20,32),(40,52))$

$(\mathbf{w},\mathbf{d}) = ((9,12,19,19,29),(38,48))$

$(\mathbf{w},\mathbf{d}) = ((9,15,23,23,37),(46,60))$

$(\mathbf{w},\mathbf{d}) = ((12,21,32,32,52),(64,84))$

19.

$(3,3t,3t+1,3t+1,3t+2),(6t+2,6t+3),\ 1 \le t$

$(6,6t+3,6t+5,6t+5,6t+7),(12t+10,12t+12),\ 0 \le t$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((2,3,4,4,5),(8,9))$

$(\mathbf{w}, \mathbf{d}) = ((3, 6, 8, 8, 10), (16, 18))$

$(\mathbf{w}, \mathbf{d}) = ((7, 8, 12, 12, 16), (24, 28))$

$(\mathbf{w}, \mathbf{d}) = ((7, 10, 15, 15, 20), (30, 35))$

$(\mathbf{w}, \mathbf{d}) = ((7, 12, 18, 18, 24), (36, 42))$

$(\mathbf{w}, \mathbf{d}) = ((9, 12, 16, 16, 20), (32, 36))$

$(\mathbf{w}, \mathbf{d}) = ((9, 12, 19, 19, 26), (38, 45))$

$(\mathbf{w}, \mathbf{d}) = ((9, 15, 20, 20, 25), (40, 45))$

$(\mathbf{w}, \mathbf{d}) = ((9, 15, 23, 23, 31), (46, 54))$

$(\mathbf{w}, \mathbf{d}) = ((9, 21, 28, 28, 35), (56, 63))$

$(\mathbf{w}, \mathbf{d}) = ((9, 24, 32, 32, 40), (64, 72))$

$(\mathbf{w}, \mathbf{d}) = ((12, 15, 25, 25, 35), (50, 60))$

20a. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((2, 2, 2, 2, 3), (4, 6))$

20b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 6, 6, 9), (12, 18))$

20c. No instances were found by computer search.

20d. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((6, 8, 9, 9, 12), (18, 24))$

20efg. No instances were found by computer search.

20h. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((12, 14, 15, 18, 21), (36, 42))$

21a.

$(\mathbf{w}, \mathbf{d}) = ((1, 1, t+1, t+1, 2t+1), (2t+2, 2t+2)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((2, 2, 2t+1, 2t+1, 4t), (4t+2, 4t+2)),\ 1 \leq t$

Computer search located no sporadic cases.

21b.

$(\mathbf{w}, \mathbf{d}) = ((1, 2, t+2, t+2, 2t+3), (2t+4, 2t+5)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((2, 4, 2t+3, 2t+3, 4t+4), (4t+6, 4t+8)),\ 0 \leq t$

Computer search located no sporadic cases.

21c. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((2, 2, 2, 3, 4), (6, 6))$

21d. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3, 4, 4, 6, 9), (12, 13))$

21e. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3, 3, 4, 6, 9), (12, 12))$

$(\mathbf{w}, \mathbf{d}) = ((6, 6, 10, 15, 24), (30, 30))$

21f.Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3, 5, 6, 8, 13), (16, 18))$

$(\mathbf{w}, \mathbf{d}) = ((3, 6, 7, 9, 15), (18, 21))$

$(\mathbf{w}, \mathbf{d}) = ((4, 5, 7, 10, 16), (20, 21))$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 8, 11, 18), (22, 24))$

$(\mathbf{w}, \mathbf{d}) = ((4, 10, 12, 15, 26), (30, 36))$

$(\mathbf{w}, \mathbf{d}) = ((5, 6, 8, 12, 19), (24, 25))$

$(\mathbf{w}, \mathbf{d}) = ((5, 7, 10, 14, 23), (28, 30))$

$(\mathbf{w}, \mathbf{d}) = ((5, 8, 9, 12, 19), (24, 27))$

$(\mathbf{w}, \mathbf{d}) = ((5, 14, 17, 21, 37), (42, 51))$

$(\mathbf{w}, \mathbf{d}) = ((6, 14, 18, 23, 40), (46, 54))$

$(\mathbf{w}, \mathbf{d}) = ((6, 18, 22, 27, 48), (54, 66))$

$(\mathbf{w}, \mathbf{d}) = ((8, 10, 16, 23, 38), (46, 48))$

$(\mathbf{w}, \mathbf{d}) = ((8, 14, 16, 21, 34), (42, 48))$

$(\mathbf{w}, \mathbf{d}) = ((8, 26, 32, 39, 70), (78, 96))$

$(\mathbf{w}, \mathbf{d}) = ((9, 11, 12, 17, 25), (34, 36))$

$(\mathbf{w}, \mathbf{d}) = ((9, 12, 17, 24, 39), (48, 51))$

$(\mathbf{w}, \mathbf{d}) = ((9, 15, 22, 30, 51), (60, 66))$

$(\mathbf{w}, \mathbf{d}) = ((9, 19, 24, 31, 53), (62, 72))$

$(\mathbf{w}, \mathbf{d}) = ((9, 23, 30, 38, 67), (76, 90))$

$(\mathbf{w}, \mathbf{d}) = ((10, 11, 15, 22, 34), (44, 45))$

$(\mathbf{w}, \mathbf{d}) = ((10, 17, 25, 34, 58), (68, 75))$

$(\mathbf{w}, \mathbf{d}) = ((11, 13, 14, 20, 29), (40, 42))$

$(\mathbf{w}, \mathbf{d}) = ((11, 17, 20, 27, 43), (54, 60))$

$$(\mathbf{w},\mathbf{d}) = ((11,21,26,34,57),(68,78))$$

$$(\mathbf{w},\mathbf{d}) = ((11,25,32,41,71),(82,96))$$

$$(\mathbf{w},\mathbf{d}) = ((11,29,38,48,85),(96,114))$$

$$(\mathbf{w},\mathbf{d}) = ((12,14,24,35,58),(70,72))$$

$$(\mathbf{w},\mathbf{d}) = ((12,18,20,27,42),(54,60))$$

$$(\mathbf{w},\mathbf{d}) = ((12,42,52,63,114),(126,156))$$

$$(\mathbf{w},\mathbf{d}) = ((14,17,27,39,64),(78,81))$$

$$(\mathbf{w},\mathbf{d}) = ((15,18,19,27,39),(54,57))$$

$$(\mathbf{w},\mathbf{d}) = ((15,18,25,36,57),(72,75))$$

$$(\mathbf{w},\mathbf{d}) = ((15,24,35,48,81),(96,105))$$

$$(\mathbf{w},\mathbf{d}) = ((15,27,40,54,93),(108,120))$$

$$(\mathbf{w},\mathbf{d}) = ((15,36,43,54,93),(108,129))$$

$$(\mathbf{w},\mathbf{d}) = ((15,48,59,72,129),(144,177))$$

$$(\mathbf{w},\mathbf{d}) = ((15,54,67,81,147),(162,201))$$

$$(\mathbf{w},\mathbf{d}) = ((16,22,24,33,50),(66,72))$$

$$(\mathbf{w},\mathbf{d}) = ((16,34,40,51,86),(102,120))$$

$$(\mathbf{w},\mathbf{d}) = ((16,46,56,69,122),(138,168))$$

$$(\mathbf{w},\mathbf{d}) = ((16,58,72,87,158),(174,216))$$

$$(\mathbf{w},\mathbf{d}) = ((18,26,30,41,64),(82,90))$$

$$(\mathbf{w},\mathbf{d}) = ((18,34,42,55,92),(110,126))$$

$(\mathbf{w}, \mathbf{d}) = ((18, 50, 66, 83, 148), (166, 198))$

$(\mathbf{w}, \mathbf{d}) = ((21, 24, 41, 60, 99), (120, 123))$

$(\mathbf{w}, \mathbf{d}) = ((24, 26, 40, 59, 94), (118, 120))$

$(\mathbf{w}, \mathbf{d}) = ((24, 30, 32, 45, 66), (90, 96))$

$(\mathbf{w}, \mathbf{d}) = ((24, 34, 56, 79, 134), (158, 168))$

$(\mathbf{w}, \mathbf{d}) = ((24, 54, 64, 81, 138), (162, 192))$

$(\mathbf{w}, \mathbf{d}) = ((24, 66, 80, 99, 174), (198, 240))$

$(\mathbf{w}, \mathbf{d}) = ((24, 90, 112, 135, 246), (270, 336))$

22a. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((6, 6, 8, 11, 16), (22, 24))$

