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Conformally Invariant Operators in Higher Spin Spaces

Chao Ding
University of Arkansas, Fayetteville

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by

Chao Ding
University of Science and Technology of China,
Bachelor of Arts in Mathematics2007
University of Science and Technology of China,
Master of Science in Mathematics2010

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University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

Dr. John Ryan
Dissertation Director

Dr. Phillip Harrington
Committee Member

Dr. Daniel Luecking
Committee Member

Abstract

In this dissertation, we complete the work of constructing arbitrary order conformally invariant operators in higher spin spaces, where functions take values in irreducible representations of Spin groups. We provide explicit formulas for them.

We first construct the Dirac operator and Rarita-Schwinger operator as Stein Weiss type operators. This motivates us to consider representation theory in higher spin spaces. We provide corrections to the proof of conformal invariance of the Rarita-Schwinger operator in [15]. With the techniques used in the second order case [7, 18], we construct conformally invariant differential operators of arbitrary order with the target space being degree-1 homogeneous polynomial spaces. Meanwhile, we generalize these operators and their fundamental solutions to some conformally flat manifolds, such as cylinders and Hopf manifolds. To generalize our results to the case where the target space is a degree k homogeneous polynomial space, we first construct third order and fourth order conformally invariant differential operators by similar techniques. To complete this work, we notice that the techniques we used previously are computationally infeasible for higher order (≥ 5) cases. Fortunately, we found a different approach to conquer this problem. This approach relies heavily on fundamental solutions of these differential operators. We also define a large class of conformally invariant convolution type operators associated to fundamental solutions. Further, their inverses, when they exist, are conformally invariant analogues of pseudo-differential operators.

We also point out that these conformally invariant differential operators with their fundamental solutions can be generalized to some conformally flat manifolds, for instance, cylinders and Hopf manifolds. This can be done with the help of Eisenstein series as in [31].

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1 Introduction

Classical Clifford analysis started as a generalization of aspects of the analysis of functions in one complex variable to m-dimensional Euclidean spaces. At the heart of this theory is the study of the Dirac operator D_x on $\mathbb{R}^m$, a first order conformally invariant differential operator which generalizes the role of the Cauchy-Riemann operator. Moreover, this operator is related to the Laplace operator by $D_x^2 = -\Delta_x$. The classical theory is centered around the study of functions on $\mathbb{R}^m$ taking values in a spinor space [1, 3], and abundant results have been found. See for instance [3, 9, 36, 21, 41, 42].

In constructing a first order relativistically covariant equation describing the dynamics of an electron, P.A.M. Dirac constructed a differential operator using Clifford modules; hence the name Dirac operator. Moreover, in the presence of an electromagnetic field, the Dirac Hamiltonian gives an additional contribution formally analogous to internal angular momentum called spin, from which the Spin group and related notions take their name; for the electron, spin has the value $\frac{1}{2}$ [28]. Indeed, in dimension four with appropriate signature, the Dirac operator reproduces the relativistically covariant dynamical equation of a massless particle of spin $\frac{1}{2}$, also called the Weyl equation.

Rarita and Schwinger [38] introduced a simplified formulation of the theory of particles of arbitrary half-integer spin $k + \frac{1}{2}$ and in particular considered its implications for particles of spin $\frac{3}{2}$. In the context of Clifford analysis, the so-called *higher spin theory* was first introduced through the Rarita-Schwinger operator [6], which is named analogously to the Dirac operator and reproduces the wave equations for a massless particle of arbitrary half-integer spin in four dimensions with appropriate signature [39]. (The solutions to these wave equations may not be physical [49, 50].) The higher spin theory studies generalizations of classical Clifford analysis techniques to higher spin spaces [7, 4, 6, 15, 17, 33]. This theory concerns the study of the operators acting on functions on $\mathbb{R}^m$, taking values in arbitrary irreducible representations of $Spin(m)$. These arbitrary representations are defined in terms of polynomial spaces that satisfy certain differential equations, such as j-homogeneous monogenic polynomials (half-integer spin) or j-homogeneous harmonic polynomials (integer spin). More generally, one can consider the highest weight vector of the spin representation as a parameter [44], but this is beyond our present scope.

In what we have discussed, two matters are emphasized. First, physics is the motivation for essential ideas in Clifford analysis and certainly for the higher spin theory. In particular, the Dirac and Rarita-Schwinger operators emerge naturally from physics. Any approach to Clifford analysis should readily include these operators. Second, the theory of group representations emerges naturally in physics when considering the quantization of angular momentum or spin, and this representation theory is needed to construct higher spin operators motivated by physics. It is desirable to construct the Dirac and Rarita-Schwinger operators using strictly representation theoretic methods. We will do this using the generalized gradient construction of Stein and Weiss [48]. The operators generated by this method are the obvious choice for studying first order differential operators on spinor spaces, which should be considered as irreducible representation spaces for Spin group. This construction naturally provides the immensely important Atiyah-Singer Dirac operator, which generalizes the Euclidean Dirac operator to a spin manifold.

In principle, all conformally invariant differential operators on locally conformally flat manifolds in higher spin theory are classified by Slovák [46]. This classification is non-constructive, showing only between which vector bundles these operators exist and what their order is. Explicit expressions of these operators are still being found. Eelbode and Roels [17] point out that the Laplace operator Δ_x is no longer conformally invariant when it acts on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, where $\mathcal{H}_1$ is the degree one homogeneous harmonic polynomial space (correspondingly $\mathcal{M}_1$ for monogenic polynomials). They construct a second order conformally invariant operator on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, called the (generalized)

Maxwell operator. In dimension four with appropriate signature it reproduces the Maxwell equation, or the wave equation for a massless spin-1 particle (the massless Proca equation) [17]. De Bie and his co-authors [7] generalize this Maxwell operator from $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$ to $C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ to provide the higher spin Laplace operators, the second order conformally invariant operators generalizing the Laplace operator to arbitrary integer spins. Their arguments also suggest that D_x^k is not conformally invariant in the higher spin theory. This raises the following question: What operators generalize kth-powers of the Dirac operator in the higher spin theory? We know these operators exist, with even order operators taking values in homogeneous harmonic polynomial spaces and odd order operators taking values in homogeneous monogenic polynomial spaces [46]. Using similar techniques as in [17, 7], we successfully discovered third order and fourth order conformally invariant differential operators in higher spin spaces, see [11]. However, in application, the computation with such a technique in [11] becomes impossible to complete when the order of the operator increases. This forces us to find a new approach.

The methods we use to construct conformally invariant operators are usually either of the following types:

1. Verify some differential operator is conformally invariant under Möbius transformations with the help of an Iwasawa decomposition, for instance as in [15].

2. Show the generalized symmetries of some differential operator generate a conformal Lie algebra, for instance as in [7, 17].

In our recent paper [14], we find a different method to solve the higher order cases. We start by applying Slovák [46] and Souček's [47] results with arguments of Bureš et al. [6] to get fundamental solutions of arbitrary order conformally invariant differential operators in higher spin spaces. Then we only need to construct differential operators with those specific fundamental solutions. In particular, from the fundamental solutions of the first and second order conformally invariant differential operators obtained from the preceding argument, we can also find the Rarita-Schwinger operators and higher spin Laplace operators [7] by verifying they have such fundamental solutions. Arguing by induction, we then complete the work of constructing conformally invariant operators in higher spin spaces by providing explicit forms of arbitrary j-th order conformally invariant operators in higher spin spaces with $j > 2$.

Notably, we discover a new analytic approach to show that a differential operator is conformally invariant. More specifically, we use its fundamental solution to define a convolution type operator, and then the fundamental solution can be realized as the inverse of the corresponding differential operator in the sense of such convolution. Hence, if we can show the fundamental solution (as a convolution operator) is conformally invariant, then as the inverse, the corresponding differential operator will also be conformally invariant. Thus the intertwining operators of the fundamental solution (as a convolution operator) are the inverses of the intertwining operators of the differential operators. This method gives us an infinite class of conformally invariant convolution type operators in higher spin spaces. Further, their inverses, if they exist, are conformally invariant pseudo-differential operators. More details can be found in Section 4.1.

Our study of conformally invariant differential operators in higher spin spaces suggests a distinct Representation-Theoretic approach to Clifford analysis, in contrast to the classical Stokes approach. In the latter approach, the motivation for Dirac-type operators is to obtain operators satisfying a Stokes-type theorem. This does not require irreducible representation theory. In contrast, in the Representation-Theoretic approach, we consider functions taking values in irreducible representations of the Spin group. This forces us to consider irreducible representation theory, as happens elsewhere in the literature where Dirac operators are used [21] and especially in spin geometry [28]. Moreover, irreducible spin representations are natural for studying spin invariance and in particular conformal invariance. However, we should not dismiss the Stokes approach—it is used, for instance, to establish the L^2 boundedness of the double layer potential operator on Lipschitz graphs [35]. Other applications are found in such works as [3]. Though the present work aims to demonstrate the value of the Representation-Theoretic approach, in future work the two distinct approaches may complement each other.

1.1 Dissertation Outline

This dissertation is organized as follows:

In Section 2, we introduce Clifford algebras with some well known properties; we then introduce some real subgroups in real Clifford algebra, in particular, the special orthogonal group and Spin group which is the double covering group of the special orthogonal group.

In Section 3, Euclidean Dirac operator is defined and some classical results of the Dirac operator are introduced, such as, Cauchy's integral formula, Cauchy's theorem and fundamental solutions. Since the main topic of this thesis is constructing conformally invariant differential operators, conformal transformations with some well known results are introduced, for instance, Möbius transformations and Ahlfors-Vahlen matrices. At the end of this section, we review several results on conformal invariance of Dirac operators in Euclidean space.

In higher spin theory, we consider functions taking values in irreducible representations of Spin groups. Some basic knowledges of representation theory are introduced in Section 4. More specifically, we first give the definitions for Lie group and representations of Lie group. In the context of Lie group, highest weights and highest weight vectors are introduced. Several irreducible representations of Spin groups are provided at the end for further use.

In sections 5 through 9, we demonstrate how to construct conformally invariant differential operators in higher spin spaces.

In Section 5, we start with a counterexample which shows that the Euclidean Dirac operator D_x is no longer conformally invariant in higher spin spaces. We then show alternative representation theoretic constructions for the Euclidean Dirac operator and Rarita-Schwinger operator as Stein Weiss type operators. It turns out that this construction fits better in this thesis, since representation theory emerges naturally in higher spin spaces. We next provide corrections to the proofs of conformal invariance and intertwining operators for the Rarita-Schwinger operator in [33]. Further, we explore these properties for other Rarita-Schwinger type operators as well.

In [7, 18], second order conformally invariant differential operator in higher spin spaces is constructed. Motivated by techniques used there, we construct arbitrary order conformally invariant differential operators in higher spin spaces with target space degree-1 homogeneous polynomial spaces in Section 6. They are named as fermionic operators for odd order and bosonic operators for even order because of their connection with particles in physics. More details can be found in this section. Fundamental solutions and ellipticity of these operators are also provided.

In [30, 31], Rarita-Schwinger operators with their fundamental solutions are generalized to cylinders and Hopf manifolds with the help of Eisenstein series [23]. In Section 7, we generalize our fermionic operators and bosonic operators to cylinders and Hopf manifolds as well with similar arguments as in [31, 30].

In the previous section, there is a restriction for our results; we require that the target space must be degree-1 polynomial space. In Section 8, we construct conformally invariant differential operators in higher spin spaces with degree-j homogeneous polynomial space as the target space. Third order fermionic operator and fourth order bosonic operator are constructed with similar techniques used in Section 7. This technique does not apply for other higher order cases due to its complicated calculations when the order of differential operator increases.

In Section 9, we provide an approach, which is different from [7, 17], to construct the other higher order conformally invariant differential operators. This approach relies heavily on fundamental solutions of these conformally invariant differential operators. We also define a convolution type operator associated to each fundamental solution to show each fundamental solution is actually the inverse of the corresponding differential operator. An explicit proof for the intertwining operators of these convolution type operators is provided here. This implies conformal invariance of these convolution type operators and conformal invariance of the corresponding differential operators is shown immediately. We also point out that this idea gives an infinite class of conformally invariant convolution type operators. Moreover, their inverses, if they exist, are conformally invariant pseudo-differential operators. We also show that the Rarita-Schwinger and higher spin Laplace operators [7] can be derived from this approach. Then we introduce bosonic operators, $\mathcal{D}_{2j}$, as the generalization of D_x^{2j} when acting on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ and fermionic operators, $\mathcal{D}_{2j-1}$, as the generalization of D_x^{2j-1} when acting on $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$. The connections between these and lower order conformally invariant operators are also revealed in the construction. Moreover, since the construction is explicitly based on the uniqueness of the operators and their fundamental solutions with the appropriate intertwining operators for a conformal transformation, the conformal invariance and fundamental solutions of the bosonic and fermionic operators arise naturally in our formalism.

In Section 10, we list several unsolved problems which are related to my work in this thesis. These are the problems for my future work.

2 Clifford algebras

2.1 Definitions and properties

Clifford algebras are the algebras that form the basis of this thesis. They are naturally associated with bilinear forms on vector spaces. A bilinear form can be considered as a generalization of an inner product and is defined as follows:

Definition 2.1. Suppose V is a vector space over $\mathbb{R}$. A bilinear form B is a map

$$\mathcal{B}: \; V \times V \longrightarrow \mathbb{R}; \; (u, v) \mapsto \mathcal{B}(u, v),$$

which is linear in both arguments:

$$\mathcal{B}(au_1 + bu_2, v) = a\mathcal{B}(u_1, v) + b\mathcal{B}(u_2, v);$$
$$\mathcal{B}(u, av_1 + bv_2) = a\mathcal{B}(u, v_1) + b\mathcal{B}(u, v_2).$$

One can associate a matrix $B = (a_{ij})_{ij} \in \mathbb{R}^{m \times m}$ to every bilinear form on an m-dimensional vector space:

$$\mathcal{B}(u, v) = \sum_{i=1}^{m} \sum_{j=1}^{m} u_i a_{ij} v_j = u^T B v,$$

where $u, v \in V$. If the matrix B is symmetric, the associated bilinear form is called symmetric and if $det(B) \neq 0$, the associated form is called non-degenerate, i.e. for all non-zero vector $u \in V$ there exists a non-zero vector $v \in V$ such that $\mathcal{B}(u, v) \neq 0$.

Definition 2.2. If V is a real vector space equipped with a symmetric, non-degenerate bilinear form $\mathcal{B}$, then $(V, \mathcal{B})$ is called a non-degenerate orthogonal space.

Note that with a proper choice of a basis for V, every non-degenerate orthogonal space can be reduced to a space $\mathbb{R}^{p,q}$ with $p + q = m = dim(V)$. (p, q) are called the signature of the orthogonal space $(V, \mathcal{B})$. The physical interpretation of the numbers p and q are the number of time-like and space-like dimensions respectively. This means that there exist a basis $\{e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q}\}$ such that:

$$\mathcal{B}(e_i, e_j) = 0,\ if\ i \neq j;$$
$$\mathcal{B}(e_i, e_i) = 1,\ if\ 1 \leq i \leq p;$$
$$\mathcal{B}(e_i, e_i) = -1,\ if\ p+1 \leq i \leq p+q.$$

For instance, the Minkowski space has signature $(1, 3)$ or $(3, 1)$, depending on the convention, while the classical Euclidean space has signature $(3, 0)$. We always assume that the basis is orthonormal. In other words, the associated matrix B is diagonal and of the type $B = diag(1, \cdots, 1, -1, \cdots, -1)$. We are now in a position to give a definition for a Clifford algebra. First, the most general definition is given. Afterwards, a more useful definition that we will continue using throughout this thesis will be given.

Definition 2.3. Suppose that $\mathcal{B}$ is a non-degenerate bilinear form on a real vector space V. The Clifford algebra $\mathcal{C}l(V, \mathcal{B})$ associated to the bilinear form $\mathcal{B}$ is a associative algebra with unit $1 \in \mathbb{R}$ defined as

$$\mathcal{C}l(V, \mathcal{B}) := T(V)/I(V, \mathcal{B}).$$

Here, $T(V)$ is the universal tensor-algebra

$$T(V) := \bigoplus_{k \in \mathbb{N}} (\bigotimes^k V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots,$$

and $I(V, \mathcal{B})$ is the two-sided ideal generated by all elements of the form $u \otimes u - \mathcal{B}(u, u)1$, with $u \in V$.

From now on, we will drop the tensor symbol $u \otimes v$, i.e. we will simply write uv instead. Moreover, we will also drop the unit because we only work with fields $\mathbb{R}$ and $\mathbb{C}$. Earlier, we showed that real non-degenerate orthogonal spaces can be classified according to their signature and that generates a universal real Clifford algebra. After a proper choice for a basis for V, $\mathcal{C}l(V,\mathcal{B})$ can be reduced to the Clifford algebra $\mathcal{C}l(\mathbb{R}^{p,q},\mathcal{B}_{p,q})$.

Lemma 2.1. *For every (p,q) with $p+q=m$, a basis for $\mathcal{C}l(\mathbb{R}^{p,q},\mathcal{B}_{p,q})$ is given by the set*

$$\{1,e_1,\cdots,e_m,e_{1,2},\cdots,e_{m-1m},\cdots,e_{12\cdots m}\},$$

where $e_{i_1\cdots e_k}$ is a shorthand for $e_{i_1}\cdots e_{i_k}$.

If $(V,\mathcal{B})=\mathbb{R}^{p,q}$, the associated Clifford algebra $\mathcal{C}l(\mathbb{R}^{p,q},\mathcal{B}_{p,q})$ will be denoted by $\mathcal{C}l_{p,q}$. An alternative and much more useful definition for this Clifford algebra is the following:

Definition 2.4. For all $(p,q)\in\mathbb{N}\times\mathbb{N}$ with $p+q=m$, the algebra $\mathbb{R}_{p,q}$ is an associative algebra (with unit) that is multiplicatively generated by the basis $\{e_1,\cdots,e_m\}$ satisfying the following multiplication rules:

$$\begin{aligned}
&e_i^2=1,\ if\ 1\le i\le p;\\
&e_i^2=-1,\ if\ p+1\le i\le p+q;\\
&e_ie_j+e_je_i=0,\ if\ i\neq j.
\end{aligned}$$

These are called the universal Clifford algebra for the space $\mathbb{R}^{p,q}$ with $dim_{\mathbb{R}}(\mathcal{C}l_{p,q})=2^m$.

It is clear that a basis for the algebra is given by

$$\mathcal{C}l_{p,q}=Span\{e_{i_1\cdots i_k}:1\le i_1<\cdots<e_k\le m\}.$$

Let $k\in\mathbb{N}$ and $A=\{i_1,\cdots,i_k\}\subset\{1,\cdots,m\}$, then every element of $\mathcal{C}l_{p,q}$ is of the form $\sum_A a_Ae_A$ with $a_A\in\mathbb{R}$. If $A=\emptyset$, we let $e_\emptyset=1$. Elements of a Clifford algebra are called Clifford numbers. We usually use $\mathcal{C}l_m$ as a shorthand notation for $\mathcal{C}l_{0,m}$. We also define the following spaces:

Definition 2.5. For all $0 \leq k \leq m$, we define the space $\mathcal{C}l_{p,q}^{(k)}$ of k-vectors as:

$$\mathcal{C}l_{p,q}^{(k)} := Span_{\mathbb{R}}\{e_A : |A| = k\},$$

with $\mathcal{C}l_{p,q}^{(0)} = \mathbb{R}$. In particular, the space of $\mathcal{C}l_{p,q}^{(1)}$ is called the space of vectors and the space of $\mathcal{C}l_{p,q}^{(2)}$ is called the space of bivectors. Hence, we have

$$\mathcal{C}l_{p,q} = \oplus \mathcal{C}l_{p,q}^{(k)}.$$

Above decomposition can also be rewritten as

$$\mathcal{C}l_{p,q} = \mathcal{C}l_{p,q}^{e} \ \oplus \mathcal{C}l_{p,q}^{o}$$

where $\mathcal{C}l_{p,q}^{e} = \oplus \mathcal{C}l_{p,q}^{(2n)}$, and $\mathcal{C}l_{p,q}^{o} = \oplus \mathcal{C}l_{p,q}^{(2n-1)}$. This tells us $\mathcal{C}l_{p,q}$ is a $\mathbb{Z}_2$-graded algebra.

To conclude this section, we introduce some (anti-)involutions on $\mathcal{C}l_{p,q}$. We first define them on the basis elements, the action on arbitrary Clifford numbers follows by linear extension.

1. The *inversion* on $\mathcal{C}l_{p,q}$ is defined as $\hat{e}_{i_1\cdots i_k} := (-1)^k e_{i_1\cdots i_k}$.

2. The *reversion* on $\mathcal{C}l_{p,q}$ is defined as $\tilde{e}_{i_1\cdots i_k} := e_{i_k\cdots i_1}$.

3. The *conjugation* on $\mathcal{C}l_{p,q}$ is defined as $\bar{e}_{i_1\cdots i_k} := \tilde{\hat{e}}_{i_1\cdots i_k} = (-1)^{\frac{k(k+1)}{2}} e_{i_1\cdots i_k}$.

In the rest of this thesis, we only deal with $\mathcal{C}l_m$ over $\mathbb{R}$ unless it is specified.

2.2 Real subgroups of real Clifford algebras

One of many applications of Clifford algebras $\mathcal{C}l_m$ is the following: they can be used to introduce some important groups which define double coverings of orthogonal group $O(m)$

and special orthogonal group $SO(m)$. These groups are crucial in the study of *the spinor representations.*

Definition 2.6. The orthogonal group $O(m)$ is the group of linear transformations on $\mathbb{R}^m$ which leave the bilinear form invariant, i.e.,

$$\{\varphi \in End(\mathbb{R}^m) : \mathcal{B}(u,v) = \mathcal{B}(\varphi(u), \varphi(v)), \forall u, v \in V\} = \{A \in \mathbb{R}^{m\times m} : A^T A = Id\}.$$

An important subgroup of $O(m)$ is the special orthogonal group

$$SO(m) = \{A \in O(m) : detA = 1\}.$$

Suppose a is a unit vector on the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$, if we consider axa, we may decompose

$$ax\tilde{a} = ax_{a_\parallel}\tilde{a} + ax_{a_\perp}\tilde{a} = -x_{a_\parallel} + x_{a_\perp}.$$

So, the action $ax\tilde{a}$ describes a reflection R_a of x in the direction of a. These reflections are the building blocks for the entire group $O(m)$:

Theorem 2.2. ***(Cartan-Dieudonné)*** *Every element of $O(m)$ is a composition of at most m reflections with respect to hyperplanes in $\mathcal{C}l_m$, i.e. For any $\varphi \in O(m)$, there exist $k \leq m$ and $a_1, \cdots, a_k \in \mathbb{S}^{m-1}$, such that*

$$\varphi = R_{a_1} \circ R_{a_2} \circ \cdots \circ R_{a_k}.$$

If k is even, then φ is a rotation and if k is odd then φ is an anti-rotation.

Hence, we are motivated to define

$$Pin(m) := \{a = y_1 \cdots y_p : \ p \in \mathbb{N} \ and \ y_1, \cdots, y_p \in \mathbb{S}^{m-1}\}$$

where for $a \in Pin(m)$ we have $ax\tilde{a} = O_a x$ for appropriate $O_a \in O(m)$. Under Clifford multiplication $Pin(m)$ is a group. Further, we have a group homomorphism as follows.

$$\theta : \ Pin(m) \longrightarrow O(m); \ a \mapsto O_a.$$

We also define

$$Spin(m) := \{a \in Pin(m) : \ for \ some \ q \in \mathbb{N}, \ a = y_1 \cdots y_{2q}\}.$$

The $Spin(m)$ group is a subgroup of $Pin(m)$ and θ is also a group homomorphism from $Spin(m)$ to $SO(m)$. Indeed, it can be shown that θ is surjective with $Ker\theta = \{-1, 1\}$. Thus, $Pin(m)$ and $Spin(m)$ are double cover of $O(m)$ and $SO(m)$ respectively. See more details in [37].

3 Clifford analysis

Now we have established Clifford algebras and some of their properties, we are concerned with defining a differential operator and performing analysis with Clifford algebras.

3.1 Dirac operators and Clifford analyticity in $\mathcal{C}l_m$

Definition 3.1. Consider $\mathbb{R}^m$ as a subset of $\mathcal{C}l_m$ and write $x \in \mathbb{R}^m$ as $x = x_1 e_1 + \cdots + x_m e_m$. Then we define

$$D_x := \sum_{j=1}^{m} e_j \partial_{x_j}$$

to be the *Dirac operator* for $\mathbb{R}^m$, where ∂_{x_j} is the partial derivative with respect to x_j.

Notice that $D_x^2 = -\Delta$, where Δ is the Laplacian in $\mathbb{R}^m$. This definition suggests we should also consider the following two differential operators in $\mathcal{C}l_m$

$$D'_x := \partial_{x_0} + \sum_{j=1}^{m} e_j \partial_{x_j};\ \overline{D'_x} := \partial_{x_0} - \sum_{j=1}^{m} e_j \partial_{x_j},$$

which have the property $D'_x \overline{D'_x} = \Delta_{m+1}$. D'_x and $\overline{D'_x}$ are also called *the Cauchy-Riemann operator* and *the conjugate Cauchy-Riemann operator*, respectively. In particular, when $m = 1$, this is the one complex variable case. This tells us that the Dirac operator is the generalization of the Cauchy-Riemann operator in analysis of one complex variable to higher dimensions.

As the generalization of analytic function in one complex variable case, a $\mathcal{C}l_m$-valued function f defined on a domain $U \subset \mathbb{R}^m$ is called *left-monogenic* if it is a solution for the Dirac equation, i.e. $D_x f(x) = 0$. Since multiplication of Clifford numbers is not commutative in general, we have a similar definition for *right-monogenic*.

3.2 Integral formulas and fundamental solutions for Euclidean Dirac operator

In complex analysis, the most important properties of analytic functions are Cauchy's integral formula and Cauchy's theorem. Since analytic functions can also be considered as solutions for the Cauchy-Riemann operator, as the generalization of Cauchy-Riemann operator to higher dimensions, the Euclidean Dirac operator also has such integral formulas.

(Cauchy's theorem) [9] *Fix a domain $U \subset \mathbb{R}^m$ and $V \subset\subset U$ with its boundary ∂V a C^1 hypersurface. Suppose $f, g : U \longrightarrow \mathcal{C}l_m$ are C^1 and $gD_x = 0 = D_x f$ on all of U. Then*

$$\int_{\partial V} g(x)n(x)f(x)d\sigma(x) = 0,$$

where $n(x)$ is the outer normal vector and $d\sigma(x)$ is the surface measure on ∂V.

Define $G : \mathbb{R}^m \backslash \{0\} \longrightarrow \mathbb{R}^m \backslash \{0\}$ by $G(x) := \dfrac{x}{||x||^m}$. Note that this function, considered as a function to $\mathcal{C}l_m$, is left and right monogenic. This is the Clifford analysis analogue of the Cauchy kernel $\dfrac{1}{z-w}$ on $\mathbb{C}$. Indeed, we have

(Cauchy's Integral Formula) [9] *Fix a domain $U \subset \mathbb{R}^m$ and $V \subset\subset U$ with its boundary ∂V a C^1 hypersurface. Suppose $f : U \longrightarrow \mathcal{C}l_m$ is C^1 and $D_x f = 0$ on all of U. Then for $y \in V$, we have*

$$f(y) = \frac{1}{\omega_{m-1}} \int_{\partial V} G(x-y) n(x) f(x) d\sigma(x),$$

where ω_{m-1} is the area of the $(m-1)$-dimensional unit sphere $\mathbb{S}^{m-1}$.

