ASPECTS OF SCALE INVARIANCE IN PHYSICS AND BIOLOGY

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF PHYSICS
Advisers: William Bialek and Juan Maldacena

September 2017

Abstract

We study three systems that have scale invariance. The first system is a conformal field theory in $d > 3$ dimensions. We prove that if there is a unique stress-energy tensor and at least one higher-spin conserved current in the theory, then the correlation functions of the stress-energy tensors and the conserved currents of higher-spin must coincide with one of the following possibilities: a) a theory of n free bosons, b) a theory of n free fermions or c) a theory of n $\frac{d-2}{2}$-forms.

The second system is the primordial gravitational wave background in a theory with inflation. We show that the scale invariant spectrum of primordial gravitational waves is isotropic only in the zero-order approximation, and it gets a small correction due to the primordial scalar fluctuations. When anisotropy is measured experimentally, our result will allow us to distinguish between different inflationary models.

The third system is a biological system. The question we are asking is whether there is some simplicity or universality underlying the complexities of natural animal behavior. We use the walking fruit fly (*Drosophila melanogaster*) as a model system. Based on the result that unsupervised flies' behaviors can be categorized into one hundred twenty-two discrete states (stereotyped movements), which all individuals from a single species visit repeatedly, we demonstrated that the sequences of states are strongly non-Markovian. In particular, correlations persist for an order of magnitude longer than expected from a model of random state-to-state transitions. The correlation function has a power-law decay, which is a hint of some kind of criticality in the system. We develop a generalization of the information bottleneck method that allows us to cluster these states into a small number of clusters. This more compact description preserves a lot of temporal correlation. We found that it is enough to use a two-cluster representation of the data to capture long-range correlations, which opens a way for a more quantitative description of the system. Usage of the maximal entropy method allowed us to find a description that closely resembles a famous inverse-square Ising model in $1d$ in a small magnetic field.

Acknowledgements

I would like to thank the Universe for this realization of our world because I was lucky to be at Princeton and I had the privilege to work with William Bialek and Juan Maldacena.

I am indebted to Juan Maldacena, my adviser, for the interesting journey to the very frontier of the High Energy Physics, as well as for a huge motivation that I got during numerous discussions with him. He exposed me to a completely new and unbelievable level of thinking about the subject.

I am indebted to William Bialek, my adviser, for introducing me to a theoretical biophysics; he taught a class in spring 2012, and as, I see it now, he planted an idea of working in the field of Biological Physics. It took a few years for the idea to grow. So, I incredibly grateful to Bill for all his help and support that lead to a smooth transition into a new field, when I decided to do it. I am very thankful to Bill for all time he spent teaching me how to work in a completely new field. Every meeting with him led to a substantial jump in my understanding of the questions we discussed. His fine taste for problems in the world of living organisms will always be the highest standard for me, that will influence my research for years.

I am very grateful to Sasha Zhiboedov for uncountable hours of discussions that we had in the Institute's "dungeons," and for all his help that made my first years at Princeton extremely pleasant.

I thank Lyman Page, who was my adviser for an experimental project. It was a really pleasant experience. Also, I would like to thank him for serving on my pre-thesis and FPO committee.

I thank Herman Verlinde for serving on my pre-thesis and FPO committee, and for numerous pieces of advice that he gave to me during these years at Princeton. I thank Igor Klebanov for being a reader of this dissertation.

I would like to thank my collaborators for a wonderful experience that I had: Kenan Diab, Gordon Berman, Joshua Shaevitz. I am thankful to Ben Machta for help with my simulations.

I would like to thank the Department of Physics for awarding me with Joseph Henry Merit Prize, and Kusaka Memorial Prize in Physics.

I am extremely grateful to Andrei Mironov and Alexei Morozov for introducing me to the beautiful world of theoretical physics, and for all their help and great support at the Institute for Theoretical and Experiment Physics (Moscow), without whom I would never reach this point.

There is not enough space in this dissertation to write about each and every remarkable person who helped me or influenced me in a good way. I am very thankful to all my friends and colleagues, especially, at the Department of Physics and at the Icahn laboratories. All of you helped me a lot in my continuous journey towards the understanding of this beautiful Nature.

Last but not least, I am extremely grateful to all my family for their constant love and support that helped me to get through all the difficulties of the graduate school.

“ *Хто думає про науку, той любить її, а хто її любить, той ніколи не перестає вчитися, хоча б зовні він і здавався бездіяльним.*

The one who thinks about science loves it, and the one who loves it will never cease to learn, even though he might seem to be outwardly idle. ”

Hryhorii Savych Skovoroda, *(1722 – 1794)*, Kharkiv

To my family.

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Introduction

"Believe you can and you are halfway there."

Theodore Roosevelt

Physics has evolved a lot over the last centuries. It started its journey as a part of philosophy and gave birth to all modern sciences. A distinguished difference that has always defined science in general and physics, in particular, was its ability to produce falsifiable predictions. Physics started as a descriptive science, but during its evolution, it has arrived at the stage when it was able to use some principles to predict new results, that had never been observed experimentally before.

As physics evolved, our approach to world exploration has been constantly changing as well. At the very beginning, science was pramarily, discovery. During this stage, scientists are looking for phenomena. It is important to catch something that is not completely random, something that can be distinguished from the background. Once we became aware of some phenomena we are eager to find not only a qualitative description but also a quantitative description for each phenomenon. The fewer assumptions are made the better, of course. Scientists usually start from a very simple phenomenological description, that allows us to quantitatively describe phenomena and maybe if we are lucky to even made some basic prediction. That is a phenomenological stage of science. During this stage, there are a lot of isolated pieces of knowledge. We usually can describe most aspects of the phenomena but cannot explain why everything happens. The next step is a breakthrough stage. Scientists are able to find some fundamental principles that allow us not only to describe phenomena but also to find connections between them and really explain what is the underlying reason that defines the behavior of many different systems. It turns out that many apparently different phenomena are only different avatars for one simple principle.

Let us consider this evolution as seen in one of the simplest examples, the law of conservation of energy. As we understand now, initially it was formulated under very strong implicit assumptions for very specific systems, and even more, the first formulations are considered wrong from the modern point of view, because at that time many concepts were not defined and therefore it was impossible to clearly state all assumptions.

Empedocles (490−430 BC) formulated his version of conservation of energy as ”nothing comes to be or perishes” [1]. His Universe had four compounds: earth, air, water, and fire, that undergo continuous rearrangement. In modern terms, his law would sound like: ”the mass is conserved”. That is also known as the Lomonosov-Lavoisier Law, that was reinvented many years later in completely different situations.

A mechanical form of the law of conservation of energy is conservation of mechanical energy. The first successful attempt to define conservation of mechanical energy was made by Leibniz during 1676−1689, who defined kinetic energy for a system of point-like particles $\sum_i m_i v_i^2$. He formulated kinetic energy conservation law for a system that does not interact with an environment. He called kinetic energy the *vis viva* or living force of the system. The first more or less complete and correct formulation of the law of conservation of mechanical energy was made by the father and son duo, Johann and Daniel Bernoulli, and was published in 1738 in Daniel Bernoulli famous work *"Hydrodynamica"* [2].

The first thermodynamical form of the law was formulated by Rudolf Clausius [3] in 1850:

> *"In all cases in which work is produced by the agency of heat, a quantity of heat is consumed which is proportional to the work done; and conversely, by the expenditure of an equal quantity of work an equal quantity of heat is produced."*

We see that during this phenomenological phase of the physics, people formulated laws in at least three different subfields. It was not clear at all that these laws are just avatars for a single one. Partially it happened because of relations between quantities, for which people formulated these laws, that were unknown. The path to a modern form of the law was long and tedious.

During the observation stage, people figured out that mechanical, chemical, and thermodynamical phenomena are restricted. During the phenomenological stage, scientists came up with a description for different, as they thought, phenomena. On the long path to the present, there were many breakthroughs that improved our understanding of Nature. However, we are going to mention only two major of them. The first is an understanding that the mass is just another form of the energy [4]. This idea led to a famous equation $E_0 = mc^2$, that allows us to use a powerful source of very cheap and clean energy. This point is very prominent on its own because this is quite unexpected that very abstract concept leads to a very practical outcome.

Another breakthrough is that the underlying reason for energy conservation is that there is an underlying theory with a symmetry - time-invariance. From Noether's theorem, developed by Emmy Noether in 1915 and first published in 1918, we know that every continuous symmetry of the system has an associated conserved current [5].

We see that as physics evolves, old concepts die and new concepts come in. We are constantly moving to more and more fundamental principles, which are the real drivers of our progress. It may seem that it is more important to focus on finding underlying principles than on a description of each particular phenomenon. However, this point of view is too radical, mainly because we need many observations before we can successfully generalize what we see.

There are three chapters (excluding the introduction) in this dissertation, and each chapter, I think, represents a part of science at a different stage. The first chapter belongs to a highly developed part of high energy physics, where we have no connection with experiment and are driven purely by the notion of beauty and naturalness. The second chapter discusses a topic in cosmology, that is also highly developed, but still has a connection to an experiment. That is why it describes particular question for a field in between second and last stage. The last chapter discusses a biological question, in more a phenomenological approach. That is why it describes a field that is in between the first and the second stage. In some sense, this dissertation is a journey from the end of the science [6] to its birth.

1.1 Symmetry of the system as a defining principle

In order to define a system one has to name all its objects, their properties, relations, and connections between all of them. Once everything is defined one may ask, for example, about the evolution of the system in time. One may define a system as a system of constraints, basic rules of dynamics, also known as, an equation of motions, and initial configuration.

One of the ways to define constraints is to define the symmetries of the system. Some symmetries are considered fundamental and we require them for most systems. Other symmetries are defining subclasses of systems. We can say that different parts of physics may be defined by specifying symmetries that are used in it. If some constrained is not representable in terms of symmetry, the constraint is considered to be unnatural.

One of the simplest, and simultaneously one of the most important symmetry is the symmetry of our space itself. An isometry group of the Minkowski space is a semidirect product of Lorentz group[1] and translations, it is called Poincare group. This covariance restricts the form of physical observables, which should be classified by the representation of the Lorentz group [7]. If we assume that we are living in more than three spatial dimensions, and we require Lorentz covariance, all observables are classified by the representation of the $SO(d-1,1)$ group. It means that observables and conserved charges can be scalars, spinors, vectors, and higher spin fields.

One may ask questions like what representations are allowed for conserved charges, or whether we can have an interacting theory at all, etc. We are going to address these questions and will show how powerful this language is.

A natural quantity to consider is the scattering matrix S, which is the evolution operator between the distant past and the distant future. Poincaré covariance restricts its form [7] a lot

$$S = \mathbb{1} + \delta^{(d)}\left(\sum_i p_i\right)\mathcal{M}(p_i \cdot p_j), \tag{1.1}$$

where δ-function represent the momentum conservation condition, p_i is a momentum of i-th particle; $p_i \cdot p_j$ are simplest scalar that one can build from momenta. From Noether's theorem, we know that symmetry corresponds to a conserved current. One may get conserved charges by integration of conserved current that is contracted with a Killing vector. Condition that charges of Poincare symmetry are conserved can be written in terms of energy-momentum conservation condition plus function should depend on Lorentz scalars. Thus, the question about allowed representation for conserved charges in the simplest situation may be formulated in terms of similar equations. For charge of spin s the corresponding

[1] We should mention, that Lorentz covariance has not appeared on its own. Initially, people assumed that we were living in $3d$ Euclidian space, that had Galilean invariance. Only after the discovery of special relativity people realized that we are living in the $(1+3)-$dimensional space, and that Lorentz covariance was a natural symmetry for the physical world.

equation is

$$\sum_i p_i^{s-1} = 0, s \geqslant 2. \tag{1.2}$$

Intuitively, it is clear that the more equations we have, the less and less likely we are to find a nontrivial solution for the system of equations. That means that we cannot have an interaction in the theory if we have higher spin conserved charge. We will give a rigorous proof of this statement for conformal field theories in the chapter 2.

These are model-independent arguments that restrict the content of any theory. These arguments are so-called ”No-Go” theorems. Interestingly, every time after the theorem was proven there was some loophole, that led to a dramatic expansion of physics' horizon.

The first ”No-Go” theorem was proven by Sidney Coleman and Jeffrey Mandula in 1967 [8]. The Coleman-Mandula theorem is valid for any Lorentz invariant quantum field theory that satisfies following conditions[2]

1. Particle-finiteness, i,e, all particles correspond to the positive-energy representation of Poincare group (there is a gap in the theory), and there is a finite number of particle types with mass less than $M, \forall M < \infty$.

2. Weak elastic analyticity, i.e. the amplitudes for elastic two body scattering are analytic functions of scattering angle at almost all energies.

3. Any two particle states interact at almost all energies.

Once these conditions are satisfied, any additional conserved charges should be Lorentz-scalars, otherwise, the $S-$matrix have to be trivial (the theory is free). The theorem makes no statement about internal symmetry, that is why one is free to add non-trivial internal symmetry, e.g. lepton number, baryon number, electric charge, etc.

There are few prominent loopholes for this theorem. If algebra of symmetries is not a usual Lie algebra, the theorem does not hold and we can have no scalar conserved charges.

[2]There is one more assumption, but we do not present it here because it is a purely technical assumption.

The most famous realization is a generalization to Lie superalgebra [9]. In this case, we can get a conserved charge that transforms in a spinor representation, i.e. fermionic symmetry charges.

Theorem does not hold If one replaces the Poincare group with an other space-time symmetry group (like the de Sitter group). The theorem is not applicable for a spontaneously broken symmetry, because it is not observable on the level of the S matrix. If all particles are massless we are allowed to have a combination of internal symmetry and space-time symmetry, that is conformal symmetry in this case [7, 10, 11].

Another reason why the Coleman-Mandula theorem does not hold for conformal field theories is the fact that the $S-$matrix is ill-defined because we cannot correctly define asymptotic states and assume that they are free if they are far apart.

The first rigorous restriction of conformal field theory was made by Juan Maldacena and Alexander Zhiboedov [12] for $d = 3$. In chapter 2 of this dissertation we generalize their argument for $\forall d > 3$. If someone can find a way around our theorem there may be a progress in understanding nature, as has happened before.

1.2 How can we look into the past?

Let us briefly discuss the present state of the Universe and ways to explore its past. Our Universe is quite isotropic and homogeneous at very large scales ($R > 100Mpc$) [14].

There are three possible realizations for an isotropic and homogeneous the Universe. At each point of time, it can be flat (zero spatial curvature), a $3-$sphere (positive spatial curvature), or a $3-$hyperboloid (negative spatial curvature). For an Universe, spatial curvature is very tiny but definitely non-zero. The spatial radius is much larger than inverse Hubble parameter, that is equal to the observable size of the Universe. This result was obtained from the study of Cosmic Microwave Background (CMB) anisotropy.

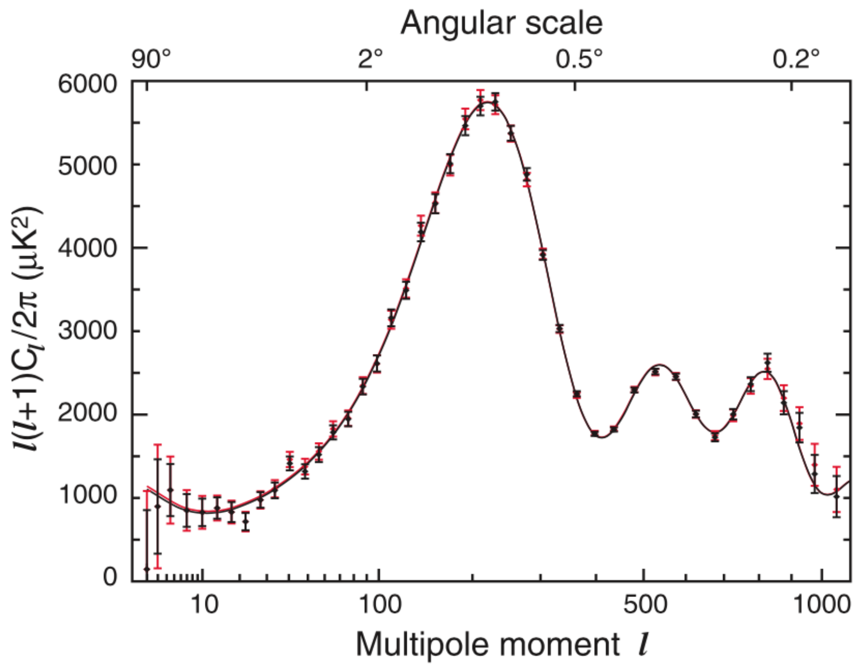

Figure 1.1: Two estimates of the WMAP nine-year power spectrum along with the best-fit model spectra obtained from each; black - the C_1-weighted spectrum and best fit model; red - the same for the MASTER spectrum and model. Plot is taken from [13]

The Cosmic Microwave Background (CMB) has a Planckian distribution with temperature $T_0 = 2.7260 \pm 0.0013K$ [15]. The temperature of photons that are coming from different directions has small angular anisotropy $\delta T/T_0 \sim 10^{-4} - 10^{-5}$ [13].

For direction dependent quantities it is convenient to introduce an expansion in terms of spherical harmonics [14]

$$\delta T(\vec{n}) = T(\vec{n}) - T_0 - \delta T_{dipole} = \sum_{l,m} a_{l,m} Y_{i,m}(\vec{n}), \tag{1.3}$$

where we explicitly remove a dipole term, that is present because of the relative movement of galaxy and its neighbors, the so-called Local Group, in the direction of the constellation Hydra with the speed $\sim 600\, km/s$. It produces an anisotropy of the temperature $\delta T = 3.346\, mK$. Temperature fluctuations turn out to be statistically independent for different l

and m, it means that

$$\left\langle a_{l,m} a^*_{l',m'} \right\rangle = \delta_{ll'} \delta_{mm'} C_l, \tag{1.4}$$

where C_l does not depend on m, because the Universe is isotropic. Standard deviation for the temperature can be written in terms of these coefficients

$$\left\langle \delta T^2 \right\rangle = \sum_l \frac{2l+1}{4\pi} C_l. \tag{1.5}$$

One can see that $l(l+1)C_j/2\pi$ is approximately the power per decade in l of the temperature anisotropies. This is the quantity is conventionally plotted fig. (1.1).

Let us briefly describe the shape of the curve on fig. 1.1. One can derive (see appendix for chapter 3, and discussions around an equation (3.12)) that temperature fluctuations are given[3] by

$$\frac{\delta T}{T}(\vec{n}, \eta_0) = \frac{1}{4}\delta_\gamma(\eta_r) + (\Phi(\eta_r) - \Phi(\eta_0)) + \tag{1.6}$$

$$+ \int_{\eta_r}^{\eta_0} (\Phi' - \Psi') \, d\eta + \tag{1.7}$$

$$+ \vec{n}\left(\vec{v}(\eta_r) - \vec{v}(\eta_0)\right), \tag{1.8}$$

where η_0 is a conformal time of an observer, η_r is a conformal time of the recombination, δ_γ is relative energy fluctuations for a photon-baryon plasma, Φ, Ψ are scalar potentials. The first line is called the Sachs-Wolfe effect(SW) [16]. We can always set $\Phi(\eta_0) = 0$ by redefining the homogeneous temperature T_0. The second line reflects that the energy of a photon is changing if the gravitational potential is changing in time along photon's trajectory; it is called the Integral Sachs-Wolfe effect(ISW). The last line comes from Doppler effect, that simply reflects the fact that during last scattering electrons had a non-zero veloc-

[3] This expression does not include contribution from gravitational waves $\frac{\delta T}{T}(\vec{n}, \eta_0) = \frac{1}{2}\int_{\eta_r}^{\eta_0} d\eta \, n_i h_{ij} n_j$, see discussion in appendices for chapter 3.

ity with respect to a conformal reference frame, as well that an observer may have non-zero velocity during observation.

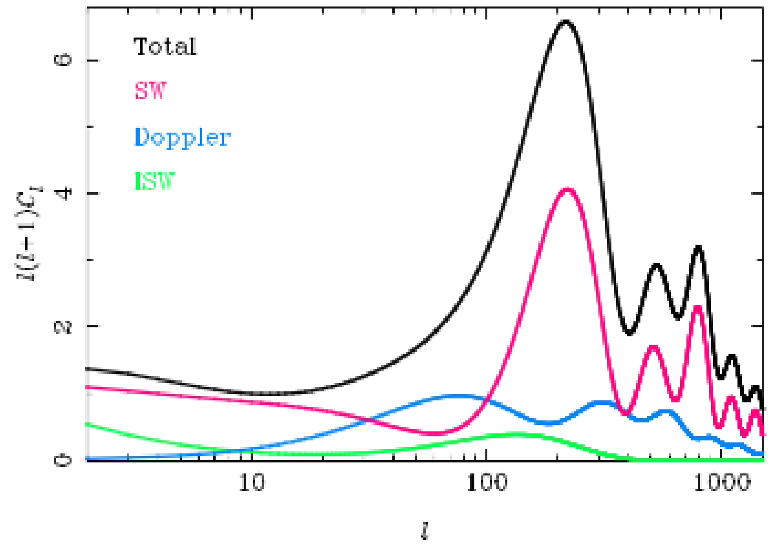

Figure 1.2: ΛCDM Model with $\Omega_\Lambda = 0.75$, $\Omega_B h^2 = 0.023$, $\Omega_{CDM} h^2 = 0.111$, $h = 0.73$, $n_s = 1$. The contribution of the various nature to temperature-anisotropy power spectrum from adiabatic initial conditions. At high l, the contributions are (from top to bottom): total power; denoted SW for Sachs-Wolfe; Doppler effect from v_b; and the integrated Sachs-Wolfe effect (ISW) coming from the evolution of the potential along the line of sight. Plot is taken from [17]

.

For large angular scales ($k\eta_r < 1$) Sachs-Wolfe effect [16] is dominant. It has a contribution from energy fluctuation of initial photons, gravitational potential and from gravitational waves, which we will discuss in details in the chapter 3.

The middle angle region $l \lesssim 1000$ corresponds to $1 \lesssim k\eta_r$. The condition means that we can use an ideal liquid approximation for the photon-baryon fluid [14]. There is a confrontation between photons and baryons in the early Universe. Photon pressure tends to erase anisotropies, while the gravitational attraction of the baryons makes them collapsed and form an over-densities. Physically, a behavior of these modes gives us a snap shot of fluid during the last scattering $\delta T^2 \sim \cos^2(k\eta_{ls}) + other\, terms$. These fluctuations are projected on the sky with $l \sim kr_{ls}$, that is why we see peaks on fig.1.1. Many acoustic peaks have now been measured definitively. The peaks contain the significant amount of physical information. For instance, the angular position of the first peak defines the curvature of the Universe [14].

The small angle region corresponds to $l \gtrsim 1000$. One can see that oscillations are suppressed for small angles. There are four effects that explain this suppresion [14]. The first one is a Silk effect. During the recombination era, both the temperature of the plasma and particles' concentration are so small, that we can consider that number of photons is conserved during an electron-photon interaction. Also, during single scattering photon's energy is almost conserved, that is why photons can propagate for a large distance without large energy dissipation. Thus, acoustic oscillations are washed away on small scales, because photons do not provide an exact snap shot of the baryon fluid. The second effect is a finiteness of the size ($2\Delta z_r \approx 150$) of the shell of the last scattering. This leads to anisotropy suppression and washes away oscillations. The third effect is a weak lensing. Cosmic structures and galaxies deflect photons. The deflection angle is very small, so this effect is important only for small angular scales. The last effect is the Sunyaev–Zel'dovich effect. Cosmic structures and galaxies have hot plasma clouds, so the probability of scattering is increased and the photon spectrum become slightly non-Planckian. So there is a correlation between the level of non-Planckianity of spectrum and galaxy positions. This is called secondary anisotropy.

The CMB is a very good way to look into the past of the Universe, and in particular the anisotropy spectrum provides a perfect way to get information about different aspects of the Universe. Given the fact that we have an unprecedented accuracy on the agreement (fig. (1.1)) between experimental results and theoretical predictions, we have a quite full and correct understanding of the physics of the Universe after recombination. Unfortunately, the CMB has almost nothing to say about physics before recombination. That is why gravitational waves are perfect candidates to look into the much farther past. It looks promising especially given the fact that gravitational waves from blackhole merging have recently been detected by LIGO [18]. This experiment measures a different type of gravitational waves, but now we know for sure that gravitational waves exist.

Primordial gravitational waves can have two natures: waves that are generated during inflation, and waves that are generated by other mechanisms between the end of inflation and the beginning of Big Bang nucleosynthesis (BBN, also known as primordial nucleosynthesis) [19, 20, 21, 22, 23, 24, 25, 26, 27]. So we can see that gravitational waves can help us not only distinguish between different inflationary models but also learn something about reheating, phase transitions, etc.

The tiny amplitude of the primordial gravitational waves makes it difficult to detect them both directly via space laser interferometers, like Japan's Decihertz Interferometer Gravitational Wave Observatory (DECIGO) [28] and NASA's Big Bang Observer (BBO) (see [29] for a review), and indirectly via their imprint on the CMB (so-called B-modes, that are curl-like pattern in polarization.)

1.3 Searching for principles

Earlier we discussed that one can consider three stages of a science. Presumably, no-one doubts that theoretical particle physics has reached the last stage quite a while ago. It is possible to argue when it happened, but one can say for sure that Z boson prediction [7, 10, 11] is a clear sign that the field had already been extremely mature at that time. The huge success of physics in the XXth century suggests that the scientific method that is widely used is the correct way to proceed.

Another important lesson from the past is that boundaries between different disciplines do not have a fundamental nature. They appear because of sociological reasons. Indeed, the broader science becomes the harder it is to follow development in each and every subfield. However, we have seen many times that concepts and ideas are traveling from one discipline to another, just like a field-theoretical approach that has developed for particle physics is widely used in condensed matter physics. Another example is that some exotic phases in condensed matter physics are described by Schwarz-type topological quantum field theory (TQFT) [30, 31], such as Chern-Simons theory of fractional quantum Hall effect [32].

One may argue that there are few nice equations and mathematical structures, so Nature has to use them again and again. However, the really surprising thing is that the description of the systems is simple. This is the case because we have a correct framework.

Biology has begun its transformation into a science that can make quantitative predictions only recently. There are many data now and there are some theories that allow us to describe simple phenomena. Some time ago it was widely believed that biological systems are very noisy, that is why it is very difficult to explore them in the usual physical way, but recently there were many studies that demonstrated that biological systems are very precise. For example, it was shown [33] that Drosophila embryo use precise control over absolute concentrations and responds reliably to small concentration differences, approaching the lower limit to the noise level possible to measure that is set by physical principles [34]. That means that apparent noisiness of the system appears only because we do not control environment and other conditions well enough. This is a very good news because it means that we can trust the data if we control all parameters and that biological systems should have a theory that describes their behavior, as in the rest of physics.

Looking back, we understand that it is important to have not only a numerical description but also a set of principles that govern the field. It looks that we cannot simply translate physical principles to systems of Living Matter. First of all, it does not work, and second, these systems are not like inanimate systems. It is not clear what makes them different, and we have to find it. It may be a key point on our way to find fundamental principles for living matter.

Biological systems are special in some sense, that is why the idea that they are near critical point appeared quite some time ago. Although the idea is very appealing for a physicist, its appearances have a different flavor for different situations. Let us consider several examples. For flocks of birds, the correlation function of velocity fluctuations extends over long distances, just as for spin-spin correlation functions in a magnet near its critical point. [35]. In the Drosophyla embryo, there is near-perfect anticorrelation of fluctuations in the expression levels of different genes at the same point [36]. Thermodynamics for a network of neurons is poised at a very unusual critical point, where the second and all higher order derivatives of entropy with respect to energy are equal to zero [37]. Maximum entropy model of the sequence repertoire of antibodies predicts that distribution of sequences obey Zipf's law which is equivalent to the statement that the entropy grows linearly with the energy. The locally linear relation between energy and entropy is characteristic of thermodynamic systems at a critical point [38].

We study the behavior of fruit fly in chapter 4. We find that it is possible to find a course grained description for the space of possible behaviors that preserves apparent critical behavior of the system. Course graining is defined by the condition that information about the next step is maximized, but this also leads to preservation of information about the distant future. This is extremely unexpected. Another unexpected feature is the existence of hierarchical structure for a set of clustering that we found. The remarkable point is that we never demanded that we should have it, that is why it is a result, not an assumption.

1.4 An Overview of the Dissertation

This dissertation has an introduction and three chapters. Each chapter contains a discussion of a separate topic.

Higher-Spin Theories

Correlation functions in conformal field theories can be written as a product of two terms: scaling factor that is completely fixed by conformal symmetry and function that is built from conformal covariants. The question we asked can be reformulated as how can we restrict this functions if our theory has at least one conserved current of spin higher than two.

We use a set of quite reasonable assumptions:

1. There is a unique conserved current of spin two, that is a stress-energy tensor. This assumption is quite natural because all known examples of the theory with non-unique stress-energy tensors can be presented as a trivial superposition of non-interaction theories.

2. Unitary bound for conformal dimensions is satisfied.

3. Cluster decomposition axiom is satisfied

Under this set of assumption, we show that if there is a conserved current of spin $s > 2$, then all correlation functions of symmetric currents of the theory are equal to correlation functions of free theory - either the theory of free bosons, or free fermions, or free forms.

Our strategy contains several non-trivial steps, that allows us to project out complications in Ward identities, and to show that only free theories satisfy them. Complete proof can be found in chapter 2, here we are presenting only conceptual outline, and ignore some important technical details for a sake of the simplicity.

The first crucial idea is that higher spin algebra requires an infinite tower of conserved charges if we have at least one conserved current of spin higher than two. It means that we have to satisfy infinitely many Ward identities. An analog of this fact is that we have to solve an infinitely large system of equations eq.(1.2). It is not enough to have infinitely many Ward identities to show that only trivial solution is possible, though.

In order to translate complicated Ward identities of the full theory into simpler ones involving only free field correlators, it is crucial to use a so-called *lightcone limit*. This allows us to transform Ward identities into easily-analyzed polynomial equations.

The third idea, that is actually a way to use the previous one. We construct certain quasi-bilocal fields, which behavior in the lightcone limit is roughly like a behavior of the product of two free fields. Higher spin charges act on these quasi-bilocal fields like derivatives.

The difference to the $d = 3$ case is that there are three possible sectors: free scalar, free fermions, and free $(d-2)/2-$forms.

There is a conjecture of Klebanov and Polyakov that Vasiliev's theory in four dimensions is dual to the critical $O(N)$ vector model in three dimensions [39] [40]. Vasiliev's higher-spin gauge theories in anti de-Sitter space [41] [42] [43] are dual to conformal field theories under the AdS/CFT correspondence [44] [45] [46]. One expects that the conformal field theory dual to Vasiliev's theory should also have a higher-spin symmetry, that is why our theorem restricts these bulk theories as well.

The natural future step is to study what happens if higher-spin symmetry is slightly broken. This may allow fixing on shell correlation functions of Vasiliev's theory on AdS_{d+1}. For the $3d$ case it was studied in [47].

Anisotropy of the Primordial Gravitational Waves

We give an extensive review of gravitational waves evolution in the appendix B. Here we would like only briefly discuss what we computed and methods we used.

At every point in time background is an exact solution that changes depending on parameters like an equation of state. Gravitational waves are considered as a perturbation for the Friedmann–Lemaitre–Robertson–Walker (FLRW) metric. These tensorial perturbations arise using a mechanism that is similar to the mechanism that produces fluctuations of the inflaton field [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. That is why gravitational waves are coming almost uniformly from all directions in the sky.

In order to compute the spectrum of gravitons today, we use two methods, and we find complete agreement between them. The first method considers the classical propagation of the gravitons on the background of slightly perturbed Universe. We found that spectrum is completely determined by times of horizon crossings and by equations of state at those moments. Also, we found that spectrum of observed gravitons is slightly anisotropic, because of the scalar perturbation of the metric. This anisotropy correlates with one we got for the CMB.

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The first method considers the classical propagation of the gravitons on the background of slightly perturbed Universe. We found that spectrum is completely determined by times of horizon crossings and by equations of state at those moments. Also, we found that spectrum of observed gravitons is slightly anisotropic, because of the scalar perturbation of the metric. This anisotropy correlates with one we got for the CMB. Sachs-Wolfe effect is the main contributor to the anisotropy and has a similar form to eq. (1.6). All detail can be found in Appendix B.1.4 and Appendix B.2.1.

Another method is to use cosmological perturbation theory. In order to get dependence of the graviton spectrum on a scalar perturbation we have to compute three-point function $\langle hh\zeta\rangle$. This computation reproduces the result of the classical evolution. All details can be found in the Appendix B.5.

Physics of Behavior

We analyze the unsupervised behavior of a fruit fly. A distinctive feature of the system is a long-range memory. The system is quite complex, and the simplest models are not able to describe complex behavior.

We attempt to simplify the system by clustering states, preserving dynamic correlator by maximizing information about the next step. This allows us to cluster states. Our simplification is very successful, and there are two prominent results that are rather unexpected.

The first result is that the maximization of the information about the next step also maximized information about a distant future. It could be the other way around. For instance, if we consider speech or written text, one can maximize information about next step, that corresponds to prediction of the next letter, but it does not give us any information about the distant future, that means that this model will not be able to predict anything about particular text on the level that exceeds random prediction.

The second result is we got a hierarchical structure. That means that if we increase the number of clusters they divide rather then complete rearrangement of the states. It is not difficult to use an algorithm that cluster data in a hierarchical way, but in our case, this arises as result.

Finally, we used maximal entropy methods to describe the simplified system. We found that some discrepancies that may be related to the fact that we used a finite system for our simulations. Further discussion will be presented in a separate work.

We found a hint of a connection to a very interesting model, well known in statistical physics, but also some discrepancies. We are trying to investigate to what extent these discrepancies result only from the finiteness of our data and simulations.

Publications and preprints

Results that are resented in this dissertation have previously been published in

- [58] Vasyl Alba and Kenan Diab, *"Constraining conformal field theories with a higher spin symmetry in* $d = 4$*"*, *arXiv* (2013) [arXiv:1307.8092];
- [59] Vasyl Alba and Kenan Diab, *"Constraining conformal field theories with a higher spin symmetry in* $d > 3$ *dimensions."*, *Journal of High Energy Physics*, **2016** (May, 2016) 044, [arXiv:1510.02535];
- [60] Vasyl Alba and Juan Maldacena, *"Primordial gravity wave background anisotropies"*, *Journal of High Energy Physics*, **2016** (Mar, 2016) 115, [arXiv:1512.01531].

Public presentations

Materials from this dissertation have been publicly presented at the following conferences and meetings

1. Workshop on Mathematical Physics, Amsterdam (26-28 May 2015);
2. TASI Summer School 2015. Student presentation section;
3. APS March Meeting 2017;
4. Mathematical Modeling of Living Systems 2017;
5. Sense to Synapse 2017.

2

Constraining conformal field theories with a higher spin symmetry in $d > 3$ dimensions

"Believe and act as if it were impossible to fail."

Charles Kettering

2.1 Introduction

This whole chapter is completely based on the paper [59], coauthored with Kenan Diab, which is an extension of the paper[58], also coauthored with Kenan Diab.

In this chapter, we will prove an analog of the Coleman-Mandula theorem for generic conformal field theories in all dimensions greater than three. We will show that in any conformal field theory that (a) satisfies the unitary bound for operator dimensions, (b) satisfies the cluster decomposition axiom, (c) contains a symmetric conserved current of spin larger than 2, and (d) has a unique stress tensor in $d > 3$ dimensions, all correlation functions of symmetric currents of the theory are equal to the correlation functions of one of the following three theories - either the theory of n free bosons (for some integer n), a theory of n free fermions, or a theory of n free $\frac{d-2}{2}$-forms.