$(\mathbf{w}, \mathbf{d}) = ((8, 8, 10, 15, 22), (30, 32))$

22b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 7, 9, 14), (18, 21))$

$(\mathbf{w}, \mathbf{d}) = ((6, 10, 12, 15, 24), (30, 36))$

$(\mathbf{w}, \mathbf{d}) = ((24, 30, 38, 53, 82), (106, 120))$

23abc. No instances were found by computer search.

23d. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((9, 10, 12, 15, 21), (30, 36))$

$(\mathbf{w}, \mathbf{d}) = ((18, 22, 27, 33, 48), (66, 81))$

$(\mathbf{w}, \mathbf{d}) = ((21, 24, 29, 36, 51), (72, 87))$

24a. Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((3,4,4,6,8),(12,12))$

$(\mathbf{w},\mathbf{d}) = ((6,10,10,15,20),(30,30))$

24b. Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((4,7,8,10,13),(20,21))$

$(\mathbf{w},\mathbf{d}) = ((6,12,14,17,22),(34,36))$

$(\mathbf{w},\mathbf{d}) = ((14,15,19,26,37),(52,56))$

$(\mathbf{w},\mathbf{d}) = ((14,19,25,32,45),(64,70))$

$(\mathbf{w},\mathbf{d}) = ((18,24,26,35,46),(70,72))$

$(\mathbf{w},\mathbf{d}) = ((18,30,34,43,56),(86,90))$

$(\mathbf{w},\mathbf{d}) = ((18,42,50,59,76),(118,126))$

$(\mathbf{w},\mathbf{d}) = ((19,20,24,36,52),(72,76))$

25a. Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((8,9,9,12,15),(24,27))$

25b. No instances were found by computer search.

26a. No instances were found by computer search.

26b. Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((6,8,9,11,13),(22,24))$

$(\mathbf{w},\mathbf{d}) = ((9,13,15,18,21),(36,39))$

$(\mathbf{w},\mathbf{d}) = ((18,20,21,27,33),(54,60))$

$(\mathbf{w},\mathbf{d}) = ((18,22,27,33,39),(66,72))$

27a.

$(\mathbf{w},\mathbf{d}) = ((1,2t+1,2t+1,3t+1,4t+1),(4t+2,6t+2)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((2,t+1,t+1,2t+1,3t+1),(3t+3,4t+2)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((2,4t+4,4t+4,6t+5,8t+6),(8t+8,12t+10)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((3,6t+5,6t+5,9t+6,12t+7),(12t+10,18t+12)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((3,6t+7,6t+7,9t+9,12t+11),(12t+14,18t+18)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((3,2t+5,2t+5,3t+6,4t+7),(4t+10,6t+12)),\ 0 \le t,\ t \ne 2 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((4,2t+3,2t+3,4t+4,6t+5),(6t+9,8t+8)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((4,4t+2,4t+2,6t+1,8t),(8t+4,12t+2)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((6,6t+2,6t+2,12t+1,18t),(18t+6,24t+2)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((6,6t+4,6t+4,12t+5,18t+6),(18t+12,24t+10)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((6,2t+2,2t+2,4t+1,6t),(6t+6,8t+2)),\ 1 \le t,\ t \ne 2 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((6,12t+4,12t+4,18t+3,24t+2),(24t+8,36t+6)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((6,12t+8,12t+8,18t+9,24t+10),(24t+16,36t+18)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((6,4t+4,4t+4,6t+3,8t+2),(8t+8,12t+6)),\ 1 \le t,\ t \ne 2 \mod 3$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((1,2,2,3,4),(5,6))$

27b.

$(\mathbf{w},\mathbf{d}) = ((1,t+1,2t+1,2t+1,3t+1),(3t+2,4t+2)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((2, 2t+3, 4t+4, 4t+4, 6t+5), (6t+7, 8t+8)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((3, 3t+2, 6t+1, 6t+1, 9t), (9t+3, 12t+2)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((3, 3t+4, 6t+5, 6t+5, 9t+6), (9t+9, 12t+10)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((3, t+2, 2t+1, 2t+1, 3t), (3t+3, 4t+2)),\ 1 \le t,\ t \ne 1 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((4, 2t+3, 4t+2, 4t+2, 6t+1), (6t+5, 8t+4)),\ 1 \le t$

Computer search located no sporadic cases.

27c.

$(\mathbf{w},\mathbf{d}) = ((u, u+2s, t(u+2s), t(u+2s)+s, 2t(u+2s)-u), (2t(u+2s), 2t(u+2s)+2s))$

$u \ge 1$

$$s \ge 1, \left\{ \begin{array}{ll} \gcd(s,u) = 1 & \text{if } u = 2v+1 \\ \gcd(2s,u) = 2 & \text{if } u = 4v \\ \gcd(s, 2v+1) = 1 & \text{if } u = 4v+2 \end{array} \right\}$$

$$t \ge 1, \left\{ \begin{array}{ll} t = v, 2v \mod (2v+1) & \text{if } u = 2v+1 \\ 2t = (2v-1), (4v-2), (4v-1) \mod (4v) & \text{if } u = 4v \\ \left\{ \begin{array}{ll} 2t = (2v-1), (2v) \mod (2v+1) & \text{if } s = 1 \mod (2) \\ 2t = (4v+1) \mod (4v+2) & \text{if } s = 0 \mod (2) \end{array} \right\} & \text{if } u = 4v+2 \end{array} \right\}$$

(t can be half integer)

$(\mathbf{w},\mathbf{d}) = ((u, u+2s, t(u+2s)+s, t(u+2s)+2s, 2t(u+2s)+2s-u), (2t(u+2s)+2s, 2t(u+2s)+4s))$

$u \ge 1$

$s \ge 1,\ \gcd(s,u) = 1$

$$t \geq 1,\ t = \begin{cases} (v-1), 2v \mod (2v+1) & \text{if } u = 2v+1 \\ 2v \mod (2v+1) & \text{if } u = 4v+2 \\ (v-1), (2v-1) \mod (2v) & \text{if } u = 4v \end{cases}$$

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 3t+1, 3t+2, 6t+1), (6t+3, 6t+4)),\ 1 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 3t+2, 3t+3, 6t+3), (6t+5, 6t+6)),\ 0 \leq t$

or $(\mathbf{w}, \mathbf{d}) = ((2, 3, t+1, t+2, 2t+1), (2t+3, 2t+4)),\ 1 \leq t,\ t \neq 2 \mod 3$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 6t+1, 6t+3, 12t), (12t+4, 12t+6)),\ 1 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 6t+5, 6t+7, 12t+8), (12t+12, 12t+14)),\ 0 \leq t$

or $(\mathbf{w}, \mathbf{d}) = ((4, 6, 2t+1, 2t+3, 4t), (4t+4, 4t+6)),\ 1 \leq t,\ t \neq 1 \mod 3$

