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Lattice points in disks and strongly convex domains

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Abstract

This thesis examines the history and some major results of the Gauss Circle Problem. The goal of the Gauss Circle Problem is to determine the best bound for the error between the number of lattice points inside a disk and that disk's area, otherwise known as the lattice point discrepancy. First we state some of the required definitions and properties from Fourier analysis that will be used throughout. In particular, we establish asymptotic results for oscillatory integrals and more specifically for Bessel functions. After examining the geometrical method for precisely counting the number of lattice points inside a disk of radius R, we use the Poisson Summation Formula and the Bessel function results to prove initial bounds on the lattice point discrepancy. We present two such results, employing a similar technique for both, and then apply oscillatory integral asymptotics to extend this method and establish a lattice point discrepancy result for strongly convex domains.

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Glossary of Notation

R	The radius of a circle in $\mathbb{R}^d$.
D_R	The disk of radius R in $\mathbb{R}^2$.
$N_d(R)$	The number of lattice points inside a sphere of radius R in $\mathbb{R}^d$.
$r_2(k)$	The number of ways that the real number k can be represented as a sum of two squares.
$\chi_R(x)$	The characteristic function of the disk of radius R.
S	The unit square centered at the origin.
$x = (x_1, x_2, ..., x_d)$	Points in $\mathbb{R}^d$.
x'	The first $d-1$ coordinates of $x \in \mathbb{R}^d$.
$\lfloor R \rfloor$	The integer part of the real number R.
$\partial\Omega$	The boundary of the region Ω.
$R\Omega$	The dilation of Ω by R. i.e. $R\Omega = \{Rx : x \in \Omega\}$.
$\nabla^2(f)$	Denotes the Hessian of f.
$m(\Omega)$	The measure of the set Ω.

Chapter 1

Introduction

The problem of finding the best bound for the error, $E(R)$, between the number of lattice points inside a circle of radius R and its area, known as the Gauss Circle Problem, remains unsolved. There is, however, a long history of incremental advancements and generalizations. Gauss first proved that $E(R) \lesssim R$, i.e. there is some uniform constant $c > 0$ so that $|E(R)| \leq cR$ for sufficiently large R. The first improvement, $E(R) \lesssim R^{2/3}$, was proved by Sierpiński in 1906 [7, Chapter 2, Section 2.6.2]. Subsequent improvements on this exponent were made by many others, including van der Corput (0.66...), Titchmarsh (0.652...), Nowak (0.648...), Iwaniec (0.636...), and Huxley (0.6301...). Hardy and Landau contributed improvements, including a proof that the exponent on the error can be no smaller than $1/2$ [7, Chapter 2, Section 2.6.2]. The bound $E(R) \lesssim R^{1/2+\varepsilon}$ for any $\varepsilon > 0$ is conjectured, but no proof has yet been published [4].

It is also natural to investigate analogues of the Gauss Circle Problem, as we do here for strongly convex domains. The most obvious case is that of the sphere. While improvements for circles are somewhat limited by current methods for working with exponential sums, the sphere problem for dimensions $d \geq 4$ is much better understood [6]. The problem is solved for $d \geq 5$ and for $d = 2$ and $d = 3$ remains a challenge [6],

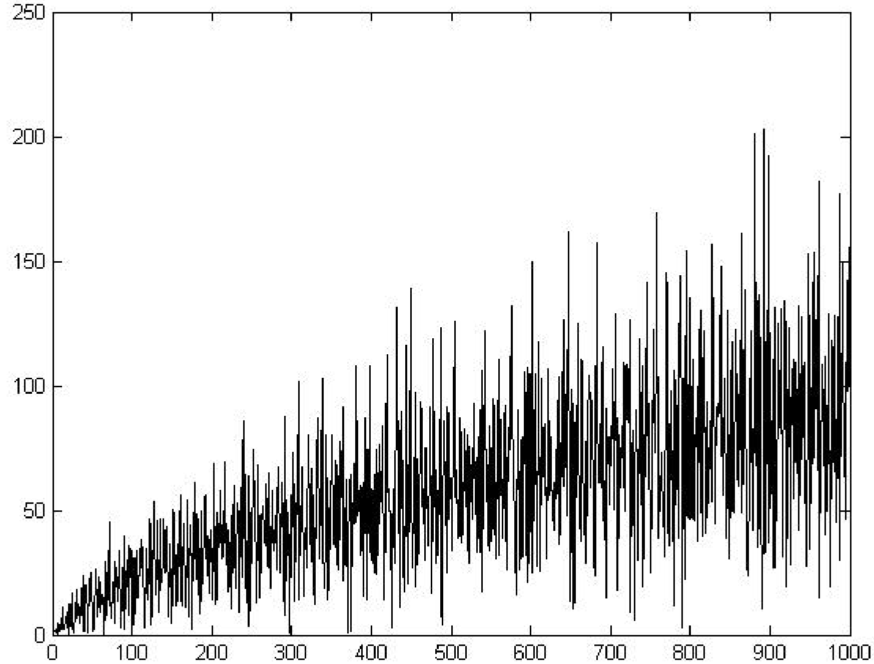
[13]. Other interesting variations include ellipsoids, conic sections and domains with fractal boundary, like the Koch snowflake [3]. We restrict our focus to the disk and strongly convex domains.

This thesis examines the history, major results, and some generalizations of the Gauss Circle Problem; that is, the problem of finding the number of lattice points inside a disk of radius R. Intuitively, the number of lattice points should be close to the area of the disk. What we find, however, is that even for large radii the error or lattice point discrepancy, $E(R)$, between the number of lattice points and the area of the disk is highly irregular. This irregularity results from those points near the boundary of the disk. Away from the boundary we can associate to each lattice point a unit square but near the boundary there are unit squares only partially contained in the disk. For some integer radii R the only lattice points on the boundary are $(0, R), (0, -R), (R, 0),$ and $(-R, 0)$. However, for some radii the number of lattice points on the boundary exceeds four. This allows the number of lattice points in the closed disk to significantly exceed the area of the disk if there are a large number of ways to represent R as a sum of two squares. Lattice points may be included by increasing the radius slightly without adding another full unit square to the area for each included lattice point, in which case the error increases.

The lattice point discrepancies for radii ranging from 1 to 1000 are plotted in Figure 1. The discrepancies, even for such a small sample size, oscillate between 0 and about 200 naturally leading us to consider an upper bound for the lattice point discrepancy.

We begin in chapter two with a brief introduction to relevant definitions, properties, and theorems from Fourier analysis that will be used throughout. With these foundations in place, we proceed in chapter three to derive geometrically the counting formula that gives precisely the number of lattice points in a circle of radius R. This formula has a natural extension to spheres in $\mathbb{R}^d$. Then we present a proof of Gauss' original result that $|E(R)| \lesssim R$. This is clearly not the best bound for $|E(R)|$, but

Figure 1.1: Lattice Point Discrepancies for Integer Radii from 1 to 1000



the proof develops further the geometrical connection between the number of lattice points inside a circle and that circle's area. We end this section with the deeper result that $|E(R)| \lesssim R^{2/3}$. Here we begin to use topics from Fourier analysis heavily, including convolution, the Fourier transform, and the Poisson summation formula.

In chapter four we employ more sophisticated results from geometry and Fourier analysis in order to examine the result for R dilates of strongly convex sets Ω containing the origin in $\mathbb{R}^d$ with boundary $(d+2)-$times continuously differentiable where d is the dimension. We conclude in chapter five with a brief review of our results.

Chapter 2

Definitions and Key Concepts

In this chapter we introduce the reader to the following definitions, properties, lemmas, and theorems which are used throughout this paper.

2.1 Basics

Definition 1. *The space of Schwartz functions on $\mathbb{R}^d$, denoted by $\mathcal{S}(\mathbb{R}^d)$, is the space of all complex-valued C^∞ functions f defined on $\mathbb{R}^d$ for which*

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty$$

for all multi-indices α and β. That is, $\mathcal{S}(\mathbb{R}^d)$ is the space of infinitely differentiable functions that, along with all of their derivatives, decay at infinity faster than any polynomial [10, Chapter 3, Section 1], [5, Chapter 2, Definition 2.2.1].

Schwartz functions, being so well-behaved at infinity, will be invaluable for us in evaluating the behavior of the lattice point discrepancy at infinity. Though the characteristic function, χ_R, of the disk of radius R lacks continuity on the boundary of D_R, we will approximate χ_R using Schwartz functions so that we may employ tools that would otherwise be unavailable to us. In particular, we will use a specific type of Schwartz function called a bump function.

Definition 2. *A function $\varphi : \mathbb{R}^d \to [0,1]$ is a bump function if it is both Schwartz and compactly supported* [8, Chapter 7, Example 7.6].

Definition 3. *A family of functions $\{\varphi_\delta\}$ with $\varphi_\delta : \mathbb{R}^d \to \mathbb{R}$ is an approximate identity if*

(i) $\int_{\mathbb{R}^d} \varphi_\delta(x)\, dx = 1$

(ii) There is a constant $C > 0$ such that $\int_{\mathbb{R}^d} |\varphi_\delta(x)|\, dx \leq C$ for all $\delta > 0$

(iii) $\lim_{\delta\to 0} \int_{|x|>\varepsilon} |\varphi_\delta(x)|\, dx = 0$ *for each $\varepsilon > 0$*

all hold [8, Chapter 7, Definition 7.21].

Definition 4. *We call a function $f : \mathbb{R}^d \to \mathbb{R}$ that is C^∞ on $\mathbb{R}^d$ a smooth function* [11, Chapter 2].

Our use of bump functions to smooth χ_R, by allowing us to apply the Fourier transform, gives us access to the tools of Fourier analysis. Ultimately this toolbox, including the Fourier Transform, convolution, and their properties, is what allows us to repeatedly utilize the Poisson Summation Formula, which is one of the most powerful results below.

Definition 5. *For $f \in \mathcal{S}(\mathbb{R}^d)$ the Fourier transform of f is defined by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

for $\xi \in \mathbb{R}^d$.

Note that if $f \in \mathcal{S}(\mathbb{R}^d)$ then $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$ [11, Chapter 6, Corollary 2.2].

Property 6. *If $f \in \mathcal{S}(\mathbb{R}^d)$ and the Fourier transform is defined as above then the Fourier inversion formula,*

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(\xi)\, d\xi$$

holds for all $x \in \mathbb{R}^d$.

Definition 7. *For f, g defined on $\mathbb{R}$ their convolution is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\, dt.$$

If $f, g \in \mathcal{S}(\mathbb{R})$, this integral converges. Similarly, the integral converges if f is compactly supported and integrable and g is a bump function.

Property 8. *We have*

(i) $f * g \in \mathcal{S}(\mathbb{R})$

(ii) $f * g = g * f$

(iii) $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$

for all $f, g \in \mathcal{S}(\mathbb{R})$. If f is integrable and compactly supported and g is C^∞ then the derivative of $f * g$ is

$$(f * g)'(x) = (f * g')(x) = (f' * g)(x)$$

and hence $f * g$ is also C^∞. i.e. we can view convolution as a smoothing operation [11, Chapter 5, Section 1], [8, Chapter 7, Section 5].

We now state the central result used in our estimates of the lattice point discrepancy both for disks of radius R and for our generalization to strongly convex domains. The Poisson Summation Formula is essential and, combined with the properties of convolution above, is what allows us to profit from smoothing χ_R with bump functions.

Property 9. [Poisson Summation Formula] *If* $f \in \mathcal{S}(\mathbb{R}^d)$, *then*

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n).$$

Note that the formula can be stated with different conditions on f, but this statement is sufficient for our purposes [10, Proposition 8.2].

In Chapter 4 our investigation shifts from disks to strongly convex domains, Ω. Here in particular we employ the fact that the boundary of Ω, denoted $\partial\Omega$, is a hypersurface. We will partition the boundary and use the implicit function theorem in order to obtain our global results by first proving them locally on each part of the partition.

Definition 10. *We say that* M *is a hypersurface of class* C^k *if for any* $x_0 \in M$ *there exists an open set* $V \subset \mathbb{R}^d$ *and a real-valued* C^k *function* ρ *defined on* V *so that* $x_0 \in V$, $|\nabla \rho(x)| > 0$ *on* $M \cap V$ *and* $M \cap V = \{x \in V : \rho(x) = 0\}$ [10, Chapter 7, Section 4].

Here ξ_k and x_k denote the kth coordinates of the vectors $\xi = (\xi_1, \xi_2, ..., \xi_d)$ and $x = (x_1, x_2, ..., x_d)$ respectively.

Definition 11. *For a hypersurface* $\Omega \subset \mathbb{R}^d$ *with* C^2 *defining function* ρ *so that* $|\nabla \rho| = 1$ *on* Ω*, the curvature form (or second fundamental form) of* Ω *at* $x \in \Omega$ *is the quadratic form*

$$\sum_{1 \leq k,j \leq d} \xi_k \xi_j \frac{\partial^2 \rho}{\partial x_k \partial x_j}(x)$$

restricted to vectors $\sum_{1 \leq k \leq d} \xi_k \frac{\partial}{\partial x_k}$ *tangent to* Ω *at* x*. By normalizing if necessary we can always arrange that* $|\nabla \rho| = 1$ *on the boundary* [10, Chapter 8, Section 3].

