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Stochastic Plume Estimation: Measurement Sampling for a Supermartingale Support

by

Steven Cutlip

B.S., Electrical Engineering, University of New Mexico, 2014

THESIS

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Dedication

To everyone who has told me to not stop moving forward

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Abstract

In this paper we use a simple a model for a stochastically moving plume center and determine sufficient measurement schemes, for three cases of measurement noise, that reduce the support of the plume center's probability distribution. We assume a multivariate gaussian plume that moves according to a stochastic discrete-time stochastic linear time-invariant model. We also assume a measurement function that is a function of proximity to the center of the plume distribution. Using both knowledge of the dynamics and the behaviour of this measurement function a recursive probability distribution was formulated. We then found sufficient measurement schemes that reduce the support of this recursive probability distribution such that the area of the support behaves like a supermartingale.

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Glossary

LTI	Linear Time Invariant
MLE	Maximum Likelihood Estimator
MEBD	Minimal Expected Branch Distance
LLN	Law of Large Numbers

Chapter 1

Introduction

Advancements in robotics, most popularly quad-copters, and a wide-variety of estimation algorithms allow us to track and understand phenomena that are either too dangerous or are unfeasible for humans to measure directly [1] [2] [3] [4] [5]. The list of phenomena that fall into this category is large and varied; natural disasters make up a large group, for example volcanic flows [6], volcanic ash dispersion [7], and dust storms [8], as well as man-made problems such as pollution plumes or chemical spills [9], [10], [11]. Accurately tracking and predicting phenomena of these types can prevent the loss of lives, property, and nature as well as provide better insight into the phenomena itself.

One of the most difficult problems with tracking phenomena, or in the specific case we will consider, plumes, is that there are a large variety of predictive models that exist, each with nuances that are application specific and depend largely on the type of plume being tracked [6] [7] [8]. Other methods focus on optimally covering fluctuating fields [12] [13] or a specific type of dispersion model [10] [14] [15] [16]. There are also many algorithms that rely on fusing a model specific to dispersion of the phenomena and satellite imagery [7] [9] and [8]. Many localization problems focus on placing stationary sensors with an environment [17] [18].

In this paper we setup a more general framework for plume tracking that is not tailored towards a specific phenomenon but instead relies on knowledge of general plume shape, LTI dynamics, and measurement noise. We then address the question of how propagating sensor information through dynamics should influence the next best measurement location. We specifically consider a multivariate Gaussian plume with a stochastic source and three different types of measurement noise: noiseless, uniform, and Gaussian. The aim is to find a measurement scheme for each case of noise such that we are guaranteed the possible locations for the plume center decrease with each measurement.

The main contributions of this thesis are: 1) Consideration of a new plume estimation framework where a physical structure of the plume is known, 2) Determine a recursive probability distribution for plume, and 3) Identify sufficient measurement schemes to reduce the search area at successive instants .

The work in this thesis will be submitted to a refereed conference:

- ”Stochastic Plume Estimation: Measurement Sampling for a Supermartingale Support”, American Control Conference 2017, to be submitted, September 2016.

and has not been published elsewhere.

The assumptions made for the dynamics and measurements are given in Section II as well as a problem statement for the distribution estimation and support reduction problem. A general expression for the probability distribution and its update is characterized in Section III. A solution for the noiseless case and infeasibility for Gaussian case are also both shown. Section IV deals specifically with the uniform noise solution. A couple of examples are shown with simulation results in Section V.

Finally, conclusions and possible extensions are briefly discussed in VI.

Chapter 2

Problem Formulation

2.1 Plume Dynamics

We presume a plume whose center moves stochastically according to the discrete-time stochastic linear time-invariant system

$$x[k+1] = Ax[k] + W[k] \tag{2.1}$$

with state $x \in \mathbb{R}^2$ that represents the spatial Cartesian coordinates of the plume center, the state matrix $A \in \mathbb{R}^{2\times 2}$, and process noise $W \in \mathbb{R}^2$. We presume that W is a weighted, uniformly distributed, discrete random variable where each element is described by the tuple $S = (\Omega_D, \mathcal{F}_D, P_D)$, with the sample space give by a discrete set of outcomes Ω_D, discrete set of events $\mathcal{F}_D$, and probability measure $P_D : \mathcal{F}_D \to [0, 1]$.Outcomes $\omega \in \Omega_D$ are integer multiples of $\psi \in \mathbb{R}$, that is,

$$\Omega_D = \{\psi, 2\psi, \ldots, n\psi\},\ n \in \mathbb{Z}^+ \tag{2.2}$$

In order to define the weighted distribution, we define a set of weights $q = \{q_i \in \mathbb{R}\}$ where $i = \{1, \ldots n\}, n \in \mathbb{Z}^+$. The likelihoods of the outcomes $P_D(\omega)$ are given by the following probabilities,

$$P_D(w = \psi i) = \frac{q_i}{n},\ i = 1, \ldots, n \tag{2.3}$$

with $\sum_{i=1}^{n} P_D w(\psi i) = 1$ and $\sum_{i=1}^{n} q_i = n$. The corresponding probability mass function p_w is then,

$$p_w(\psi i) = \begin{cases} P_D(w = \psi i) & i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

2.2 Plume Measurements

The plume is observed through measurement y that may be noisy. In particular,

$$y[k] = h(x_m[k], x[k]) + V[k] \tag{2.5}$$

The measurement function

$h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is determined not only by the true location of the plume, but also the location $x_m \in \mathbb{R}^2$ of the measurement, and may be corrupted by measurement noise $V \in \mathbb{R}^2$.

We presume that h takes the form of a multivariate Gaussian function in $\mathbb{R}^2$ given by $\mathcal{N}(x[k], \Sigma)$ with mean x and a known positive definite variance $\Sigma = \sigma^2 I$, where $\sigma \in \mathbb{R}$ and I is the identity matrix in $\mathbb{R}^{2x2}$.

$$h(x_m, x) = \frac{1}{\sqrt{(2\pi^2|\Sigma|}} e^{(\frac{1}{2}(x_m - x)^T \Sigma^{-1}(x_m - x))} \tag{2.6}$$

Three types of measurement noise $V[k]$ are considered whose value is unknown in all cases.