$(\mathbf{w}, \mathbf{d}) = ((1, 3t+2, 4t+2, 6t+3, 9t+4), (9t+5, 12t+6)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((2, 6t+7, 8t+8, 12t+12, 18t+17), (18t+19, 24t+24)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((3, 9t+3, 12t+2, 18t+3, 27t+3), (27t+6, 36t+6)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((3, 9t+9, 12t+10, 18t+15, 27t+21), (27t+24, 36t+30)),\ 0 \leq t$

or $(\mathbf{w}, \mathbf{d}) = ((3, 3t+3, 4t+2, 6t+3, 9t+3), (9t+6, 12t+6)),\ 0 \leq t,\ t \neq 1 \mod 3$

$(\mathbf{w}, \mathbf{d}) = ((4, 6t+5, 8t+4, 12t+6, 18t+7), (18t+11, 24t+12)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((6, 18t+9, 24t+8, 36t+12, 54t+15), (54t+21, 72t+24)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((6, 18t+15, 24t+16, 36t+24, 54t+33), (54t+39, 72t+48)),\ 0 \leq t$

or $(\mathbf{w}, \mathbf{d}) = ((6, 6t+9, 8t+8, 12t+12, 18t+15), (18t+21, 24t+24)),\ 0 \leq t,\ t \neq 2$ mod 3

$(\mathbf{w}, \mathbf{d}) = ((8, 6t+7, 8t+4, 12t+6, 18t+5), (18t+13, 24t+12)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((9, 9t+6, 12t+2, 18t+3, 27t), (27t+9, 36t+6)),\ 1 \leq t$

$(\mathbf{w},\mathbf{d}) = ((9, 9t+12, 12t+10, 18t+15, 27t+18), (27t+27, 36t+30)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((9, 3t+6, 4t+2, 6t+3, 9t), (9t+9, 12t+6)),\ 1 \le t,\ t \ne 1 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((12, 18t+9, 24t+4, 36t+6, 54t+3), (54t+15, 72t+12)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((12, 18t+21, 24t+20, 36t+30, 54t+39), (54t+51, 72t+60)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((12, 6t+9, 8t+4, 12t+6, 18t+3), (18t+15, 24t+12)),\ 1 \le t,\ t \ne 1$ mod 3

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((1,3,4,6,9),(10,12))$

$(\mathbf{w},\mathbf{d}) = ((1,4,5,8,12),(13,16))$

$(\mathbf{w},\mathbf{d}) = ((1,5,8,12,19),(20,24))$

$(\mathbf{w},\mathbf{d}) = ((1,6,10,15,24),(25,30))$

$(\mathbf{w},\mathbf{d}) = ((1,7,11,17,27),(28,34))$

$(\mathbf{w},\mathbf{d}) = ((1,8,13,20,32),(33,40))$

$(\mathbf{w},\mathbf{d}) = ((2,4,6,9,14),(16,18))$

$(\mathbf{w},\mathbf{d}) = ((2,5,6,9,13),(15,18))$

$(\mathbf{w},\mathbf{d}) = ((2,7,8,13,19),(21,26))$

$(\mathbf{w},\mathbf{d}) = ((5,9,12,20,31),(36,40))$

$(\mathbf{w},\mathbf{d}) = ((6,9,13,21,33),(39,42))$

$(\mathbf{w},\mathbf{d}) = ((6,12,16,27,42),(48,54))$

$(\mathbf{w},\mathbf{d}) = ((9,12,17,27,42),(51,54))$

$(\mathbf{w},\mathbf{d}) = ((9,15,22,36,57),(66,72))$

$(\mathbf{w},\mathbf{d}) = ((10,12,16,25,38),(48,50))$

$(\mathbf{w},\mathbf{d}) = ((10,24,32,55,86),(96,110))$

$(\mathbf{w},\mathbf{d}) = ((11,14,21,33,52),(63,66))$

$(\mathbf{w},\mathbf{d}) = ((11,15,20,32,49),(60,64))$

$(\mathbf{w},\mathbf{d}) = ((11,18,27,44,70),(81,88))$

$(\mathbf{w},\mathbf{d}) = ((11,21,28,47,73),(84,94))$

$(\mathbf{w},\mathbf{d}) = ((11,27,36,62,97),(108,124))$

$(\mathbf{w},\mathbf{d}) = ((13,14,19,29,44),(57,58))$

$(\mathbf{w},\mathbf{d}) = ((13,17,24,38,59),(72,76))$

$(\mathbf{w},\mathbf{d}) = ((13,20,29,47,74),(87,94))$

$(\mathbf{w},\mathbf{d}) = ((13,23,34,56,89),(102,112))$

$(\mathbf{w},\mathbf{d}) = ((15,21,28,45,69),(84,90))$

$(\mathbf{w},\mathbf{d}) = ((15,33,44,75,117),(132,150))$

$(\mathbf{w},\mathbf{d}) = ((15,39,52,90,141),(156,180))$

$(\mathbf{w},\mathbf{d}) = ((18,21,29,45,69),(87,90))$

$(\mathbf{w},\mathbf{d}) = ((18,21,35,54,87),(105,108))$

$(\mathbf{w},\mathbf{d}) = ((18,24,40,63,102),(120,126))$

$(\mathbf{w},\mathbf{d}) = ((18,33,49,81,129),(147,162))$

$(\mathbf{w},\mathbf{d}) = ((22,24,32,49,74),(96,98))$

$(\mathbf{w},\mathbf{d}) = ((22,24,36,55,86),(108,110))$

$(\mathbf{w},\mathbf{d}) = ((22,32,48,77,122),(144,154))$

$(\mathbf{w},\mathbf{d}) = ((22,36,48,79,122),(144,158))$

$(\mathbf{w},\mathbf{d}) = ((22,40,60,99,158),(180,198))$

$(\mathbf{w},\mathbf{d}) = ((22,48,64,109,170),(192,218))$

$(\mathbf{w},\mathbf{d}) = ((22,60,80,139,218),(240,278))$

$(\mathbf{w},\mathbf{d}) = ((26,30,50,77,124),(150,154))$

$(\mathbf{w},\mathbf{d}) = ((26,36,60,95,154),(180,190))$

$(\mathbf{w},\mathbf{d}) = ((30,48,64,105,162),(192,210))$

$(\mathbf{w},\mathbf{d}) = ((30,84,112,195,306),(336,390))$

28a.

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((2,2,3,4,5),(7,8))$

$(\mathbf{w},\mathbf{d}) = ((2,2,4,5,6),(8,10))$

$(\mathbf{w},\mathbf{d}) = ((4,4,5,6,7),(11,12))$

$(\mathbf{w},\mathbf{d}) = ((4,4,7,10,13),(17,20))$

28b.