In analogy to complex analysis, the Clifford analysis version of Cauchy's integral formula immediately gives one a great deal of results, such as the analyticity (interpreted in the appropriate sense) of monogenic functions.

Now given $f(x)$, a C^1 function defined in a neighborhood of a bounded domain V, we define its *Cauchy transform* by the convolution integral

$$\frac{1}{\omega_{m-1}} \int_V G(x-y) f(x) dx^m.$$

Theorem 3.1. *[40] Suppose f and V are as above. Then for each $y \in V$, it holds that*

$$f(y) = \frac{1}{\omega_{m-1}} D_y \int_V G(x-y) f(x) dx^m.$$

Usually one writes $Tf(y)$ for $\int_V G(x-y) f(x) dx^m$. Note that the previous theorem solves the following problem: Given g, a function C^1 in a neighborhood of V, find f on V so that $Df = g$ and $f = Tg$.

Corollary 3.2. *[40] Suppose $f \in C_0^\infty(\mathbb{R}^m, \mathcal{C}l_m)$. Then $DTf = f$. In particular, if we* *rewrite Tf as $G *_{\mathbb{R}^m} f$, the previous theorem says that*

$$DG *_{\mathbb{R}^m} f = f,$$

$$G *_{\mathbb{R}^m} Df = f.$$

In this sense, D and $G*_{\mathbb{R}^m}$ are inverses of each other over $C_0^\infty(\mathbb{R}^m, \mathcal{C}l_m)$ and G is the fundamental solution of D.

3.3 Möbius transformations and Ahlfors-Vahlen matrices

In analysis of one complex variable, a function f sending a region in $\mathbb{R}^2 = \mathbb{C}$ into $\mathbb{C}$ is conformal at z if it is complex analytic and has a non-zero derivative, $f'(z) \neq 0$ (we only consider sense-preserving conformal mappings). The only conformal transformations of the whole plane $\mathbb{C}$ are affine linear transformations: compositions of rotations, dilations and translations. The *Möbius mapping*

$$f(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0,$$

is affine linear when $c = 0$; otherwise, it is conformal at each $z \in \mathbb{C}$ except $z = -\frac{d}{c}$. The Möbius mapping f sends $\mathbb{C} \backslash \{-\frac{d}{c}\}$ onto $\mathbb{C} \backslash \{\frac{a}{c}\}$. If we agree that $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$, then f becomes a (one-to-one) transformation of $\mathbb{C} \cup \infty$, the complex plane compactified by the point at infinity. These transformations are called *Möbius transformations* of $\mathbb{C} \cup \{\infty\}$. Möbius transformations are compositions of rotations, translations, dilations and inversions. Möbius transformations send circles (and affine lines) to circles (or affine lines). The derivative of a Möbius transformation is a composition of a rotation and a dilation.

In the higher dimensional case, a conformal mapping preserves angles between intersecting curves. Formally, a differomorphism $\phi : U \longrightarrow \mathbb{R}^m$ is said to be *conformal* if for each $x \in U \subset \mathbb{R}^m$ and each $\mathbf{u}, \mathbf{v} \in TU_x$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is preserved under $D\phi_x$. When the dimension $m > 2$, Liouville's Theorem states that any smooth conformal mapping on a domain of $\mathbb{R}^m$ can be expressed as compositions of translations, dilations, orthogonal transformations and inversions: they are Möbius transformations. Ahlfors and Vahlen find a connection between Möbius transformations and a particular matrix group, when the dimension $m > 2$. They show that given a Möbius transformation on $\mathbb{R}^m \cup \{\infty\}$ it can be expressed as $y(x) = (ax+b)(cx+d)^{-1}$ where $a,\ b,\ c,\ d \in Cl_m$ and satisfy the following conditions:

1. $a,\ b,\ c,\ d$ are all products of vectors in $\mathbb{R}^m$;
2. $a\tilde{b},\ c\tilde{d},\ \tilde{b}c,\ \tilde{d}a$ in $\mathbb{R}^m$;
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

The associated matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a *Vahlen matrix* of the Möbius transformation $y(x)$ of $\mathbb{R}^m$, see more details in [37]. All Vahlen matrices form a group under matrix multiplication, the Vahlen group. Notice that $y(x) = (ax+b)(cx+d)^{-1} = ac^{-1} + (b - ac^{-1}d)(cx+d)^{-1}$, this suggests that a conformal transformation can be decomposed as compositions of translation, dilation, reflection and inversion. This is called the *Iwasawa decomposition* for the Möbius transformation $y(x)$.

3.4 Conformal invariance of Dirac operators

One important fact about the conformal mapping is that it preserves monogenic functions, which also means the conformal invariance of the Dirac equation. This has been established for many years, see [36, 41, 42].

Theorem 3.3. ***(Conformal invariance of Dirac equation)****[40]*

Assume $f \in C^1(\mathbb{R}^m, \mathcal{C}l_m)$ and $D_y f(y) = 0$. If $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation, then

$$D_x \frac{\widetilde{cx+d}}{||cx+d||^m} f(M(x)) = 0.$$

In other words, the kernel of the Dirac operator is invariant under Möbius transformations.

Further, we have intertwining operators for the Euclidean Dirac operator, i.e., it is conformally invariant.

Proposition 3.4. *[36] If $y = (ax+b)(cx+d)^{-1}$, then we have*

$$\frac{\widetilde{cx+d}}{||cx+d||^{m+2}} D_y f(y) = D_x \frac{\widetilde{cx+d}}{||cx+d||^m} f((ax+b)(cx+d)^{-1}).$$

We just reviewed the first order conformally invariant differential operator in classical Clifford analysis with some properties. Recall that, in harmonic analysis, as a second order differential operator, the Laplacian Δ is also conformally invariant, and we already knew that $-\Delta = D_x^2$. Hence, we expect that D_x^j is conformally invariant as well for $j > 2$. This has been confirmed and similar results on fundamental solutions and intertwining operators have also been established. First, let $y = M(x) = (ax+b)(cx+d)^{-1}$ be a Möbius transformation, we denote

$$\begin{aligned}
&G_j(x) = \frac{x}{||x||^{m-2n}},\ if\ j = 2n+1;\ G_j(x) = ||x||^{m-2n},\ if\ j = 2n;\\
&J_k(M,x) = \frac{\widetilde{cx+d}}{||cx+d||^{m-2n}},\ if\ j = 2n+1,\ J_k(M,x) = ||cx+d||^{m-2n},\ if\ j = 2n;\\
&J_{-k}(M,x) = \frac{\widetilde{cx+d}}{||cx+d||^{m+2n}},\ if\ j = 2n+1,\ J_k(M,x) = ||cx+d||^{m+2n},\ if\ j = 2n.
\end{aligned}$$

Then we have

Proposition 3.5. *[36]**(Intertwining operators for j-Dirac operator)***

If $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation, then

$$J_{-k}(M,x)D_y^j f(y) = D_x^j J_k(M,x) f((ax+b)(cx+d)^{-1}).$$

Notice that conformal invariance of the j-Dirac equation $D_x^j f(x) = 0$ can be deduced from this easily.

Proposition 3.6. *[36]**(Fundamental solutions for D_x^j)***

The fundamental solution of D_x^j is $G_j(x)$ (up to a multiplicative constant), where $G_j(x)$ is defined as above. However, when the dimension m is even, we require that $j < m$.

Notice that, for instance, when the dimension m is even and $m = j$, then the candidate for the fundamental solution $G_j(x)$ is a constant, which can not be a fundamental solution.

4 Representation theory of Spin group

In this section, we will discuss some classical results of representation theory of Lie groups, which play an important role in later sections. First, we start with some general definitions and notations, then we will review some basic knowledges of weight theory. Finally, some well-known representations of Spin group are given as examples. These representation spaces are often considered as the target spaces of functions in Clifford analysis. This also points out that representation theory emerges naturally in Clifford analysis.

4.1 General definitions and notations

A *Lie group* is a smooth manifold G which is also a group such that multiplication $(g,h) \mapsto gh: \ G \times G \longrightarrow G$ and inversion $g \mapsto g^{-1}: \ G \longrightarrow G$ are both smooth. A *representation* of a Lie group G on a finite dimensional complex vector space V is a homomorphism $\rho: \ G \longrightarrow GL(V)$ of G to the group of automorphisms of V; we say that such a map *gives V the structure of a G-module.* We sometimes call V itself a representation of G and write $g\dot{v}$ or gv instead of $\rho(g)v$. The dimension of V is sometimes called the *degree (or dimension)* of ρ.

A *G-module homomorphism* φ between two representations V and W of G is a vector space homomorphism $\varphi : V \longrightarrow W$ such that

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes for every $g \in G$.

A *subrepresentation* of a representation V is a vector subspace W of V, which is invariant under G action. A representation V is called *irreducible* if its only subrepresentations are $\{0\}$ and itself.

If V and W are both representations, the *direct sum* $V \oplus W$ and the *tensor product* $V \otimes W$ are also representations, the latter via

$$g(v \otimes w) = gv \otimes gw.$$

The *dual* $V^* = Hom(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: we want the two representations of G to respect the natural pairing (denoted $\langle \cdot, \cdot \rangle$) between V^* and V, so that if $\rho : G \longrightarrow GL(V)$ is a representation and $\rho^* : G \longrightarrow GL(V^*)$ is the dual, we should have

$$\langle \rho^*(g)(v^*, \rho(g)(v)) \rangle = \langle v^*, v \rangle$$

for all $g \in G,\ v \in V$ and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g) = {}^t\rho(g)^{-1} : V^* \longrightarrow V^*$$

for all $g \in G$.

In this thesis, we only deal with finite dimensional representations of compact Lie groups, which have the following important property.

Proposition 4.1. ***(Complete Reducibility)*** *Any finite dimensional representation of a compact Lie group is a direct sum of irreducible representations.*

Thanks to the following Schur's lemma, the above decomposition is unique.

(Schur's Lemma)[22] *If V and W are irreducible representations of G and φ is a G-module homomorphism, then*

1. *Either φ is a G-module isomorphism, or $\varphi = 0$;*

2. *If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity.*

4.2 Weight and weight vectors

Given a compact Lie group G, we can consider the closed, connected, abelian sub-groups of G. Each of these groups is a direct product of copies of the unit circle S^1. A maximal abelian subgroup of G is called a maximal torus of G, and it is denoted by T. The maximal torus has the form $T = \{(e^{i\theta_1}, \cdots, e^{i\theta_k}) : \ \theta_j \in \mathbb{R}, \ \forall j\}$. Since T is abelian, each irreducible complex representation of T must be one-dimensional, hence has the form

$$(e^{i\theta_1}, \cdots, e^{i\theta_k}) \mapsto e^{i(\theta_1 m_1 + \theta_2 m_2 + \cdots + \theta_k m_k)}.$$

Given a representation ρ of G, we may write the restriction $\rho|_T$ as a direct sum of irreducible (i.e. one-dimensional) representations

$$(e^{i\theta_1}, \cdots, e^{i\theta_k}) \mapsto e^{i(\theta_1 m_1 + \theta_2 m_2 + \cdots + \theta_k m_k)}.$$

Definition 4.1. Given a representation ρ of G, a *weight* of ρ is a k-tuple $(m_1, \cdots, m_k)$ of integers corresponding to an irreducible representation

$$(e^{i\theta_1}, \cdots, e^{i\theta_k}) \mapsto e^{i(\theta_1 m_1 + \theta_2 m_2 + \cdots + \theta_k m_k)}$$

of T when ρ is restricted to T. The vectors in each (one-dimensional) irreducible representation space are called *weight vectors*.

The set of weights of a representation can be given the lexicographic ordering. That is $(m_1, \cdots, m_k) > (n_1, \cdots, n_k)$ if the first nonzero difference of $m_i - n_i$ is positive. The weight which is largest with respect to this ordering is called the *highest weight* of representation. and its weight vectors are also called *highest weight vectors*. These two terminologies play an important role in the representation theory of compact Lie groups, since every representation of a compact Lie group possesses a unique highest weight, and the representation space can be constructed from its highest weight vector by applying group elements to it successively. See more details in [20].

Maximal torii play an integral role in the representation theory of Lie groups. Every element in a compact Lie group is conjugate to an element in a maximal torus, and any two maximal tori are themselves conjugate by an automorphism of G. If we let $N(T) = \{g \in G \mid gTg^{-1} = T\}$ denote the normalizer of T, then we define the *Weyl group* associated to the Lie group G by

$$W = N(T)/T.$$

The Weyl group of a Lie group acts on a maximal torus by conjugation. Hence, one has an action on the set of weights by conjugation. See more details in [22].

4.3 Irreducible representations of Spin groups

We now introduce three representations of $Spin(m)$. The first representation space of the Spin group is used as the target space in spinor-valued theory and the other two representation spaces of the Spin group are frequently used as the target spaces of functions in higher spin theory.

4.3.1 Spinor representation space $\mathcal{S}$

The most commonly used representation of the Spin group in $\mathcal{C}l_m(\mathbb{C})$-valued function theory is the spinor space. To this end, consider the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ with odd dimension $m = 2n+1$. The space of vectors $\mathbb{C}^m$ is embedded in $\mathcal{C}lm(\mathbb{C})$ as

$$(x_1, x_2, \cdots, x_m) \mapsto \sum_{j=1}^{m} x_j e_j : \ \mathbb{C}^m \hookrightarrow \mathcal{C}l_m(\mathbb{C}).$$

We denote x for a *vector* in both interpretations. The space of $k-$*vectors* is defined as

$$\mathcal{C}l_m(\mathbb{C})^{(k)} = span_{\mathbb{C}}\{e_{i_1} \dots e_{i_k} : \ 1 \leq i_1 < \cdots < i_k \leq m\}.$$

The Clifford algebra $\mathcal{C}l_m$ can be rewritten as a direct sum of the even subalgebra and the odd subalgebra:

$$\mathcal{C}l_m(\mathbb{C}) = \mathcal{C}l_m(\mathbb{C})^+ \oplus \mathcal{C}l_m(\mathbb{C})^-,$$

where

$$\mathcal{C}l_m(\mathbb{C})^+ = \oplus_{j=1}^{m} \mathcal{C}l_m(\mathbb{C})^{(2j)}; \ \mathcal{C}l_m(\mathbb{C})^- = \oplus_{j=1}^{m} \mathcal{C}l_m(\mathbb{C})^{(2j-1)}.$$

The *Witt basis* elements of $\mathbb{C}^{2m}$ are defined by

$$f_j := \frac{e_j - ie_{j+n}}{2}, \quad f_j^\dagger := -\frac{e_j + ie_{j+n}}{2}.$$

Let $I := f_1 f_1^\dagger \dots f_n f_n^\dagger$. The space of *Dirac spinors* is defined as

$$\mathcal{S} := \mathcal{C}lm(\mathcal{C})^+ I.$$

This is a representation of $Spin(m)$ under the following action

$$\rho(s)\mathcal{S} := s\mathcal{S},\ \ for\ s \in Spin(m).$$

Note $\mathcal{S}$ is a left ideal of $\mathcal{C}lm(\mathbb{C})$. For more details, see [9]. An alternative construction of spinor spaces is given in the classic paper of Atiyah, Bott and Shapiro [1].

4.3.2 Homogeneous harmonic polynomials on $\mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$

It is well known that the space of harmonic polynomials is invariant under action of $Spin(m)$, since the Laplacian Δ_m is an $SO(m)$ invariant operator. It is not irreducible for $Spin(m)$, however, and can be decomposed into the infinite sum of k-homogeneous harmonic polynomials, $0 \leq j < \infty$. Each of these spaces is irreducible for $Spin(m)$. This brings the most familiar representations of $Spin(m)$: spaces of j-homogeneous harmonic polynomials on $\mathbb{R}^m$, denoted by $\mathcal{H}_j$. The following action has been shown to be an irreducible representation of $Spin(m)$ [27]:

$$\rho\ :\ Spin(m) \longrightarrow Aut(\mathcal{H}_j),\ s \longmapsto \big(f(x) \mapsto f(sy\tilde{s})\big)$$

with $x = sy\tilde{s}$. This can also be realized as follows

$$Spin(m) \xrightarrow{\theta} SO(m) \xrightarrow{\rho} Aut(\mathcal{H}_j),\ a \longmapsto O_a \longmapsto \big(f(x) \mapsto f(O_a y)\big),$$

where $x = O_a y$, θ is the double covering map, and ρ is the standard action of $SO(m)$ on a function $f(x) \in \mathcal{H}_j$ with $x \in \mathbb{R}^m$. The function $\phi(z) = (z_1 + iz_m)^j$ is the highest weight vector for $\mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$ having highest weight $(k, 0, \cdots, 0)$. See [21] for details. Accordingly, we say the spin representations given by $\mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$ have integer spin j; we can either specify an integer spin j or the degree of homogeneity j of harmonic polynomials.

4.3.3 Homogeneous monogenic polynomials on $\mathcal{C}l_m$

Let $\mathcal{M}_k$ denote the space of $\mathcal{C}l_m$-valued monogenic polynomials, homogeneous of degree k. Note that if $h_k \in \mathcal{H}_k$, the space of $\mathcal{C}l_m$-valued harmonic polynomials homogeneous of degree k, then $Dh_k \in \mathcal{M}_{k-1}$, but $Dup_{k-1}(u) = (-m - 2k + 2)p_{k-1}u$, so

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1},\ h_k = p_k + up_{k-1}.$$

This is an *Almansi-Fischer decomposition* of $\mathcal{H}_k$. See [15] for more details. From the above decomposition, we have another important representation of $Spin(m)$: the space of j-homogeneous monogenic polynomials on $\mathbb{R}^m$, denoted by $\mathcal{M}_j$. Specifically, the following action has been shown to be an irreducible representation of $Spin(m)$:

$$\pi\ :\ Spin(m) \longrightarrow Aut(\mathcal{M}_j),\ s \longmapsto f(x) \mapsto sf(sy\tilde{s})$$

with $x = sy\tilde{s}$. When m is odd, in terms of complex variables $z_s = x_{2s-1} + ix_{2s}$ for all $1 \leq s \leq \frac{m-1}{2}$, the highest weight vector is $\omega_k(x) = (\bar{z_1})^k I$ for $\mathcal{M}_j(\mathbb{R}^m, \mathcal{S})$ having highest weight $(k + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, where $\bar{z_1}$ is the conjugate of z_1; $\mathcal{S}$ is the Dirac spinor space; and I is defined as in *Section 2.2.1*. For details, see [27]. Accordingly, the spin representations given by $\mathcal{M}_j(\mathbb{R}^m, \mathcal{S})$ are said to have half-integer spin $j + \frac{1}{2}$; we can either specify a half-integer spin $j + \frac{1}{2}$ or the degree of homogeneity j of monogenic polynomials.

In classical Clifford analysis, the Euclidean Dirac operator and its j-th power have been proved as conformally invariant differential operators. These operators act on functions taking values in Clifford numbers, see [36, 41, 42] etc. In contrast, operator theory in higher spin spaces considers functions taking values in irreducible representations of the Spin group, for instance, j-homogeneous monogenic or j-homogeneous harmonic polynomial spaces. In the following several sections, we will complete the work of constructing conformally invariant differential operators in higher spin spaces.

5 First order conformally invariant differential operators in Euclidean space

In 1941, Rarita and Schwinger [38] introduced a simplified formulation of the theory of fermions and in particular its implications for spin $\frac{3}{2}$. Bures et al. [6] systematically studied the first order conformally invariant differential operator, named the Rarita-Schwinger operator, in 2002. It has the following analytic construction.

Recall the Almansi-Fischer decomposition

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}.$$

We define P_k as the projection map

$$P_k : \mathcal{H}_k \longrightarrow \mathcal{M}_k.$$

Suppose U is a domain in $\mathbb{R}^m$. Consider $f : U \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$, such that for each $x \in U$, $f(x,u)$ is a left monogenic polynomial homogeneous of degree k in u. The Rarita-Schwinger operator is defined as follows

$$R_k := P_k D_x f(x,u) = (\frac{uD_u}{m+2k-2} + 1) D_x f(x,u).$$

We also have a right projection $P_{k,r} :\ \mathcal{H}_k \longrightarrow \overline{\mathcal{M}}_k$, and a right Rarita-Schwinger operator $R_{k,r} = D_x P_{k,r}$. See [6, 15].

In classical Clifford analysis, the Euclidean Dirac operator was initially motivated from

Stokes' Theorem [40] and Clifford algebras were used to study it. When we consider function theory in higher spin spaces, since these functions take values in irreducible representations of Spin groups, it turns out representation theory provides a quite different approach for operator theory in higher spin spaces. Abundant results have been found with this approach: for instance, [7, 4, 8, 17]. In 1968, Stein and Weiss [48] pointed out that many first order differential operators can be constructed as projections of generalized gradients with the help of representation theory. Fegan [19] showed that such operators are conformally invariant with certain conditions. Since this construction generalizes further to representations of principal bundles over oriented Riemannian spin manifolds, by which one constructs the Atiyah-Singer Dirac operator, we argue the Stein and Weiss construction is the natural way to construct other Dirac type operators as in [45, 21]. Hence, we will show the constructions of the Euclidean Dirac and Rarita-Schwinger operators as Stein-Weiss type operators. First, let us see the definition for Stein-Weiss type operators.

5.1 Stein Weiss type operators

Let U and V be m-dimensional inner product vector spaces over a field $\mathbb{F}$. Denote the groups of all automorphisms of U and V by $GL(U)$ and $GL(V)$, respectively. Suppose $\rho_1: G \longrightarrow GL(U)$ and $\rho_2: G \longrightarrow GL(V)$ are irreducible representations of a compact Lie group G. We have a function $f: U \longrightarrow V$ which has continuous derivative. Taking the gradient of the function $f(x)$, we have

$$\nabla f \in Hom(U,V) \cong U^* \otimes V \cong U \otimes V, \text{ where } \nabla := (\partial_{x_1}, \cdots, \partial_{x_m}).$$

Denote by $U[\times]V$ the irreducible subrepresentation of $U \otimes V$ whose representation space has largest dimension. This is known as the Cartan product of ρ_1 and ρ_2 [16]. Using the inner products on U and V, we may write

$$U \otimes V = (U[\times]V) \oplus (U[\times]V)^{\perp}.$$

If we denote by E and $E^{\perp}$ the orthogonal projections onto $U[\times]V$ and $(U[\times]V)^{\perp}$, respectively, then we define differential operators D and $D^{\perp}$ associated to ρ_1 and ρ_2 by

$$D = E\nabla;\ D^{\perp} = E^{\perp}\nabla.$$

These are named *Stein-Weiss type operators* after [48]. The importance of this construction is that one can reconstruct many first order differential operators with it by choosing proper representation spaces U and V for a Lie group G, such as Euclidean Dirac operators and Rarita-Schwinger operators.

1. Euclidean Dirac operators

Here we only show the odd dimension case, but the even dimensional case is similar.

Theorem 5.1. *Let ρ_1 be the representation of the spin group given by the standard representation of $SO(m)$ on $\mathbb{R}^m$*

$$\rho_1 :\ Spin(m) \longrightarrow SO(m) \longrightarrow GL(\mathbb{R}^m)$$

and let ρ_2 be the spin representation on the spinor space $\mathcal{S}$. Then the Euclidean Dirac operator is the differential operator given by projecting the gradient onto $(\mathbb{R}^m[\times]\mathcal{S})^{\perp}$ when $m = 2n + 1$.

Outline proof: Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of $\mathbb{R}^m$ and $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$. For a function $f(x)$ having values in $\mathcal{S}$, we must show that the system

$$\sum_{i=1}^{m} e_i \frac{\partial f}{\partial x_i} = 0$$

is equivalent to the system

$$D^{\perp} f = E^{\perp} \nabla f = 0.$$

We have

$$\mathbb{R}^m \otimes \mathcal{S} = \mathbb{R}^m[\times]\mathcal{S} \oplus (\mathbb{R}^m[\times]\mathcal{S})^{\perp}$$

and [48] provides an embedding map

$$\begin{aligned} &\eta : \mathcal{S} \hookrightarrow \mathbb{R}^m \otimes \mathcal{S}, \\ &\omega \mapsto \frac{1}{\sqrt{m}}(e_1\omega, \cdots, e_m\omega). \end{aligned}$$

Actually, this is an isomorphism from $\mathcal{S}$ into $\mathbb{R}^m \otimes \mathcal{S}$. For the proof, we refer the reader to *page 175* of [48]. Thus, we have

$$\mathbb{R}^m \otimes \mathcal{S} = \mathbb{R}^m[\times]\mathcal{S} \oplus \eta(\mathcal{S}).$$

Consider the equation $D^{\perp} f = E^{\perp} \nabla f = 0$, where f has values in $\mathcal{S}$. So ∇f has values in $\mathbb{R}^m \otimes \mathcal{S}$, and the condition $D^{\perp} f = 0$ is equivalent to ∇f being orthogonal to $\eta(\mathcal{S})$. This is precisely the statement that

$$\sum_{i=1}^{m} (\frac{\partial f}{\partial x_i}, e_i\omega) = 0, \ \forall \omega \in \mathcal{S}.$$

Notice, however, that as an endomorphism of $\mathbb{R}^m \otimes \mathcal{S}$, we have $-e_i$ as the dual of e_i. Hence the equation above becomes

$$\sum_{i=1}^{m} (e_i \frac{\partial f}{\partial x_i}, \omega) = 0, \ \forall \omega \in \mathcal{S},$$

which says precisely that f must be in the kernel of the Euclidean Dirac operator. This completes the proof. $\square$

2. Rarita-Schwinger operators

We first introduce decomposition of the following tensor product representation of $Spin(m)$.

Theorem 5.2. *Let ρ_1 be defined as above and ρ_2 be the representation of $Spin(m)$ on $\mathcal{M}_k$. Then as a representation of $Spin(m)$, we have the following decomposition*

$$\mathcal{M}_k \otimes \mathbb{R}^m \cong \mathcal{M}_k[\times]\mathbb{R}^m \oplus \mathcal{M}_k \oplus \mathcal{M}_{k-1} \oplus \mathcal{M}_{k,1}.$$

Proof. In this theorem, $\mathcal{M}_{k,1}$ stands for the simplicial-$(k,1)$ homogeneous monogenic polynomial space, which is an irreducible representation of $Spin(m)$. More details can be found in [4, 27]. To prove the above theorem, we need several facts of representations of $Spin(m)$ as follows, which can also be found in [21, 22]. Here we only treat the odd dimensional case for the Lie group $Spin(m)$. The even dimensional case is similar.

The fundamental weights for $Spin(m)$, which are the 'basis' for highest weights of irreducible representations of $Spin(m)$, are denoted by m entries as follows

$$\big\{(1,0,\cdots,0),(1,1,0,\cdots,0),\cdots,(1,\cdots,1,0),(\frac{1}{2},\cdots,\frac{1}{2})\big\},$$

where the 'basis' means that all highest weights of any irreducible representation of $Spin(m)$ can be written as linear combinations of the above fundamental weights. $\delta = (k-\frac{1}{2},k-\frac{3}{2},\cdots,\frac{1}{2})$ is the sum of all fundamental weights of $Spin(m)$. The Weyl group W of $Spin(m)$ acts on the set of weights by permutations and sign reversals of entries. The length of an element $\omega \in W$ will be denoted by $|\omega|$. The set of dominant integral weights Λ_W consists of weights $\mu = (\mu_1,\cdots,\mu_m)$ satisfying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0$. For each $\mu \in \Lambda_W$ we denote by V_μ the irreducible representation of $Spin(m)$ with highest weight μ and by Π_μ its set of all weights. Now, we introduce the Brauer-Klimyk formula [8].