Note that in odd dimensions, the free $\frac{d-2}{2}$-form does not exist, and the status of our result is somewhat complicated. We do not show that there exists any solution to the conformal Ward identities that corresponds to this possibility in odd dimensions, although we do show that if one exists, it is unique. For every odd dimension $d \geq 7$, we know that an infinite tower of higher-spin currents must be present [61]. In $d = 5$, the unitarity forbids finite tower case, so only infinite tower is possible. Assuming that the solution exists and there are an infinite number of higher spin currents, we show that the correlation functions of the conserved currents of the theory may be understood as the analytic continuation of the correlation functions of the currents of the even-dimensional free $\frac{d-2}{2}$-form theory to odd dimensions. Then, even under all these assumptions, we do not show that there exists any conformal field theory that realizes this solution. That is, it is possible that this structure may have no good microscopic interpretation for other reasons. For example, in odd dimensions, it could be possible that some correlation function of some operator is not consistent with the operator product expansion in the sense that it cannot be decomposed into a sum over conformal blocks with non-negative coefficients (i.e. consistent with unitarity[1]). Such questions are not explored in this work.

[1]There is an example of this phenomenon. If one considers a theory of N scalar fields ϕ_i and computes the four-point function of the operator $\phi^2 = \sum_i \phi_i\phi_i$, it turns out that N should be greater than 1, otherwise the theory is nonunitary.

The paper by Boulanger, Ponomarev, Skvortsov, and Taronna [61] strongly indicates that all the algebras of higher-spin charges that are consistent with conformal symmetry are not only Lie algebras but associative. Hence, they are all reproduced by the universal enveloping construction of [62] with the conclusion that any such algebra must contain a symmetric higher-spin current. This implies that our result should be true even after relaxing our assumption that the higher-spin current is symmetric. The argument is structured as follows:

In section 2.2, we will present the main technical tool of the chapter: we will define a particular limit of three-point functions of symmetric conserved currents called *lightcone limits*. We will show that such correlation functions behave essentially like correlation functions of a free theory in these limits, enabling us to translate complicated Ward identities of the full theory into simpler ones involving only free field correlators. We will also compute the Fourier transformation of these correlation functions; this will ultimately allow us to simplify certain Ward identities into easily-analyzed polynomial equations.

The rest of the chapter will then carry out proof of our main statement. The steps are as follows:

In section 2.3, we will solve the Ward identity arising from the action of the charge Q_s arising from a spin s current j_s on the correlator $\langle j_2 j_2 j_s \rangle$ in the lightcone limit, where j_2 is the stress tensor. We will show that the only possible solution is given by the free-field solution. This implies the existence of infinitely many conserved currents of arbitrarily high spin,[2] thereby giving rise to infinitely many charge conservation laws which powerfully constrain the theory.

[2]The fact that the existence of a higher-spin current implies the existence of infinitely many other higher-spin currents has been proven before in the four-dimensional case [63] under the additional assumptions that the theory flows to a theory with a well defined S-matrix in the infrared, that the correlation function $\langle j_2 j_2 j_s \rangle \neq 0$, and that the scattering amplitudes of the theory have a certain scaling behavior. This statement was also proven for $d \neq 4, 5$ in [61] by classifying all the higher-spin algebras in all dimensions other than 4 and 5. We give a proof for the sake of completeness, and also because our techniques differ from those two papers.

In section 2.4, we will construct certain quasi-bilocal fields which roughly behave like products of free fields in the lightcone limit, yet are defined for any CFT. We will establish that all the higher-spin charges (whose existence was proven in the previous step) act on these quasi-bilocals in a particularly simple way.

In section 2.5, we will translate the action of the higher-spin charges on the quasi-bilocals into constraints on correlation functions of the quasi-bilocals. We will then show that these constraints are so powerful that they totally fix every correlation function of the quasi-bilocals to agree with the corresponding correlation function of a particular biprimary operator in free field theory on the lightcone.

In section 2.6, we show how the quasi-bilocal correlation functions can be used to prove that the three-point function of the stress tensor must be equal to the three-point function of either the free boson, the free fermion, or the free $\frac{d-2}{2}$-form, even away from the lightcone limit. This is then used to recursively constrain every correlation function of the CFT to be equal to the corresponding correlation function in the free theory, finishing the proof.

This strategy is similar to the argument in the three-dimensional case given in [12]. There are two main differences between the three-dimensional case and the higher-dimensional cases that we must account for:

First, the Lorentz group in $d > 3$ admits asymmetric representations, but the three-dimensional Lorentz group does not. By asymmetric, we mean that a current $J_{\mu_1 \dots \mu_n}$ is not invariant with respect to interchange of its indices. For example, in in the standard (j_1, j_2) classification of representations of the four-dimensional Lorentz group induced from the isomorphism of Lie algebras $\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, these are the representations with $j_1 \neq j_2$. The existence of these representations means that many more structures are possible in $d > 3$ dimensions than in three dimensions (the asymmetric structures), and so many more coefficients have to be constrained in order to solve the Ward identities.

We restrict our attention to Ward identities arising from the action of a symmetric charge to correlation functions of only symmetric currents; we will then show that asymmetric structures cannot appear in these Ward identities, making the exact solution of the identities possible.

Second, the space of possible correlation functions consistent with conformal symmetry is larger in $d > 3$ dimensions than in three dimensions. For example, consider the three-point function of the stress tensor $\langle j_2 j_2 j_2 \rangle$. It has long been known (see, e.g. [64][65][66][67]) that this correlation function factorizes into three structures in $d > 3$ dimensions, as opposed to only two structures in three dimensions (ignoring a parity-violating structure which is eliminated in three dimensions by the higher-spin symmetry). These three structures correspond to the correlation functions that appear in the theories of free bosons, free fermions, and free $\frac{d-2}{2}$-forms. We will show that even though more structures are possible in four dimensions and higher, the Ward identities we need can still be solved.

We note that our work is related to a paper by Stanev [68], in which the four, five, and six-point correlation functions of the stress tensor were constrained in CFT's with a higher spin current in four dimensions. It was also shown that the pole structure of the general n-point function of the stress tensor coincides with that of a free field theory. Though this chapter reaches the same conclusions, we do not make the rationality assumption [69] of that paper.

By investigating the Jacobi identity for the charges of the conformal algebra combined with at least one HS charge it was proved in [61] that in d>5 there is a unique HS algebra that such charges can form. This algebra corresponds to the free boson and contains HS charges of all (at least even) spins. One of the assumptions was that only HS charges built out of the symmetric HS currents can contribute to the Lie bracket. In the present chapter, we relax this assumption, which results in two more solutions: free fermion and the $\frac{d-2}{2}$-form field.

Finally, the paper [70] appeared in which they showed that unitary "Cauchy conformal fields", which are fields that satisfy a certain first-order differential equation, are free in the sense that their correlation functions factorize on the 2-point function. Their result may be understood as establishing a similar result that applies even to certain fields which are not symmetric traceless, which we say nothing about.

2.2 Definition of the lightcone limits

The fundamental technical tool we need to extend into four dimensions and higher is the *lightcone limit*. In order to constrain the correlation functions of the theory to be equal to free field correlators, we will show that the three-point function of the $\langle j_2 j_2 j_2 \rangle$ must be equal to $\langle j_2 j_2 j_2 \rangle$ for a free boson, a free fermion, or a free $\frac{d-2}{2}$-form field - it cannot be some linear combination of these three structures. To this end, it will be helpful to split up the Ward identities of the theory into three different identities, each of which involves only one of the three structures separately. To do this, we will need to somehow project all the three-point functions of the theory into these three sectors. The lightcone limits accomplish this task.

Before defining the lightcone limits, we will set up some notation. As in [12], we are writing the flat space metric $ds^2 = dx^+ dx^- + d\vec{y}^2$ and contracting each current with lightline polarization vectors whose only nonzero component is in the minus direction: $j_s \equiv J_{\mu_1 \ldots \mu_s} \epsilon^{\mu_1} \ldots \epsilon^{\mu_s} = J_{-- \ldots -}$. We will also denote $\partial_1 \equiv \partial / \partial x_1^-$ and similarly for ∂_2 and ∂_3. Thus, in all expressions where indices are suppressed, those indices are taken to be minus indices. There are two things we will establish:

1. We need to define an appropriate limit for each of the three cases, which, when applied to a three-point function of conserved currents $\left\langle \underline{j_{s_1} j_{s_2}} j_{s_3} \right\rangle$, yields an expression proportional to an appropriate correlator of the free field theory. For example, in the

bosonic case where all the currents are symmetric, we would like the lightcone limit to give us $\partial_1^{s_1}\partial_2^{s_2}\langle\phi\phi^* j_{s_3}\rangle_{\text{free}}$.

2. Second, we need to explicitly compute the free field correlator which we obtain from the lightcone limits. In the bosonic case where all currents are symmetric, this would mean that we need to compute the three-point function $\langle\phi\phi^* j_{s_3}\rangle$ in the free theory.

For the first task, we claim that the desired lightcone limits are:

$$\left\langle \underline{j_{s_1} j_{s_2}}_b j_{s_3}\right\rangle \equiv \lim_{|y_{12}|\to 0}|y_{12}|^{d-2}\lim_{x_{12}^+\to 0}\langle j_{s_1}j_{s_2}j_{s_3}\rangle \propto \partial_1^{s_1}\partial_2^{s_2}\langle\phi\phi^* j_{s_3}\rangle_{\text{free}} \tag{2.1}$$

$$\left\langle \underline{j_{s_1} j_{s_2}}_f j_{s_3}\right\rangle \equiv \lim_{|y_{12}|\to 0}|y_{12}|^{d}\lim_{x_{12}^+\to 0}\frac{1}{x_{12}^+}\langle j_{s_1}j_{s_2}j_{s_3}\rangle \propto \partial_1^{s_1-1}\partial_2^{s_2-1}\left\langle\psi\gamma_-\bar{\psi} j_{s_3}\right\rangle_{\text{free}} \tag{2.2}$$

$$\left\langle \underline{j_{s_1} j_{s_2}}_t j_{s_3}\right\rangle \equiv \lim_{|y_{12}|\to 0}|y_{12}|^{d+2}\lim_{x_{12}^+\to 0}\frac{1}{(x_{12}^+)^2}\langle j_{s_1}j_{s_2}j_{s_3}\rangle \propto \partial_1^{s_1-2}\partial_2^{s_2-2}\left\langle F_{-\{\alpha\}}F_{-\{\alpha\}} j_{s_3}\right\rangle_{\text{free}} \tag{2.3}$$

Here, the subscript *b, f,* and *t* denote the bosonic, fermionic, and tensor lightcone limits. ϕ is a free boson, ψ is a free fermion, and F is the field tensor for a free $\frac{d-2}{2}$-form field; the repeated $\{\alpha\}$ indices indicate Einstein summation over all other indices. For example, in four dimensions, the "tensor" structure is just the ordinary free Maxwell field. For conciseness, we will often refer to the free $\frac{d-2}{2}$-form field as simply the "tensor field" or the "tensor structure". Again, we emphasize that in odd dimensions, the free $\frac{d-2}{2}$-form field does not exist. In odd dimensions, our claim is that the only possible structure with the scaling behavior captured by the tensor lightcone limit is the one which coincides with the naive analytic continuation of the correlation functions of the free $\frac{d-2}{2}$ form field to odd d.

The justification for the first two equations comes from the generating functions obtained in [66][67]; in those references, the three-point functions for correlation functions of conserved currents with y_{12} and x_{12}^+ dependence of those types was uniquely characterized, and so taking the limit of those expressions as indicated gives us the claimed result. In the tensor case, [67] did not find a unique structure, but rather, a one-parameter family of possible structures. Nevertheless, all possible structures actually coincide in the lightcone limit, as is proven in Appendix A.2.

We note that parity-violating structures cannot appear after taking these lightcone limits. This is because the all-minus component of every parity violating structure allowed by conformal invariance in $d > 3$ dimensions is identically zero. To see this, observe that all parity-violating structures for three-point functions consistent with conformal symmetry must have exactly one $\epsilon_{\mu_1\mu_2\cdots\mu_d}$ tensor contracted with polarization vectors and differences in coordinates. Only two of these differences are independent of each other, and all polarization vectors in the all-minus components are set to be equal. Thus, there are only three unique objects that can be contracted with the ϵ tensor, but we need d unique objects to obtain a nonzero contraction. Thus, all parity-violating structures have all-minus components equal to zero.

Later in our argument, we will need expressions for the Fourier transformation of the lightcone-limit three point function of two free fields and a spin s current with respect to the variables x_1^- and x_2^- in the theories of a free boson, a free fermion, and a free $\frac{d-2}{2}$-form field. The computation for each of the three cases is straightforward and is given explicitly in appendix A.1. The results are as follows:

$$F_s^b \equiv \left\langle \underline{\phi\phi^*} j_s \right\rangle \propto (p_2^+)^s {}_2F_1\left(2 - \frac{d}{2} - s, -s, \frac{d}{2} - 1, p_1^+/p_2^+\right) \tag{2.4}$$

$$F_s^f \equiv \left\langle \underline{\psi\gamma_-\bar{\psi}} j_s \right\rangle \propto (p_2^+)^{s-1} {}_2F_1\left(1 - \frac{d}{2} - s, -s, \frac{d}{2}, p_1^+/p_2^+\right) \tag{2.5}$$

$$F_s^b \equiv \left\langle \underline{F_{-\{\alpha\}}F_{-\{\alpha\}}} j_s \right\rangle \propto (p_2^+)^{s-2} {}_2F_1\left(-\frac{d}{2} - s, -s, \frac{d}{2} + 1, p_1^+/p_2^+\right) \tag{2.6}$$

Here, ${}_2F_1$ is the hypergeometric function, and the proportionality sign in each formula indicates that we have omitted an overall nonsingular function which we are not interested in. That they are indeed nonsingular is also proven in appendix A.1.

Before continuing, we emphasize that the three lightcone limits we have defined do not cover all possible lightcone behaviors which can be realized in a conformal field theory. We define only these three limits because one crucial step in our proof is to constrain the three-point function of the stress tensor $\langle j_2 j_2 j_2 \rangle$, which has only these three scaling behaviors.

Furthermore, though we have discussed only symmetric currents, one could hope that similar expressions could be generated for asymmetric currents - that is, lightcone limits of correlation functions of asymmetric currents are generated by one of the three free field theories discussed here. Unfortunately, running the same argument in [67] fails in the case of asymmetric currents in multiple ways. Consider the current $\langle j_2 j_s \bar{j}_s \rangle$, where j_s is some asymmetric current and $\bar{j}_s$ is its conjugate. To determine how such a correlator could behave the lightcone limit, one could write out all the allowed conformally invariant structures consistent with the spin of the fields and seeing how each one behaves in the lightcone limits. Unlike the symmetric cases, one finds that in the lightcone limit many independent structures exist, and these structures behave differently depending on which pair of coordinates we take the lightcone limit. To put it another way, for a symmetric current s, one has the decomposition:

$$\langle j_2 j_s j_s \rangle = \sum_{j \in \{b,f,t\}} \langle j_2 j_s j_s \rangle_j \tag{2.7}$$

where the superscript j denotes the result after taking corresponding lightcone limit in any of the three pairs of coordinates (all of which yield the same result), and the corresponding structures can be understood as arising from some free theory. In the case of asymmetric j_s, this instead becomes a triple sum

$$\langle j_2 j_s \bar{j}_s \rangle = \sum_{j,k,l \in \{b,f,t\}} \langle j_2 j_s \bar{j}_s \rangle_{(j,k,l)} \tag{2.8}$$

where each sum corresponds to taking a lightcone limit in each of the three different pairs of coordinates and we do not know how to interpret the independent structures in terms of a free field theory. This tells us that for asymmetric currents, the lightcone limit no longer achieves its original goal of helping us split up the Ward identities into three identities which can be analyzed independently; each independent structure could affect multiple different

Ward identities. Again, we emphasize that this does not exclude the possibility of a different lightcone limit reducing the correlators of asymmetric currents to those of some other free theory. It simply means that our techniques are not sufficient to constrain correlation functions involving asymmetric currents, so we will restrict our attention to correlation functions that involve only symmetric currents.

2.3 Charge conservation identities

We will now use the results of the previous section to prove that every CFT with a higher-spin current contains infinitely many higher-spin currents of arbitrarily high (even) spin. We note that this result was proven in a different way in [61] for all dimensions other than $d = 4$ and $d = 5$, wherein they showed that there is a unique higher-spin algebra in $d \neq 4, 5$ and showed that there are infinitely many higher-spin currents. The discussion below is a different proof of this statement based on analysis of the constraints that conservation of the higher-spin charge imposes, and the techniques we develop here will be used later. As before, we treat the bosonic, fermionic, and tensor cases separately.

Before beginning, we will tabulate a few results about commutation relations that we will use freely throughout from this section onwards. Their proofs are identical to those in [12], and are therefore omitted:

1. If a current j' appears (possibly with some number of derivatives) in the commutator $[Q_s, j]$, then j appears in $[Q_s, j']$.

2. Three-point functions of a current with odd spin with two identical currents of even spin are zero: $\langle j_s j_s j_{s'} \rangle = 0$ if s is even and s' is odd.

3. The commutator of a symmetric current with a charge built from another symmetric current contains only symmetric currents and their derivatives:

$$[Q_s, j_{s'}] = \sum_{s''=\max[s'-s+1,0]}^{s'+s-1} \alpha_{s,s',s''} \partial^{s'+s-1-s''} j_{s''} \tag{2.9}$$

The proof of this statement requires an additional step since one needs to exclude asymmetric currents contracted with invariant symbols like the ϵ tensor. For example, consider what structures could appear in $[Q_2, j_2]$ in four dimensions. In $SU(2)$ indices, this object has three dotted and three undotted spinor indices, so one could imagine that a structure like $\epsilon_{ab} j^{abcde\dot{c}\dot{d}\dot{e}}$ could appear in $[Q_2, j_2]$. However, $[Q_2, j_2]$ has conformal dimension 5, and the unitarity bound constrains the current j, which transforms in the $(5/2, 3/2)$ representation, to have conformal dimension at least $d - 2 + s = 6$, which is impossible. The proof for a general commutator $[Q_s, j_{s'}]$ follows in an identical manner.

4. $[Q_s, j_2]$ contains ∂j_s. This was actually proven for all dimensions in appendix A of [12]. Item 1 then implies that $[Q_s, j_s]$ contains $\partial^{2s-3} j_2$.

In these statements, we are implicitly ignoring the possibility of parity violating structures. For example, the three-point function $\langle 221 \rangle$, which is related to the $U(1)$ gravitational anomaly, may not be zero in a parity violating theory. As mentioned in section 2.2, however, the all-minus components of every parity-violating structure consistent with conformal symmetry is identically zero, so they will not appear in any of our identities here.

Let's start with the bosonic case. Consider the charge conservation identity arising from the action of Q_s on $\langle \underline{22}_b s \rangle$:

$$0 = \left\langle \underline{[Q_s, 2]2}_b s \right\rangle + \left\langle \underline{2[Q_s, 2]}_b s \right\rangle + \langle \underline{22}_b [Q_s, s] \rangle \tag{2.10}$$

If s is symmetric, we may use the general commutation relation (2.9) and the lightcone limit (2.1) to expand this equation out in terms of free field correlators:

$$0 = \partial_1^2 \partial_2^2 \left(\gamma(\partial_1^{s-1} + (-1)^s \partial_2^{s-1}) \left\langle \underline{\phi\phi^*} s \right\rangle_{free} + \sum_{2 \le k < 2s-1 \text{ even}} \tilde{\alpha}_k \partial_3^{2s-1-k} \left\langle \underline{\phi\phi^*} k \right\rangle_{free} \right) \tag{2.11}$$

Note that the sum over k is restricted to even currents since $\langle 22k \rangle = 0$ for odd k. In addition, the fact that the coefficient in front of the ∂_2^{s-1} term is constrained to be $(-1)^s$ times the coefficient for the ∂_1^{s-1} term arises from the symmetry of $\langle \phi(x_1)\phi^*(x_2)j_s(x_3) \rangle$ under interchange of x_1 and x_2.

Now, we apply our Fourier space expressions for the three-point functions given in section 2.2. In the Fourier transformed variables, derivatives along the minus direction turn into multiplication by the momenta in the plus direction. After ”cancelling out” the overall derivatives, which just yields an overall factor of $(p_1^+)^2(p_2^+)^2$, the relevant equation is:

$$0 = \gamma((p_1^+)^{s-1} + (-1)^s (p_2^+)^{s-1}) F_s(p_1^+, p_2^+) + \sum_{2 \le k < 2s-1 \text{ even}} \tilde{\alpha}_k (p_1^+ + p_2^+)^{2s-1-k} F_k(p_1^+, p_2^+) \tag{2.12}$$

The solution of (2.12) is not easy to obtain by direct calculation. We can make two helpful observations, however. First, not all coefficients can be zero. This is because we know 2 appears in $[Q_s, s]$, so at least $\tilde{\alpha}_2$ is not zero. Second, we know that the free boson exists (there is a CFT with higher spin symmetry), and therefore, the coefficients one obtains from that theory would exactly solve this equation. We will show that this solution is unique.

Suppose we have two sets of coefficients $(\gamma, \{\tilde{\alpha}_k\})$ and $(\gamma', \{\tilde{\beta}_k\})$ that solve this equation. First, suppose $\gamma \neq 0$ and $\gamma' \neq 0$. Then, we can normalize the coefficients so that $\gamma = \gamma'$ are equal for the two solutions. Then, subtract the two solutions from each other so that the γ terms vanish. If we evaluate the result at some arbitrary nonzero value of p_2^+, we may absorb all overall p_2^+ factors into the coefficients and re-express the equation as a polynomial identity in a single variable $z \equiv p_1^+/p_2^+$:

$$0 = \sum_{2 \le k < 2s-1 \text{ even}} \tilde{\delta}_k (1+z)^{2s-1-k} {}_2F_1(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -z) \tag{2.13}$$

Then, the entire right hand side is divisible by $1+z$ since s is even, so we may divide both sides by $1+z$. Setting $z = -1$, since ${}_2F_1(a, a, 1, 1) \neq 0$ for all negative half-integers a, we conclude that $\tilde{\delta}_{2s-2} = 0$. Then, the entire right hand side is proportional to $(1+z)^2$, so we may divide it out. Then, setting $z = -1$ again, we find $\tilde{\delta}_{2s-4} = 0$. Repeating this procedure, we conclude that all coefficients are zero, and therefore, that the two solutions are identical. On the other hand, suppose one of the solutions has $\gamma = 0$. Then, the same argument establishes that all the coefficients $\tilde{\alpha}_k$ are zero. As noted earlier, however, the trivial solution is disallowed. Therefore, the solution is unique and coincide with one for free boson. Thus, we have infinitely many even conserved currents, as desired.

In the fermionic case, precisely the same analysis works. The action of Q_s on $\left\langle \underline{22}_f s \right\rangle$ for symmetric s leads to

$$0 = \partial_1^2 \partial_2^2 \Big(\gamma (\partial_1^{s-2} + (-1)^{s-1} \partial_2^{s-2}) \left\langle \psi \bar{\psi} s \right\rangle + \sum_{2 \le k < 2s-2 \text{ even}} \tilde{\alpha}^k \partial_3^{2s-2-k} \left\langle \psi \bar{\psi} k \right\rangle \Big), \tag{2.14}$$

Then, converting this expression to form factors and running the same analysis from the bosonic case verbatim establishes that the unique solution to this equation is the one arising in the theory of a free fermion.

In the tensor case, the argument again passes through exactly as before, except for two subtleties:

First, unlike in the bosonic and fermionic case, we do not have unique expressions for the three-point functions of currents with the tensor-type coordinate dependence, so this only demonstrates that the free-field solution is an admissible solution, but not necessarily the unique solution. Nevertheless, in the lightcone limit, all possible structures for three-point functions coincide with the free-field answer.[3] This was proven in Appendix A.2.

Second, there may not exist a solution to the Ward identities in odd dimensions, because the free $\frac{d-2}{2}$-form does not exist in odd dimensions. However, if any solution exists, our argument shows that it is unique. In $d \geq 7$, it is known that there is a unique higher-spin algebra containing the tower of higher-spin currents described in the bosonic case [61]. In $d = 5$, our technique shows that if there is a solution for the Ward identity in the tensor lightcone limit, then it is unique. We do not prove, however, that there is an infinite tower of higher spin currents or that there is exactly one current of every spin. Finite dimensional representations would be inconsistent with unitarity. We do not explore this question further in this work. Henceforth, we assume that our theory does indeed contain the infinite tower of higher-spin currents necessary for our analysis.

2.4 Quasi-bilocal fields: basic properties

In this section, we will define a set of quasi-bilocal operators, one for each of the three lightcone limits, and characterize the charge conservation identities arising from the action of the higher-spin currents. As we will explain in section 2.5, these charge conservation identities will turn out to be so constraining that the correlation functions of the quasi-bilocal operators are totally fixed. This will then enable us to recursively generate all the correlation functions of the theory and prove that the three-point function of the stress tensor can exhibit only one of the three possible structures allowed by conformal symmetry. As in the three-dimensional case, we define the quasi-bilocal operators on the lightcone as

[3] Actually, we proved that correlators of the form $\langle 22s \rangle$ have a unique tensor structure even away from the lightcone limit. The proof, however, is very technical, and it is given in Appendix A.3.

operator product expansions of the stress tensor with derivatives "integrated out":

$$\underline{22}_b = \partial_1^2 \partial_2^2 B(\underline{x_1, x_2}) \tag{2.15}$$

$$\underline{22}_f = \partial_1 \partial_2 F_-(\underline{x_1, x_2}) \tag{2.16}$$

$$\underline{22}_t = V_{--}(\underline{x_1, x_2}) \tag{2.17}$$

The motivation behind these definitions can be understood by appealing to what these expressions look like in free field theory. There, they will be given by simple products of free fields:

$$B(x_1, x_2) \sim : \phi(x_1)\phi^*(x_2) : + : \phi(x_2)\phi^*(x_1) : \tag{2.18}$$

$$F_-(x_1, x_2) \sim : \bar{\psi}(x_1)\gamma_-\psi(x_2) : - : \bar{\psi}(x_2)\gamma_-\psi(x_1) : \tag{2.19}$$

$$V_{--} \sim : F_{-\{\alpha\}}(x_1) F_{-\{\alpha\}}(x_2) : \tag{2.20}$$

It is clear from the basic properties of our lightcone limits that when they are inserted into correlation functions with another conserved current j_s, they will be proportional to an appropriate free field correlator. Since $\langle \underline{22} s \rangle = 0$ for odd s, only the correlation functions with even s will be nonzero:

$$\left\langle B(\underline{x_1, x_2}) j_s \right\rangle \propto \langle \phi(x_1)\phi^*(x_2) j_s(x_3) \rangle_{\text{free}} \tag{2.21}$$

$$\left\langle F_-(\underline{x_1, x_2}) j_s \right\rangle \propto \left\langle \psi(x_1)\gamma_-\bar{\psi}(x_2) j_s(x_3) \right\rangle_{\text{free}} \tag{2.22}$$

$$\left\langle V_{--}(\underline{x_1, x_2}) j_s \right\rangle \propto \left\langle F_{-\{\alpha\}}(x_1) F_{-\{\alpha\}}(x_2) j_s(x_3) \right\rangle_{\text{free}} \tag{2.23}$$

Of course, away from the lightcone, things will not be so simple: we have not even defined the quasi-bilocal operators there, and their behavior there is the reason why they are not true bilocals. In fact, even on the lightcone, these expressions are not fully conformally invariant: the contractions of indices performed in equations 2.22 and 2.23 are only invariant under the action of the collinear subgroup of the conformal group defined by the line connecting x_1 and x_2. For now, however, the lightcone properties enumerated above are enough to establish the commutator of Q_s with the bilocals. As usual, we begin with the bosonic case:

Assume that $\langle \underline{22}_b 2\rangle \neq 0$. Our goal is to show that

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})B(\underline{x_1, x_2}). \tag{2.24}$$

This can be shown using the same arguments as [12]. To begin, notice that the action of Q_s commutes with the lightcone limit. Thus,

$$\langle [Q_s, B] j_k\rangle = \left\langle \underline{[Q_s, j_2] j_2} j_k \right\rangle + \left\langle \underline{j_2 [Q, j_2]} j_k \right\rangle = -\left\langle \underline{j_2 j_2} [Q_s, j_k]\right\rangle = \left\langle [Q_s, \underline{j_2 j_2}] j_k \right\rangle \tag{2.25}$$

This immediately leads to:

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(\underline{x_1, x_2}) + (\partial_1^{s-1} - \partial_2^{s-1})B'(\underline{x_1, x_2}), \tag{2.26}$$

Here, $\tilde{B}$ is built from even currents, while B' is built from odd currents. This makes the whole expression symmetric. We would like to show that $B' = 0$. Therefore, suppose otherwise so that some current $j_{s'}$ has nontrivial overlap with B'. Then, the charge conservation identity $0 = \langle [Q_{s'}, B' j_2]\rangle$ yields

$$0 = \left\langle \left[Q_{s'}, B'(\underline{x_1, x_2})\right] j_2 \right\rangle + \left\langle B'(\underline{x_1, x_2}) \left[Q_{s'}, j_2\right] \right\rangle, \tag{2.27}$$

$$\Rightarrow 0 = \gamma \left(\partial_1^{s'-1} - \partial_2^{s'-1}\right) \left\langle \underline{\phi\bar{\phi}} j_2 \right\rangle + \sum_{k=0}^{s'+1} \tilde{\alpha}_k \partial^{s'+1-k} \left\langle \underline{\phi\bar{\phi}} j_k \right\rangle. \tag{2.28}$$

Using the same techniques as the previous section, we obtain

$$0 = \gamma((p_1^+)^{s'-1} - (p_2^+)^{s'-1})F_2(p_1^+, p_2^+) + \sum_{k=0}^{s'+1} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s'+1-k} F_k(p_1^+, p_2^+). \tag{2.29}$$

In this sum, $\tilde{\alpha}_{s'} \neq 0$ because $j_{s'} \subset [Q_{s'}, 2]$. Therefore, we can use the same procedure as before to show that all $\tilde{\alpha}_k$ are nonzero if they are nonzero for the free field theory. In particular, since $\tilde{\alpha}_1$ is not zero for the complex free boson, the overlap between j_1 and B' is not zero. Now, let's consider

$$0 = \langle [Q_s, Bj_1] \rangle = \left(\partial_1^{s-1} - \partial_2^{s-1} \right) \langle B' j_1 \rangle + \langle B \left[Q_s, j_1 \right] \rangle \,, \tag{2.30}$$

where Q_s is a charge corresponding to any even higher-spin current appearing in the operator product expansion of $\underline{j_2 j_2}_b$. We have shown the first term is not zero. We will prove that the second term must be equal to zero to get a contradiction. Specifically, we will show that there are no even currents in $[Q_s, j_1]$. Since B is proportional to $\underline{22}$, and since $\langle 22s \rangle = 0$ for all odd s, this yields the desired conclusion.

Consider the action of Q_s on $\langle 221 \rangle$. We obtain the now-familiar form:

$$0 = \gamma((p_1^+)^{s-1} - (p_2^+)^{s-1}) F_1(p_1^+, p_2^+) + \sum_{k=0}^{s} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s-k} F_k(p_1^+, p_2^+) \tag{2.31}$$

We want to show that $\alpha_k = 0$ for even k. Recall the definition of F_k:

$$F_k = (p_2^+)^k {}_2F_1 \left(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -\frac{p_1^+}{p_2^+} \right) \tag{2.32}$$

$$= \sum_{i=0}^{k} c_i^k (p_1^+)^i (p_2^+)^{s-i} \tag{2.33}$$

The hypergeometric coefficients c_i^k have the property that $c_i^k = (-1)^k c_{k-i}^k$. Now, we collect terms in equation (2.31) proportional to $(p_1^+)^s$ and $(p_2^+)^s$ - each sum must vanish separately for the entire polynomial to vanish. We obtain

$$\gamma + \sum_{0\le k\le s \text{ odd}} \alpha_k u_k + \sum_{0\le k\le s \text{ even}} \alpha_k v_k = 0 \tag{2.34}$$

$$-\gamma - \sum_{0\le k\le s \text{ odd}} \alpha_k u_k + \sum_{0\le k\le s \text{ even}} \alpha_k v_k = 0 \tag{2.35}$$

Here, u_k and v_k are sums of products of coefficients of the hypergeometric function and the binomial expansion of $(p_1^+ + p_2^+)^{s-k}$; we do not care about their properties except that, with the signs indicated above, they are strictly positive, as can be verified by direct calculation. By adding and subtracting these equations, we obtain two separate equations that must be satisfied by the odd and even coefficients separately

$$\gamma + \sum_{0\le k\le s \text{ odd}} \alpha_k u_k = 0 \tag{2.36}$$

$$\sum_{0\le k\le s \text{ even}} \alpha_k v_k = 0 \tag{2.37}$$

Exactly analogously, we may do the same procedure to every other pair of monomials $(p+1^+)^a(p_2^+)^{s-a}$ and $(p_1^+)^{s-a}(p_2^+)^a$ to turn the constraints for the two monomials into constraints for the even and odd coefficients (where we're considering γ as an odd coefficient) separately. Hence, by multiplying each term by the monomial from which it was computed and then resumming, we find that the original identity (2.31) actually splits into two separate identities that must be satisfied. For the even terms, this identity is:

$$0 = \sum_{0\le k\le s \text{ even}} \alpha_k (p_1^+ + p_2^+)^{s-k}(p_2^+)^k {}_2F_1\left(2-\frac{d}{2}-k, -k, \frac{d}{2}-1, -\frac{p_1^+}{p_2^+}\right) \tag{2.38}$$

Then, we may again use the argument from section 2.3 to conclude that all $\alpha_k = 0$ for even k, which is what we wanted. Thus, $B' = 0$.

