5-2014

Closed-Range Composition Operators on Weighted Bergman Spaces and Applications

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctorate of Philosophy in Mathematics

by

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May 2014
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This dissertation is approved for recommendation to the Graduate Council.

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Abstract

We will discuss necessary and sufficient conditons for the composition operator C_φ to be closed range on the weighted Bergman space $\mathbb{A}^p_\alpha$ for $1 \leq p < \infty$ with weights of the form $(1-|z|^2)^\alpha$ for $\alpha > -1$. The function φ is an analytic self-map of the unit disk $\mathbb{D}$ and our results extend those previously intended for the classical Bergman space $\mathbb{A}^2$. We will also give applications.

Acknowledgements

I would like to express my deepest gratitude to my advisor, Dr. John Akeroyd. His amazing patience, invaluable guidance, and constant encouragement made this project possible. I would also like to express my gratefulness to my committee members, Dr. Maria Tjani and Dr. Dan Luecking, for all their helpful advice and support. Special thanks are due to Dr. Kristen Karber for her help with Advanced Grapher and with inserting images in Latex. Many thanks to Dr. Jennifer Paulk and Dr. Brad Lutes for listening and for offering the wisdom of their own experiences. Last, but certainly not least, I want to thank my fiancé, Tim Hood, for always being there.

Contents

1 Introduction

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let m denote normalized Lebesgue measure on $\mathbb{T}$, and let A denote normalized two-dimensional Lebesgue measure on the unit disk $\mathbb{D}$. For a point a in $\mathbb{D}$ and r, $0 < r < 1$, let $D(a, r) := \{z \in \mathbb{D} : |z - a| < r\}$ and let $\Delta(a, r) := \{z \in \mathbb{D} : \rho(z, a) < r\}$ where $\rho(z, a)$ denotes the *pseudohyperbolic metric* defined for z and a in $\mathbb{D}$ by

$$\rho(z, a) = \frac{|z - a|}{|1 - \bar{z}a|}.$$

For a in $\mathbb{D}$ and $0 \leq b \leq 1$, let $D_b(a)$ denote a disk centered at a with radius $b(1 - |a|)$. We let $\mathcal{H}(\mathbb{D})$ be the set of all functions f which are analytic in $\mathbb{D}$. A function φ is said to be an *analytic self-map of the unit disk* $\mathbb{D}$ if $\varphi \in \mathcal{H}(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If φ is an analytic self-map of $\mathbb{D}$, then the composition operator C_φ is defined on $\mathcal{H}(\mathbb{D})$ by $C_\varphi(f) = f \circ \varphi$. If X is a Banach space of analytic functions in $\mathbb{D}$, then we say that a composition operator C_φ on a space X is *compact* if every bounded set in X is mapped to a set whose closure is compact. The composition operator C_φ is said to be *closed-range* on X if $C_\varphi(X)$ is a closed subspace of X. By the Open Mapping Theorem, for nontrivial φ, this occurs when there exists a constant $c > 0$ such that $||f \circ \varphi||_X \geq c||f||_X$ for all f in X. For $1 \leq p < \infty$, the Hardy space H^p is the set of all functions f in $\mathcal{H}(\mathbb{D})$ such that

$$||f||_{H^p}^p := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^p dm(\xi) < \infty$$

and H^∞ is the set of all functions f in $\mathcal{H}(\mathbb{D})$ such that

$$||f||_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

For $\alpha > -1$ we define dA_α by $dA_\alpha := c_\alpha \cdot (1 - |z|^2)^\alpha dA(z)$, where $c_\alpha = \alpha + 1$. The weighted Bergman space $\mathbb{A}^p_\alpha$ is given by:

$$\mathbb{A}^p_\alpha := \{f : f \in \mathcal{H}(\mathbb{D}) \text{ and } ||f||^p_{p,\alpha} = \int_{\mathbb{D}} |f|^p dA_\alpha < \infty\}.$$

In this chapter, we will discuss several classical Banach spaces of analytic functions on the unit disk. We will review standard results describing when a composition operator is bounded, compact, and closed-range on these spaces. In chapter 2, we will give a necessary and sufficient condition for when C_φ is closed-range on the weighted Bergman space A^p_α. Note that in [28], Nina Zorboska gives conditions for when C_φ is closed-range on the Hardy space H^2 and the weighted Bergman space A^2_α. These conditions involve the Nevanlinna counting function, which can be difficult to work with.