Note that this normalization will decrease the regularity of ρ. If ρ is C^k then the normalization of ρ will be C^{k-1}. Thus, if a normalization is necessary, the hypersurface must have a defining function that is C^3.

Definition 12. *We say* $\Omega \subset \mathbb{R}^d$ *with* C^k *boundary* $\partial\Omega$ *for* $k \geq 2$ *is strongly convex when the curvature form of* Ω *is strictly positive definite for all* $x \in \partial\Omega$ [10, Chapter 8, Section 8].

Theorem 13. [Implicit Function Theorem.]
Let E *be an open subset of* $\mathbb{R}^d$*, let* $f : E \to \mathbb{R}$ *be continuously differentiable, and let* $y = (y_1, ..., y_d)$ *be a point in* E *such that* $f(y) = 0$ *and* $\frac{\partial f}{\partial x_d}(y) \neq 0$*. Then there exists an open subset* U *of* $\mathbb{R}^{d-1}$ *containing* $(y_1, ..., y_{d-1})$*, an open subset* $V \subset E$ *containing* y*, and a function* $\varphi : U \to \mathbb{R}$ *such that* $\varphi(y_1, ..., y_{d-1}) = y_d$*, and*

$$\begin{aligned}&\{(x_1, ..., x_d) \in V : f(x_1, ..., x_d) = 0\}\\ &= \{(x_1, ..., x_{d-1}, \varphi(x_1, ..., x_{d-1})) : (x_1, ..., x_{d-1}) \in U\}.\end{aligned}$$

In other words, we can parameterize $\{x \in V : f(x) = 0\}$ *by* $\varphi(y_1, ..., y_{d-1}) = y_d$ *so we can view the set as a graph of a function over* U*. Moreover,* φ *is differentiable at* $(y_1, ..., y_{d-1})$ *with*

$$\frac{\partial \varphi}{\partial x_j}(y_1, ..., y_{d-1}) = -\frac{\partial f}{\partial x_j}(y) / \frac{\partial f}{\partial x_d}(y)$$

for all $1 \leq j \leq d-1$, [12, Chapter 17, Theorem 17.8.1].

2.2 Bessel Functions and Asymptotics

Bessel functions, because of their similarity to Fourier transforms, naturally arise in our calculations of $\widehat{\chi_R}(n)$ in Chapter 3 Section 3.2. We present here a useful identity and asymptotic result that will be used in that section.

Definition 14. *The Bessel function* $J_m(r)$ *of order* m *is*

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\sin(\theta)} e^{-im\theta} \, d\theta$$

for $m \in \mathbb{Z}^+$ [9].

Lemma 15. *For $r \geq 0$ we can rewrite $rJ_1(r)$ as*

$$rJ_1(r) = \int_0^r \sigma J_0(\sigma)\, d\sigma \tag{2.1}$$

Proof. We begin, as suggested in [10, Chapter 8, Exercise 23], by calculating two identities,

a) $J_1'(r) = \frac{1}{2}(J_0(r) - J_2(r))$

b) $J_1(r) = \frac{r}{2}(J_0(r) + J_2(r))$

which we will then use to get (2.1). For the first identity, using integration by parts, calculate

$$\begin{aligned}
\frac{1}{2}\Big(J_0(r) - J_2(r)\Big) &= \frac{1}{4\pi}\int_0^{2\pi} \Big(e^{ir\sin(\theta)} - e^{ir\sin(\theta)}e^{-2i\theta}\Big)\, d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} e^{ir\sin(\theta)}\Big(1 - e^{-2i\theta}\Big)\, d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} e^{ir\sin(\theta)}e^{-i\theta}\Big(e^{i\theta} - e^{-i\theta}\Big)\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} i\sin(\theta)e^{ir\sin(\theta)}e^{-i\theta}\, d\theta \qquad (2.2)
\end{aligned}$$

By taking the derivative of J_1 with respect to r we have

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin(\theta)}e^{-i\theta}\, d\theta\right) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dr}\Big(e^{ir\sin(\theta)}e^{-i\theta}\Big)\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} i\sin(\theta)e^{ir\sin(\theta)}e^{-i\theta}\, d\theta
\end{aligned}$$

which is precisely (2.2). We are able to interchange differentiation and integration above because we are integrating an exponential function, so continuity of the integrand and its derivative is clear.

For the second identity we calculate

$$\begin{aligned}
\frac{r}{2}\Big(J_0(r) + J_2(r)\Big) &= \frac{r}{4\pi}\int_0^{2\pi} e^{ir\sin(\theta)} + e^{ir\sin(\theta)}e^{-2i\theta}\,d\theta \\
&= \frac{r}{4\pi}\int_0^{2\pi} e^{ir\sin(\theta)}(1+e^{-2i\theta})\,d\theta \\
&= \frac{r}{4\pi}\int_0^{2\pi} e^{ir\sin(\theta)}e^{-i\theta}(e^{i\theta}+e^{-i\theta})\,d\theta \\
&= \frac{r}{2\pi}\int_0^{2\pi} \cos(\theta)\,e^{ir\sin(\theta)}e^{-i\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} ir\cos(\theta)\,e^{ir\sin(\theta)}\frac{1}{i}e^{-i\theta}\,d\theta \\
&= \frac{e^{-i\theta}}{i}e^{ir\sin(\theta)}\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin(\theta)}\Big(-e^{-i\theta}\Big)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin(\theta)}e^{-i\theta}\,d\theta = J_1(r)
\end{aligned}$$

which gives us the second identity. We then use the product rule to find the derivative of $rJ_1(r)$ and apply our identities to get

$$\begin{aligned}
\frac{d}{dr}\Big(rJ_1(r)\Big) &= rJ_1'(r) + J_1(r) \\
&= \frac{r}{2}\Big(J_0(r) - J_2(r)\Big) + \frac{r}{2}\Big(J_0(r) + J_2(r)\Big) \\
&= rJ_0(r)
\end{aligned}$$

So we have that the antiderivative of $\frac{d}{dr}\Big(rJ_1(r)\Big)$ is $\int_0^r \sigma J_0(\sigma)\,d\sigma$, so that (2.1) holds.

$\bowtie$

Finally we present some asymptotic results for oscillatory integrals of the form $\int e^{i\lambda\phi}\psi(x)\,dx$ with certain conditions on the phase function, ϕ, and the amplitude function, ψ. We do not use these lemmas directly in our lattice point discrepancy proof for strongly convex sets, Ω. We need them in order to prove a preliminary result, Theorem 25, which we will apply to Ω. By establishing these results we will have the machinery we need to approach the strongly convex domain problem analogously to the disk problem.

Lemma 16. *If $I(\lambda) = \int_{\mathbb{R}} e^{i\lambda\phi(x)}\psi(x)\,dx$ is an oscillatory integral with ϕ and ψ smooth real-valued functions and $\phi'(x) \neq 0$ for all $x \in \operatorname{supp}(\psi)$, then*

$$|I(\lambda)| \lesssim \lambda^{-N}$$

for any $N \in \mathbb{N}$. i.e. there exists a uniform constant $c > 0$ so that $|I(\lambda)| \leq c\lambda^{-N}$ for any $N \in \mathbb{N}$ and for λ sufficiently large [5, Chapter 2, Section 2.6].

Proof. This proof follows that in [5, Chapter 2, 2.6.a]. First note that $\phi'(x) \neq 0$ for all $x \in \operatorname{supp}(\psi)$ means that ϕ' is either strictly positive or strictly negative on $\operatorname{supp}(\psi)$, since ϕ' is also smooth. Hence ϕ is either monotonically decreasing or monotonically increasing so we can change variables with $u = \phi(x)$. Then $du = \phi'(x)\,dx$ so we have $dx = (\phi'(x))^{-1}\,du$ and hence $dx = (\phi^{-1})'(u)\,du$. This substitution gives us

$$I(\lambda) = \int e^{i\lambda u}\psi(\phi^{-1}(u))(\phi^{-1})'(u)\,du = \int e^{i\lambda u}\Psi(u)\,du$$

with $\Psi(u) = \psi(\phi^{-1}(u))(\phi^{-1})'(u)$, so Ψ inherits smoothness and compact support from ψ and ϕ. Consider the nth derivatives of $e^{i\lambda u}$. We know by properties of exponential functions that $e^{i\lambda u} = \frac{1}{(i\lambda)^N}\frac{d^N}{du^N}(e^{i\lambda u})$. Making this replacement in $I(\lambda)$ and integrating by parts we have

$$\begin{aligned} I(\lambda) &= \frac{1}{(i\lambda)^N}\int \frac{d^N}{du^N}(e^{i\lambda u})\Psi(u)\,du \\ &= \frac{1}{(i\lambda)^N}\left[\Psi(u)e^{i\lambda u}\Big|_{-\infty}^{\infty} - \int e^{i\lambda u}\frac{d}{du}(\Psi(u))\,du\right] \\ &= -\frac{1}{(i\lambda)^N}\int e^{i\lambda u}\frac{d}{du}(\Psi(u))\,du \end{aligned}$$

where the third line follows because Ψ is compactly supported so $\Psi(u)e^{i\lambda u}$ vanishes at positive and negative infinity.

If we continue to integrate by parts N times we can write $I(\lambda)$ as

$$I(\lambda) = \frac{(-1)^N}{(i\lambda)^N}\int e^{i\lambda u}\frac{d^N}{du^N}(\Psi(u))\,du$$

so that

$$|I(\lambda)| \leq \frac{1}{(\lambda)^N} \int \left| e^{i\lambda u} \Psi^{(N)}(u) \right| du.$$

Since the integral above is finite we have $|I(\lambda)| \lesssim \lambda^{-N}$. $\bowtie$

Lemma 17. *If $I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} \psi(x)\, dx$ is an oscillatory integral with ψ smooth and compactly supported and ϕ smooth with no critical points in the support of ψ, then*

$$I(\lambda) \lesssim \lambda^{-N}$$

for any $N \in \mathbb{N}$.

Proof. We prove this lemma as in [9, Chapter 8, Proposition 4]. We begin by noting the condition that ϕ has no critical points in the support of ψ is equivalent to ϕ having non-zero gradient on the support of ψ, $|\nabla\phi(x)| \neq 0$ for all $x \in \operatorname{supp}(\psi)$. So we know that for any $y \in \operatorname{supp}(\psi)$ there exists a ball $B(y)$ centered at y and a unit vector ξ so that $\xi \cdot (\nabla\phi)(x) \geq c > 0$ for all $x \in B(y)$ and some constant c.

To establish the claim we want to decompose $I(\lambda)$ and consider the integral locally on each of these balls. The support of ψ is compact so it can be partitioned using finitely many such balls where we denote the restriction of ψ to the kth ball in the partition by ψ_k. Then each ψ_k is smooth and compactly supported. We can now rewrite $I(\lambda)$ as

$$I(\lambda) = \sum_k \int e^{i\lambda\phi(x)} \psi_k(x)\, dx \tag{2.3}$$

where the sum is over finitely many terms.

Now we must prove the claim for any arbitrary integral in this sum. By writing the integral as an iterated integral we can apply the one-dimensional case for the desired result. Consider the integral for some fixed k and take a local coordinate system $x_1, ..., x_d$ so that ξ lies along the x_1 axis. We can now integrate first in x_1

and then in x_2 etc., so we can write the integral as

$$\int e^{i\lambda\phi(x)}\psi_k(x)\,dx = \int_{\mathbb{R}^{d-1}} \left(\int e^{i\lambda\phi(x_1,\dots,x_d)}\psi_k(x_1,...,x_d)\,dx_1 \right) dx_2 \dots dx_d$$

and apply Lemma 16 to the inner integral. Hence when we integrate in all variables we have that $|\int e^{i\lambda\phi(x)}\psi_k(x)\,dx| \lesssim \lambda^{-N}$ for any $N \in \mathbb{N}$. If we repeat this process for all terms in the sum (2.3) then $|I(\lambda)| \lesssim \lambda^{-N}$ as required. $\bowtie$

Lemma 18. *For the oscillatory integral $I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\,dx$, if ϕ is real-valued and smooth on (a,b) with $|\phi^{(k)}(x)| \geq c > 1$ for all $x \in (a,b)$ and for $k \in \{1,2\}$ then $\left|\int_a^b e^{i\lambda\phi(x)}\right| \lesssim \lambda^{-1/k}$ when*

(i) k=1 and $\phi'(x)$ is monotonic on (a,b), or

(ii) k=2

Proof. This proof follows those in [9, Chapter 8, Proposition 2] and [10, Chapter 8, Proposition 2.3]. First consider the case $k = 1$. We know $e^{i\lambda\phi(x)}$ can be written as $e^{i\lambda\phi(x)} = \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}(e^{i\lambda\phi(x)})$, which allows us to rewrite $I(\lambda)$ as

$$I(\lambda) = \int_a^b \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}(e^{i\lambda\phi(x)})\,dx$$

and integrate by parts. This gives us

$$I(\lambda) = \frac{1}{i\lambda\phi'(x)}e^{i\lambda\phi(x)}\bigg|_a^b - \int_a^b e^{i\lambda\phi(x)}\frac{d}{dx}(i\lambda\phi'(x))^{-1}\,dx.$$

We can now apply our lower bound on $|\phi'(x)|$ so that

$$\begin{aligned}
|I(\lambda)| &\leq \frac{1}{c\lambda} + \int_b^a \left|e^{i\lambda\phi(x)}\right|\left|\frac{d}{dx}(i\lambda\phi'(x))^{-1}\right| dx \\
&\leq \frac{1}{c\lambda} + \int_b^a \left|\frac{d}{dx}(i\lambda\phi'(x))^{-1}\right| dx \\
&\leq \frac{1}{c\lambda} + \frac{1}{\lambda}\int_b^a \left|\frac{d}{dx}(\phi'(x))^{-1}\right| dx \\
&\leq \frac{1}{c\lambda} + \frac{1}{\lambda}\left|\int_b^a \frac{d}{dx}(\phi'(x))^{-1}\,dx\right|
\end{aligned}$$

where the last line follows because $\phi'(x)$ is monotonic, so $\frac{d}{dx}(\phi'(x))^{-1}$ is either non-positive for every $x \in (a,b)$ or non-negative for every $x \in (a,b)$. Since the integral converges, we have $|I(\lambda)| \lesssim \lambda^{-1}$.