Proposition 5.3. *For every pair $\mu, \rho \in \Lambda_W$,*

$$V_\mu \otimes V_\rho = \bigoplus_{\mu' \in \Pi_\mu} \sigma(\rho + \mu' + \delta) m_\mu(\mu') V_{[\rho+\mu'+\delta]-\delta},$$

where $\sigma(\omega) = 0$ if there is some $\lambda \in W$ that fixes ω and $\sigma(\omega) = (-1)^{|\lambda(\omega)|}$ where $\lambda(\omega)$ means the unique element of W such that $[\omega] := \lambda(\omega)\omega \in \Lambda_W$ and $m_\mu(\mu')$ is the multiplicity of the weight μ' in V_μ.

The highest weight of $\mathcal{M}_k$ is $\rho = (k + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and the highest weight of the standard representation $\mathbb{R}^m$ is $\mu = (1, 0, \cdots, 0)$. Let

$$\Pi_\mu = \{(\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1), (0, \cdots, 0)\}.$$

Hence, for $\mu' \in \Pi_\mu$, we have

$$\rho + \delta + \mu' = (2k, k-1, k-2, \cdots, 1) + \mu'.$$

Notice that if $\mu' \in \{(0, -1, 0, \cdots, 0),\ (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1, 0),\ (0, \cdots, 0, 1)\}$, then there are two identical entries in $\rho + \delta + \mu'$. Since the Weyl group of $Spin(m)$ acts on the set of weights by permutations and sign reversals of entries, there exists an element of the Weyl group which fixes $\rho + \sigma + \mu'$. Hence, in the Brauer-Klimyk formula, $\sigma(\rho + \delta + \mu') = 0$ for these choices of μ'. In other words, μ' can only be chosen from $\{(\pm 1, 0, \cdots, 0),\ (0, 1, 0 \cdots, 0),\ (0, \cdots, 0, -1),\ (0, \cdots, 0)\}$. However, we notice that on the right side of the Brauer-Klimyk formula, $V_{[\rho+\mu'+\delta]-\delta} = V_{\rho+\mu'}$ in our circumstance, since entries of $\rho + \mu' + \delta$ are all integers. We also require $\rho + \mu'$ to be a dominant weight (all entries must be positive), which rules out $(0, \cdots, 0, -1)$. Thus, the only remaining weights in Π_μ are

$$\{(1, 0, \cdots, 0),\ (-1, 0, \cdots, 0),\ (0, 1, 0, \cdots, 0),\ (0, \cdots, 0)\}.$$

These give the following highest weights in the decomposition

$$(k+\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2}),\ (k-\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2}),\ (k+\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2}),\ (k+\frac{1}{2},\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2}),$$

which correspond to the following irreducible representations of $Spin(m)$ [21, 27]

$$\mathcal{M}_k[\times]\mathbb{R}^m,\ \mathcal{M}_{k-1},\ \mathcal{M}_k,\ \mathcal{M}_{k,1}.$$

The multiplicities of these irreducible representations in the decomposition are 1, which can be obtained easily from the Kostant's weight multiplicity formula. Since this requires more details of representation theory of Lie groups, which are beyond the scope of this thesis, we refer the readers to [5] for more details. □

Given the previous theorem, we have a construction for the Rarita-Schwinger operator as a Stein Weiss type operator as follows.

Theorem 5.4. *The Rarita-Schwinger operator is the differential operator given by projecting the gradient from $\mathcal{M}_k\otimes\mathbb{R}^m$ onto the $\mathcal{M}_k$ component of the decomposition given in the previous theorem.*

Proof. Consider $f(x,u)\in C^\infty(\mathbb{R}^m,\mathcal{M}_k)$. We observe that the gradient of $f(x,u)$ satisfies

$$\nabla f(x,u)=(\partial_{x_1},\cdots,\partial_{x_m})f(x,u)=(\partial_{x_1}f(x,u),\cdots,\partial_{x_m}f(x,u))\in\mathcal{M}_k\otimes\mathbb{R}^m.$$

Since

$$\mathcal{M}_k\otimes\mathbb{R}^m=\mathcal{M}_k[\times]\mathbb{R}^m\oplus V_1\oplus V_2\oplus V_3,$$

where $V_1\cong\mathcal{M}_k$, $V_2\cong\mathcal{M}_{k-1}$ and $V_3\cong\mathcal{M}_{k,1}$ as $Spin(m)$ representations. Similar arguments as on *page 175* of [48] show

$$\theta: \ \mathcal{M}_k \longrightarrow \mathcal{M}_k \otimes \mathbb{R}^m, \ q_k(u) \mapsto (q_k(u)e_1, \cdots, q_k(u)e_m)$$

is an isomorphism from $\mathcal{M}_k$ into $\mathcal{M}_k \otimes \mathbb{R}^m$. Hence, we have

$$\mathcal{M}_k \otimes \mathbb{R}^m = \mathcal{M}_k[\times]\mathbb{R}^m \oplus \ \theta(\mathcal{M}_k) \oplus V_2 \oplus V_3.$$

Let P_k' be the projection map from $\mathcal{M}_k \otimes \mathbb{R}^m$ to $\theta(\mathcal{M}_k)$. Consider the equation $P_k'\nabla f(x,u) = 0$ for $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$. Then, for each fixed x, $\nabla f(x,u) \in \mathcal{M}_k \otimes \mathbb{R}^m$ and the condition $P_k'\nabla f(x,u) = 0$ is equivalent to ∇f being orthogonal to $\theta(\mathcal{M}_k)$. This says precisely

$$\sum_{i=1}^m (q_k(u)e_i, \partial_{x_i} f(x,u))_u = 0, \ \forall q_k(u) \in \mathcal{M}_k,$$

where $(p(u), q(u))_u = \displaystyle\int_{\mathbb{S}^{m-1}} \overline{p(u)}q(u)dS(u)$ is the Fischer inner product for any pair of $\mathcal{C}l_m$-valued polynomials. Since $-e_i$ is the dual of e_i as an endomorphism of $\mathcal{M}_k \otimes \mathbb{R}^m$. The previous equation becomes

$$\sum_{i=1}^m (q_k(u), e_i\partial_{x_i} f(x,u)) = (q_k(u), D_x f(x,u))_u = 0.$$

Since $f(x,u) \in \mathcal{M}_k$ for fixed x, then $D_x f(x,u) \in \mathcal{H}_k$. According to the Almansi-Fischer decomposition, we have

$$D_x f(x,u) = f_1(x,u) + u f_2(x,u), \ f_1(x,u) \in \mathcal{M}_k \text{ and } f_2(x,u) \in \mathcal{M}_{k-1}.$$

We then obtain $(q_k(u), f_1(x,u))_u + (q_k(u), uf_2(x,u))_u = 0$. However, the Clifford-Cauchy theorem [15] shows $(q_k(u), uf_2(x,u))_u = 0$. Thus, the equation $P_k'\nabla f(x,u) = 0$ is equivalent to

$$(q_k(u), f_1(x,u))_u = 0, \ \forall q_k(u) \in \mathcal{M}_k.$$

Hence, $f_1(x,u) = 0$. We also know, from the construction of the Rarita-Schwinger operator, that $f_1(x,u) = R_k f(x,u)$. Therefore, the Stein-Weiss type operator $P_k'\nabla$ is precisely the Rarita-Schwinger operator in this context. □

We have demonstrated one application of the Representation-Theoretic approach to Clifford analysis: the Stein-Weiss generalized gradient construction of the Euclidean Dirac and Rarita-Schwinger operators. The operators are realized on irreducible representations of the Spin group. In higher spin theory, we consider operators on functions taking values in irreducible spin representations that have higher spin. Seeing our success already, we will use the Representation-Theoretic approach to extend the higher spin theory to arbitrary order conformally invariant differential operators of arbitrary spin in the next several sections. Now, we prefer to introduce a counterexample, which actually motivates our work. Then we provide corrections for some proofs in [15]. Finally, we introduce all first order conformally invariant differential operators in higher spin spaces with some of their properties.

5.2 Properties of the Rarita-Schwinger operator

5.2.1 A counterexample

We know that the Dirac operator D_x is conformally invariant in $\mathcal{C}l_m$-valued function theory [42]. But in the Rarita-Schwinger setting, D_x is not conformally invariant anymore. In other words, in $\mathcal{C}l_m$-valued function theory, the Dirac operator D_x has the following conformal invariance property under inversion: If $D_x f(x) = 0$, $f(x)$ is a $\mathcal{C}l_m$-valued function and $x = y^{-1}$, $x \in \mathbb{R}^m$, then $D_y \dfrac{y}{||y||^m} f(y^{-1}) = 0$. In the Rarita-Schwinger setting, if $D_x f(x,u) = D_u f(x,u) = 0$, $f(x,u)$ is a polynomial for any fixed $x \in \mathbb{R}^m$ and let

$x = y^{-1},\ u = \frac{ywy}{||y||^2}$, $x \in \mathbb{R}^m$, then $D_y \frac{y}{||y||^m} f(y^{-1}, \frac{ywy}{||y||^2}) \neq 0$ in general.

A quick way to see this is to choose the function $f(x,u) = u_1e_1 - u_2e_2$, and use $u = \frac{ywy}{||y||^2} = w - 2\frac{y}{||y||^2}\langle w, y\rangle$, $u_i = w_i - 2\frac{y_i}{||y||^2}\langle w, y\rangle$, where $i = 1, 2, \ldots, m$. A straightforward calculation shows that

$$D_y \frac{y}{||y||^m} f(y^{-1}, \frac{ywy}{||y||^2}) = \frac{-2wy(y_1e_1 - y_2e_2)}{||y||^{m+2}} \neq 0,$$

for $m > 2$. However, $P_1 D_y \frac{y}{||y||^m} f(y^{-1}, \frac{ywy}{||y||^2}) = \big(\frac{wD_w}{m} + 1\big) w \frac{-2y(y_1e_1 - y_2e_2)}{||y||^{m+2}} = 0.$

5.2.2 Conformal Invariance

In [15], the conformal invariance of the equation $R_k f = 0$ is proved and some other properties under the assumption that D_x is still conformally invariant in the Rarita-Schwinger setting. This is incorrect, as we just showed. In this section, we will use the Iwasawa decomposition of Möbius transformations and some integral formulas to correct this. As observed earlier, according to this Iwasawa decomposition, a conformal transformation is a composition of translation, dilation, reflection and inversion. A simple observation shows that the Rarita-Schwinger operator is conformally invariant under translation and dilation and the conformal invariance under reflection can be found in [27]. Hence, we only show it is conformally invariant under inversion here.

Theorem 5.5. *For any fixed $x \in U \subset \mathbb{R}^m$, let $f(x,u)$ be a left monogenic polynomial homogeneous of degree k in u. If $R_{k,u} f(x,u) = 0$, then $R_{k,w} G(y) f(y^{-1}, \frac{ywy}{||y||^2}) = 0$, where $G(y) = \frac{y}{||y||^m},\ x = y^{-1},\ u = \frac{ywy}{||y||^2} \in \mathbb{R}^m$.*

To establish the conformal invariance of R_k, we need *Stokes′ Theorem* for R_k.

Theorem 5.6 ([15])**.** ***(Stokes' theorem for*** R_k)

Let Ω' and Ω be domains in $\mathbb{R}^m$ and suppose the closure of Ω lies in Ω'. Further suppose *the closure of* Ω *is compact and* $\partial\Omega$ *is piecewise smooth. Let* $f, g \in C^1(\Omega', \mathcal{M}_k)$*. Then*

$$\begin{aligned}
& \int_\Omega \big[(g(x,u)R_k, f(x,u))_u + (g(x,u), R_k f(x,u))\big] dx^m \\
= \quad & \int_{\partial\Omega} (g(x,u), P_k d\sigma_x f(x,u))_u \\
= \quad & \int_{\partial\Omega} (g(x,u) d\sigma_x P_{k,r}, f(x,u))_u,
\end{aligned}$$

where P_k *and* $P_{k,r}$ *are the left and right projections,* $d\sigma_x = n(x)d\sigma(x)$*,* $d\sigma(x)$ *is the area element.* $(P(u), Q(u))_u = \int_{\mathbb{S}^{m-1}} P(u)Q(u)dS(u)$ *is the inner product for any pair of* Cl_m*-valued polynomials.*

If both $f(x,u)$ and $g(x,u)$ are solutions of R_k, then we have *Cauchy's theorem.*

Corollary 5.7 ([15]). ***(Cauchy's theorem for*** R_k)

If $R_k f(x,u) = 0$ *and* $g(x,u)R_k = 0$ *for* $f, g \in C^1(, \Omega', \mathcal{M}_k)$*, then*

$$\int_{\partial\Omega} (g(x,u), P_k d\sigma_x f(x,u))_u = 0.$$

We also need the following well-known result.

Proposition 5.8 ([41]). *Suppose that* S *is a smooth, orientable surface in* R^m *and* $f,\ g$ *are integrable* Cl_m*-valued functions. Then if* $M(x)$ *is a conformal transformation, we have*

$$\int_S f(M(x))n(M(x))g(M(x))ds = \int_{S^{-1}} f(M(x))\tilde{J}_1(M,x)n(x)J_1(M,x)g(M(x))dS^{-1},$$

where $M(x) = (ax+b)(cx+d)^{-1}$*,* $S^{-1} = \{x \in \mathbb{R}^m : M(x) \in S\}$*,* $J_1(M,x) = \dfrac{\widetilde{cx+d}}{||cx+d||^m}$.

Now we are ready to prove *Theorem* 5.5.

Proof. First, in Cauchy's theorem, we let $g(x,u)R_{k,r} = R_k f(x,u) = 0$. Then we have

$$0 = \int_{\partial\Omega}\int_{\mathbb{S}^{m-1}} g(x,u)P_k n(x)f(x,u)dS(u)d\sigma(x)$$

Let $x = y^{-1}$. According to *Proposition* 5.8, we have

$$= \int_{\partial\Omega^{-1}}\int_{\mathbb{S}^{m-1}} g(u)P_{k,u}G(y)n(y)G(y)f(y^{-1},u)dS(u)d\sigma(y),$$

where $G(y) = \frac{y}{||y||^m}$. Set $u = \frac{ywy}{||y||^2}$. Since $P_{k,u}$ interchanges with $G(y)$ [33], we have

$$\begin{aligned} &= \int_{\partial\Omega^{-1}}\int_{\mathbb{S}^{m-1}} g(\frac{ywy}{||y||^2})G(y)P_{k,w}n(y)G(y)f(y^{-1},\frac{ywy}{||y||^2})dS(w)d\sigma(y) \\ &= \int_{\partial\Omega^{-1}} (g(\frac{ywy}{||y||^2})G(y), P_{k,w}d\sigma_y G(y)f(y^{-1},\frac{ywy}{||y||^2}))_w, \end{aligned}$$

According to Stokes' theorem,

$$\begin{aligned} = & \int_{\Omega^{-1}} (g(\frac{ywy}{||y||^2})G(y), R_{k,w}G(y)f(y^{-1},\frac{ywy}{||y||^2}))_w \\ & + \int_{\Omega^{-1}} (g(\frac{ywy}{||y||^2})G(y)R_{k,w}, G(y)f(y^{-1},\frac{ywy}{||y||^2}))_w. \end{aligned}$$

Since $g(x,u)$ is arbitrary in the kernel of $R_{k,r}$ and $f(x,u)$ is arbitrary in the kernel of R_k, we get $g(\frac{ywy}{||y||^2})G(y)R_{k,w} = R_{k,w}G(y)f(y^{-1},\frac{ywy}{||y||^2}) = 0$. $\square$

5.2.3 Intertwining operators of R_k

In $\mathcal{C}l_m$-valued function theory, if we have the Möbius transformation $y = \phi(x) = (ax+b)(cx+d)^{-1}$ and D_x is the Dirac operator with respect to x and D_y is the Dirac operator with respect to y then $D_x = J_{-1}^{-1}(\phi,x)D_yJ_1(\phi,x)$, where $J_{-1}(\phi,x) = \frac{cx+d}{||cx+d||^{m+2}}$ and $J_1(\phi,x) = \frac{cx+d}{||cx+d||^m}$ [41]. In the Rarita-Schwinger setting, we have a similar result as follows.

Theorem 5.9. *([15]) For any fixed $x \in U \subset \mathbb{R}^m$, let $f(x,u)$ be a left monogenic polynomial homogeneous of degree k in u. Then*

$$J_{-1}^{-1}(\phi,y)R_{k,y,\omega}J_1(\phi,y)f(\phi(y),\frac{\widetilde{(cy+d)}\omega(cy+d)}{||cy+d||^2}) = R_{k,x,u}f(x,u),$$

where $x = \phi(y) = (ay+b)(cy+d)^{-1}$ is a Möbius transformation., $u = \dfrac{\widetilde{(cy+d)}\omega(cy+d)}{||cy+d||^2}$, $R_{k,x,u}$ and $R_{k,y,\omega}$ are Rarita-Schwinger operators.

Proof. We use the techniques in [17] to prove this Theorem. Let
$f(x,u),\ g(x,u) \in C^\infty(\Omega', \mathcal{C}l_m)$ and Ω and Ω' are as in Theorem 5.6. We have

$$\begin{aligned}
&\int_{\partial\Omega}(g(x,u), P_k n(x) f(x,u))_u dx^m \\
= &\int_{\phi^{-1}(\partial\Omega)}\big(g(\phi(y),\frac{y\omega y}{||y||^2})P_k J_1(\phi,y)n(y)J_1(\phi,y)f(\phi(y),\frac{y\omega y}{||y||^2})\big)_\omega dy^m \\
= &\int_{\phi^{-1}(\partial\Omega)}\big(g(\phi(y),\frac{y\omega y}{||y||^2})J_1(\phi,y), P_k n(y)J_1(\phi,y)f(\phi(y),\frac{y\omega y}{||y||^2}))_\omega dy^m
\end{aligned}$$

Then we apply the Stokes' Theorem for R_k,

$$\begin{aligned}
&\int_{\phi^{-1}(\Omega)}\big(g(\phi(y),\frac{y\omega y}{||y||^2})J_1(\phi,y)R_k, J_1(\phi,y)f(\phi(y),\frac{y\omega y}{||y||^2})\big)_\omega \\
+ &\big(g(\phi(y),\frac{y\omega y}{||y||^2})J_1(\phi,y), R_k J_1(\phi,y)f(\phi(y),\frac{y\omega y}{||y||^2})\big)_\omega dy^m, \qquad (1)
\end{aligned}$$

where $u = \frac{y\omega y}{||y||^2}$. On the other hand,

$$\begin{aligned}
&\int_{\partial\Omega} (g(x,u), P_k n(x) f(x,u))_u dx^m \\
= &\int_{\Omega} \left[\big(g(x,u)R_k, f(x,u)\big)_u + \big(g(x,u), R_k f(x,u)\big)_u\right] dx^m \\
= &\int_{\phi^{-1}(\Omega)} \left[\big(g(x,u)R_k, f(x,u)\big)_u + \big(g(x,u), R_k f(x,u)\big)_u\right] j(y) dy^m \\
= &\int_{\phi^{-1}(\Omega)} \left[\big(g(x,u)R_k, f(x,u)j(y)\big)_u + \big(g(x,u), J_1(\phi,y)J_{-1}(\phi,y)R_k f(x,u)\big)_u\right] dy^m, \quad (2)
\end{aligned}$$

where $j(y) = J_{-1}(\phi,y)J_1(\phi,y)$ is the Jacobian. Now, we let arbitrary $g(x,u) \in ker R_{k,r}$ and since $J_1(\phi,y)g(\phi(y), \frac{y\omega y}{||y||^2})R_{k,r} = 0$, then from (1) and (2), we get

$$\begin{aligned}
&\int_{\phi^{-1}(\Omega)} \big(g(\phi(y), \frac{y\omega y}{||y||^2})J_1(\phi,y)R_k J_1(\phi,y) f(\phi(y), \frac{y\omega y}{||y||^2})\big)_\omega dy^m \\
= &\int_{\phi^{-1}(\Omega)} \big(g(\phi(y), \frac{y\omega y}{||y||^2}), J_1(\phi,y)J_{-1}(\phi,y)R_k f(x,u)\big)_u dy^m \\
= &\int_{\phi^{-1}(\Omega)} \big(g(\phi(y), \frac{y\omega y}{||y||^2})J_1(\phi,y)J_{-1}(\phi,y)R_k f(x,u)\big)_\omega dy^m
\end{aligned}$$

Since Ω is an arbitrary domain in $\mathbb{R}^m$, we have

$$\begin{aligned}
&\big(g(\phi(y), \frac{y\omega y}{||y||^2})J_1(\phi,y)R_k J_1(\phi,y) f(\phi(y), \frac{y\omega y}{||y||^2})\big)_\omega \\
= &\big(g(\phi(y), \frac{y\omega y}{||y||^2})J_1(\phi,y)J_{-1}(\phi,y)R_k f(x,u)\big)_\omega
\end{aligned}$$

Also, $g(x,u)$ is arbitrary, we get

$$J_1(\phi,y)R_k J_1(\phi,y) f(\phi(y), \frac{y\omega y}{||y||^2}) = J_1(\phi,y)J_{-1}(\phi,y)R_k f(x,u).$$

Theorem 5.9 follows immediately. $\square$

5.3 Rarita-Schwinger type operators

5.3.1 Constructions

In the construction of the Rarita-Schwinger operator above, we notice that the Rarita-Schwinger operator is actually a projection map P_k followed by the Dirac operator D_x, where in the Almansi-Fischer decomposition,

$$\begin{aligned}
&\mathcal{M}_k \xrightarrow{D_x} \mathcal{H}_k \otimes \mathcal{S} = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}\\
&P_k:\ \mathcal{H}_k \otimes \mathcal{S} \longrightarrow \mathcal{M}_k;\\
&I - P_k:\ \mathcal{H}_k \otimes \mathcal{S} \longrightarrow \mathcal{M}_{k-1}.
\end{aligned}$$

If we project to the $u\mathcal{M}_{k-1}$ component after we apply D_x, we get a Rarita-Schwinger type operator from $\mathcal{M}_k$ to $u\mathcal{M}_{k-1}$.

$$\mathcal{M}_k \xrightarrow{D_x} \mathcal{H}_k \otimes \mathcal{S} \xrightarrow{I-P_k} u\mathcal{M}_{k-1}.$$

Similarly, starting with $u\mathcal{M}_{k-1}$, we get another two Rarita-Schwinger type operators.

$$\begin{aligned}
&u\mathcal{M}_{k-1} \xrightarrow{D_x} \mathcal{H}_k \otimes \mathcal{S} \xrightarrow{P_k} \mathcal{M}_k;\\
&u\mathcal{M}_{k-1} \xrightarrow{D_x} \mathcal{H}_k \otimes \mathcal{S} \xrightarrow{I-P_k} u\mathcal{M}_{k-1}.
\end{aligned}$$

In a summary, there are three further Rarita-Schwinger type operators as follows:

$$\begin{aligned}
&T_k^*:\ C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}), \quad T_k^* = (I - P_k)D_x = \frac{uD_u}{m+2k-2}D_x;\\
&T_k:\ C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k), \quad T_k = P_k D_x = (\frac{uD_u}{m+2k-2} + 1)D_x;\\
&Q_k:\ C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}), \quad Q_k = (I - P_k)D_x = \frac{uD_u}{m+2k-2}D_x,
\end{aligned}$$

T_k^* and T_k are also called the *dual-twistor operator* and *twistor operator*. See [6]. We also have

$$T_{k,r}^* : \ C^\infty(\mathbb{R}^m, \overline{\mathcal{M}}_k) \longrightarrow C^\infty(\mathbb{R}^m, \overline{\mathcal{M}}_{k-1}u), \quad T_{k,r}^* = D_x(I - P_{k,r});$$
$$T_{k,r} : \ C^\infty(\mathbb{R}^m, \overline{\mathcal{M}}_{k-1}u) \longrightarrow C^\infty(\mathbb{R}^m, \overline{\mathcal{M}}_k), \quad T_k = D_x P_{k,r};$$
$$Q_{k,r} : \ C^\infty(\mathbb{R}^m, \overline{\mathcal{M}}_{k-1}u) \longrightarrow C^\infty(\mathbb{R}^m, \overline{\mathcal{M}}_{k-1}u), \quad Q_k = D_x(I - P_{k,r}).$$

5.3.2 Conformal Invariance

We cannot prove conformal invariance and intertwining operators of Q_k with the assumption that D_x is conformally invariant. Here we correct this using similar techniques that we used in Section 3 for the Rarita-Schwinger operators.

Following our Iwasawa decomposition we only need to show the conformal invariance of Q_k under inversion. We also need Cauchy's theorem for the Q_k operator.

Theorem 5.10 ([33]). ***(Stokes' theorem for Q_k operator)***

Let Ω' and Ω be domains in $\mathbb{R}^m$ and suppose the closure of Ω lies in Ω'. Further, suppose the closure of Ω is compact and the boundary of Ω, $\partial\Omega$, is piecewise smooth. Then for $f,\ g \in C^1(\Omega', \mathcal{M}_{k-1})$, we have

$$\begin{aligned} & \int_\Omega [(g(x,u)uQ_{k,r}, uf(x,u))_u + (g(x,u)u, Q_k uf(x,u))_u] dx^m \\ = & \int_{\partial\Omega} (g(x,u)u, (I - P_k) d\sigma_x uf(x,u))_u \\ = & \int_{\partial\Omega} (g(x,u)u d\sigma_x (I - P_{k,r}), uf(x,u))_u \end{aligned}$$

where P_k and $P_{k,r}$ are the left and right projections, $d\sigma_x = n(x)d\sigma(x)$, $d\sigma(x)$ is the area element. $(P(u), Q(u))_u = \int_{\mathbb{S}^{m-1}} P(u)Q(u)dS(u)$ is the inner product for any pair of Cl_m-valued polynomials.

When $g(x,u)uQ_{k,r} = Q_k uf(x,u) = 0$, we get Cauchy's theorem for Q_k.

Corollary 5.11 ([33]). ***(Cauchy's theorem for Q_k operator)***

If $Q_k uf(x,u) = 0$ and $ug(x,u)Q_{k,r} = 0$ for $f, g \in C^1(,\Omega', \mathcal{M}_{k-1})$, then

$$\int_{\partial\Omega} (g(x,u)u, (I - P_k)d\sigma_x uf(x,u))_u = 0$$

The conformal invariance of the equation $Q_k uf = 0$ under inversion is as follows.

Theorem 5.12. *For any fixed $x \in U \subset \mathbb{R}^m$, let $f(x,u)$ be a left monogenic polynomial homogeneous of degree $k-1$ in u. If $Q_{k,u}uf(x,u) = 0$, then*
$Q_{k,w}G(y)\frac{ywy}{||y||^2}f(y^{-1}, \frac{ywy}{||y||^2}) = 0$*, where* $G(y) = \frac{y}{||y||^m},\ x = y^{-1},\ u = \frac{ywy}{||y||^2} \in \mathbb{R}^m$.