$(\mathbf{w},\mathbf{d}) = ((u,2s+u,t(2s+u),t(2s+u)+s,t(2s+u)+2s),(t(2s+u)+2s+u,2t(2s+u)+2s))$

$u \geq 1$,

$s \geq 1$, $\gcd(s,u) = 1$,

$$t \geq 1,\ t = \begin{cases} v, 2v \mod (2v+1) & \text{if } u = 2v+1 \text{ or } u = 4v+2 \\ (2v-1) \mod (2v) & \text{if } u = 4v \end{cases}$$

$(\mathbf{w},\mathbf{d}) = ((u, 2s+u, t(2s+u)-s, t(2s+u), t(2s+u)+s), (t(2s+u)+s+u, 2t(2s+u)))$

$u \geq 1$,

$$s \geq 1,\ \begin{cases} \gcd(s,u) = 1 & \text{if } u = 2v+1 \\ \gcd(s,u) = 1 & \text{if } u = 4v \\ \text{any } s \geq 1 & \text{if } u = 2 \\ s \neq 0 \mod (2v+1) & \text{if } u = 4v+2,\ v \geq 1 \end{cases}$$

$$2t \geq 3,\ \begin{cases} t = 0, v \mod (2v+1) & \text{if } u = 2v+1 \\ 2t = 0, (2v-1), 2v \mod (4v) & \text{if } u = 4v \\ 2t = \left\{ \begin{array}{ll} r(2v+1) & \text{if } s = 1 \mod (2) \\ (2q+1)(2v+1) & \text{if } s = 0 \mod (2) \end{array} \right\} & \text{if } u = 4v+2 \end{cases}$$

(t can be half integer)

$(\mathbf{w},\mathbf{d}) = ((1, 4t+2, 6t+3, 9t+4, 12t+5), (12t+6, 18t+8)),\ 0 \leq t$

$(\mathbf{w},\mathbf{d}) = ((2, t+1, 2t+2, 3t+2, 4t+2), (4t+4, 6t+4)),\ 0 \leq t$

$(\mathbf{w},\mathbf{d}) = ((2, 8t+8, 12t+12, 18t+17, 24t+22), (24t+24, 36t+34)),\ 0 \leq t$

$(\mathbf{w},\mathbf{d}) = ((3, 4t+2, 6t+3, 9t+3, 12t+3), (12t+6, 18t+6)),\ 0 \leq t,\ t \neq 1 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((4, 2t+1, 4t+2, 6t+1, 8t), (8t+4, 12t+2)),\ 1 \leq t$

$(\mathbf{w},\mathbf{d}) = ((6, t+2, 2t+4, 3t+3, 4t+2), (4t+8, 6t+6)),\ 1 \leq t,\ t \neq 1 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((6, 8t+8, 12t+12, 18t+15, 24t+18), (24t+24, 36t+30)),\ 0 \leq t,\ t \neq 2 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((7, 4t+6, 6t+9, 9t+10, 12t+11), (12t+18, 18t+20)),\ 0 \leq t,\ t \neq 2 \mod 7$

$(\mathbf{w},\mathbf{d}) = ((9, 4t+6, 6t+9, 9t+9, 12t+9), (12t+18, 18t+18)),\ 0 \le t,\ t \ne 0 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((10, 2t+4, 4t+8, 6t+7, 8t+6), (8t+16, 12t+14)),\ 0 \le t,\ t \ne 3 \mod 5$

$(\mathbf{w},\mathbf{d}) = ((14, 8t+8, 12t+12, 18t+11, 24t+10), (24t+24, 36t+22)),\ 0 \le t,\ t \ne 6 \mod 7$

$(\mathbf{w},\mathbf{d}) = ((18, 8t+8, 12t+12, 18t+9, 24t+6), (24t+24, 36t+18)),\ 1 \le t,\ t \ne 2 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((2, 4, t, t+1, t+2), (t+4, 2t+2)),\ 1 \le t,\ t \ne 2 \mod 4$

$(\mathbf{w},\mathbf{d}) = ((2, t+1, t+2, 2t+2, 3t+2), (3t+4, 4t+4)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((3, t+1, t+2, 2t+1, 3t), (3t+3, 4t+2)),\ 1 \le t,\ t \ne 2 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((4, 2t+1, 2t+3, 4t+2, 6t+1), (6t+5, 8t+4)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((6, 6t+7, 6t+9, 12t+12, 18t+15), (18t+21, 24t+24)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((6, t+1, t+4, 2t+2, 3t), (3t+6, 4t+4)),\ 1 \le t,\ t \ne 2 \mod 3$

$(\mathbf{w},\mathbf{d}) = ((9, 3t+5, 3t+8, 6t+7, 9t+6), (9t+15, 12t+14)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((12, 6t+5, 6t+9, 12t+6, 18t+3), (18t+15, 24t+12)),\ 1 \le t$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((1,3,4,6,8),(9,12))$