Now consider the case $k = 2$ and assume $|\phi''(x)| \geq c > 1$ for some real number c. Since $\phi''(x)$ is nonzero, we know by continuity that either $\phi''(x) \geq c$ on (a,b) or $\phi''(x) \leq -c$ on (a,b). Assume without loss of generality that $\phi''(x) \geq c$. This means that $\phi'(x)$ is strictly increasing on (a,b). Hence, $\phi'(x) = 0$ for at most one $x \in [a,b]$. If $\phi'(x)$ is non-zero on $[a,b]$ then the previous claim applies and we have a better bound for $|I(\lambda)|$. Here we consider the case $\phi'(t) = 0$ for some $t \in [a,b]$. We assume without loss of generality that $t \in (a,b)$. If $t = a$ or $t = b$, the same proof applies by splitting $[a,b]$ into two intervals rather than three, as we do below.

Take some $\delta > 0$ so that $t+\delta, t-\delta \in (a,b)$. We will fix δ later in the proof. Using δ we split (a,b) into three intervals so that

$$\int_a^b e^{i\lambda\phi(x)}\,dx = \int_a^{t-\delta} e^{i\lambda\phi(x)}\,dx + \int_{t-\delta}^{t+\delta} e^{i\lambda\phi(x)}\,dx + \int_{t+\delta}^{b} e^{i\lambda\phi(x)}\,dx$$

On $(t-\delta, t+\delta)$ we calculate

$$\left|\int_{t-\delta}^{t+\delta} e^{i\lambda\phi(x)}\,dx\right| \leq \int_{t-\delta}^{t+\delta} \left|e^{i\lambda\phi(x)}\right| dx \leq \int_{t-\delta}^{t+\delta} dx \leq 2\delta.$$

On $(a, t-\delta)$ and $(t+\delta, b)$ the conditions on $\phi'(x)$ for the $k = 1$ case are satisfied. Since $\phi'(x)$ is strictly increasing on (a,b), $\phi'(t) = 0$ and $\phi''(x) > 1$ for all $x \in (a,b)$ we know $|\phi'(x)| > \delta$ for all $x \in (a, t-\delta) \cup (t+\delta, b)$. So, on $(t+\delta, b)$ and $(a, t-\delta)$, by the previous case we have

$$\left|\int_{t+\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| \lesssim (\delta\lambda)^{-1} \text{ and } \left|\int_{a}^{t+\delta} e^{i\lambda\phi(x)}\,dx\right| \lesssim (\delta\lambda)^{-1}.$$

Applying our results from each interval we conclude that

$$|I(\lambda)| \lesssim (\lambda\delta)^{-1} + 2\delta + (\lambda\delta)^{-1}.$$

If we take $\delta = \lambda^{-1/2}$ we can combine these terms to obtain

$$|I(\lambda)| \lesssim 2\lambda^{-1}\lambda^{1/2} + 2\lambda^{-1/2} \lesssim \lambda^{-1/2}.$$

$\bowtie$

Theorem 19. *We have the following result for the asymptotics for an oscillatory integral of the form* $I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx$ *as* λ *tends to infinity. If* ϕ *and* ψ *are* C^∞ *and real-valued and* $|\phi^{(k)}(x)| \geq c > 0$ *for all* $x \in (a,b)$ *and* $k \in \{1,2\}$ *then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx\right| \lesssim \lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right]$$

Proof. This proof follows the approach suggested in [9, Chapter 8]. We will rewrite $I(\lambda)$ so that we may apply Lemma 18. First we denote $F(x) = \int_a^x e^{i\lambda\phi(t)}\,dt$. Then using the Fundamental Theorem of Calculus we can write $I(\lambda) = \int_a^b F'(x)\psi(x)\,dx$. Fix $k = 1$ or $k = 2$. Integrate by parts so that

$$\begin{aligned}
|I(\lambda)| &= \left|\int_a^b F'(x)\psi(x)\,dx\right| = \left|F(b)\psi(b) - \int_a^b F(x)\psi'(x)\,dx\right| \\
&= \left|\psi(b)\int_a^b e^{i\lambda\phi(t)}\,dt - \int_a^b F(x)\psi'(x)\,dx\right| \\
&\leq |\psi(b)|\left|\int_a^b e^{i\lambda\phi(t)}\,dt\right| + \left|\int_a^b F(x)\psi'(x)\,dx\right| \\
&\lesssim \lambda^{-1/k}|\psi(b)| + \left|\int_a^b F(x)\psi'(x)\,dx\right| \\
&\lesssim \lambda^{-1/k}|\psi(b)| + \lambda^{-1/k}\int_a^b |\psi'(x)|\,dx
\end{aligned}$$

where the last line follows because Lemma 18 applies to $F(x)$. Factor out $\lambda^{-1/k}$ to conclude that $I(\lambda) \lesssim \lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right]$. $\bowtie$

Property 20. *If* ϕ *and* ψ *are* C^∞ *on* $\mathbb{R}^d$*,* ψ *has compact support, and* $\det\{\nabla^2\phi\}$ *is nonzero on the support of* ψ *then*

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)\,dx\right| \lesssim \lambda^{-d/2}$$

.

This property and the following proof can be found in [10, Chapter 8, Proposition 2.5].

Proof. Denote $I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)\,dx$. To estimate $|I(\lambda)|$ we will use the relationship $|I(\lambda)|^2 = \overline{I(\lambda)}I(\lambda)$. Additionally, we assume without loss of generality that the support of ψ is a sufficiently small ball of radius $\varepsilon > 0$ where ε will be chosen using ϕ. Once we have proved the property for ψ with sufficiently small support, we can extend the result for ψ supported on any compact set by taking a partition of unity so that $\operatorname{supp}(\psi) = \cup_{j=1}^M \operatorname{supp}(\psi_j)$ where each ψ_j has sufficiently small support and M is finite.

Consider $|I(\lambda)|^2$ and apply the change of variables $y = x + u$. This gives us

$$\begin{aligned}\overline{I(\lambda)}I(\lambda) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\lambda[\phi(y)-\phi(x)]}\psi(y)\overline{\psi}(x)\,dx\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\lambda[\phi(x+u)-\phi(x)]}\psi(x+u)\overline{\psi}(x)\,dx\,du. \end{aligned} \tag{2.4}$$

Let $\Psi(x,u) = \psi(x+u)\overline{\psi}(x)$. Since ψ is smooth and compactly supported, so is Ψ. Since $u = y - x$ and both x and y are restricted to a ball of radius ε, we must have that $|u| \leq 2\varepsilon$. Hence, the support of Ψ is the ball of radius 2ε.

We proceed by first proving the asymptotics for the inner integral in (2.4). We will show that

$$\left| \int_{\mathbb{R}^d} e^{i\lambda[\phi(x+u)-\phi(x)]}\Psi(x,u)\,dx \right| \lesssim (\lambda|u|)^{-N} \tag{2.5}$$

for every $N \geq 0$. To this end, consider the vector field $L = \frac{1}{i\lambda}(a\cdot\nabla)$ and its transpose $L^t(f) = \frac{-1}{i\lambda}\nabla\cdot(af)$ where

$$a = \frac{\nabla_x(\phi(x+u)-\phi(x))}{|\nabla_x(\phi(x+u)-\phi(x))|^2}.$$

To simplify notation denote $\nabla_x(\phi(x+u)-\phi(x)) = b$ so that $a = \frac{b}{|b|^2}$.

We will show that $|b| \approx |u|$ for $|u| \leq 2\varepsilon$ by proving that $|b| \lesssim |u|$ and $|b| \gtrsim |u|$. Since ϕ is smooth, clearly $|b| \lesssim |u|$ for $|u| \leq 2\varepsilon$. To show the other inequality expand

b using a Taylor series centered at $u = 0$. This gives us

$$\begin{aligned}\nabla_x \phi(x+u) - \nabla_x \phi(x) &= \nabla_x \phi(x) + \nabla^2 \phi(x) \cdot u + R(x,u) - \nabla_x \phi(x) \\ &= \nabla^2 \phi(x) \cdot u + R(x,u) \end{aligned} \tag{2.6}$$

where $R(x,u)$ is a remainder term with $R(x,u) \lesssim |u|^2$. If we take $\varepsilon > 0$ to be sufficiently small, then $|u| \geq |u|^2$ and $|\nabla^2 \phi(x) \cdot u| \gtrsim |u|$, since $\det \nabla^2 \phi \neq 0$. Hence, (2.6) gives us $|b| \geq c_1|u| + c_2|u|^2 \geq 2\max(c_1, c_2)|u|$, where c_1 and c_2 are positive constants. Therefore, $|b| \approx |u|$.

Observe that $|\partial_x^\alpha b| \leq c_\alpha |u|$ for all α with c_α a positive constant dependent on α. We can combine this with $|b| \approx |u|$ to write $|\partial_x^\alpha a| \lesssim |u|^{-1}$ for all α, using our definition of a and differentiation rules. Recall that $L^t(\Psi) = \frac{-1}{i\lambda} \nabla \cdot (a\Psi)$ so, for every $N \in \mathbb{Z}$ with $N > 0$, we have that

$$|(L^t)^N(\Psi(x,u))| \lesssim (\lambda^{-1}|u|^{-1})^N \lesssim (\lambda|u|)^{-N}.$$

We can now apply this property to the left side of (2.5) using the invariance of $e^{i\lambda[\phi(x+u)-\phi(x)]}$ under L. This gives us

$$\begin{aligned}\left| \int_{\mathbb{R}^d} e^{i\lambda[\phi(x+u)-\phi(x)]} \Psi(x,u)\, dx \right| &= \left| \int_{\mathbb{R}^d} L^N(e^{i\lambda[\phi(x+u)-\phi(x)]}) \Psi(x,u)\, dx \right| \\ &= \left| \int_{\mathbb{R}^d} e^{i\lambda[\phi(x+u)-\phi(x)]} (L^t)^N \Psi(x,u)\, dx \right| \\ &\leq \int_{\mathbb{R}^d} \left| e^{i\lambda[\phi(x+u)-\phi(x)]} \right| \left| (L^t)^N \Psi(x,u) \right| dx \\ &\lesssim (\lambda|u|)^{-N}. \end{aligned}$$

Returning to (2.4), let $N = 0$ in (2.5) to establish a bound for u near 0, and $N = d+1$ for u away from zero. We then have

$$\begin{aligned}\overline{I(\lambda)} I(\lambda) = |I(\lambda)|^2 &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} L^N(e^{i\lambda[\phi(x+u)-\phi(x)]}) \Psi(x,u)\, dx \right| du \\ &\lesssim \int_{\mathbb{R}^d} \frac{du}{(1+\lambda|u|)^{d+1}} \\ &\lesssim \lambda^{-d}. \end{aligned}$$

Taking the square root of $|I(\lambda)|^2$ we conclude that $|(\lambda)| \lesssim \lambda^{-d/2}$. $\bowtie$

Lemma 21. *For χ_R the characteristic function of the disk in $\mathbb{R}^2$ with radius R ,*

$$|\hat{\chi}_R(n)| = \frac{R}{|n|}|J_1(2\pi|n|R)| \lesssim R^{1/2}|n|^{-3/2}.$$

Proof. To show the equality, take the Fourier transform of χ_R evaluated at n,

$$|\hat{\chi}_R(n)| = \left|\int_{B_R} e^{-2\pi i x\cdot n}\,dx\right|.$$

Without loss of generality, because the quantity above is rotationally invariant, assume that n is aligned in the negative direction along the second axis with respect to the usual basis e_1, e_2. Recall by Definition 14 that $J_0(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin(\theta)}\,d\theta$. We make the replacement $x \cdot n = |n| \cdot e_2 \cdot (r\cos(\theta), r\sin(\theta))$ to get

$$\begin{aligned}|\hat{\chi}_R(n)| &= \left|\int_0^R \int_0^{2\pi} e^{2\pi i r|n|\sin(\theta)}\,d\theta\, r\,dr\right| \\ &= \left|\int_0^R 2\pi r J_0(2\pi|n|r)\,dr\right|.\end{aligned}$$

Now applying the substitution $u = 2\pi r|n|$ we have

$$\begin{aligned}|\hat{\chi}_R(n)| &= \left|\int_0^{2\pi R|n|} \frac{r}{|n|} J_0(u)\,du\right| \\ &= \left|\int_0^{2\pi R|n|} \left(\frac{2\pi|n|}{2\pi|n|}\right)\frac{r}{|n|} J_0(u)\,du\right| \\ &= \left|\frac{1}{2\pi|n|^2}\int_0^{2\pi R|n|} uJ_0(u)\,du\right| \\ &= \left|\frac{2\pi R|n|}{2\pi|n|^2} J_1(2\pi|n|R)\right| = \frac{R}{|n|}|J_1(2\pi|n|R)|\end{aligned}$$

where the last line follows from Lemma 15.