1. No noise case:

$$V[k] = 0 \tag{2.7}$$

2. Uniform noise case:

$$V[k] \sim U[x[k] - \Delta, x[k] + \Delta] \tag{2.8}$$

3. Gaussian noise case:

$$V[k] \sim \mathcal{N}(x[k], \sigma_N^2)N \tag{2.9}$$

Note that h is a many-to-one function, meaning that its inverse will (in general) return a set of possible values. The shape of these sets differs by measurement noise. A symmetric multivariate gaussian has radial symmetry. Given x_m and y a radial distance r from x_m to a possible x can be calculated. The inverse map can then be described as the a set satisfying a circle.

$$h^{-1}(x_m, y) = \{x|(x_1 - x_{m_1})^2 + (x_2 - x_{m_2} = (r + V[k])^2\} \tag{2.10}$$

Then for the noiseless case (2.7), the mapping is a single circle. For the uniform case (2.8), the map returns a set of equally likely circles with a range of radii. The gaussian case (2.9) the inverse map yields a set of gradually less likely circles that span all of $\mathbb{R}^2$ and decrease in probability as distance is increased from the mean. These sets are described informally as a 'ring' for (2.7), as an 'annulus' for (2.8), and as a 'cinder-cone' for for (2.9).

2.3 Plume Estimate

In order to define probabilities of sets of x that are subsets of $\mathbb{R}^2$ we define the probability space $S_2 = (\Omega, \mathcal{F}, \mathbb{P})$. Where $\Omega = \mathbb{R}^2$ is the sample space , $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra on $\mathbb{R}^2$, and $\mathbb{P}$ is a probability measure given by the map $\mathbb{P} : B \in \mathcal{F} \to [0, 1]$. The density p associated with $\mathbb{P}$ is given by the Lebesque measure defined by $\mathbb{P}(B) = \int_B p(x)dx$.

We denote the set of all possible locations for the plume center x, given current measure y and no previous information, as the set $\hat{X} \subset \mathbb{R}^2$,

$$\hat{X} = \{x \mid h(x_m, x) + V = y\} \tag{2.11}$$

such that, $\mathbb{P}(x \in \hat{X}) = 1$. The density is uniform about the set. We indicate the probability density function of this set as $f_{\hat{X}}$. This estimate will be coupled in later sections with dynamic information to contruct a probability density function for the system.

Definition 1. $f_{x,k}(x)$*: Probability Density Function of x[k]*

Given space S_2 *we denote the probability density function of all possible centers, at time* k*, as* $f_{X,k}(x)$*.*

If conditioned against current measurement information x_m*,* y *and knowledge of how the measurement information maps to* $\hat{X}$*, the pdf is given by* $f_{x,k+1}(x|Y = y)$*.*

Definition 2. *supp(g(x)): Support of a Function We define the support of a function* g *as:*

$$\mathit{supp}(g(x)) \triangleq \{x \in X | g(x) \neq 0\} \tag{2.12}$$

Definition 3. $Q(f_x(x))$*: Area function*

We define the metric Q *to be the area of a distribution* $f_x(x)$*, given by:*

$$Q(f_x(x)) \triangleq \int_{\{x|f_x(x)\neq 0\}} 1dx \tag{2.13}$$

Definition 4. *Martingale Criteria*

We define a function g *that satisfies the supermartingale criteria as one that satisfies for all* $n \in \mathbb{Z}^+$*:*

$$\mathbb{E}[g_{n+1}|g_1, ..., g_n] \leq g_n \tag{2.14}$$

2.4 Problem Statement

Problem 1. *Given plume dynamics (2.1) with measurement process described by (2.7), (2.8), or (2.9), determine*

1. *The recursive probability density function of plume locations* $f_{x,k+1}(x|Y = y)$.
2. *A measurement scheme that causes the area of the support of* $f_{x,k+1}$ *to satisfy the supermartingale criteria,*

$$\mathbb{E}[Q(supp(f_{x,k+1}(x)))] \leq Q[supp(f_{x,k}(x))] \tag{2.15}$$

We denote subproblems for 2) for the three types of noise as 2a) for ()2.7), 2b) for (2.8), and 3c) for (2.9).

A solution to this problem yields a representation of possible plume locations that is guaranteed to shrink with time making it easier to triangulate the true location of x as time elapses.

Chapter 3

Methods

In this section we determine the probability density function for the plume, develop a recursion given the previous distribution and measurement information, and finally discuss the distribution over three steps for the three types of measurement noise given by (2.7), (2.8), and (2.9). A sufficient measurement scheme satifying (2.15) is determined for (2.7), a solution for (2.8) is discussed in the next session, and (2.9) is shown to be infeasible for (2.15).

3.1 Estimating the Probability Distribution

In order to describe the distribution of $x[k+1]$ we need to know which states are possible to map to at the next instant, or the support of the distribution. We call the support of the distribution at the next time instant the reachable set. The probability distribution at the next instant is then defined by the new support as well as knowledge of how the probability measure is assigned to elements in the new support or reachable set.

We assume that the true center x in contained in a support set $\Phi \subset \mathbb{R}^2$. We then suppose, given no other information, there is a uniform density function defined on Φ given by $f_{k,x} : \mathbb{R}^2 \rightarrow [0, \infty]$. The support of this probability distribution function is then given by the set Φ. We then have the following where probability density p is as previously defined in the space S_2,

$$f_{k,x} = \begin{cases} p(x), & for\ x \in \Phi \\ 0, & elsewhere \end{cases} \tag{3.1}$$

To generate the support of $f_{k,x}$ at the next instant, or $supp(f_{k+1,x})$ we start by choosing a possible $x \in \Phi$ given by $\tilde{x}$. We can generate a possible $\tilde{x}[k+1]$ by propagating the point $\tilde{x}$ through the dynamics for each choice of $W[k] = w_i$. Suppose $i = 1, 2, \cdots n$ choices for $W[k]$, where $n \in \mathbb{Z}^+$.

Updating a point to the next instant given only a single choise of $W[k] = w_i$ is given by,

$$\tilde{x}_i[k+1] = A\tilde{x}[k] + w_i \tag{3.2}$$

where $w_i \in \Omega_D$.Updating all points in a set given only a single choice of $W[k] = w_i$ is defined as branch given by,

$$b_i[k+1] = \{\tilde{x}_i[k+1] |\ \forall \tilde{x}[k] \in \Phi\} \tag{3.3}$$

The support of the distribution $f_{k+1,x}$ or the reachable set is then given by the union of all branches, or the evolution of all points given all choices of $W[k]$, this is given by,

$$\begin{aligned} supp(f_{k+1,x}) &= \{\bigcup_{i=1}^{n} \{b_i[k+1]\}\} \\ &= \Phi[k+1] \end{aligned} \tag{3.4}$$

We have now determined the support update but we now have to consider the assignment of measure in order to fully describe the distribution $f_{k+1,x}(x)$. Using the probability space defined previously by S_D, if we consider the update of a singular discrete point whose probability of being the true x is already known, $P(\tilde{x} = x)$, its next state has a probability induced by the probability of w_i.

$$\begin{aligned} P_D(\tilde{x}_i[k+1] = x) &= P_D(W[k] = w_i)P(\tilde{x} = x) \\ \sum_{i=1}^{n} P_D(\tilde{x}_i[k+1] = x) &= P(\tilde{x} = x) \end{aligned} \tag{3.5}$$

If try to assign probabilities to future sets rather than future points we have to then map exisiting probability density rather than existing probability mass, we then consider the space S_2.