Now we would like to show that $B = \tilde{B}$. First of all, we will show that $\tilde{B}$ is nonzero. Consider the charge conservation identity

$$0 = \langle [Q_s, Bj_2] \rangle = \left(\partial_1^{s-1} + \partial_2^{s-1}\right) \left\langle \tilde{B} 2 \right\rangle + \langle B, [Q_s, 2] \rangle \tag{2.39}$$

Since $[Q_s, j_2] \supset \partial j_s$, and since $\langle Bs \rangle \neq 0$, the second term in that identity is nonzero, and so $\tilde{B}$ must be nonzero. Now we can normalize the currents in such a way that j_2 has the same overlap with $\tilde{B}$ and B. After normalization, we know that $B - \tilde{B}$ does not contain any spin 2 current because the stress tensor is unique, by hypothesis. Now, we will show that $B - \tilde{B}$ is zero by contradiction. Suppose $B - \tilde{B}$ is nonzero. Then, there is a current j_s whose overlap with $B - \tilde{B}$ is nonzero. Then, the charge conservation identity for the case $s > 2$ is

$$0 = \left\langle \left[Q_s, \left(B - \tilde{B}\right) j_2\right] \right\rangle, \tag{2.40}$$

$$0 = \gamma \left(\left(p_1^+\right)^{s-1} + \left(p_2^+\right)^{s-1} \right) F_2(p_1^+, p_2^+) + \sum_{k=0}^{s+1} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s+1-k} F_k\left(p_1^+, p_2^+\right), \tag{2.41}$$

where we assume that $\tilde{\alpha}_s \neq 0$. Then, we can again run the same analysis as section 2.3 to conclude that since $\tilde{\alpha}_s \neq 0$, we must have $\tilde{\alpha}_2 \neq 0$ - that is, j_2 has nonzero overlap with $B - \tilde{B}$, which is a contradiction. It means that $B - \tilde{B}$ has no overlap with any currents j_s for $s > 2$. The only possibility is to overlap only with spin zero currents. Suppose that there is a current j_0' that overlaps with $B - \tilde{B}$, where the prime distinguishes it from a spin 0 current j_0 that could appear in B. We first show that $\langle j_0 j_0' \rangle = 0$. Consider the charge conservation identity the action Q_4 on $\left\langle (B - \tilde{B}) j_0 \right\rangle$. The action of the charge is $[Q_4, 0] = \partial^3 j_0 + \partial j_2 + \ldots$, where the $\ldots$ represent terms that cannot overlap with $\underline{22}$ (from which B is constructed) or the even currents that appear in $\tilde{B}$. By hypothesis, $B - \tilde{B}$ has no overlap with j_2, so the identity simplifies to $\langle j_0 j_0' \rangle = 0$. Then, since j_0' is nonzero, it should have nontrivial overlap with some Q_s. Now, recall the fact that if a current j' appears (possibly with some number of derivatives) in the commutator of $[Q_s, j]$, then j appears in $[Q_s, j']$. Thus, there should be a current current of spin $s'' < s$ such that $[Q_s, j_{s''}] = j'_0 + \dots$. The action Q_s on $\left\langle \left(B - \tilde{B}\right) j_{s''} \right\rangle$ is

$$\left\langle \left[Q_s, \left(B - \tilde{B}\right) j_{s''}\right] \right\rangle = \partial_3^3 \left\langle \left(B - \tilde{B}\right) j'_0 \right\rangle + \partial \left\langle \left(B - \tilde{B}\right) j_2 \right\rangle, \tag{2.42}$$

Here, we have used that the action of Q_s on B and $\tilde{B}$ is identical because $B' = 0$. Then, since the second term is zero, thus the first term is equal to zero as well. Thus, $B - \tilde{B}$ has no overlap with any currents and is equal to zero, as desired.

In the fermionic case, we can run almost the same argument as in the bosonic case, except there is no discussion of a possible j_0, since there is no conserved spin zero current in the free fermion theory. We obtain the action of the charge on the fermionic quasi-bilocal is

$$[Q_s, F_-(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) F_-(\underline{x_1, x_2}). \tag{2.43}$$

In the tensor case, we again can repeat the argument to obtain

$$[Q_s, V_{--}(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) V_{--}(\underline{x_1, x_2}) \tag{2.44}$$

In this case, there is neither a conserved spin 0 or spin 1 current in the free tensor theory. The argument works the same way, however, if we consider j_3 instead of j_1 in the steps of the argument that requires it.

2.5 Quasi-bilocal fields: correlation functions

In this section, we will discuss how to precisely define the quasi-bilocal operators in a way that makes their symmetries manifest. In particular, each of the three bilocals will be bi-primary operators in some sense. This will allow us to argue that the correlation functions of the bilocals should agree with an appropriate corresponding free-field result. We will then explore what this implies for the full theory in section 2.6.

2.5.1 Symmetries of the quasi-bilocal operators

Let us first consider the case of the bosonic bilocal operator $B(x_1, x_2)$. Recall that, on the lightcone, the bilocals should imitate products of the appropriate free fields. In the bosonic free-field theory, the operator product expansion of $\phi(x_1)\phi^*(x_2)$ is composed of all of the even-spin currents of the theory with appropriate numbers of derivatives and factors of $(x_1 - x_2)$ so that the expression has the correct conformal dimension. More explicitly, we may write:

$$\phi(x_1)\phi^*(x_2) = \sum_{\text{even } s \geq 0} b_s^{\text{free}}(x_1, x_2) \tag{2.45}$$

$$b_s^{\text{free}}(x_1, x_2) = \sum_{(k,l)|s+l-k=0} c_{skl}(x_1 - x_2)^k \partial^l j_s\left(\frac{x_1 + x_2}{2}\right) \tag{2.46}$$

All the coefficients c_{skl} may be computed exactly in the free theory just by Taylor expansion. We have shown that all the currents j_s exist in our theory for all even s. So we may define an analogous quantity in our theory as follows:

$$B(x_1, x_2) = \sum_{\text{even } s \geq 0} b_s(x_1, x_2) \tag{2.47}$$

$$b_s(x_1, x_2) = \sum_{(k,l)|s+l-k=0} c'_{skl}(x_1 - x_2)^k \partial^l j_s\left(\frac{x_1 + x_2}{2}\right) \tag{2.48}$$

Here, the c' are some other coefficients which are to be determined by demanding that this definition of B coincide with the definition given on the lightcone in the previous section, i.e. that $\partial_1^2\partial_2^2 B(\underline{x_1, x_2}) = \underline{22}_b$. We claim that this can be accomplished by choosing the c' coefficients such that $\left\langle B(\underline{x_1, x_2}) j_s\right\rangle \propto \left\langle \underline{\phi(x_1)\phi^*(x_2)} j_s\right\rangle_{\text{free}}$. To see that there exists such a choice of c' which can achieve this condition, we explicitly compare $\langle B j_s\rangle$ and $\langle \phi\phi^* j_s\rangle_{\text{free}}$

term by term using 2.46 and 2.48. Each term in both of these correlation functions has the structure $(x_1 - x_2)^k \partial^l \langle j_{s'} j_s \rangle$ with coefficient $c_{s'kl}$ and $c'_{s'kl}$, respectively. Two-point functions of primary operators in CFT's are determined up to a constant, so each term is identical up to a possible scaling, which can be eliminated by choosing the c' coefficient appropriately. Then, by applying $\partial_1^2 \partial_2^2$ to both sides of $\left\langle B(\underline{x_1, x_2}) j_s \right\rangle \propto \left\langle \underline{\phi(x_1)\phi^*(x_2)} j_s \right\rangle_{\text{free}}$, we find that our definition coincides on the lightcone, as desired. This construction works the same way for the fermionic and tensor quasi-bilocals with analogous results, except that the quasi-bilocals in those cases carry the appropriate spin structure.

Since the conformal transformation properties of a conserved current j_s is theory-independent in the sense that it is completely fixed by its spin and conformal dimension, it is manifest from this definition that the bosonic quasi-bilocal $B(\underline{x_1, x_2})$ has the same transformation properties under the full conformal group as a product of free bosons. That is, it is a scalar bi-primary field with a conformal dimension of 1 with respect to each argument.

On the other hand, consider the fermionic and tensor quasi-bilocals F_- and V_{--}. The same line of reasoning tells us that they will transform like products of free fields contracted in a particular way: F_- will transform like $: \psi \gamma_- \bar{\psi} :$ does in the free fermionic theory, and V_{--} will transform like $: F_{-\{\alpha\}} F_-^{\{\alpha\}} :$ does in the theory of a free $\frac{d-2}{2}$-form[4]. These contractions, however, are not preserved by the full conformal group - the special conformal transformations orthogonal to the $-$ direction will ruin the structure of the Lorentz contraction. Thus, even in the free theory, these objects are not preserved by the full conformal group. They are only preserved by the so-called collinear conformal group generated by K_-, P_+, J_{+-}, and D, where K, P, J, and D are the generators of special conformal transformations, translations, boosts, and dilatations, respectively. It is clear from the structure of the conformal algebra that the commutation relations of this subset of conformal gen-

[4]Technically, the argument given above for the symmetries of the bosonic quasi-bilocal only works for even dimensions in the tensorial case since the free $\frac{d-2}{2}$-form exists only in even dimensions, so the matching procedure can't be carried out naively in odd dimensions. On the other hand, it is evident from the definition 2.17 that it has at least the collinear conformal symmetry since there are no derivatives to be "integrated out."

erators close, so it forms a proper subalgebra. Thus, what we are allowed to conclude is that F_- and V_{--} are bi-primary operators with respect to this collinear subgroup, not the conformal group. Nevertheless, this will still be enough symmetry for our purposes.

The key fact which is still true for this more restricted set of symmetries is that under K_-, the special conformal transformation in the $-$ direction, the n-point function of fermionic and tensor quasi bi-primaries should scale separately in each variable. That is, under K_-, if $x \to x'$ and $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$, we have

$$\begin{aligned} &\left\langle F_-(\underline{x'_1, x'_2}), \cdots, F_-(\underline{x'_{2n-1}, x'_{2n}}) \right\rangle \\ &\qquad = \Omega(x_1)^{d/2-1} \ldots \Omega(x_{2n})^{d/2-1} \left\langle F_-(\underline{x_1, x_2}), \cdots, F_-(\underline{x_{2n-1}, x_{2n}}) \right\rangle \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} &\left\langle V_{--}(\underline{x'_1, x'_2}), \cdots, V_{--}(\underline{x'_{2n-1}, x'_{2n}}) \right\rangle \\ &\qquad = \Omega(x_1)^{d/2-1} \ldots \Omega(x_n)^{d/2-1} \left\langle V_{--}(\underline{x_1, x_2}), \cdots, V_{--}(\underline{x_{2n-1}, x_{2n}}) \right\rangle \end{aligned} \tag{2.50}$$

The proof of these two statements is given in appendix A.4.

2.5.2 Correlation functions of the bosonic quasi-bilocal

Now we will discuss the structure of the n-point functions of the quasi-bilocals. Again, let's begin with the bosonic case. We wish to constrain $\left\langle B(\underline{x_1, x_2}) \ldots B(\underline{x_{2n-1}, x_{2n}}) \right\rangle$. We established that $B(x_1, x_2)$ has the transformation properties of a product of two free fields under the full conformal group - i.e. it is a bi-primary field with dimension $\frac{d-2}{2}$ in each variable. That means that the n-point function can only depend on distances between coordinates d_{ij} and have conformal dimension $\frac{d-2}{2}$ with respect to each variable. Since x_1 and x_2 are lightlike separated, d_{12} cannot appear, and similarly for every pair of arguments of the same bilocal. There is also a permutation symmetry: B is symmetric in its two arguments, and the n point function must be symmetric under interchange of any pair of the identical B's. Finally, there is the higher-spin symmetry. In the bosonic case, the charge conservation identity (2.24) imposes the simple relation

$$\sum_{i=1}^{2n} \partial_i^{s-1} \left\langle B(\underline{x_1, x_2}) \ldots B(\underline{x_{2n-1} x_{2n}}) \right\rangle , \quad \text{for all even s} \tag{2.51}$$

As shown in appendix E of [12], this fixes the x^- dependence of the n-point function to have the particular form:

$$\sum_{\sigma \in S^{2n}} g_\sigma \left(x^-_{\sigma(1)} - x^-_{\sigma(2)}, x^-_{\sigma(3)} - x^-_{\sigma(4)}, \ldots, x^-_{\sigma(2n-1)} - x^-_{\sigma(2n)} \right) \tag{2.52}$$

where S^{2n} is the set of permutations of $2n$ elements. The point is that the x_i^- dependence of the n-point function is constrained such that, for each g_σ, x_i^- can only appear in a difference with one and only one other coordinate. This is a very strong constraint. The conformal symmetry tells us that each g_σ in the above series can be written as a product of a dimensionful function of distances with the correct dimension in each variable times a smooth, dimensionless function of conformal cross-ratios. The constraint on the functional form of g_σ, however, forbids all such functions except the trivial function 1, because each cross ratio separately violates the constraint.

Putting it all together, we conclude that the n-point function has to be proportional to a sum of terms with equal coefficients, each of which is a product $\prod d_{ij}^{-(d-2)}$, where the product has n terms corresponding to some partition of the $2n$ points into pairs where no pair contains two arguments of the same bilocal. For example, the two-point function is:

$$\left\langle B(\underline{x_1, x_2}) B(\underline{x_3, x_4}) \right\rangle = \tilde{N}_b \left(\frac{1}{d_{13}^{d-2} d_{24}^{d-2}} + \frac{1}{d_{14}^{d-2} d_{23}^{d-2}} \right) , \tag{2.53}$$

where $\tilde{N}_b$ is a constant of proportionality. One immediately notes that the expressions one obtains this way for all n-point functions of the quasi-bilocals are proportional to the n-point function of : $\phi(x_1)\phi(x_2)$: in a theory of free bosons.

2.5.3 Correlation functions of the fermionic and tensor quasi-bilocal

In the fermionic and tensor cases, we claim that the correlation functions of the quasi-bilocals still coincide with the correlation functions of the corresponding free field theories, despite the fact that the fermionic and tensor quasi-bilocals have less symmetry than the bosonic quasi-bilocal. The argument, however, is somewhat more complicated due to the reduced amount of symmetry. The proof is essentially the same for both the fermionic and tensor cases, so we will only present the argument for the tensor case. Our general strategy will be to compare the constraints that one obtains from the definition of V_{--} as the lightcone limit $\underline{22}_t$ with the constraints one obtains from the symmetries of V_{--} as established by its definition away from the lightcone given at the beginning of this section. In the bosonic case, we only used the latter, but in the fermionic case and tensor case, we will need the former as well.

First, we consider what the $2n$-point function of T_{--} is away from the lightcone. We know from the definition of $V_{--} = \underline{22}_t$ that if we take n lightcone limits of this $2n$ point function in each pair of adjacent arguments $(x_1, x_2), (x_3, x_4), \ldots (x_{2n-1}, x_{2n})$, we will obtain the n point function of $V_{--}(\underline{x_1, x_2})$. It may be the case that the definition of V_{--} given earlier as a sum of currents and descendants (with appropriate derivatives and powers of x) will yield a different result away from the lightcone, but nevertheless, it must agree with the $2n$-point function of T_{--} in the lightcone limit.

Generically, the $2n$ point function of T_{--} with arguments in arbitrary locations can be decomposed as a polynomial in some basis of conformally invariant structures. One convenient basis is the $\{H_{ij}, V_i\}$ space defined in [71]. In this basis, we may write

$$\langle T_{--}(x_1)\cdots T_{--}(x_{2n})\rangle = \frac{\langle\langle T_{--}(x_1)\cdots T_{--}(x_{2n})\rangle\rangle}{d_{12}^{d-2}d_{23}^{d-2}\ldots d_{2n-1,2n}^{d-2}d_{2n,1}^{d-2}} \tag{2.54}$$

where

$$\langle\langle T_{--}(x_1)\cdots T_{--}(x_{2n})\rangle\rangle = \sum_i f_i(\{u_j\})\left(\prod_{k<l} H_{kl}^{h_i^{(k,l)}}\right)\left(\prod_{k<l<m} V_{k,lm}^{v_i^{(k,lm)}}\right) \tag{2.55}$$

where $f_i(\{u_j\})$ is an arbitrary function of cross-ratios $\{u_j\}$, the h_{kl} and v_i coefficients satisfy

$$\sum_{l,m|k<l<m} v_i^{(k,lm)} + \sum_{n|k<n} h_i^{(k,n)} = 2 \text{ for all } i,k \tag{2.56}$$

and the conformal invariants are

$$V_{k,lm} = \frac{x_{kl}^+}{d_{kl}^2} + \frac{x_{km}^+}{d_{km}^2} \tag{2.57}$$

$$H_{kl} = \frac{-2(x_{kl}^+)^2}{d_{kl}^4} \tag{2.58}$$

Note that this decomposition omits structures which contain the epsilon tensor, which all vanish in our formalism because we contract all free indices with the same polarization vector in the $-$ direction.

We would like to understand the properties of this decomposition under the tensor lightcone limit 2.3. First, note that the universal dimensionful factor of distances that is factored out of $\langle\langle T\ldots T\rangle\rangle$ in 2.54 is conventional. In principle, one could choose it to be something different and compensate by appropriate redefinitions of f_i. We have chosen it as shown in order to simplify the structure of this function under the lightcone limit. More precisely, the distances corresponding to pairs of points that become lightlike separated d_{12}, d_{34}, …, $d_{2n-1,2n}$ vanish in the lightcone limit, so they cannot explicitly appear in the correlation function, and we have chosen the universal factor so that this property is manifest. To see this, note that when we take the lightcone limit 2.3 of this general structure, the part of this universal factor corresponding to the distances between points that become lightlike separated - i.e. $d_{12}^{-d+2}d_{34}^{-d+2}\ldots d_{2n-1,2n}^{-d+2}$ - becomes $d_{12}^{4}d_{34}^{4}\ldots d_{2n-1,2n}^{4}$. This residual factor is exactly canceled out by the V and H terms corresponding to the x^+ factors stripped away in 2.3. To see this, recall that the light-cone limits of correlation functions are well-defined and non-divergent[5], so any structure consistent with conformal symmetry needs to appear with enough V's and H's with appropriate indices to cancel out the factors of $(x_{12}^+)^{-2}, (x_{34}^+)^{-2}, \ldots, (x_{2n-1,2n}^+)^{-2}$ that appear in the lightcone limit. As noted earlier, these factors of V's and H's come with exactly two powers each of $d_{12}^2, \ldots, d_{2n-1,2n}^2$, which is exactly what is needed to cancel out the residual term.

Thus, after the lightcone limit, the most general structure that can appear in the n-point function of V_{--} is:

$$\Big\langle V_{--}(\underline{x_1,x_2})\ldots V_{--}(\underline{x_{2n-1},x_{2n}})\Big\rangle = \Big\langle \underline{T_{--}(x_1)T_{--}(x_2)}_t \cdots \underline{T_{--}(x_{2n-1})T_{--}(x_{2n})}_t\Big\rangle \tag{2.59}$$

$$= \sum_i \frac{f_i(\{u_j\})}{d_{23}^{d-2}\ldots d_{2n,1}^{d-2}} \prod_{k,l}\left(\frac{x_{kl}^+}{d_{kl}^2}\right)^{c_{kl}} \tag{2.60}$$

where the product over k and l is understood to be restricted to pairs (k,l) not corresponding to x_k, x_l lightlike separated, and $\sum c_{kl} = 2n$.

We can determine which terms of this form are consistent with the symmetries of V_{--}. Consider the n-point correlation function of V_{--}. Its transformation properties under Lorentz transformations and dilatations tell us that we must have $2n$ $+$ indices in the numerator of the correlation function and that the overall scaling dimension of the n-point function should be $2n \times d/2 = dn$. Then, as mentioned before, since V_{--} is a bi-primary under the collinear conformal group, the n-point function should scale appropriately in

[5]As we remarked before, this is only true a priori if we subtract off the bosonic and fermionic pieces, but we will show in section 2.6 that if any one of the three lightcone limits are nonzero, it follows that the other two are zero, so this subtraction procedure is not actually necessary.

each variable separately after acting with K_- according to 2.50. In order to satisfy this constraint, for each independent structure appearing in the correlation function and each index i, we must have 2 factors of x^+_{ij} in the numerator (not necessarily the same j for each of the 2 factors) and $d+2$ powers of d_{ik} in the denominator for some k (again, not necessarily the same k for each of the $d+2$ factors). Once we have picked such a denominator, there is still some ambiguity since conformally invariant functions f_i can still appear after imposing this constraint (since they are fixed by K_-), and such functions can change the denominator. What is tightly constrained here is the numerator - i.e. the spin structure. "Imbalanced" structures with that would otherwise be allowed by Lorentz symmetry, scaling symmetry, and permutation symmetry cannot appear. For example, for the two-point function $\langle V_{--}V_{--}\rangle$ in four dimensions, structures such as $\frac{(x^+_{13})^4}{d^{12}_{13}} + \frac{(x^+_{24})^4}{d^{12}_{24}}$ do not satisfy 2.50. Note that the numerators which are allowed by this constraint are precisely the ones that appear in free-field correlation functions (i.e. the ones arising from Wick contractions of free fields) and no others.

Now, let's impose the higher-spin constraint, which stipulates that the correlation function must be a sum of terms g_σ which have the functional form given by 2.52. Since that constraint only involves the dependence in the x^- direction, it does not constrain the numerator, which involves only terms involving the x^+_i variables. However, it does restrict the denominator to only have each index i involved in a power d_{ik} for one specific k since d_{ik} does depend nontrivially on x^-_{ik}. That is, the denominator is built out of terms like d^{d+2}_{ik}. This constrains the f_i powerfully. Since each cross ratio separately violates the higher spin constraint, the only f_i that can appear are the ones whose product with a denominator satisfying the higher-spin constraint is another denominator satisfying that constraint. That is, once we have picked a denominator, the f_i can only be very specific kinds of rational functions. We can still generate terms that don't appear in the free-field result, however, because the spin structure in the numerator doesn't have to match the index structure of the denominator. For example, the following structure could in principle appear in the four-point function of V_{--}, but obviously this structure is not generated in the free theory:

$$\frac{(x_{14}^{+})^2(x_{27}^{+})^2(x_{36}^{+})^2(x_{58}^{+})^2}{(d_{13}d_{24}d_{57}d_{68})^{d+2}} \tag{2.61}$$

This structure has a numerator which is consistent with free field theory but a denominator that does not match the result one would obtain from the free field propagator. Another possibility is to write a structure where the numerator corresponds to the connected part of the free-field correlator - i.e. the two factors of x_{ij}^{+} appear with different j for some i.

$$\frac{x_{13}^{+}x_{32}^{+}x_{28}^{+}x_{86}^{+}x_{67}^{+}x_{75}^{+}x_{54}^{+}x_{41}^{+}}{(d_{13}d_{24}d_{57}d_{68})^{d+2}} \tag{2.62}$$

Purely on symmetry considerations, these terms are consistent with the general structure 2.60. Indeed, one can set 2.61 and 2.62 equal to 2.60 to explicitly solve for the function $f_i(\{u_j\})$ that generates it, and one can check that this f_i is indeed conformally invariant, as required. These structures are inconsistent, however, with cluster decomposition. To see this, we examine the tensor analogue of 2.48:

$$V_{--}(x_1, x_2) = \sum_{\text{even } s\geq 2} v_{--}^{s}(x_1, x_2) \tag{2.63}$$

$$v_{--}^{s}(x_1, x_2) = \sum_{k,l} c_{kl}(x_1 - x_2)^k \partial^l j_s\left(\frac{x_1 + x_2}{2}\right) \tag{2.64}$$

Comparing the conformal dimension of the left and right hand side yields the constraint that $s + l - k = 2$. Hence, by setting $x_1 = x_2$, we extract the $k = 0$ piece, forcing $l = 0$ and $s = 2$ (since $s = 1$ is not realized in the tensor sector). That is, $V_{--}(x, x) = T_{--}(x)$. By performing this projection on each factor of V_{--} in the correlation function (i.e. setting $x_1 = x_2$, $x_3 = x_4$, etc.), we obtain an expression for the n-point function of T_{--}, which we know must satisfy cluster decomposition since T is a local operator. Then, by taking the points to be separated very far apart from each other, we obtain constraints on how the structures must simplify. For example, we know that if we take x_1 and x_3 to be very far from all the other points, we must have that

$$\langle T_{--}(x_1)T_{--}(x_3)\dots T_{--}(x_{2n-1})\rangle \Longrightarrow \langle T_{--}(x_1)T_{--}(x_3)\rangle \langle T_{--}(x_5)\dots T_{--}(x_{2n-1})\rangle \tag{2.65}$$

This factorization property is not satisfied by the structure 2.61, for example. Indeed, the only way to satisfy all such constraints arising from cluster decomposition is to have all powers of x_{ij}^+ appear with the corresponding factor of d_{ij}^{d-2} in the denominator, modulo trivial equalities such as $x_{13}^+ = x_{14}^+$ (which arise since points which are taken to be $-$ separated in the lightcone limit have the same difference in the $+$ direction). If it appears with the wrong d_{ij} factor in the denominator (again, modulo the trivial relabeling of the spin structure), it cannot satisfy the cluster decomposition identity arising from taking the two points appearing in that factor to be very far from all the other points. The spin structure required by the factorization will simply not be present.

Hence, the only allowed terms are the ones that are built from free-field propagators $(x_{ij}^+)^2/d_{ij}^{d+2}$. Permutation symmetry implies that the coefficients of all the structures that can appear are the same up to disconnected terms which are fixed, as before, by cluster decomposition. This implies that the n-point function of bilocals V_{--} are exactly the same as the n-point function of stress tensors in free field theory up to a possible overall constant.

Clearly, this entire argument works for the fermionic case as well with only minor modifications - the projection procedure that isolates the contribution from the stress tensor is slightly more complicated since it appears at first order, not zeroth order, in x_{12} in the fermionic analogue of 2.64, and the correlation function is permutation anti-symmetric instead of symmetric because fermions anticommute. All other steps are the same, and we conclude that in the fermionic case, the n-point functions of bilocals are also given by the free field result. For example, the two-point functions of fermionic and tensor quasi-bilocals are given by

$$\left\langle F_-(\underline{x_1, x_2})F_-(\underline{x_3, x_4})\right\rangle = \tilde{N}_f \left(\frac{x_{13}^+ x_{24}^+}{d_{13}^d d_{24}^d} - \frac{x_{14}^+ x_{23}^+}{d_{14}^d d_{23}^d}\right) \tag{2.66}$$

$$\left\langle V_{--}(\underline{x_1, x_2})V_{--}(\underline{x_3, x_4})\right\rangle = \tilde{N}_t \left(\frac{(x_{13}^+)^2 (x_{24}^+)^2}{d_{13}^{d+2} d_{24}^{d+2}} + \frac{(x_{14}^+)^2 (x_{23}^+)^2}{d_{14}^{d+2} d_{23}^{d+2}}\right) \tag{2.67}$$

where $\tilde{N}_f$ and $\tilde{N}_t$ are overall constants that we will presently analyze.

2.5.4 Normalization of the quasi-bilocal correlation functions

Now, let's fix the the overall constants $\tilde{N}_b$, $\tilde{N}_f$, and $\tilde{N}_t$ in front of each n-point function. We claim that they all are fixed by the normalization of the two-point function of the bilocals. This can be seen by considering how one can obtain the n-point function of quasi-bilocals from the $n-1$ point function. We know the n-point function of some quasi-bilocal $\mathcal{A}$ is:

$$\underbrace{\langle \mathcal{A} \ldots \mathcal{A} \rangle}_{n \text{ copies of } \mathcal{A}} = \tilde{N}_n g(d_{ij}) \tag{2.68}$$

where g is some known function which agrees with the result for the n-point function of the corresponding free theory bilocal. Each bilocal contains the stress tensor j_2 in its OPE, so we can consider acting on both sides with the projector P which isolates the contribution of j_2 from the first bilocal. We have already seen, for example, that for the tensor bilocal, this projector just sets $x_1 = x_2$. Then, we can integrate over the coordinate x_1. This yields the action of the dilatation operator on the $n-1$ point function, whose eigenvalue will be some multiple of the conformal dimension of the appropriate free field. So by this procedure, we can fix the coefficient in front of the n-point function in terms of the $n-1$ point function. So by recursion, all the coefficients of the correlation functions are fixed by the coefficient $\tilde{N}$ appearing in front of the two-point function.

2.6 Constraining all the correlation functions

We have shown now that the n-point functions of all the quasi-bilocal fields exactly coincide with the corresponding free-field result for a theory of N free fields of appropriate spin for some N (which we will show later must be an integer). Now, we will explain how to use these facts to constrain all the other correlation functions of the theory. We will start by proving that the three point function $\langle 222 \rangle$ must be either equal to the result for a free boson, a free fermion, or a free $\frac{d-2}{2}$ form. That is, if we write the most general possible form:

$$\langle 222 \rangle = c_b \langle 222 \rangle_{\text{free boson}} + c_f \langle 222 \rangle_{\text{free fermion}} + c_t \langle 222 \rangle_{\text{free tensor}}, \tag{2.69}$$

then the result will be consistent with higher-spin symmetry only if $(c_b, c_f, c_t) \propto (1,0,0)$ or $(0,1,0)$ or $(0,0,1)$.

We first show that if $\langle \underline{22}_b 2 \rangle \neq 0$ then $\langle \underline{22}_f 2 \rangle = 0 = \langle \underline{22}_t 2 \rangle$. Consider the action of Q_4 on $\langle \underline{22}_b 2 \rangle$. By exactly the same analysis as the charge conservation identities of section 2.3, we obtain exactly the same expression as equation (2.11), except the summation starts from $j = 0$. Thus, the existence of the spin 4 current implies the existence of a spin 0 current with $\langle \underline{22}_b 0 \rangle \neq 0$. The action of charge Q_4 on j_0 is

$$[Q_4, j_0] = \partial^3 j_0 + \partial j_2 + \text{no overlap with } \underline{22}_b \tag{2.70}$$

Now consider the charge conservation identities arising from the action of Q_4 on $\langle \underline{22}_f 0 \rangle$ and $\langle \underline{22}_t 0 \rangle$. Since $\langle \underline{22}_f 0 \rangle = 0 = \langle \underline{22}_t 0 \rangle$, we conclude $\langle \underline{22}_f 2 \rangle = 0 = \langle \underline{22}_t 2 \rangle$, as desired.

Now, assume that $\langle \underline{22}_b 2 \rangle = 0$. It suffices to show that if $\langle \underline{22}_t 2 \rangle \neq 0$, then $\langle \underline{22}_f 2 \rangle = 0$. In this case, by hypothesis, the quasi-bilocal V_{--} is nonzero. The results of the previous section tell us that the three point function of the tensor quasi-bilocal is proportional to:

$$\left\langle V_{--}(\underline{x_1, x_2}) V_{--}(\underline{x_3, x_4}) V_{--}(\underline{x_5, x_6}) \right\rangle \propto \frac{\left(x_{13}^+\right)^2 \left(x_{25}^+\right)^2 \left(x_{46}^+\right)^2}{d_{13}^{\frac{d}{2}+2} d_{25}^{\frac{d}{2}+2} d_{46}^{\frac{d}{2}+2}} + perm. \tag{2.71}$$

and this precisely coincides with the three-point function of the free field operator $v_{--}(x_1, x_2) =: F_{-\{\alpha\}}(x_1) F_{-\{\alpha\}}(x_2) :$

$$\langle V_{--}(x_1,x_2) V_{--}(x_3,x_4) V_{--}(x_5,x_6) \rangle \propto \langle v_{--}(x_1,x_2) v_{--}(x_3,x_4) v_{--}(x_5,x_6) \rangle \quad (2.72)$$

Now, take x_1 and x_2 very close together and expand both sides of this equation in powers of $(x_1 - x_2)$. The zeroth order term of v is clearly the normal ordered product $: F_{-\alpha}(\frac{x_1+x_2}{2}) F_{-\alpha}(\frac{x_1+x_2}{2}) :$ - this is precisely the free field stress tensor. On the other hand, we know from the previous section that the term in V_{--} which is zeroth order in $(x_1 - x_2)$ - i.e. the term that arises from setting $x_1 = x_2$, is just the stress tensor of the theory T_{--}. Repeating the same procedure for the pairs of coordinates (x_3, x_4) and (x_5, x_6), we obtain the desired result:

$$\langle 222 \rangle = \langle 222 \rangle_{\text{free tensor}} \quad (2.73)$$

$$\Rightarrow \left\langle \underline{22}_f 2 \right\rangle = \langle \underline{22}_b 2 \rangle = 0 \quad (2.74)$$

as required. Therefore, since the stress-energy tensor is unique,

$$\langle 222 \rangle_b \neq 0 \Rightarrow \quad \langle 222 \rangle_f = 0, \quad \langle 222 \rangle_t = 0, \quad \underline{j_2 j_2}_b = \sum_{k=0}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_f = 0, \quad \underline{j_2 j_2}_t = 0, \quad (2.75)$$

$$\langle 222 \rangle_f \neq 0 \Rightarrow \quad \langle 222 \rangle_b = 0, \quad \langle 222 \rangle_t = 0, \quad \underline{j_2 j_2}_f = \sum_{k=1}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_b = 0, \quad \underline{j_2 j_2}_t = 0, \quad (2.76)$$

$$\langle 222 \rangle_t \neq 0 \Rightarrow \quad \langle 222 \rangle_b = 0, \quad \langle 222 \rangle_f = 0, \quad \underline{j_2 j_2}_t = \sum_{k=1}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_b = 0, \quad \underline{j_2 j_2}_f = 0, \quad (2.77)$$

where square brackets denotes currents and their descendants. This establishes the claim that the three-point function of the stress tensor coincides with the answer for some free theory.

At this point, we would like to stress that the factorization property we have proven here holds only for conformal field theories that satisfy the unitarity bound for the dimensions of operators. Clearly, all unitary CFT's have this property, but it is possible to conceive of non-unitary CFT's which also satisfy it. Without the unitarity bound's constraint on operator dimensions, however, various operators we have not considered could appear in all the charge conservation identities we have written. These operators make it possible to construct theories where the three-point function of the stress tensor decomposes as a nontrivial superposition of the bosonic, fermionic, and tensor sectors. For example, we show in Appendix A.6 that the free five-dimensional Maxwell field is a non-unitary conformal field theory whose stress tensor decomposes into a superposition of all three sectors.

Returning to the main argument, we may now obtain all the other correlation functions, we may expand equation (2.72) to higher orders in $x_1 - x_2$, and use the correlation functions obtained at lower orders to fix the ones that appear at higher orders. For example, at second order in $x_1 - x_2$, we have:

$$v_{--} = (x_1 - x_2)^2 \Bigg(: \partial^2 F_{-\alpha}\left(\frac{x_1 + x_2}{2}\right) F_{-\alpha}\left(\frac{x_1 + x_2}{2}\right) : \\ + : \partial F_{-\alpha}\left(\frac{x_1 + x_2}{2}\right) \partial F_{-\alpha}\left(\frac{x_1 + x_2}{2}\right) : \Bigg), \quad (2.78)$$

and V_{--} contains terms involving only the spin 2, 3, and 4 currents. Using our answers for $\langle 222 \rangle$ and our knowledge that $\langle 223 \rangle = 0$, we can then fix $\langle 224 \rangle$ to agree with the free field theory. This procedure recursively fixes all the correlators in the free tensor sector. The argument flows identically for the free bosonic and free fermionic sectors, except that the zeroth order term will not fix $\langle 222 \rangle$, but some lower-order current. For example, in the bosonic theory, the zeroth order term will fix $\langle 000 \rangle$, and one will need to carry out the power series expansion to higher orders in order to fix the correlators of the higher-spin conserved currents.

Then, one could consider correlation functions that have indices set to values other than a minus. This works in exactly the same way since the operator product expansion of two currents with minus indices will contain currents with other indices. This has the effect of doubling the number of bilocals required to build a correlation function since we need to take an extra OPE to fix the index structure. Thus, an n-point function with non-minus indices can be fixed from $2n$ bilocals. Thus, we have fixed every correlation function from currents at appearing in successive OPE's of two stress tensors, including those of every higher-spin current.

The last thing we will argue is that the normalization of the correlation functions matches the normalization for some free theory. For example, in the theory of N free bosons, the two-point function of $\sum_{i=1}^{N} : \phi_i \phi_i^* :$ will have overall coefficient N. The same is true for the fermionic and tensor cases. One might wonder if the overall coefficient $\tilde{N}$ of the quasi-bilocal could be non-integer, which would imply that it could not coincide with any theory of N free bosons. We will now argue that this is not possible. We start with the bosonic case, which works similarly to the argument presented in [12]:

In a theory of N free bosons, consider the operator

$$\mathcal{O}_{q,free} = \delta^{[i_1,\ldots,i_q]}_{[j_1,\ldots,j_q]} (\phi^{i_1} \partial \phi^{i_2} \ldots \partial^{q-1} \phi^{i_q})(\phi^{j_1} \partial \phi^{j_2} \ldots \partial^{q-1} \phi^{j_q}) \tag{2.79}$$

Here, δ is the totally antisymmetric delta function that arises from a partial contraction of ϵ symbols:

$$\delta^{[i_1,\ldots,i_q]}_{[j_1,\ldots,j_q]} \propto \epsilon^{i_1 \ldots, i_q, i_{q+1} \ldots i_N} \epsilon_{j_1 \ldots, j_q, i_{q+1} \ldots i_N} \tag{2.80}$$

We claim that in the full theory, there exists an operator $\mathcal{O}_q$ whose correlation functions coincide with the correlation functions of $\mathcal{O}_{q,free}$ in the free theory. The proof of this is given in Appendix A.5.