$(\mathbf{w},\mathbf{d}) = ((1,4,7,10,13),(14,20))$

$(\mathbf{w},\mathbf{d}) = ((1,5,9,13,17),(18,26))$

$(\mathbf{w},\mathbf{d}) = ((1,6,10,15,20),(21,30))$

$(\mathbf{w},\mathbf{d}) = ((1,7,12,18,24),(25,36))$

$(\mathbf{w},\mathbf{d}) = ((2,4,8,11,14),(16,22))$

$$(\mathbf{w},\mathbf{d}) = ((2,5,8,11,14),(16,22))$$

$$(\mathbf{w},\mathbf{d}) = ((2,7,10,15,20),(22,30))$$

$$(\mathbf{w},\mathbf{d}) = ((3,4,5,6,7),(10,12))$$

$$(\mathbf{w},\mathbf{d}) = ((3,4,5,7,9),(12,14))$$

$$(\mathbf{w},\mathbf{d}) = ((3,5,7,9,11),(14,18))$$

$$(\mathbf{w},\mathbf{d}) = ((3,7,8,12,16),(19,24))$$

$$(\mathbf{w},\mathbf{d}) = ((4,5,6,7,8),(12,14))$$

$$(\mathbf{w},\mathbf{d}) = ((4,5,8,12,16),(20,24))$$

$$(\mathbf{w},\mathbf{d}) = ((5,6,10,15,20),(25,30))$$

$$(\mathbf{w},\mathbf{d}) = ((5,6,15,20,25),(30,40))$$

$$(\mathbf{w},\mathbf{d}) = ((6,7,9,12,15),(21,24))$$

$$(\mathbf{w},\mathbf{d}) = ((6,8,9,12,15),(21,24))$$

$$(\mathbf{w},\mathbf{d}) = ((6,8,20,27,34),(40,54))$$

$$(\mathbf{w},\mathbf{d}) = ((8,9,12,20,28),(36,40))$$

$$(\mathbf{w},\mathbf{d}) = ((8,12,13,18,23),(31,36))$$

$$(\mathbf{w},\mathbf{d}) = ((8,12,19,30,41),(49,60))$$

$$(\mathbf{w},\mathbf{d}) = ((8,13,20,32,44),(52,64))$$

$$(\mathbf{w},\mathbf{d}) = ((9,10,12,15,18),(27,30))$$

$$(\mathbf{w},\mathbf{d}) = ((9,14,15,21,27),(36,42))$$

$$(\mathbf{w},\mathbf{d}) = ((10,16,25,40,55),(65,80))$$

$$(\mathbf{w}, \mathbf{d}) = ((10, 16, 40, 55, 70), (80, 110))$$

$$(\mathbf{w}, \mathbf{d}) = ((12, 20, 21, 30, 39), (51, 60))$$

$$(\mathbf{w}, \mathbf{d}) = ((11, 14, 21, 33, 45), (56, 66))$$

$$(\mathbf{w}, \mathbf{d}) = ((11, 18, 27, 44, 61), (72, 88))$$

$$(\mathbf{w}, \mathbf{d}) = ((13, 14, 35, 46, 57), (70, 92))$$

$$(\mathbf{w}, \mathbf{d}) = ((13, 18, 45, 61, 77), (90, 122))$$

$$(\mathbf{w}, \mathbf{d}) = ((13, 22, 55, 76, 97), (110, 152))$$

$$(\mathbf{w}, \mathbf{d}) = ((15, 22, 55, 75, 95), (110, 150))$$

$$(\mathbf{w}, \mathbf{d}) = ((15, 26, 65, 90, 115), (130, 180))$$

$$(\mathbf{w}, \mathbf{d}) = ((16, 21, 28, 48, 68), (84, 96))$$

$$(\mathbf{w}, \mathbf{d}) = ((22, 24, 36, 55, 74), (96, 110))$$

$$(\mathbf{w}, \mathbf{d}) = ((22, 32, 48, 77, 106), (128, 154))$$

$$(\mathbf{w}, \mathbf{d}) = ((22, 40, 60, 99, 138), (160, 198))$$

$$(\mathbf{w}, \mathbf{d}) = ((26, 30, 40, 67, 94), (120, 134))$$

$$(\mathbf{w}, \mathbf{d}) = ((26, 32, 80, 107, 134), (160, 214))$$

$$(\mathbf{w}, \mathbf{d}) = ((26, 36, 48, 83, 118), (144, 166))$$

$$(\mathbf{w}, \mathbf{d}) = ((26, 40, 100, 137, 174), (200, 274))$$

$$(\mathbf{w}, \mathbf{d}) = ((26, 48, 120, 167, 214), (240, 334))$$

$$(\mathbf{w}, \mathbf{d}) = ((30, 32, 80, 105, 130), (160, 210))$$

$$(\mathbf{w}, \mathbf{d}) = ((30, 56, 140, 195, 250), (280, 390))$$

29.

$(\mathbf{w}, \mathbf{d}) = ((4, 2t+2, 2t+3, 2t+4, 2t+5), (4t+7, 4t+8)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((6, 3t+1, 3t+3, 3t+4, 3t+5), (6t+6, 6t+8)),\ 1 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((8, 4t+1, 4t+3, 4t+5, 4t+7), (8t+8, 8t+10)),\ 1 \leq t$

Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 4, 5, 6), (9, 10))$

$(\mathbf{w}, \mathbf{d}) = ((2, 3, 5, 6, 7), (10, 12))$

$(\mathbf{w}, \mathbf{d}) = ((2, 4, 5, 6, 7), (11, 12))$

$(\mathbf{w}, \mathbf{d}) = ((2, 4, 6, 7, 8), (12, 14))$

$(\mathbf{w}, \mathbf{d}) = ((2, 6, 8, 9, 10), (16, 18))$

$(\mathbf{w}, \mathbf{d}) = ((3, 4, 5, 6, 7), (11, 12))$

$(\mathbf{w}, \mathbf{d}) = ((3, 4, 6, 7, 8), (12, 14))$

$(\mathbf{w}, \mathbf{d}) = ((3, 5, 6, 8, 10), (15, 16))$

$(\mathbf{w}, \mathbf{d}) = ((3, 5, 7, 9, 11), (16, 18))$

$(\mathbf{w}, \mathbf{d}) = ((3, 8, 10, 12, 14), (22, 24))$

$(\mathbf{w}, \mathbf{d}) = ((3, 8, 12, 14, 16), (24, 28))$

$(\mathbf{w}, \mathbf{d}) = ((4, 5, 7, 10, 13), (18, 20))$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 8, 9, 10), (16, 18))$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 8, 11, 14), (20, 22))$

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 9, 12, 15), (21, 24))$

$$(\mathbf{w},\mathbf{d}) = ((4,6,12,15,18),(24,30))$$

$$(\mathbf{w},\mathbf{d}) = ((4,8,11,14,17),(25,28))$$

$$(\mathbf{w},\mathbf{d}) = ((4,9,15,18,21),(30,36))$$

$$(\mathbf{w},\mathbf{d}) = ((4,10,12,15,18),(28,30))$$

$$(\mathbf{w},\mathbf{d}) = ((4,12,15,18,21),(33,36))$$

$$(\mathbf{w},\mathbf{d}) = ((4,14,20,23,26),(40,46))$$

$$(\mathbf{w},\mathbf{d}) = ((4,18,24,27,30),(48,54))$$

$$(\mathbf{w},\mathbf{d}) = ((5,6,8,10,12),(18,20))$$

$$(\mathbf{w},\mathbf{d}) = ((5,6,10,12,14),(20,24))$$

$$(\mathbf{w},\mathbf{d}) = ((5,6,14,18,22),(28,36))$$

$$(\mathbf{w},\mathbf{d}) = ((5,7,8,11,14),(21,22))$$

$$(\mathbf{w},\mathbf{d}) = ((5,7,10,14,18),(25,28))$$

$$(\mathbf{w},\mathbf{d}) = ((5,8,12,14,16),(24,28))$$

$$(\mathbf{w},\mathbf{d}) = ((5,9,12,15,18),(27,30))$$

$$(\mathbf{w},\mathbf{d}) = ((5,9,15,18,21),(30,36))$$

$$(\mathbf{w},\mathbf{d}) = ((5,11,14,18,22),(33,36))$$

$$(\mathbf{w},\mathbf{d}) = ((5,12,16,20,24),(36,40))$$

$$(\mathbf{w},\mathbf{d}) = ((5,12,18,21,24),(36,42))$$

$$(\mathbf{w},\mathbf{d}) = ((5,12,20,24,28),(40,48))$$

$$(\mathbf{w},\mathbf{d}) = ((5,16,24,28,32),(48,56))$$

$$(\mathbf{w}, \mathbf{d}) = ((6, 8, 18, 23, 28), (36, 46))$$

$$(\mathbf{w}, \mathbf{d}) = ((6, 14, 18, 23, 28), (42, 46))$$

$$(\mathbf{w}, \mathbf{d}) = ((6, 14, 19, 24, 29), (43, 48))$$

$$(\mathbf{w}, \mathbf{d}) = ((6, 20, 25, 30, 35), (55, 60))$$

$$(\mathbf{w}, \mathbf{d}) = ((6, 20, 30, 35, 40), (60, 70))$$

$$(\mathbf{w}, \mathbf{d}) = ((7, 8, 12, 16, 20), (28, 32))$$

$$(\mathbf{w}, \mathbf{d}) = ((7, 9, 15, 21, 27), (36, 42))$$

$$(\mathbf{w}, \mathbf{d}) = ((7, 9, 21, 27, 33), (42, 54))$$

$$(\mathbf{w}, \mathbf{d}) = ((7, 10, 15, 20, 25), (35, 40))$$

$$(\mathbf{w}, \mathbf{d}) = ((7, 12, 18, 24, 30), (42, 48))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 10, 15, 20, 25), (35, 40))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 10, 16, 19, 22), (32, 38))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 10, 16, 23, 30), (40, 46))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 10, 17, 24, 31), (41, 48))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 10, 20, 25, 30), (40, 50))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 14, 16, 21, 26), (40, 42))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 14, 21, 28, 35), (49, 56))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 14, 28, 35, 42), (56, 70))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 18, 24, 27, 30), (48, 54))$$