From (3.4) the reach set is a mapping from one set Φ to n sets or branches whose union represents $\Phi[k+1]$. This more simply corresponds to a shift of the entire set by the LTI dynamics for each possible $W = w_i$. In probability, a re-scaled density of each of these new shifted sets (branches) corresponding to the probability of the branch occuring is required. This given by $P_D(W = w_i)$, this is a scaling factor on the existing probability density in similar fashion to 3.5. This shifting and scaling is then given by the following,

$$f_{k+1,x}(x) = \sum P(W = w_i) f_{k,x}(A^{-1}(x - w_i)) \tag{3.6}$$

We now have an update for the distribution of possible centers given a starting set and dynamics but this update is not considering the available measurement information. We incorportate this additional information by using a Bayes update in the next part of this section.

3.2 Recursive Estimate of Plume center

In this section the general Bayes estimate is used to develop a recursive estimate for all three cases of measurement noise. The general Bayes Rule is as follows, where x is a location and y is measurement information as before,

$$f_X(x|Y = y[k]) = \frac{f_Y(y|X = x[k])f_X(x)}{\int_{-\infty}^{\infty} f_Y(y|X = x[k])f_X(x)dx} \tag{3.7}$$

Suppose now we don't have access to the current distribution at time k, we only have access to the distribution at the previous time step $k-1$. We then use the R operator, defined by the update (3.6) discussed in the previous section, on the nonzero valued states of $f_X(x)$.

$$f_{k+1,x}(x) = R(f_{k,x}(x)) \tag{3.8}$$

Lemma 1. ***Solution for Distribution Update - Problem 1, part 1:*** *The recursive update for the probability distribution of possible plume centers is given by*

$$f_{k,X}(x|Y = y[k]) = \frac{f_{\hat{X}[k]}R(f_{k-1,X}(x))}{\int_{-\infty}^{\infty} f_{\hat{X}[k]}R(f_{k-1,X}(x))dx} \tag{3.9}$$

Proof 1. ***Recursive Estimate of Plume Center***

Using 3.7 and the substitution given by 3.8 we have,

$$f_{k,X}(x|Y = y[k]) = \frac{f_{k,Y}(y|X = x)R(f_{k-1,X}(x))}{\int_{-\infty}^{\infty} f_{k,Y}(y|X = x)R(f_{k-1,X}(x))dx} \tag{3.10}$$

The first term $f_{k,Y}(y|X = x)$ describes the measurement distribution. This was previously given by $\hat{X}[k]$. Making this substitution yields,

$$f_{k,X}(x|Y = y[k]) = \frac{f_{\hat{X}[k]}R(f_{k-1,X}(x))}{\int_{-\infty}^{\infty} f_{\hat{X}[k]}R(f_{k-1,X}(x))dx} \tag{3.11}$$

We now have our recursive update for the probability of possible plume centers. Problem 1) is solved.

The significance behind the other two terms is the following:

- The second term $R(f_{k-1,X}(x))$ is the reach of the support of distribution with appropriate probability as discussed in the previous section.
- The term $\int_{-\infty}^{\infty} f_{k,Y}(y|X = x)R(f_{k-1,X}(x))dx$ describes the total probability that was induced from the previous instant; if there are events no longer possible due to the measurement information this is a normalizing term ensuring $\int f_X(x) = 1$.

Figure (X) shows three measurements of this process for both the no noise and uniform noise cases.

If we consider (3.10) and assume again that $f_{x,0} = \mathbb{R}^2$ (we initially have no idea where in $\mathbb{R}^2$) for three steps in the noise free case we have the following:

$$\begin{aligned} f_{k=1,X}(x|Y = y[1]) &= f_{\hat{X}[1]} \\ f_{k=2,X}(x|Y = y[2]) &= \frac{f_{\hat{X}[2]}R(X[1])}{\int_X f_{\hat{X}[2]}R(X[1])dx} \\ f_{k=3,X}(x|Y = y[3]) &= \frac{f_{\hat{X}[3]}R(f_{k=2,X}(x|Y = y[2]))}{\int_X f_{\hat{X}[3]}R(f_{k=2,X}(x|Y = y[2])dx} \end{aligned} \tag{3.12}$$

Analytical Minimization: If we were to solve for the best measurement scheme analytically we would need to maximize our chances to satisfy the inequality. We then seek to minimize the expected value of Q. The optimization for this problem is then the following:

$$(x_m^*) = arg_{(x_m)}(min(\mathbb{E}[Q(f_{t_n,X}(x|Y = y))) \tag{3.13}$$

Then is necessary to compare $Q(m_n^*)$ to $Q(m_{n+1}^*)$ to see if Q is a martingale, $\mathbb{E}[Q(m_{n+1}^*)] \leq Q(m_n^*)$. This is in general a hard optimization.
We instead seek to find a sufficient scheme and first take a look at how the distribution changesd for each case. We consider the three cases of measurement noise and describe briefly the shape of each distribution under random measurements in reference to the three step distribution given by (3.12). That is, each step corresponds to a distribution update and then conditioning the distribution against a measurement y.

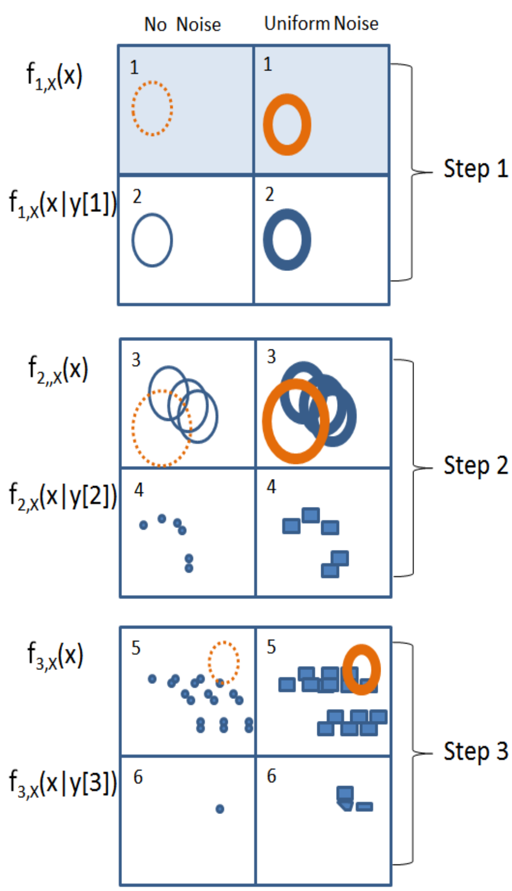

Figure 3.1: This figure shows three possible steps of example distributions for the no noise and uniform noise cases. Each step is broken down into the distribution (shown in blue) before and after measurement information (shown in orange).