Let $\lambda = 2\pi|n|r$, $\phi(x) = \sin(x)$, and $\psi(x) = e^{-ix}$. Then we can write $I(\lambda) = |J_1(\lambda)| = \int_0^{2\pi} e^{i\lambda\sin(x)}e^{-ix}\,dx = \int_0^{2\pi} e^{i\lambda\phi(x)}\psi(x)\,dx$. Then $|\phi'(x)| = |\cos(x)|$ and $|\phi''(x)| = |\sin(x)|$. We know that $|\cos(x)| = |\sin(x)| = \frac{1}{\sqrt{2}}$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. We proceed as in [10, Chapter 8, Corollary 2.4] and [9]. Using these points to split the interval $[0, 2\pi]$ into subintervals with $[0, 2\pi] = [0, \frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi]$, we know that $|\cos x| \geq \frac{1}{\sqrt{2}}$ on $[0, \frac{\pi}{4}]$, $[\frac{3\pi}{4}, \frac{5\pi}{4}]$, and $[\frac{7\pi}{4}, 2\pi]$, and $|\sin x| \geq \frac{1}{\sqrt{2}}$

on $[\frac{\pi}{4}, \frac{3\pi}{4}]$ and $[\frac{7\pi}{4}, 2\pi]$. Splitting $I(\lambda)$ into five integrals over these subintervals we have,

$$\begin{aligned}
I(\lambda) &= \int_0^{\pi/4} e^{i\lambda\phi(x)}\psi(x)\,dx + \int_{\pi/4}^{3\pi/4} e^{i\lambda\phi(x)}\psi(x)\,dx + \int_{3\pi/4}^{5\pi/4} e^{i\lambda\phi(x)}\psi(x)\,dx \\
&+ \int_{5\pi/4}^{7\pi/4} e^{i\lambda\phi(x)}\psi(x)\,dx + \int_{7\pi/4}^{2\pi} e^{i\lambda\phi(x)}\psi(x)\,dx \\
&= I_1(\lambda) + I_2(\lambda) + I_3(\lambda) + I_4(\lambda) + I_5(\lambda).
\end{aligned}$$

First consider $I_1(\lambda)$. Since the first derivative of the phase, $\phi(x)$, is non-vanishing on $[0, \pi/4]$, we can apply Theorem 19 with $k = 1$. This gives us some constant $c_1 > 0$ so that

$$\begin{aligned}
|I_1(\lambda)| &\leq c_1\lambda^{-1}\left[|\psi(\pi/4)| + \int_0^{\pi/4} |\psi'(x)|\,dx\right] \\
&\leq c_1\lambda^{-1}\left[\left|e^{-i\pi/4}\right| + \int_0^{\pi/4} \left|e^{-ix}\right|\,dx\right] \\
&\leq c_1\lambda^{-1}[1 + \pi/4]
\end{aligned}$$

and hence $|I_1(\lambda)| \lesssim \lambda^{-1}$. The first derivative of the phase is similarly non-vanishing on $[\frac{3\pi}{4}, \frac{5\pi}{4}]$ and $[\frac{7\pi}{4}, 2\pi]$ so a similar application of Theorem 19 shows that we also have $|I_3(\lambda)| \lesssim \lambda^{-1}$ and $|I_5(\lambda)| \lesssim \lambda^{-1}$.

Now consider $I_2(\lambda)$ and $I_4(\lambda)$. We know $|\cos(x)| = \phi'(x)$ is zero at a point inside $[\frac{\pi}{4}, \frac{3\pi}{4}]$ and $[\frac{5\pi}{4}, \frac{7\pi}{4}]$, but $|\sin(x)| = |\phi''(x)|$ is non-zero at every point in these intervals. So, we can apply Theorem 19 once more with $k = 2$. For I_2 this gives us

$$\begin{aligned}
|I_2(\lambda)| &\leq c_2\lambda^{-1/2}\left[\left|\psi(3\pi/4)\right| + \int_{\pi/4}^{3\pi/4} |\psi'(x)|\,dx\right] \\
&\leq c_2\lambda^{-1/2}\left[\left|e^{-i3\pi/4}\right| + \int_{\pi/4}^{3\pi/4} |e^{-ix}|\,dx\right] \\
&\leq c_2\lambda^{-1/2}[1 + \pi/2]
\end{aligned}$$

and hence $|I_2(\lambda)| \lesssim \lambda^{-1/2}$. Applying the same procedure to $I_4(\lambda)$ we also have that $I_4(\lambda) \lesssim \lambda^{-1/2}$.

Combining our estimates for all five integrals to estimate the decay of $|I(\lambda)|$ requires us to take the slower decay estimate from $I_2(\lambda)$ and $I_4(\lambda)$ and sacrifice the faster decay of $I_1(\lambda)$, $I_3(\lambda)$ and $I_5(\lambda)$. We conclude that $|I(\lambda)| = |J_1(\lambda)| \lesssim \lambda^{-1/2}$. Returning to $|\widehat{\chi}_R(n)|$ with the replacement $\lambda = 2\pi|n|R$, we have

$$|\widehat{\chi}_R(n)| \lesssim \frac{R}{|n|}(2\pi|n|R)^{-1/2} \lesssim R^{1/2}|n|^{-3/2}$$

as required. $\bowtie$

Chapter 3

First Results

The following claims and more succinct versions of the included proofs can be found in [10, Chapter 8], upon which this chapter relies heavily.

3.1 Counting Formulae

Theorem 22. *The number of lattice points inside a disk, D_R, of radius R centered at the origin in $\mathbb{R}^2$ is*

$$N_2(R) = \sum_{x=-\lfloor R \rfloor}^{\lfloor R \rfloor} 2\left\lfloor \sqrt{R^2 - x^2} \right\rfloor + 1 \tag{3.1}$$

Proof. Recall that $\lfloor R \rfloor$ denotes that largest natural number smaller than R, if $R \notin \mathbb{N}$, and denotes R itself if $R \in \mathbb{N}$. We begin by first counting the lattice points on the $x-$axis. This is clearly $2\lfloor R \rfloor + 1$, since there are exactly $\lfloor R \rfloor$ lattice points on either side of the origin. The remaining lattice points are directly below and above these points. We construct the line perpendicular to the $x-$axis, intersecting at each lattice point $(n, 0)$ inside the disk. This segment intersects the circle and together with the radius forms a right triangle, allowing us to easily calculate the height of the segment.

Thus, the number of lattice points inside the circle that fall directly above $(n, 0)$ is $\left\lfloor \sqrt{R^2 - n^2} \right\rfloor$. By symmetry we have that the same number of lattice points also fall directly below $(n, 0)$, so there are $2\left\lfloor \sqrt{R^2 - n^2} \right\rfloor + 1$ lattice points inside the disk along the perpendicular line through $(n, 0)$. This process is illustrated in Figure 3.1. We

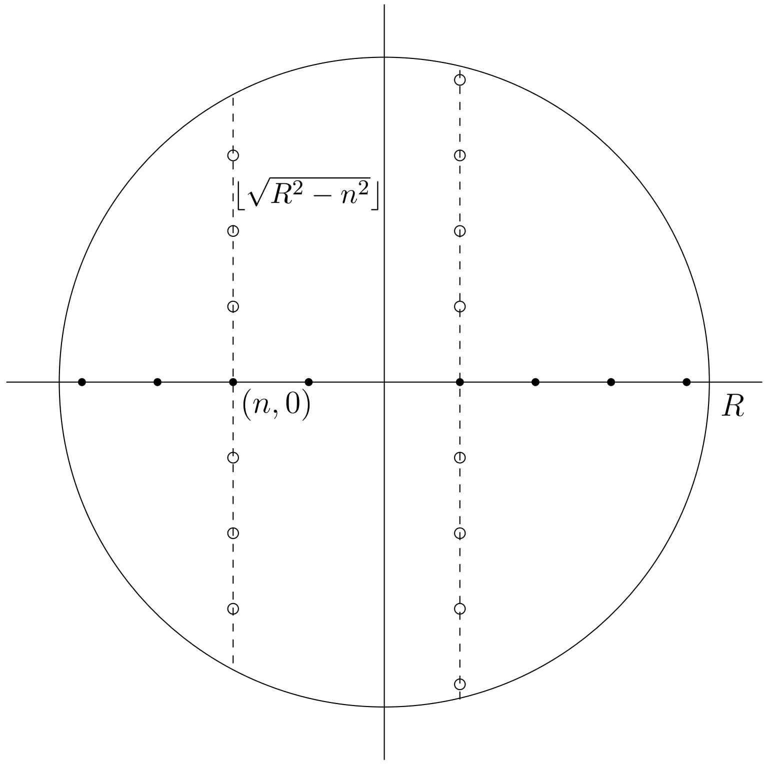


Figure 3.1: Method for finding the lattice points inside a circle using lines.

repeat this process and sum over all the lattice points in $[-R, R]$, to obtain equation (3.1). $\bowtie$

This method naturally extends to higher dimensions. In $\mathbb{R}^2$ we calculate the number of lattice points in D_R by adding the number of lattice points on intervals within D_R. Essentially, we add the lattice points falling on one-dimensional circles inside our two-dimensional disk. This suggests a natural relationship between the number of lattice points inside a sphere in $\mathbb{R}^d$ and its volume; we add the number of lattice points in each $(n-1)$-sphere centered at $(x_1, 0, ..., 0)$ with $-\lfloor R \rfloor \leq x_1 \leq \lfloor R \rfloor$. This gives us the recursive formula for $N_d(R)$, the number of lattice points in a d-dimensional sphere of radius R. We have

$$N_d(R) = \sum_{x_1=-\lfloor R \rfloor}^{\lfloor R \rfloor} N_{n-1}\Big(2\Big\lfloor\sqrt{R^2 - x_1^2}\Big\rfloor\Big) + 1$$

with $N_1(R) = 2\lfloor R \rfloor + 1$. For notational simplicity when the dimension is clear we will omit the subscript and write $N(R)$ for $N_d(R)$.

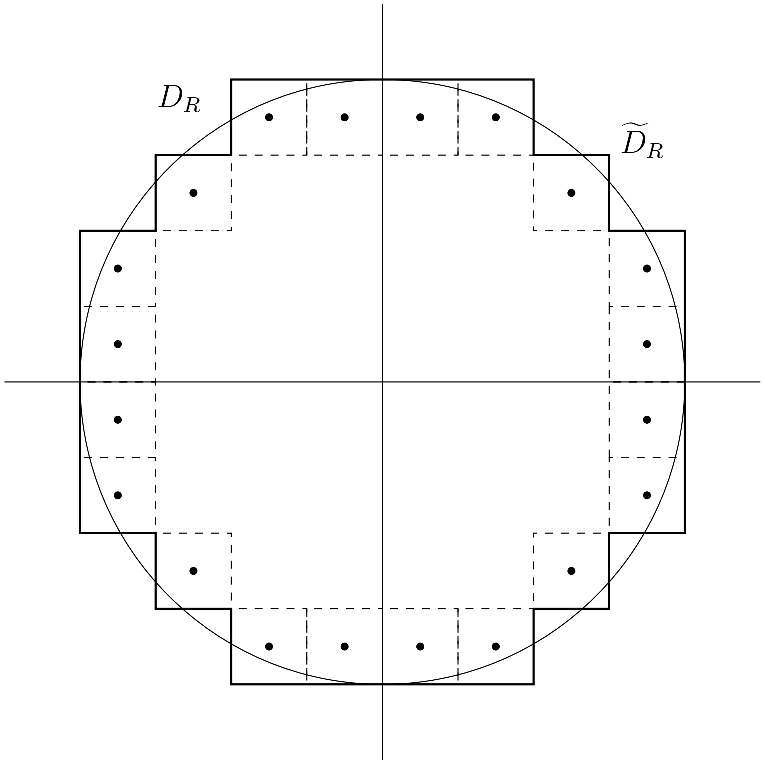
3.2 Early Bounds

In this section we prove two early bounds for the lattice point discrepancy, $|E(R)|$. The initial result, that $|E(R)| \lesssim R$, was first proved by Gauss. We approximate the area of the disk using cubes and then use geometry to bound this error. For the second result we proceed in a similar fashion by sandwiching the characteristic function of the disk between two continuous approximations.