$$(\mathbf{w}, \mathbf{d}) = ((8, 22, 32, 37, 42), (64, 74))$$

$(\mathbf{w},\mathbf{d}) = ((8,26,32,39,46),(72,78))$

$(\mathbf{w},\mathbf{d}) = ((8,30,40,45,50),(80,90))$

$(\mathbf{w},\mathbf{d}) = ((8,34,48,55,62),(96,110))$

$(\mathbf{w},\mathbf{d}) = ((8,42,56,63,70),(112,126))$

$(\mathbf{w},\mathbf{d}) = ((10,11,15,22,29),(40,44))$

$(\mathbf{w},\mathbf{d}) = ((10,12,21,30,39),(51,60))$

$(\mathbf{w},\mathbf{d}) = ((10,12,30,39,48),(60,78))$

$(\mathbf{w},\mathbf{d}) = ((10,13,25,31,37),(50,62))$

$(\mathbf{w},\mathbf{d}) = ((10,16,30,37,44),(60,74))$

$(\mathbf{w},\mathbf{d}) = ((10,17,25,34,43),(60,68))$

$(\mathbf{w},\mathbf{d}) = ((10,19,35,43,51),(70,86))$

$(\mathbf{w},\mathbf{d}) = ((10,21,28,35,42),(63,70))$

$(\mathbf{w},\mathbf{d}) = ((10,21,35,42,49),(70,84))$

$(\mathbf{w},\mathbf{d}) = ((10,22,40,49,58),(80,98))$

$(\mathbf{w},\mathbf{d}) = ((10,27,36,45,54),(81,90))$

$(\mathbf{w},\mathbf{d}) = ((10,27,45,54,63),(90,108))$

$(\mathbf{w},\mathbf{d}) = ((11,12,15,18,21),(33,36))$

$(\mathbf{w},\mathbf{d}) = ((11,13,19,25,31),(44,50))$

$(\mathbf{w},\mathbf{d}) = ((11,16,20,24,28),(44,48))$

$(\mathbf{w},\mathbf{d}) = ((11,17,24,31,38),(55,62))$

$(\mathbf{w}, \mathbf{d}) = ((11, 20, 25, 30, 35), (55, 60))$

$(\mathbf{w}, \mathbf{d}) = ((11, 21, 29, 37, 45), (66, 74))$

$(\mathbf{w}, \mathbf{d}) = ((11, 24, 30, 36, 42), (66, 72))$

$(\mathbf{w}, \mathbf{d}) = ((11, 25, 34, 43, 52), (77, 86))$

$(\mathbf{w}, \mathbf{d}) = ((11, 28, 35, 42, 49), (77, 84))$

$(\mathbf{w}, \mathbf{d}) = ((11, 29, 39, 49, 59), (88, 98))$

$(\mathbf{w}, \mathbf{d}) = ((11, 32, 40, 48, 56), (88, 96))$

$(\mathbf{w}, \mathbf{d}) = ((11, 36, 45, 54, 63), (99, 108))$

$(\mathbf{w}, \mathbf{d}) = ((11, 40, 50, 60, 70), (110, 120))$

$(\mathbf{w}, \mathbf{d}) = ((12, 14, 24, 35, 46), (60, 70))$

$(\mathbf{w}, \mathbf{d}) = ((12, 16, 23, 30, 37), (53, 60))$

$(\mathbf{w}, \mathbf{d}) = ((12, 20, 25, 30, 35), (55, 60))$

$(\mathbf{w}, \mathbf{d}) = ((12, 28, 35, 42, 49), (77, 84))$

$(\mathbf{w}, \mathbf{d}) = ((12, 32, 43, 54, 65), (97, 108))$

$(\mathbf{w}, \mathbf{d}) = ((12, 44, 55, 66, 77), (121, 132))$

$(\mathbf{w}, \mathbf{d}) = ((14, 15, 25, 35, 45), (60, 70))$

$(\mathbf{w}, \mathbf{d}) = ((14, 15, 35, 45, 55), (70, 90))$

$(\mathbf{w}, \mathbf{d}) = ((14, 16, 42, 55, 68), (84, 110))$

$(\mathbf{w}, \mathbf{d}) = ((14, 17, 29, 41, 53), (70, 82))$

$(\mathbf{w}, \mathbf{d}) = ((15, 16, 20, 32, 44), (60, 64))$

$$(\mathbf{w},\mathbf{d}) = ((16,18,24,27,30),(48,54))$$

$$(\mathbf{w},\mathbf{d}) = ((16,18,48,63,78),(96,126))$$

$$(\mathbf{w},\mathbf{d}) = ((16,22,24,33,42),(64,66))$$

$$(\mathbf{w},\mathbf{d}) = ((16,26,40,47,54),(80,94))$$

$$(\mathbf{w},\mathbf{d}) = ((16,30,40,45,50),(80,90))$$

$$(\mathbf{w},\mathbf{d}) = ((16,34,40,51,62),(96,102))$$

$$(\mathbf{w},\mathbf{d}) = ((16,38,56,65,74),(112,130))$$

$$(\mathbf{w},\mathbf{d}) = ((16,42,56,63,70),(112,126))$$

$$(\mathbf{w},\mathbf{d}) = ((16,46,56,69,82),(128,138))$$

$$(\mathbf{w},\mathbf{d}) = ((16,50,72,83,94),(144,166))$$

$$(\mathbf{w},\mathbf{d}) = ((16,54,72,81,90),(144,162))$$

$$(\mathbf{w},\mathbf{d}) = ((16,58,72,87,102),(160,174))$$

$$(\mathbf{w},\mathbf{d}) = ((16,62,88,101,114),(176,202))$$

$$(\mathbf{w},\mathbf{d}) = ((16,66,88,99,110),(176,198))$$

$$(\mathbf{w},\mathbf{d}) = ((16,74,104,119,134),(208,238))$$

$$(\mathbf{w},\mathbf{d}) = ((16,78,104,117,130),(208,234))$$

$$(\mathbf{w},\mathbf{d}) = ((16,90,120,135,150),(240,270))$$

$$(\mathbf{w},\mathbf{d}) = ((17,20,35,50,65),(85,100))$$

$$(\mathbf{w},\mathbf{d}) = ((24,26,32,51,70),(96,102))$$

$$(\mathbf{w},\mathbf{d}) = ((24,26,60,77,94),(120,154))$$

$(\mathbf{w}, \mathbf{d}) = ((24, 34, 40, 63, 86), (120, 126))$

$(\mathbf{w}, \mathbf{d}) = ((24, 38, 84, 107, 130), (168, 214))$

30.

$\mathbf{w} = (u, u + s, u + 2s, t(u + 2s) - u - s, t(2s + u) - u)$

$\mathbf{d} = (t(u + 2s), 2t(u + 2s))$

$u \geq 1$

$s \geq 1$

$$t \geq 2 \;\; t = \begin{cases} 0, v \mod (2v+1) & \text{if } u = 2v+1 \\ rv & \text{if } u = 2v \end{cases}$$

$\mathbf{w} = (u, u + s, u + 2s, (t - 1)(u + 2s), (t - 1)(2s + u) + s)$

$\mathbf{d} = (t(u + 2s) - s, t(u + 2s))$

$u \geq 1$

$s \geq 1$

$$t \geq 2 \;\; t = \begin{cases} 0, (v+1) \mod (2v+1) & \text{if } u = 2v+1 \\ rv & \text{if } u = 2v \end{cases}$$

Computer search located no sporadic cases.

31a.