No Measurement Noise

For the noise free case the set represented by $\hat{X}[k]$ forms a ring with zero thickness. The area of this set is zero. The probability is distributed radially such that the line integral about the ring yields 1 in probability.

- Step 1) A single measurement ring with uniform probability about it.
- Step 2) A collection of points. This comes from the intersection of two collections of rings: branches from the distribution in step 1 and a new measurement from step 2. The intersection forms disjoint points that each have an associated probability mass.
- Step 3) A single point. This comes about from the intersection of the branches of the collection of points from step 2) and a new measurement ring.

The reasoning behind step 3 is the following: We are guaranteed to intersect with at least one point because each measurement provides all possible valid locations. The likelihood we happen to intersect with another point however is zero. This is because the sets of points that lie equidistantly between every two branch points (points the would yield intersections with two points) form lines that have measure zero in $\mathbb{R}^2$ and therefore have zero chance of occurring.

Therefore, given random measurements we are able to identify the true center of the plume after three steps. The distribution for the noiseless case satisfies the martingale trivially as at all time steps the distribution has a support with zero area.

Lemma 2. ***Sufficient Scheme for Noiseless Case - Problem 1, 2a)***

Random measurement locations are sufficient for the noise case given by (2.7) the area of the support of (3.12) has zero area.

Proof 2. ***Sufficient Scheme for Noiseless Case:***

Given the measurement function (2.11), where measurement noise is given by case (2.7), we have the following:

$$\begin{aligned} Q(supp(\hat{X}[k])) &= 0 \; \forall k \\ Q(supp(f_{X,k+1}(x)))] &= Q[supp(f_{X,k}(x))] \\ &= 0, \; \forall k \end{aligned} \tag{3.14}$$

The conditions required by (2.15) are then automatically satisfied. A random measurement scheme is therefore sufficient. Problem 2a) is solved.

Uniform Measurement Noise

For the uniform noise case the set represented by $\hat{X}[k]$ forms an annulus with thickness proportional to Δ and the relative distance from the true center x. The probability is distributed uniformly about this annulus such that the integral about the annulus yields 1 in probability. If we consider (3.12) and assume again that $f_{x,0} = \mathbb{R}^2$ (we initially have no idea where in $\mathbb{R}^2$) for three steps in the noise free case we have the following:

- Step 1) A single annulus with uniform probability about it.
- Step 2) A collection of regions. This comes from the intersection of two collections of annuli: branches from the distribution in step 1 and a new measurement from step 2. The intersection forms regions that each have an associated probability density.
- Step 3) Another collection of regions. This comes from the intersection of two collections of annuli: branches from the distribution in step 2 and a new measurement from step 3. The intersection forms regions that each have an associated probability density.

This is a distribution that has a complicated interaction with its measurements that will be discussed in detail the next section. A sufficient measurement scheme for supermartingale support will be shown there.

Gaussian Measurement Noise

Lemma 3. ***Infeasibility for Gaussian Noise Case - Problem 1, 3C)***

No measurement scheme is sufficient for the gaussian noise case given by (2.9); the area of the support of (3.12) increases for all k.

Proof 3. ***Infeasibility for Gaussian Case:***

Given the measurement function (2.11), where measurement noise is given by case (2.8), we have the following:

$$Q(supp(\hat{X}[k])) = \mathbb{R}^2 \; \forall k$$

Because the support of the measurement spans all of $\mathbb{R}^2$ *successive measurements will not reduce the area of the support.*

$$Q(supp(f_{X,k+1}(x)))] > Q[supp(f_{X,k}(x))] \; \forall k \tag{3.15}$$

Therefore no scheme is sufficient to satisfy criteria for (2.15) for measurement noise of type (2.9); the Gaussian noise case Problem 2c) is infeasible.

In this section we have shown solutions for part 1 of problem 1 as well as parts 2a) and 2c), the next section focuses on solving the final portion part 2c).

Chapter 4

Methods: Uniform Noise Solution

In this section we focus on solving the uniform noise case. We describe a simple measurement scheme, two modes of intersections, introduce the idea of minimal expected branch distance tracking, use a maximum likelihood estimator (MLE) argument for each mode, and finally show convergence of the support under this proposed measurement scheme.

4.1 Proposed Measurement Scheme and Intersection Modes

The support for the conditioned distribution is equal to the intersection of the measurement support and the distribution's previous support. Therefore, we will refer to the remaining support after conditioning as an intersection.

The basis of the proposed sub-optimal but sufficient scheme measurement scheme is the following observation: If reoccurring measurements are made at a point in the distribution such that the intersection contains the same amount of area as a previous branch, or sufficiently intersects with only a single branch, then the distribution cannot grow.

For the Gaussian plume and uniform noise case we have measurements in the form of an annulus. The measurement has a measurable radius returned by the function $r : \mathbb{R} \to \mathbb{R}$. For the Gaussian plume case the r function is inversely proportional to signal strength this can be understood by noting that the highest signal value y for a gaussian distribution corresponds to the smallest x. This yields the following observations and expressions:
For a gaussian r is inversely related to y:

$$\forall y, r(y[k+1] - \Delta)] > r(y[k+1) + \Delta) \tag{4.1}$$

The outermost radii of the annuli is given by distance

$$r(y[k+1] - \Delta)] \tag{4.2}$$

The innermost radii of the annuli is given by distance

$$r(y[k+1] + \Delta)] \tag{4.3}$$

The diameter of a branch is then given by distance

$$2r(y[k+1] - \Delta)] \tag{4.4}$$

The thickness of a measurement annulus is given by

$$r(y[k+1] - \Delta) - r(y[k+1) + \Delta) \tag{4.5}$$

Which is illustrated in figure 4.1.