Consider the norm of the state that $\mathcal{O}_q$ generates. This is computed by the two point function $\langle\mathcal{O}_q\mathcal{O}_q\rangle$. It is obvious from the definition of $\mathcal{O}_q$ that it arises from the contraction of q bilocal fields, so this correlator is a polynomial in N of order q. The antisymmetry of the totally antisymmetric function in the definition of $\mathcal{O}_{q,free}$ enforces that the correlation function vanishes at $q > N$. So we know all the roots of the polynomial, and hence the correlation function is proportional to $N(N-1)\dots(N-(q-1))$. Now, consider an analytic continuation of this correlator to non-integer $\tilde{N}$. By taking $q = \lfloor N\rfloor + 2$, we find that this product is negative, which is impossible for the norm of a state. Since the correlators of $\mathcal{O}_q$ are forced to agree with the correlators of some operator in the full CFT, we conclude that the normalization $\tilde{N}$ of the scalar quasi-bilocals must be an integer.

The same argument can be ran in the tensor case for an operator defined similarly:

$$\mathcal{O}_q = \delta^{[i_1,\dots,i_q]}_{[j_1,\dots,j_q]}(F^{i_1}_{-\{\alpha_1\}}\partial F^{i_2}_{-\{\alpha_2\}}\dots\partial^{q-1}F^{i_q}_{-\{\alpha_q\}})(F^{j_1}_{-\{\alpha_1\}}\partial F^{j_2}_{-\{\alpha_2\}}\dots\partial^{q-1}F^{j_q}_{-\{\alpha_q\}}) \quad (2.81)$$

We again conclude that the normalization constant $\tilde{N}$ must be an integer.

The construction in the fermionic case is somewhat simpler. We know j_2 appears in F_-, and we can define an operator $\mathcal{O}_q = (j_2)^q$ by extracting the term in the operator product expansion of q copies of j_2 whose correlation functions coincide with the free fermion operator $(j_2)^q_{\text{free}}$. In the theory of N free fermions, $j_2 = \sum_i(\partial\psi_i)\gamma_-\bar{\psi}_i - \psi_i\gamma_-(\partial\bar{\psi}_i)$, where here i is the flavor index for the N fermions. By antisymmetry of the fermions, we know that $\mathcal{O}_q$ will be zero if $q \geq N$. Then, as in the bosonic case, we can consider the norm of the state that $\mathcal{O}_q$ generates, which is computed by $\langle\mathcal{O}_q\mathcal{O}_q\rangle$, and the rest of the argument runs as before. Thus, the normalization $\tilde{N}$ of the fermionic bilocals must be an integer.

It is worth noting the relationship between this result and one of the primary motivations for studying higher-spin CFT's - holographic dualities involving Vasiliev gravity in an anti-de Sitter space. As mentioned earlier, it has been conjectured that Vasiliev gravity is conjectured to be dual to a theory of N free scalar fields in the $O(N)$ singlet sector. This implies a relationship between the vacuum energy of Vasiliev gravity at tree-level and the free energy of a scalar field, namely, that $F_{\text{Vasiliev}}/G_N \sim NF_{\text{scalar}}$, where G_N is the Newton constant. Our result implies that this normalization constant N, and therefore, the Newton constant G_N is quantized in the Vasiliev theory in any dimension.

It must be noted, however, that we cannot claim that this quantization can be seen perturbatively in N. Recent work of Giombi and Klebanov [72] have shown that the one-loop correction to the vacuum energy of minimally coupled type A Vasiliev gravity in anti-de Sitter background does not vanish as expected. This was interpreted as a shift of $N \to N-1$ in the tree-level calculation of the vacuum free energy. Our result cannot predict such a shift or any other $1/N$ corrections that appear in higher orders in perturbation theory. We claim only that the exact result, after summing all loop corrections, must be quantized.

2.7 Discussion and conclusions

In this chapter, we have shown that in a unitary conformal field theory in $d > 3$ dimensions with a unique stress tensor and a symmetric conserved current of spin higher than 2, the three-point function of the stress tensor must coincide with the three-point function of the stress tensor in either a theory of free bosons, a theory of free fermions, or a theory of free $\frac{d-2}{2}$-forms. This implies that all the correlation functions of symmetric currents of the theory coincide with the those in the corresponding free field theory.

Our technique was to use a set of appropriate lightcone limits to transform the data of certain key Ward identities into simple polynomial equations. Even though we could not directly solve for the coefficients in these identities like in three dimensions, we were nevertheless able to show that the only solution these Ward identities admit is the one furnished by the appropriate free field theory. This was the key step that allowed us to defined bilocal operators which were used to show that the three-point function of the stress tensor must agree with a free field theory. This, in turn, fixed all the other correlators of the theory to agree with those in the same free field theory. These results can be understood as an extension of the techniques and conclusions of [12] from three dimensions to all dimensions higher than three.

We stress that our classification into the bosonic, fermionic, and tensor free field theories depends somewhat sharply on our assumption that a unique stress tensor exists. Other free field theories with higher spin symmetry exist in $d > 3$ dimensions, such as a theory of free gravitons. This theory, however, does not have a stress tensor, and we make no statement about how the correlation functions of such theories are constrained, and analogously for theories with many stress tensors. On the other hand, we may consider the possibility of multiple stress tensors. It was argued in [12] that the result holds if there are two stress tensors $\{j_2, \hat{j}_2\}$ instead of just one. The idea is that one can define a basis $\{J_2, \hat{J}_2\}$ of vectors $\{j_2, \hat{j}_2\}$ such that each vector generates action along itself. As a result correlation function of n currents J_2 and n currents $\hat{J}_2$ factorizes into product of correlation function of n currents J_2 and correlation function of n currents $\hat{J}_2$. So one can argue that in some sense theories are dynamically decoupled. This argument carries over to our result totally unchanged, and so our result also holds in the case of two stress tensors. In the case of more than two stress tensors, one has to find an analogous basis. We do not comment on the possibility of more than two stress tensors.

In [47], it was shown that for $3d$ conformal theories with slightly broken higher-spin symmetry it is possible to constrain the three point functions in the leading order in large parameter N. The similar analysis for D dimensional case is out of the scope of the present chapter. But we expect that similar conclusion should be valid.

Moreover, we have not computed correlation functions or commutators for asymmetric currents and charges. In [62], it was shown that if one considers the possible algebras of charges in theories that contain asymmetric currents in four dimensions, a one-parameter family of algebras exists. This may suggest the existence of nontrivial higher-spin theories, though our result indicates that at least the subalgebra generated by the symmetric currents must agree with free field theory.

We also stress that the tensor structure is not well understood in all dimensions. In even dimensions, it corresponds to the theory of a free $\frac{d-2}{2}$-form field, which does not exist in odd dimensions. In odd dimensions, the structure may not exist, and even it does, there may not exist a conformal field theory which realizes it. Our argument only tells us that if there is a solution of the conformal and higher-spin Ward identities corresponding to this structure, then it is unique. If the structure exists, we only know for a fact that it contains an infinite tower of higher-spin currents for $d \geq 7$ and in this case, the theory, if it exists, has the correlation functions we claimed. In $d = 5$, it is not known if all the higher-spin currents must be present. Assuming they are present, our results also flow through in $d = 5$. Even then, the tensor structure in odd dimensions could fail to have a good microscopic interpretation for many other reasons. For example, the four-point function of the stress tensor in this sector may not be consistent with the operator product expansion in the sense that it may not be decomposable as a sum over conformal blocks - i.e. it may be possible to continue all the correlation functions to odd dimensions, but not the blocks. We have not explored this question.

3

Anisotropy of gravitational waves

"Every moment is a fresh beginning."

Thomas Stearns Eliot

3.1 Introduction

This whole chapter is completely based on paper [60], coauthored with Juan Maldacena.

The theory of inflation predicts a stochastic background of gravity waves, which were produced by quantum effects [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. To leading order, this is a statistically homogeneous and isotropic background. However, at higher orders, we expect some influence from the scalar fluctuations. In the same way that the cosmic microwave radiation is not isotropic but has small anisotropies of order 10^{-5}, the gravity wave background is also not isotropic but has small anisotropies of the same order, also due to the scalar fluctuations. At relatively large angles, which correspond to scalar fluctuations entering the horizon during matter domination, the main effect is due to the Sachs-Wolfe effect [16]. Namely, we start locally with the same spectrum of gravity waves, which are then red or blue shifted by an amount proportional to the amplitude of the scalar fluctuations. The overall observable effect then depends on the frequency dependence of the spectrum of the gravity waves, which is determined by the properties of the universe when the gravity wave mode crosses the horizon, both exiting and reentering.

Of course, given that we have not yet measured the leading order gravity waves, the effects that we discuss in this chapter are not going to be measured in the near future. However, it is a theoretically interesting effect, and we trust in the ingenuity of current and future experimentalists!

This chapter is organized as follows. We first give a quick review of the leading order gravity wave background. Then, we discuss its anisotropies, focusing on relatively large angles which correspond to scalar fluctuations that entered the horizon after matter domination. We also focus on gravity waves that entered the horizon in the radiation dominated era, which would correspond to wavelengths that could be measured directly by gravity wave detectors.

Results of this chapter were obtained with the help of cumbersome calculations. That is why we put most of them into the Appendix, but it should be considered as an essential part of chapter.

3.2 Gravity waves from inflation at leading order

We can view the leading order effect as the result of a Bogoliubov transformation between the vacuum in the very early inflationary period and the vacuum today [48]. In other words, expanding the Einstein action to quadratic order we get an action for each of the two polarization components of the gravity waves. Each polarization component obeys an equation equal to that of a minimally coupled scalar field in the spatially uniform time dependent cosmological solution. This background metric can be written as

$$ds^2 = a(\eta)^2(-d\eta^2 + d\vec{x}^2), \tag{3.1}$$

where η is conformal time. Due to translation symmetry, we can focus on one comoving momentum mode $\vec{k}$ at a time. Each of these modes undergoes the following history. It starts its life well inside the horizon in the adiabatic vacuum. It then exits the horizon during inflation at some time $\eta_{*,k}$, where

$$\eta_{*,k} k = 1. \tag{3.2}$$

It then remains with constant (time independent) amplitude until it reenters the horizon. When the mode is well inside the horizon it is oscillating and redshifting as any other massless particle. See figure 3.1. The solution well inside the horizon is given by the WKB approximation. In order to match the solution well inside to the one well outside the horizon, we need to solve the equation during horizon crossing. During horizon crossing we can approximate the evolution of the background in terms of a fluid with a constant w, where $p = w\rho$, so that $a(\eta) \propto \eta^{\frac{2}{1+3w}}$. Then the solution becomes a combination of Bessel functions. When the mode is exiting the horizon during inflation, the appropriate solution is proportional to

$$\eta^{\mu} H^{(2)}_{-\mu}(-k\eta), \text{ with } \mu = \frac{3}{2} + \epsilon, \tag{3.3}$$

where ϵ is the standard slow roll parameter $\left(\epsilon = -\dot{H}/H^2\right)$ and $k = \left|\vec{k}\right|$. Similarly, when the mode reenters the horizon when the universe is dominated by a fluid with an equation of state with fixed w, then the solution is proportional to

$$\eta^{-\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}(k\eta), \text{ with } \nu = 2/(1+3w). \tag{3.4}$$

We will call the time of horizon reentry, $\eta_{\times,k}$, defined by setting

$$k = a(\eta_{\times,k})H(\eta_{\times,k}) \text{ or } k\eta_{\times,k} = \nu. \tag{3.5}$$

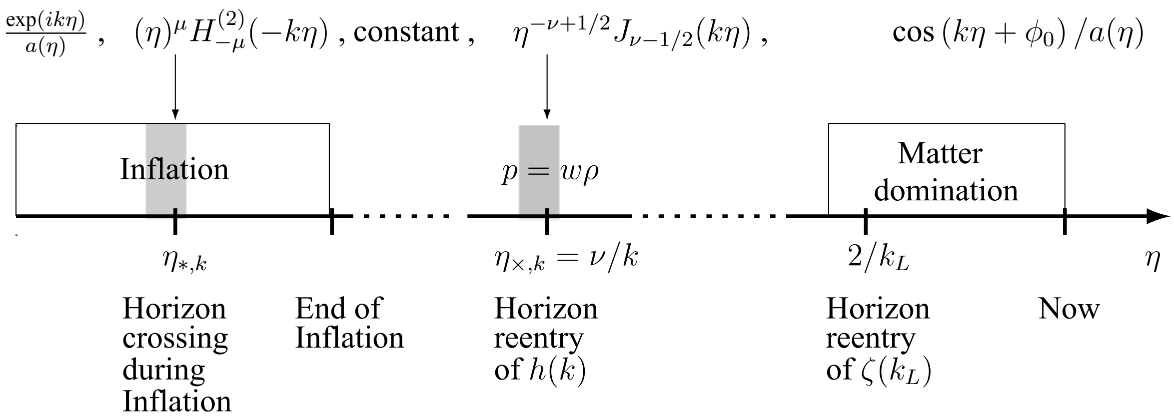

Figure 3.1: Summary of the evolution of a single mode during the history of the universe. We need to know the equation of state only in the shaded regions. The first shaded region occurs during slow-roll inflation. The second shaded region takes place during an epoch with the equation of state $p = w\rho$. Both in the far past and present time the WKB approximation is valid. Outside the horizon, the solution has a constant amplitude. Long wavelength scalar fluctuation modes entered the horizon during the matter dominated era that is assumed to last until now. On the top line, we have written the form of the solution in each region.

Matching the solutions we find that the normalized solution has a late time behavior of the form (see appendix B.1 for a complete derevation, equation B.31)

$$h(k,\eta) = \frac{H(\eta_{*,k_0})}{\sqrt{2}k^{\frac{3}{2}}M_{pl}}\left(\frac{k_0}{k}\right)^{\epsilon+\nu}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}\left(\frac{2}{\nu}\right)^{\nu}\frac{a(\eta_{\times,k_0})}{a(\eta)}\cos\left(k\eta+\phi_0\right), \tag{3.6}$$

Here k_0 is a reference comoving momentum[1]. The factor of $(k_0/k)^{\epsilon+\nu}$ arises because, as we move k away from k_0, we are crossing the horizon earlier or later. Here ϕ_0 is the initial phase of the wave, which is $-\pi\nu/2$, but is unmeasurable for the gravity waves we consider. The redshift after horizon reentry is encoded in the factor $1/a(\eta)$ in (3.6).

[1]In other words, if we are interested in a certain range of values of k, we can choose k_0 to be somewhere in that range. There is no dependence on k_0 due to the dependence on η_{*,k_0} and $\eta_{\times,k_0}$.

The final density of gravity waves is proportional to the square of (3.6). We can express it in terms of the energy density or number density as follows

$$
\begin{aligned}
d\rho_g &= 2\omega\frac{\omega^2 d\omega}{(2\pi)^3}d\Omega\left\langle N_{\vec{k}}\right\rangle, \quad \text{with} \\
\left\langle N_{\vec{k}}\right\rangle &= \frac{H(\eta_{*,k})^2}{H(\eta_{\times,k})^2}C_\nu = \frac{H(\eta_{*,k_0})^2}{H(\eta_{\times,k_0})^2}\left(\frac{k_0}{k}\right)^{2(\nu+1+\epsilon)} C_\nu \\
C_\nu &\equiv \frac{\Gamma(\nu+\frac{1}{2})^2}{4\pi}\left(\frac{2}{\nu}\right)^{2\nu}
\end{aligned}
\tag{3.7}
$$

where $\left\langle N_{\vec{k}}\right\rangle$ is a density of states as a function of k and $\omega = k/a$ is the physical energy of the waves (see appendix B.2 for a detailed derivation). The factor of two comes from the two polarization states. It is useful to think in terms of N_k because this is conserved after horizon reentry. Notice that N_k is essentially given by the ratio of the Hubble constants at horizon crossing. We have also made more explicit the form of the spectrum by defining an arbitrary reference momentum k_0. In the case of photons, we would get the usual thermal distribution for N_k. We see that the spectrum is determined by the equation of state at the horizon crossing times. More generally, the spectrum of gravity waves encodes the whole expansion history of the universe [73, 74, 75, 76, 77].

3.3 Sachs–Wolfe effect

In this section we are going to explain a simple way to compute an energy of a massless particle with 4-momentum P^μ that came from a distant past as it is seen by an observer with four velocity U^μ. We write geodesic equation for a massless particle

$$
\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{3.8}
$$

it is enough for us to consider this equation for zero component only. Then we can define momentum $P^\mu = \frac{dx^\mu}{d\lambda}$ and $\frac{dP^\mu}{d\lambda} = P^0\frac{dP^\mu}{d\eta}$. Using this notation one can write the geodesic equation for a zero component in a form

$$\frac{1}{P^0}\frac{dP^0}{d\eta} + \Gamma^0_{00} + 2\Gamma^0_{0i}n^i + \Gamma^0_{ij}n^i n^j = 0, \tag{3.9}$$

where $n^i := P^i/P^0$ is a unit vector. Once this equation is integrated, we can obtain an expression for the energy $\omega = P_\mu U^\mu$, that is measured by an observer that is moving with a four-velocity U^μ. Sachs–Wolfe effect [16] is an anisotropy contribution to the CMB that comes from the inhomogeneities of the scalar fluctuations. Originally, computation was done in the Newtonian gauge, but for our purposes, it is more convenient to work in $\zeta-$gauge.

3.3.1 Geodesic Equations in Different Gauges

We can remove scale factor from metric because an action for a massless particle

$$S = \int ds \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} g_{\alpha\beta}, \tag{3.10}$$

is invariant under the rescaling [14]

$$g_{\alpha\beta} \to a^2(\tau)\bar{g}_{\alpha\beta}, \quad ds \to a^2(\tau)d\tau. \tag{3.11}$$

Therefore geodesics for the massless particle are the same in the original metric and in rescaled one.

Newtonian Gauge

Geodesic equation (B.51) in the first order becomes

$$\frac{1}{P^0}\frac{dP^0}{d\eta} + \left(\Phi' + \Psi' n^i n^j \delta_{ij} + 2n^i \partial_i \Phi\right) = 0. \tag{3.12}$$

After integration along the geodesic one gets equation (1.6).

$\zeta-$Gauge

Geodesics equation (B.51) for zero component becomes

$$\frac{1}{P^0}\frac{dP^0}{d\eta} + \frac{2}{5}\frac{a}{a'}n^i n^j \partial_i \partial_j \zeta_0 = 0. \tag{3.13}$$

We see that charges changes of P^0 is proportional to the second derivative of a long wave perturbation, thus we can assume that at the lowest order, P^0 is a constant. It is convenient to use another form of the geodesic equation

$$\frac{dP_\mu}{d\lambda} - \frac{1}{2}\partial_\mu g_{\alpha\beta}P^\alpha P^\beta = 0. \tag{3.14}$$

In $\zeta-$gauge g_{00}, g_{ij} are η independent therefore only g_{0i} gives contribution for the equation for zero component

$$\frac{dP_0}{d\eta} = P^0 \partial_0 g_{0i} n^i = -\frac{2}{5}P^0\left(\frac{a}{a'}\right)' n^i \partial_i \zeta_0 \overset{MD}{=} -\frac{1}{5}P^0 \frac{d}{d\eta}\zeta_0. \tag{3.15}$$

so we get

$$P_0(\eta_2) - P_0(\eta_1) = -\frac{1}{5}\bar{P}^0\left(\zeta_0(\eta_2) - \zeta_0(\eta_1)\right), \tag{3.16}$$

where $\bar{P}^0$ is the lowest order expression for a P^0 that is constant. Measured frequency is

$$\Omega(\eta) = P_\mu(\eta)U^\mu = P_0(\eta)\left(1 + n_i v^i(\eta)\right). \tag{3.17}$$

So, the relative change of frequency is

$$\frac{\Omega(\eta_2) - \Omega(\eta_1)}{\Omega(\eta_1)} = -\frac{1}{5}\frac{\bar{P}^0}{P_0(\eta_1)}\left(\zeta_0(\eta_2) - \zeta_0(\eta_1)\right) + n_i v^i(\eta_2) - n_i v^i(\eta_1). \tag{3.18}$$

3.4 Anisotropy

We are interested in the anisotropy of the gravity waves that come from the interaction with the scalar fluctuations. We will concentrate on the anisotropy at relatively large angular scales, which is produced by long wavelength scalar fluctuations. More precisely, we consider a scalar fluctuation mode with a momentum k_L which reenters the horizon during the matter dominated era. In order to find the effect on the gravity waves, it is important to solve the equation of motion for the mode ζ_{k_L} after it reenters the horizon. We then consider the propagation of the gravity waves through that perturbed universe. This computation is essentially identical to one done for photons, which is the so called Sachs-Wolfe effect [16]. See appendices B.3 and B.4 for a complete derivation. In this section we are going to provide only schematic description.

We can work in the gauge where the matter density is constant so that the fluctuation is purely on the scale factor of the spatial part of the metric of the surfaces with constant density. In that gauge ζ_{k_L} continues to be constant after crossing the horizon, but the metric develops a g_{0i} component that should be computed by first considering $w > 0$ and taking $w \to 0$ at the end of the computation [78]. Then the perturbed metric has the form

$$ds^2 = a^2 \left[-d\eta^2 + (1+2\zeta) d\vec{x} d\vec{x} - \frac{4}{5aH} \partial_i \zeta d\eta dx^i \right]. \tag{3.19}$$

Solving the geodesic equation on this perturbed metric we find that the extra redshift for a massless particle emitted at an early time and observed today is (3.12)

$$\frac{\delta\omega}{\omega} = \frac{1}{5} \left[\zeta_{\text{today}} - \zeta_{\text{emitted}} \right] \tag{3.20}$$

where ζ_{today} is the value at the location of the observations today, while ζ_{emitted} is the value of ζ at the location where the massless particle was emitted, when the mode ζ_{k_L} was outside the horizon (so that we can neglect the "doppler term"[2]). Since ζ is time independent, $\zeta_{\text{emitted}} = \zeta(\vec{x}_e = \vec{n}\eta_0)$ where η_0 is the present value of the conformal time and $\vec{n}$ is the direction that the massless particle is coming from[3]. We then assume that at the emission time the local physics is completely independent of the long scalar mode, which is well outside the horizon. As usual, when we compare the energy coming from different directions ζ_{today} cancels out, see figure 3.2.

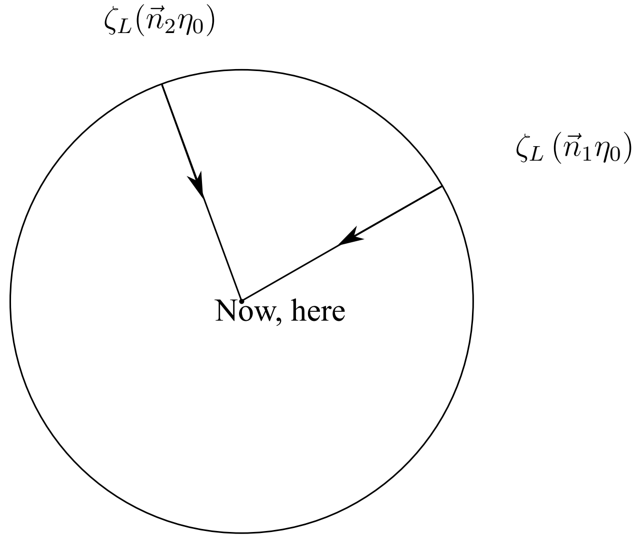

Figure 3.2: Gravitons that are coming from different directions have different redshift, because values of a scalar perturbation ζ_L are different at different emission points.

The discussion so far is identical to the discussion of the Sachs Wolfe effect for the CMB. The only difference is that the spectrum of gravitons is not thermal, but it is instead given by a power law distribution $(\omega_0/\omega)^{2(\nu+1+\epsilon)}$, (3.7). Instead of

$$(\delta T/T)_{SW} = -\frac{\zeta}{5}, \tag{3.21}$$

we simply vary

$$\delta\omega_0/\omega_0 = -\frac{\zeta}{5}. \tag{3.22}$$

[2]We have also ignored the doppler term today. It is trivial to put it back in.

[3]Note that $\vec{x}_e = \vec{n}(\eta_0 - \eta_e) \sim \vec{n}\eta_0$, and $\vec{x}_{\text{today, here}} = \vec{0}$.

More explicitly, if we look at the spectrum of gravity waves as a function of frequency ω coming from the direction $\vec{n}$, we obtain

$$d\rho(\omega, \vec{n}) = d\rho(\omega) \left\{ 1 + \frac{2}{5}(\nu + 1 + \epsilon)\left[\zeta_{\text{today}} - \zeta_L(\vec{n}\,\eta_0)\right] \right\} \tag{3.23}$$

were $d\rho(\omega)$ is the isotropic part of the spectrum, given in (3.7). This result can be derived also using cosmological perturbation theory, performing the computation of the three point function as in [79] and following the evolution to the present, the details will be presented separately [80]. Here we have neglected the Integrated Sachs Wolfe effect which is due to the cosmological constant. This can be taken into account, as in the case of photons. Since the same effect is giving rise to fluctuations in the CMB and in the gravity wave spectrum, we can also write the formula as

$$d\rho(\omega, \vec{n}) = d\rho(\omega) \left\{ 1 - \left(\frac{\delta T}{T}\right)_{SW+ISW} \times k\partial_k \log\langle N_k \rangle \right\} \tag{3.24}$$

where the subindex indicates the contribution from the Sachs-Wolfe and the Integrated Sachs Wolfe effect. The discussion so far has not included the damping effect of the neutrinos [81]. This is expected to be a local effect which will not depend on the long mode. For that reason, the final formula as written in (3.24) would also be correct if one inserts the full $\langle N_k \rangle$ expression that includes the effects of the neutrinos. We can view this equation as a consistency condition for a single field inflation. Usually, the consistency condition is discussed for the wavefunction of the universe outside the horizon [79, 82] (see also [83]), but as emphasized in [84] the physical content of that condition is that a local observer cannot notice the long fluctuation. It is a manifestation of the equivalence principle. Furthermore, the final answer could be viewed as arising from a projection effect due to the propagation of the massless particles through the perturbed universe. Furthermore, since the whole effect comes from late time projection effects, the final formula (3.24) is valid also for gravity waves that are generated by any process that happened before the long mode crossed the horizon, such as phase transitions in the early universe, for example.

Any deviation away from this expression (3.24) would be evidence of a second field which would be affecting the relative densities of gravity waves relative to everything else. We think that this would occur in the curvaton models [85, 86, 87].

Since the fluctuations are very small, of order 10^{-5}, these are rather difficult to measure. In addition, we should remember that we are dealing with a stochastic background, so that the observed gravity wave over a small number of cycles is fairly random, and the statement in (3.24) is about the deviation in the variance of that random variable. This means that we need to measure this random variable many times. For the case of random waves, we can view each cycle of the wave as one instance of the random distribution. This means that to reach this accuracy we need to observe of order 10^{10} cycles in each angular direction. For waves, a frequency of f[Hz] we need about $300/f$ years[4].

Anisotropies also could be present due to individual sources [88, 89] and it might be possible to discriminate them since perturbation that is discussed in the present chapter are frequency independent.

[4]This estimate has been made under the assumption that detector has enough sensitivity to measure gravitational wave for each frequency bin.

4

Exploring a strongly non-Markovian behavior

"The true sign of intelligence is not knowledge, but imagination."

Albert Einstein

4.1 Introduction

This chapter is based on work in progress[90] that will be coauthored with Gordon Berman, William Bialek, and Joshua Shaevitz.

Animals are able to perform a large verity of motions and stereotyped behaviors. However, their complexity was a serious challenge on the way to build a mathematically rigorous theory that has any predictive power. Historically, there were two main approaches to solve this problem.

The one is a pure description. People were observing animals in the natural environment and were describing what they saw. They were trying to systematize their observation and if possible generalized them using a known biological structure of the animal population. The advantage of this approach is that animal's behavior is unrestricted and natural. An obvious disadvantage is the lack of the comprehensive quantitative framework to describe it. That is surprising how far people managed to proceed in this direction, especially given the number of degrees of freedom in the system, and complete disability to control the environment as well as a diversity of species. The result of this approach strongly depends on our ability to observe and to find a correlation. We are very good at this, but if one thinks about the situation itself it is weird because our ability to observe and find the correlation is part of our behavior. So, we are trying to describe a black box using Blackbox itself. The success of this approach is rather a miracle than expected result.

Another approach is a controlled experiment. In this approach, people were trying to isolate perturbation of the system. The advantage of this approach, that we indeed have full control over environment, perturbations, and diversity of species. We can control every single detail that we know about, and this is a problem. Because we have to know what to control and what is important, etc. Another problem is set of all possible outcomes is defined by us, and therefore it is limited by our imagination.

The ideal solution is to marry these two approaches. This challenge was addressed in [91], where authors managed to come up with a description of the behaviors of the fruit flies as a trajectory in a high-dimensional space. The behavior was unsupervised, but in the controlled environment though. The experimental setup was as close to the real life as possible given all limitations.

It turned out that despite the naive expectation behavior of the fruit fly can be presented as a relatively small set of stereotyped behaviors. These stereotyped behaviors were determined algorithmically, so most of our previous concerns were addressed. Let us discuss results of [91] because this is a crucial ingredient of our analysis.

4.2 Experimental setup and initial analysis

Experimental data were obtained and analyzed in [91]. Flies were enclosed in a cylindrical clear PET-G (*Polyethylene terephthalate glycol-modified*) plastic dome 100 mm in diameter and 2 mm in height with sloping sides at the edge clamped to a flat glass plate. Flies were enclosed to this arena to make sure that it stays in the focal plane of the camera. We are not going into full details of the experimental setup, but we would like to notice that all precautions were made to make sure that flies are walking only on the horizontal plane and there was no upside-down walking.

There were 59 male *D. melanogaster* (Oregon-R strain). The motion of each fly was filmed by 100Hz high-speed camera; each frame had a resolution of 1088×1088 pixels, and flies were fully inside 200×200 square. Total imaging time was 1hr, that produced 3.6×10^5 frames.

Each clip went through complicated analysis. We are going to give only brief description of it to give an understanding of the huge progress that was achieved [92]. Each frame was cropped, aligned, rotated, and scaled to make sure that each fly occupies the same number of pixels. So, the only remaining information was information about dynamics of postures. Instead of supervised analysis, that is when one identifies particular stereotyped behaviors and then translate each clip into a sequence of these behaviors, a much clever approach was taken. Each image was refined by performing Radon transformation, that reduced image from 18090 pixels to 6763 pixels while retaining approximately 95% of the total variation in the images. This simplification allowed to perform PCA analysis, that showed that is was sufficient to have only 50 modes to keep information about 93% of the observed variation of the data. In other words, a movie of unsupervised motion of the fly was projected into a 50-dimensional time series.

Next step was to retrieve information about multi-time scale dynamics of postural modes. Morlet continuous wavelet transformation was used to get this information. They used a dyadically spaced set of frequencies in the range $f \in [1, 50]\, Hz$. Obtained spectrogram is 1250 dimensional feature vector at each time. Presumably, dynamic does not fill all space, so it is possible to look for lower dimensional features. The $t-$distributed stochastic neighbor embedding (*t-SNE*) was used to search for lower dimensional structure by embedding into $2d$ space. In this approach set of invariants are preserved as best as possible. In particular, preserved invariants are related to the local probability for a Markov transition probability of a random walk is performed in the high-dimensional space, that means that probability of transitions between images of the point should be preserved as best as possible. The result of this analysis is $2d-$space with dynamics on it.

The estimate of local density appears to have many local maxima, in the course grained image of this space. Trajectories in the space pause near local maxima of the probability density [92]. These pauses are identified as stereotyped behavioral states, they recur many times for an individual fly and can be identified across multiple individuals. Duration of these behaviors varies from 0.05 s to more than 10 s.

In order to define regions in this space rigorously, ”watershed transformation” was used [93]. For probability distribution that came from a physical system in thermal equilibrium ”watershed transform” is equivalent to finding the valleys in the free energy landscape. The result is the map that is presented on the fig.(4.1).

So, in [91] they managed to transform a video movie of fly’s unsupervised motion into a sequence of stereotyped behavior completely algorithmically, where each stereotyped behavior was identified automatically. The number of states depends on the thresholds that were used in PCA analysis and during the watersheded transform. We work with

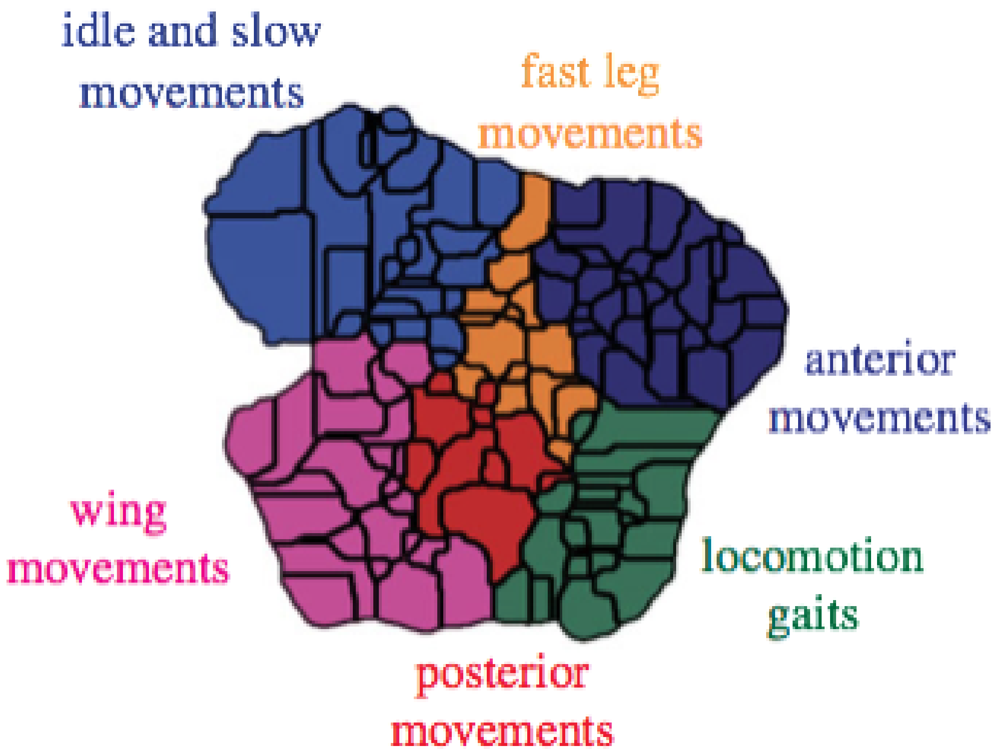

Figure 4.1: Maps of stereotyped behavior [91]. Each region on the plot is particular stereotyped behavior. This map defines space of behaviors.

122-behavior realization, but one can move up to five behaviors into other nearby ones because they are almost statistically unresolvable in the analyses.

Each algorithmically identified ”stereotyped behavior” indeed corresponds to real stereotyped behavior, that can be checked easily. One can notice that similar behaviors are neighbors on the $2d-$map, that is a nice feature.