Proof. First, in Cauchy's theorem, we let $ug(x,u)Q_{k,r} = Q_k uf(x,u) = 0$. Then we have

$$0 = \int_{\partial\Omega}\int_{\mathbb{S}^{m-1}} g(u)u(I - P_k)n(x)uf(x,u)dS(u)d\sigma(x)$$

Letting $x = y^{-1}$, we have

$$= \int_{\partial\Omega^{-1}}\int_{\mathbb{S}^{m-1}} g(u)u(I - P_{k,u})G(y)n(y)G(y)uf(y^{-1},u)dS(u)d\sigma(y),$$

where $G(y) = \frac{y}{||y||^m}$. Set $u = \frac{ywy}{||y||^2}$, since $I - P_{k,u}$ interchanges with $G(y)$ [15], we have

$$\begin{aligned} &= \int_{\partial\Omega^{-1}}\int_{\mathbb{S}^{m-1}} g(\frac{ywy}{||y||^2})\frac{ywy}{||y||^2}G(y)(I - P_{k,w})n(y)G(y)\frac{ywy}{||y||^2}f(y^{-1}, \frac{ywy}{||y||^2})dS(w)d\sigma(y) \\ &= \int_{\partial\Omega^{-1}} \big(g(\frac{ywy}{||y||^2})\frac{ywy}{||y||^2}G(y), (I - P_{k,w})d\sigma_y G(y)\frac{ywy}{||y||^2}f(y^{-1}, \frac{ywy}{||y||^2})\big)_w. \end{aligned}$$

According to Stokes' theorem for Q_k,

$$
\begin{aligned}
= & \int_{\Omega^{-1}} \big(g(\frac{ywy}{||y||^2})\frac{ywy}{||y||^2}G(y), Q_{k,w}G(y)\frac{ywy}{||y||^2}f(y^{-1}, \frac{ywy}{||y||^2})\big)_w \\
& + \int_{\Omega^{-1}} \big(g(\frac{ywy}{||y||^2})\frac{ywy}{||y||^2}G(y)Q_{k,w}, G(y)\frac{ywy}{||y||^2}f(y^{-1}, \frac{ywy}{||y||^2})\big)_w.
\end{aligned}
$$

Since $ug(x,u)$ is arbitrary in the kernel of $Q_{k,r}$ and $uf(x,u)$ is arbitrary in the kernel of Q_k, we get $g(\frac{ywy}{||y||^2})\frac{ywy}{||y||^2}G(y)Q_{k,w} = Q_{k,w}G(y)\frac{ywy}{||y||^2}f(y^{-1}, \frac{ywy}{||y||^2}) = 0$. $\square$

To complete this section, we provide the *Stokes' theorem* for other Rarita-Schwinger type operators as follows:

Theorem 5.13. ***(Stokes' theorem for*** T_k)

Let Ω' and Ω be domains in $\mathbb{R}^m$ and suppose the closure of Ω lies in Ω'. Further, suppose the closure of Ω is compact and $\partial\Omega$ is piecewise smooth. Let $f, g \in C^1(\Omega', \mathcal{M}_k)$. Then

$$
\begin{aligned}
& \int_{\Omega} \big[(g(x,u)T_k, f(x,u))_u + (g(x,u), T_k f(x,u))\big] dx^m \\
= & \int_{\partial\Omega} (g(x,u), P_k d\sigma_x f(x,u))_u \\
= & \int_{\partial\Omega} (g(x,u) d\sigma_x P_{k,r}, f(x,u))_u,
\end{aligned}
$$

where P_k and $P_{k,r}$ are the left and right projections, $d\sigma_x = n(x)d\sigma(x)$ and $(P(u), Q(u))_u = \int_{\mathbb{S}^{m-1}} P(u)Q(u)dS(u)$ is the inner product for any pair of $\mathcal{C}l_m$-valued polynomials.

Theorem 5.14. ***(Stokes' theorem for*** T_k^*)

Let Ω' and Ω be domains in $\mathbb{R}^m$ and suppose the closure of Ω lies in Ω'. Further suppose *the closure of Ω is compact and $\partial\Omega$ is piecewise smooth. Let $f, g \in C^1(\Omega', u\mathcal{M}_{k-1})$. Then*

$$\begin{aligned}
& \int_\Omega \left[(g(x,u)T_k^*, f(x,u))_u + (g(x,u), T_k^* f(x,u))\right] dx^m \\
= & \int_{\partial\Omega} (g(x,u), (I - P_k) d\sigma_x f(x,u))_u \\
= & \int_{\partial\Omega} (g(x,u) d\sigma_x (I - P_{k,r}), f(x,u))_u,
\end{aligned}$$

where P_k and $P_{k,r}$ are the left and right projections, $d\sigma_x = n(x)d\sigma(x)$ and $(P(u), Q(u))_u = \int_{\mathbb{S}^{m-1}} P(u)Q(u)dS(u)$ is the inner product for any pair of $\mathcal{C}l_m$-valued polynomials.

Theorem 5.15. ***(Alternative form of Stokes' Theorem)***

Let Ω and Ω' be as in the previous theorem. Then for $f \in C^1(\mathbb{R}^m, \mathcal{M}_k)$ and $g \in C^1(\mathbb{R}^m, \mathcal{M}_{k-1})$, we have

$$\begin{aligned}
& \int_{\partial\Omega} \big(g(x,u) u d\sigma_x f(x,u)\big)_u \\
= & \int_\Omega \big(g(x,u) u T_k, f(x,u)\big)_u dx^m + \int_\Omega \big(g(x,u)u, T_k^* f(x,u)\big)_u dx^m.
\end{aligned}$$

Further

$$\begin{aligned}
& \int_{\partial\Omega} \big(g(x,u) u d\sigma_x f(x,u)\big)_u = \int_{\partial\Omega} \big(g(x,u)u, (I - P_k) f(x,u)\big)_u \\
= & \int_{\partial\Omega} \big(g(x,u) u d\sigma_x P_k, f(x,u)\big)_u.
\end{aligned}$$

6 Higher order fermionic and bosonic operators

6.1 Motivation

We have mentioned that the Laplace operator (acting on a $\mathcal{C}$-valued field) is related to the Dirac operator (acting on a spinor-valued field) and they are both conformally invariant operators [42]. Moreover, the kth-power of the Dirac operator D_x^k for k a positive integer, is shown also to be conformally invariant in the spinor-valued function theory [42]. However, the Dirac operator D_x and the Laplace operator are not conformally invariant anymore in the higher spin spaces [7, 17], and for the Dirac operator case in the previous section. The first generalization of the Dirac operator to higher spin spaces is instead the so-called Rarita-Schwinger operator [6, 15], and the generalization of the Laplace operator to higher spin spaces is the so-called higher spin Laplace or Maxwell operator given in [7, 17].

Let us look deeper into this lack of conformal invariance of the Dirac operator D_x. Given a function $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_j)$, we apply inversion $x = y^{-1}$ to it. There is also a reflection of u in the direction y given by $\dfrac{yuy}{||y||^2}$; this reflection involves y, which changes the conformal invariance of D_x. This explanation also applies for the Laplace operator Δ_x in the higher spin spaces. The explanation we just mentioned further implies that the kth-power of the Dirac operator D_x^k is not conformally invariant in the higher spin spaces. In this section, we will provide the generalization of D_x^k when it acts on $C^\infty(\mathbb{R}^m, U)$, where $U = \mathcal{H}_1$ or $U = \mathcal{M}_1$, depending on the order. More generally, we provide nomenclature for higher order operators in higher spin theory.

6.2 Construction and conformal invariance

By arguments of Slovák [46], for integers $j \geq 0$ and $k > 0$ there exist conformally invariant differential operators in the higher spin setting

$$\mathcal{D}_{j,k}: \ C^\infty(\mathbb{R}^m, U) \longrightarrow C^\infty(\mathbb{R}^m, U),$$

where $U = \mathcal{H}_j$ if k is even and $U = \mathcal{M}_j$ if k is odd. As a Spin representation $\mathcal{H}_j$ is associated with integer spin j and particles of integer spin are called bosons, so the operators $\mathcal{D}_{j,k}: \ C^\infty(\mathbb{R}^m, \mathcal{H}_j) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ are named *bosonic operators*. Thus in the spin 0 case we have the Laplace operator and its k-powers, the spin 1 case the Maxwell operator and its generalization to order $k = 2n$, and general higher spin Laplace operators and their generalization to order $k = 2n$. Correspondingly, as a Spin representation $\mathcal{M}_j$ is associated with half-integer spin $j + \frac{1}{2}$ and particles of half-integer spin are called fermions, so the operators $\mathcal{D}_{j,k}: C^\infty(\mathbb{R}^m, \mathcal{M}_j) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_j)$ are named *fermionic operators.* Thus in the spin $\frac{1}{2}$ case we have the Dirac operator and its $k = 2n+1$ powers, the spin $\frac{3}{2}$ case the simplest Rarita-Schwinger operator and its generalization to order $k = 2n+1$, and general Rarita-Schwinger operators and their generalization to order $k = 2n+1$. Note our notation indexes according to degree of homogeneity of the target space j and differential order k, so fractions are not used in the notation; if we indexed according to spin, fractional spins would need to be used for odd order operators.

We will consider the higher order spin 1 and spin $\frac{3}{2}$ operators $\mathcal{D}_{1,k}: C^\infty(\mathbb{R}^m, U) \longrightarrow C^\infty(\mathbb{R}^m, U)$, where $U = \mathcal{H}_1$ for k even and $U = \mathcal{M}_1$ for k odd. Note that the target space U here is a function space. That means any element in $C^\infty(\mathbb{R}^m, U)$ is of the form $f(x, u)$ with $f(x, u) \in U$ for each fixed $x \in \mathbb{R}^m$ and x is the variable which $\mathcal{D}_{1,k}$ acts on. The construction and conformal invariance of these two operators are considered as follows.

k even, $k = 2n$, $n > 1$ (The bosonic case)

Theorem 6.1. *For positive integer n, the unique $2n$-th order conformally invariant differential operator of spin-1 $\mathcal{D}_{1,2n}: C^\infty(\mathbb{R}^m, \mathcal{H}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_1)$ has the following form, up to a multiplicative constant:*

$$\mathcal{D}_{1,2n} = \Delta_x^n - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}.$$

For the case $n = 1$, we retrieve the Maxwell operator from [17].

According to the Iwasawa decomposition of Möbius transformations, to show this operator is conformally invariant we only must show it is invariant under translation, dilation, reflection, and inversion. From the expression of $\mathcal{D}_{1,2n}$, it is obvious that it is invariant under translation and dilation. This operator consists of an inner product operator and a power of Laplace operator, which are both invariant under reflection. So $\mathcal{D}_{1,2n}$ is invariant under reflection. Hence to prove conformal invariance, we only show invariance under inversion. To do so, we need the concept of harmonic inversion ([2]), which is an involution mapping solutions for $\mathcal{D}_{1,2n}$ to solutions for $\mathcal{D}_{1,2n}$.

Definition 6.1. The harmonic inversion is a conformal transformation defined as

$$\mathcal{J}_{2n} : C^\infty(\mathbb{R}^m, \mathcal{H}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_1) : f(y, v) \mapsto \mathcal{J}_{2n}[f](y, v) := ||x||^{2n-m} f(\frac{x}{||x||^2}, \frac{xux}{||x||^2}),$$

with $y = -x^{-1}$ and $v = \dfrac{xux}{||x||^2}$.

Note that this inversion consists of the classical Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u \omega$ acting on the dummy variable u (where $x = ||x||\omega$), and satisfies $\mathcal{J}_{2n}^2 = 1$.

Then we have the following lemma.

Lemma 6.2. *The special conformal transformation is given as follows.*

$$\mathcal{C}_{2n} := \mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n} = 2\langle u, x\rangle\partial_{u_j} - 2u_j\langle x, D_u\rangle + ||x||^2\partial_{x_j} - x_j(2\mathbb{E}_x + m - 2n).$$

Proof. As similar calculation as in *Proposition A.1* in [7] will show the conclusion. □

Further we need the concept of generalized symmetry (see [7, 17]):

Definition 6.2. An operator η_1 is a generalized symmetry for a differential operator $\mathcal{D}$ if and only if there exists another operator η_2 such that $\mathcal{D}\eta_1 = \eta_2\mathcal{D}$. Note that for $\eta_1 = \eta_2$, this reduces to a definition of a symmetry in the sense that $\mathcal{D}\eta_1 = \eta_1\mathcal{D}$.

Proposition 6.3. *The special conformal transformations $\mathcal{C}_{2n}$, with $j \in \{1, 2, \dots, m\}$ are generalized symmetries of $\mathcal{D}_{1,2n}$. More specifically,*

$$[\mathcal{D}_{1,2n}, \mathcal{C}_{2n}] = -4nx_j\mathcal{D}_{1,2n}.$$

Note: In particular, this shows that $\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = ||x||^{4n}\mathcal{D}_{1,2n}$, which is the generalization of the case of the classical higher order Laplace operator Δ_x^n. This also implies $\mathcal{D}_{1,2n}$ is invariant under inversion.

If one can show $\mathcal{C}_{2n}$ is a generalized symmetry of $\mathcal{D}_{1,2n}$, then applying $\mathcal{J}_{2n}$ will give the intertwining operator for $\mathcal{D}_{1,2n}$ under inversion. This reveals that $\mathcal{D}_{1,2n}$ is invariant under harmonic inversion. See more details in the proof of the following proposition. It states that the special conformal transformations induce generalized symmetries of the operator $\mathcal{D}_{1,2n}$. For the case n=1, similar results to the following proposition and Lemmas 2 and 3 can be found in [7, 17].

First, let us prove following technical lemmas. It is worth pointing out that since we are dealing with degree-1 homogeneous polynomials in u, terms involving second derivative with respect to u disappear.

Lemma 6.4. *For all $1 \le j \le m$, we have*

$$[\Delta_x^n, \mathcal{C}_{2n}] = -4nx_j\Delta_x^n + 4n\langle u, D_x\rangle\partial_{u_j}\Delta_x^{n-1} - 4nu_j\langle D_u, D_x\rangle\Delta_x^{n-1}.$$

Proof. We prove this by induction. First, we have ([7])

$$[\Delta_x, \mathcal{C}_2] = -4x_j\Delta_x + 4\langle u, D_x\rangle\partial_{u_j} - 4u_j\langle D_u, D_x\rangle.$$

Assuming the lemma is true for Δ^{n-1}, applying the fact that for general operators A, B

and C, we have $[AB, C] = A[B, C] + [A, C]B$ and $\mathcal{C}_{2n} = \mathcal{C}_2 + (2n-2)x_j$, we have

$$[\Delta_x^n, \mathcal{C}_{2n}] = \Delta_x^{n-1}[\Delta_x, \mathcal{C}_{2n}] + [\Delta_x^{n-1}, \mathcal{C}_{2n}]\Delta_x.$$

Since $\mathcal{C}_{2n} = \mathcal{C}_{2n-2} + 2x_j$, a straightforward calculation leads to the conclusion. $\square$

Lemma 6.5. *For all* $1 \leq j \leq m$, *we have*

$$\begin{aligned} & [\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}, \mathcal{C}_{2n}] \\ = \; & -4nx_j\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1} + (m+2n-2)\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big)\Delta_x^{n-1}. \end{aligned}$$

Proof. First, we have [7]:

$$\begin{aligned} & [\langle u, D_x\rangle\langle D_u, D_x\rangle, \mathcal{C}_2] \\ = \; & 2||u||^2\partial_{u_j}\langle D_u, D_x\rangle - 4x_j\langle u, D_x\rangle\langle D_u, D_x\rangle + \big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big)(2\mathbb{E}_u + m - 2) \\ = \; & -4x_j\langle u, D_x\rangle\langle D_u, D_x\rangle + m\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big), \end{aligned}$$

Then

$$\begin{aligned} & [\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}, \mathcal{C}_{2n}] \\ = \; & \langle u, D_x\rangle\langle D_u, D_x\rangle[\Delta_x^{n-1}, \mathcal{C}_{2n}] + [\langle u, D_x\rangle\langle D_u, D_x\rangle, \mathcal{C}_{2n}]\Delta_x^{n-1} \end{aligned}$$

together with the previous lemma proves the conclusion. $\square$

With the help of *Lemma* 6.4 and 6.5, a straightforward calculation shows that

$$[\mathcal{D}_{1,2n}, \mathcal{C}_{2n}] = -4nx_j\mathcal{D}_{1,2n}.$$

We rewrite the previous equation as

$$\mathcal{D}_{1,2n}\mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n} - \mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n}\mathcal{D}_{1,2n} = -4nx_j\mathcal{D}_{1,2n}.$$

Then we apply $\mathcal{J}_{2n}$ to both sides with the fact that $\mathcal{J}_{2n}^2 = 1$ to write

$$\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n}\partial_{x_j} - \partial_{x_j}\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = -4n\frac{x_j}{||x||^2}\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n}.$$

In Section 4, we will provide the fundamental solution of this conformally invariant operator. Since the proof there does not rely on the specific expression of the operator, the fundamental solution is unique (up to a multiplicative constant). Therefore, this conformally invariant operator is also unique (up to a multiplicative constant). This suggests that $\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = ||x||^{4n}\mathcal{D}_{1,2n}$ is our only option. This can also be rewritten as

$$\mathcal{D}_{1,2n,y,w}||x||^{2n-m}f(y,w) = ||x||^{-m-2n}\mathcal{D}_{1,2n,x,u}f(x,u),\ \forall f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{H}_1),$$

where $y = x^{-1}$ and $w = \dfrac{xux}{||x||^2}$. Therefore, we have proven $\mathcal{D}_{1,2n}$ is invariant under inversion and, more generally, is conformally invariant.

k odd, $k = 2n-1$, $n > 1$ (The fermionic case)

Theorem 6.6. *For positive integer n, the unique $(2n-1)$-th order conformally invariant differential operator of spin-$\frac{3}{2}$ $\mathcal{D}_{1,2n-1} : C^\infty(\mathbb{R}^m, \mathcal{M}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_1)$ has the following form, up to a multiplicative constant:*

$$\mathcal{D}_{1,2n-1} = D_x\Delta_x^{n-1} - \frac{2}{m+2n-2}u\langle D_u, D_x\rangle\Delta_x^{n-1} - \frac{4n-4}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x.$$

When $n = 1$, we have the Rarita-Schwinger operator appearing in [6, 15] and elsewhere. The same strategy in the even case applies: we only must show invariance under inversion. We have the definition for monogenic inversion as follows.

Definition 6.3. The monogenic inversion is a conformal transformation defined as

$$\begin{aligned} \mathcal{J}_{2n+1} \ &: \ C^{\infty}(\mathbb{R}^m, \mathcal{M}_1) \longrightarrow C^{\infty}(\mathbb{R}^m, \mathcal{M}_1); \\ f(y,v) \mapsto \mathcal{J}_{2n+1}[f](y,v) &:= \frac{x}{||x||^{m-2n}} f\left(\frac{x}{||x||^2}, \frac{xux}{||x||^2}\right), \end{aligned}$$

with $y = -x^{-1}$ and $v = \dfrac{xux}{||x||^2}$.

Note that this inversion also consists of the classical Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u \omega$ acting on the dummy variable u (where $x = ||x||\omega$), but it satisfies $\mathcal{J}_{2n+1}^2 = -1$ instead.

Similarly, the monogenic inversion is an involution mapping solutions for $\mathcal{D}_{1,2n-1}$ to solutions for $\mathcal{D}_{1,2n-1}$ ([39]). Then we have the following lemma:

Lemma 6.7. *The special conformal transformation is given as follows.*

$$\begin{aligned} \mathcal{C}_{2n-1} &:= \mathcal{J}_{2n-1}\partial_{x_j}\mathcal{J}_{2n-1} \\ &= -e_j x - 2\langle u, x\rangle\partial_{u_j} + 2u_j\langle x, D_u\rangle - ||x||^2\partial_{x_j} + x_j(2\mathbb{E}_x + m - 2n), \end{aligned}$$

$\mathcal{C}_{2n-1} = \mathcal{C}_{2n-3} - 2x_j = -\mathcal{C}_{2n-2} - e_j x - 2x_j.$

Proof. A similar calculation as in *Proposition A.1* in [7] will show the conclusion. □

Then we arrive at the main proposition, stating that the special conformal transformations are generalized symmetries of operator $\mathcal{D}_{1,2n-1}$.

Proposition 6.8. *The special conformal transformations $\mathcal{C}_{2n-1}$, with $j \in \{1, 2, \ldots, m\}$ are generalized symmetries of $\mathcal{D}_{1,2n-1}$. More specifically,*

$$[\mathcal{D}_{1,2n-1}, \mathcal{C}_{2n-1}] = (4n-2)x_j\mathcal{D}_{1,2n-1}.$$

Note: In particular, this shows that $\mathcal{J}_{2n-1}\mathcal{D}_{1,2n-1}\mathcal{J}_{2n-1} = ||x||^{4n-2}\mathcal{D}_{1,2n-1}$, which is the generalization of the case of the classical higher order Dirac operator D_x^{2n-1}. This also implies $\mathcal{D}_{1,2n-1}$ is invariant under inversion.

To prove this proposition, we need the following technical lemmas as in the even case:

Lemma 6.9. *For all* $1 \leq j \leq m$, *we have*

$$\begin{aligned} & [D_x\Delta_x^{n-1}, \mathcal{C}_{2n-1}] \\ = \ & (4n-2)x_jD_x\Delta_x^{n-1} + (4n-4)\big(u_j\langle D_u, D_x\rangle - \langle u, D_x\rangle\partial_{u_j}\big)D_x\Delta_x^{n-2} - 2u\partial_{u_j}\Delta_x^{n-1}. \end{aligned}$$

Lemma 6.10. *For all* $1 \leq j \leq m$, *we have*

$$\begin{aligned} & [u\langle D_u, D_x\rangle\Delta_x^{n-1}, \mathcal{C}_{2n-1}] \\ = \ & (4n-2)x_ju\langle D_u, D_x\rangle\Delta_x^{n-1} - (m+2n-2)u\partial_{u_j}\Delta_x^{n-1} - (2n-2)ue_j\langle D_u, D_x\rangle\Delta_x^{n-2}. \end{aligned}$$

Lemma 6.11. *For all* $1 \leq j \leq m$, *we have*

$$\begin{aligned} & [\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x, \mathcal{C}_{2n-1}] = (4n-2)x_j\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x \\ & -(m+2n-2)\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big)\Delta_x^{n-2}D_x + ue_j\langle D_u, D_x\rangle\Delta_x^{n-2}D_x. \end{aligned}$$

We combine *Lemma* 6.9, 6.10 and 6.11 to get

$$[\mathcal{D}_{1,2n-1}, \mathcal{C}_{2n-1}] = (4n-2)x_j\mathcal{D}_{1,2n-1}.$$

This implies $\mathcal{J}_{2n-1}\mathcal{D}_{1,2n-1}\mathcal{J}_{2n-1} = ||x||^{4n-2}\mathcal{D}_{1,2n-1}$ and $\mathcal{D}_{1,2n-1}$ is invariant under inversion according to an explanation similar to that for the even case.

Let $\mathcal{D}_{1,k,x,u}$ and $\mathcal{D}_{1,k,y,w}$ be the higher order higher spin operators with respect to $x,\ u$

and $y,\ w$, respectively and $y = \phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation. Let

$$J_k = \frac{\widetilde{cx+d}}{||cx+d||^{m-2n}}, \quad for\ k = 2n+1;$$
$$J_k = \frac{1}{||cx+d||^{m-2n}}, \quad for\ k = 2n;$$
$$J_{-k} = \frac{\widetilde{cx+d}}{||cx+d||^{m+2n+2}}, \quad for\ k = 2n+1;$$
$$J_{-k} = \frac{1}{||cx+d||^{m+2n}}, \quad for\ k = 2n,$$

with $n = 1, 2, 3, \cdots$. See [36].

Then we claim that

Theorem 6.12. *Let $y = \phi(x) = (ax+b)(cx+d)^{-1}$ be a Möbius transformation, then*

$$J_{-k}\mathcal{D}_{1,k,y,w}f(y,w) = \mathcal{D}_{1,k,x,u}J_k f(\phi(x), \frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $w = \dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$.

We only prove the bosonic (order $k = 2n$) case, as the fermionic (order $k = 2n+1$) case can be done similarly. As we observed earlier, according to the Iwasawa decomposition, we only prove this with respect to orthogonal transformation and inversion, since translation and dilation cases are trivial.

Orthogonal transformations $a \in Pin(m)$

Lemma 6.13. *If $x = ay\tilde{a}$, $u = aw\tilde{a}$, then $\mathcal{D}_{1,2n,x,u}f(x,u) = a\mathcal{D}_{1,2n,y,w}\tilde{a}f(y,w)$.*

Proof.

$$\mathcal{D}_{1,2n,x,u}f(x,u) = \left(\triangle_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\right)\Delta_x^{n-1}f(x,u)$$
$$= \left(a\triangle_y\tilde{a} - \frac{4n}{m+2n-2}a\langle w, D_y\rangle\tilde{a}a\langle D_w, D_y\rangle\tilde{a}\right)a\Delta_y^{n-1}\tilde{a}f(y,w)$$
$$= a\left(\triangle_y - \frac{4n}{m+2n-2}\langle w, D_y\rangle\langle D_w, D_y\rangle\right)\Delta_y^{n-1}\tilde{a}f(y,w)$$
$$= a\mathcal{D}_{1,2n,y,w}\tilde{a}f(y,w).$$

□

Inversions

Lemma 6.14. *Let $x = y^{-1}$ and $u = \dfrac{ywy}{||y||^2}$, then*

$$\mathcal{D}_{1,2n,y,w}||x||^{m-2n}f(y,w) = ||x||^{m+2n}\mathcal{D}_{1,2n,x,u}f(x,u).$$

Proof. Recall that $[\mathcal{D}_{1,2n}, \mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n}] = -4nx_j\mathcal{D}_{1,2n}$, where $\mathcal{J}_{2n}$ is the harmonic inversion. This implies that $\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = ||x||^{4n}\mathcal{D}_{1,2n}$. It can also be written as

$$\mathcal{D}_{1,2n,y,w}||x||^{m-2n}f(y,w) = ||x||^{m+2n}\mathcal{D}_{1,2n,x,u}f(x,u).$$

□

Theorem 3 now follows using the Iwasawa decomposition. See [11] for the first order case.