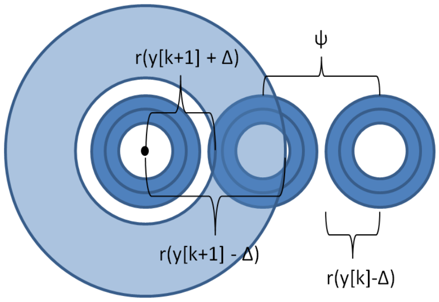

Figure 4.1: This figure has three branches with measurement and branch distances shown

Lemma 4. ***Sufficient Condition for Support Area Reduction***

A sufficient condition for area reduction is given by,

$$r(y[k] - \Delta) \geq \frac{n_o}{2}(r(y[k+1] - \Delta) - r(y[k+1) + \Delta)) \tag{4.6}$$

Where $n_o \in \mathbb{Z}^+$ is the number of possible intersection branches. For the linear case $n_0 = 2$. For the grid case of 9 possible dynamics $n_0 = 8$.

Proof 4. ***Lemma: Sufficient Condition for Support Area Reduction***

Given a starting distribution f_0 and an update, $k \to k+1$, we have now a support whose area is given by $n_0 Q(supp(f_0))$ where n_0 is the number of possible intersection branches. For the area to remain constant we require a measurement whose support overlaps only one branch worth of area.

A sufficient condition is then that the diameter of a branch (4.4) exceeds that of n_0 multiples of the thickness of the next measurement annuli (4.5). This is given by,

$$2r(y[k]-\Delta) \geq n_o(r(y[k+1]-\Delta) - r(y[k+1) + \Delta))$$

$$or$$

$$r(y[k]-\Delta) \geq \frac{n_o}{2}(r(y[k+1]-\Delta) - r(y[k+1) + \Delta))$$

Therefore 4.6 is a sufficient criteria.

4.2 Types of Measurement Outcomes

We introduce two types outcomes measurements can have and the notion of branch distance number that will be used throughout the next two sections:

1. Thinning - making a measurement on the center of the correct branch such that the noise is reduced.
2. Localization - making a measurement on the center of the incorrect branch such that section(s) of correct branch(es) are identified.
3. 'Branch Distance Number', is an integer $\mathbb{Z}^+$ that indicates how many branches away the correct branch was identified.

Thinning then corresponds to a branch distance of 0 and localizing corresponds to any other integer in $\mathbb{Z}^+$. Branch number is an important way of differentiating measurements that will be a useful later when we separate measurement outcomes into collections

In the figure below we illustrate both modes. thinning with branch distance 0 and localizing with a branch distance of 1. The blue indicates the existing distribution and red indicates the new measurement distribution. The new valid region after the intersection is considered is indicated with black arrows. The black dot is the location of x_m.

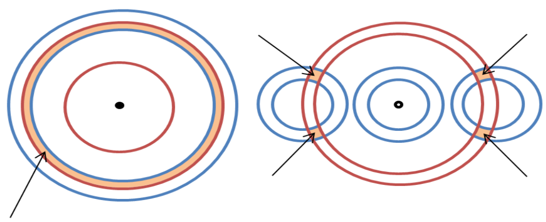

Figure 4.2: This figure shows two intersection modes, thinning on the left and localization on the right. The tan regions are the remaining support of the pre-existing distribution (blue) given the intersection of the new measurement (red).

4.3 Minimal Expected Branch Distance Tracking

In this section the point in the distribution that has the minimal expected branch distance (MEBD) is identified. The motivation for this section is to identify a scheme with the best chances of satisfying the sufficient condition given by (4.6).

Lemma 5. ***Measurement Scheme to Satisfy Sufficient Criteria***

A measurement scheme that gives the sufficient conditions (4.6) a better chance of succeeding is given by the MEBD method.

Proof 5. ***Measurement Scheme to Satisfy Sufficient Criteria***

Given a gaussian mapping, the radius measure 'r' increases as in inverse function of y.

$$r(y + \Delta) < r(y - \Delta)$$

Then, necessarily the radius of the measurement annulus increases in thickness inversely with y.

$$(r(y[k+1]-\Delta)-r(y[k+1)+\Delta)$$

The outcome of 'y' can be measured using the branching distance metric. To minimize the expected overlap we can then minimize the expected branch distance. Therefore the MEBD method gives the sufficient condition a better chance of succeeding.

Determining the optimal branch for minimizing expected branch length can be accomplished by considering the grid of dynamics as a simply connected graph and evaluating the closeness centrality of each point while weighting the edges by the probability of occurrence (probabilistic centrality). The centrality of node $p \in \mathbb{Z}^+$ with respect to other nodes $q \in \mathbb{Z}^+$ and distance function $d : (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$,

$$d = \sqrt{(x_{1p} - x_{1q})^2 + (x_{2p} - x_{2q})^2} \tag{4.7}$$

where x_{1p}, x_{2p} are the spatial coordinates of node p and x_{1q}, x_{2q} are the spatial coordinates of node p. The centrality measure is then given by given by

$$C(p) = 1/(\sum_q d(p,q)) \tag{4.8}$$

The probabilistic centrality, $PC : \mathbb{Z} \to \mathbb{R}$, is then given by,

$$PC(p) = 1/(\sum_q P(q)d(p,q)) \tag{4.9}$$

The node corresponding to the minimum expected branching distance (MEBD) is then,

$$MEBD = argmax(PC(p)) \tag{4.10}$$

The spatial coordinates of this node are indicated by the point

$$MEBD = (min\,\mathbb{E}_b[x_1], min\,\mathbb{E}_b[x_2]) \tag{4.11}$$

where $min\,\mathbb{E}_b$ indicates the minimal expected branch.

If an intersection is generated such that the measurement location is no longer the center of probability mass (denoted x_c) the distance from the measurement to the center of mass of the intersection region can also be calculated. The next desired measurement is then the combination of both distances.

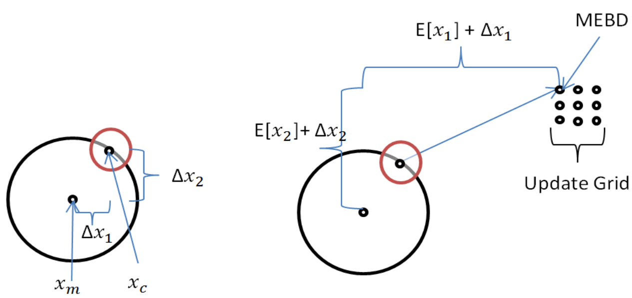

Figure 4.3: This figure shows base idea behind minimal expected dynamic tracking, where the expectation shown is with respect to minimal branch distance. The update grid is the gird of possible update measurement locations given all possible W.

4.4 Maximum Likelihood Estimation

We have now provided a method for selecting a measurement point that gives condition (4.6) the best chance for success. However, we have not yet shown that the area of the support of the distribution will decrease. In this section we assume (4.6) holds, due to Δ being small and MEBD being utilized. We then show that by decomposing a sequence of measurements into sets based on branching distance and using maximum likelihood estimators (MLE) on each set we can prove the support of the distribution will decrease.