4.3 Formalization of behavior description

Representation of the behavior as a set of stereotyped behavior looks promising and like a correct framework for the quantitative analysis. Our objective is to describe dynamics of this system. We do not think that it is possible to come up with a deterministic description of the behavior and we do not think that this is correct approach anyway. So, our way to go is to describe fly's behavior in a probabilistic way.

One can say that system is defined by $2-$dimensional patch, that has 122 regions, and sequence of transitions between these regions. In this representation information about real time is not present. We care only about transitions between different states. That is why we never have transition $S \to S$, where S is any stereotyped behavior.

Every time when we are building a theory that describes some system we have to face several challenges. One is that we can not measure every quantity that may appear in the theory. Even if this quantity is very natural or fundamental from a theoretical point of view. We can measure only observables. One of the simplest examples is vector potential and electric or magnetic field, the latter pair is observable but is less fundamental than vector potential.

Another challenge is not a problem, but rather a philosophical decision. We have to choose a framework to describe phenomena. We would like to have as simple and as clear description as possible, but we would like to maximize the predictive power. Field-theoretical formalism is currently the most powerful and promising one. Correlation functions of n operators are the fundamental object in this approach, but they are not directly observable. All observable quantities are related to some correlation functions. For instance, the differential cross section is related to the S matrix that itself can be computed from the time-ordered correlation function via Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. In Ising model, susceptibility χ can be written as a sum of truncated correlation functions $\chi = \beta \sum_{i,j\in\Lambda} \langle \sigma_i \sigma_j \rangle$. There are many more examples. The main idea is clear. We have to choose a quantity to measure carefully not only to make sure that we can achieve reasonable precision, but also to make sure that we can relate it to some quantity that can be (easily) computed in the theory.

Being inspired by the field-theoretic approach, we would like to consider quantities that are analogous to correlation functions, because these functions have proven to be a natural and correct way to analyze theory. For a system with many states, correlation function has indices because we can consider the correlation between any two fields. This matrix may have complicated index structure, because it may be the case that ”fundamental” field is some superposition of the apparent ones. That is why we can start by analyzing some invariant of this matrix, for instance, trace, that is analog to the probability that system will be in the same state[1] after τ transitions. This quantity allows us to capture some basic properties and to check some hypothesis.

For each fly, n, one can measure distribution of states $P_n(S)$. One can define transition probability to go from state $S(n) = i$ to state $S(n+\tau) = j$ after τ transitions

$$(T_{n,\tau})_{i,j} = \text{Prob}\,(\text{fly } n \text{ goes from state } S = i \text{ to state } S' = j \text{ after } \tau \text{ transitions})\,. \quad (4.1)$$

The quantity we would like to consider is probability to be in the same state after τ transitions. It can be expressed as

$$P_c(\tau) = \frac{1}{n_f}\sum_{n=1}^{n_f}\sum_S T_{n,\tau}\,(S \leftarrow S)\,P_n(S) = \frac{1}{n_f}\,\text{tr}\,(T_{n,\tau}P_n)\,, \quad (4.2)$$

where we sum over every state S and we average over n_f flies, because each fly may have individual features that we are not important right now. This quantity is an analog of the complete correlation function, so it contains information about the lower-order disconnected $1-$point correlation function that describes asymptotic behavior at $\tau \to \infty$. In other words, $P_c(\tau)$ does not decay to zero as $\tau \to \infty$. Asymptotical value of the function eq.(4.2) can be expressed in terms of state distribution

$$P_c(\infty) = \sum_S P(S)^2. \quad (4.3)$$

[1]We use ”system” as a shortcut for behavior of a fly at large, and ”state” is a shortcut for a stereotyped behavior. In this text, we tend to adopt physical terminology, and there is a chance that we are using it excessively. So, authors apologise for any inconveniences that might be caused by this, but we can assure that there is detailed dictionary between physical slang and biological terminology.

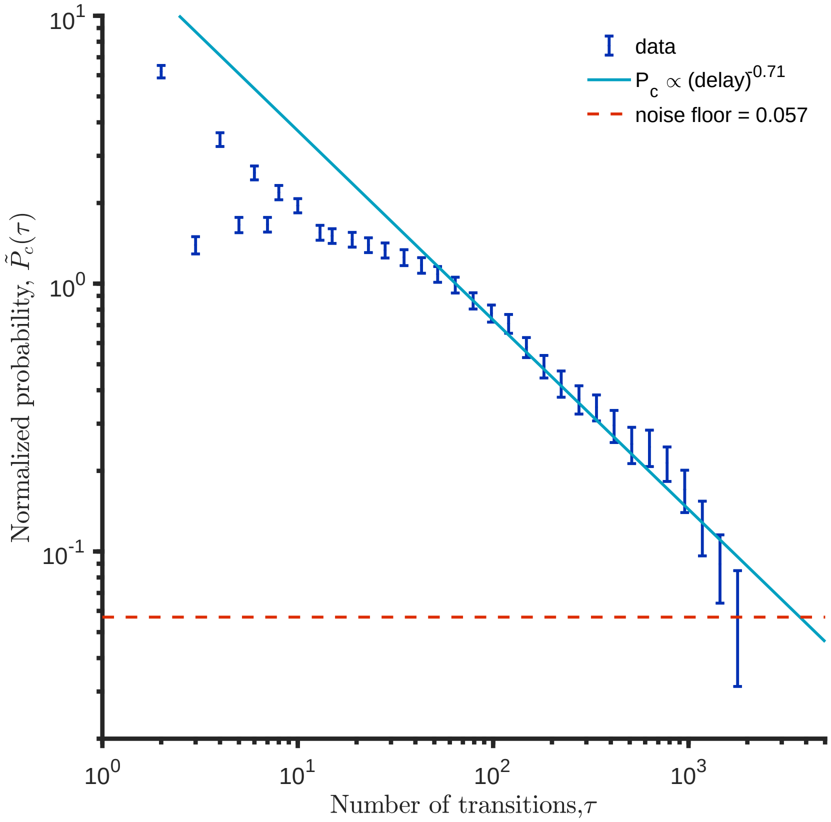

Figure 4.2: Normalized probability to stay in the same state after τ transitions averaged over all flies.

Asymptotic behavior does not capture dynamics, so we can subtract it

$$\tilde{P}_c(\tau) = P_c(\tau)/P_c(\infty) - 1. \tag{4.4}$$

We would like to point out the attention of the reader, that this probability includes the possibility that system visited the same state several times in between, exactly how it happens for correlation function. The plot of this function for the experimental data is presented on fig.(4.2). Function eq.(4.4) for two-state system is an analog of connected part of correlation function. Indeed, if we have two states $\sigma \in \{1, -1\}$, then

$$\langle \sigma(x)\sigma(x+\tau)\rangle = (+1)\, P_c(\tau) + (-1)\,(1 - P_c(\tau)) = 2P_c(\tau) - 1. \tag{4.5}$$

Asymptotic of this function looks like

$$\langle\sigma\rangle^2 = \lim_{\tau\to\infty} \langle\sigma(x)\sigma(x+\tau)\,\rangle = 2P_c(\infty) - 1. \tag{4.6}$$

Connected part of correlation function is

$$\langle\sigma(x)\sigma(x+\tau)\rangle - \langle\sigma\rangle^2 = 2\left(P_c(\tau) - P_c(\infty)\right) = 2P_c(\infty)\tilde{P}_c(\tau). \tag{4.7}$$

One can notice (fig.4.2) a prominent feature of $\tilde{P}_c(\tau)$, a power-law decay that lasts for more than an order of magnitude for large τ. As one may know power law decay is usually a sign of some kind of criticality. We are not going to fantasize what criticality is present and how we should understand it. There are few reasons for it. The first and the most important reason is that we do not have a theory that describes a behavior of the system and therefore we do not have a particular definition of phase space. Also no need to say about other concepts that are usually used to describe critical phenomena in condensed matter physics or high energy physics. We would like to make a point that power-law decay usually is a sign of something interesting that happens in the theory and usually is associated with some sort of scale invariance. So, the surprising part is that fly has this large scale in the description of its behavior.

4.4 Is there any memory in the system or can we use HMM/MC?

Dynamics of the system looks complicated, so we would like to figure out if this complexity is a real underlying property of the system, or it is just a result of the large size of the state-space.

The simplest possibility is that this system is completely random, that means that there is no memory in the system. This approach is used sometimes in the field, regardless the fact that it is quite primitive.

4.4.1 Markov process

Here we briefly describe what happens if we assume that system is Markovian. For Markovian system probability to go from state S to state S' depends only on present state and we can actually measure the transition matrix $T(S' \leftarrow S)$ from the experimental data. The transition matrix is a square matrix that has size 122×122. In this framework, one can easily write down probability to be in the same state after τ transitions

$$P_c(\tau) = \sum_S \left(T^\tau\right)(S \leftarrow S)\, P(S) = \sum_S \sum_{\mu=1}^{N} (\lambda_\mu)^\tau\, u_\mu(S) \otimes v_\mu(S) P(S), \tag{4.8}$$

where we presented transition matrix T in terms of u_μ, v_μ are eigenvectors and λ_μ is the corresponding eigenvalue. Transition matrix has all positive elements, so maximal eigenvalue is finite. The transition matrix is a stochastic matrix that is why maximal eigenvalue is equal to one. A spectrum of eigenvalues define the time dynamics of the function. The second to maximal eigenvalue, λ_2, determines the slowest decay with characteristic decay time $\tau_2 = -1/\ln\lambda_2$. For experimental data $\lambda_2 = 0.95 \pm 0.03$, where the error bar is the standard deviation across individual flies. This determines a characteristic decay time $\tau_2 = -1/\ln\lambda_2 = 19^{+30}_{-7}$ transitions. We see that any "memory" (apparent decay rate) that extended beyond 30 is a direct sign of non-Markovian behavior. One can also notice that in this model probability to be in the same state, P_c has an exponential decay rather than a power-law decay. So, we can see that Markov model can not describe this system in any naive interpretation [92]. To finalize comparison we would like to plot prediction of the Markov model and the real data results on fig.4.3.

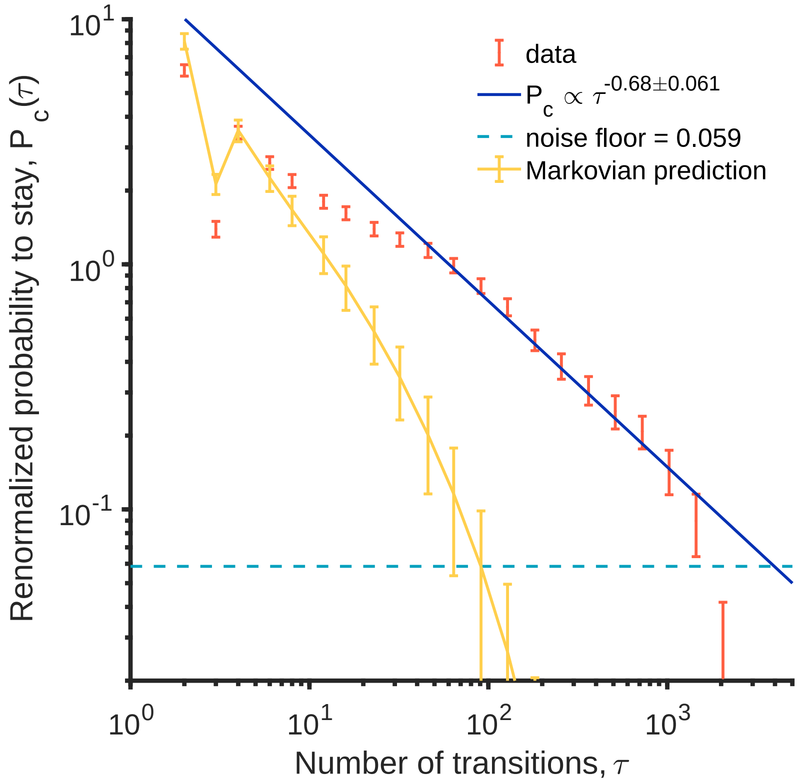

Figure 4.3: Normalized probability to stay in the same state after τ transitions averaged over all flies vs Markov model prediction.

The fact that Markov model does not work is actually a very good sign because this model has too many parameters. Transition matrix has $122 \times 122 - 122 = 14762$ parameters. This is more than the length of the maximal sequence that we have. So this is quite surprising that we managed to get any result from experimental data at all. The reason for this is the fact that transition matrix has close to block form. That means that many elements are almost zero, so effectively there are fewer parameters than possible.

4.4.2 Hidden Markov Model

The internal structure of the brain is much more complex than a system with 122 states. So, in some sense, it was very naive to expect to get such simple effective theory without a memory for fly's behavior. Hidden Markov Models is a more realistic approach, because fly may have many internal unobservable states.

Let us briefly discuss Hidden Markov Model. More details we will provide in the appendix. There are N hidden states and k observable states. The prior distribution of hidden states is π. Transitions between internal state are random and described by Markov process with transition matrix A

$$A_{ij} = \text{Prob(that system goes from state } i \text{ to state } j), \tag{4.9}$$

that has size $N \times N$. After system went from one state to another it emits some observable state O_k with probability

$$B_{jk} = \text{Prob(we observe state } O_k \text{ if system is in hidden state } j). \tag{4.10}$$

One can easily write an expression for probability to be in the same state after τ transitions.

$$P_c(\tau) = \text{tr}\left(B^T A^\tau B\right). \tag{4.11}$$

We fitted many model HMM with a number of internal states from 2 to 122, but we have neither found anything interesting nor any agreement between fitted models and experimental data. Hidden Markov Model tends to fit better a few points in the vicinity of $\tau \sim 10^2$ (middle range) but fails to keep agreement for small and large scale.

One may notice that usual way of using HMM implies using a large number of hidden states, but the number of degrees of freedom increases N^2, that already gives $\sim 10^4$ for $N = 122$. We do not have enough data to train much larger model, because of the size of datasets.

Another remark about this model is that time dependence of the model has the same form as in a regular Markov model, so most discussions can be translated here with very mild modification.

One can argue that if we increase the number of hidden layers it is possible to have several timescales in the system. It is possible to demonstrate equivalence of multiple layers Hidden Markov Model to larger Hidden Markov Model with one Hidden layer. So, we do not expect it work as well. We would like to stress attention that the main limitation to make these model work is the size of the data set.

One can try to use some more complex models to produce the same output for the system with 122 states. For instance, in [94] authors built a complicated network that provided transitions between attractors, that represents Markov chain. Maybe something similar could be done for our system, even though our system is non-Markovian. This idea requires more investigation and we prefer to focus on a different approach.

4.5 Clustering

We have a very large system, and there is some complexity that is associated with its size. The simplest class of models for complex system failed to provide a description of experimental data. That is why we would like to simplify our system to use more complicated model. Our goal is to simplify system as much as we can but leave as much information and features as possible. One of the possible approaches is to cluster our data and study dynamics for clusters. In this case, we do not have a prohibition for staying in the same state anymore.

There are many ways to cluster data in data science. For instance, we can preserve some structures, or we can demand that clustering is hierarchical, etc. We would like to use a method that has ”physical” motivation, and such that its clustering is reasonable. On of the reasonable approaches, the bottleneck algorithm [95] was studied in the paper [92] to obtained a clustering of the states. Their finding shows that clusters are meaningful and there is a hierarchical structure or in other words, there is a nested structure. That means that if we increase the number of clusters, old clusters divide to new clusters. In the case of two clusters, one cluster corresponds to idle while another to everything else [92]. In the information bottleneck algorithm, mutual information between clustering now and actual states in future $I(Z_t, X_{t+\tau})$ is maximized. In this paper, we would like to use a different approach, because division for idle and everything else is not particularly interesting.

We consider more symmetric mutual information. Let us discuss it in details. We can say that there is a signal $x \in X$, and another signal $z \in Z$. The set of all possible observable states $X = \{1, \ldots, 122\}$ is constant in time. The set of all clusters $Z = \{1, \ldots, n\}$, where n is a number of clusters. We define two types of clustering soft and hard. Hard clustering means that each state x belongs to one and only one cluster, while soft clustering means that each state x belongs to any cluster z with some probability $p(z|x)$. Normalization condition is $\sum_{z=1}^{n} p(z|x) = 1, \forall x$.

We assume that clustering is the same at every point of the time, that means that $p(z|x)$ does not depend on time. We would like to maximize mutual information between clustering now and clustering after τ transitions. It means that we would like to maximize the amount of information about the cluster that system will be in if we know in what cluster it is now. The difference between standard bottleneck approach is that we maximize symmetric mutual information $I(Z_t, Z_{t+\tau})$, instead of $I(Z_t, X_{t+\tau})$. Graphical comparison looks like

$$\begin{tikzcd} X_t \arrow[r] \arrow[d, "F"'] & X_{t+\tau} \\ Z_t \arrow[ur, "I_{Z_t, X_{t+\tau}}"'] & \end{tikzcd} \qquad \text{vs} \qquad \begin{tikzcd} X_t \arrow[r] \arrow[d, "F"'] & X_{t+\tau} \arrow[d, "F"'] \\ Z_t \arrow[r, "I_{Z_t, Z_{t+\tau}}"'] & Z_{t+\tau} \end{tikzcd}$$

Mathematical formulation of out procedure requires solution of the optimization problem that is defined by the Lagrangian

$$\mathcal{L} = I(\hat{Z}, \hat{W}) - TI(\hat{Z}, X) - \sum_{z} \lambda(x) p(\hat{z}|x), \tag{4.12}$$

where mutual information I is given by

$$I(\hat{Z}, \hat{W}) = \sum_{\hat{z},\hat{w}} p(\hat{z}, \hat{w}) \log \frac{p(\hat{w}, \hat{z})}{p(\hat{z})p(\hat{w})}. \tag{4.13}$$

For convenience we introduced a shorthand notation $\hat{Z} = Z(t)$, $\hat{W} = Z(t+\tau)$, $p(\hat{z}|x) = p(z(t) = \hat{z}|x(t) = x)$, $p(\hat{z}) = p(z(t) = \hat{z})$, $p(\hat{w}) = p(z(t+\tau) = \hat{w})$, and $p(\hat{w}, \hat{z}) = p\left(z(t+\tau) = \hat{w}, z(t) = \hat{z}\right)$.

We would like to find conditional probability $p(\hat{z}|x)$, such that amount of information about $\hat{W}$ is maximized, while information about clustering is kept the same. We will call this function $Q(\hat{a}|b), \hat{a} \in \hat{Z}$, $b \in X$.

One can get that variation of mutual information with respect to conditional probability

$$\frac{\delta I(\hat{Z}, \hat{W})}{\delta Q(\hat{a}|b)} = \sum_{\hat{z},\hat{w}} \log \frac{p(\hat{z}, \hat{w})}{p(\hat{z})p(\hat{w})} \frac{\delta p(\hat{z}, \hat{w})}{\delta Q(\hat{a}|b)}, \tag{4.14}$$

where

$$p(\hat{z}, \hat{w}) = \sum_{x,y} Q(\hat{z}|x) Q(\hat{w}|y) P(x, y). \tag{4.15}$$

So, one gets,

$$\frac{\delta p(\hat{z}, \hat{w})}{\delta Q(\hat{a}|b)} = \sum_{y} Q(\hat{w}|y) p(y|b) p(b) + \sum_{x} Q(\hat{z}|x) p(x|b) p(b) = \tag{4.16}$$

$$= p(b)\Big(p(\hat{w}|b)\delta_{\hat{z},\hat{a}} + p(\hat{z}|b)\delta_{\hat{w},\hat{a}}\Big). \tag{4.17}$$

Thus, the variation of the mutual information is

$$
\begin{aligned}
\frac{\delta I(\hat{Z},\hat{W})}{\delta Q(\hat{a}|b)} &= p(b)\sum_{\hat{z},\hat{w}} \log\frac{p(\hat{z},\hat{w})}{p(\hat{z})p(\hat{w})}\Big(p(\hat{w}|b)\delta_{\hat{z},\hat{a}} + p(\hat{z}|b)\delta_{\hat{w},\hat{a}}\Big) = \\
&= p(b)\left(\sum_{\hat{w}} p(\hat{w}|b)\log\frac{p\left(\hat{w}|\hat{a}\right)}{p(\hat{w})} + \sum_{\hat{z}} p(\hat{z}|b)\log\frac{p(\hat{z}|\hat{a})}{p(\hat{z})}\right) = \\
&= -p(b)\left(\sum_{\hat{w}} p(\hat{w}|b)\log\frac{p(\hat{w}|b)}{p(\hat{w}|\hat{a})}\frac{p(\hat{w})}{p(\hat{w}|b)} + \sum_{\hat{z}} p(\hat{z}|b)\log\frac{p(\hat{z}|b)}{p(\hat{z}|\hat{a})}\frac{p\left(\hat{z}\right)}{p(\hat{z}|b)}\right) = \\
&= -p(b)\left(D_{KL}\big(p(\hat{w}|b)||p(\hat{w}|\hat{a})\big) + D_{KL}\big(p(\hat{z}|b)||p(\hat{z}|\hat{a})\big)\right) + f(b).
\end{aligned}
\tag{4.18}
$$

variation of the second mutual information is

$$
\frac{\delta I(\hat{Z},X)}{\delta Q(\hat{a}|b)} = p(b)\log\frac{Q(\hat{a}|b)}{p(\hat{a})}. \tag{4.19}
$$

One can write the variation of the Lagrangian

$$
\begin{aligned}
&\frac{\delta \mathcal{L}}{\delta Q(\hat{a}|b)} = \\
&-p(b)\left(D_{KL}\big(p(\hat{w}|b)||p(\hat{w}|\hat{a})\big) + D_{KL}\big(p(\hat{z}|b)||p(\hat{z}|\hat{a})\big) + T\log\frac{Q(\hat{a}|b)}{p(\hat{a})} + \tilde{\lambda}(b)\right) = 0.
\end{aligned}
\tag{4.20}
$$

$$
Q(\hat{a}|b) = \frac{p(\hat{a})}{Z(b,T)}\exp\left(-\frac{1}{T}\Big[D_{KL}\big(p(\hat{w}|b)\parallel p(\hat{w}|\hat{a})\big) + D_{KL}\big(p(\hat{z}|b)\parallel p(\hat{z}|\hat{a})\big)\Big]\right). \tag{4.21}
$$

In order to solve this equation numerically we introduce the following iteration prescription, where we use subscript n as an iteration counter

$$p_n(z(t) = \hat{a}) = \sum_x Q_n(z(t) = \hat{a}|x)p(x), \tag{4.22}$$

$$Z_{n+1}(b, T) = \sum_{\hat{z}} Q_n(\hat{z}|b) Z_n(b, T), \tag{4.23}$$

$$p_n(z(t+\tau) = \hat{w}|z(t) = \hat{a}) = \\ = \sum_{x,y} Q_n(z(t+1) = \hat{w}|y)p(y|x)Q_n(z(t) = \hat{a}|x)p(x)/p_n(z(t) = \hat{a}), \tag{4.24}$$

$$p_n(z(t) = \hat{z}|z(t+1) = \hat{a}) = \\ = \sum_{x,y} Q_n(z(t) = \hat{z}|x)p(y|x)p(x)Q_n(z(t+1) = a|y)/p(z(t+1) = \hat{a}). \tag{4.25}$$

$$Q_{n+1}\Big(z(t) = \hat{a}|x(t) = b\Big) = \\ \frac{p_t(\hat{a})}{Z_{n+1}(b,T)} \exp\left(-\frac{1}{T}\Big[D_{KL}\big(Q_n(\hat{w}|b) \parallel p_n(\hat{w}|\hat{a})\big) + D_{KL}\big(Q_n\,(\hat{z}|b) \parallel p_n(\hat{z}|\hat{a})\big)\Big]\right). \tag{4.26}$$

In the case of soft clustering, this system finds local maximum very well, but some playing with initial conditions is required to make sure that that solution represents global maximum.

In the case of hard clustering, another way to solve this problem exists. One can also use simple Monte Carlo simulations. We start from some random clustering and then we slightly change clustering if the mutual information is getting larger we accept it. In this way, we can get to a local maximum of mutual information that is compatible with initial distribution. In order to make sure that we get to the global maximum we can effectively thermalize system by introducing the probability of acceptance, that means that we can accept configuration with lower mutual information with some probability. This allows us to explore the vicinity of local maximum, that leads to average movement to the global maximum.

For each $n \in [2; 30]$ we performed a few dozens of simulations with 10^8 iterations and got clusterings, that maximize information about clustering at next (that is $\tau = 1$). Almost every clustering has the nice property that each cluster is simply connected, with a few very minor exceptions. For instance, for $n = 4$, one can notice that the third cluster is almost simply connected, because there are two regions that should belong to other clusters, or at least naively it looks like more natural choice to more on of them to the second (blue on the picture) cluster and another to the first (grey) or to the fourth (red). However, if one does it, mutual information slightly decreases from $I_{max} = 0.8512$ to $I_{simply\,connected} = 0.8510$, that is 0.0235% difference and this is definitely much smaller than any error in the data. It may be the case, that if we slightly modify initial analysis we can get simply connected clustering in this case.

We present out results on color $2d$ map fig.(4.5), such that each color corresponds to particular cluster.

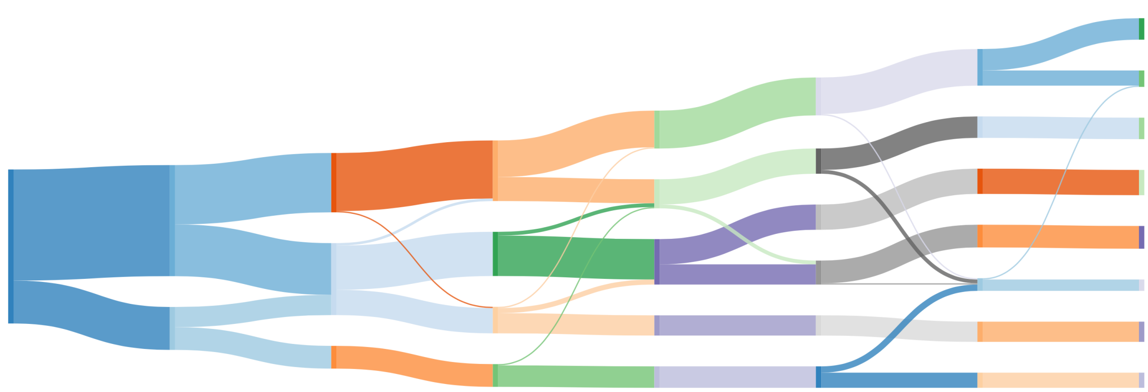

Figure 4.4: Hierarchy Map for clustering that we obtained from modified bottleneck approach.

There are several observations that one can make. The first one is that clusters do make sense. We can see that in most case we can define types of states that correspond to a

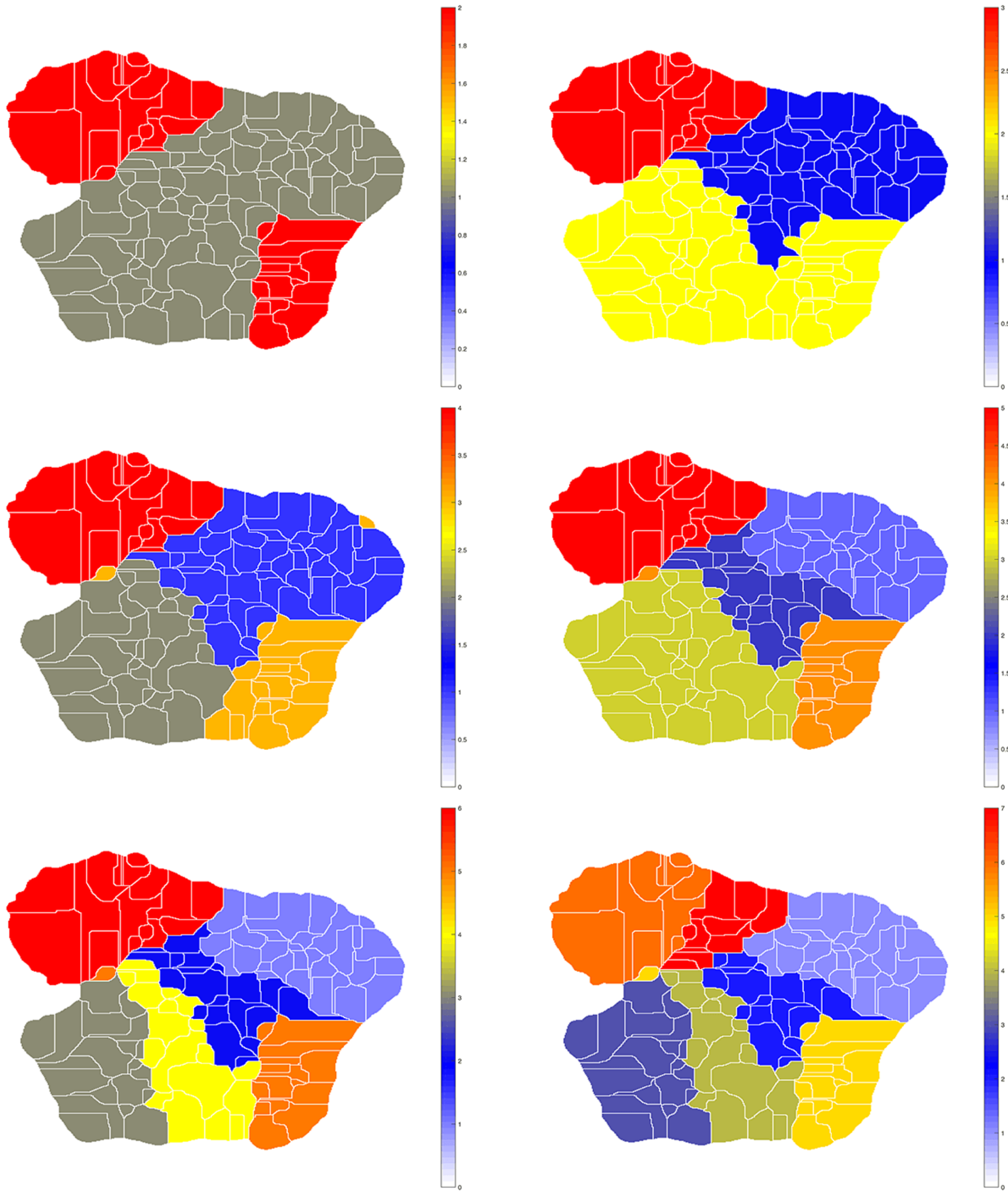

Figure 4.5: Clustering maps for $n = 2, 3, 4, 5, 6, 7$

particular cluster and most states of particular type belong to one or two clusters. This is not obvious at all, it could be the case that we got salt and pepper pattern that does not make much sense.

Another observation is that there is a hierarchical structure, that means that clusters tend to divide if we increase the number of the cluster rather than completely rearrange. The fact that we got this structure is a result, not a condition that we imposed. There are different clustering algorithms that require this property, but the only thing we demanded was that mutual information is maximized.

Now it is time to ask if we can use this clustering to simplify our system. The particular question is whether we can replace dynamics of complete system by dynamics of the system that has only two states. That means that instead of the sequence where each element takes value in $S = \{1, ..., 122\}$, we have a binary sequence. In this description, we are losing information about dynamics with in each cluster, but preserve information about transitions between clusters. The simplest way to check if this dynamics is similar to the original one is to compute probability to be in the same cluster after τ transitions.

One can easily see that the normalized probability to be in the same state has the decay law for large τ similar to one for the original sequence. One can compare fig.(4.6) and fig.(4.2) and find that the power–law decay has the same exponent $\alpha_{122} = \alpha_2$ in both cases. This is not a trivial fact, because the first of all, for a random clustering we could have got any decay law. Another reason to be surprised is that we maximized information about clustering in next step ($\tau = 1$), but we managed to effectively preserve information about system's behavior at large scale (up to $\tau \sim 10^3$). This is a very good news because we can study this simple system instead of the original complicated system. Once we explore its features we can make next step and try to restore information about dynamics within each cluster.

Before we move to beautiful results, let us discuss a possibility of Markovian description for a binary system. We have simple Markov model with only two states. In general, one can easily use the previous argument, but to make the point completely obvious we can fit Markov model to the experimental data and then plot probability that was computed for Markov process.

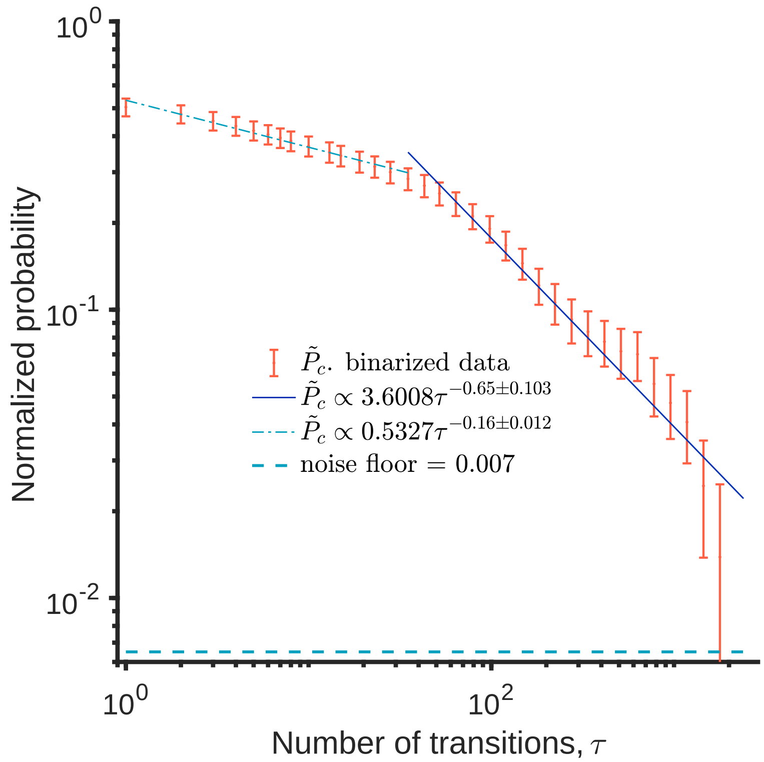

Figure 4.6: Probability to be in the same state after τ transitions for a binary sequence

One can notice that agreement became much worse than we had for the full system. So, despite the fact that qualitatively behavior of the system remains the same, but non-Markovianity increased dramatically $\lambda_2 = 0.88 \pm 0.01$, that gives characteristic decay time $\tau_2 = -1/\ln(\lambda_2) = 8^{+20}_{-3}$. It is somehow expected change, because Markov model with only two states has much fewer degrees of freedom, and it is much more difficult to mask complicated non-Markovian dynamics.