6.3 Fundamental solutions of $\mathcal{D}_{1,k}$

To get the fundamental solutions of either $\mathcal{D}_{1,k}$, we use the technique used in [6].

k even, $k = 2n$ (The bosonic case)

The identity of $End(\mathcal{H}_1)$ can be represented by the reproducing kernel $Z_1(u,v)$ for the zonal spherical harmonics of degree 1. The zonal spherical harmonics satisfy

$$P_1(v) = (Z_1(u,v), P_1(u))_u = \int_{S^{m-1}} \overline{Z_1(u,v)} P_1(u) dS(u),$$

where $(\ ,\ )_u$ denotes the Fischer inner product with respect to u. A homogeneous $End(\mathcal{H}_1)$-valued C^∞-function $x \to E(x)$ on $\mathbb{R}^m \backslash \{0\}$ satisfying $\mathcal{D}_{1,2n} E(x) = \delta(x) Z_1(u,v)$ will be referred to as a fundamental solution for the operator $\mathcal{D}_{1,2n}$. We will show that such a fundamental solution has the form $E_{1,2n}(x,u,v) = c_1 ||x||^{2n-m} Z_1(\frac{xux}{||x||^2}, v)$. Since $Z_1(u,v)$ is a trivial solution of $\mathcal{D}_{1,2n}$, then according to the invariance of $\mathcal{D}_{1,2n}$ under inversion, we obtain a non-trivial solution $\mathcal{D}_{1,2n} E_{1,2n}(x,u,v) = 0$ in $\mathbb{R}^m \backslash \{0\}$. Clearly this function is homogeneous of degree $2n - m$ in x and belongs to $L_1^{loc}(\mathbb{R}^m)$. Because $\delta(x)$ is the only (up to a multiple) distribution homogeneous of degree $-m$ having its support at the origin, we have in the sense of distributions:

$$\mathcal{D}_{1,2n} E_{1,2n}(x,u,v) = \delta(x) P_1(u,v)$$

for some $P_1(u,v) \in \mathcal{H}_1 \otimes \mathcal{H}_1^*$. Thus for all $Q_1 \in \mathcal{H}_1$,

$$\begin{aligned}
&\int_{\mathbb{S}^{m-1}} \mathcal{D}_{1,2n}\overline{E_{1,2n}(x,u,v)}Q_1(v)dS(v) \\
= \;& \mathcal{D}_{1,2n}\int_{\mathbb{S}^{m-1}} c_1||x||^{2n-m}\overline{Z_1(\frac{xux}{||x||^2},v)}Q_1(v)dS(v) \\
= \;& \mathcal{D}_{1,2n}\int_{\mathbb{S}^{m-1}} c_1||x||^{2n-m}\overline{Z_1(\frac{xux}{||x||^2},\frac{xv'x}{||x||^2})}Q_1(\frac{xv'x}{||x||^2})dS(v') \\
= \;& \mathcal{D}_{1,2n}\int_{\mathbb{S}^{m-1}} c_1\overline{Z_1(u,v')}||x||^{2n-m}Q_1(\frac{xv'x}{||x||^2})dS(v') \\
= \;& c_1\mathcal{D}_{1,2n}||x||^{2n-m}Q_1(\frac{xux}{||x||^2}) \\
= \;& \delta(x)\int_{\mathbb{S}^{m-1}} \overline{P_1(u,v)}Q_1(v)dS(v).
\end{aligned}$$

As the reproducing kernel $Z_1(u,v)$ is invariant under the $Spin(m)$-representation $H:\ f(u,v) \mapsto sf(su\tilde{s}, sv\tilde{s})\tilde{s}$, the kernel $E_{1,2n}(x,u,v)$ is also $Spin(m)$-invariant:

$$sE_{1,2n}(sx\tilde{s}, su\tilde{s}, sv\tilde{s})\tilde{s} = E_{1,2n}(x,u,v).$$

From this it follows that $P_1(u,v)$ must be also invariant under H. Let now ϕ be a test function with $\phi(0) = 1$. Let L be the action of $Spin(m)$ given by $L: f(u) \mapsto sf(su\tilde{s})\tilde{s}$. Then

$$\begin{aligned}
&\langle \mathcal{D}_{1,2n}\big(c_1||x||^{2n-m}L(\frac{x}{||x||})L(s)Q_1(u)\big), \phi(x)\rangle \\
= \;& \int_{\mathbb{S}^{m-1}} \overline{P_1(u,v)}L(s)Q_1(v)dS(v) \\
= \;& L(s)\int_{\mathbb{S}^{m-1}} \overline{P_1(u,v)}Q_1(v)dS(v) \\
= \;& \langle L(s)\big(\mathcal{D}_{1,2n}c_1||x||^{2n-m}L(\frac{x}{||x||})Q_1(u)\big), \phi(x)\rangle.
\end{aligned}$$

In this way we have constructed an element of $End(\mathcal{H}_1)$ commuting with the L-representation of $Spin(m)$ that is irreducible; see Section 2.2.2. By Schur's Lemma

([20]), it follows that $P_1(u,v)$ must be the reproducing kernel $Z_1(u,v)$ if we choose c_1 properly. Hence

$$\mathcal{D}_{1,2n}E_{1,2n}(x,u,v) = \delta(x)Z_1(u,v).$$

k odd, $k = 2n-1$ (The fermionic case)

Let $Z_1(u,v)$ be the zonal spherical monogenic polynomial, which is the reproducing kernel of $\mathcal{M}_1$. With similar arguments and the fact that $Z_1(u,v)$ is also $Spin(m)$-invariant under the same $Spin(m)$-action as in the even case, one can show that

$$E_{1,2n-1}(x,u,v) = c_1' \frac{x}{||x||^{m-2n+2}} Z_1(\frac{xux}{||x||^2}, v)$$

is the fundamental solutions of $\mathcal{D}_{1,2n-1}$, where c_1' is a non-zero, real constant.

Since $E_{1,k}(x,u,v)$ is the fundamental solution of $\mathcal{D}_{1,k}$, we have

$$\int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} E_{1,k}(x-y,u,v)\mathcal{D}_{1,k}\psi(x,u)dS(u)dx^m = \psi(y,v),$$

where $\psi(x,u) \in C^\infty(\mathbb{R}^m, U)$ with compact support in x for each $u \in \mathbb{R}^m$, $U = \mathcal{M}_1$ when k is odd and $U = \mathcal{H}_1$ when k is even. Hence, we have $\mathcal{D}_{1,k}E_{1,k} = Id$ and $E_{1,k} = \mathcal{D}_{1,k}^{-1}$ in the above sense. On the other hand,

$$J_{-k}\mathcal{D}_{1,k,y,w}\psi(y,w) = \mathcal{D}_{1,k,x,u}J_k\psi(\phi(x), \frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $y = \phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and $w = \dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$ as in Theorem 3. We get

$$J_k^{-1}\mathcal{D}_{1,k,x,u}^{-1}J_{-k} = \mathcal{D}_{1,k,y,w}^{-1}.$$

Alternatively,

$$J_k^{-1} E_{1,k,x,u} J_{-k} = E_{1,k,y,w}.$$

This gives us the intertwiners of the fundamental solution $E_{1,k}$ under Möbius transformations, which also reveals that the fundamental solutions are conformally invariant under Möbius transformations.

6.4 Ellipticity of the operator $\mathcal{D}_{1,k}$

Notice that the bases of the target space $\mathcal{H}_1$ and $\mathcal{M}_1$ have simple expressions. We can use techniques similar to those in [7, 17] to show that the operators $\mathcal{D}_{1,k}$ are elliptic. First, we introduce the definition for an elliptic operator.

Definition 6.4. A linear homogeneous differential operator of k-th order $\mathcal{D}_{1,k} : C^\infty(\mathbb{R}^m, V_1) \longrightarrow C^\infty(\mathbb{R}^m, V_2)$ is elliptic if for every non-zero vector $x \in \mathbb{R}^m$ its principal symbol, the linear map $\sigma_x(\mathcal{D}_{1,k}) : V_1 \longrightarrow V_2$ obtained by replacing its partial derivatives ∂_{x_j} with the corresponding variables x_j, is a linear isomorphism.

Then we prove ellipticity of $\mathcal{D}_{1,k}$ in the even and odd cases individually.

k even, $k = 2n$ (The bosonic case)

Theorem 6.15. *The operator* $\mathcal{D}_{1,2n} := \left(\triangle_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\right)\Delta_x^{n-1}$ *is an elliptic operator.*

Proof. In [17] it was shown that the operator $\triangle_x - \frac{4}{m}\langle u, D_x\rangle\langle D_u, D_x\rangle$ is elliptic. In our case, the term in the parentheses is the same as the previous one up to a constant coefficient, so a similar argument shows $\triangle_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle$ is elliptic. Since the symbol of Δ_x^{n-1} is non-negative, $\left(\triangle_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\right)\Delta_x^{n-1}$ is elliptic. □

k odd, $k = 2n-1$ (The fermionic case)

Theorem 6.16. *The operator*

$$\mathcal{D}_{1,2n-1} := D_x\Delta_x^{n-1} - \frac{2}{m+2n-2}u\langle D_u, D_x\rangle\Delta_x^{n-1} - \frac{4n-4}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x$$

is an elliptic operator.

Proof. To prove the theorem, we will show that for fixed $x \in \mathbb{R}^m$ the symbol of the operator $\mathcal{D}_{1,2n-1}$, which is given by

$$x||x||^{2n-2} - \frac{2}{m+2n-2}u\langle D_u, x\rangle||x||^{2n-2} - \frac{4n-4}{m+2n-2}\langle u, x\rangle\langle D_u, x\rangle||x||^{2n-4}x,$$

is a linear isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_1$. As the symbol is clearly a linear map, it remains to be proven that the map is injective. An arbitrary element of $\mathcal{M}_1$ can be written as $\sum_{j=1}^m \alpha_j(e_ju_m + e_mu_j)$ with $\alpha_j \in \mathbb{C}$ for all $1 \le j \le m$. We have to show that the following system of equations has an unique solution:

$$\big(x||x||^2 - \frac{2u\langle D_u, x\rangle||x||^2}{m+2n-2} - \frac{4n-4}{m+2n-2}x\langle u, x\rangle\langle D_u, x\rangle\big)\big(\sum_{j=1}^m \alpha_j(e_ju_m + e_mu_j)\big) = 0.$$

With $c_1 = \frac{2}{m+2n-2}$, $c_2 = \frac{4n-4}{m+2n-2}$, $a_i = (c_1e_i||x||^2 + c_2xx_i)$, $b_j = x_me_j + x_je_m$, and $1 \le i, j \le m-1$, this equation system can be written in matrix notation as follows:

$$\begin{bmatrix} -x||x||^2e_m - a_1b_1 & -a_1b_2 & \dots & -a_1b_{m-1} \\ -a_2b_1 & -x||x||^2e_m - a_2b_2 & \dots & -a_2b_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m-1}b_1 & -a_{m-1}b_2 & \dots & -x||x||^2e_m - a_{m-1}b_{m-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} = 0.$$

In order to show that this system has an unique solution, it suffices to prove that

$$\begin{vmatrix} -x||x||^2e_m - a_1b_1 & -a_1b_2 & \dots & -a_1b_{m-1} \\ -a_2b_1 & -x||x||^2e_m - a_2b_2 & \dots & -a_2b_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m-1}b_1 & -a_{m-1}b_2 & \dots & -x||x||^2e_m - a_{m-1}b_{m-1} \end{vmatrix} \neq 0.$$

Using the notation $\vec{a} = (a_1, a_2, \dots, a_{m-1})^T$ and $\vec{b} = (b_1, b_2, \dots, b_{m-1})^T$, the determinant can be written more compactly as

$$P(x) = det\big(- x||x||^2e_m\mathbb{I}_{m-1} - \vec{a}\cdot\vec{b^T}\big) \neq 0.$$

As a function,

$$\begin{aligned} P(x) &= det\big(- x||x||^2e_m\mathbb{I}_{m-1} - \vec{a}\cdot\vec{b^T}\big) = det\big(- x||x||^2e_m\mathbb{I}_{m-1} - \vec{a}\cdot\vec{b^T}\big)^T \\ &= -x||x||^2e_m - \vec{a}^T\cdot\vec{b} = -x||x||^2e_m - \sum_{j=1}^{m-1}(c_1e_j||x||^2 + c_2xx_j)(x_me_j + x_je_m) \\ &= -x||x||^2e_m + (c_1+c_2)||x||^2x_m - (c_1+c_2)x||x||^2e_m \end{aligned}$$

Checking each e_j-th component with $1 \leq j \leq m$, it is easy to see $P(x)$ is non-zero if x is non-zero. This completes the proof. □

7 Higher order fermionic and bosonic operators on cylinders and Hopf manifolds

Conformally flat manifolds are manifolds with atlases whose transition functions are Möbius transformations. They can be constructed by factoring out a subdomain U of either the sphere $\mathbb{S}^m$ or $\mathbb{R}^m$ by a Kleinian group Γ of the Möbius group, where Γ acts strongly discontinuously on U and Γ is not cyclic. This gives rise to a conformally flat manifold U/Γ. Cylinders are examples of conformally flat manifolds of type $\mathbb{R}^m/\mathbb{Z}^l$ where $\mathbb{Z}^l$ is an integer lattice and $1 \le l \le m$.

In this section, we will follow the strategy in [30, 31] to define kth order higher spin operators on cylinders and Hopf manifolds, where k is a positive integer. We also construct fundamental solutions of these operators by applying translation groups or dilation groups.

7.1 The higher order higher spin operator on cylinders

For an integer l, $1 \le l \le m$, we define the l-cylinder C_l to be the m-dimensional manifold $\mathbb{R}^m/\mathbb{Z}^l$, where $\mathbb{Z}^l$ denote the l-dimensional lattice defined by $\mathbb{Z}^l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$. We denote its members $m_1e_1 + \cdots + m_le_l$ for each $m_1, \cdots, m_l \in \mathbb{Z}$ by a bold letter $\mathbf{m}$. When $l = m$, C_l is the m-torus, T_m. For each l the space $\mathbb{R}^m$ is the universal covering group of the cylinder C_l. Hence there is a projection map $\pi_l : \ \mathbb{R}^m \longrightarrow C_l$.

An open subset U of the space $\mathbb{R}^m$ is called l-fold periodic if for each $x \in U$ the point $x + \mathbf{m} \in U$. So $U' := \pi_l(U)$ is an open subset of the l-cylinder C_l.

Suppose that $U \in \mathbb{R}^m$ is a l-fold periodic open set. Let $f(x, u)$ be a function defined on $U \times \mathbb{R}^m$ with values in $\mathcal{C}l_m$. Then we say that $f(x, u)$ is a l-fold periodic function if for each $x \in U$ we have that $f(x, u) = f(x + \mathbf{m}, u)$. Moreover, we will assume throughout that f is a monogenic polynomial homogeneous of degree j in u.

Now, if $f : \ U \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$ is an l-fold periodic function, then the projection π_l induces a well defined function

$$f' : \ U' \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m,$$

where $f'(x', u) = f(x, u)$ for each $x' \in U'$ and x an arbitrary representative of $\pi_l^{-1}(x')$. Moreover, any function $f' : \ U' \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$ lifts to an l-fold periodic function $f : \ U \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$, where $U = \pi_l^{-1}(U')$.

The projection map π_l induces a projection of the higher order higher spin operator

$\mathcal{D}_{1,k}$ to an operator $\mathcal{D}_{1,k}^{C_l}$ acting on domains on $C_l \times \mathbb{R}^m$ which is defined by replacing D_x and Δ_x with D'_x and Δ'_x in $\mathcal{D}_{1,k}$, respectively, where D'_x is the projection of the Dirac operator D_x and Δ'_x is the projection of the Laplace operator Δ_x. Essentially this is the method in [30]. That is

$$\begin{aligned}
\mathcal{D}_{1,2n}^{C_l} &= \Delta_x'^n - \frac{4n}{m+2n-2}\langle u, D'_x\rangle\langle D_u, D'_x\rangle\Delta_x'^{n-1};\\
\mathcal{D}_{1,2n-1}^{C_l} &= D'_x\Delta_x'^{n-1} - \frac{2}{m+2n-2}u\langle D_u, D'_x\rangle\Delta_x'^{n-1} - \frac{4n-4}{m+2n-2}\langle u, D'_x\rangle\langle D_u, D'_x\rangle\Delta_x'^{n-2}D'_x.
\end{aligned}$$

We call the operator $\mathcal{D}_{1,2n}^{C_l}$ an l-cylindrical higher order bosonic operator of spin 1 and $\mathcal{D}_{1,2n-1}^{C_l}$ an l-cylindrical higher order fermionic operator of spin $\frac{3}{2}$.

Fundamental solutions of $\mathcal{D}_{1,k}^{C_l}$

We follow the techniques in [30] and [11], requiring that the order of the operator $k < m$ when the dimension m is even. Let U be a domain in $\mathbb{R}^m$. We recall the fundamental solution of the higher order higher spin operators $\mathcal{D}_{1,k}$ in $\mathbb{R}^m$:

$$E_{1,k}(x,u,v) = c_1 G_k(x) Z_1(\frac{xux}{||x||^2}, v),$$

where c_1 is a non-zero real constant, $Z_1(u,v)$ is the reproducing kernel of degree-1 homogeneous harmonic (respectively monogenic) polynomials if k is even (resp. odd), and

$$G_k(x) := \begin{cases} \dfrac{1}{||x||^{m-2n}}, & \text{if } k = 2n;\\ \dfrac{x}{||x||^{m-2n+2}}, & \text{if } k = 2n-1.\end{cases}$$

Now we consider sums of the following form:

$$cot_{l,1,k}(x,u,v) = \sum_{(m_1,\cdots,m_l)\in\mathbb{Z}^l} E_{1,k}(x + m_1e_1 + \cdots + m_le_l, u, v), \qquad (3)$$

for all $1 \leq p \leq m - k - 1$.

We will show these functions are defined on the l-fold periodic domain $\mathbb{R}^m/\mathbb{Z}^l$ for fixed u and v in $\mathbb{R}^m$ and are $\mathcal{C}l_m$-valued. They are also l-fold periodic functions.

To prove the locally uniform convergence of the series (3), use the locally normal convergence of the series

$$\sum_{\mathbf{m}\in\mathbb{Z}^l} G_k(x + \mathbf{m})$$

established by the following proposition [26].

Proposition 7.1. *Let $p \in \mathbb{N}$ with $1 \leq p \leq m - k - 1$. Let $\mathbb{Z}^p$ be the p-dimensional lattice. Then the series*

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^p} q_0^k(x + \boldsymbol{m})$$

converges normally in $\mathbb{R}^m \backslash \mathbb{Z}^p$.

Here q_0^k is in the kernel of the kth-power of D_x and q_0^k is exactly our G_k above. The proof follows from Proposition 2.2 appearing in [26].

Returning to the series defined by (3),

$$\begin{aligned} cot_{l,1,k}(x,u,v) &= \sum_{(m_1,\cdots,m_l)\in\mathbb{Z}^l} E_{1,k}(x + m_1e_1 + \cdots + m_le_l, u, v) \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^l} G_k(x+\mathbf{m})Z_1(\frac{(x+\mathbf{m})u(x+\mathbf{m})}{||x+\mathbf{m}||^2}, v),\ 1 \leq l \leq n-k-1, \end{aligned}$$

we observe that $Z_1(\frac{(x+\mathbf{m})u(x+\mathbf{m})}{||x+\mathbf{m}||^2}, v)$ is a bounded function on a bounded domain in $\mathbb{R}^m$ because its first variable, $\frac{(x+\mathbf{m})u(x+\mathbf{m})}{||x+\mathbf{m}||^2}$, is a reflection in the direction $\frac{(x+\mathbf{m})}{||x+\mathbf{m}||}$ for each $\mathbf{m}$, and hence is a linear transformation which is a continuous function. Furthermore, the norm of $\frac{(x+\mathbf{m})u(x+\mathbf{m})}{||x+\mathbf{m}||^2}$ is equal to the norm of u, so the bound of Z_1

with respect to the first variable does not depend on $\mathbf{m}$. On the other hand, we would get with respect to the second variable, bounded homogeneous functions of degree 1.

Consequently, applying the former proposition, the series

$$cot_{l,1,k}(x,u,v) = \sum_{(m_1,\cdots,m_l)\in\mathbb{Z}^l} E_{1,k}(x + m_1e_1 + \cdots + m_le_l, u, v),$$

for $1 \le l \le m-k-1$, is a uniformly convergent series and represents a kernel for the higher order higher spin operators under translations by $\mathbf{m} \in \mathbb{Z}^l$, with $1 \le l \le m-k-1$.

Using similar argument to those in [30], we define the $(m-k)$-fold periodic cotangent. In order to do that, we decompose the lattice $\mathbb{Z}^l$ into three parts: the origin $\{0\}$ and a positive and a negative part. The last two parts are equal and disjoint:

$$\begin{aligned}\Lambda_l \;\; &= \;\; \{m_1e_1 : \; m_1 \in \mathbb{N}\} \cup \{m_1e_1 + m_2e_2 : \; m_1, m_2 \in \mathbb{Z}, m_2 > 0\} \\ &\quad \cup \cdots \cup \{m_1e_1 + \cdots + m_le_l : \; m_1, \cdots, m_l \in \mathbb{Z}, m_l > 0\}\end{aligned}$$

and

$$-\Lambda_l = (\mathbb{Z}^l \backslash \{0\}) \backslash \Lambda_l.$$

For $l = m-k$, we define

$$cot_{m-k,1,k}(x,u,v) = E_{1,k}(x,u,v) + \sum_{\mathbf{m}\in\Lambda_{m-k}} \big[E_{1,k}(x+\mathbf{m},u,v) + E_{1,k}(x-\mathbf{m},u,v)\big].$$

The following proposition [26] and a similar argument as in [30] shows the above series converges normally.

Proposition 7.2. *Let $\mathbb{Z}^{m-k}$ be the $(m-k)$-dimensional lattice. Then the series*

$$q_0^k(x) + \sum_{\boldsymbol{m}\in\mathbb{Z}^{m-k}\backslash\{0\}} \big(q_0^k(x+\boldsymbol{m}) - q_0^k(\boldsymbol{m})\big)$$

converges normally.

The proof follows Proposition 2.2 appearing in [26]. Hence, it is a kernel for the higher order higher spin operator under translation by $\mathbf{m} \in \Lambda_{m-k}$.

For $x, y \in \mathbb{R}^m$, the function $cot_{l,1,k}(x-y,u,v)$ induces functions on C_l:

$$cot'_{l,1,k}(x',y',u,v) = cot_{l,1,k}(x-y,u,v).$$

for each $x', y' \in U'$ and x, y arbitrary representatives of $\pi_l^{-1}(x')$ and $\pi_l^{-1}(y')$. These functions are defined on $(C_l \times C_l)\backslash diagonal(C_l \times C_l)$ for each fixed $u, v \in \mathbb{R}^m$, where

$$diagonal(C_l \times C_l) = \{(x', x') : \ x' \in C_l\}$$

and they satisfy

$$\mathcal{D}_{1,k}^{C_l} cot'_{l,1,k}(x',y',u,v) = 0.$$

Conformally inequivalent spinor bundles on C_l

Previously the spinor bundle over C_l was trivial, $C_l \times \mathcal{C}l_m$. However, there are 2^l spinor bundles on C_l. See more details in [33]. The following construction is for some of the spinor bundles over C_l and all the others can be constructed similarly.

First let p be an integer in the set $\{1, 2 \cdots, l\}$ and consider the lattice $\mathbb{Z}^p := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_p$. We also consider the lattice $\mathbb{Z}^{l-p} := \mathbb{Z}e_{p+1} + \cdots + \mathbb{Z}e_l$. In this case $\mathbb{Z}^l = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \mathbb{Z}^p, \ \mathbf{n} \in \mathbb{Z}^{l-p}\}$. Suppose that $\mathbf{m} = m_1e_1 + \cdots + m_pe_p$. Let us make the identification (x, X) with $(x + \mathbf{m} + \mathbf{n}, (-1)^{m_1+\cdots+m+p}X)$ where $x \in \mathbb{R}^m$ and $X \in \mathcal{C}l_m$. This identification gives rise to a spinor bundle E^p over C_l.

We adapt functions in previous section as follows. For $1 \le l \le m-k-1$ we define

$$cot_{l,1,k,p}(x,u,v) = \sum_{m\mathbb{Z}^p,\ n\in\mathbb{Z}^{l-p}} (-1)^{m_1+\cdots+m_p} E_{1,k}(x+\mathbf{m}+\mathbf{n},u,v).$$

These are well defined functions on $\mathbb{R}^m\backslash\mathbb{Z}^l$. Therefore, we obtain from these functions the cotangent kernels

$$cot_{l,1,k,p}(x,y,u,v) = \sum_{m\mathbb{Z}^p,\ n\in\mathbb{Z}^{l-p}} (-1)^{m_1+\cdots+m_p} E_{1,k}(x-y+\mathbf{m}+\mathbf{n},u,v).$$

Again applying the projection map π_l these kernels give rise to the kernels

$$cot'_{l,1,k,p}(x',y',u,v).$$

In the case $l = m-k$, by considering the series

$$cot_{m-k,1,k}(x,y,u,v) = E_{1,k}(x,u,v) + \sum_{\mathbf{m}\in\Lambda_{m-k}} \big[E_{1,k}(x+\mathbf{m},u,v) + E_{1,k}(x-\mathbf{m},u,v)\big],$$

we obtain the kernel

$$\begin{aligned} & cot_{m-k,1,k}(x,y,u,v) \\ = \; & E_{1,k}(x-y,u,v) + \sum_{\mathbf{m}\in\Lambda_{m-k}} \big[E_{1,k}(x-y+\mathbf{m},u,v) + E_{1,k}(x-y-\mathbf{m},u,v)\big], \end{aligned}$$

which in turn using the projection map induces kernels

$$cot'_{m-k,1,k}(x',y',u,v).$$

Defining

$$
\begin{aligned}
&cot_{m-k,1,k,p}(x,u,v) = E_{1,k}(x+\mathbf{m}+\mathbf{n},u,v)\\
&+\sum_{\substack{\mathbf{m}\in\mathbb{Z}^p,\mathbf{n}\in\mathbb{Z}^{m-k-p}\\ \mathbf{m}+\mathbf{n}\in\Lambda_{m-k}}}(-1)^{m_1+\cdots+m_p}\big[E_{1,k}(x+\mathbf{m}+\mathbf{n},u,v)+E_{1,k}(x-\mathbf{m}-\mathbf{n},u,v)\big],
\end{aligned}
$$

we obtain the cotangent kernels

$$
\begin{aligned}
&cot_{m-k,1,k,p}(x,y,u,v) = E_{1,k}(x-y+\mathbf{m}+\mathbf{n},u,v)\\
&+\sum_{\substack{\mathbf{m}\in\mathbb{Z}^p,\mathbf{n}\in\mathbb{Z}^{m-k-p}\\ \mathbf{m}+\mathbf{n}\in\Lambda_{m-k}}}(-1)^{m_1+\cdots+m_p}\big[E_{1,k}(x-y+\mathbf{m}+\mathbf{n},u,v)+E_{1,k}(x-y-\mathbf{m}-\mathbf{n},u,v)\big],
\end{aligned}
$$

and by π_l the kernels

$$cot'_{m-k,1,k,p}(x',y',u,v).$$

7.2 The higher order higher spin operator on Hopf manifolds

In this section, we use similar arguments as in [33] and [11] to get the generalization of higher order higher spin operators to Hopf manifolds. It is worth pointing out that the two spin structures introduced there also apply for our fermionic and bosonic operators. Let $U=\mathbb{R}^m$ and $\Gamma=\{t^i:\ i\in\mathbb{Z}\}$, where t is an arbitrary strictly positive real number distinct from 1. Then by factoring out U by Γ we have the conformally flat spin manifold $\mathbb{S}^1\times\mathbb{S}^{m-1}$ which we denote by H_m and call a Hopf manifold. As $\prod_1(\mathbb{S}^1\times\mathbb{S}^{m-1})=\mathbb{Z}$ for $m>2$, it follows that the Hopf manifold H_m has two distinct spin structures. On this space, we can define the higher order higher spin operators and construct their kernels over the two different spinor bundles over $\mathbb{S}^1\times\mathbb{S}^{m-1}$.

In all that follows $\mathbb{R}^m\backslash\{0\}$ will be a universal covering space of the conformally flat manifold $\mathbb{S}^1\times\mathbb{S}^{m-1}$ ([25]). So there is a projection map $p:\ \mathbb{R}^m\backslash\{0\}\longrightarrow\mathbb{S}^1\times\mathbb{S}^{m-1}$. Further, for each $x\in\mathbb{R}^m\backslash\{0\}$ we shall denote $p(x)$ by x'. Further, if V is a subset of

$\mathbb{R}^m\backslash\{0\}$, then we denote $p(V)$ by V'.

One spinor bundle F_1 over $\mathbb{S}^1\times\mathbb{S}^{m-1}$ can be constructed by identifying the pair (x,X) with $(t^ix, t^{\frac{i(m-1)}{2}}X)$ for every $k\in\mathbb{Z}$, where $x\in\mathbb{R}^m\backslash\{0\}$ and $X\in\mathcal{C}l_m$.