Let D be a sequence of d_i that are measurement outcomes defined on the set of branching distances $\{0, 1, \ldots n\}$ where the $n \in \mathcal{Z}^+$. The outcome indicates the measurement's branch distance in branch lengths where 0 corresponds to the thinning, and $1, \ldots n$ are localizations at different branch lengths.

Suppose D consists of a sequence of only 0, this means the region is thinned forever. Let Q be the area function as previously defined. We then would like to know what the following evaluates to,

$$Q\{\bigcap_{k=0}^{N} \hat{X}[k|x_m]\} \tag{4.12}$$

Now suppose rather than the support of $\hat{X}[k|x_m]$ in $\mathbb{R}^2$ we consider only a slice since $\hat{X}[k|x_m]$ has radial symmetry. We now have a uniform interval in $\mathbb{R}$ where we don't know the true center.

This problem can be cast as an estimation problem, specifically a Maximum Likelihood Estimator problem (MLE) where we are trying to estimate the true location of the interval center (s^*) of a uniform random variable $U \sim U[s^* - \Delta, s^* + \Delta]$. Maximum Likelihood Estimators (MLE) for uniform intervals are well known. Existing theory of MLEs through the law of large numbers (LLN) tells us that the exact location for the true center can be found almost surely in the limit. However, instead of returning the center of the uniform interval we would like to know the estimator's support, or the band of all possible centers.

Lemma 6. ***Convergence of Single Mode Support Area***

The area of D consisting of a sequence of the same branch length outcomes tends towards an area of zero.

$$Q\{\bigcap_{k=0}^{N} \hat{X}[k|x_m]\} \to 0, \ As\ N \to \infty$$

Proof 6. ***Convergence of Single Mode Support Area***

$$\begin{aligned}
&\textit{Let } s_k, \textit{ where } k = 1 \ldots n, \textit{ be samples of } U[s^* - \Delta, s^* + \Delta] \\
&\textit{and let } MLE(s^*) := \hat{s^*} \\
&\textit{Then } supp(MLE(s^*)) = \bigcap_{k=1}^{N} [s_k - \Delta, s_k + \Delta] \\
&\hat{s^*} \in \bigcap_{k=1}^{N} [s_k - \Delta, s_k + \Delta] \\
&\textit{For some } k, \ supp(MLE(s^*)) = [s_k - \epsilon_1, s_k + \epsilon_2] \\
&\lim_{N \to \infty} supp(MLE(s^*)) \ \textit{causes } \epsilon_1, \epsilon_2 \to 0 \textit{ as } N \to \infty \\
&\textit{Then } Q\{\bigcap_{k=0}^{N} \hat{X}[k|x_m]\} \to 0 \textit{ as } N \to \infty
\end{aligned} \tag{4.13}$$

This corresponds to eventually converging to the no noise case because we tend towards a singular valued support. The support can then be visualized as a decreasing annulus that eventually converges to a ring. This same argument can also made for the localization annuli (ex. sequence D of only 1 or only 2 etc.).

4.5 Proof of Support Area Convergence

We have now shown for cases where D consists of only one type of branching distance outcome the support converges to zero. We now suppose an alternate sequence $D = \{d_i\}$ that consists of random elements.

We define a collection $C(i)$ as being all measurement information, that is, the support and measure, associated with the branch number i. For example, if a process is defined from k = 1 to k = 3 and all three measurements result in a branch distance of 1. Then the collection $C(1)$ contains the entire distribution with the proper measure and all other collections are the null set with measure zero.

It can be shown (see Appendix A) that changing propagation and intersection order such that all intersections occur last causes an upper bound.

Lemma 7. ***Sufficient Scheme for the Uniform Noise Case***
Given sufficient condition (4.6) is satisfied through the combination of Δ being sufficiently small and choosing measurements according to the MEBD measurement scheme the area of the support satisfies the martingale criteria.

Proof 7. ***Sufficient Scheme for the Uniform Noise Case***
This follows from the one mode MLE argument (4.13) combined with the fact propagation before intersection forms an upper bound (see appendix).

Suppose the distribution is decomposed into collections given by $C(i)$.

$$f(x,k) = \bigcap_{i=0}^{n} C\{i\}$$

$$\begin{aligned} &\textit{From (4.13), } Q(supp(C\{i\})) \to 0 \textit{ as } \#C\{i\} \to \infty \\ &\textit{Therefore, } Q(supp(\bigcap_{i=0}^{n} C\{i\})) \to 0 \textit{ as } \#C\{i\} \to \infty \\ &\textit{or, } Q(supp(f(x,k))) \to 0 \textit{ as } \#C\{i\} \to \infty \\ &\textit{Then } \mathbb{E}[Q(supp(f_{x,k+1}(x)))] \leq Q[supp(f_{x,k}(x))] \end{aligned} \tag{4.14}$$

We have now shown solutions for 1,2a, and 2b, and infeasibility for 2c. We have addressed of all parts of our problem and move on towards some implemented examples.

Chapter 5

Example

The sufficient dynamic expected tracking measurement scheme was coded for the noiseless case using two different methodologies - 'intersect, branch, iterate' and 'branch all then intersect'.

For the noisy case only the latter was used to avoid needing to represent intersection regions with polytopes. For the noiseless case these methods are equivalent. For the noisy case the branch all then intersect forms an upper bound (see proof in appendix).

For examples 1 through 3 the state matrix A is the identity to make the figures easier to visualize.

5.1 Example 1: Noiseless Case

The noiseless case converges to a point using both methods in three steps. The diamond is the true and correctly identified center.

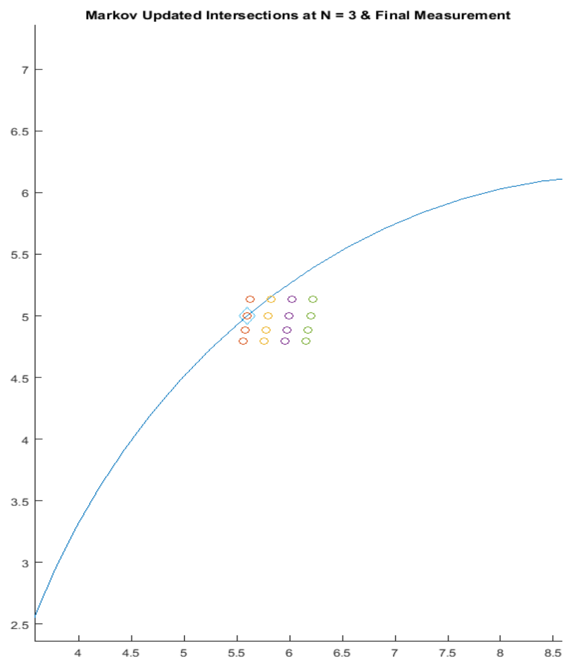

Figure 5.1: This figure shows the second and third steps for the noiseless case. The collection of points is the distribution at k = 2, the measurement at k = 3 is the ring shown, and finally the diamond is the singular point representing the distribution at k = 3.