4.6 Maximal entropy model

Simplest clustering with two clusters is sufficient to catch long-range behavior of the system. So let us describe it with some model. We transform our system into the two-state

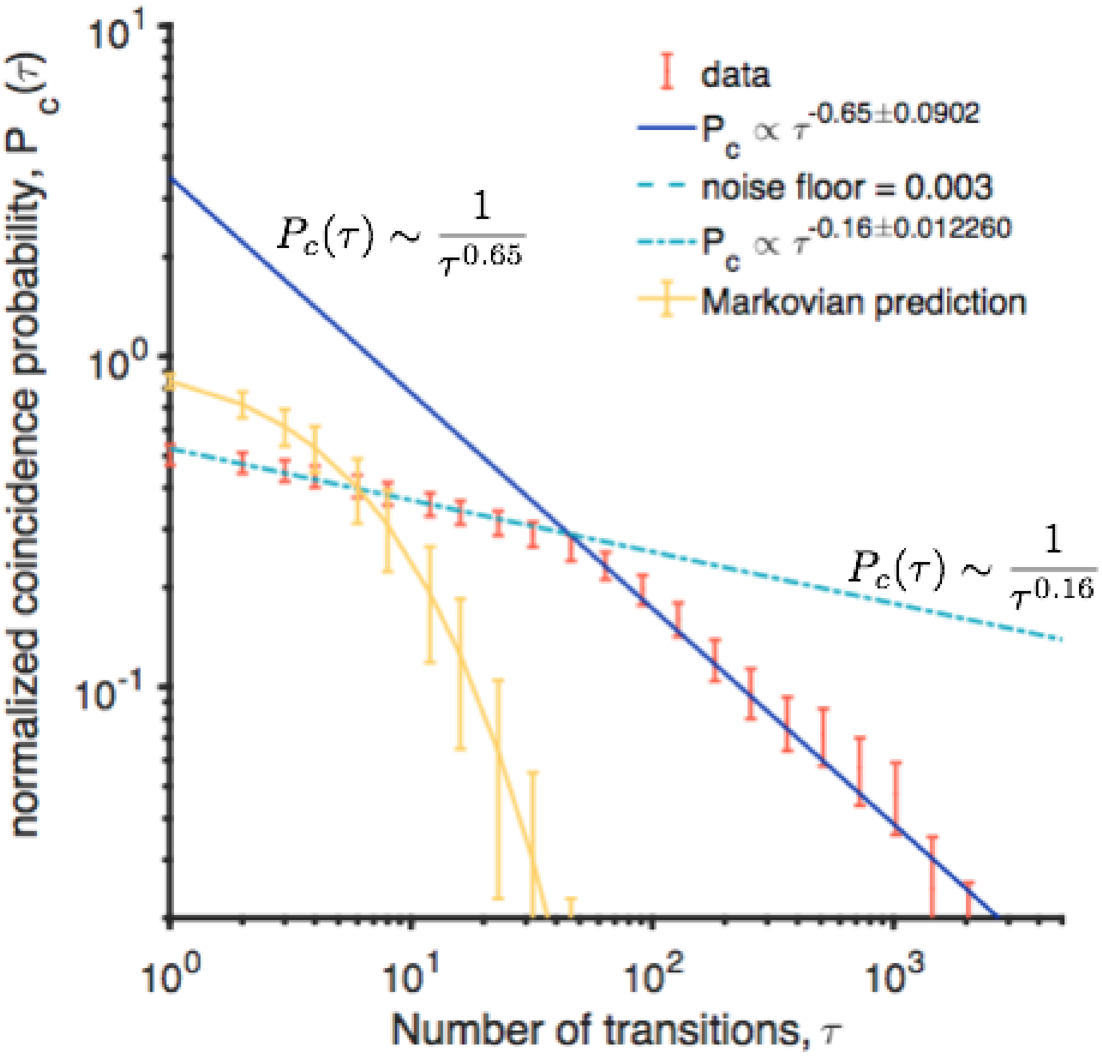

Figure 4.7: Probability to be in the same state after τ transitions for binarized experimental data and prediction for Markov Model.

system. There is a representation of a two state system via spin chain, that is an Ising model. It means that one state corresponds to $+1$ and another state corresponds to -1. For Ising models, it is natural to consider correlation functions. Many important observable quantities are defined in terms of correlation functions. Simplest correlation function, the one-point function is simply a magnetization of the system. The two-point function has connected and disconnected part. Disconnected part of $n-$point correlation function is trivially related to the connected part of correlation functions of lower orders. So, we will focus on the connected part of the correlation function only. One can say that connected part of $n-$point correlation function capture information about the interaction of order n, while disconnected part corresponds to the effective contribution that comes from lower-order interactions.

Our ultimate goal is to come up with a model that has descriptive and predictive power for fly's behaviors. In order to check descriptive power, we have to be able to tune our parameters in such way that we can reproduce known some quantities. We choose to compare connected part of the two-point spin-spin correlation function, that is the simplest quantity that has any dynamics in it.

We assume that we have approximate translational symmetry, that is not correct for finite-size chains, but we will ignore it for a while.

Any real chain that we can consider has a finite length. There are two consequences. The first one is that we have to define boundary conditions for a spin chain. We choose open boundary conditions. The second consequence is that if we have a spin chain of length L, we have to consider scales that are (much) smaller than L. In our simulations, we chose $L = 10^4$, and we defined cut-off length $R = 1600$.

Connected part of the 2-point correlation function can be simply computed numerically

$$\begin{aligned} \langle\langle \sigma\sigma(r) \rangle\rangle &= \langle \sigma\sigma(r) \rangle - \langle \sigma \rangle^2 = \\ &= \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{L}\sum_{i=1}^{L}\sigma_i\sigma_{i+r}\right) - \left(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{L}\sum_{i=1}^{L}\sigma_i\right)^2, \end{aligned} \tag{4.27}$$

where we $t \in [1, T]$, and T is number of steps in Montecarlo simulations. We performed Monte Carlo simulation using Metropolis algorithm for or spin chain that has length $L =$

10^4, cut-off $R = 1600$, with energy function

$$E = \sum_{i=1}^{L-1} \sum_{r=1}^{\min(L-i,R)} J(r)\sigma_i(t)\sigma_{i+r}(t) - h\sum_{i=1}^{L}\sigma_i(t), \tag{4.28}$$

This model is called maximal entropy model. Let us that the entropy is indeed maximal. The constrain that we fit theoretical expectation value and experimental results is

$$\langle f_i \rangle_{exp} = \langle f_i \rangle_{th} \Longrightarrow \sum_x p_{th}(x) f_i(x) - \langle f_i \rangle_{exp} = 0. \tag{4.29}$$

We have to maximize entropy under the condition eq.(4.29), That means that we have to maximize following functional

$$\tilde{S} = S + \sum_i \lambda_1 \left(\sum_x p_{th}(x) f_i(x) - \langle f_i \rangle_{exp} \right). \tag{4.30}$$

Variation with respect to probability distribution gives

$$\frac{\partial \tilde{S}}{\partial p_{th}(x)} = \log\Big(p(x)\Big) + \lambda_i f_i(x) = 0. \tag{4.31}$$

That gives us our cost function.

It is sufficient to run MC simulation for 500 sweeps, that gives 2×10^6 absolute steps. we use only about 20% sweeps ($4 \cdot 10^5$ steps) to compute time average because we have to wait until system thermalizes. The cut-off R defines, that is maximal distance at which we compute correlation function (also $J(r > R) = 0$). There are $1600 + 1$ degrees of freedom for this model ($J(r > R) = 0$).

The way we proceed is following. We have 59 experimental sequences of length $L \sim 10000$. We computed magnetization and correlation functions for $r \in [1, R]$ for each sequence, then we averaged over all of them, and we use what we got as a reference magnetization and correlation functions that we have to match.

For each theoretical model, we perform MC simulation, and we compute magnetization and 2−point spin-spin correlation functions for this model. Then for each $r \in [1, R]$, we compare $\langle\langle\sigma\sigma(r)\rangle\rangle_{th}$ and $\langle\langle\sigma\sigma(r)\rangle\rangle_{exp}$. We decrease $J(r)$ slightly, if $\langle\langle\sigma\sigma(r)\rangle\rangle_{th} > \langle\langle\sigma\sigma(r)\rangle\rangle_{exp}$ and increase $J(r)$ otherwise. Once we have agreement between all correlation functions we compare magnetizations and adjust magnetic field correspondingly.

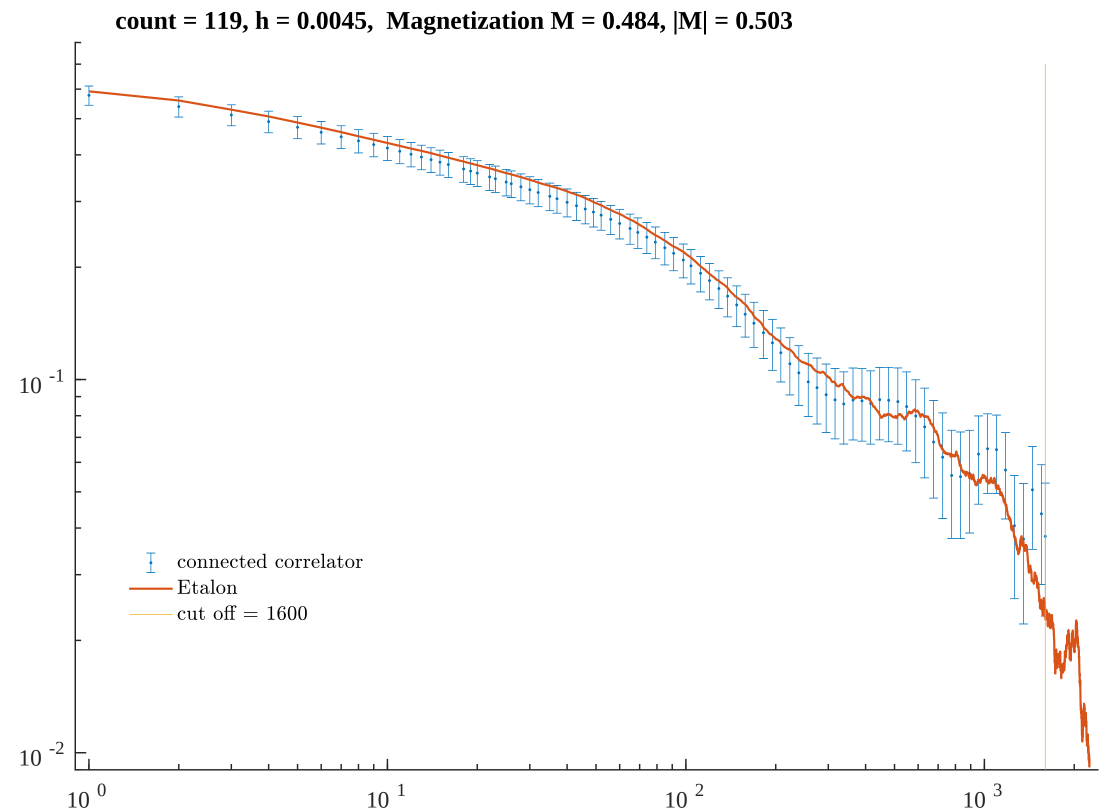

Figure 4.8: Comparison between two-point correlation function for MC simulation and experimental data

4.7 Simulation results

For a given magnetic field, we adjust $J(r)$ until theoretical correlation function is equal to experimental one. If theoretical magnetization is not equal to the experimental one we adjust magnetic field, h. Latest results are presented on the fig.(4.8) where we should compare connected part of the correlation function for experimental data and connected part of the correlation function for simulation. We see quite a good agreement.

Interaction is given on the fig.(4.9) Interaction looks like $1/r^2$, that is very interesting

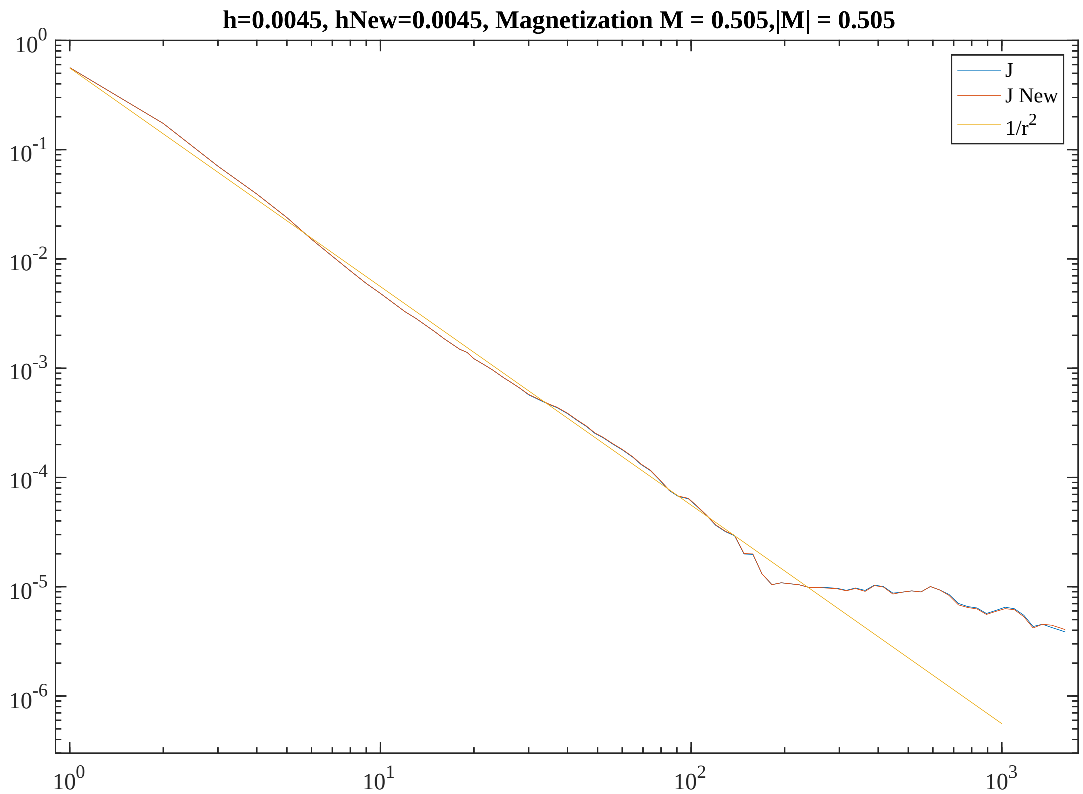

Figure 4.9: Profile of a long-range interaction for Maximal Entropy Model.

since this model is a special one [96, 97] (also see [98] for a review). Another feature is that magnetic field is quite small, that removes model only slightly from critical point. Also it is rather unexpected, because magnetization is rather large $m_{exp} = 0.424 \pm 0.048$. Our model is in quite good agreement with two point functions. Let us see how three point function looks like. We see that there are some similarities between experimental correlation function and one that we got from Montecarlo simulations, but these functions are different. One of the possible reason for the discrepancy is that we have not tuned our model completely.

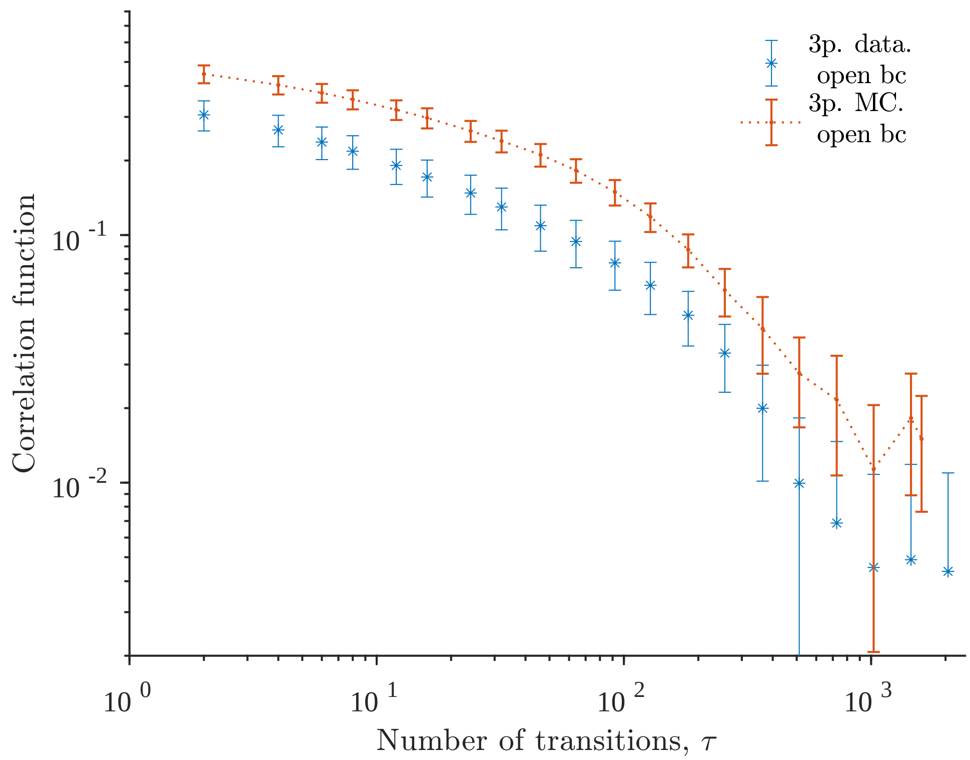

Figure 4.10: Comparison between three-point correlation function $\langle\langle ss(r)s(2r)\rangle\rangle$ for MC simulation and experimental data

One can see on the fig.(4.8), that there is an agreement between experimental results and simulations only within error bars, that means that we have some freedom for parameters in the model. We can fix profile of interaction and magnetic field only with some precision. Maybe this freedom can be used to remove discrepancy between connected part of $3-$point correlation function. Another point to notice is that we should use another, smaller, cut-off $R_2 < R$ for three-point function, because of the finite size of the chain.

We do not have enough data to compare correlation functions of higher order, so we would like to consider some non-perturbative function. The simplest one is domain size distribution n_l i.e., the number of finite domains of l connected sites, per lattice site fig.(4.11) For experimental data, we see that there is a power-law decay in distribution of domain size,

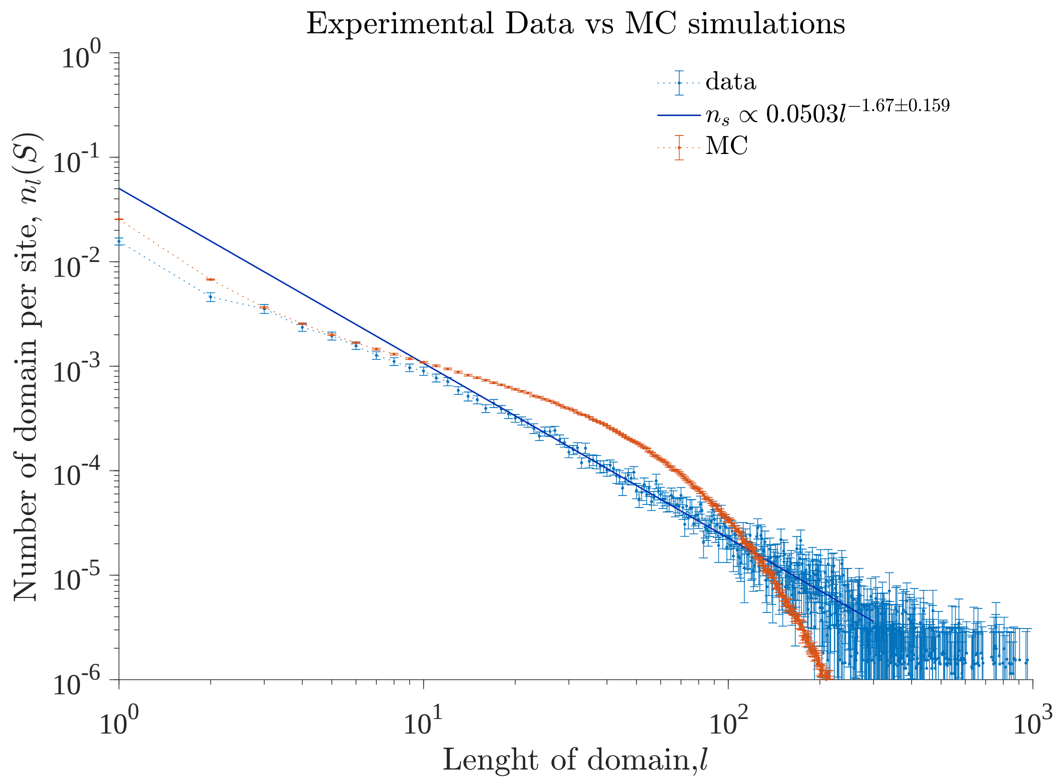

Figure 4.11: Cluster size distribution n_l i.e., the number of finite domains of l connected sites, per lattice site for experimental data and for simulations.

with exponent that is related to decay of power law, in particular, $\alpha_{n_l} = 1 + \alpha_{cor}$.

One can notice, that for domain sizes from $l = 2$ up to $l \sim 10^1$, there is some similarity in experimental and theoretical distributions. For larger l discrepancy is huge. However, it looks like discrepancy decreases if we fit our model better and better.

Interaction profile looks almost like $1/r^2$. but there is some irregularity at very large τ. We think that there are several reasons for this irregularity. First of all, our chain is of finite size, that is extremely crucial for a model near critical point because correlation length is large. Another reason is that long-range interaction has a cut-off, and that is why there may be some need to compensate for a tail.

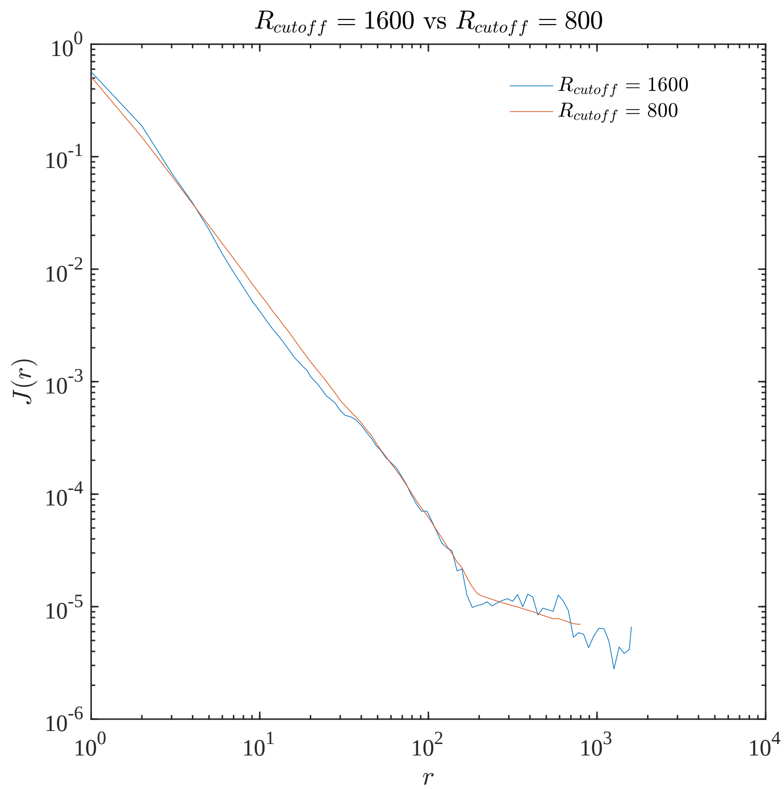

Figure 4.12: Size of the system is the main limitation for our simulations. We explore an impact of cut-off length on result for interaction profile $J(r)$. Error bars are not drawn on this plot, but they obviously are present.

The question of this irregularity is quite tricky and requires peculiar consideration. This question will definitely be explored in a future work [90].

We fixed the size of the lattice to be 10^4 spins, but we changed cut-off length $R_{cut-off}$. We performed simulations for $R_{cut-off} = 1600, 800$. We found the $1/r^2$ behavior for an interaction potential is consistent for a $r < 200$. Behavior for larger distances, $r > 200$, has much slower decay rate with power $\alpha \sim 0.5$

A

Appendices for Chapter 2

This whole chapter is completely based on the paper [59], coauthored with Kenan Diab, which is an extension of the paper[58], also coauthored with Kenan Diab.

A.1 Form factors as Fourier transforms of correlation functions

In this appendix, we will explicitly calculate the Fourier-transformed, lightcone-limit three-point functions F_s^b, F_s^f, and F_s^v cited in section 2.2. Let's start with the bosonic case. We want to compute the relevant Fourier transformation of the three-point function $\langle\phi(x_1)\phi^*(x_2)j_s(x_3)\rangle$. The explicit form of $j_s(x_3)$ is given in [99] as:

$$j_s = \sum_{k=0}^{s} c_k \partial^k \phi \partial^{s-k} \phi^*, \quad c_k = \frac{(-1)^k}{2} \frac{\binom{s}{k}\binom{s+d-4}{k+\frac{d}{2}-2}}{\binom{s+d-4}{\frac{d}{2}-2}} \tag{A.1}$$

Wick's theorem and translation invariance of the correlatiors yields that::

$$\langle \phi(x_1)\phi^*(x_2)j_s(x_3)\rangle = \sum c_i(\partial_3^i \langle \phi(x_1)\phi^*(x_3)\rangle)(\partial_3^{s-i} \langle \phi(x_3)\phi^*(x_2)\rangle) \tag{A.2}$$

$$= \sum c_i(\partial_1^i \langle \phi(x_1)\phi^*(x_3)\rangle)(\partial_2^{s-i} \langle \phi(x_3)\phi^*(x_2)\rangle) \tag{A.3}$$

Then, we may Fourier transform term by term with respect to x_1^- and x_2^-. Recalling that the propagator of a scalar field is $(x^2)^{\frac{2-d}{2}}$ and that in the lightcone limit, $x_1^+ = x_2^+$ and $\vec{y}_1 = \vec{y}_2$, we obtain:

$$\partial_1^{s-i}\partial_2^i \langle \phi(x_1)\phi^*(x_3)\rangle \langle \phi(x_3)\phi^*(x_2)\rangle$$

$$\longrightarrow i^s(p_1^+)^{s-i}(p_2^+)^i \int dx_1^- dx_2^- e^{ip_1^+x_1^-} e^{ip_2^+x_2^-} \frac{1}{\left(x_{13}^+x_{13}^- + \vec{y}_{13}^2\right)^{\frac{d-2}{2}}} \frac{1}{\left(x_{23}^+x_{23}^- + \vec{y}_{23}^2\right)^{\frac{d-2}{2}}} \tag{A.4}$$

$$= \frac{i^s(p_1^+)^{s-i}(p_2^+)^i}{(x_{13}^+)^{d-2}} \int dx_1^- dx_2^- e^{ip_1^+x_1^-} e^{ip_2^+x_2^-} \frac{1}{\left(x_{13}^- + \frac{\vec{y}_{13}^2}{x_{13}^+}\right)^{\frac{d-2}{2}}} \frac{1}{\left(x_{23}^- + \frac{\vec{y}_{13}^2}{x_{13}^+}\right)^{\frac{d-2}{2}}} \tag{A.5}$$

$$= \frac{i^s(p_1^+)^{s-i}(p_2^+)^i}{(x_{13}^+)^{d-2}} \left(\int dx_1^- e^{ip_1^+x_1^-} \frac{1}{\left(x_1^- - \bar{x}\right)^{\frac{d-2}{2}}}\right) \left(\int dx_2^- e^{ip_2^+x_2^-} \frac{1}{\left(x_2^- - \bar{x}\right)^{\frac{d-2}{2}}}\right) \tag{A.6}$$

Here, we have defined $\bar{x} = x_3^- - \frac{\vec{y}_{13}^2}{x_{13}^+}$. Depending on the parity of d, each integral has either a pole of order $\frac{d-2}{2}$ at $\bar{x}$ or a branch point at $\bar{x}$. Our prescription for evaluating this integral is as follows: First, we shift x_1^- and x_2^- by $\bar{x}$ so that the singularity is at 0, and then we will move move the singularity from 0 to sign$(p)i\epsilon$. Then, the integral can be evaluated by Schwinger parameterization. For example, suppose p_1^+ and p_2^+ are positive. Following our procedure, the x_1 integral becomes:

$$\int_{-\infty}^{\infty} dx_1^- e^{ip_1^+ x_1^-} \frac{1}{\left(x_1^- - \bar{x}\right)^{\frac{d-2}{2}}} = e^{ip_1^+ \bar{x} + p_1^+ \epsilon} \int_{-\infty}^{\infty} dy e^{ip_1^+ y} \frac{1}{\left(y - i\epsilon\right)^{\frac{d-2}{2}}} \tag{A.7}$$

$$= e^{ip_1^+ \bar{x} + p_1^+ \epsilon} \int_{-\infty}^{\infty} dy \int_0^{\infty} ds \frac{i}{\Gamma(\frac{d-2}{2})} e^{ip_1^+ y} s^{\frac{d-4}{2}} e^{-is(y-i\epsilon)} \tag{A.8}$$

$$= \frac{ie^{ip_1^+ \bar{x} + p_1^+ \epsilon}}{\Gamma(\frac{d-2}{2})} \int_0^{\infty} ds 2\pi \delta(s - p_1^+) e^{ip_1^+ y} s^{\frac{d-4}{2}} e^{-s\epsilon} \tag{A.9}$$

$$= \frac{2\pi i e^{ip_1^+ \bar{x}}}{\Gamma(\frac{d-2}{2})} (p_1^+)^{\frac{d-4}{2}} \tag{A.10}$$

This function is indeed nonsingular, as required. The x_2 integral has exactly the same form, and so gives the same answer. Hence, we obtain that the Fourier transform of $\langle \phi\phi^* j_s \rangle$ is indeed proportional to $\sum c_i (p_1^+)^i (p_2^+)^{s-i}$, where the proportionality factor is a nonsingular function. The, noting that the coefficients c_i are the coefficients for the hypergeometric function with appropriate arguments, we obtain the answer cited in the text:

$$F_s^b \equiv \langle \underline{\phi\phi^*} j_s \rangle \propto (p_2^+)^s {}_2F_1 \left(2 - \frac{d}{2} - s, -s, \frac{d}{2} - 1, p_1^+/p_2^+\right) \tag{A.11}$$

The fermionic and tensor cases can be tackled in exactly the same way. There are only two differences. First, the propagator in the free fermion and free tensor theories are $(x^2)^{\frac{1-d}{2}}$ and $(x^2)^{\frac{-d}{2}}$, respectively, as compared with the free scalar propagator $(x^2)^{\frac{2-d}{2}}$. Second, the coefficients in the expression for j_s are different, as can be checked from the expressions in [100] [101] or in [99]. The end result is that the arguments of the hypergeometric function are different in the way claimed in the text.

A.2 Uniqueness of three-point functions in the tensor lightcone limit

Our goal in this section is to show that the free tensor solution for the lightcone limit of three-point functions explained in section 2.2 is indeed unique, at least in the lightcone limit.

Note that Lorentz symmetry constrains the propagator of spin j field to be of the form

$$\langle \psi_{-j}(x)\bar{\psi}_{-j}(0)\rangle \propto (x^+)^{2j}. \tag{A.12}$$

Generically, according to [67], the most generic conformally invariant expression one can write down for a three-point function of symmetric conserved currents with tensor-type coordinate dependence is:

$$\begin{aligned}\langle j_{s_1} j_{s_2} j_{s_3}\rangle = \frac{1}{x_{12}^{d-2}x_{23}^{d-2}x_{13}^{d-2}}\sum_{a,b,c}\Big(&\left(\Lambda_1^2\alpha_{a,b,c}+\Lambda_2\beta_{a,b,c}\right)\left(P_{12}P_{21}\right)^{a}Q_1^b\\ &\left(P_{23}P_{32}\right)^{c}\left(P_{13}P_{31}\right)^{-a-b+s_1}Q_2^{-a-c+s_2}Q_3^{a+b-c-s_1+s_3}\Big)\end{aligned} \tag{A.13}$$

where the $\alpha_{a,b,c}$ and $\beta_{a,b,c}$ are free coefficients, and the Λ_i are defined as:

$$\Lambda_1 = Q_1Q_2Q_3 + \left[Q_1P_{23}P_{32}+Q_2P_{13}P_{31}+Q_3P_{12}P_{21}\right], \tag{A.14}$$

$$\Lambda_2 = 8P_{12}P_{21}P_{23}P_{32}P_{13}P_{31}. \tag{A.15}$$

Here, the P and Q invariants are defined as in [102] and [103]. However, for the choice of polarization vector $\epsilon^\mu = \epsilon^-$ there is a nontrivial relation:

$$\Lambda_2\big|_{\epsilon_i^\mu=\epsilon^-} = -2\Lambda_1^2\big|_{\epsilon_i^\mu=\epsilon^-}, \quad \Lambda_1\big|_{\epsilon_i^\mu=\epsilon^-} = \frac{1}{4}\frac{x_{12}^+x_{23}^+x_{13}^+}{x_{12}^2x_{23}^2x_{13}^2}(\epsilon^-)^3. \tag{A.16}$$

Therefore, in the case $\epsilon^\mu = \epsilon^-$ the expression for this three-point function greatly simplifies. Instead of having two sets of undetermined coefficients c_a and d_a, one can combine the Λ_i's into a single prefactor $\alpha_1\Lambda_1^2 + \alpha_2\Lambda_2$, where the α_i are arbitrary and can be chosen to be convenient; to produce exact agreement with the canonically normalized free-tensor theory, we will choose $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2(d-2)}$. Now, we take the lightcone limit, which corresponds to the point where

$$P_{23}P_{32} = 0, \quad Q_1 = -\left(\frac{P_{13}P_{31}}{Q_3} + \frac{P_{12}P_{21}}{Q_2}\right) \tag{A.17}$$

in P_{ij}, Q_i space. Then, the three-point function reduces to

$$\left\langle j_{s_1} \underline{j_{s_2} j_{s_3}}_t \right\rangle = \frac{\Lambda_1^2 + \Lambda_2/(2(d-2))}{x_{12}^{d-2} x_{23}^{d-2} x_{13}^{d-2}} \sum_{a=0}^{s_1-2} c_a \left(P_{12}P_{21}\right)^a \left(P_{13}P_{31}\right)^{s_1-2-a} Q_2^{s_2-a} Q_3^{S_3-s_1+a}, \tag{A.18}$$

Now, the c_a can be fixed demanding that all currents are conserved. The result is given by the following recurrence relation, with $c_0 = 1$:

$$\frac{c(a+1)}{c(a)} = \frac{(s_1 - 2 - a)(s_1 + \frac{d-4}{2} - a)(s_2 + a + \frac{d-2}{2})}{(a+1)(a + \frac{d-2}{2} + 2)(s_1 + s_3 + \frac{d-4}{2} - 2 - a)}$$

This solution exactly coincides with the free tensor solution, as required.