Theorem 23. *If $N(R)$ denotes the number of lattice points inside the disk of radius R in $\mathbb{R}^2$ then*

$$|N(R) - \pi R^2| \lesssim R.$$

Denote the disk of radius R by $D_R = \{x \in \mathbb{R}^2 : |x| \leq R\}$ and denote the region formed by unit squares, Q, centered at $n \in \mathbb{Z}^2 \cap D_R$ by $\widetilde{D}_R = \bigcup_{|n| \leq R,\, n \in \mathbb{Z}^2}(Q + n)$. The shifted unit squares that make up $\widetilde{D}_R$ are almost disjoint, sharing only boundary points, and each has unit area. Since each lattice point in D_R corresponds to exactly one of these unit squares we know that $m(\widetilde{D}_R) = N(R)$. Figure 3.2 shows D_R and the region $\widetilde{D}_R$.



Proof.

Figure 3.2: The region $\widetilde{D}_R$ containing all of the lattice points of D_R (see [10, Chapter 8, Figure 2]).

If we then take the disks $D_{R-2^{-1/2}}$ and $D_{R+2^{-1/2}}$ we can bound the area of $\widetilde{D}_R$ as R tends to infinity using the area of these disks. Essentially, because the area of disks in $\mathbb{R}^2$ is easy to calculate, we will sandwich the boundary of $\widetilde{D}_R$ between two disks and use their areas to control the area of $\widetilde{D}_R$ as R tends to infinity. This is illustrated in Figure 3.3.

The disk $D_{R-2^{-1/2}}$ contains all of the unit squares inside $\widetilde{D}_R$ except those that intersect the boundary of the disk D_R because the maximum distance between a boundary point of D_R and a boundary point of $\widetilde{D}_R$ is $2^{-1/2}$. Similarly, the disk $D_{R+2^{-1/2}}$ includes all of the squares inside $\widetilde{D}_R$. Hence we have that $D_{R-2^{-1/2}} \subset \widetilde{D}_R \subset D_{R+2^{-1/2}}$ and thus $m(D_{R-2^{-1/2}}) \leq m(\widetilde{D}_R) \leq m(D_{R+2^{-1/2}})$. These areas are

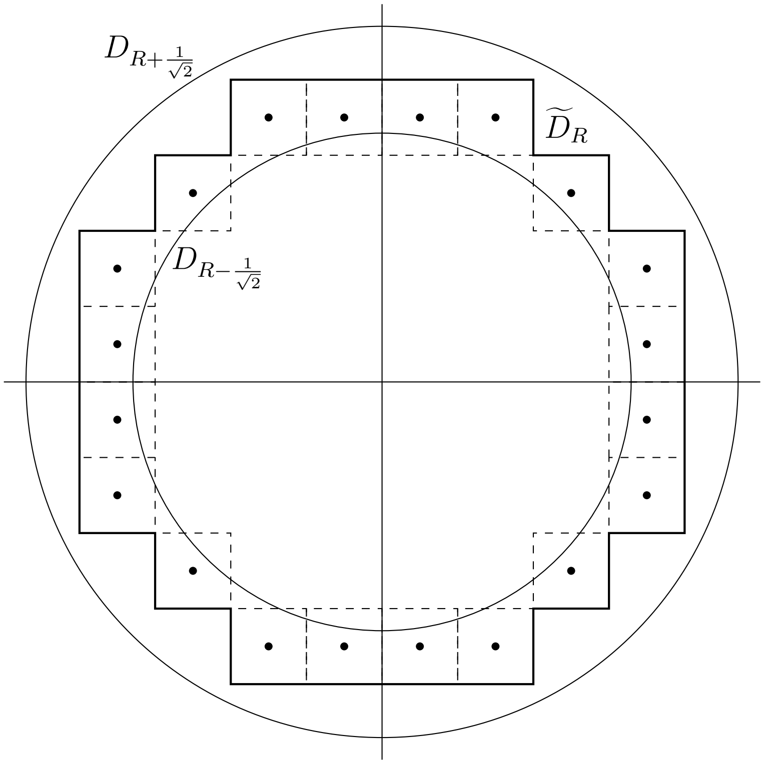


Figure 3.3: The region $\widetilde{D}_R$ contained in $D_{R+\frac{1}{\sqrt{2}}}$ and containing $D_{R-\frac{1}{\sqrt{2}}}$.

simple to calculate:

$$m(D_{R-2^{-1/2}}) = \pi\Big(R - \frac{1}{\sqrt{2}}\Big)^2 = \pi\Big(R^2 - \frac{2}{\sqrt{2}}R + \frac{1}{2}\Big) \text{ and}$$
$$m(D_{R+2^{-1/2}}) = \pi\Big(R + \frac{1}{\sqrt{2}}\Big)^2 = \pi\Big(R^2 + \frac{2}{\sqrt{2}}R + \frac{1}{2}\Big).$$

So we have

$$|m(D_{R-2^{-1/2}}) - \pi R^2| \lesssim R \text{ and } |m(D_{R+2^{-1/2}}) - \pi R^2| \lesssim R.$$

Therefore $\widetilde{D}_R$ is trapped between $D_{R-2^{-1/2}}$ and $D_{R+2^{-1/2}}$ and we have

$$|m(\widetilde{D}_R) - \pi R^2| = |N(R) - \pi R^2| \lesssim R.$$

$\bowtie$

Theorem 24. *For the disk of radius R in $\mathbb{R}^2$*

$$|N(R) - \pi R^2| \lesssim R^{2/3}. \tag{3.2}$$

Proof. Take the characteristic function χ_R of D_R. To improve our estimate in Theorem 23 we appeal to the Poisson Summation Formula, Property 9, which holds for all Schwartz functions [10, Chapter 8, Section 2]. We cannot apply the formula directly to χ_R however, because χ_R is not smooth, and hence not Schwartz. To smooth χ_R, take a C^∞ bump function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ so that $\int_{\mathbb{R}^2} \varphi \, dx = 1$ and φ is supported in the unit disk. Define $\varphi_\delta(x) = \delta^{-2}\varphi(x/\delta)$ and $\chi_{R,\delta} = \chi_R * \varphi_\delta$. We will fix the parameter δ later in the proof. Because χ_R and φ_δ are non-negative $\chi_{R,\delta}$ is also non-negative. We know φ is a Schwartz function, so the normalization φ_δ is also Schwartz, and because of our choice of φ the family $\{\varphi_\delta\}$ is an approximate identity. Then since χ_R is integrable, χ_R inherits smoothness from φ_δ by Property 8 in Section 2.1, so that $\chi_{R,\delta}$ is C^∞ and compactly supported [8, Chapter 7, Section 5]. We proceed by applying the Poisson Summation Formula, Property 9, to $N_\delta(R) = \sum_{n\in\mathbb{Z}} \chi_{R,\delta}(n)$.

$$N_\delta(R) = \sum_{n\in\mathbb{Z}} \chi_{R,\delta}(n) = \sum_{n\in\mathbb{Z}} \widehat{\chi}_{R,\delta}(n) = \sum_{n\in\mathbb{Z}} \widehat{\chi_R * \varphi_\delta}(n) = \sum_{n\in\mathbb{Z}} \widehat{\chi}_R(n)\widehat{\varphi}_\delta(n)$$

The $n = 0$ term of this sum is

$$\begin{aligned}
\hat{\chi}_R(0)\hat{\varphi}_\delta(0) &= \int_{\{x:|x|\le R\}} \chi_R(x)e^{-2\pi i x\cdot 0}\, dx \int \varphi_\delta(x)e^{-2\pi i x\cdot 0}\, dx \\
&= \int_{\{x:|x|\le R\}} \chi_R(x)\, dx \int \varphi_\delta(x)\, dx \\
&= \int_0^{2\pi}\int_0^R r\, dr d\theta \\
&= \pi R^2
\end{aligned}$$

Hence the $n = 0$ term is the volume of the disk of radius R. We now have

$$N_\delta(R) - \pi R^2 = \sum_{n\neq 0} \widehat{\chi}_R(n)\widehat{\varphi}_\delta(n) = \sum_{n\neq 0} \widehat{\chi}_R(n)\widehat{\varphi}(\delta n),$$

where the second equality follows because $\delta^{-2}\widehat{\varphi}(n/\delta) = \widehat{\varphi}(\delta n)$, [11, Chapter 6, Proposition 2.1]. It is convenient to estimate this sum by separating it into two sums, one over a region around zero, and one over the remaining region away from zero,

$$\sum_{n\neq 0} \widehat{\chi}_R(n)\widehat{\varphi}(\delta n) = \sum_{0<|n|\leq\frac{1}{\delta}} \widehat{\chi}_R(n)\widehat{\varphi}(\delta n) + \sum_{|n|>\frac{1}{\delta}} \widehat{\chi}_R(n)\widehat{\varphi}(\delta n). \tag{3.3}$$

For the first sum we appeal to Lemma 21

$$|\widehat{\chi}_R(n)| = \frac{R}{|n|}|J_1(2\pi|n|R)| = O(R^{1/2}|n|^{-3/2}), \tag{3.4}$$

which is proved in Section 2.2. By our choice of φ, we also have a good bound for $|\widehat{\varphi}(\delta n)|$,

$$|\widehat{\varphi}(\delta n)| \leq \int |\varphi(x)||e^{-2\pi i\delta x\cdot n}|\,dx \leq \int |\varphi(x)|\,dx \lesssim 1.$$

Hence, for the first sum in equation (3.3) we have

$$\begin{aligned}
\sum_{0<|n|\leq\frac{1}{\delta}} \left|\widehat{\chi}_R(n)\widehat{\varphi}(\delta n)\right| &\lesssim R^{1/2} \sum_{0<|n|\leq\frac{1}{\delta}} |n|^{-3/2} \\
&\lesssim R^{1/2} \int_{\{x\,:\,0<|x|<\frac{1}{\delta}\}} |x|^{-3/2}\,dx \\
&\lesssim R^{1/2} \int_0^{2\pi} \left(\int_0^{1/\delta} r^{-3/2} r\,dr \right) d\theta \\
&\lesssim R^{1/2} \int_0^{2\pi} \left(\int_0^{1/\delta} r^{-1/2}\,dr \right) d\theta \\
&\lesssim R^{1/2} \int_0^{2\pi} (2r^{1/2})\Big|_0^{1/\delta} d\theta \\
&\lesssim R^{1/2} \int_0^{2\pi} 2\delta^{-1/2}\,d\theta \\
&\lesssim R^{1/2}\delta^{-1/2}
\end{aligned} \tag{3.5}$$

For the second sum in equation (3.3) we apply a similar strategy, utilizing (3.4) and the rapid decay of $\widehat{\varphi}$ away from zero, which gives us $|\widehat{\varphi}(n\delta)| \lesssim (1+|n\delta|)^{-1} \lesssim |n\delta|^{-1}$

for $|n| \geq \frac{1}{\delta}$. We now estimate the sum

$$\begin{aligned}
\sum_{|n|>\frac{1}{\delta}} \left|\widehat{\chi}_R(n)\widehat{\varphi}(\delta n)\right| &\lesssim R^{1/2}\delta^{-1} \sum_{|n|>\frac{1}{\delta}} |n|^{-3/2}|n|^{-1} \\
&\lesssim R^{1/2}\delta^{-1} \int_{\{x\,:\,|x|\geq\frac{1}{\delta}\}} |x|^{-5/2}\,dx \\
&\lesssim R^{1/2}\delta^{-1} \int_0^{2\pi}\int_{\frac{1}{\delta}}^{\infty} r^{-5/2} r\,dr\,d\theta \\
&\lesssim R^{1/2}\delta^{-1} \int_0^{2\pi}\int_{\frac{1}{\delta}}^{\infty} r^{-3/2}\,dr\,d\theta \\
&\lesssim R^{1/2}\delta^{-1} \int_0^{2\pi} (-2r^{-1/2})\Big|_{1/\delta}^{\infty}\,d\theta \\
&\lesssim R^{1/2}\delta^{-1}\delta^{1/2} \\
&\lesssim R^{1/2}\delta^{-1/2} \qquad (3.6)
\end{aligned}$$

We combine our estimates (3.5) and (3.6) to get

$$N_\delta(R) - \pi R^2 \lesssim R^{1/2}\delta^{-1/2}. \qquad (3.7)$$

Note that because πR^2 is the $n = 0$ term of $N_\delta(R)$, a sum of positive terms, $N_\delta(R) - \pi R^2$ is always positive.