$(\mathbf{w}, \mathbf{d}) = ((2, t + 1, t + 1, t + 2, 2t + 1), (2t + 3, 3t + 3)),\ 0 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((4, 2t + 1, 2t + 1, 2t + 3, 4t), (4t + 4, 6t + 3)),\ 1 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((6, 6t + 2, 6t + 2, 6t + 5, 12t + 1), (12t + 7, 18t + 6)),\ 1 \leq t$

$(\mathbf{w}, \mathbf{d}) = ((6, 6t + 4, 6t + 4, 6t + 7, 12t + 5), (12t + 11, 18t + 12)),\ 0 \leq t$

or $(\mathbf{w},\mathbf{d}) = ((6,2t+2,2t+2,2t+5,4t+1),(4t+7,6t+6)),\ 1 \le t,\ t \ne 2 \mod 3$

Computer search located no sporadic cases.

31b.

$(\mathbf{w},\mathbf{d}) = ((4,4t+1,4t+2,4t+3,8t),(8t+4,12t+3)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((4,4t+4,4t+5,4t+6,8t+6),(8t+10,12t+12)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((4,t+1,t+2,t+3,2t),(2t+4,3t+3)),\ 1 \le t,\ t = 0,3 \mod 4$

$(\mathbf{w},\mathbf{d}) = ((8,4t+5,4t+7,4t+9,8t+6),(8t+14,12t+15)),\ 0 \le t$

$(\mathbf{w},\mathbf{d}) = ((9,3t+2,3t+5,3t+8,6t+1),(6t+10,9t+9)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((12,12t+4,12t+7,12t+10,24t+2),(24t+14,36t+12)),\ 1 \le t$

$(\mathbf{w},\mathbf{d}) = ((12,12t+8,12t+11,12t+14,24t+10),(24t+22,36t+24)),\ 0 \le t$

or $(\mathbf{w},\mathbf{d}) = ((12,4t+4,4t+7,4t+10,8t+2),(8t+14,12t+12)),\ 1 \le t,\ t \ne 2 \mod 3$

Computer search located the following:

$(\mathbf{w},\mathbf{d}) = ((4,5,7,10,11),(15,21))$

$(\mathbf{w},\mathbf{d}) = ((4,6,8,11,13),(17,24))$

$(\mathbf{w},\mathbf{d}) = ((4,8,11,14,18),(22,32))$

$(\mathbf{w},\mathbf{d}) = ((5,7,10,14,16),(21,30))$

$(\mathbf{w},\mathbf{d}) = ((5,9,12,16,20),(25,36))$

$(\mathbf{w},\mathbf{d}) = ((6,8,12,17,19),(25,36))$

$(\mathbf{w},\mathbf{d}) = ((6,12,16,21,27),(33,48))$

$(\mathbf{w},\mathbf{d}) = ((8,12,17,22,26),(34,48))$

$$(\mathbf{w},\mathbf{d}) = ((8,18,24,31,41),(49,72))$$

$$(\mathbf{w},\mathbf{d}) = ((8,20,27,34,46),(54,80))$$

$$(\mathbf{w},\mathbf{d}) = ((9,10,15,22,23),(32,45))$$

$$(\mathbf{w},\mathbf{d}) = ((9,12,17,24,27),(36,51))$$

$$(\mathbf{w},\mathbf{d}) = ((9,14,21,29,34),(43,63))$$

$$(\mathbf{w},\mathbf{d}) = ((9,15,22,30,36),(45,66))$$

$$(\mathbf{w},\mathbf{d}) = ((10,11,15,22,23),(33,45))$$

$$(\mathbf{w},\mathbf{d}) = ((10,12,20,29,31),(41,60))$$

$$(\mathbf{w},\mathbf{d}) = ((10,17,25,34,41),(51,75))$$

$$(\mathbf{w},\mathbf{d}) = ((11,13,19,25,27),(38,52))$$

$$(\mathbf{w},\mathbf{d}) = ((11,14,21,30,33),(44,63))$$

$$(\mathbf{w},\mathbf{d}) = ((11,17,24,31,37),(48,68))$$

$$(\mathbf{w},\mathbf{d}) = ((11,18,27,37,44),(55,81))$$

$$(\mathbf{w},\mathbf{d}) = ((11,21,29,37,47),(58,84))$$

$$(\mathbf{w},\mathbf{d}) = ((11,25,34,43,57),(68,100))$$

$$(\mathbf{w},\mathbf{d}) = ((11,29,39,49,67),(78,116))$$

$$(\mathbf{w},\mathbf{d}) = ((12,30,40,51,69),(81,120))$$

$$(\mathbf{w},\mathbf{d}) = ((13,14,23,32,33),(46,65))$$

$$(\mathbf{w},\mathbf{d}) = ((13,17,27,37,41),(54,78))$$

$$(\mathbf{w},\mathbf{d}) = ((13,20,31,42,49),(62,91))$$

$(\mathbf{w}, \mathbf{d}) = ((13, 23, 35, 47, 57), (70, 104))$

$(\mathbf{w}, \mathbf{d}) = ((14, 17, 29, 41, 44), (58, 85))$

$(\mathbf{w}, \mathbf{d}) = ((15, 18, 25, 36, 39), (54, 75))$

$(\mathbf{w}, \mathbf{d}) = ((15, 21, 28, 39, 45), (60, 84))$

$(\mathbf{w}, \mathbf{d}) = ((15, 24, 35, 48, 57), (72, 105))$

$(\mathbf{w}, \mathbf{d}) = ((15, 27, 40, 54, 66), (81, 120))$

$(\mathbf{w}, \mathbf{d}) = ((15, 33, 44, 57, 75), (90, 132))$

$(\mathbf{w}, \mathbf{d}) = ((15, 39, 52, 66, 90), (105, 156))$

$(\mathbf{w}, \mathbf{d}) = ((16, 18, 24, 35, 37), (53, 72))$

$(\mathbf{w}, \mathbf{d}) = ((16, 20, 29, 38, 42), (58, 80))$

$(\mathbf{w}, \mathbf{d}) = ((16, 28, 39, 50, 62), (78, 112))$

$(\mathbf{w}, \mathbf{d}) = ((16, 30, 40, 53, 67), (83, 120))$

$(\mathbf{w}, \mathbf{d}) = ((16, 36, 49, 62, 82), (98, 144))$

$(\mathbf{w}, \mathbf{d}) = ((16, 42, 56, 71, 97), (113, 168))$

$(\mathbf{w}, \mathbf{d}) = ((16, 44, 59, 74, 102), (118, 176))$

$(\mathbf{w}, \mathbf{d}) = ((18, 21, 35, 51, 54), (72, 105))$

$(\mathbf{w}, \mathbf{d}) = ((18, 32, 48, 65, 79), (97, 144))$

$(\mathbf{w}, \mathbf{d}) = ((20, 24, 41, 58, 62), (82, 120))$

$(\mathbf{w}, \mathbf{d}) = ((20, 28, 47, 66, 74), (94, 140))$

$(\mathbf{w}, \mathbf{d}) = ((24, 30, 40, 57, 63), (87, 120))$

$(\mathbf{w}, \mathbf{d}) = ((24, 42, 56, 75, 93), (117, 168))$

$(\mathbf{w}, \mathbf{d}) = ((24, 66, 88, 111, 153), (177, 264))$

$(\mathbf{w}, \mathbf{d}) = ((30, 42, 70, 99, 111), (141, 210))$

32a. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((4, 4, 6, 7, 9), (13, 16))$

$(\mathbf{w}, \mathbf{d}) = ((6, 6, 8, 11, 13), (19, 24))$

32b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((8, 12, 18, 19, 29), (37, 48))$