A.3 Uniqueness of $\langle s22\rangle$ for $s \geq 4$

Define

$$\langle j_{s_1} j_{s_2} j_{s_3}\rangle = \frac{\langle\langle j_{s_1} j_{s_2} j_{s_3}\rangle\rangle}{{x_{12}}^{d-2} {x_{23}}^{d-2} {x_{13}}^{d-2}}. \tag{A.19}$$

Using the previous defined V and H conformal invariants, we can write the most general expression for a conformally invariant correlation function as follows:

$$\begin{aligned}\langle\langle j_s j_2 j_2\rangle\rangle = V_1^{s-4}\Big[&a_1 H_{1,2}^2 H_{1,3}^2 + a_2\left(V_1 V_2 H_{1,2} H_{1,3}^2 + V_1 V_3 H_{1,2}^2 H_{1,3}\right) + a_3 V_1^2 H_{1,2} H_{1,3} H_{2,3} + \\ &+ a_4\left(V_1^2 V_3^2 H_{1,2}^2 + V_1^2 V_2^2 H_{1,3}^2\right) + a_5 V_1^2 V_2 V_3 H_{1,2} H_{1,3} + \\ &+ a_6\left(V_1^3 V_2 H_{1,3} H_{2,3} + V_1^3 V_3 H_{1,2} H_{2,3}\right) + a_7\left(V_1^3 V_2 V_3^2 H_{1,2} + V_1^3 V_2^2 V_3 H_{1,3}\right) + \\ &a_8 V_1^4 H_{2,3}^2 + a_9 V_1^4 V_2 V_3 H_{2,3} + a_{10} V_1^4 V_2^2 V_3^2\Big].\end{aligned} \tag{A.20}$$

The coefficients can be solved by imposing charge conservation. For example, in $d = 4$ we obtain:

$$a_1 = -\frac{a_7(s-3)(s-1)(s-2)^2}{32(s+1)(s+4)} + \frac{a_4(s-5)(s-3)s(s-2)}{8(s+1)(s+4)} + \frac{a_5(s-3)(s-2)}{8(s+4)}, \tag{A.21}$$

$$a_2 = -\frac{a_4(s-2)^2}{s+4} + \frac{a_7(s-1)(s-2)}{4(s+4)} - \frac{a_5(s-2)}{2(s+4)}, \tag{A.22}$$

$$a_3 = -\frac{8a_4\left(s^2-3s-1\right)}{(s+1)(s+4)} + \frac{a_5(s-8)}{2(s+4)} + \frac{a_7(s-1)(2s-1)}{(s+1)(s+4)}, \tag{A.23}$$

$$a_6 = \frac{12a_4(s-2)}{(s-1)(s+4)} + \frac{6a_5}{(s-1)(s+4)} + \frac{a_7(s-2)}{2(s+4)}, \tag{A.24}$$

$$a_8 = \frac{a_7(s-2)\left(s^2+11s-2\right)}{4s(s+1)(s+4)} - \frac{6a_4(s-5)}{(s+1)(s+4)} + \frac{a_5(s-2)}{s(s+4)}, \tag{A.25}$$

$$a_9 = \frac{a_7\left(s^2+8s-8\right)}{s(s+4)} - \frac{24a_4(s-2)}{(s-1)(s+4)} + \frac{4a_5(s-2)(s+2)}{(s-1)s(s+4)}, \tag{A.26}$$

$$a_{10} = \frac{a_7\left(s^2+8s+4\right)}{s(s+4)} - \frac{24a_4(s+1)}{(s-1)(s+4)} + \frac{4a_5(s+1)(s+2)}{(s-1)s(s+4)}. \tag{A.27}$$

Therefore, $\langle\langle j_s j_2 j_2\rangle\rangle_t$ depends only on three parameters. The bosonic light-cone limit of this function is zero if

$$a_5 = \frac{a_7(s-2)(s-1)}{4(s+1)} - \frac{a_4(s-5)s}{s+1}. \tag{A.28}$$

The fermionic light-cone limit of this function is also zero if

$$a_4 = \frac{a_7}{4}. \tag{A.29}$$

Therefore, $\langle\langle s22\rangle\rangle_t$ depends only on one parameter or in other words it is unique up to a rescaling[1]

$$\langle\langle j_s j_2 j_2\rangle\rangle_t \propto V_1^{s-2}\Big[H_{12}^2V_3^2+\left(H_{23}V_1+V_2\left(H_{13}+2V_1V_3\right)\right)^2+H_{12}\left(H_{13}+2V_1V_3\right)\left(H_{23}+2V_2V_3\right)\Big], \tag{A.30}$$

In arbitrary dimension $d>3$, the full expression is:

$$\begin{aligned}\langle\langle j_s j_2 j_2\rangle\rangle_t &= V_1^{s-2}\left[\left(H_{23}V_1+H_{13}V_2+H_{12}V_3+2V_2V_3V_1\right)^2+\frac{2}{(d-2)}H_{12}H_{13}H_{23}\right]\\ &= V_1^{s-2}\left[\Lambda_1^2+\frac{1}{2(d-2)}\Lambda_2\right].\end{aligned} \tag{A.31}$$

This formula coincides with the expression that was proposed in [67], and we have proven that this structure is unique.

A.4 Transformation properties of bilocal operators under K_-

In this appendix, we will prove 2.49 and 2.50 by computing the action of a finite conformal transformation on them. The same results can be proven using the infinitesimal transforma-

[1] In [66] it was proven that there are only three structures for $\langle\langle 22s\rangle\rangle$ in $d=4$.

tions, e.g. by using equation (3) of [104] and supplying the correct representation matrices for the Lie algebra of the Lorentz group. One can then check that the two computations agree by expanding our results to first order in b (remembering that only b^- is nonzero for K_-).

A.4.1 Fermionic case

Consider a special conformal transformation

$$x^\mu \to y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2} \tag{A.32}$$

Under K_-, the parameter $b^\mu = b^- \delta^\mu_-$. We know that F_- has the same transformation properties as the contraction of free fields $\bar{\psi}\gamma_-\psi$ on the lightcone. Since K_- sends the lightcone into the lightcone, V_{--} transforms the same way as $\bar{\psi}\gamma_-\psi$ under K_-. Using the well-known expression for the finite conformal transformation of a Dirac spinor (e.g. [105])

$$\psi(y) = \left|\frac{\partial y}{\partial x}\right|^{\Delta - 1/2} (1 - b_\mu x_\nu \gamma^\mu \gamma^\nu)\, \psi(x) \tag{A.33}$$

$$\bar{\psi}(y) = \left|\frac{\partial y}{\partial x}\right|^{\Delta - 1/2} \bar{\psi}(x)(1 - b_\mu x_\nu \gamma^\nu \gamma^\mu) \tag{A.34}$$

we may therefore compute:

$$F_-(\underline{y_1, y_2}) \sim \bar{\psi}(y_1)\gamma^+\psi(y_2) \tag{A.35}$$

$$= \left|\frac{\partial y_1}{\partial x_1}\right|^{\Delta-1/2} \left|\frac{\partial y_2}{\partial x_2}\right|^{\Delta-1/2} \bar{\psi}(x_1)(1 - b_\mu(x_1)_\nu\gamma^\nu\gamma^\mu)\gamma^+(1 - b_\mu(x_2)_\nu\gamma^\mu\gamma^\nu)\psi(x_2) \tag{A.36}$$

$$= \left|\frac{\partial y_1}{\partial x_1}\right|^{\Delta-1/2} \left|\frac{\partial y_2}{\partial x_2}\right|^{\Delta-1/2} \bar{\psi}(x_1)$$
$$\times [\gamma^+ - b_+(x_1)_\nu\gamma^\nu\gamma^+\gamma^+ - \gamma^+ b_+(x_2)_\nu\gamma^+\gamma^\nu + b_+(x_1)_\nu\gamma^\nu\gamma^+\gamma^+ b_+(x_2)_\mu\gamma^+\gamma^\mu]\psi(x_2) \tag{A.37}$$

$$= \left|\frac{\partial y_1}{\partial x_1}\right|^{\Delta-1/2} \left|\frac{\partial y_2}{\partial x_2}\right|^{\Delta-1/2} \bar{\psi}(x_1)\gamma^+\psi(x_2) \tag{A.38}$$

$$= \Omega^{d/2-1}(x_1)\Omega^{d/2-1}(x_2)F_-(\underline{x_1, x_2}) \tag{A.39}$$

The cancellations occur because $\gamma^+\gamma^+ = \eta^{++} = 0$. This is exactly equation 2.49.

A.4.2 Tensor case

We'll start with the four-dimensional case for ease of notation and then at the end, we'll describe how one can generalize the computation to all dimensions. Consider a special conformal transformation

$$x^\mu \to y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2} \tag{A.40}$$

Under K_-, the parameter $b^\mu = b^- \delta^\mu_-$. We know that V_{--} has the same transformation properties as the contraction of free fields $F_{-\mu}F^\mu_-$ on the lightcone. Since K_- sends the lightcone into the lightcone, V_{--} transforms the same way as $F_{-\mu}F^\mu_-$ under K_-. We therefore compute:

$$V_{--}(\underline{y_1, y_2}) = \left|\frac{\partial y_1}{\partial x_1}\right|^{-\tau_F/d} \left|\frac{\partial y_2}{\partial x_2}\right|^{-\tau_F/d} \frac{\partial x_1^\mu}{\partial y_1^-}\frac{\partial x_1^\nu}{\partial y_1^\alpha}\frac{\partial x_2^\lambda}{\partial y_2^-}\frac{\partial x_2^\rho}{\partial y_2^\beta}\eta^{\alpha\beta}F_{\mu\nu}(x_1)F_{\lambda\rho}(x_2) \tag{A.41}$$

$$= (1-b^- x_1^+)^{\tau_F}(1-b^- x_2^+)^{\tau_F}(1-b^- x_1^+)^2\eta^{\alpha\beta}F_{-\alpha}(x_1)F_{-\beta}(x_2) \tag{A.42}$$

$$= (1-b^- x_1^+)(1-b^- x_2^+)V_{--}(\underline{x_1, x_2}) \tag{A.43}$$

$$= \Omega(x_1)\Omega(x_2)V_{--}(\underline{x_1, x_2}) \tag{A.44}$$

In the above manipulations, $\tau_F = \Delta - s = 0$ is the twist of F, and in the second to last line, we used that $x_1^+ = x_2^+$ (because the points x_1 and x_2 are $-$ separated by hypothesis). This immediately implies 2.50 in the four-dimensional case. In general dimensions, the twist of F will not be 0, but rather $\Delta - s = d/2 - s$, and we will have a corresponding number of extra factors of $\partial x/\partial y$ to contract with the additional indices of F. This will make the exponent of the Ω factors equal to $\frac{d}{2} - 1$ instead of 1.

A.5 Proof that $\mathcal{O}_q$ exists

In this appendix, we will prove that an operator $\mathcal{O}_q$ whose correlation functions agree with the corresponding free field operator $\mathcal{O}_{q,free}$ defined in 2.79 exists in the operator spectrum of every conformal field theory with higher-spin symmetry. As usual, we will consider the bosonic case, since the tensor case works almost in precisely the same way. To prove our statement, we will show that in the free theory, for any $q \leq N$

$$A_{q,N}(x_1, x_2, \ldots, x_{q+1}) \equiv \left\langle \underbrace{\phi^2\phi^2\ldots\phi^2}_{\text{q copies}}\mathcal{O}_{q,free}\right\rangle \neq 0 \tag{A.45}$$

Here, $\phi^2 = \sum_i \phi_i^2$, which is known to appear in the OPE of two stress tensors. Thus, if we prove A.45, then we would know that $\mathcal{O}_{q,free}$ appears in the operator product expansion of $2q$ copies of the free field stress tensor j_2. Then, just as knowing the OPE structure of products of free field stress tensors allowed us to obtain conserved currents from products of the quasi-bilocal fields, we can obtain $\mathcal{O}_q$ in the full theory by defining it to be the operator appearing in the operator product expansion of $2q$ copies of j_2 in the full theory whose correlation functions coincide with the correlation functions of $\mathcal{O}_{q,free}$ in the free theory. Thus, it suffices to prove A.45.

First, note that we can immediately reduce to the $q = N$ case. This follows from the structure of the Wick contractions in $A_{q,N}$. To see this, note that every term in $\mathcal{O}_{q,free}$ involves exactly q of the N bosons, each of which appears twice for a total of $2q$ fields. Since ϕ^2 is bilinear in the fields, the product of q copies of ϕ^2 will also contain $2q$ fields. Hence, we will need all the ϕ^2 fields to be contracted with the $\mathcal{O}_{q,free}$ fields in order to obtain a nonzero answer. Thus, for each term in $\mathcal{O}_{q,free}$, none of the $N-q$ flavors appearing in that term will contribute, and so we can partition the terms in $A_{q,N}$ according to which of the q flavors appear. Since the correlation function is manifestly symmetric under relabeling of the N ϕ_i fields, this implies that each group of terms in this partition will equally contribute to the total correlation function an amount exactly equal to $A_{q,q}$. Hence, $A_{q,N} = \binom{N}{q} A_{q,q}$, so it suffices to show $A_{q,q}$ is nonzero.

Then, note that since $\mathcal{O}_{q,free}$ contains exactly two copies of each of the q ϕ_i fields, each of the q factors of ϕ^2 must contribute a different ϕ_i field for the contraction to be nonzero. Since $\mathcal{O}_{q,free}$ is manifestly invariant under arbitrary relabelings of the ϕ_i fields, we may relabel each term so that the first copy of ϕ^2 contributes ϕ_1^2, the second copy of ϕ^2 contributes ϕ_2^2 and so on. That is, we have

$$A_{q,q} = q! \left\langle \phi_1^2(x_1)\phi_2^2(x_2)\dots\phi_q^2(x_q)\mathcal{O}_{q,free}(x_{q+1})\right\rangle \tag{A.46}$$

The correlator on the right-hand side can be easily computed by direct evaluation of the Wick contractions. To illustrate, consider the result given by the term in $\mathcal{O}_{q,free}$ corresponding to setting the internal indices $i_k = j_k = k$ for all $k \in \{1, 2, \dots, q\}$. The contribution of this term is, up to a sign, given by:

$$\prod_{k=1}^{q} \partial_{q+1}^{k-1} x_{k,q+1}^{2-d} \tag{A.47}$$

This is a rational function whose numerator is an integer. All other terms in the correlation function will be generated by permuting the powers of the partial derivatives that appear. Hence, each term in the overall sum will depend on differently only each x_i, and the overall sum cannot cancel because the numerators have no x_i dependence. Thus, the correlation we wanted to show is non-zero is indeed non-zero, completing the proof.

A.6 The free Maxwell field in five dimensions

Consider the theory of a free Maxwell field in d dimensions. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 - \frac{1}{2\xi}\left(\partial A\right)^2 \tag{A.48}$$

where $\xi = \frac{d}{d-4}$. As was noted in [106], this theory is a conformal field theory with higher spin symmetry, but it is non-unitary in dimension d > 4. We claim that this theory is an example of a conformal, non-unitary theory where the three-point function of the stress tensor does not coincide with one of the three free structures described in the body of the paper. This can be checked by explicit calculation. The canonical stress energy tensor is not trace-free, and it may be improved using the procedure of [107]. The result is

$$\begin{aligned} T^{--} &= 4\partial_+ A^\rho \partial_+ A_\rho + \partial^\rho A^- \partial_\rho A^- - 4\partial_+ A^\rho \partial_\rho A^- + 4\frac{(d-4)}{d} A^- \partial_+ (\partial A) + \\ &\quad + \frac{1}{(d-2)}\Big[4a(\partial A)\partial_+ A^- + 4a\, A^- \partial_+(\partial A) + 4a\partial_+ A^\rho \partial_\rho A^- + 4a A^\rho \partial_\rho \partial_+ A^- + \\ &\qquad + 16b A_\rho \partial_+^2 A^\rho + 16b\partial_+ A_\rho \partial_+ A^\rho - 2a A^- \partial^2 A^- - 2a\partial^\rho A^- \partial_\rho A^-\Big] - \\ &\qquad\qquad - 2\frac{(d-4)}{(d-1)}\Big[\partial_+ A_\rho \partial_+ A^\rho + A_\rho \partial_+^2 A^\rho\Big], \end{aligned} \tag{A.49}$$

where $a = 2 - d/2, b = d/4 - 1$. Now, the three point function $\langle T_{--}T_{--}T_{--}\rangle$ can be evaluated by Wick contraction, and the result can be decomposed as follows:

$$\langle T_{--}T_{--}T_{--}\rangle = c_s \langle T_{--}T_{--}T_{--}\rangle_s + c_f \langle T_{--}T_{--}T_{--}\rangle_f + c_t \langle T_{--}T_{--}T_{--}\rangle_t \, , \quad \text{(A.50)}$$

where $c_s = \frac{12125}{576}, c_f = -\frac{1000}{9}, c_t = \frac{54179}{576}$. This demonstrates that unitarity is a necessary assumption for our result; the three-point function of the stress tensor is not the same as the result for an appropriate free field theory. It is a superposition of the three possible structures.

B

Appendices for Chapter 3

In the first part of the appendix we are going to describe a classical propagation of the graviton in the expanding Friedmann–Lemaitre–Robertson–Walker (FLRW) Universe with a small scalar perturbations. In the second part of the appendix, we are going to demonstrate field-theoretical computation, which reproduces the same result.

B.1 Evolution

Tensor perturbations are defined as following

$$ds^2 = a^2 \left(d\eta^2 - (\delta_{ij} - h_{ij})dx^i dx^j\right) , \tag{B.1}$$

where scale factor can be parametrized as

$$a(\eta) = a(\eta - \eta_i)^q,\ q = \frac{2}{1+3w}, \quad \eta \in \left(\eta_i^{tr}, \eta_{i+1}^{tr}\right) , \tag{B.2}$$

where w is defined through an equation of state $p = w\rho$ and η_i^{tr} is a time of transition from stage $i-1$ to stage i. Equation of motion for gravitational waves is

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \triangle h_{ij} = 16\pi G a^2 \pi_{ij}^{TT}, \tag{B.3}$$

where π_{ij}^{TT} is a part of stress-energy tensor. For a perfect fluid $\pi_{ij}^{TT} = 0$.

There are two limits: a superhorizon limit ($k^2 \ll |a''/a|$), the modes are frozen, and a subhorizon limit ($k^2 \gg |a''/a|$), the gradient term is negligible.

There are two solutions in the superhorizon limit. The first solution is a constant $h = h_i$. The second solution decreases in time $h(\eta) \sim \int^{\eta} a^{-2}(\tau)d\tau$. Thus, only constant solution is physical.

For our analysis it is convenient to decompose solution for a tensor perturbation in the following form

$$h_{ij}(\mathbf{k}, x) = A(\mathbf{k}) \underbrace{\frac{\phi(k,\eta)}{a(\eta)}}_{=h(\eta)} e_{ij}(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}} + A^{\dagger}(\mathbf{k}) \underbrace{\frac{\bar{\phi}(k,\eta)}{a(\eta)}}_{=\bar{h}(\eta)} e_{ij}(\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}}, \tag{B.4}$$

where $e_{ij}(\mathbf{k})$ is a time-independent polarization tensor, and $A(\mathbf{k})$, $A^{\dagger}(\mathbf{k})$ are annihilation and creation operators with canonical commutation relation $[A(\mathbf{k}), A^{\dagger}(\mathbf{p})] = \delta(\mathbf{k}-\mathbf{p})$. Equation of motion for $\phi(k,\eta)$ looks like an equation of motion for a scalar

$$\phi''(k,\eta) + \left(k^2 - \frac{a''}{a}\right)\phi(k,\eta) = 0. \tag{B.5}$$

Solution for this equation is a superposition of Bessel or Hankel functions

$$\begin{aligned}\phi(k,\eta) &= \frac{b_i}{k^{1/2}}\sqrt{k(\eta-\eta_i)}J_{q-1/2}(k(\eta-\eta_i)) + \frac{c_i}{k^{1/2}}\sqrt{k(\eta-\eta_i)}Y_{q-1/2}(k(\eta-\eta_i)) = \\ &= \frac{\tilde{b}_i}{k^{1/2}}\sqrt{k(\eta-\eta_i)}H^{(1)}_{q-1/2}(k(\eta-\eta_i)) + \frac{\tilde{c}_i}{k^{1/2}}\sqrt{k(\eta-\eta_i)}H^{(2)}_{q-1/2}(k(\eta-\eta_i)),\end{aligned} \tag{B.6}$$

B.1.1 Solution during inflation

During slow-roll inflation stage scale factor and Hubble constant in the first order in epsilon are

$$a(\eta) = \frac{1+\epsilon}{H_\epsilon(\eta_\epsilon - \eta)^{1+\epsilon}}, \quad H(\eta) = H_\epsilon(\eta_\epsilon - \eta)^\epsilon, \quad \eta < 0. \tag{B.7}$$

In order to fix coefficients we have to use asymptotic behavior for a solution $e^{-ik\eta}$, as $\eta \to -\infty$, for the inflationary phase, that gives us

$$\phi_\epsilon(k,\eta) = \frac{\sqrt{\pi}}{2k^{1/2}}\sqrt{k(\eta_\epsilon - \eta)}H^{(2)}_{-3/2-\epsilon}(k(\eta_\epsilon - \eta)), \quad \eta \in (-\infty, 0), \tag{B.8}$$

where we used normalization condition

$$\phi(k,\eta)\bar{\phi}'(k,\eta) - \bar{\phi}(k,\eta)\phi'(k,\eta) = i. \tag{B.9}$$

Limit $\eta \to -\eta_\epsilon$ for the complete solution

$$\begin{aligned}\lim_{\eta\to-\eta_\epsilon} h(k,\eta) &= \frac{-k^{\nu+1/2}}{a_\epsilon k^{1/2}2^\nu\Gamma(\nu+1)}\left(1 - i\frac{\cos\nu\pi}{\sin\nu\pi}\right) = \\ &= \frac{H_\epsilon}{\sqrt{2}k^{3/2+\epsilon}}\left[1+\epsilon\left(-1-i\pi+\log(2)+\psi^{(0)}\left(-\frac{1}{2}\right)\right)\right]\end{aligned} \tag{B.10}$$

is finite for negative $\nu = q - 1/2$ that is the case for slow roll inflation ($q_\epsilon = -1-\epsilon, \nu_\epsilon = -3/2-\epsilon$)

B.1.2 Solution during the first non-inflationary era

Next era has positive q_i so in order to have finite amplitude of the gravity wave at $\eta = 0, k\eta_1 \to 0$, we should have only $J_{q_1-1/2}(k(\eta+\eta_1))$[1]

$$\lim_{k\eta_1\to 0} h_1(k,0) = \frac{b_1 k^{q_1-1/2}}{a_1 2^{q_1-1/2}\Gamma(q_1+1)} = \frac{H_\epsilon}{\sqrt{2}k^{-\nu_\epsilon}} \Rightarrow b_1 = \frac{a_1 H_\epsilon 2^{q_1-1}\Gamma\left(q_1+\frac{1}{2}\right)}{k^{1+\epsilon+q_1}} \quad \text{(B.11)}$$

So the solution for the first era after inflation[2] is

$$\phi_1(k,\eta) = \frac{a_1 H_\epsilon 2^{q_1-1}\Gamma\left(q_1+\frac{1}{2}\right)}{k^{-\nu_\epsilon+q}}\sqrt{k(\eta+\eta_1)}J_{q-1/2}(k(\eta+\eta_1)), \quad \eta\in(0,\eta_0). \quad \text{(B.13)}$$

Boundary conditions

$$a_\epsilon(0) = a_1(0), \quad a'_\epsilon(0) = a'_1(0) \quad \text{(B.14)}$$

gives us

$$\eta_\epsilon = \frac{(1+\epsilon)}{q_1}\eta_1, \quad \text{(B.15)}$$

$$a_1 = \frac{(1-\epsilon)q_1^{1+\epsilon}}{H_\epsilon \eta_\epsilon^{q+1+\epsilon}}. \quad \text{(B.16)}$$

The solution during second era is

$$\phi_1(k,\eta) = \frac{q_1^{1+\epsilon}2^{q_1-1}\Gamma\left(q_1+\frac{1}{2}\right)}{k^{1/2}(k\eta_1)^{q_1+1+\epsilon}}\sqrt{k(\eta+\eta_1)}J_{q-1/2}(k(\eta+\eta_1)). \quad \text{(B.17)}$$

---

[1] Actually as one can see in appendix B.6 we always should have mix of J_ν and Y_ν with complex coefficient to be able to satisfy normalization condition and to properly match solutions. But an error in the assumption is of order $(k\eta_\epsilon)^5 \ll 1$, so we can neglect other terms for our purposes.

[2] Suppose now there is still radiation-dominated era. Then gravity wave has an amplitude

$$h(\eta_0) = h_i \frac{H_0\sqrt{\Omega_{rad}}}{q_0}\left(\frac{g_{*,0}}{g_*(\eta_*)}\right)^{1/6} \sin q_0\eta_0/a_0, \quad \eta_0 = \frac{1}{a_0 H_0}. \quad \text{(B.12)}$$

where $q_0 = k/a_0$ is current physical momentum, $g_*(\eta)$ is an effective number of degrees of freedom at time η, subscript 0 correspond to the current time.

B.1.3 Solutions during consecutive eras

During consecutive eras solution is also given by (B.6), where coefficients (b_i, c_i) or $(\tilde{b}_i, \tilde{c}_i)$ should be fixed from the condition that solution $h(k,\eta)$ and its derivative $h'(k,\eta)$ are continuous. An example of three stage solution is given in appendix (B.6). In next section we are going to present a solution for an arbitrary history of the Universe under some assumptions.

B.1.4 Solutions for an arbitrary history

We are going to divide history of the Universe into five periods for each mode

1. $\eta \ll 0$. Mode is deep below horizon. WKB approximation is valid.
2. $\eta \approx \eta_\times$. Mode is crossing horizon. The solution is exact under slow-roll approximation.
3. $\eta_\times \ll \eta \ll \eta_{cr}$. Superhorizon perturbations. The mode is constant.
4. $\eta \approx \eta_{cr}$. Mode is crossing horizon, during matter-eradominated era with an equation of state $p = w\rho$.
5. $\eta \gg \eta_{cr}$. Mode is deep below horizon. WKB approximation is valid.

Lets build solution in each era and match them. Modes obey an equation of motion

$$h_k'' + 2\frac{a'}{a}h_k' + k^2 h_k = 0. \tag{B.18}$$

In order to use WKB approximation we have to make a change of variables

$$x = \int \frac{d\eta}{a^2(\eta)}, \tag{B.19}$$

then the equation of motion reads

$$\frac{d^2 h_k}{dx^2} + k^2 a^4 h_k = 0. \tag{B.20}$$

Solution is

$$h_k = \frac{1}{\sqrt{2k}}\frac{1}{a(x)}\exp\left(\pm ik\int a^2 dx\right) = \frac{1}{\sqrt{2k}}\frac{1}{a(\eta)}\exp\left(\pm i(k\eta + c)\right). \tag{B.21}$$

During horizon crossing, we assume that Hubble constant H changes slowly. Solution mod irrelevant phase is given by

$$h_k(\eta) = \frac{\sqrt{\pi}}{2k^{1/2}}\frac{(k\eta)^{1/2}}{a(\eta)}H^{(2)}_{\nu_\epsilon}(k\eta) \tag{B.22}$$

$$h_k(\eta)^* = \frac{\sqrt{\pi}}{2k^{1/2}}\frac{(k\eta)^{1/2}}{a(\eta)}H^{(1)}_{\nu_\epsilon}(k\eta), \quad \text{where } \nu_\epsilon = -\frac{3-\epsilon}{2(1-\epsilon)} \approx -\frac{3}{2} - \epsilon. \tag{B.23}$$

This solution should be matched with WKB solution in the limit $\eta \ll 0$ ($k\eta \to 0$)

$$\lim_{\eta\to -0} h_k = -\frac{H_\times}{k^{3/2+\epsilon}}\frac{2^{1/2+\epsilon}\sqrt{\pi}}{\Gamma\left(-\frac{1}{2}-\epsilon\right)} = \frac{H_\times}{2^{1/2-\epsilon}k^{3/2+\epsilon}}\left(1 + \epsilon\psi^{(0)}\left(-\frac{1}{2}\right)\right), \tag{B.24}$$

where we should take a value of Hubble constant at the time of horizon crossing. The second horizon crossing happens during matter-dominated era with an equation of state $p = w\rho$.There are two solutions

$$h_k(\eta) = \frac{b_1}{k^{1/2}}\frac{(k\eta)^{1/2}J_{q-\frac{1}{2}}(k\eta)}{a(\eta)} + \frac{b_2}{k^{1/2}}\frac{(k\eta)^{1/2}Y_{q-\frac{1}{2}}(k\eta)}{a(\eta)}, \quad q = \frac{2}{(1+3w)}. \tag{B.25}$$

We have to match this solution with the constant at $k\eta \ll 1$. We can choose (b_1, b_2) such that singularity disappears and b_1 should be chosen such that constant value is equal to one that we found. The solution is

$$h_k(\eta) = -\frac{H_\times}{k^{3/2+\epsilon}}\frac{\sqrt{\pi}2^{q+\epsilon}a_w\Gamma\left(q+\frac{1}{2}\right)}{k^q\Gamma\left(-\epsilon-\frac{1}{2}\right)}\frac{\sqrt{k\eta}J_{q-1/2}(k\eta)}{a(\eta)} \tag{B.26}$$

In the limit $k\eta \gg 1$ Bessel function behaves as

$$J_\alpha(z) \underset{z\to\infty}{=} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right), \tag{B.27}$$

so solution at late time in WKB approximation is

$$h_k(\eta) = -\frac{H_\times}{k^{q+\frac{3}{2}+\epsilon}} \frac{2^{q+\frac{1}{2}+\epsilon}\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}-\epsilon\right)} \frac{a_\times}{a(\eta)} \frac{k^q}{(k\eta_\times)^q} \cos\left(k\eta - \frac{\pi q}{2}\right), \tag{B.28}$$

where $a_\times := a(\eta_\times)$ is a scale factor at the horizon crossing time $\eta_\times$, and

$$\alpha = \frac{3}{2}\frac{(1-w)}{1+3w} = q - \frac{1}{2}, \quad \nu = \frac{3-\epsilon}{2(1-\epsilon)} \approx \frac{3}{2} + \epsilon. \tag{B.29}$$

In order to write expression in coordinate independent way we should do some gymnastics

$$H_{w,\times} = \frac{a'_\times}{a^2_\times} = \frac{q a_w \eta_\times^{q-1}}{a_w^2 \eta_\times^{2q}} = \frac{q}{a_w \eta_\times^{q+1}} = \frac{q}{a_\times \eta_\times} \Rightarrow \eta_\times = \frac{q}{a_\times H_{w,\times}}. \tag{B.30}$$

The final answer is

$$h(k,\eta) = -\frac{H_\times}{k^{q+\frac{3}{2}+\epsilon}} \frac{2^{q+\frac{1}{2}+\epsilon}\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}-\epsilon\right)} \left(\frac{a_\times H_{w,\times}}{q}\right)^q \frac{a_\times}{a(\eta)} \cos\left(k\eta - \frac{\pi q}{2}\right). \tag{B.31}$$

The solution depends only on time of the horizon crossing and on the equation of state at that time.

B.2 Particle creation

B.2.1 The first method to calculate Bogolyubov coefficients

Here we consider an approximation of several phase transition. We assume that phase transition is almost instant in comparison with wave period[3]. In the opposite case of adiabatic changes, there is on particle production.

Let us discuss a mechanism of a particle creation. Vacuum depends on the equation of state, and therefore if phase transition is instant then vacuum from one phase may become an excited state during other phase. Solutions for each phase can be expanded in terms of creation and annihilation operators

$$\phi_i(k,\eta) = \sum_m A_m \mu_{m,i}(k,\eta) + A_m^\dagger \bar{\mu}_{m,i}(k,\eta), \tag{B.32}$$

$$\phi_{i+1}(k,\eta) = \sum_n a_n \mu_{n,i+1}(k,\eta) + a_n^\dagger \bar{\mu}_{n,i+1}(k,\eta) \tag{B.33}$$

where $\{A_m, A_m^\dagger\}$, $\{a_n, a_n^\dagger\}$ are creation and annihilation operators in the phase i and $i+1$ correspondingly. Both sets form a complete basis, so we can decompose one basis in terms of another basis

$$a_n = \sum_m \alpha_{nm} A_m + \bar{\beta}_{nm} A_m^\dagger, \tag{B.34}$$

$$a_n^\dagger = \sum_m \beta_{nm} A_m + \bar{\alpha}_{nm} A_m^\dagger. \tag{B.35}$$

In order to get an expression for α, β, we substitute these expressions into the solution for the second phase, we demand that at the transition time function and its derivative are

[3] In particular, if the time of phase transition is $a\Delta\eta$ then we assume that $\omega \ll 2\pi/(a\Delta\eta)$.

continuous $h_i(\eta_t) = h_{i+1}(\eta_t), h'_i(\eta_t) = h'_{i+1}(\eta_t)$. We get[4]

$$\alpha_{i+1} = i\left[\bar{\mu}_{i+1}(\eta_t)\mu'_i(\eta_t) - \bar{\mu}'_{i+1}(\eta_t)\mu_i(\eta_t)\right], \tag{B.36}$$

$$\beta_{i+1} = i\left[\mu_i(\eta_t)\mu'_{i+1}(\eta_t) - \mu'_i(\eta_t)\mu_{i+1}(\eta_t)\right], \tag{B.37}$$

where we used $\mu_i(\eta)\bar{\mu}'_i(\eta) - \bar{\mu}_i(\eta)\mu'_i(\eta) = i$, that follows from the normalization condition $\left[\hat{\phi}, \hat{\pi}\right] = i$. Solutions are chosen to be

$$\mu_i(\eta) = \frac{\sqrt{\pi}}{2k^{1/2}}\sqrt{k(\eta - \eta_i)}H^{(2)}_{q-1/2}(k(\eta - \eta_i)), \tag{B.38}$$

these modes tend to be positive frequency modes in the far past.

We would like to stress attention that there is no contradiction between this choice and the solution that we found for the first non-inflationary mode. Indeed the solution is

$$\phi_i = \alpha_i\mu_i + \beta\bar{\mu}_i. \tag{B.39}$$

In this approach coefficient β indeed coincide with the coefficient that is defined usually as a coefficient in front of $(1/\sqrt{k})e^{ik\eta}$, because e^{iz} is an asymptotic of Hankel function of the first type $H^{(1)}_\nu(z)$.

If there are several phase transition we have to take into account several Bogolyubov transformations. It is easy to get a recurrent relation

$$\begin{pmatrix} \alpha_{i\to f} & \beta_{i\to f} \\ \bar{\beta}_{i\to f} & \bar{\alpha}_{i\to f} \end{pmatrix} = \begin{pmatrix} \alpha_{i\to f-1} & \beta_{i\to f-1} \\ \bar{\beta}_{i\to f-1} & \bar{\alpha}_{i\to f-1} \end{pmatrix}\begin{pmatrix} \alpha_f & \beta_f \\ \bar{\beta}_f & \bar{\alpha}_f \end{pmatrix}. \tag{B.40}$$

One should take into account that everything here was derived under an assumption of an instant phase transition. This is important because for low-frequency modes phase transition is adiabatic, and there is no particle production.

[4]We dropped dependance on k.