Take $x \in D_R$. Then $x - y \in D_{R+\delta}$ whenever $|y| \leq \delta$. So, $\chi_R(x) \leq \chi_{R+\delta}(x - y)$ for such a y. We also know that $\int \varphi_\delta(y)\,dy = 1$ and φ_δ is supported on the unit disk so we have $\chi_R(x) \leq \int \chi_{R+\delta}(x - y)\varphi_\delta(y)\,dy$. By definition the right hand side of this inequality is $\chi_{R+\delta,\delta}(x)$, so $\chi_R(x) \leq \chi_{R+\delta,\delta}(x)$. Similarly, if $x - y \in D_{R-\delta,\delta}$ with $|y| < \delta$, then $x \in D_R$ and we have $\int \chi_{R-\delta}(x - y)\varphi_\delta(y)\,dy \leq \chi_R(x)$. So by definition $\chi_{R-\delta,\delta}(x) \leq \chi_R(x)$. Altogether, we have $\chi_{R-\delta,\delta}(x) \leq \chi_R(x) \leq \chi_{R+\delta,\delta}(x)$. This means that $N_\delta(R - \delta)$ underestimates the number of lattice points in D_R, while $N_\delta(R + \delta)$ overestimates the number of lattice points. We can find estimates bounding $N(R)$ from above and below by using this relation

$$N_\delta(R - \delta) - \pi R^2 \leq N(R) - \pi R^2 \leq N_\delta(R + \delta) - \pi R^2. \qquad (3.8)$$

We apply (3.7) to $N_\delta(R - \delta)$ to get

$$N_\delta(R - \delta) - \pi(R - \delta)^2 \lesssim (R - \delta)^{1/2}\delta^{-1/2} \lesssim R^{1/2}\delta^{-1/2}.$$

Expanding the left hand side and rearranging terms we have

$$N_\delta(R-\delta) - \pi R^2 \lesssim R^{1/2}\delta^{-1/2} - 2\pi R\delta + \delta^2 \lesssim R^{1/2}\delta^{-1/2} + R\delta$$

since $\delta^2 < R\delta$ for $\delta < R$. A similar calculation gives us the same result for $N_\delta(R+\delta)$ so we have the estimates

$$|N_\delta(R-\delta) - \pi R^2| \lesssim R^{1/2}\delta^{-1/2} + R\delta$$

and

$$|N_\delta(R+\delta) - \pi R^2| \lesssim R^{1/2}\delta^{-1/2} + R\delta$$

which we can apply to (3.8). This gives us

$$|N(R) - \pi R^2| \lesssim R^{1/2}\delta^{-1/2} + R\delta. \tag{3.9}$$

Previously, our only requirement for δ was that $\frac{1}{\delta}$ is large enough that $\widehat{\varphi}(n\delta)$ decays rapidly for $|n| > \frac{1}{\delta}$. We now choose $\delta = R^{-1/3}$ so that $R^{1/2}\delta^{-1/2} = R\delta$ to optimize the bound in (3.9). We are able to specify this δ because for R large, $1/\delta$ is large so we can still appeal to the rapid decay of $\widehat{\varphi}$. Combine terms to obtain

$$|N(R) - \pi R^2| \lesssim R^{2/3}$$

as required. $\bowtie$

Chapter 4

Generalization to Strongly Convex Domains in $\mathbb{R}^d$

In this chapter we generalize the methods used for disks to strongly convex domains in $\mathbb{R}^d$. The basic premise of smoothing the characteristic function, applying the Fourier transform and then using the Poisson Summation Formula will again ultimately lead to the main result. However, in order to apply this tactic we must first prove some preliminary results for strongly convex domains in $\mathbb{R}^d$, requiring us to delve deeper into the geometry of strongly convex domains.

4.1 Preliminaries

To prove the main theorem of this chapter we first need the following two results.

Theorem 25. *Suppose Ω is a bounded region so that $M = \partial\Omega$ is a smooth hypersurface with non-vanishing Gaussian curvature at each point. Then*

$$|\widehat{\chi}_\Omega(\xi)| \lesssim (1 + |\xi|)^{-\frac{d+1}{2}}. \tag{4.1}$$

Proof. The proof of this theorem follows the proof outlined in [10, Chapter 8, Corollary 3.3]. Recall that we denote the dth coordinates of the vectors x and ξ by x_d and ξ_d respectively and we denote the first $d-1$ components of each vector by x' and ξ'.

It is sufficient to consider $|\xi| \gg 1$, as the claim is clear for $|\xi|$ bounded. Since $\Omega \cup \partial\Omega$ is compact, we can take a partition of unity so that $\chi_\Omega = \sum_{j=0}^{N} \psi_j \chi_\Omega$, with $N \in \mathbb{N}$, where we have the following conditions: each ψ_j is C^∞ and has compact support, in particular ψ_0 is supported in the interior of $\Omega(\text{supp}(\psi_0) \subset \Omega)$, and each ψ_j is supported in a neighborhood of the boundary.

Since $\text{supp}(\psi_0) \subset \Omega$, we know that $\psi_0\chi_\Omega = \psi_0$, and thus $\widehat{\psi_0\chi_\Omega} = \widehat{\psi_0}$. Then because ψ_0 is C^∞ and compactly supported, we know ψ_0 is a Schwartz function, and hence so is $\widehat{\psi_0}$. So $\widehat{\psi_0}$ decays rapidly as $|\xi| \to \infty$ and we concern ourselves now with the decay of $\widehat{\psi_j\chi_\Omega}$ for $j \neq 0$.

To consider $\widehat{\psi_j\chi_\Omega}$ for each $j \neq 0$ we need a convenient way to characterize Ω and $\partial\Omega$ locally on the support of ψ_j. So instead of using a global defining function for the hypersurface $\partial\Omega$, we will use, for each ψ_j, a local defining function for $\partial\Omega$. To this end, take a finite covering $V = \cup_{j=1}^{n} V_j$ of $\partial\Omega$ so that $\partial\Omega \subseteq V$ and there is a defining function ρ_j for $\partial\Omega$ defined on each V_j. Without loss of generality, assume for each j that $\text{supp}(\psi_j) \subseteq V_j$. Were this not the case, we could simply take a finer partition of unity to arrange it. For every j take the defining function $\rho_j : V_j \to \mathbb{R}$ of $\partial\Omega$ so that

$$\begin{cases} \rho_j(x) > 0 & \text{if } x \in V_j \cap \Omega \\ \rho_j(x) = 0 & \text{if } x \in V_j \cap \partial\Omega \\ \rho_j(x) < 0 & \text{if } x \in V_j \cap (\overline{\Omega})^c \end{cases}$$

with $\rho_j \in C^\infty(V_j)$. We can further arrange for $|\nabla\rho_j(x)| = 1$ whenever $\rho_j(x) = 0$, i.e. whenever $x \in V_j \cap \partial\Omega$. Were this not the case we could take a new defining function given by $\rho_j(x)/|\nabla\rho_j(x)|$ so the defining function is normalized and still satisfies the above conditions. If $\partial\Omega$ were C^k this normalization would result in a C^{k-1} defining function. However, because $\partial\Omega$ is C^∞ there is no cost to this normalization.

Now consider ψ_j, V_j, and ρ_j for some fixed $j \neq 0$. We can assume, via an appropriate translation and rotation if necessary, that $0 \in \operatorname{supp}(\psi_j)$, $\rho_j(0) = 0$, and $\nabla\rho_j(0) = e_d$ where e_d is the unit vector $(0, ..., 0, 1)$. Then clearly $\partial_d\rho_j(0) = 1$ and $\partial_i\rho_j(0) = 0$ for all $i \neq d$.

By the Implicit Function Theorem, shrinking the V_j if necessary, there exists a function $\varphi_j : \mathbb{R}^{d-1} \to \mathbb{R}$ that is C^k such that for every $(x', x_d) \in V_j \cap \partial\Omega$ we have that $x_d = \varphi_j(x')$ and $\rho_j(x', \varphi_j(x')) = 0$. This allows us to realize any $x \in \partial\Omega \cap V_j$ as $(x', \varphi_j(x')) = (x', x_d)$, so within V_j we have that Ω is given by $x_d > \varphi_j(x')$ and $\partial\Omega$ is given by $x_d = \varphi_j(x')$. Thus, we can characterize $\Omega \cap V_j$ and $\partial\Omega \cap V_j$ locally on each V_j as

$$\begin{cases} \Omega \cap V_j & \text{if } x_d > \varphi_j(x'), \\ \partial\Omega \cap V_j & \text{if } x_d = \varphi_j(x'). \end{cases}$$

We can now calculate $\partial_i\varphi_j(x')$ as follows

$$\begin{aligned} \partial_i\rho_j(x', \varphi_j(x')) &= 0 \\ \partial_i\rho_j(x', \varphi_j) + \partial_d\rho_j(x', \varphi_j)\partial_i\varphi_j(x') &= 0 \\ -\frac{\partial_i\rho_j(x', \varphi_j)}{\partial_d\rho_j(x', \varphi_j)} &= \partial_i\varphi_j(x') \end{aligned}$$

so we know $\partial_i\varphi_j(0) = 0$ for all $i \neq d$, since $\partial_i\rho_j(0) = 0$ for all $i \neq d$. Hence, we also have that $\nabla_{x'}\varphi(x')|_{x'=0} = 0$.

Consider $\widehat{\psi_j\chi_\Omega}$ for $1 \leq j \leq N$. With our change of coordinates and the change of variables given by $x_d = u + \varphi_j(x')$ we have

$$
\begin{aligned}
\widehat{\psi_j\chi_\Omega}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i(x'\cdot\xi' + x_d\xi_d)}(\psi_j\chi_\Omega)(x', x_d)\,dx'\,dx_d \\
&= \int_\Omega e^{-2\pi i(x'\cdot\xi' + x_d\xi_d)}\psi_j(x', x_d)\,dx'\,dx_d \\
&= \int_{\mathbb{R}^{d-1}}\int_{\varphi_j(x')}^\infty e^{-2\pi i(x'\cdot\xi' + x_d\xi_d)}\psi_j(x', x_d)\,dx_d\,dx' \\
&= \int_{\mathbb{R}^{d-1}}\int_0^\infty e^{-2\pi i(x'\cdot\xi' + (u+\varphi_j(x'))\xi_d)}\psi_j(x', u+\varphi_j(x'))\,du\,dx' \\
&= \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x'\cdot\xi' + \varphi_j(x')\xi_d)}\Big(\int_0^\infty e^{-2\pi i u\xi_d}\psi_j(x', u+\varphi_j(x'))\,du\Big)\,dx' \\
&= \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x'\cdot\xi' + \varphi_j(x')\xi_d)}\Psi_j(x', \xi_d)\,dx'
\end{aligned}
$$

where $\Psi_j(x', \xi_d) = \int_0^\infty e^{-2\pi i u\xi_d}\psi_j(x', u+\varphi_j(x'))\,du$. Now because the exponential function is C^∞ and ψ_j is C^∞ and compactly supported for all $1 \le j \le N$, we also have that $\Psi_j(x', \xi_d)$ is C^∞ with compact support in x'.

In order to find the behavior of $\widehat{\psi_j\chi_\Omega}(\xi)$ as $|\xi| \to \infty$ we will consider two cases. Because $|\nabla_{x'}\varphi_j(x')|_{x'=0} = 0$, and is otherwise positive, we can take a constant $c > 0$ sufficiently small so that $c|\nabla_{x'}\varphi_j| \le 1/2$ on the support of ψ_j, for all $1 \le j \le N$. With c now fixed we can consider the two cases $|\xi_d| < c|\xi'|$ and $|\xi_d| \ge c|\xi'|$.

For the first region, $|\xi_d| < c|\xi'|$, let $\lambda = 2\pi|\xi'|$ and $\Phi(x') = -\frac{x'\cdot\xi'}{|\xi'|} - \varphi_j(x')\frac{\xi_d}{|\xi'|}$. Note that

$$
i\lambda\Phi(x') = 2\pi i|\xi'|\Big(-\frac{x'\cdot\xi'}{|\xi'|} - \varphi_j(x')\frac{\xi_d}{|\xi'|}\Big) = -2\pi i x'\cdot\xi' - 2\pi i\xi_d\varphi_j(x')
$$

is precisely the exponent in $\widehat{\psi_j\chi_\Omega}(\xi)$, so we can rewrite the Fourier transform as $\int_{\mathbb{R}^{d-1}} e^{i\lambda\Phi(x')}\Psi_j(x', \xi_d)\,dx'$.