Consider a domain $V\subset\mathbb{R}^m\backslash\{0\}$ satisfying $t^ix\in V$ for each $i\in\mathbb{Z}$ and $x\in V$. We will call such a domain a k-factor dilation domain. Further, we define an i-factor dilation function as a function $f(x,u):\ V\times\mathbb{R}^m\longrightarrow\mathcal{C}l_m$ such that f is a monogenic polynomial homogeneous of degree l in u satisfying $f(x,u)=t^{\frac{i(m-1)}{2}}f(t^ix,u)$ for each $x\in V$ and each integer i.

The projection map p induces a well defined function

$$f':\ V'\times\mathbb{R}^m\longrightarrow F_1,$$

where $f'(x,u)=t^{\frac{i(m-1)}{2}}f(x,u)$ for each $x'\in V'$ and x an arbitrary representative of $\pi_l^{-1}(x')$.

The higher order higher spin operator over $\mathbb{R}^m\backslash\{0\}$ induces a higher order higher spin operator acting on sections of the bundle F_1 over H_m. We will denote this operator by $\mathcal{D}_{1,k}^{H_m}$. If $\mathcal{D}_{1,k}^{H_m}(f')=0$ then f' is called an F_1-left higher order higher spin section. Moreover, any F_1-left higher order higher spin section $f':\ V'\times\mathbb{R}^m\longrightarrow F_1$ lifts to a k-factor dilation function $f:\ V\times\mathbb{R}^m\longrightarrow\mathcal{C}l_m$, where $V=p^{-1}(V')$ and $\mathcal{D}_{1,k}(f)=0$.

Now we consider the series

$$\begin{aligned}
E_{1,k,1}^{H_m}(x,y,u,v) &= \sum_{i=-\infty}^{0}G_k(t^ix-t^iy)Z_1(\frac{(t^ix-t^iy)u(t^ix-t^iy)}{||(t^ix-t^iy)||^2},v)\\
+\quad & t^{2(k-m)}G_k(x)Z_1(\frac{xux}{||x||^2},v)\big[\sum_{i=1}^{\infty}G_k(t^{-i}x^{-1}-t^{-i}y^{-1})\\
& Z_1(\frac{(t^{-i}x^{-1}-t^{-i}y^{-1})u(t^{-i}x^{-1}-t^{-i}y^{-1})}{||t^{-i}x^{-1}-t^{-i}y^{-1}||^2},v)\big]G_k(y)Z_1(\frac{yuy}{||y||^2},v),
\end{aligned}$$

where $x,y\in\mathbb{R}^m\backslash\{0\}$ and $y\neq t^ix$ for all $i\in\mathbb{Z}$. From the definition of $Z_1(u,v)$ and the homogeneity of $G_k(x)$, it is easy to obtain that $E_{1,k,1}^{H_m}$ converges normally on any compact subset K not containing the points $y=t^ix$ where $i\in\mathbb{Z}$. Since $E^{H_m}_{1,k,1}(tx,ty,u,v)=E^{H_m}_{1,k,1}(x,y,u,v)$, this kernel is periodic with respect to the Kleinian group $\{t^i:\ i\in\mathbb{Z}\}$. The kernel for the higher order higher spin operators on $(\mathbb{S}^1\times\mathbb{S}^{m-1})\times(\mathbb{S}^1\times\mathbb{S}^{m-1})\backslash diagonal(\mathbb{S}^1\times\mathbb{S}^{m-1})$ is then the projection of $E^{H_m}_{1,k,1}(x,y,u,v)$ on $(\mathbb{S}^1\times\mathbb{S}^{m-1})\times(\mathbb{S}^1\times\mathbb{S}^{m-1})\backslash diagonal(\mathbb{S}^1\times\mathbb{S}^{m-1})$:

$$E^{H_m}_{1,k,1}(x',y',u,v)=E^{H_m}_{1,k,1}(x,y,u,v),$$

for x,y representatives of $\pi_l^{-1}(x')$ and $\pi_l^{-1}(y')$ as earlier.

The second spinor bundle F_2 over $\mathbb{S}^1\times\mathbb{S}^{m-1}$ can be constructed by identifying the pair $(x,\ X)$ with $(t^ix,(-1)^it^{\frac{i(m-1)}{2}}X)$.

Now we introduce the normally convergent series:

$$\begin{aligned}E^{H_m}_{1,k,2}(x,y,u,v)&=\sum_{i=-\infty}^{0}(-1)^iG_k(t^ix-t^iy)Z_1(\frac{(t^ix-t^iy)u(t^ix-t^iy)}{||(t^ix-t^iy)||^2},v)\\+&\quad(-1)^it^{2(k-m)}G_k(x)Z_1(\frac{xux}{||x||^2},v)\big[\sum_{i=1}^{\infty}G_k(t^{-i}x^{-1}-t^{-i}y^{-1})\\&\quad Z_1(\frac{(t^{-i}x^{-1}-t^{-i}y^{-1})u(t^{-i}x^{-1}-t^{-i}y^{-1})}{||t^{-i}x^{-1}-t^{-i}y^{-1}||^2},v)\big]G_k(y)Z_1(\frac{yuy}{||y||^2},v)\end{aligned}$$

where $x,y\in\mathbb{R}^m\backslash\{0\}$ and $y\neq t^ix$ for all $i\in\mathbb{Z}$. This function induces through the projection map p on the variable $x,y\in\mathbb{R}^m\backslash\{0\}$, the higher order higher spin kernel associated with F_2 :

$$E^{H_m}_{1,k,2}(x',y',u,v)=E^{H_m}_{1,k,2}(x,y,u,v),$$

for x and y again as earlier.

8 Third order fermionic and fourth order bosonic operators

In Section 5, we used the concept of generalized symmetry as in [7, 18] to construct arbitrary order conformally invariant differential operators. Unfortunately, we require the target spaces of functions to be degree one polynomial spaces. With the same technique, we start constructing conformally invariant differential operators acting on functions taking values in arbitrary k-homogeneous harmonic (or monogenic) polynomial spaces. More specifically, in this section, we will give specific expressions for third order fermionic and fourth order bosonic operators with some of their properties. Then, we plan to solve the other higher order cases by induction. Unfortunately, it does not work as we expected because of the complicated calculation for generalized symmetries. However, this will be solved in the next section with a different approach.

8.1 3rd order higher spin operator $\mathcal{D}_3$

Our main result in the $3rd$ order higher spin case is the following theorem.

Theorem 8.1. *Up to a multiplicative constant, the unique 3rd-order conformally invariant differential operator is* $\mathcal{D}_{3,k} : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k)$, *where*

$$\begin{aligned}\mathcal{D}_3 \;=\;& D_x^3 + \frac{4}{m+2k}\langle u, D_x\rangle\langle D_u, D_x\rangle D_x - \frac{4||u||^2\langle D_u, D_x\rangle^2 D_x}{(m+2k)(m+2k-2)} - \frac{2u\langle D_u, D_x\rangle D_x^2}{m+2k}\\ &- \frac{8u\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k)(m+2k-2)} - \frac{8u^3\langle D_u, D_x\rangle^3}{(m+2k)(m+2k-2)(m+6k-10)}.\end{aligned}$$

Hereafter we may suppress the k index for the operator since there is little risk of confusion. Note the target space $\mathcal{M}_k$ is a function space, so any element in $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ has the form $f(x,u) \in \mathcal{M}_k$ for each fixed $x \in \mathbb{R}^m$ and x is the variable on which $\mathcal{D}_3$ acts.

The theorem is proved by a strategy similar to that used in Section 5. According to the Iwasawa decomposition, a Möbius transformation can be decomposed into a composition of reflections, translations, dilations, and inversions. It is obvious that all these six terms in the operator are invariant under translation, dilation, and reflection. We need only show it is invariant under inversion here. We have the definition for monogenic inversion as follows.

Definition 8.1. Monogenic inversion is a (conformal) transformation defined as

$$\mathcal{J}_3 : C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k) : f(x,u) \mapsto \mathcal{J}_3[f](x,u) := \frac{x}{||x||^{m-2}} f(\frac{x}{||x||^2}, \frac{xux}{||x||^2}).$$

Note that this inversion also consists of the classical Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u \omega$ acting on the dummy variable u (where $x = ||x||\omega$), but it satisfies $\mathcal{J}_3^2 = -1$ instead. Then we have the following lemma:

Lemma 8.2. *The special conformal transformation is defined as*

$$\mathcal{C}_3 := \mathcal{J}_3 \partial_{x_j} \mathcal{J}_3 = x e_j - 2\langle u, x\rangle \partial_{u_j} + 2u_j \langle x, D_u\rangle - ||x||^2 \partial_{x_j} + x_j(2\mathbb{E}_x + m - 2),$$

Proof. A similar calculation as in *Proposition A.1* in [7] will show the conclusion. □

Further we need the concept of generalized symmetry (see [7, 17]):

Definition 8.2. An operator η_1 is a generalized symmetry for a differential operator $\mathcal{D}$ if and only if there exists another oprator η_2 such that $\mathcal{D}\eta_1 = \eta_2 \mathcal{D}$. Note that for $\eta_1 = \eta_2$, this reduces to a definition of a symmetry in the sense that $\mathcal{D}\eta_1 = \eta_1 \mathcal{D}$.

Then we arrive at the main proposition, stating that the special conformal transformations are generalized symmetries of operator $\mathcal{D}_3$.

Proposition 8.3. *The special conformal transformations $\mathcal{C}_3$, with $j \in \{1, 2, \dots, m\}$ are generalized symmetries of $\mathcal{D}_3$. More specifically,*

$$[\mathcal{D}_3, \mathcal{C}_3] = 6x_j \mathcal{D}_3.$$

In particular, this shows $\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3 = ||x||^6\mathcal{D}_3$*, which generalizes the case of the classical higher order Dirac operator* D_x^3*. This also implies* $\mathcal{D}_3$ *is invariant under inversion.*

To prove this proposition, we first introduce the following technical lemmas:

Lemma 8.4. *For all* $1 \leq j \leq m$*, we have*

$$[D_x^3, \mathcal{C}_3] = 4\langle u, D_x\rangle D_x\partial_{u_j} - 2u\partial_{u_j}D_x^2 - 4u_jD_x\langle D_u, D_x\rangle + 6x_jD_x^3.$$

Lemma 8.5. *For all* $1 \leq j \leq m$*, we have*

$$\begin{aligned}&[\langle u, D_x\rangle\langle D_u, D_x\rangle D_x, \mathcal{C}_3] = -(m+2k)\langle u, D_x\rangle D_x\partial_{u_j} - e_ju\langle D_u, D_x\rangle D_x\\&+(m+2k-2)u_j\langle D_u, D_x\rangle D_x - 2u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j} - 2|u|^2\langle D_u, D_x\rangle D_x\partial_{u_j}\\&+6x_j\langle u, D_x\rangle\langle D_u, D_x\rangle D_x.\end{aligned}$$

Lemma 8.6. *For all* $1 \leq j \leq m$*, we have*

$$\begin{aligned}&[|u|^2\langle D_u, D_x\rangle^2D_x, \mathcal{C}_3] = 2|u|^2\langle D_u, D_x\rangle^2e_j - (2m+4k-4)|u|^2\langle D_u, D_x\rangle D_x\partial_{u_j}\\&-2u|u|^2\langle D_u, D_x\rangle^2\partial_{u_j} + 6x_j|u|^2\langle D_u, D_x\rangle^2D_x.\end{aligned}$$

Lemma 8.7. *For all* $1 \leq j \leq m$*, we have*

$$\begin{aligned}&[u\langle D_u, D_x\rangle D_x^2, \mathcal{C}_3] = -2e_ju\langle D_u, D_x\rangle D_x - 4u_j\langle D_u, D_x\rangle D_x - (m+2k)uD_x^2\partial_{u_j}\\&+4u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j} - 4u_ju\langle D_u, D_x\rangle^2 + 6x_ju\langle D_u, D_x\rangle D_x^2.\end{aligned}$$

Lemma 8.8. *For all* $1 \leq j \leq m$*, we have*

$$\begin{aligned}&[u\langle u, D_x\rangle\langle D_u, D_x\rangle^2, \mathcal{C}_3] = -e_j|u|^2\langle D_u, D_x\rangle^2 - (2m+4k-4)u\langle u, D_x\rangle\langle D_u, D_x\rangle\partial_{u_j}\\&-2u|u|^2\langle D_u, D_x\rangle^2\partial_{u_j} + (m+2k-2)u_ju\langle D_u, D_x\rangle^2 + 6x_ju\langle u, D_x\rangle\langle D_u, D_x\rangle^2.\end{aligned}$$

Lemma 8.9. *For all $1 \leq j \leq m$, we have*

$$[u^3\langle D_u, D_x\rangle^3, \mathcal{C}_3] = -(m+6k-10)u^3\langle D_u, D_x\rangle^2\partial_{u_j} + 6x_j u^3\langle D_u, D_x\rangle^3.$$

To prove these lemmas, we calculate the commutators of our operator and each component of $\mathcal{C}_3$, then combining them gives the results. We use these lemmas to obtain

$$[\mathcal{D}_3, \mathcal{C}_3] = 6x_j\mathcal{D}_3.$$

We rewrite the previous equation as

$$\mathcal{D}_3\mathcal{J}_3\partial_{x_j}\mathcal{J}_3 - \mathcal{J}_3\partial_{x_j}\mathcal{J}_3\mathcal{D}_3 = 6x_j\mathcal{D}_3;$$

then we apply $\mathcal{J}_3$ to both sides with the fact that $\mathcal{J}_3^2 = -1$:

$$-\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3\partial_{x_j} + \partial_{x_j}\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3 = 6\frac{x_j}{||x||^2}\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3.$$

This gives that $\mathcal{J}_3\mathcal{D}_3\mathcal{J}_3 = ||x||^6\mathcal{D}_3$, which can be rewritten as

$$\mathcal{D}_{3,y,w}\frac{x}{||x||^{m-2}}f(y,w) = \frac{x}{||x||^{m+2}}\mathcal{D}_{3,x,u}f(x,u),\ \forall f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k),$$

where $y = x^{-1}$ and $w = \dfrac{xux}{||x||^2}$. Therefore, we have proved $\mathcal{D}_3$ is invariant under inversion and, by earlier arguments, is conformally invariant.

8.2 4th order higher spin operator $\mathcal{D}_4$

Now for the main result in the $4th$ order higher spin case.

Theorem 8.10. *Up to a multiplicative constant, the unique $4th$-order conformally* *invariant differential operator is* $\mathcal{D}_4 : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k)$*, where*

$$\mathcal{D}_4 = \mathcal{D}_2^2 - \frac{8}{(m+2k-2)(m+2k-4)}\mathcal{D}_2\Delta_x.$$

Hereafter we may suppress the k index for the operator since there is little risk of confusion. The strategy is similar to that used above. It is sufficient to show only invariance under inversion. We have the definition for harmonic inversion as follows.

Definition 8.3. Harmonic inversion is a (conformal) transformation defined as

$$\mathcal{J}_4 : C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k) : f(x,u) \mapsto \mathcal{J}_4[f](x,u) := ||x||^{4-m} f(\frac{x}{||x||^2}, \frac{xux}{||x||^2}).$$

Note this inversion consists of the classical Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u \omega$ acting on the dummy variable u (where $x = ||x||\omega$). It satisfies $\mathcal{J}_4^2 = 1$. Then a similar calculation as in *Proposition A.1* in [7] provides the following lemma.

Lemma 8.11. *The special conformal transformation is defined as*

$$\mathcal{C}_4 := \mathcal{J}_4 \partial_{x_j} \mathcal{J}_4 = 2\langle u, x\rangle \partial_{u_j} - 2u_j\langle x, D_u\rangle + ||x||^2 \partial_{x_j} - x_j(2\mathbb{E}_x + m - 4).$$

Proposition 8.12. *The special conformal transformations* $\mathcal{C}_4$*, with* $j \in \{1, 2, \ldots, m\}$ *are generalized symmetries of* $\mathcal{D}_4$*. More specifically,*

$$[\mathcal{D}_4, \mathcal{C}_4] = -8x_j\mathcal{D}_4.$$

In particular, this shows $\mathcal{J}_4\mathcal{D}_4\mathcal{J}_4 = ||x||^8\mathcal{D}_4$*, which generalizes the case of the classical higher order Dirac operator* D_x^4*. This also implies* $\mathcal{D}_4$ *is invariant under inversion and hence conformally invariant.*

This proposition follows immediately with the help of the following two lemmas.

Lemma 8.13.

$$\begin{aligned}
\left[\mathcal{D}_2^2, \mathcal{C}_4\right] &= -8x_j\mathcal{D}_2^2 + \frac{32\langle u, D_x\rangle\Delta_x\partial_{u_j}}{(m+2k-2)^2} - \frac{32u_j\langle D_u, D_x\rangle\Delta_x}{(m+2k-2)^2} \\
&\quad - \frac{128\langle u, D_x\rangle^2\langle D_u, D_x\rangle\partial_{u_j}}{(m+2k-2)^2(m+2k-4)} + \frac{128||u||^2\langle D_u, D_x\rangle\Delta_x\partial_{u_j}}{(m+2k-2)^2(m+2k-4)^2} \\
&\quad - \frac{128||u||^2\langle D_u, D_x\rangle^2\partial_{x_j}}{(m+2k-2)^2(m+2k-4)^2} + \frac{128u_j\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k-2)^2(m+2k-4)} \\
&\quad + \frac{128||u||^2\langle u, D_x\rangle\langle D_u, D_x\rangle^2\partial_{u_j}}{(m+2k-2)^2(m+2k-4)^2} - \frac{128u_j||u||^2\langle D_u, D_x\rangle^3}{(m+2k-2)^2(m+2k-4)^2}.
\end{aligned}$$

Lemma 8.14.

$$\begin{aligned}
\left[\mathcal{D}_2\Delta_x, \mathcal{C}_4\right] &= -8x_j\mathcal{D}_2\Delta_x + \frac{4m+8k-16}{m+2k-2}\langle u, D_x\rangle\Delta_x\partial_{u_j} - \frac{16\langle u, D_x\rangle^2\langle D_u, D_x\rangle\partial_{u_j}}{m+2k-2} \\
&\quad + \frac{16||u||^2\langle D_u, D_x\rangle\Delta_x\partial_{u_j}}{(m+2k-2)(m+2k-4)} + \frac{16||u||^2\langle u, D_x\rangle\langle D_u, D_x\rangle^2\partial_{u_j}}{(m+2k-2)(m+2k-4)} \\
&\quad - \frac{4m+8k-16}{m+2k-2}u_j\langle D_u, D_x\rangle\Delta_x + \frac{16u_j\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{m+2k-2} \\
&\quad - \frac{16||u||^2\langle D_u, D_x\rangle^2\partial_{x_j}}{(m+2k-2)(m+2k-4)} - \frac{16u_j||u||^2\langle D_u, D_x\rangle^3}{(m+2k-2)(m+2k-4)}.
\end{aligned}$$

Proof. With the help of $[AB, C] = A[B, C] + [A, C]B$, where A, B and C are operators, a straightforward calculation leads to the result that

$$[\mathcal{D}_4, \mathcal{C}_4] = -8x_j\mathcal{D}_4.$$

Similar arguments as in the 3rd order case complete the proof. □

8.3 Connection with lower order conformally invariant operators

To construct higher order conformally invariant operators, one possible method is by composing and combining lower order conformally invariant operators. In this section, we will rewrite our operators $\mathcal{D}_3$ and $\mathcal{D}_4$ in terms of first order and second order conformally invariant operators. We expect this will help us when we consider other higher order conformally invariant operators.

Recall $\mathcal{D}_3$ maps $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ to $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$. If we fix $x \in \mathbb{R}^m$, then for any $f(x, u) \in \mathcal{M}_k$, we have $\mathcal{D}_3 f(x, u) \in \mathcal{M}_k$. In other words, $\mathcal{D}_3$ should be equal to the sum of contributions to $\mathcal{M}_k$ of all terms in $\mathcal{D}_3$. Notice that if we apply each term of $\mathcal{D}_3$ to $f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$, we will get a k-homogeneous polynomial in u that is in the kernel of Δ_u^2. Hence, we can decompose it by harmonic decomposition as follows

$$\mathcal{P}_k = \mathcal{H}_k \oplus u^2 \mathcal{H}_{k-2}.$$

where $\mathcal{P}_k$ is the k-homogeneous polynomial space and $\mathcal{H}_k$ is the k-homogeneous harmonic polynomial space. The Almansi-Fischer decomposition provides further

$$\mathcal{H}_k = \mathcal{M}_k \oplus u \mathcal{M}_{k-1},$$

where $\mathcal{M}_k$ is the k-homogeneous monogenic polynomial space; therefore, the contribution of each term to $\mathcal{M}_k$ can be written with two projections. For instance, the contribution of $u^3 \langle D_u, D_x \rangle^3 f(x, u)$ to $\mathcal{M}_k$ is $P_k P_1 u^3 \langle D_u, D_x \rangle^3 f(x, u)$, where

$$\mathcal{P}_k \xrightarrow{P_1} \mathcal{H}_k \xrightarrow{P_k} \mathcal{M}_k,$$

and

$$P_1 = 1 + \frac{u^2 \Delta_u}{2(m + 2k - 4)}, \ P_k = 1 + \frac{u D_u}{m + 2k - 2}.$$

We also notice that for fixed $x \in \mathbb{R}^m$ and $f(x,u) \in \mathcal{M}_k$,

$$u^3\langle D_u, D_x\rangle^3 f(x,u),\ \ |u|^2\langle D_u, D_x\rangle^2 D_x f(x,u) \in u^2\mathcal{H}_{k-2},$$

and $u\langle D_u, D_x\rangle D_x^2 \in u\mathcal{M}_{k-1}$. Hence, their contributions to $\mathcal{M}_k$ are all zero. Therefore,

$$\mathcal{D}_3 = P_k P_1\bigg(D_x^3 + \frac{4}{m+2k}\langle u, D_x\rangle\langle D_u, D_x\rangle D_x - \frac{8u\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k)(m+2k-2)}\bigg).$$

It is useful to recall some first and second order conformally invariant operators in higher spin spaces [7, 6]:

$$\begin{aligned}
&R_k:\ C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k),\ R_k = P_k D_x = (1 + \frac{uD_u}{m+2k-2})D_x;\\
&T_k:\ C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k),\ T_k = P_k D_x = (1 + \frac{uD_u}{m+2k-2})D_x;\\
&T_k^*:\ C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}),\ T_k^* = (I - P_k)D_x = \frac{uD_u}{m+2k-2}D_x;\\
&\mathcal{D}_2:\ C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k),\ \mathcal{D}_2 = P_1(\Delta_x - \frac{4}{m+2k-2}\langle u, D_x\rangle\langle D_u, D_x\rangle D_x).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{D}_3 &= P_k P_1\bigg(D_x^3 + \frac{4\langle u, D_x\rangle\langle D_u, D_x\rangle D_x}{m+2k-2}\bigg)\\
&\quad - P_k P_1\bigg(\frac{8\langle u, D_x\rangle\langle D_u, D_x\rangle D_x}{(m+2k)(m+2k-2)} - \frac{8u\langle u, D_x\rangle\langle D_u, D_x\rangle^2}{(m+2k)(m+2k-2)}\bigg)\\
&= -P_k P_1\mathcal{D}_2 D_x - \frac{8P_k P_1}{(m+2k)(m+2k-2)}\big(\langle u, D_x\rangle\langle D_u, D_x\rangle D_x + u\langle u, D_x\rangle\langle D_u, D_x\rangle^2\big).
\end{aligned}$$

Since for $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$, we have [7]:

$$\mathcal{D}_2 = -R_k^2 + \frac{4u\langle D_u, D_x\rangle}{(m+2k-2)(m+2k-4)}R_k.$$

A straightforward calculation leads to

$$\mathcal{D}_3 = R_k^3 + \frac{4}{(m+2k)(m+2k-4)} T_k T_k^* R_k.$$

Recall these conformally invariant second order twistor and dual-twistor operators [7]:

$$\begin{aligned}
&T_{k,2} = \langle u, D_x \rangle - \frac{||u||^2 \langle D_u, D_x \rangle}{m+2k-4} : \ C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k),\\
&T_{k,2}^* = \langle D_u, D_x \rangle : \ C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}), \text{ and}\\
&\mathcal{D}_2 = \Delta_x - \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{D}_4 &= \mathcal{D}_2^2 - \frac{8\mathcal{D}_2 \Delta_x}{(m+2k-2)(m+2k-4)}\\
&= \mathcal{D}_2^2 - \frac{8\mathcal{D}_2}{(m+2k-2)(m+2k-4)} \big(\mathcal{D}_2 + \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}\big)\\
&= \frac{(m+2k)(m+2k-6)}{(m+2k-2)(m+2k-4)} \mathcal{D}_2^2 - \frac{32\mathcal{D}_2 T_{k,2} T_{k,2}^*}{(m+2k-2)^2(m+2k-4)}.
\end{aligned}$$

8.4 Fundamental solutions and Intertwining operators

Using similar arguments as in [12], we obtain the fundamental solutions and intertwining operators of $\mathcal{D}_3$ and $\mathcal{D}_4$ as follows.

Theorem 8.15. ***(Fundamental solutions of*** $\mathcal{D}_3$***)***

Let $Z_k(u,v)$ be the reproducing kernel of $\mathcal{M}_k$. Then the fundamental solutions of $\mathcal{D}_3$ are

$$c_1 \frac{x}{||x||^{m-2}} Z_k\Big(\frac{xux}{||x||^2}, v\Big),$$

where c_1 is a constant.

Theorem 8.16. ***(Fundamental solutions of*** $\mathcal{D}_4$***)***

Let $Z_k(u,v)$ be the reproducing kernel of $\mathcal{H}_k$. Then the fundamental solutions of $\mathcal{D}_4$ are

$$c_2||x||^{4-m}Z_k(\frac{xux}{||x||^2},v),$$

where c_2 is a constant.

Theorem 8.17. ***(Intertwining operators)***

Let $y=\phi(x)=(ax+b)(cx+d)^{-1}$ be a Möbius transformation. Then

$$\frac{\widetilde{cx+d}}{||cx+d||^{m+4}}\mathcal{D}_{3,y,\omega}f(y,\omega)=\mathcal{D}_{3,x,u}\frac{\widetilde{cx+d}}{||cx+d||^{m-2}}f(\phi(x),\frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $\omega=\dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$ and $f(y,\omega)\in C^\infty(\mathbb{R}^m,\mathcal{M}_k)$;

$$||cx+d||^{-m-4}\mathcal{D}_{4,y,\omega}f(y,\omega)=\mathcal{D}_{4,x,u}||cx+d||^{4-m}f(\phi(x),\frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $\omega=\dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$ and $f(y,\omega)\in C^\infty(\mathbb{R}^m,\mathcal{H}_k)$.

It is worth pointing out that our above results generalize to conformally flat manifolds according to the method in our paper on cylinders and Hopf manifolds [13].

9 Construction of arbitrary order conformally invariant differential operators in higher spin spaces

As mentioned in the previous section, the approach with the concept of generalized symmetries does not work for the rest of the higher order (≥ 5) cases. That is because when the order increases, it becomes impossible to find the generalized symmetries of our conformally invariant differential operators. In this section, we will introduce a different representation theoretic approach to solve the rest of the higher order cases. This approach relies heavily on fundamental solutions of these conformally invariant differential operators. See more details below.