$(\mathbf{w}, \mathbf{d}) = ((11, 14, 21, 23, 33), (44, 56))$

$(\mathbf{w}, \mathbf{d}) = ((11, 18, 27, 28, 44), (55, 72))$

$(\mathbf{w}, \mathbf{d}) = ((12, 15, 20, 26, 34), (46, 60))$

$(\mathbf{w}, \mathbf{d}) = ((13, 14, 19, 23, 29), (42, 52))$

$(\mathbf{w}, \mathbf{d}) = ((13, 17, 24, 27, 38), (51, 65))$

$(\mathbf{w}, \mathbf{d}) = ((13, 20, 29, 31, 47), (60, 78))$

$(\mathbf{w}, \mathbf{d}) = ((13, 23, 34, 35, 56), (69, 91))$

$(\mathbf{w}, \mathbf{d}) = ((14, 15, 19, 26, 31), (45, 57))$

$(\mathbf{w}, \mathbf{d}) = ((14, 19, 25, 32, 43), (57, 75))$

$(\mathbf{w}, \mathbf{d}) = ((16, 20, 30, 33, 47), (63, 80))$

$(\mathbf{w}, \mathbf{d}) = ((16, 21, 28, 36, 48), (64, 84))$

$(\mathbf{w}, \mathbf{d}) = ((16, 28, 42, 43, 69), (85, 112))$

$(\mathbf{w}, \mathbf{d}) = ((18, 19, 24, 33, 39), (57, 72))$

$(\mathbf{w}, \mathbf{d}) = ((18, 23, 30, 39, 51), (69, 90))$

$(\mathbf{w}, \mathbf{d}) = ((18, 24, 32, 41, 55), (73, 96))$

33. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3, 5, 6, 7, 8), (13, 15))$

$(\mathbf{w}, \mathbf{d}) = ((4, 8, 10, 11, 13), (21, 24))$

$(\mathbf{w}, \mathbf{d}) = ((5, 7, 10, 11, 14), (21, 25))$

$(\mathbf{w}, \mathbf{d}) = ((6, 14, 18, 19, 23), (37, 42))$

$(\mathbf{w}, \mathbf{d}) = ((8, 10, 16, 17, 23), (33, 40))$

$(\mathbf{w}, \mathbf{d}) = ((10, 11, 15, 18, 22), (33, 40))$

$(\mathbf{w}, \mathbf{d}) = ((10, 17, 25, 26, 34), (51, 60))$

$(\mathbf{w}, \mathbf{d}) = ((11, 13, 14, 19, 20), (33, 39))$

$(\mathbf{w}, \mathbf{d}) = ((11, 17, 20, 24, 27), (44, 51))$

$(\mathbf{w}, \mathbf{d}) = ((11, 21, 26, 29, 34), (55, 63))$

$(\mathbf{w}, \mathbf{d}) = ((11, 25, 32, 34, 41), (66, 75))$

$(\mathbf{w}, \mathbf{d}) = ((11, 29, 38, 39, 48), (77, 87))$

$(\mathbf{w}, \mathbf{d}) = ((12, 16, 18, 23, 25), (41, 48))$

$(\mathbf{w}, \mathbf{d}) = ((12, 32, 42, 43, 53), (85, 96))$

$(\mathbf{w}, \mathbf{d}) = ((14, 17, 27, 29, 39), (56, 68))$

$(\mathbf{w}, \mathbf{d}) = ((15, 16, 20, 28, 32), (48, 60))$

35a. No instances were found by computer search.

35b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3,3,4,4,6),(7,12))$

$(\mathbf{w}, \mathbf{d}) = ((6,6,10,10,15),(16,30))$

35cd. No instances were found by computer search.

35e. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((6,8,8,10,15),(16,30))$

35f. No instances were found by computer search.

35g. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((9,12,13,16,24),(25,48))$

36abc. No instances were found by computer search.

37ab. No instances were found by computer search.

38abc. No instances were found by computer search.

38d. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((4,6,6,8,11),(12,22))$

38e. No instances were found by computer search.

38f. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((3,4,5,6,8),(9,16))$

$(\mathbf{w}, \mathbf{d}) = ((4,6,10,12,15),(16,30))$

$(\mathbf{w}, \mathbf{d}) = ((6,10,14,18,23),(24,46))$

$(\mathbf{w}, \mathbf{d}) = ((8, 14, 26, 32, 39), (40, 78))$

$(\mathbf{w}, \mathbf{d}) = ((12, 14, 18, 20, 27), (32, 54))$

39a. No instances were found by computer search.

39b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((6, 9, 10, 13, 18), (19, 36))$

40ab. No instances were found by computer search.

40c. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((4, 6, 6, 7, 9), (13, 18))$

40d. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((6, 7, 9, 11, 14), (18, 28))$

$(\mathbf{w}, \mathbf{d}) = ((8, 12, 13, 14, 18), (26, 36))$

$(\mathbf{w}, \mathbf{d}) = ((8, 20, 23, 26, 30), (46, 60))$

$(\mathbf{w}, \mathbf{d}) = ((12, 18, 22, 27, 33), (45, 66))$

41. No instances were found by computer search.

42. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((1, 3, 3, 4, 6), (7, 9))$

43a. No instances were found by computer search.

43b. Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((1, 4, 6, 8, 11), (12, 17))$

$(\mathbf{w}, \mathbf{d}) = ((1, 5, 7, 10, 14), (15, 21))$

$(\mathbf{w}, \mathbf{d}) = ((1, 7, 12, 17, 23), (24, 35))$

$(\mathbf{w}, \mathbf{d}) = ((1, 8, 14, 20, 27), (28, 41))$

$(\mathbf{w}, \mathbf{d}) = ((1, 9, 15, 22, 30), (31, 45))$

$(\mathbf{w}, \mathbf{d}) = ((1, 10, 17, 25, 34), (35, 51))$

$(\mathbf{w}, \mathbf{d}) = ((2, 7, 10, 13, 18), (20, 28))$

$(\mathbf{w}, \mathbf{d}) = ((2, 9, 12, 17, 24), (26, 36))$

$(\mathbf{w}, \mathbf{d}) = ((3, 7, 8, 9, 13), (16, 21))$

$(\mathbf{w}, \mathbf{d}) = ((3, 8, 10, 12, 17), (20, 27))$

$(\mathbf{w}, \mathbf{d}) = ((3, 10, 11, 15, 22), (25, 33))$

45abcd. No instances were found by computer search.

46abc. No instances were found by computer search.

47 Computer search located the following:

$(\mathbf{w}, \mathbf{d}) = ((1, 4, 5, 7, 11), (12, 15))$

$(\mathbf{w}, \mathbf{d}) = ((1, 5, 6, 9, 14), (15, 19))$

$(\mathbf{w}, \mathbf{d}) = ((1, 8, 13, 19, 31), (32, 39))$

$(\mathbf{w}, \mathbf{d}) = ((1, 9, 15, 22, 36), (37, 45))$

$(\mathbf{w}, \mathbf{d}) = ((1, 10, 16, 24, 39), (40, 49))$

$(\mathbf{w}, \mathbf{d}) = ((1, 11, 18, 27, 44), (45, 55))$

$(\mathbf{w}, \mathbf{d}) = ((2, 9, 12, 17, 27), (29, 36))$

$(\mathbf{w}, \mathbf{d}) = ((2, 11, 14, 21, 33), (35, 44))$

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