By the reverse triangle inequality we have

$$\begin{aligned}
\Big|\nabla_{x'}\Phi(x')\Big| &= \Big|\nabla_{x'}\Big(-\frac{x'\cdot\xi'}{|\xi'|} - \varphi_j(x')\frac{\xi_d}{|\xi'|}\Big)\Big| \\
&= \Big|\nabla_{x'}\Big(-\frac{x'\cdot\xi'}{|\xi'|}\Big) - \nabla_{x'}\Big(\varphi_j(x')\frac{\xi_d}{|\xi'|}\Big)\Big| \\
&\geq \Big|\nabla_{x'}\Big(\frac{x'\cdot\xi'}{|\xi'|}\Big)\Big| - \Big|\nabla_{x'}\Big(\varphi_j(x')\frac{\xi_d}{|\xi'|}\Big)\Big| \\
&\geq 1 - \frac{|\xi_d|}{|\xi'|}\Big|\nabla_{x'}(\varphi_j(x'))\Big| \geq 1 - 1/2 = 1/2.
\end{aligned}$$

The last inequality follows because $\frac{|\xi_d|}{|\xi'|} < c$ and we chose c sufficiently small so that $c|\nabla_{x'}(\varphi_j(x'))| \leq 1/2$ on the support of ψ_j. Since $|\nabla_{x'}\Phi(x')|$ is bounded away from zero, by Lemma 17 in Section 2.2 we have that

$$\begin{aligned}
\Big|\widehat{\psi_j\chi_\Omega}(\xi)\Big| = \Big|\int_{\mathbb{R}^{d-1}} e^{i\lambda\Phi(x')}\Psi_j(x',\xi_d)\,dx'\Big| &\lesssim \lambda^{-K} \\
&\lesssim |\xi'|^{-K} \\
&\lesssim |\xi|^{-K}
\end{aligned}$$

for any $K \geq 0$.

Now consider the second case when $|\xi_d| \geq c|\xi'|$. We rewrite $\Psi_j(x',\xi_d)$ and integrate by parts so that

$$\begin{aligned}
\Psi_j(x',\xi_d) &= -\frac{1}{2\pi i\xi_d}\int_0^\infty \frac{d}{du}e^{-2\pi iu\xi_d}\psi_j(x',u+\varphi_j(x'))\,du \\
&= \frac{1}{2\pi i\xi_d}\Big[-\psi_j(x',u+\varphi_j(x'))e^{-2\pi iu\xi_d}\Big|_0^\infty \\
&\quad + \int_0^\infty e^{-2\pi iu\xi_d}(\partial_d\psi_j)(x',u+\varphi_j(x'))\,du\Big]
\end{aligned}$$

At infinity $\psi_j(x',u+\varphi_j(x'))$ vanishes because ψ_j is compactly supported. Only the $u=0$ term remains and hence

$$\Psi_j(x',\xi_d) = \frac{1}{2\pi i\xi_d}\Big[-\psi_j(x',\varphi_j(x')) + \int_0^\infty e^{-2\pi iu\xi_d}(\partial_d\psi_j)(x',u+\varphi_j(x'))\,du\Big].$$

This gives us $\Psi_j(x',\xi_d) = \frac{1}{2\pi i\xi_d}\widetilde{\Psi}_j(x',\xi_d)$, where $\widetilde{\Psi}_j$ is C^∞ and compactly supported in x'. Thus, Ψ_j contributes $|\xi_d|^{-1} \approx |\xi|^{-1}$ decay to $\widehat{\psi_j\chi_\Omega}$.

Take $\det\{\nabla^2_{x'}\varphi_j\}$, the determinant of the Hessian matrix of φ_j with respect to x', on the support of ψ_j. For simplicity, we omit the subscripts on ρ_j and φ_j in the following calculations. Using $0 = \partial_i\rho(x', \varphi(x'))$ we calculate the second partial derivatives for $1 \leq i, k \leq d-1$,

$$0 = \partial_k(\partial_i\rho + \partial_d\rho\partial_i\varphi) = \partial_k(\partial_i\rho) + \partial_k(\partial_d\rho\partial_i\varphi).$$

By the chain rule the first term is

$$\partial_k(\partial_i\rho) = \partial_k\partial_i\rho + \partial_d\partial_i\rho\partial_k\varphi \tag{4.2}$$

and by applying the product rule and then the chain rule the second term is

$$\partial_k(\partial_d\rho\partial_i\varphi) = \partial_i\varphi\partial_k(\partial_d\rho) + \partial_d\rho\partial_k\partial_i\varphi = \partial_i\varphi[\partial_k\partial_d\rho + \partial_d^2\rho\partial_k\varphi] + \partial_d\rho\partial_k\partial_i\varphi. \tag{4.3}$$

Combining (4.2) and (4.3) we have

$$0 = \partial_k\partial_i\rho + \partial_d\partial_i\rho\partial_k\varphi + \partial_i\varphi[\partial_k\partial_d\rho + \partial_d^2\rho\partial_k\varphi] + \partial_d\rho\partial_k\partial_i\varphi. \tag{4.4}$$

Since for $j \neq d$ we know $\partial_j\varphi$ vanishes and $\partial_d\rho = 1$ at $x' = 0$, only the first and last terms in (4.4) are nonzero at $x' = 0$ and we have that $\partial_k\partial_i\rho = -\partial_k\partial_i\varphi$ at $x' = 0$. Within a neighborhood of zero $\partial\Omega$ has nonzero Gaussian curvature. Thus at 0 we have that $\nabla^2_{x'}\rho$ is diagonalizable with nonzero eigenvalues so that $\det\{\nabla^2_{x'}\rho_j\} \neq 0$. Since $-\nabla^2_{x'}\rho = \nabla^2_{x'}\varphi$ at 0 we have $\det\{-\nabla^2_{x'}\rho\} = \det\{\nabla^2_{x'}\varphi\}$ so that $\det\{\nabla^2_{x'}\varphi_j\} \neq 0$ in a neighborhood of the origin. So $\widehat{\psi_j\chi_\Omega}$ decays like $|\xi|^{-\frac{d+1}{2}}$ when $|\xi_d| \geq c|\xi'|$, since Ψ_j decays like $|\xi|^{-1}$ and the nonzero Hessian determinant of the phase ensures that $\int_{\mathbb{R}^{d-1}} e^{-2\pi i(x'\cdot\xi' + \varphi_j(x')\xi_d)}\,dx'$ decays like $|\xi|^{-\frac{d-1}{2}}$ as $|\xi| \to \infty$ by Property 20 in Section 2.2. Hence

$$\begin{aligned}\left|\widehat{\psi_j\chi_\Omega}(\xi)\right| &= \left|\int_{\mathbb{R}^{d-1}} e^{-2\pi i(x'\cdot\xi' + \varphi_j(x')\xi_d)}\Psi_j(x', \xi_d)\,dx'\right| \\ &\lesssim |\xi|^{-\frac{d-1}{2}}|\xi|^{-1} \lesssim |\xi|^{-\frac{d+1}{2}}.\end{aligned}$$

Thus $\widehat{\psi_j\chi_\Omega}(\xi)$ has arbitrarily fast decay like $|\xi|^{-K}$ for any $K > 0$ on $|\xi_d| < c|\xi'|$ and decays like $|\xi|^{-\frac{d+1}{2}}$ on $|\xi_d| \geq c|\xi'|$. Since we can repeat this process for all ψ_j in the partition of unity, $|\widehat{\chi}_\Omega(\xi)| \lesssim (1 + |\xi|)^{-\frac{d+1}{2}}$. $\bowtie$

Theorem 26. *Suppose Ω is a bounded open convex set with $0 \in \Omega$ and C^2 boundary $\partial\Omega$. Then there is a constant $\mu > 0$ so that if $R \geq 1$ is sufficiently large and $\delta \leq 1$ then $x \in R\Omega$ and $|y| \leq \delta$ implies $x + y \in (R + \mu\delta)(\Omega)$.*

Proof. We proceed in this proof with the method suggested in [10, Chapter 8, Excercise 21]. Consider $x + y \in (R + \mu\delta)\Omega = \{(R + \mu\delta)z : z \in \Omega\}$, which is equivalent to $\frac{x}{R} + \frac{y}{R} \in (1 + \mu\frac{\delta}{R})\Omega = \{(1 + \mu\frac{\delta}{R})z : z \in \Omega\}$. Relabeling $\widetilde{\delta} = \frac{\delta}{R}$, $\widetilde{x} = \frac{x}{R}$, and $\widetilde{y} = \frac{y}{R}$ and taking δ sufficiently small allows us to reduce to the case $R = 1$ without loss of generality. We must now show that there exists a constant $\mu > 0$ independent of R so that if $\widetilde{x} \in \Omega$ and $|\widetilde{y}| \leq \widetilde{\delta}$ then $\widetilde{x} + \widetilde{y} \in (1 + \widetilde{\delta}\mu)\Omega$. We assume that $R = 1$ and proceed with x, y and δ.

Take any $x \in \partial\Omega$. Apply a local change of coordinates as in the proof of Theorem 25 that maps x to $(0,0) \in \mathbb{R}^{d-1} \times \mathbb{R}$ so that Ω is defined near $(0,0)$ by $x_d > \varphi(x')$ with $\varphi(0) = 0$ and $x_d = \varphi(x')$ for $x \in \partial\Omega$. Because this local change of coordinates is given by a rotation and translation, we can denote it by the transformation $T(y) = Ay + z$, where A is the rotation matrix and z is the point corresponding to the original origin.

To establish a condition for membership in $T(\Omega)$ consider any point $t \in T((1 + \mu\delta)\Omega)$. This is equivalent to saying that $\frac{T^{-1}(t)}{1+\mu\delta} \in \Omega$. Applying the definition of T^{-1} we have

$$\frac{T^{-1}(t)}{1 + \mu\delta} = \frac{A^{-1}(t - z)}{1 + \mu\delta} = A^{-1}\Big(\frac{t - z}{1 + \mu\delta}\Big).$$

Now we need to show that $T\Big(\frac{T^{-1}(t)}{1+\mu\delta}\Big)$ is in $T(\Omega)$. By applying the definition of T we have

$$AA^{-1}\Big(\frac{t - z}{1 + \mu\delta}\Big) + z = \frac{t - z}{1 + \mu\delta} + z = \frac{t + \mu\delta z}{1 + \mu\delta},$$

which is contained in $T(\Omega)$ if $\frac{t_d - z_d}{1+\mu\delta} + z_d = \frac{t_d + \mu\delta z_d}{1+\mu\delta} > \varphi\Big(\frac{(t-z)'}{1+\mu\delta} + z'\Big)$.

First we assume $x = 0$. Our change of coordinates ensures that Ω is tangent to the hyperplane $x_d = 0$ with the interior of Ω defined above the graph of $\varphi(x')$. The convexity of Ω implies that since (z', z_d) is an interior point, we must have z_d bounded away from zero. If z_d is not bounded away from zero but (z', z_d) is an interior point, then either Ω is either not convex, or Ω is not tangent to the hyperplane $x_d = 0$. So we know there is some uniform constant $c > 0$ such that $z_d \geq c$ for (z', z_d) the point corresponding to the original origin. Since we will take y so that $|y| < \delta$ for sufficiently small δ we can write

$$\begin{aligned}
\frac{y_d + \mu\delta z_d}{1+\mu\delta} &\geq \frac{\mu\delta|z_d| - |y_d|}{1+\mu\delta} \\
&\geq \frac{\mu\delta c - \delta}{1+\mu\delta} \\
&= \frac{\mu\delta\left(c - \frac{1}{\mu}\right)}{1+\mu\delta} \\
&\geq \frac{\mu\delta\left(c - \frac{c}{2}\right)}{1+\mu\delta} \\
&= \frac{c}{2}\left(\frac{\mu\delta}{1+\mu\delta}\right)
\end{aligned} \tag{4.5}$$

when μ is taken large enough so $\mu \geq \frac{2}{c}$.

With μ now fixed, recall that $\varphi \in C^k$ with $k \geq 2$, so by applying Taylor's Theorem there exists a constant $C > 0$ dependent on the second derivatives of φ so that

$$\begin{aligned}
\varphi\left(\frac{y' + \mu\delta z'}{1+\mu\delta}\right) &\leq C\left|\frac{y' + \mu\delta z'}{1+\mu\delta}\right|^2 \\
&\leq C\left(\frac{\delta + \mu\delta|z'|}{1+\mu\delta}\right)^2 \\
&\leq C\left(\frac{\delta^2 + 2\delta^2\mu|z'| + (\mu\delta|z'|)^2}{(1+\mu\delta)^2}\right).
\end{aligned}$$

Since $(\delta - \mu\delta|z'|)^2 \geq 0$ we know $\delta^2 + (\mu\delta|z'|)^2 \geq 2\mu\delta^2|z'|$ so we can simplify the inequality further to obtain

$$\begin{aligned}\varphi\left(\frac{y'+\mu\delta z'}{1+\mu\delta}\right) &\leq 2C\left(\frac{\delta^2+(\mu\delta|z'|)^2}{(1+\mu\delta)^2}\right)\\ &= 2C\left(\frac{\delta}{1+\mu\delta}\right)\left(\frac{\delta+\mu^2\delta|z'|^2}{1+\mu\delta}\right)\\ &< \frac{c}{2}\left(\frac{\mu\delta}{1+\mu\delta}\right), \end{aligned} \tag{4.6}$$

where the last inequality follows by taking δ small enough so that $2C\left(\frac{\delta+\mu^2\delta|z'|^2}{1+\mu\delta}\right) < \frac{\mu c}{2}$. We know such a δ exists because the limit as δ tends to zero of $2C\left(\frac{\delta+\mu^2\delta|z'|^2}{1+\mu\delta}\right)$ is zero and we have uniform upper bounds on $|z|$.