9.1 Motivation

The arbitrary t-th-order conformally invariant differential operator is denoted by

$$\mathcal{D}_t : \ C^\infty(\mathbb{R}^m, V) \longrightarrow C^\infty(\mathbb{R}^m, V),$$

where the target space V is $\mathcal{M}_k$ or $\mathcal{H}_k$. Thanks to results in [46, 47], the existence and uniqueness (up to a multiplicative constant) of $\mathcal{D}_t$ are already established. More specifically, even order conformally invariant differential operators only exist when $V = \mathcal{H}_k$ and odd order conformally invariant differential operators only exist when $V = \mathcal{M}_k$. This can be easily obtained by taking $\mathcal{M}_k$ or $\mathcal{H}_k$ as the irreducible representation of $Spin(m)$ in Theorems 2 and 3 in [47]; these theorems also give the conformal weights of $\mathcal{D}_t$, which provides the intertwining operators of $\mathcal{D}_t$. Recall the fundamental solution of the Rarita-Schwinger operator is $c\dfrac{x}{||x||^m}Z_k(\dfrac{xux}{||x||^2}, v)$, where $Z_k(u, v)$ is the reproducing kernel of $\mathcal{M}_k$ and c is a non-zero constant [6]. The fundamental solution of D_x^k is [36]

$$c_{2j+1}\frac{x}{||x||^{m-2j}}, \ if \ k = 2j+1; \quad c_{2j}||x||^{2j-m}, \ if \ k = 2j;$$

where c_{2j+1} and c_{2j} are both non-zero constants. However, when dimension m is even, we also require that $k < m$, because for instance, when $m = k = 2j$, the only candidate for a fundamental solution is a constant. We expect the fundamental solutions of our higher order higher spin conformally invariant differential operators $\mathcal{D}_t$ to have a conformal weight factor and a reproducing kernel part, behaving as follows.

1. The conformal weight changes with increasing order similar to the powers of the Dirac operator, differing in the even and odd cases.

2. The reproducing kernel factor changes with increasing degree of homogeneity of the target polynomial space similar to the Rarita-Schwinger operator, differing according to whether it is the space of harmonic or monogenic polynomials.

Thus we guess candidates for the fundamental solutions as follows.

1. For $\mathcal{D}_{2j}$, $c||x||^{2j-m}Z_k(\frac{xux}{||x||^2},v)$, where $Z_k(u,v)$ is the reproducing kernel of $\mathcal{H}_k$.
2. For $\mathcal{D}_{2j-1}$, $c\frac{x}{||x||^{m-2j+2}}Z_k(\frac{xux}{||x||^2},v)$, where $Z_k(u,v)$ is the reproducing kernel of $\mathcal{M}_k$.

These can be verified with intertwining operators of $\mathcal{D}_t$ and arguments similar to those in [6, 12]. We initially expect when the dimension m is even, we must restrict order $2j$ or $2j-1$ to be less than m, analogously to the powers of the Dirac operator. However, the reproducing kernel factor renders this restriction on the order unnecessary for even dimensions. Therefore, constructing a conformally invariant differential operator becomes finding an operator which has a particular fundamental solution.

In the rest of this section, we first introduce convolution type operators associated to fundamental solutions, then we point out fundamental solutions are actually the inverses of the corresponding differential operators in the sense of the previous type of convolution. Further, we show these convolution type operators are conformally invariant. Therefore, operators with such fundamental solutions should also be conformally invariant, considering they are the inverses of their fundamental solutions in the sense of convolution. This also brings us a class of conformally invariant convolution type operators and their inverses, if they exist, are conformally invariant pseudo-differential operators. Then we explain how the Rarita-Schwinger operator and higher spin Laplace operator can also be derived from the representation-theoretic approach. Then, since even and odd order conformally invariant differential operators have different target spaces, we will show the constructions in even and odd order cases separately. The even order operators, which have integer spin, are named bosonic operators in analogy with bosons in physics, which are particles of integer spin. Correspondingly, the odd order operators, which have half-integer spin, are named fermionic operators after fermions, which are particles of half-integer spin.

9.2 Convolution type operators

Assume $E_k(x,u,v)$ is the fundamental solution of $\mathcal{D}_k$. Then we define a convolution operator as follows.

$$\Phi(f)(y,v) = E_k(x-y,u,v) * f(x,u) := \int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} E_k(x-y,u,v)f(x,u)dS(u)dx^m.$$

Notice this is not the usual convolution operator, as it has an integral over the unit sphere with respect to the variable u. Since $E_k(x,u,v)$ is the fundamental solution of $\mathcal{D}_k$, we have

$$\mathcal{D}_{k,x,u}E_k(x-y,u,v) * f(x,u) := \int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} \mathcal{D}_{k,x,u}E_k(x-y,u,v)f(x,u)dS(u)dx^m = f(y,v),$$

where $f(y,v) \in C^\infty(\mathbb{R}^m, U)$ $(U = \mathcal{H}_k \ or \ \mathcal{M}_k)$ with compact support in y for each $v \in \mathbb{R}^m$. Hence, we have $\mathcal{D}_k E_k = Id$ and $E_k^{-1} = \mathcal{D}_k$ in the sense above. This implies that if we can show our convolution operator Φ is conformally invariant, then its corresponding differential operator should also be conformally invariant by taking its inverse.

Denote

$$E_{2j}(x,u,v) = ||x||^{2j-m}Z_k(\frac{xux}{||x||^2},v) \text{ and } E_{2j-1}(x,u,v) = \frac{x}{||x||^{m-2j+2}}Z_k(\frac{xux}{||x||^2},v),$$

where $Z_k(u,v)$ is the reproducing kernel of $\mathcal{H}_k$ in the even case and the reproducing kernel of $\mathcal{M}_k$ in the odd case.

Next, we will show the above convolution operator Φ is conformally invariant under Möbius transformations. Thanks to the Iwasawa decomposition, it suffices to verify it is conformally invariant under orthogonal transformation, inversion, translation, and dilation.

Conformal invariance under translation and dilation is trivial; hence, we only show the orthogonal transformation and inversion cases here.

Proposition 9.1. ***(Orthogonal transformation)*** *Suppose $a \in Spin(m)$. If $x' = ax\tilde{a}$, $y' = ay\tilde{a}$, $u' = au\tilde{a}$, and $v' = av\tilde{a}$, then*

1. $E_{2j}(x' - y', u', v') * f(x', u') = E_k(x - y, u, v) * f(ax\tilde{a}, au\tilde{a})$,

2. $E_{2j-1}(x' - y', u', v') * f(x', u') = aE_{2j-1}(x - y, u, v)\tilde{a} * f(ax\tilde{a}, au\tilde{a})$

Proof. Case 1. Let $f(x, u) \in C^\infty(\mathbb{R}^m, \mathcal{H}_k)$. Since the reproducing kernel of $\mathcal{H}_k$ is rotationally invariant, $ax\tilde{a}$ is a rotation of x in the direction of a for $a \in Spin(m)$, and $a\tilde{a} = 1$, we have

$$
\begin{aligned}
& E_{2j}(x' - y', u', v') * f(x', u') \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} ||x' - y'||^{2j-m} Z_k(\frac{(x' - y')u'(x' - y')}{||x' - y'||^2}, v') f(x', u') dS(u') dx'^m \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} ||a(x - y)\tilde{a}||^{2j-m} Z_k(\frac{a(x - y)\tilde{a}au\tilde{a}a(x - y)\tilde{a}}{||ax\tilde{a}||^2}, av\tilde{a}') f(ax\tilde{a}, au\tilde{a}) dS(u) dx^m \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} ||x - y||^{2j-m} Z_k(\frac{a(x - y)u(x - y)\tilde{a}}{||x - y||^2}, av\tilde{a}) f(ax\tilde{a}, au\tilde{a}) dS(u) dx^m \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} ||x - y||^{2j-m} Z_k(\frac{(x - y)u(x - y)}{||x - y||^2}, v) f(ax\tilde{a}, au\tilde{a}) dS(u) dx^m \\
= & E_{2j}(x - y, u, v) * f(ax\tilde{a}, au\tilde{a})
\end{aligned}
$$

Case 2. Since the reproducing kernel of $\mathcal{M}_k$ has the property

$$Z_(u, v) = aZ_k(au\tilde{a}, av\tilde{a})\tilde{a}$$

for $a \in Spin(m)$, similar arguments as in *Case 1* give the result. □

Proposition 9.2. ***(Inversion)*** *Let $x' = x^{-1} = -\frac{x}{||x||^2}$, $y' = y^{-1} = -\frac{y}{||y||^2}$, $u' = \frac{yuy}{||y||^2}$ and $v' = \frac{xvx}{||x||^2}$. Then*

1. $E_{2j}(x'-y',u',v') * f(x',u') = ||y||^{m-2j}E_{2j}(x-y,u,v)||x||^{-m-2j} * f(x^{-1}, \frac{yuy}{||y||^2})$,

2. $E_{2j-1}(x'-y',u',v') * f(x',u') = \left(\frac{y}{||y||^{m-2j+2}}\right)^{-1}E_{2j-1}(x-y,u,v)\frac{x}{||x||^{m-2j}} * f(x^{-1}, \frac{yuy}{||y||^2})$.

Proof. Case 1. Suppose $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{H}_k)$. Notice

$$x^{-1} - y^{-1} = -x^{-1}(x-y)y^{-1} = -y^{-1}(x-y)x^{-1} = -\frac{x}{||x||^2}(x-y)\frac{y}{||y||^2} = -\frac{y}{||y||^2}(x-y)\frac{x}{||x||^2}.$$

Recall that, as the reproducing kernel of $\mathcal{H}_k$, $Z_k(u,v)$ has the property

$$Z_k(u,v) = Z_k(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2})$$

for $x \in \mathbb{R}^m$. Hence, we have

$$\begin{aligned}
& E_{2j}(x'-y',u',v') * f(x',u') \\
= & \int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} ||x'-y'||^{2j-m} Z_k(\frac{(x'-y')u'(x'-y')}{||x'-y'||^2}, v')f(x',u')dS(u')dx'^m \\
= & \int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} ||x^{-1}(x-y)y^{-1}||^{2j-m} Z_k(\frac{x(x-y)yu'y(x-y)x}{|y(|x-y)x||^2}, v') \\
& \quad \cdot f(x^{-1},u')j(x^{-1})dS(u')dx^m \\
= & \int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} ||x^{-1}(x-y)y^{-1}||^{2j-m} Z_k(\frac{(x-y)u(x-y)}{||x-y||^2}, v) \\
& \quad \cdot f(x^{-1}, \frac{yuy}{||y||^2})j(x^{-1})dS(u)dx^m
\end{aligned}$$

where $j(x^{-1}) = ||x||^{-2m}$ is the Jacobian. Hence,

$$\begin{aligned}
= & \int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} ||y||^{m-2j}||x-y||^{2j-m} Z_k(\frac{(x-y)u(x-y)}{||x-y||^2}, v)||x||^{-m-2j} \\
& \quad \cdot f(x^{-1}, \frac{yuy}{||y||^2})dS(u)dx^m \\
= & ||y||^{m-2j}E_{2j}(x-y,u,v)||x||^{-m-2j} * f(x^{-1}, \frac{yuy}{||y||^2}).
\end{aligned}$$

Case 2. Recall that, as the reproducing kernel of $\mathcal{M}_k$, $Z_k(u,v)$ has the property

$$Z_k(u,v) = \frac{x}{||x||} Z_k(\frac{xux}{||x||^2}, \frac{xvx}{||x||^2}) \frac{x}{||x||}$$

for $x \in \mathbb{R}^m$. Then, by arguments similar to those above, we have

$$\begin{aligned}
& E_{2j-1}(x'-y', u', v') * f(x', u') \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \frac{x'-y'}{||x'-y'||^{m-2j+2}} Z_k(\frac{(x'-y')u'(x'-y')}{||x'-y'||^2}) f(x', u') dS(u') dx'^m \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} \frac{y^{-1}(x-y)x^{-1}}{||y^{-1}(x-y)x^{-1}||^{m-2j+2}} \cdot Z_k(\frac{x(x-y)yu'y(x-y)x}{|x(x-y)y||^2}, v') \\
& \cdot f(x^{-1}, u') j(x^{-1}) dS(u') dx^m \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} (\frac{y}{||y||^{m-2j+2}})^{-1} \frac{x-y}{||x-y||^{m-2j+2}} (\frac{x}{||x||^{m-2j+2}})^{-1} \\
& \cdot \frac{x}{||x||} Z_k(\frac{(x-y)u(x-y)}{||x-y||^2}, v) \frac{x}{||x||} f(x^{-1}, \frac{yuy}{||y||^2}) ||x||^{-2m} dS(u) dx^m \\
= & \int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} (\frac{y}{||y||^{m-2j+2}})^{-1} E_{2j-1}(x-y, u, v) \frac{x}{||x||^{m-2j}} f(x^{-1}, \frac{yuy}{||y||^2}) dS(u) dx^m \\
= & (\frac{y}{||y||^{m-2j+2}})^{-1} E_{2j-1}(x-y, u, v) \frac{x}{||x||^{m-2j}} * f(x^{-1}, \frac{yuy}{||y||^2}).
\end{aligned}$$

$\square$

Hence, the intertwining operators for the convolution operators are as follows.

Proposition 9.3. *Suppose* $x' = \varphi(x) = (ax+b)(cx+d)^{-1}$ *is a Möbius transformation,* $u' = \frac{(cy+d)u(\widetilde{cy+d})}{||cy+d||^2}$, *and* $v' = \frac{(cx+d)v(\widetilde{cx+d})}{||cx+d||^2}$. *Recall*

$$\begin{aligned}
J_k(\varphi, x) &= \frac{\widetilde{cx+d}}{||cx+d||^{m-2j+2}}, \ if \ k = 2j-1, \\
J_k(\varphi, x) &= ||cx+d||^{2j-m}, \ if \ k = 2j; \\
J_{-k}(\varphi, x) &= \frac{\widetilde{cx+d}}{||cx+d||^{m+2j}}, \ if \ k = 2j-1, \\
J_{-k}(\varphi, x) &= ||cx+d||^{-m-2j}, \ if \ k = 2j.
\end{aligned}$$

Then

$$\begin{aligned}
& E_k(x'-y',u',v') * f(x',u') \\
= \;& J_k^{-1}(\varphi,y)E_k(x-y,u,v)J_{-k}(\varphi,x) * f(\varphi(x), \frac{(cy+d)u(\widetilde{cy+d})}{||cy+d||^2}).
\end{aligned}$$

Recall that $\mathcal{D}_k$ is the inverse of its fundamental solution in the sense of convolution. Hence, we obtain the intertwining operators of $\mathcal{D}_k$ as follows.

Proposition 9.4. *Suppose* $y' = (ay+b)(cy+d)^{-1}$ *is a Möbius transformation and* $u' = \frac{(cy+d)u(\widetilde{cy+d})}{||cy+d||^2}$. *Then*

$$\mathcal{D}_{k,y',u'} = J_{-k}^{-1}(\varphi,y)\mathcal{D}_{k,y,u}J_k(\varphi,y),$$

where J_k *and* J_{-k} *are defined as above.*

It is worth pointing out that for general $\alpha \in \mathbb{R}$, if we denote

$$E_k^{\alpha,1}(x-y,u,v) = \frac{x}{||x||^\alpha} Z_k(\frac{xux}{||x||^2}, v)$$

where $Z_k(u,v)$ is the reproducing kernel of $\mathcal{M}_k$, and

$$E_k^{\alpha,2}(x-y,u,v) = ||x||^\alpha Z_k(\frac{xux}{||x||^2}, v)$$

where $Z_k(u,v)$ is the reproducing kernel of $\mathcal{H}_k$, then we can define a class of convolution type operators

$$\int_{\mathbb{R}^m}\int_{\mathbb{S}^{m-1}} E_k^{\alpha,i}(x-y,u,v)f_i(x,u)dS(u)dx^m$$

where $f_i(x,u) \in C^\infty(\mathbb{R}^m, U_i)$ with $U_1 = \mathcal{M}_k$ and $U_2 = \mathcal{H}_k$. More importantly, these convolution type operators are conformally invariant with similar arguments as above and their inverses, if they exist, are conformally invariant pseudo-differential operators.

9.3 Rarita-Schwinger and higher spin Laplace operators

From the preceding argument, we expect the fundamental solution of the first-order conformally invariant differential operator is $c\frac{x}{||x||^m}Z_k(\frac{xux}{||x||^2},v)$, where c is a non-zero constant and $Z_k(u,v)$ is the reproducing kernel of $\mathcal{M}_k$. To show $R_k=(1+\frac{uD_u}{m+2k-2})D_x$ has such a fundamental solution, all we need are the following two theorems.

Theorem 9.5. *[15] For each $\psi\in C^\infty(\mathbb{R}^m,\mathcal{M}_k)$ with compact support,*

$$\iint_{\mathbb{R}^m}-(E_k(x-y,u,v),R_k\psi(x,v))_v dx^m=\psi(y,u),$$

where

$$(f(v),g(v))_v=\iint_{\mathbb{S}^{m-1}}f(v)g(v)dS(v)$$

is the Fischer-inner product for two Clifford valued polynomials,

$E_k(x,u,v)=\frac{1}{\omega_{m-1}c_k}\frac{x}{||x||^m}Z_k(\frac{xux}{||x||^2},v)$, *$Z_k(u,v)$ is the reproducing kernel of $\mathcal{M}_k$, the constant c_k is $\frac{m-2}{m+2k-2}$, and ω_{m-1} is the area of $(m-1)$-dimensional unit sphere.*

Theorem 9.6. *[15] Let Ω and Ω' be domains in $\mathbb{R}^m$ and suppose the closure of Ω lies in Ω'. Further suppose the closure of Ω is compact and $\partial\Omega$ is piecewise smooth. Let $f,g\in C^1(\mathbb{R}^m,\mathcal{C}l_m)$. Then*

$$\int_{\partial\Omega}(g(x,u)d\sigma_x f(x,u))_u=\int_\Omega(g(x,u)R_k,f(x,u))_u dx^m+\int_\Omega(g(x,u),R_kf(x,u))_u dx^m,$$

where $d\sigma_x=n(x)d\sigma(x)$, σ is is scalar Lebegue measure on $\partial\Omega$, and $n(x)$ is unit outer normal vector to $\partial\Omega$.

Notice that our function $\psi(x,u)$ has compact support in x, so the above theorem also applies in our case with $\Omega = \mathbb{R}^m$. Therefore, one can have

$$\begin{aligned} & \iint_{\mathbb{R}^m} (R_k E_k(x-y,u,v), \psi(x,v))_v dx^m \\ = & \iint_{\mathbb{R}^m} -(E_k(x-y,u,v), R_k\psi(x,v))_v dx^m = \psi(y,u) \end{aligned}$$

for each $\psi \in C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ with compact support. This exactly means $E_k(x,u,v)$ is the fundamental solution of R_k. Hence, R_k is the first order conformally invariant differential operator, which is the Rarita-Schwinger operator.

Similarly, we know the fundamental solution of the second order conformally invariant differential operator should be $c||x||^{2-m}Z_k(\frac{xux}{||x||^2}, v)$, where $Z_k(u,v)$ is the reproducing of $\mathcal{H}_k$ and c is a non-zero constant. In [7], it is showed that the differential operator

$$\mathcal{D}_2 = \Delta_x - \frac{4\langle u, D_x\rangle\langle D_u, D_x\rangle}{m+2k-2} + \frac{||u||^2\langle D_u, D_x\rangle^2}{(m+2k-2)(m+2k-4)}$$

has fundamental solution

$$\frac{(m+2k-4)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}}||x||^{2-m}Z_k(\frac{xux}{||x||^2}, v).$$

Therefore, $\mathcal{D}_2$ is the second order conformally invariant differential operator, which is the higher spin Laplace operator in [7].

There is, however, a much simpler way to recover the higher spin Laplace operator instead of using generalized symmetries as in [7]. Consider the second order twistor and dual twistor operators from the same reference:

$$\begin{aligned} T_{k,2} &= \langle u, D_x\rangle - \frac{||u||^2\langle D_u, D_x\rangle}{m+2k-4} : \ C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k), \\ T_{k,2}^* &= \langle D_u, D_x\rangle : \ C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}). \end{aligned}$$

Any second order operator $C^\infty(\mathbb{R}^m,\mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m,\mathcal{H}_k)$ reduces to a linear combination of the second order operators Δ_x and $T_{k,2}T_{k,2}^*$, since these two are the only second order differential operators mapping from $C^\infty(\mathbb{R}^m,\mathcal{H}_k)$ to $C^\infty(\mathbb{R}^m,\mathcal{H}_k)$ which also do not change the degree of the variable u; more details can be found in [7]. These are scalar-valued as desired, since $\mathcal{H}_k$ is a scalar-valued function space. A linear combination of Δ_x and $T_{k,2}T_{k,2}^*$ that annihilates $c||x||^{2-m}Z_k(\frac{xux}{||x||^2},v)$, where $Z_k(u,v)$ is the reproducing kernel of $\mathcal{H}_k$ and c is a non-zero constant, is the higher spin Laplace operator up to a multiplicative constant:

$$\mathcal{D}_2 = \Delta_x - \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}.$$

9.4 Bosonic operators: even order, integer spin

With a similar strategy as in the previous section and arguing by induction, we now construct higher order conformally invariant differential operators in higher spin spaces. We start with the even order case. Denote the $2j$-th order bosonic operator by

$$\mathcal{D}_{2j}:\ C^\infty(\mathbb{R}^m,\mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m,\mathcal{H}_k).$$

As the generalization of D_x^{2j} in Euclidean space to higher spin spaces, it is conformally invariant and has the following intertwining operators:

$$||cx+d||^{2j+m}\mathcal{D}_{2j,y,\omega}f(y,\omega) = \mathcal{D}_{2j,x,u}||cx+d||^{2j-m}f(\phi(x),\frac{(cx+d)u\widetilde{(cx+d)}}{||cx+d||^2}),$$

where $y=\phi(x)=(ax+b)(cx+d)^{-1}$ is a Möbius transformation and
$\omega = \dfrac{(cx+d)u\widetilde{(cx+d)}}{||cx+d||^2}$.

As mentioned above, the uniqueness (up to a multiplicative constant) and existence of $\mathcal{D}_{2j}$ having the above intertwining operators can be justified by Theorems 2, 3 and 4 of [47] and Chapter 8 of [46], where the irreducible representation of $Spin(m)$ is $\mathcal{H}_k$ with highest weight $\lambda = (k, 0, \cdots, 0)$.

Recall we expect the fundamental solution of $\mathcal{D}_{2j}$ to be

$$c||x||^{2j-m} Z_k(\frac{xux}{||x||^2}, v),$$

where c is a non-zero real constant and $||x||^{2j-m}$ is the conformal weight J_{2j}. Indeed, this is proved by similar arguments as in [6, 12]. Therefore, to find the $2j$-th order conformally invariant differential operator, we need only find a $2j$-th order differential operator whose fundamental solution is $c||x||^{2j-m} Z_k(\frac{xux}{||x||^2}, v)$. Here is our first main theorem.

Theorem 9.7. *Let $Z_k(u, v)$ be the reproducing kernel of $\mathcal{H}_k$. When $j > 1$, the $2j$-th order conformally invariant differential operator on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$ is the $2j$-th bosonic operator*

$$\mathcal{D}_{2j} = \mathcal{D}_2 \prod_{s=2}^{j} (\mathcal{D}_2 - \frac{(2s)(2s-2)}{(m+2k-2)(m+2k-4)} \Delta_x)$$

that has the fundamental solution

$$a_{2j}||x||^{2j-m} Z_k(\frac{xux}{||x||^2}, v),$$

where

$$\mathcal{D}_2 = \Delta_x - \frac{4T_{k,2}T_{k,2}^*}{m+2k-2}$$

is the higher spin Laplace operator [7],

$$T_{k,2} = \langle u, D_x \rangle - \frac{||u||^2 \langle D_u, D_x \rangle}{m+2k-4} \text{ and } T_{k,2}^* = \langle D_u, D_x \rangle$$

are the second order twistor and dual twistor operators, and a_{2j} is a non-zero real constant whose expression is given later in this section.

To prove the previous theorem, we start with the following proposition.

Proposition 9.8. *For every $H_k(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathcal{C})$, when $\alpha > 2 - m$,*

$$(\mathcal{D}_2 - \frac{(m+\alpha)(m+\alpha-2)}{(m+2k-2)(m+2k-4)}\Delta_x)||x||^{\alpha} H_k(\frac{xux}{||x||^2}) = c_{\alpha+m}||x||^{\alpha-2} H_k(\frac{xux}{||x||^2}),$$

in the distribution sense, where

$$c_{\alpha+m} = -(m+\alpha)(m+\alpha-2)\frac{(\alpha-2k)(\alpha-2k-2)+2k(m+2\alpha-2k-4)}{(m+2k-2)(m+2k-4)}.$$

Proof. In order to prove the above proposition with an arbitrary function $H_k(u) \in \mathcal{H}_k$, as stated in [7], we can rely on the fact $\mathcal{H}_k$ is an irreducible $Spin(m)$-representation generated by the highest weight vector $\langle u, 2\mathfrak{f}_1\rangle^k$. As $\mathcal{D}_2$ and Δ_x are both $Spin(m)$-invariant operators, it suffices to prove the statement for

$$||x||^{\alpha}\langle \frac{xux}{||x||^2}, 2\mathfrak{f}_1\rangle^k = ||x||^{\alpha-2k}\langle xux, 2\mathfrak{f}_1\rangle^k = ||x||^{\alpha-2k}\langle u||x||^2 - 2\langle u, x\rangle x, 2\mathfrak{f}_1\rangle^k.$$

First, we assume $x \neq 0$. On the one hand, we have

$$\begin{aligned}
&\Delta_x||x||^{\alpha-2k}\langle xux, 2\mathfrak{f}_1\rangle^k = \Delta_x||x||^{\alpha-2k}\langle u||x||^2 - 2\langle u, x\rangle x, 2\mathfrak{f}_1\rangle^k \\
= \ &\Delta_x(||x||^{\alpha-2k})\langle xux, 2\mathfrak{f}_1\rangle^k + ||x||^{\alpha-2k}\Delta_x(\langle xux, 2\mathfrak{f}_1\rangle^k) + \sum_{j=1}^{m}\partial_{x_j}(||x||^{\alpha-2k})\partial_{x_j}(\langle xux, 2\mathfrak{f}_1\rangle^k).
\end{aligned}$$

Since

$$\begin{aligned}
&\partial_{x_j}\langle xux, 2\mathfrak{f}_1\rangle^k = \partial_{x_j}\langle u||x||^2 - 2\langle u, x\rangle x, 2\mathfrak{f}_1\rangle^k \\
= \ &k\langle xux, 2\mathfrak{f}_1\rangle^{k-1}\langle 2ux_j - 2u_j x - 2\langle u, x\rangle e_j, 2\mathfrak{f}_1\rangle,
\end{aligned}$$

and from [7]

$$
\begin{aligned}
& \Delta_x \langle xux, 2\mathfrak{f}_1\rangle^k \\
= \quad & 4k(k-1)||u||^2 \langle x, 2\mathfrak{f}_1\rangle^2 \langle xux, 2\mathfrak{f}_1\rangle^{k-2} + 2k(m+2k-4)\langle u, 2\mathfrak{f}_1\rangle \langle xux, 2\mathfrak{f}_1\rangle^{k-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \Delta_x ||x||^{\alpha-2k} \langle xux, 2\mathfrak{f}_1\rangle^k \\
= \quad & \big[(\alpha-2k)(\alpha-2k-2) + 2k(m-2k+2\alpha-4)\big] ||x||^{\alpha-2k-2} \langle xux, 2\mathfrak{f}_1\rangle^k \\
& +4k(m+2k-4)\langle u, x\rangle \langle x, 2\mathfrak{f}_1\rangle ||x||^{\alpha-2k-2} \langle xux, 2\mathfrak{f}_1\rangle^{k-1} \\
& +4k(k-1)||u||^2 \langle x, 2\mathfrak{f}_1\rangle^2 ||x||^{\alpha-2k} \langle xux, 2\mathfrak{f}_1\rangle^{k-2}.
\end{aligned}
$$

On the other hand, we have [7]

$$
\begin{aligned}
& \mathcal{D}_2 ||x||^{\alpha-2k} \langle xux, 2\mathfrak{f}_1\rangle^k = (m+\alpha-2)(\alpha + \frac{4k}{m+2k-2}) ||x||^{\alpha-2k-2} \langle xux, 2\mathfrak{f}_1\rangle^k \\
& +(m+\alpha-2)(m+\alpha)\frac{4k}{m+2k-2} \langle u, x\rangle \langle x, 2\mathfrak{f}_1\rangle ||x||^{\alpha-2k-2} \langle xux, 2\mathfrak{f}_1\rangle^{k-1} \\
& + \frac{4k(k-1)(m+\alpha)(m+\alpha-2)}{(m+2k-2)(m+2k-4)} ||u||^2 \langle x, 2\mathfrak{f}_1\rangle^2 ||x||^{\alpha-2k} \langle xux, 2\mathfrak{f}_1\rangle^{k-2}.
\end{aligned}
$$

Combining the above two equalities completes the proof when $x \neq 0$. Next, we consider the singularity of $\phi(x, u)$ at $x = 0$. Notice that singularity only occurs in the $||x||^\alpha$ part and that $||x||^\alpha$ is weak differentiable if $\alpha > -m+1$ with weak derivative $\partial_{x_i} ||x||^\alpha = \alpha x_i ||x||^{\alpha-2}$. Hence, with the assumption that $\alpha > 2-m$, every differentiation in the process above is also correct in the distribution sense. This completes the proof. □

Now, we can prove the following proposition immediately.