Bogolyubov coefficient for a transition from Slow-roll inflation to arbitrary era

As an example, we may calculate Bogolyubov coefficients for a transition from slow-roll inflation to non-inflationary stage in the lowest order in $(k\eta_1)$ is

$$\begin{aligned}\beta = & -\frac{i\pi(k\eta_1)(\epsilon+1)}{4\sqrt{q(\epsilon+1)}}\left[H^{(2)}_{q-\frac{1}{2}}(k\eta_1)H^{(2)}_{-\epsilon-\frac{1}{2}}\left(\frac{(\epsilon+1)(k\eta_1)}{q}\right)+\right.\\ & \left.+H^{(2)}_{q+\frac{1}{2}}(k\eta_1)H^{(2)}_{-\epsilon-\frac{3}{2}}\left(\frac{(\epsilon+1)(k\eta_1)}{q}\right)\right]\\ & =\frac{(1-\epsilon)2^{q+\epsilon}q^{1+\epsilon}\Gamma\left(q+\frac{1}{2}\right)\left[1-i\tan(\pi\epsilon)\right](k\eta_1)^{-q-\epsilon-1}}{\Gamma\left(-\epsilon-\frac{1}{2}\right)}=\\ & =\frac{q2^{q-1}\Gamma\left(q+\frac{1}{2}\right)}{\sqrt{\pi}(k\eta_1)^{1+q+\epsilon}}\left[1+\epsilon\left(-\log(q)+1+i\pi-\log(2)-\psi^{(0)}\left(-\frac{1}{2}\right)\right)\right]\end{aligned} \tag{B.41}$$

An expectation value of number of gravitons is

$$\langle N_{\omega(k)}\rangle = |\beta(k)|^2 = \frac{q^2 2^{2q-2}\Gamma^2\left(q+\frac{1}{2}\right)}{\pi(k\eta_1)^{2+2q+2\epsilon}} \sim \begin{cases} \dfrac{1}{(k\eta_1)^{4+2\epsilon}}, & \text{radiation domination,} \\ \\ \dfrac{1}{(k\eta_1)^{6+2\epsilon}}, & \text{matter domination.} \end{cases} \tag{B.42}$$

B.2.2 More conventional method of Bogolyubov coefficient calculation

In this method, we consider only asymptotical behavior of the modes in the past and in the future. The solution for the present time was found in the previous section. So, from (B.31) one get that Bogolyubov coefficient is

$$\beta \sim \frac{1}{k^{q+\frac{3}{2}+\epsilon-\frac{1}{2}}} = \frac{1}{k^{q+1+\epsilon}} \tag{B.43}$$

B.2.3 Graviton density

The energy density of the gravitational waves with frequencies between ω and $\omega + d\omega$ is

$$d\rho_g = 2\hbar\omega \frac{\omega^2 d\omega}{(2\pi)^3} d\Omega \left\langle N_{\omega(k)} \right\rangle = \frac{\omega^3}{4\pi^3} |\beta_{i+1}|^2 d\omega d\Omega. \tag{B.44}$$

Lets stress our attention to the functional behavior of the density

$$d\rho_g \sim \frac{\omega^3}{k^{2q+2+2\epsilon}\eta_1^{2q+2+2\epsilon}} d\omega d\Omega = \frac{d\omega d\Omega}{a(\eta)^{2q+2+2\epsilon}\omega(\eta)^{2q-1+2\epsilon}} \tag{B.45}$$

After integration we have

$$\frac{d\rho_g}{d\Omega} \sim \frac{1}{a(\eta)^{2q+2+2\epsilon}} \left(\frac{1}{\omega_{min}^{2q-2+2\epsilon}} - \frac{1}{\omega_{max}^{2q-2+2\epsilon}} \right), \tag{B.46}$$

where $\omega_{max} \approx 2\pi H(\eta_1)$ and $\omega_{\min} = 2\pi H(\eta)$. At present time $a \approx \eta^q$ therefore $H(\eta) = a'/a^2 \approx a^{-2/q}$.

$$\frac{d\rho}{d\Omega} \approx \frac{1}{a^{2q+2+\epsilon}a^{-2(2q-2+\epsilon)/q}} = \frac{1}{a^{\frac{4-4\epsilon}{q}+2q+2\epsilon-2}} = \frac{1}{a^{\frac{4}{3w+1}-6w(\epsilon-1)}} =$$
$$= \begin{cases} \dfrac{1}{a^{4-2\epsilon}}, & w = 1/3, \text{ RD}, \\ \\ \dfrac{1}{a^4}, & w = 0, \text{ Dust} \end{cases} \tag{B.47}$$

For matter $p = w\rho$

$$\rho \sim \frac{1}{a^{3(1+\omega)}} = \frac{1}{a^{2+\frac{2}{q}}} = \begin{cases} \dfrac{1}{a^4}, & w = 1/3, \text{ RD}, \\ \\ \dfrac{1}{a^3}, & w = 0, \text{ Dust} \end{cases} \tag{B.48}$$

B.2.4 Spectrum

A spectrum is defined as

$$d\rho = P(\omega) d\omega d\Omega, \tag{B.49}$$

So, one can get that the spectrum is

$$P(\omega) = \frac{\omega^3}{4\pi^3}|\beta_i|^2 \sim \frac{1}{a(\eta)^{2+2q+2\epsilon}\omega(\eta)^{2q-1+2\epsilon}}. \tag{B.50}$$

B.3 Anisotropy imprint to energy

Let us consider a simple way to relate anisotropy and gravitational perturbation. An energy of a particle that has 4-momentum P^μ that is measured by an observer with four velocity U^μ is $\Omega = P^\mu U_\mu$. In order to find a 4-velocity one can solve a geodesic equation for massless particle

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{B.51}$$

it is enough for us to consider this equation for zero component only. Then we can define momentum $P^\mu = \frac{dx^\mu}{d\lambda}$ and $\frac{dP^\mu}{d\lambda} = P^0 \frac{dP^\mu}{d\eta}$. Using this notation one can write geodesic equation for zero component in a form

$$\frac{1}{P^0}\frac{dP^0}{d\eta} + \Gamma^0_{00} + 2\Gamma^0_{0i} n^i + \Gamma^0_{ij} n^i n^j = 0, \tag{B.52}$$

where $n^i := P^i/P^0$ is a unit vector.

B.3.1 Isotropic case. Exact FRW Universe

FRW Universe is a homogeneous Universe with a metric

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\mathbf{x}^2\right). \tag{B.53}$$

Christoffel symbols are

$$\Gamma^0_{00} = \frac{a'}{a}, \tag{B.54}$$

$$\Gamma^0_{ij} = \delta_{ij}\frac{a'}{a}, \tag{B.55}$$

$$\Gamma^i_{0j} = \delta^i_j\frac{a'}{a}. \tag{B.56}$$

Equation (B.52) become

$$\frac{(P^0)'}{P^0} + 2\frac{a'}{a} = 0. \tag{B.57}$$

It is easy to integrate this equation

$$P^0 \propto \frac{1}{a^2}. \tag{B.58}$$

Frequency that is measured by an observer with 4-velocity U_μ is

$$\Omega = P^\mu U_\mu = P^0 U_0 + P^i U_i, \tag{B.59}$$

where we are going to ignore U_i because it corresponds to a Doppler effect, but here we are interested in gravitational red-shift only, so we put $U_i = 0$. From the normalization condition $U^\mu U_\mu = 1$, we get $U_0 = a$, so the measured frequency is

$$\Omega = P^0 U_0 = \frac{const}{a^2} a \propto \frac{1}{a}. \tag{B.60}$$

We got an answer that we expected to get $(T \propto 1/a)$.

B.3.2 Perturbed FRW in ζ-gauge

Here for a simplicity we would like to get rid of $a^2(\eta)$ factor. An action

$$S = \int ds \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} g_{\alpha\beta}, \tag{B.61}$$

is invariant under the rescaling

$$g_{\alpha\beta} \to a^2(\tau)\bar{g}_{\alpha\beta}, \quad ds \to a^2(\tau)d\tau. \tag{B.62}$$

Therefore, geodesics for the massless particle are the same for the original metric and for rescaled one. Thus, rescaled metric in a ζ-gauge looks like

$$ds^2 = -d\eta^2 - \frac{4}{5}\frac{a}{a'}\partial_i\zeta_0(\mathbf{x})dx^i d\eta + (1 + 2\zeta_0(\mathbf{x}))\, d\mathbf{x}^2. \tag{B.63}$$

Geodesic equation (B.52) can be written in another form

$$\frac{dP_\mu}{d\lambda} - \frac{1}{2}\partial_\mu g_{\alpha\beta}P^\alpha P^\beta = 0. \tag{B.64}$$

In the $g_{00} = -1$ it gives

$$\frac{dP_\mu}{d\eta} = P^0 n^i \partial_\mu g_{0i} + \frac{1}{2}P^0 n^i n^j \partial_\mu g_{ij}. \tag{B.65}$$

In $\zeta-$gauge for the dust dominated era g_{00}, g_{ij} are η independent therefore only g_{0i} gives contribution for zero component of the equation

$$\frac{dP_0}{d\eta} = P^0\partial_0 g_{0i}n^i = -\frac{2}{5}P^0\left(\frac{a}{a'}\right)' n^i\partial_i\zeta_0 = -\frac{1}{5q}P^0\frac{d}{d\eta}\zeta_0, \tag{B.66}$$

where we used that a derivative along geodesic is

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + n^i\partial_i. \tag{B.67}$$

We use $P_0 = \bar{P}_0 + \delta P_0(\eta)$ and $\bar{P}_0 \approx -\bar{P}^0$ to integrate this equation[5]

$$P_0(\eta_2) = P_0(\eta_1) \left(1 + \frac{1}{5q} \left(\zeta_0(\eta_2) - \zeta_0(\eta_1) \right) \right), \tag{B.69}$$

where $\bar{P}^0$ is the lowest order expression for a P^0 that is a constant. Thus, measured frequency is

$$E(\eta) = P_\mu(\eta) U^\mu = P_0(\eta) \left(1 - n^i v^i(\eta) \right). \tag{B.70}$$

So relative change of frequency is

$$\frac{E(\eta_2) - E(\eta_1)}{E(\eta_1)} = \frac{1}{5q} \left(\zeta_0(\eta_2) - \zeta_0(\eta_1) \right) - n^i v^i(\eta_2) + n^i v^i(\eta_1), \tag{B.71}$$

where last two terms correspond to Doppler shift. So we have a relative shift of the massless particle that was emitted at time η_1 and was absorbed at time η_2. The spatial component of four-momentum is constant in the lowest order because its change is proportional to the fluctuation.

We should mention that this consideration is valid for small w (vicinity of $q = 2$). For the case of general w, we should take into account equation of motion for ζ and solve a geodesic equation for time dependent $\zeta(\eta)$.

[5] In the case if ζ depends on a conformal time

$$P_0(\eta_2) - P_0(\eta_1) = \frac{2}{5q} \bar{P}_0 \left(\zeta(\eta_2) - \zeta(\eta_1) \right) - \bar{P}_0 \left(1 + \frac{2}{5q} \right) \int_{\eta_1}^{\eta_2} d\eta \partial_\eta \zeta(\eta) + \\ + \frac{2}{5q} \bar{P}_0 \int_{\eta_1}^{\eta_2} d\eta \eta n^i \partial_i \partial_\eta \zeta(\eta), \tag{B.68}$$

where we integrate along geodesic.

B.4 Anisotropy

Now we would like to take into account that our background is not isotropic, and we have some perturbation. In section B.3 we derived the behavior of the energy of the graviton that is measured by a local observer as a function of time and of the perturbation. The result (B.71) is

$$\frac{E(\eta_2) - E(\eta_1)}{E(\eta_1)} \sim \frac{1}{5} \left(\zeta_0(\eta_2) - \zeta_0(\eta_1)\right) + \ldots , \tag{B.72}$$

where we neglected Doppler terms for the moment. The conformal energy that it measured gets a shift in such a way that shift is proportional to the energy. So one can conclude that the shape of the spectrum is not changed. The spectrum is

$$P(\omega) = \frac{\omega^3}{4\pi^3}|\beta_i|^2 \sim \frac{1}{a(\eta)^{2+2q+2\epsilon}\omega(\eta)^{2q-1+2\epsilon}} \left(1 + \frac{2}{5}\left(q+1+\epsilon\right)\left(\zeta_0(\eta_\times, \mathbf{x}_\times) - \zeta_0(\eta, \mathbf{x})\right)\right), \tag{B.73}$$

where $\eta_\times$ is a time when mode enters horizon. As was mentioned before this result is valid for the vicinity of $w = 0$ ($q = 2$). In next section, we will consider another approach to the problem that will let us reproduce answer for gravitational wave anisotropy in this case and will let us extend this result to other cases.

B.5 Field theoretical calculation

B.5.1 Matter-Dominated Era

The action for a massless scalar field that propagates on the background (B.63) is

$$S = \frac{1}{2}\int d^{D-1}\mathbf{x}\, d\eta \times \\ \times\, a^{D-2}\left(1 + (D-1)\zeta_0(\mathbf{x})\right)\left(-(\partial_\eta h)^2 + \left(\vec{\partial} h\right)^2 - 2\zeta_0(\mathbf{x})\left(\vec{\partial} h\right)^2 - \frac{4h'}{5}\frac{\vec{\partial}\zeta_0.\vec{\partial} h}{aH}\right), \tag{B.74}$$

where D is the dimension of the space-time. An equation of motion for his field for the case $\zeta = 0$ is

$$h'' + (D-2)\frac{a'}{a}h' - \triangle h = 0. \tag{B.75}$$

From now on we are going to assume that the dimension of the space-time $D = 4$. It is convenient to work in Fourier space because that let us make a clear distinction between short and long wavelength modes. Now we can compute two point function

$$\langle h(\eta)h(\eta)\rangle = \left\langle \Psi \middle| S^{\dagger}(-\infty,\eta)h(\eta,\mathbf{x})h(\eta,\mathbf{x})S(-\infty,\eta)\middle| \Psi \right\rangle, \tag{B.76}$$

where

$$S = T\exp\left(i\int\limits_{-\infty}^{\eta} d\eta' L_{int}(\eta')\right), \tag{B.77}$$

$$-\mathcal{L}_{int} = \underbrace{a^2\zeta_0(\mathbf{x})\left[3\left(\partial_\eta h\right)^2 - \left(\vec{\partial}h\right)^2\right]}_{diagonal} + \underbrace{\frac{4}{5}\frac{a\partial_\eta h}{H}\vec{\partial}\zeta_0.\vec{\partial}h}_{off-diagonal}. \tag{B.78}$$

Zero order contribution is

$$\langle h(\eta,\mathbf{k}), h(\eta,\mathbf{k}')\rangle^{(0)} = \delta(\mathbf{k}+\mathbf{k}')\frac{9H^2}{4k^{3+2\epsilon}}\frac{1}{(k\eta)^4} \tag{B.79}$$

where we neglected all terms that are proportional to sin $2k\eta$, cos $2k\eta$. Leading contribution comes from the period of horizon crossing. Contribution from MD interval to the first order correction is

$$\langle h(\eta,\mathbf{k})h(\eta,\mathbf{k}')\rangle^{(1)}_{MD} = -\delta\left(\mathbf{k}+\mathbf{k}'+\mathbf{k}_L\right)\zeta(\mathbf{k}_L)\frac{9H^2}{20k^{3+2\epsilon}}\frac{(11+6\cos(\eta\hat{\mathbf{k}}.\mathbf{k}_L))}{(k\eta)^4} + O(\epsilon) \tag{B.80}$$

So we find that in x space anisotropic contribution is proportional to the difference of ζ at the crossing horizon and now

$$\langle h(\eta)h(\eta)\rangle_{MD} = \langle h(\eta)h(\eta)\rangle^0 \left(1 + \frac{2}{5}(2+1)\left(\zeta(\eta_\times) - \zeta(\eta)\right)\right) + O(\epsilon). \tag{B.81}$$

This result coincides with the result from the classical calculation. Computations for a different equation of state can be done in a similar way. We do not present it here for a sake of saving space, because it requires a long gymnastics with Bessel functions. restriction that we have for an equation of state is $w \in \left(-\frac{1}{3}, 1\right)$.

B.6 Solutions Matching

We consider space-time

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + d\mathbf{x}^2\right), \tag{B.82}$$

where scale factor $a(\eta)$ is determined by an equation of state of the matter. We are going to consider two scenarios. In the first scenario de Sitter era lasted form $\eta = -\infty$ to $\eta = 0$ then Matter-Dominated era lasts form $\eta = 0$ to present time. In the second one de Sitter era lasted from $\eta = -\infty$ to $\eta = 0$. Radiation-dominated era lasted from $\eta = 0$ to $\eta = \eta_{eq}$ and Matter-dominated era lasts from $\eta = \eta_{eq}$ to present times.

B.6.1 de Sitter - MD

During inflation we have

$$a_\Lambda(\eta) = -\frac{1}{H(\eta - \eta_\Lambda)}, \quad H_\Lambda(\eta) = H, \quad \eta \in (-\infty, 0) \tag{B.83}$$

while during matter-dominated era (equation of state $p = 0$)

$$a_{MD}(\eta) = a_{0m}(\eta+\eta_m)^2, \quad H_{MD}(\eta) = \frac{2}{a_{0m}(\eta+\eta_m)^3}, \quad \eta \in (0,\eta_0). \tag{B.84}$$

In order to relate end of inflation time η_Λ and time of the begging of the reheating η_m we have two conditions

$$a_\Lambda(0) = a_{MD}(0), \quad H_\Lambda(0) = H_{MD}(0). \tag{B.85}$$

From these conditions we found that

$$\eta_m = \left(\frac{2}{a_{0m}H}\right)^{\frac{1}{3}}, \qquad \eta_\Lambda = \frac{1}{2}\eta_m. \tag{B.86}$$

We use the following normalization of solution

$$\phi_\Lambda(\eta) = \frac{H}{\sqrt{2k^3}}\left(1-k(\eta-\eta_\Lambda)\right)e^{ik(\eta-\eta_\Lambda)}, \tag{B.87}$$

$$\phi_{MD}(\eta) = \frac{H}{\sqrt{2k^3}}\sqrt{\frac{\pi}{2}}\left((3u_1+iu_2)\frac{J_{\frac{3}{2}}(k(\eta+\eta_m))}{(k(\eta+\eta_m))^{3/2}} + (v_1+iv_2)\frac{Y_{\frac{3}{2}}(k(\eta+\eta_m))}{(k(\eta+\eta_m))^{3/2}}\right) \tag{B.88}$$

Solutions should be equal

$$\left.\left(1-ik(\eta-\eta_\Lambda)\right)e^{ik(\eta-\eta_\Lambda)}\right|_{\eta=0} = \\ = \sqrt{\frac{\pi}{2}}\left((3u_1+iu_2)\frac{J_{\frac{3}{2}}(k(\eta+\eta_m))}{(k(\eta+\eta_m))^{3/2}} + (v_1+iv_2)\frac{Y_{\frac{3}{2}}(k(\eta+\eta_m))}{(k(\eta+\eta_m))^{3/2}}\right)\Big|_{\eta=0}, \tag{B.89}$$

And the first derivatives should be equal

$$\left.\frac{d}{d(k\eta)}\left(1-ik(\eta-\eta_\Lambda)\right)e^{ik(\eta-\eta_\Lambda)}\right|_{\eta=0} = \\ = \sqrt{\frac{\pi}{2}}\frac{d}{d(k\eta)}\left((3u_1+iu_2)\frac{J_{\frac{3}{2}}(k(\eta+\eta_m))}{(k(\eta+\eta_m))^{3/2}} + (v_1+iv_2)\frac{Y_{\frac{3}{2}}(k(\eta+\eta_m))}{(k(\eta+\eta_m))^{3/2}}\right)\Big|_{\eta=0}. \tag{B.90}$$

The solution for this system of equation is

$$u_1 = 1 + \frac{\theta^2}{8} - \frac{\theta^4}{128} + \frac{257\theta^6}{9216}, u_2 = \frac{\theta^3}{8} + \frac{31\theta^5}{320}, v_1 = -\frac{\theta^5}{10}, v_2 = \frac{\theta^6}{12} \tag{B.91}$$

where $\theta = k\eta_m =: k\eta_{reh}$.

For this solution canonical commutator is

$$\left[\hat{\phi}(\eta, \mathbf{k}), \hat{\pi}(\eta, \mathbf{p})\right] = \delta(\mathbf{k} - \mathbf{p}) i \left(1 + \frac{7\theta^2}{40} + \frac{99\theta^4}{3200} + \frac{257\theta^6}{9216}\right). \tag{B.92}$$

B.6.2 de Sitter-RD-MD

Scale factors matching

The most general solutions for scale factors for these eras are

$$a_\Lambda(\eta) = -\frac{1}{H(\eta - \eta_\Lambda)}, \qquad \eta \in (-\infty, 0), \tag{B.93}$$

$$a_{RD}(\eta) = a_{0r}(\eta + \eta_r), \qquad \eta \in (0, \eta_{eq}) \tag{B.94}$$

$$a_{MD}(\eta) = a_{0m}(\eta + \eta_m)^2, \qquad \eta \in (\eta_{eq}, \eta_0). \tag{B.95}$$

Constants in these solutions can be fixed my requiring that scale function and its derivative are continuous at at every point.

Lets consider transition from de Sitter era to Radiation-dominated era

$$a_\Lambda(0) = a_{RD}(0), \quad H_\Lambda(0) = H_{RD}(0). \tag{B.96}$$

From these two conditions we have

$$\eta_r = \frac{1}{\sqrt{a_{0r}H}} = \eta_\Lambda \tag{B.97}$$

Considering transition from radiation-dominated to matter dominated era we get

$$a_{0m} = \frac{a_{0r}}{4(\eta_{eq} + \eta_r)}, \quad \eta_m = \eta_{eq} + 2\eta_r \tag{B.98}$$

Solutions matching

First we should match solution for de Sitter era and for radiation-dominated era. We demand that value of solution at the end of de Sitter era is equal to the value of the solution at the beginning of the radiation-dominated era

$$\frac{H}{\sqrt{2k^3}} (1 - ik\eta) e^{ik\eta}\Big|_{\eta=-\eta_r} = \frac{H}{\sqrt{2k^3}} \left((u_1 + iu_2) \frac{\sin k\eta_r}{k\eta_r} + (v_1 + iv_2) \frac{\cos k\eta_r}{k\eta_r} \right), \tag{B.99}$$

end that value of the first derivatives are equal as well

$$\frac{d}{d(k\eta)} (1 - ik\eta) e^{ik\eta}\Big|_{\eta=-\eta_r} = \frac{d}{d(k\eta_r)} \left((u_1 + iu_2) \frac{\sin k\eta_r}{k\eta_r} + (v_1 + iv_2) \frac{\cos k\eta_r}{k\eta_r} \right). \tag{B.100}$$

Due to the existence of symmetry $\mathbf{x} \to \lambda\mathbf{x}, \eta \to \lambda\eta, k \to k/\lambda$ invariant quantity is $k\eta_{reh} =: \theta$. We assume that θ is small and we can expand in Taylor series r.h.s. and l.h.s.. in other words we should solve the system

$$1 + \frac{\theta^2}{2} - i\frac{\theta^3}{3} - \frac{\theta^4}{8} = (u_1 + iu_2) \left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} \right) + (v_1 + iv_2) \left(\frac{1}{\theta} - \frac{\theta}{2} + \frac{\theta^3}{24} \right), \tag{B.101}$$

$$-\theta + i\theta^2 + \frac{\theta^3}{2} - i\frac{\theta^4}{6} = (u_1 + iu_2) \left(-\frac{\theta}{3} + \frac{\theta^3}{30} \right) + (v_1 + iv_2) \left(-\frac{1}{\theta^2} - \frac{1}{2} + \frac{\theta^2}{8} - \frac{\theta^4}{144} \right). \tag{B.102}$$

Solution for this system is

$$(u_1 + iu_2, v_1 + iv_2) = \left(1 + \theta^4 + i\frac{2}{3}\theta^3, \frac{2}{3}\theta^3 - i\theta^4 \right). \tag{B.103}$$

So solution for radiation-dominated era is

$$\phi_k(\eta) = \frac{H}{\sqrt{2k^3}} \left(1 + (k\eta_{reh})^4 + i\frac{2}{3}(k\eta_{reh})^3 \right) \frac{\sin k(\eta + \eta_{reh})}{k(\eta + \eta_{reh})} + \\ + \frac{H}{\sqrt{2k^3}} \left(\frac{2}{3}(k\eta_{reh})^3 - i(k\eta_{reh})^4 \right) \frac{\cos k(\eta + \eta_{reh})}{k(\eta + \eta_{reh})}, \quad \eta \in (0, \eta_{eq}). \quad \text{(B.104)}$$

The commutator of ϕ and canonical momentum it is equal

$$[\phi(\eta, \mathbf{k}), \pi(\eta, \mathbf{p})] = i \left(1 + \frac{4}{9}\theta^2 + \theta^4 \right) \delta(\mathbf{k} - \mathbf{p}). \quad \text{(B.105)}$$

On can check that the more orders in theta are taken into account the higher precision of equality commutator to i (e.g. if one take into account terms $\sim O(\theta^{24})$, then the commutator is equal $i\,(1 + O(\theta^{22}))$).

From previous consideration, we know the solution for the radiation-dominated era. And we know that general solution for the matter-dominated era is

$$\phi_k(\eta) = \frac{H}{\sqrt{2k^3}} \sqrt{\frac{\pi}{2}} \left((3f_1 + if_2) \frac{J_{\frac{3}{2}}(k(\eta + \eta_m))}{(k(\eta + \eta_m))^{3/2}} + (g_1 + ig_2) \frac{Y_{\frac{3}{2}}(k(\eta + \eta_m))}{(k(\eta + \eta_m))^{3/2}} \right), \quad \text{(B.106)}$$

$$\phi_k^*(\eta) = \frac{H}{\sqrt{2k^3}} \sqrt{\frac{\pi}{2}} \left((3f_1 - if_2) \frac{J_{\frac{3}{2}}(k(\eta + \eta_m))}{(k(\eta + \eta_m))^{3/2}} + (g_1 - ig_2) \frac{Y_{\frac{3}{2}}(k(\eta + \eta_m))}{(k(\eta + \eta_m))^{3/2}} \right). \quad \text{(B.107)}$$

We demand that solutions and the first derivatives are continuos at the equality time.

There are two limits that are very easy to analyze. The first one is $k\eta_{eq} \ll 1$ and the second limit is $k\eta_{eq} \gg 1$. In the the first limit solution is

$$f_1 = 1 + \frac{5}{18}(k\eta_r)^2 + \frac{2}{9}\frac{(k\eta_r)^3}{k\eta_{eq}} - \frac{1}{5}(k\eta_r)^4 - \frac{2}{9}\frac{(k\eta_r)^4}{(k\eta_{eq})^2} + k\eta_{eq}\left(\frac{5(k\eta_r)}{9} + \frac{8}{15}(k\eta_r)^3\right) \tag{B.108}$$

$$+ (k\eta_{eq})^2\left(\frac{5}{18} + \frac{22}{15}(k\eta_r)^2 + \frac{254}{45}(k\eta_r)^4\right), \tag{B.109}$$

$$f_2 = 2\frac{(k\eta_r)^4}{k\eta_{eq}} + 4k\eta_{eq}(k\eta_r)^4, \tag{B.110}$$

$$g_1 = -\frac{16}{3}(k\eta_r)^4 k\eta_{eq}, \tag{B.111}$$

$$g_2 = \frac{16}{3}(k\eta_r)^4(k\eta_{eq})^2. \tag{B.112}$$

This solution allows to write two point function in the first order in η_{eq} and η_{reh}

$$\langle\phi(\eta,\mathbf{k})\phi(\eta,\mathbf{p})\rangle^{(0)} = \delta(\mathbf{k}+\mathbf{p})\frac{9H^2}{2k^3}\frac{1}{(k\eta)^4}\left[1 - \frac{8\eta_{reh}}{\eta} - 4\frac{\eta_{eq}}{\eta}\right]. \tag{B.113}$$

In the second limit $k\eta_{eq} \gg 1$ we have zero order solution

$$f_1 = \frac{1}{3}k\eta_{eq}\sin k\eta_{eq}, \quad f_2 = 0, \quad g_1 = -k\eta_{eq}\cos k\eta_{eq}, \quad g_2 = 0. \tag{B.114}$$

$$\langle\phi(\eta,\mathbf{k})\phi(\eta,\mathbf{p})\rangle^{(0)} = \delta(\mathbf{k}+\mathbf{p})\frac{H^2}{2k^3}\frac{\eta_{eq}^2}{k^2(\eta+\eta_{eq})^4}. \tag{B.115}$$

B.6.3 Quintessence $p = w\rho$

During inflation we have

$$a_\Lambda(\eta) = -\frac{1}{H\eta}, \quad H_\Lambda(\eta) = H, \tag{B.116}$$

while during matter-dominated era with equation of state $p = w\rho$

$$a_w(\eta) = \beta\eta^{2/(1+3w)}, \quad H_w(\eta) = \frac{2}{(1+3w)\beta\eta^{3(1+w)/(1+3w)}}. \tag{B.117}$$

In order to relate end of inflation time η^* and η_{reh} time of the begging of the reheating we have two conditions

$$a_\Lambda(\eta^*) = a_w(\eta_{reh}), \quad H_\Lambda(\eta^*) = H_w(\eta_{reh}). \tag{B.118}$$

From these conditions we found that

$$\eta_{reh} = \left(\frac{2}{\beta H(1+3w)}\right)^{\frac{1+3w}{3(1+w)}}, \qquad \eta^* = -\frac{1}{2}(1+3w)\eta_{reh}. \tag{B.119}$$

B.7 Some remarks about solutions

For matter-dominated era scaling factor is $a \sim \eta^{\alpha+1/2}$, where $\alpha = (3(1-w))/(2(1+3w))$. Solutions for modes are given by

$$\phi_k(\eta) = (u_1 + iu_2)\sqrt{\frac{\pi}{2}}\sqrt{k\eta}\frac{J_\alpha(k\eta)}{a(\eta)} + (v_1 + iv_2)\sqrt{\frac{\pi}{2}}\sqrt{k\eta}\frac{Y_\alpha(k\eta)}{a(\eta)}. \tag{B.120}$$

Modes normalization condition gives

$$u_1v_2 - u_2v_1 = \frac{1}{2k}. \tag{B.121}$$

Propagator for field is

$$\left[\hat{\phi}(k,\eta_1), \hat{\phi}(k,\eta_2)\right] = i\frac{\pi}{2}\frac{\sqrt{\eta_1\eta_2}}{a(\eta_1)a(\eta_2)}\left(J_\alpha(k\eta_1)Y_\alpha(k\eta_2) - J_\alpha(k\eta_2)Y_\alpha(k\eta_1)\right) \tag{B.122}$$

B.8 Connection between time and number of e-foldings

Physical momentum at present

$$q_0 = \frac{k}{a_0}. \tag{B.123}$$

Definition of the number of $e-$foldings

$$\frac{a_e}{a_k} = e^{N_e}. \tag{B.124}$$

Condition that mode crosses the horizon

$$H_k = \frac{k}{a_k} = q_0 \frac{a_0}{a_e} e^{N_e}. \tag{B.125}$$

Let's estimate a_0/a_e

$$\frac{a_0}{a_e} = \frac{a_0}{a_{reh}} \frac{a_{reh}}{a_e} = \frac{T_{reh}}{T_0} \left(\frac{\rho_e}{\rho_{reh}}\right)^{1/\beta}, \tag{B.126}$$

where we assume that $\rho(a) \sim a^{-\beta}$, $\beta = const$.

B.9 Connection between conformal time and scalar factor

From conservation of stress-energy tensor we have entropy conservation $g_* a^3 T^3 = const$. For radiation-dominated era Friedmann equation is

$$H^2 = \frac{8\pi}{3} G g_* \frac{\pi^2}{30} T^4 = \left(\frac{g_{*,0}}{g_*}\right)^{1/3} \Omega_{rad} H_0^2 \left(\frac{a_0}{a}\right)^4. \tag{B.127}$$

From this equation we get

$$\eta = \int_0^t \frac{d\tau}{a(\tau)} = \int_0^a \frac{da}{a^2 H(a)} = \left(\frac{g_*}{g_{*,0}}\right)^{1/6} \frac{1}{a_0 H_0 h^{-1} \sqrt{\Omega_{rad} h^2}} \frac{a}{a_0}, \tag{B.128}$$

where $\Omega_{rad} h^2 \approx 4 \cdot 10^{-5}$

B.10 Asymptotics of Bessel functions

It is convenient to know the behavior of Bessel functions at small arguments (non integer order ν)

$$J_\nu(z) = z^\nu \frac{1}{2^\nu \Gamma(1+\nu)} + \dots, \tag{B.129}$$

$$Y_\nu(z) = -\frac{1}{z^\nu} \frac{2^\nu}{\Gamma(1-\nu)\sin \pi\nu} + \dots, \tag{B.130}$$

$$H_\nu^{(1)}(z) = -i \frac{1}{z^\nu} \frac{2^\nu}{\Gamma(1-\nu)\sin \pi\nu} + \dots, \tag{B.131}$$

$$H_\nu^{(2)}(z) = i \frac{1}{z^\nu} \frac{2^\nu}{\Gamma(1-\nu)\sin \pi\nu} + \dots. \tag{B.132}$$

For a large values of $|z|$

$$J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[1 - \frac{1}{z^2} \frac{\Gamma(\nu+5/2)}{8\Gamma(\nu-3/2)} + \dots\right] - \right. \tag{B.133}$$

$$\left. - \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\frac{1}{z} \frac{4\nu^2-1}{8} + \dots\right] \right\}, \tag{B.134}$$

$$Y_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[1 - \frac{1}{z^2} \frac{\Gamma(\nu+5/2)}{8\Gamma(\nu-3/2)} + \dots\right] - , \right. \tag{B.135}$$

$$\left. - \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\frac{1}{z} \frac{4\nu^2-1}{8} + \dots\right] \right\} \tag{B.136}$$

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \left(1 + \frac{1}{z} \frac{i(1+2\nu)}{4} + \dots\right), \tag{B.137}$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \left(1 - \frac{1}{z} \frac{i(1+2\nu)}{4} + \dots\right), \tag{B.138}$$

C

Appendices for Chapter 4

C.1 HMM training

Hidden Markov Model (HMM) is a statistical Markov model in which system is being modeled by a Markov process with unobservable states [108, 109] and random output for each state. There are several questions that are usually asked in the context of HMM, however, we are going to focus on only one of them: What is an optimal set of parameters such that likelihood of an observed data is maximized?

A standard way to find an answer for this question is to use an expectation–maximization (EM) algorithm, that is an iterative method to find maximum a posteriorly estimates of parameters in a statistical model. There are two steps in this algorithm. The first one is ”E” step, that implies computation of the expected log-likelihood for current values of parameters. The second step, ”M” step, computes values of parameters that maximize the expected log-likelihood found on the ”E” step. In the case of HMM, realization of the ”EM” algorithm is called Baum–Welch algorithm, which was first obtained in a series of papers [110, 111, 112].

We are following [109]. First we would like to define few convenient concepts. Hidden states are S_t , $O = \{O_t\}$ is an observable sequence, $\alpha_i(t)$ is a forward probability

$$\alpha_t(i) = \text{Prob}\left(O_1, O_2, \ldots, O_t, S_t = i|\theta\right), \tag{C.1}$$

where, $S_t = j$ means ”the $t-$th state in the sequence of states is state j” [109]. It is convenient to compute this probability recursively

$$\alpha_1(i) = a_{1j} b_j \left(O_i\right), \; 1 \leq i \leq N, \tag{C.2}$$

$$\alpha_t(i) = \sum_{j=1}^{N} \alpha_{t-1}(i) a_{ij} b_j \left(O_t\right). \tag{C.3}$$

A backward probability

$$\beta_t(i) = \text{Prob}\left(O_{t+1}, O_{t+2}, \ldots, O_T, S_t = 0|\theta\right) \tag{C.4}$$

usually is computed recursively

$$\beta_T(i) = a_{iF}, \; 1 \leq i \leq N, \tag{C.5}$$

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(j), \; 1 \leq i \leq N, \; 1 \leq t \leq T, \tag{C.6}$$

where F is a final state.

We are ready to define E and M steps. The first step is E-step:

$$\gamma_t(i) = \text{Prob}\left(S_t = i|O,\theta\right) = \frac{\alpha_t(i)\beta_t(i)}{\sum_j \alpha_t(j)\beta_t(j)}, \tag{C.7}$$

$$\xi_t(i,j) = \text{Prob}\left(S_{t-1} = 1, S_t = j|O,\theta\right) = \frac{\alpha_t(i)a_{ij}b_j\left(O_{t+1}\right)\beta_{t+1}(j)}{\sum_j \alpha_t(j)\beta_t(j)}. \tag{C.8}$$

The second step is $M-$step

$$\hat{\pi}_i = \gamma_i(1), \tag{C.9}$$

$$\hat{a}_{ij} = \frac{\sum\limits_{t=1}^{T-1} \xi_t(i,j)}{\sum\limits_{t=1}^{T-1} \gamma_t(i)}, \tag{C.10}$$

$$\hat{b}_j(\nu_k) = \frac{\sum\limits_{t=1,O_t=\nu_k}^{T} \gamma_t(j)}{\sum\limits_{t=1}^{T} \gamma_t(j)}. \tag{C.11}$$

We start with a random initial parameters and repeat these steps until parameters are quite stable.

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