5.2 Example 2: Uniform Noise Case, Thinning

The following shows three steps where all measurements correspond to a branch distance of zero and both the line and grid disturbances are considered.

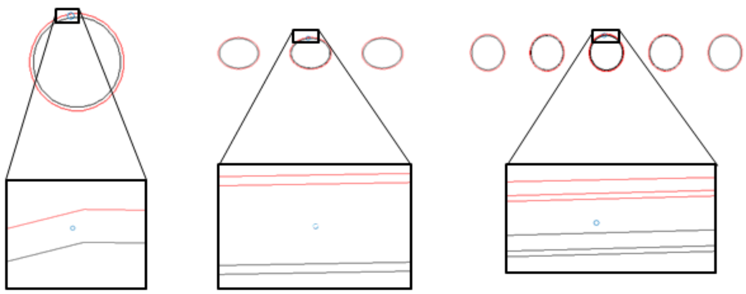

Figure 5.2: This figure shows three steps from left to right for the line case you can see thinning occur.

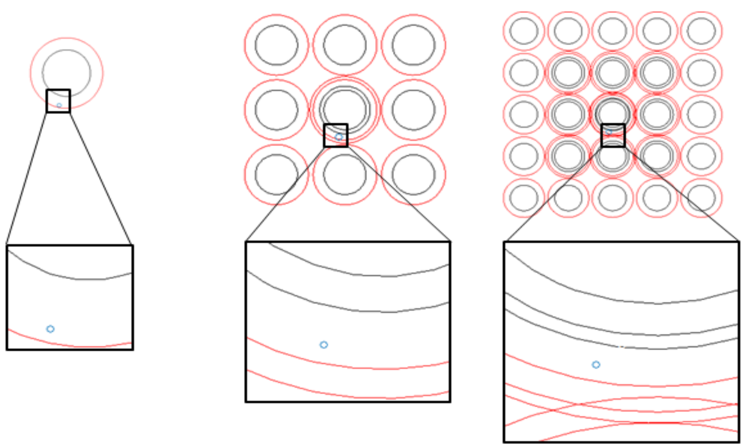

Figure 5.3: This figure shows three steps from left to right for the grid case you can see thinning occur.

5.3 Example 3: Color Coded Mixed Thinning and Localization

The following are figures for an example for the 3x3 grid noise case. In this example a mix of thinning and localization occurs. The color code is as follows.

- Red and Black both correspond to information according to the first measurement where Red is the outer bound of a branch and Black is the inner bound.
- Green and Pink both correspond to information according to the first measurement where Green is the outer bound of a branch and Pink is the inner bound.
- Blue and Cyanbboth correspond to information according to the first measurement where Blue is the outer bound of a branch and Cyan is the inner bound.

The blue circle corresponds to the true center location.

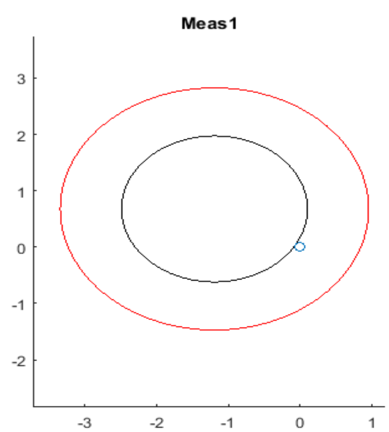

Figure 5.4: Step 1: First Measurement and Intersection

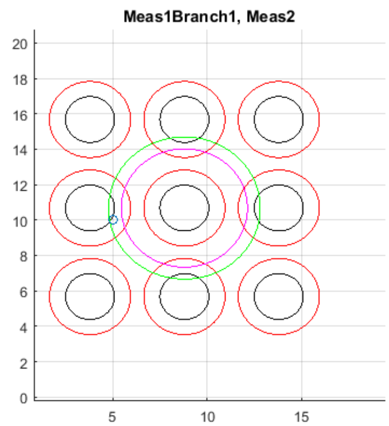

Figure 5.5: Step 2: Second Measurement and Intersection

5.4 Example 4: Elliptical Example

The following shows three steps for the elliptical case where A is not the identity matrix. The blue