Proposition 9.9. *For integers $j > 1$,*

$$\mathcal{D}_{2j} a_{2j} ||x||^{2j-m} H_k(\frac{xux}{||x||^2}) = \delta(x) H_k(u),$$

where $H_k(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathcal{C})$ and

$$a_{2j} = \frac{(m+2k-4)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}} \prod_{s=2}^{j} c_{2s}^{-1},$$

c_{2s} defined by Proposition 9.8 for $\alpha = 2s - m$.

Proof. We prove this proposition by induction. First, when $j = 2$,

$$\begin{aligned}
& \mathcal{D}_4 a_4 ||x||^{4-m} H_k(\frac{xux}{||x||^2}) \\
= \; & (\mathcal{D}_2 - \frac{8}{(m+2k-2)(m+2k-4)} \Delta_x) \mathcal{D}_2 a_4 ||x||^{4-m} H_k(\frac{xux}{||x||^2}) \\
= \; & \mathcal{D}_2 (\mathcal{D}_2 - \frac{8}{(m+2k-2)(m+2k-4)}) \Delta_x a_4 ||x||^{4-m} H_k(\frac{xux}{||x||^2}) \\
= \; & \mathcal{D}_2 \frac{(m+2k-4)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}} ||x||^{2-m} H_k(\frac{xux}{||x||^2}),
\end{aligned}$$

where the last line follows using $\alpha = 4 - m$ in Proposition 9.8. Thanks to Theorem 5.1 in [7], this last equation is equal to $\delta(x) H_k(u)$.

Assume when $j = s$ that the proposition is true. Then for $j = s + 1$, we have

$$\begin{aligned}
& \mathcal{D}_{2s+2} a_{2s+2} ||x||^{2s+2-m} H_k(\frac{xux}{||x||^2}) \\
= \; & (\mathcal{D}_2 - \frac{2s(2s+2)}{(m+2k-2)(m+2k-4)} \Delta_x) \mathcal{D}_{2s} a_{2s} c_{2s+2}^{-1} ||x||^{6-m} H_k(\frac{xux}{||x||^2}) \\
= \; & \mathcal{D}_{2s} (\mathcal{D}_2 - \frac{24}{(m+2k-2)(m+2k-4)} \Delta_x) c_{2s+2}^{-1} a_{2s} ||x||^{2s+2-m} H_k(\frac{xux}{||x||^2}) \\
= \; & \mathcal{D}_{2s} a_{2s} ||x||^{2s-m} H_k(\frac{xux}{||x||^2}) = \delta(x) H_k(u),
\end{aligned}$$

where the penultimate equality follows using $\alpha = 2s + 2 - m$ in Proposition 9.8. This last equation comes from our assumption $j = s$. Therefore, our proposition is proved. $\square$

In particular, from the above proposition, we have

$$\mathcal{D}_{2j} a_{2j} ||x||^{2j-m} Z_k(\frac{xux}{||x||^2}, v) = \delta(x) Z_k(u, v),$$

where $Z_k(u, v)$ is the reproducing kernel of $\mathcal{H}_k$. Hence, Theorem 9.7 is proved and the even order case is resolved.

9.5 Fermionic operators: odd order, half-integer spin

We denote the $(2j-1)$-th fermionic operator

$$\mathcal{D}_{2j-1} : \ C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k)$$

as the generalization of D_x^{2j-1} in Euclidean space to higher spin spaces. With similar arguments as in the bosonic case, it is conformally invariant with the following intertwining operators

$$\frac{\widetilde{cx+d}}{||cx+d||^{m+2j}} \mathcal{D}_{2j-1,y,\omega} f(y,\omega) = \mathcal{D}_{2j-1,x,u} \frac{\widetilde{cx+d}}{||cx+d||^{m-2j+2}} f(\phi(x), \frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}),$$

where $y = \phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and
$\omega = \dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$. Furthermore, its fundamental solution is

$$c \frac{x}{||x||^{m-2j+2}} Z_k(\frac{xux}{||x||^2}, v),$$

where c is a non-zero real constant and $Z_k(u, v)$ is the reproducing kernel of $\mathcal{M}_k$. Here is our second main theorem.

Theorem 9.10. *Let $Z_k(u, v)$ be the reproducing kernel of $\mathcal{M}_k$. When $j > 1$, the $(2j-1)$-th* *order conformally invariant differential operator on $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ is the $(2j-1)$-th order fermionic operator*

$$\mathcal{D}_{2j-1} = R_k \prod_{s=1}^{j-1} \Big(-R_k^2 - \frac{4s^2 T_k T_k^*}{(m+2k-2s-2)(m+2k+2s-2)} \Big)$$

that has the fundamental solution

$$\lambda_{2s} \frac{x}{||x||^{m-2j+2}} Z_k(\frac{xux}{||x||^2}, v),$$

where

$$T_k = (1 + \frac{uD_u}{m+2k-2})D_x \text{ and } T_k^* = \frac{uD_u D_x}{m+2k-2}$$

are the twistor and dual twistor operators defined in [10] and λ_{2s} is a non-zero real constant whose expression is given later in this section.

To prove Theorem 9.10, we start with the following proposition.

Proposition 9.11. *For any $f_k(u) \in \mathcal{M}_k$, we denote*

$$B_{m-\beta} = \Delta_x + a_{m-\beta}||u||^2 \langle D_u, D_x\rangle^2 + b_{m-\beta}\langle u, D_x\rangle\langle D_u, D_x\rangle + c_{m-\beta} u \langle D_u, D_x\rangle D_x.$$

When $\beta \leq m-2$, we have

$$B_{m-\beta} \frac{x}{||x||^\beta} f_k(\frac{xux}{||x||^2}) = d_{m-\beta} \frac{x}{||x||^{\beta+2}} f_k(\frac{xux}{||x||^2})$$

in the distribution sense, where

$$\begin{aligned}
a_{m-\beta} &= \frac{4}{(\beta+2k-2)(2m+2k-\beta-2)};\\
b_{m-\beta} &= -\frac{4(m+2k-2)}{(\beta+2k-2)(2m+2k-\beta-2)};\\
c_{m-\beta} &= -\frac{4}{(\beta+2k-2)(2m+2k-\beta-2)};\\
d_{m-\beta} &= (\beta+2k)(\beta+2k-m)+2k(m-2\beta-2k-2)+\frac{4k(m+2k-2)}{\beta+2k-2}.
\end{aligned}$$

It is worth pointing out that if $\beta = m-2s$, then B_{2s} is exactly the term in the parenthesis in Theorem 9.10. Details can be found later in this section.

In order to prove the above proposition with arbitrary functions $f_k(u) \in \mathcal{M}_k$, as stated in [7], we can rely on the fact that $\mathcal{M}_k$ is an irreducible $Spin(m)$-representation generated by the highest weight vector $\langle u, 2\mathfrak{f}_1\rangle^k I$, where I is defined in *Section 2.2.1*. It suffices to prove the statement for

$$\frac{x}{||x||^{\beta}}\langle\frac{xux}{||x||^2}, 2\mathfrak{f}_1\rangle^k I = \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I = \frac{x}{||x||^{\beta+2k}}\langle u||x||^2-2\langle u,x\rangle x, 2\mathfrak{f}_1\rangle^k I.$$

First, we assume that $x\neq 0$, and we have the following technical lemmas.

Lemma 9.12.

$$\begin{aligned}
&\Delta_x \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I\\
&= \big[(\beta+2k)(\beta+2k-m)+2k(m-2\beta-2k-2)\big]\frac{x}{||x||^{\beta+2k+2}}\langle xux, 2\mathfrak{f}_1\rangle^k I\\
&-4k\frac{u\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I + 4k(m+2k-2)\frac{x}{||x||^{\beta+2k+2}}\langle u,x\rangle\langle x, 2\mathfrak{f}_1\rangle\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I\\
&+4k(k-1)||u||^2\langle x, 2\mathfrak{f}_1\rangle^2\frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-2} I.
\end{aligned}$$

Proof. Since

$$\Delta_x \frac{x}{||x||^{\beta+2k}} = (\beta+2k)(\beta+2k-m)\frac{x}{||x||^{\beta+2k+2}}$$

and [7] gives

$$\begin{aligned} & \Delta_x \langle xux, 2\mathfrak{f}_1\rangle^k \\ = \;& 4k(k-1)||u||^2\langle x, 2\mathfrak{f}_1\rangle^2\langle xux, 2\mathfrak{f}_1\rangle^{k-2}I + 2k(m+2k-4)\langle u, 2\mathfrak{f}_1\rangle\langle xux, 2\mathfrak{f}_1\rangle^{k-1}I, \end{aligned}$$

we have

$$\begin{aligned} & \Delta_x \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I \\ = \;& \Delta_x(\frac{x}{||x||^{\beta+2k}})\langle xux, 2\mathfrak{f}_1\rangle^k I + \frac{x}{||x||^{\beta+2k}}\Delta_x(\langle xux, 2\mathfrak{f}_1\rangle^k)I \\ & +2\sum_{i=1}^m \partial_{x_i}\frac{x}{||x||^{\beta+2k}}\partial_{x_i}\langle xux, 2\mathfrak{f}_1\rangle^k I \\ = \;& (\beta+2k)(\beta+2k-m)\frac{x}{||x||^{\beta+2k+2}}\langle xux, 2\mathfrak{f}_1\rangle^k I \\ & +\frac{x}{||x||^{\beta+2k}}\big(4k(k-1)||u||^2\langle x, 2\mathfrak{f}_1\rangle^2\langle xux, 2\mathfrak{f}_1\rangle^{k-2}I \\ & +2k(m+2k-4)\langle u, 2\mathfrak{f}_1\rangle\langle xux, 2\mathfrak{f}_1\rangle^{k-1}\big)I \\ & +2k\sum_{i=1}^m(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}})\langle 2ux_i - 2u_i x - 2\langle u, x\rangle e_i, 2\mathfrak{f}_1\rangle\langle xux, 2\mathfrak{f}_1\rangle^{k-1}I. \end{aligned}$$

Notice that $I = \mathfrak{f}_1\mathfrak{f}_1^\dagger\mathfrak{f}_2\mathfrak{f}_2^\dagger\cdots\mathfrak{f}_n\mathfrak{f}_n^\dagger$ and $\mathfrak{f}_1^2 = 0$. Therefore, we obtain

$$\begin{aligned} & = (\beta+2k)(\beta+2k-m)\frac{x}{||x||^{\beta+2k+2}}\langle xux, 2\mathfrak{f}_1\rangle^k I \\ & +4k(k-1)||u||^2\langle x, 2\mathfrak{f}_1\rangle^2\frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-2}I \\ & +2k(m-2\beta-2k-2)\frac{x}{||x||^{\beta+2k}}\langle u, 2\mathfrak{f}_1\rangle\langle xux, 2\mathfrak{f}_1\rangle^{k-1}I \\ & -4k\frac{u\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1}I + 8k(\beta+2k)\frac{x\langle u, x\rangle\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k+2}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1}I. \end{aligned}$$

With the help of $\langle u||x||^2, 2\mathfrak{f}_1\rangle = \langle xux, 2\mathfrak{f}_1\rangle + 2\langle u, x\rangle\langle x, 2\mathfrak{f}_1\rangle$, this lemma is proved immediately. $\square$

Lemma 9.13.

$$
\begin{aligned}
& ||u||^2\langle D_u, D_x\rangle^2 \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I \\
= \;& k(k-1)(2m-\beta+2k-2)(2m-\beta+2k-4)\frac{||u||^2 x}{||x||^{\beta+2k}}\langle x, 2\mathfrak{f}_1\rangle^2\langle xux, 2\mathfrak{f}_1\rangle^{k-2} I; \qquad (4)\\
& u\langle D_u, D_x\rangle D_x \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I \\
= \;& -k(2m-\beta+2k-2)\Bigg[(\beta-m)\frac{u\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
& +2(k-1)||u||^2\frac{x}{||x||^{\beta+2k}}\langle x, 2\mathfrak{f}_1\rangle^2\langle xux, 2\mathfrak{f}_1\rangle^{k-2} I\Bigg]; \qquad (5)\\
& \langle u, D_x\rangle\langle D_u, D_x\rangle \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I \\
= \;& -k(2m-\beta+2k-2)\Bigg[\frac{x\langle xux, 2\mathfrak{f}_1\rangle^k}{||x||^{\beta+2k+2}} I - (\beta+2k-2)\frac{x\langle u, x\rangle\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k+2}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
& +\frac{u\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I - 2(k-1)\frac{||u||^2 x}{||x||^{\beta+2k}}\langle x, 2\mathfrak{f}_1\rangle^2\langle xux, 2\mathfrak{f}_1\rangle^{k-2} I\Bigg] \qquad (6)
\end{aligned}
$$

Proof. Since these three operators on the left contain $\langle D_u, D_x\rangle$, first let us check:

$$
\begin{aligned}
& \langle D_u, D_x\rangle \frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^k I = \sum_{i=1}^m \partial_{u_i}\Bigg(\Big(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}}\Big)\langle xux, 2\mathfrak{f}_1\rangle^k I \\
& +k\frac{x}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1}\langle 2ux_i - 2u_i x - 2\langle u, x\rangle e_i, 2\mathfrak{f}_1\rangle I\Bigg) \\
= \;& \sum_{i=1}^m k\Big(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}}\Big)\langle xux, 2\mathfrak{f}_1\rangle^{k-1}\langle e_i||x||^2 - 2x_i x, 2\mathfrak{f}_1\rangle I \\
& +\sum_{i=1}^m k(k-1)\frac{x\langle xux, 2\mathfrak{f}_1\rangle^{k-2}}{||x||^{\beta+2k}}\langle e_i||x||^2 - 2x_i x, 2\mathfrak{f}_1\rangle\langle 2ux_i - 2u_i x - 2\langle u, x\rangle e_i, 2\mathfrak{f}_1\rangle I \\
& +\sum_{i=1}^m k\frac{x\langle xux, 2\mathfrak{f}_1\rangle^{k-1}}{||x||^{\beta+2k}}\langle 2e_i x_i - 2x - 2x_i e_i, 2\mathfrak{f}_1\rangle I.
\end{aligned}
$$

The last expression simplifies as

$$-k(2m-\beta+2k-2)\frac{x\langle x,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux,2\mathfrak{f}_1\rangle^{k-1}I.$$

Hence, to verify Eq. (4), we only need to check

$$\begin{aligned}
&\langle D_u,D_x\rangle\frac{x\langle x,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux,2\mathfrak{f}_1\rangle^{k-1}I\\
=\;&\sum_{i=1}^{m}\partial_{u_i}\Big(\big(\frac{e_i}{||x||^{\beta+2k}}-\frac{(\beta+2k)x_ix}{||x||^{\beta+2k+2}}\big)\langle x,2\mathfrak{f}_1\rangle\langle xux,2\mathfrak{f}_1\rangle^{k-1}I+\frac{x\langle e_i,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux,2\mathfrak{f}_1\rangle^{k-1}I\\
&+(k-1)\frac{x\langle x,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux,2\mathfrak{f}_1\rangle^{k-2}\langle 2ux_i-2u_ix-2\langle u,x\rangle e_i,2\mathfrak{f}_1\rangle I\Big)\\
=\;&\sum_{i=1}^{m}\big(\frac{e_i}{||x||^{\beta+2k}}-\frac{(\beta+2k)x_ix}{||x||^{\beta+2k+2}}\big)\langle x,2\mathfrak{f}_1\rangle(k-1)\langle xux,2\mathfrak{f}_1\rangle^{k-2}\langle e_i||x||^2-2x_ix,2\mathfrak{f}_1\rangle I\\
&+\sum_{i=1}^{m}\frac{x\langle e_i,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}(k-1)\langle xux,2\mathfrak{f}_1\rangle^{k-2}\langle e_i||x||^2-2x_ix,2\mathfrak{f}_1\rangle I\\
&+\sum_{i=1}^{m}(k-1)\frac{x\langle x,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux,2\mathfrak{f}_1\rangle^{k-2}\langle 2e_ix_i-2x-2x_ie_i,2\mathfrak{f}_1\rangle I\\
&+\sum_{i=1}^{m}(k-1)\frac{x\langle x,2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}(k-2)\langle xux,2\mathfrak{f}_1\rangle^{k-3}\\
&\quad\cdot\langle 2ux_i-2u_ix-2\langle u,x\rangle e_i,2\mathfrak{f}_1\rangle\langle e_i||x||^2-2x_ix,2\mathfrak{f}_1\rangle I.
\end{aligned}$$

This last expression simplifies as

$$-(k-1)(2m-\beta+2k-4)\frac{x\langle x,2\mathfrak{f}_1\rangle^2}{||x||^{\beta+2k}}\langle xux,2\mathfrak{f}_1\rangle^{k-2}I.$$

Hence, Eq. (4) is verified.

For Eq. (5), we check

$$\begin{aligned}
& uD_x \frac{x\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
= \;& u\sum_{i=1}^{m} e_i\Big(\big(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}}\big)\langle x, 2\mathfrak{f}_1\rangle \langle xux, 2\mathfrak{f}_1\rangle^{k-1} I + \frac{x\langle e_i, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
& +(k-1)\frac{x\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-2}\langle 2ux_i - 2u_i x - 2\langle u, x\rangle e_i, 2\mathfrak{f}_1\rangle I\Big) \\
= \;& u\Big[\frac{(\beta+2k-m)\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I - 2\frac{\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
& -2(k-1)\frac{\langle x, 2\mathfrak{f}_1\rangle\langle u||x||^2, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-2} I \\
& -2(k-1)\frac{ux}{||x||^{\beta+2k}}\langle x, 2\mathfrak{f}_1\rangle^2 \langle xux, 2\mathfrak{f}_1\rangle^{k-2} I + 4(k-1)\frac{\langle u, x\rangle\langle x, 2\mathfrak{f}_1\rangle^2}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-2} I\Big] \\
= \;& (\beta - m)\frac{u\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I + 2(k-1)\frac{||u||^2 x}{||x||^{\beta+2k}}\langle x, 2\mathfrak{f}_1\rangle^2 \langle xux, 2\mathfrak{f}_1\rangle^{k-2} I.
\end{aligned}$$

For Eq. (6), we check

$$\begin{aligned}
& \langle u, D_x\rangle \frac{x\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
= \;& \sum_{i=1}^{m} u_i\Big(\big(\frac{e_i}{||x||^{\beta+2k}} - \frac{(\beta+2k)x_i x}{||x||^{\beta+2k+2}}\big)\langle x, 2\mathfrak{f}_1\rangle \langle xux, 2\mathfrak{f}_1\rangle^{k-1} I + \frac{x\langle e_i, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} \\
& +(k-1)\frac{x\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-2}\langle 2ux_i - 2u_i x - 2\langle u, x\rangle e_i, 2\mathfrak{f}_1\rangle I\Big) \\
= \;& \frac{x\langle xux, 2\mathfrak{f}_1\rangle^k}{||x||^{\beta+2k+2}} I - (\beta+2k-2)\frac{x\langle u, x\rangle\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k-2}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I \\
& +\frac{u\langle x, 2\mathfrak{f}_1\rangle}{||x||^{\beta+2k}}\langle xux, 2\mathfrak{f}_1\rangle^{k-1} I - 2(k-1)\frac{||u||^2 x}{||x||^{\beta+2k}}\langle x, 2\mathfrak{f}_1\rangle^2 \langle xux, 2\mathfrak{f}_1\rangle^{k-2} I.
\end{aligned}$$

Therefore, Eqs. (5) and (6) are verified. $\square$

Recall the fact mentioned in the $2j$-th order bosonic operator case, $||x||^\alpha$ is weak differentiable if $\alpha > -m+1$ with weak derivative $\partial_{x_i}||x||^\alpha = \alpha x_i ||x||^{\alpha-2}$. Hence, when $\beta \leq m-2$, Lemmas 9.12 and 9.13 are both true in the distribution sense. Combining them completes the proof of Proposition 9.11. With the help of Proposition 9.11 and similar arguments as in Proposition 9.9, we have the following proposition by induction.

Proposition 9.14. *Let $f_k(u) \in \mathcal{M}_k$. For integers $j > 1$,*

$$\left[\prod_{s=1}^{j-1} B_{2s} d_{2s}^{-1}\right] \frac{x}{||x||^{m-2j+2}} f_k(\frac{xux}{||x||^2}) = \frac{x}{||x||^m} f_k(\frac{xux}{||x||^2})$$

in the distribution sense, where $a_{2s},\ b_{2s},\ c_{2s},\ d_{2s}$ are defined as in Proposition 9.11 with $\beta = m - 2s$.

In the above proposition, it is worth pointing out that

$$B_{2s_1} B_{2s_2} = B_{2s_2} B_{2s_1},$$

where $s_1 \neq s_2$. Indeed, with a straightforward calculation, one can get

$$R_k^2 = -\Delta_x + \frac{4\langle u, D_x\rangle\langle D_u, D_x\rangle}{m+2k-2} - \frac{4||u||^2\langle D_u, D_x\rangle^2}{(m+2k-2)^2} + \frac{4u\langle D_u, D_x\rangle D_x}{(m+2k-2)^2}.$$

Then

$$\begin{aligned} B_{2s} &= \Delta_x + \frac{4\big(||u||^2\langle D_u, D_x\rangle^2 - (m+2k-2)\langle u, D_x\rangle\langle D_u, D_x\rangle^2 - u\langle D_u, D_x\rangle D_x\big)}{(m+2k-2s-2)(m+2k+2s-2)} \\ &= \Delta_x - \frac{(m+2k-2)^2}{(m+2k-2s-2)(m+2k+2s-2)}(R_k^2 + \Delta_x). \end{aligned} \tag{7}$$

So B_{2s} is a linear combination of R_k^2 and Δ_x. This is no surprise, since [8] points out

$$\{R_k^i \Delta_x^j,\ 0 \leq i \leq min(2p+1, 2k+1),\ 0 \leq j,\ i + 2j = p\}$$

is the basis of the space of $Spin(m)$-invariant constant coefficient differential operators of order p on $\mathcal{M}_k$. $\mathcal{D}_{2j-1}$ is conformally invariant, so it is also $Spin(m)$-invariant and hence can be expressed in this basis. Furthermore, with the help of $-\Delta_x = R_k^2 - T_kT_k^*$ and Eq. (7), we can also rewrite B_{2s} in terms of first order conformally invariant operators:

$$B_{2s} = -R_k^2 - \frac{4s^2 T_k T_k^*}{(m+2k-2s-2)(m+2k+2s-2)}.$$

Now, we have fundamental solution of $\mathcal{D}_{2j-1}$ restated as follows.

Theorem 9.15. *Let $Z_k(u,v)$ be the reproducing kernel of $\mathcal{M}_k$. When $j>1$, the $(2j-1)$-th order fermionic operator $\mathcal{D}_{2j-1}$ has fundamental solution*

$$\lambda_{2s}\frac{x}{||x||^{m-2j+2}}Z_k(\frac{xux}{||x||^2},v),\ \ \lambda_{2s} = \frac{-(m+2k-2)}{(m-2)\omega_{m-1}}\prod_{s=1}^{j-1} d_{2s}^{-1},$$

where d_{2s} is defined in Proposition 9.11 with $\beta = m-2s$ and ω_{m-1} is the area of the $(m-1)$-dimensional unit sphere.

Proof. With the help of Proposition 9.14 and noticing that

$$\frac{-(m+2k-2)}{(m-2)\omega_{m-1}}\frac{x}{||x||^m}Z_k(\frac{xux}{||x||^2},v)$$

is the fundamental solution of R_k [6], the above theorem follows immediately. □

Hence, Theorem 9.10 is proved and the odd order case is resolved.

On a concluding note, using similar arguments as in our previous paper [13], one can generalize our conformally invariant differential operators to conformally flat spin manifolds in the fermionic case and conformally flat Riemannian manifolds in the bosonic case. Further, if M is a conformally flat manifold with spin structure then the conformal weight structure allows us to lift the fermionic differential operators to act on sections on certain bundles. This will be developed more formally elsewhere.

10 Future work

We have finished the work of constructing conformally invariant differential operators in higher spin spaces. However, there are still many interesting problems related to it. I list some of them below as my future work.

1. In [43], one can find that Cauchy Green type formulas exist for the kth-power of the Dirac operator D_x^k. Since our fermionic and bosonic operators are generalizations of D_x^k to higher spin spaces, they should also have Cauchy Green type formulas. Such integral formulas for the first order conformally invariant differential operators (the Rarita-Schwinger operators) have already been found [6, 15]. For the other higher order cases, integral formulas are still undiscovered.

2. At the end of Section 9, we mentioned that our differential operators can be generalized to conformally spin (Riemannian) flat manifolds in the fermionic (bosonic) case. This should be clarified in detail. Further, intertwining operators and fundamental solutions should also be explored in these circumstances.

3. The first order conformally invariant differential operators in higher spin spaces are called Rarita-Schwinger operators, which map from $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$ to $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$. In Section 5, we also point out that there are other conformally invariant Rarita-Schwinger type operators:

$$Twistor\ operator : T_k :\ C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_k);$$
$$Dual\ Twistor\ operator : T_k^* :\ C^\infty(\mathbb{R}^m, \mathcal{M}_k) \longrightarrow C^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}).$$

   De Bie et al. [7] also find the following conformally invariant twistor and dual twistor operators:

$$Twistor\ operator : T_k :\ C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_k);$$

$$Dual\ Twistor\ operator : T_k^* :\ C^\infty(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_{k-1}).$$

Hence, higher order twistor and dual twistor operators should be investigated in the future.

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