Combining our results from (4.5) and (4.6) we have

$$\frac{y_d+\mu\delta z_d}{1+\mu\delta} \geq \left(\frac{c}{2}\right)\frac{\mu\delta}{1+\mu\delta} > \varphi\left(\frac{y'+\mu\delta z'}{1+\mu\delta}\right),$$

and hence there is a $\mu > 0$ and $\delta > 0$ such that for y with $|y| < \delta$ we have that $y \in (1+\mu\delta)\Omega$. $\bowtie$

We now address the case when $x \neq 0$. If the distance between x and $\partial\Omega$ is greater than δ, the result is clear. So assume that x is δ-close to the boundary. We take our $t \in T((1+\mu\delta)\Omega)$ as before, but now we have $t = T(x) + y$. The proof that $\frac{t_d+\mu\delta z_d}{1+\mu\delta} > \varphi\left(\frac{t'+\mu\delta z'}{1+\mu\delta}\right)$ is unchanged and the proof that $\varphi\left(\frac{t'+\mu\delta z'}{1+\mu\delta}\right) < \left(\frac{c}{2}\right)\left(\frac{\mu\delta}{1+\mu\delta}\right)$ for appropriate δ follows similarly with $|t| \leq |T(x)| + |y| \leq 2\delta$.

4.2 Strongly Convex Domains

The previous section established the tools we need to generalize our process for the lattice point discrepancy of disks in Theorem 24 to the lattice point discrepancy for strongly convex domains (see Definitions 10, 11, and 12). We proceed in a similar manner, using convolution to smooth the characteristic function of the domain in order to use the Poisson Summation Formula.

Theorem 27. *Suppose Ω is a strongly convex bounded domain in $\mathbb{R}^d$ containing 0, with C^{d+2} boundary $\partial\Omega$. Then with $N_R(\Omega)$ denoting the number of lattice points inside $R\Omega$, we have $\left|N_R(\Omega) - R^d m(\Omega)\right| \lesssim R^{\left(d-\frac{2d}{d+1}\right)}$.*

Proof. Let χ_Ω denote the characteristic function of Ω and χ_R denote the characteristic function of $R\Omega$, which is defined as $R\Omega = \{Rx : x \in \Omega\}$. Based on this definition it is clear that $\chi_R(x) = \chi_\Omega(x/R)$. Analogously to our approach for the circle, we take a C^∞ function φ that is supported in the unit ball with $\int \varphi(x)dx = 1$ and again define $\varphi_\delta(x) = \delta^{-d}\varphi(x/\delta)$. Denote $\chi_{R,\delta} = \chi_R * \varphi_\delta$ and $N_{R,\delta} = \sum_{n\in\mathbb{Z}^d} \chi_{R,\delta}(n)$.

The Poisson Summation Formula (Property 9) applies again so that

$$N_{R,\delta} = \sum_{n\in\mathbb{Z}^d} \chi_{R,\delta}(n) = \sum_{n\in\mathbb{Z}^d} \widehat{\chi}_{R,\delta}(n) = \sum_{n\in\mathbb{Z}^d} \widehat{\chi_R * \varphi_\delta}(n) = \sum_{n\in\mathbb{Z}^d} \widehat{\chi}_R(n)\widehat{\varphi}_\delta(n).$$

First consider the $n = 0$ term. Note that $\widehat{\varphi}_\delta(n) = \widehat{\varphi}(\delta n)$ and $\widehat{\chi}_R(n) = R^d\widehat{\chi}(Rn)$, and recall that $\int \varphi(x)\,dx = 1$ so we have

$$\begin{aligned}
\widehat{\chi}_R(0)\widehat{\varphi}_\delta(0) &= \int_{R\Omega} \chi_R(x)e^{-2\pi i x\cdot 0}\,dx \int \varphi_\delta(x)e^{-2\pi i x\cdot 0}\,dx \\
&= \int_{R\Omega} \chi_R(x)\,dx \int \varphi(x)\,dx \\
&= \int_{\Omega} R^d\chi_\Omega(x)\,dx \\
&= R^d\int_\Omega 1\,dx = R^d m(\Omega).
\end{aligned}$$

We can rewrite $N_{R,\delta} = R^d m(\Omega) + \sum_{n\neq 0} \widehat{\chi}_{R,\delta}(n)$ and, using Theorem 25, we have

$$\begin{aligned}
|\widehat{\chi}_R(n)| &= |R^d\widehat{\chi}(Rn)| \\
&\lesssim R^d R^{-\frac{d+1}{2}}|n|^{-\frac{d+1}{2}} \\
&\lesssim R^{\frac{d-1}{2}}|n|^{-\frac{d+1}{2}},
\end{aligned}$$

which we can apply to $\widehat{\chi}_{R,\delta}(n)$ to get

$$\begin{aligned}
|\widehat{\chi}_{R,\delta}(n)| = |\widehat{\chi}_R(n)\widehat{\varphi}_\delta(n)| &= |\widehat{\chi}_R(n)\widehat{\varphi}(\delta n)| \\
&\lesssim R^{\frac{d-1}{2}}|n|^{-\frac{d+1}{2}}|\widehat{\varphi}(\delta n)|.
\end{aligned}$$

We now use this estimate to consider the split sum

$$|N_{R,\delta} - R^d m(\Omega)| = \sum_{n\neq 0} |\widehat{\chi}_{R,\delta}(n)| = \sum_{1\leq |n|\leq \frac{1}{\delta}} |\widehat{\chi}_{R,\delta}(n)| + \sum_{\frac{1}{\delta}<|n|} |\widehat{\chi}_{R,\delta}(n)|. \tag{4.7}$$

For the first sum, with $d\sigma$ denoting the surface measure of S^{d-1}, the $d-1$-sphere, we have

$$\begin{aligned}
\sum_{1\leq |n|\leq \frac{1}{\delta}} |\widehat{\chi}_{R,\delta}(n)| &\lesssim \sum_{1\leq |n|\leq \frac{1}{\delta}} R^{\frac{d-1}{2}} |n|^{-\frac{d+1}{2}} \\
&\lesssim R^{\frac{d-1}{2}} \int_{|x|\leq \frac{1}{\delta}} |x|^{-\frac{d+1}{2}}\, dx \\
&\lesssim R^{\frac{d-1}{2}} \int_{S^{d-1}} \int_0^{\frac{1}{\delta}} r^{-\frac{d+1}{2}} r^{d-1}\, dr\, d\sigma \\
&\lesssim R^{\frac{d-1}{2}} \int_{S^{d-1}} \int_0^{\frac{1}{\delta}} r^{\frac{d-3}{2}}\, dr\, d\sigma \\
&\lesssim R^{\frac{d-1}{2}} r^{\frac{d-1}{2}} \Big|_{r=0}^{\delta^{-1}} \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d-1}{2}}.
\end{aligned} \tag{4.8}$$

For the second sum in (4.7) we use the rapid decay that $\widehat{\varphi}$ inherits from φ, which gives us $|\widehat{\varphi}| \lesssim (1+|n\delta|)^{-t} \lesssim |n\delta|^{-t}$ for any $t \geq 0$. Choose $t = d/2$ so that

$$\begin{aligned}
\sum_{\frac{1}{\delta}<|n|} |\widehat{\chi}_{R,\delta}(n)| &\lesssim \sum_{\frac{1}{\delta}<|n|} R^{\frac{d-1}{2}} |n|^{-\frac{d+1}{2}} |n\delta|^{-\frac{d}{2}} \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d}{2}} \int_{|x|>\delta} |x|^{\frac{-2d-1}{2}}\, dx \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d}{2}} \int_{S^{d-1}} \int_{\frac{1}{\delta}}^{\infty} |r|^{\frac{-2d-1}{2}} r^{d-1}\, dr\, d\sigma \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d}{2}} \int_{S^{d-1}} \int_{\frac{1}{\delta}}^{\infty} |r|^{-\frac{3}{2}}\, dr\, d\sigma \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d}{2}} r^{-\frac{1}{2}} \Big|_{r=\delta^{-1}}^{\infty} \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d}{2}} \delta^{\frac{1}{2}} \\
&\lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d-1}{2}}.
\end{aligned} \tag{4.9}$$

Hence, combining (4.8) and (4.9) we have

$$|N_{R,\delta} - R^d m(\Omega)| \lesssim R^{\frac{d-1}{2}} \delta^{-\frac{d-1}{2}}. \tag{4.10}$$

By Theorem 26 we know that if R is sufficiently large and $\delta \le 1$, then $x \in R\Omega$ and $|y| \le \delta$ imply that $x - y \in (R + c\delta)\Omega$ for some $c > 0$. In particular, if $y = 0$ then for $x \in R\Omega$ there is a $c > 0$ so that $x \in (R + c\delta)\Omega$, hence $R\Omega \subset (R + c\delta)\Omega$. This containment allows us to relate the characteristic functions of these dilates of Ω. Thus, $\chi_R(x) \le \chi_{R+c\delta}(x) \le \int \chi_{R+c\delta}(x-y)\varphi_\delta(y)\,dy = \chi_{R+c\delta,\delta}(x)$, and similarly $\chi_{R-c\delta,\delta}(x) \le \chi_R(x)$. So we have $\chi_{R-c\delta,\delta}(x) \le \chi_R(x) \le \chi_{R+c\delta,\delta}(x)$.

Taking the sum of these characteristic functions over all $n \in \mathbb{Z}^d$ we obtain $N_{R-c\delta,\delta} \le N_R \le N_{R+c\delta,\delta}$, so that

$$N_{R-c\delta,\delta} - R^d\, m(\Omega) \le N_R - R^d\, m(\Omega) \le N_{R+c\delta,\delta} - R^d\, m(\Omega). \tag{4.11}$$

Applying our estimate in (4.10) to $N_{R-c\delta,\delta}$ we have,

$$|N_{R-c\delta,\delta} - (R-\delta)^d\, m(\Omega)| \lesssim (R-\delta)^{\frac{d-1}{2}}\delta^{-\frac{d-1}{2}},$$

and expanding the left hand side and rearranging yields

$$|N_{R-c\delta,\delta} - R^d\, m(\Omega)| \lesssim R^{\frac{d-1}{2}}\delta^{-\frac{d-1}{2}} + R^{d-1}\delta.$$

A similar calculation for $N_{R+c\delta,\delta}$ shows that $N_{R+c\delta,\delta} - R^d m(\Omega) \lesssim R^{\frac{d-1}{2}}\delta^{-\frac{d-1}{2}} + R^{d-1}\delta$ and thus by taking absolute values in (4.11) we have $|N_R - R^d m(\Omega)| \lesssim R^{\frac{d-1}{2}}\delta^{-\frac{d-1}{2}} + R^{d-1}\delta$. Choose $\delta = R^{-\frac{d-1}{d+1}}$ so that

$$\begin{aligned} |N_R - R^d\, m(\Omega)| &\lesssim R^{\frac{d-1}{2}} R^{\frac{(d-1)^2}{2(d+1)}} + R^{d-1}R^{-\frac{d-1}{d+1}} \\ &\lesssim R^{\frac{d^2-d}{d+1}} + R^{\frac{d^2-d}{d+1}} \lesssim R^{(d-\frac{2d}{d+1})} \end{aligned}$$

as required. $\bowtie$

Chapter 5

Conclusion

In this thesis, we first explained how to geometrically count the number of lattice points inside a disk of radius R. We then presented a proof of Gauss' original result that $E(R) \lesssim R$, with $E(R)$ the error between the number of lattice point inside the disk and its area. By appealing instead to Fourier analysis, we proved the first improvement of this bound; $E(R) \lesssim R^{2/3}$. This established our basic protocol for finding such bounds by smoothing characteristic functions so the Poisson Summation Formula could be applied.

Our investigations of the disk naturally led to a generalization to strongly convex domains in $\mathbb{R}^d$. In order to apply a similar tactic, we first showed that, for $\partial\Omega$ a hypersurface of class C^∞ with everywhere non-vanishing Gaussian curvature, $|\widehat{\chi_\Omega}(\xi)| \lesssim (1+|\xi|)^{-\frac{d+1}{2}}$, which we needed to apply in our proof of the main theorem for strongly convex domains. Additionally, in order to bound the number of lattice points inside Ω by estimates above and below, we proved a geometric result for strongly convex domains. Thus we were able to show, using a similar strategy to that employed for the disk, that the error term for the number of lattice points inside a strongly convex domain is $\left|N_R(\Omega) - R^d m(\Omega)\right| \lesssim R^{(d-\frac{2d}{d+1})}$.

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