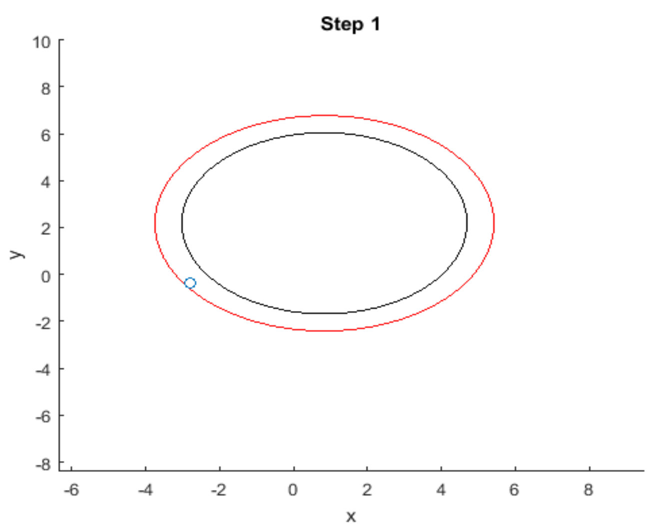

Figure 5.6: Step 1

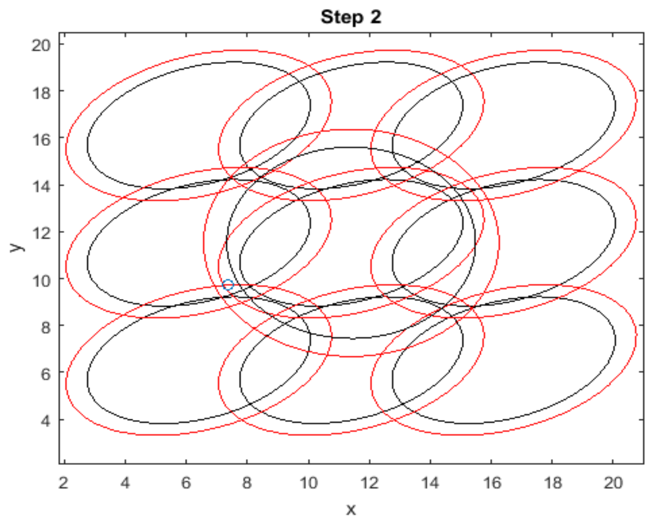

Figure 5.7: Step 2

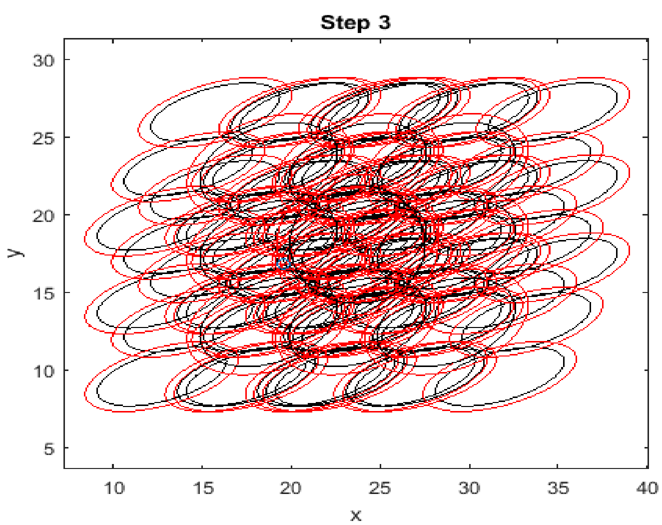

Figure 5.8: Step 3

Chapter 6

Conclusion and Future Directions

In this Thesis a variation of the plume tracking problem was explored. The variation assumed a known mapping mapping from sensor data to a known plume structure allowing a distribution of all possible potential plume centers to be constructed. The question we successfully answered was, given this known mapping and assuming known discrete stochastic dynamics, can we make measurements in such a fashion to reduce the support of possible plume centers.

To explore this problem we assumed a plume given by a multivariate Gaussian structure and discrete stochastic LTI dynamics and determined a recursive update (3.10) for the probability distribution and three types of noise were considered. It was shown that a random measurement scheme satisfied the noiseless case (3.14), the gaussian case was found to be infeasible (3.15), and a sufficient scheme was derived for the uniform measurement case (4.14).

The main practical result here is give the particular problem set-up and assumptions an simple algorithm was shown that allows a robot or a network of robots to hone in on a discrete set of points for their target (for the noiseless and uniform cases) rather than disjoint sets that potentially grow in size.

6.1 Possible Extensions

This was an initial investigation of a problem structure of this type but from this preliminary work it is clear that there are many potential directions for future work that have a wide variety of practical applications.

- Volume minimization: Instead of optimizing over area, bring in probability as an optimization parameter.
- Gaussian Mixture: Consider an h function given by a Gaussian mixture. This problem becomes equivalent to this problem in a parallel fashion. Each measurement made can be interpreted with respect to a different Gaussian plume.
- A time varying h function
- Other h functions
- Reach constraints on x_m: Suppose the measurements are made by quadopters or some drones with some dynamics that limit the feasible measurement spaces at each k.
- Quadcopter dispersion problem: Given a distribution at time k optimally assign several quad-copters to different regions.
- Non-discrete inputs - consider disturbances given by continuous intervals rather than discrete values. This yields branching connected tubes rather than branching sets.
- Consider the Uniform noise case bounds as two separate noiseless problems. Each noiseless problem will yield a point - something may be able to be concluded about the convex hull.

Appendix A

Intersection Order Lemma and Proof

Let b be a branching function that maps a branch x_1 to $x_{11} \cup x_{12} ... \cup x_{1N}$. A branching process b with $n = 3$ then has the following properties:

$$b(r_1) = r_{11} \cup r_{12} \cup r_{13}$$

$$b(b(r_1) = r_{111} \cup r_{121} \cup r_{131} \cup r_{112} \cup r_{122} \cup r_{132} \cup r_{113} \cup r_{123} \cup r_{133}$$

$$b(r_2) = r_{21} \cup r_{22} \cup r_{23}$$

Lemma 8. ***Propogation Before Intersection Forms Upper Bound***

The support associated with propagating all measurement information through a branching process before taking their intersection forms an upper bound on intersecting the support between measurements during each propagation step.

Proof 8. ***Propogation Before Intersection Forms Upper Bound***

Suppose a branching process b with $N = 3$. Let A be a process that has measurement intersections between each branch step. Let B be a process that has a single *measurement intersection after completion of all branch steps.*

$$Dist_A = b(b(r_1) \cap r_2) \cap r_3$$
$$Dist_B = b(b(r_1)) \cap f(r_2) \cap r_3$$

$$\begin{aligned}Dist_A &= f(f(r_1) \cap r_2) \cap r_3\\ &= f((r_{11} \cup r_{12} \cup r_{13}) \cap r_2) \cap r_3\\ &= (((r_{111} \cup r_{121} \cup r_{131}) \cap r_{21}) \cup ((r_{112} \cup r_{122} \cup r_{132})\\ &\cap r_{22}) \cup ((r_{113} \cup r_{123} \cup r_{133}) \cap r_{23})) \cap r_3\end{aligned}$$

$$\begin{aligned}Dist_B &= b(b(r_1)) \cap b(r_2) \cap r_3\\ &= (r_{111} \cup r_{121} \cup r_{131} \cup r_{112} \cup r_{122} \cup r_{132} \cup r_{113}\\ &\cup r_{123} \cup r_{133}) \cap (r_{21} \cup r_{22} \cup r_{23}) \cap r_3\end{aligned}$$

Let

$$R1 = (r_{111} \cup r_{121} \cup r_{131})$$
$$R2 = (r_{112} \cup r_{122} \cup r_{132})$$
$$R3 = (r_{113} \cup r_{123} \cup r_{133})$$

Then

$$Dist_A = ((R_1 \cap r_{21}) \cup (R_2 \cap r_{22}) \cup (R_3 \cap r_{23})) \cap r_3 \tag{A.1}$$
$$Dist_B = (R_1 \cup R_2 \cup R_3) \cap (r_{21} \cup r_{22} \cup r_{23}) \cap r_3 \tag{A.2}$$

Using the distributive property $(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *of intersections on* $Dist_B$ *yields:*

$$Dist_B = ((R_1 \cap (r_{21} \cup r_{22} \cup r_{23})) \cup (R_2 \cap (r_{21} \cup r_{22} \cup r_{23})) \cup ((R_3 \cap r_{23}) \cap (r_{21} \cup r_{22} \cup r_{23}))) \tag{A.3}$$

Clearly, comparing (A.3) to (A.1), $Dist_B$ *is a larger collection and* $Dist_A$ *and* $Dist_B$ *are not equivalent.*

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