A good reference for the following discussion is [19]. For $0 < r < 1$ and a point ξ in $\mathbb{T}$, let $S(\xi, r)$ denote the interior of the convex hull of the union of $\{\xi\}$ and $\{z \in \mathbb{D} : |z| \leq r\}$. We call $S(\xi, r)$ the *Stolz region* based at $\xi \in \mathbb{T}$ with contact angle $2 \arctan(r)$.

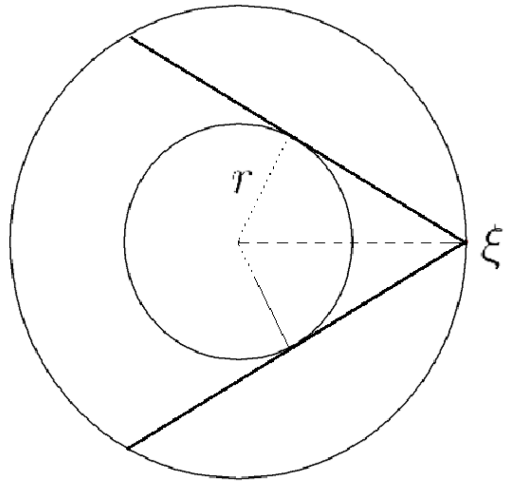

Figure 1: A stolz region

Note that if a curve approaches ξ from inside the region $S(\xi, r)$, then this curve cannot be tangent to the unit circle. For f in H^p, we say f has nontangential limit L at the point ξ if for all r in $(0, 1)$ and for every sequence $\{z_n\}$ in $S(\xi, r)$ that converges to the point ξ, we have $\lim_{n\to\infty} f(z_n) = L$. Also, for r in $(0, 1)$ and for any complex function f defined on $\mathbb{D}$, we define the *nontangential maximal function* $N_r f$ on $\mathbb{T}$ by

$$N_r f(\xi) = \sup\{|f(z)| : z \in S(\xi, r)\}.$$

For any f in H^p, $0 < p < \infty$ and any r in $(0, 1)$, we have $N_r f \in L^p(\mathbb{T})$. It is well known (see [19]) that the nontangential limits of f in H^p, denoted $f^*(e^{i\theta})$, exist almost everywhere $[m]$ on $\mathbb{T}$ and $f^* \in L^p(\mathbb{T})$. Furthermore, $||f^*||_p = ||f||_p$ for all f in H^p. An *inner function* is a function M in H^∞ such that $|M^*| = 1$ a.e. $[m]$. A function of the form

$$S_\mu(z) = \exp\{-\int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z} d\mu(t)\},$$

where μ is a positive Borel measure on $\mathbb{T}$ that is singular with respect to m, is known as a *singular inner function.* Note that such a function does not have any zeros in $\mathbb{D}$. Let $\{a_n\}$ be a sequence of points in $\mathbb{D}$ such that $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$. For such sequences, there is an associated *Blaschke product* B defined on $\mathbb{D}$ by

$$B(z)=\prod_{n=1}^{\infty}\frac{a_n-z}{1-\bar{a_n}z}\frac{|a_n|}{a_n},$$

where $\frac{|a_n|}{a_n}$ is taken to be 1 if $a_n=0$. The function B is in H^∞ and $|B^*(e^{i\theta})|=1$ almost everywhere on $\mathbb{T}$. Hence, each Blaschke product is an inner function, as is each singular inner function. Now, every inner function M can be factored uniquely as the product of a Blaschke product and a singular inner function. That is, every inner function M may be written in the form

$$M(z)=c\cdot B(z)\cdot S_\mu(z)$$

where c is a constant such that $|c|=1$. An *outer function* is a function of the form

$$G(z)=c\cdot\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\varphi(e^{it})dt\right\}$$

where c is a constant such that $|c|=1$, and φ is a positive measurable function on $\mathbb{T}$ such that $\log\varphi\in L^1(\mathbb{T})$. For $0<p\leq\infty$ and f in H^p such that f is not identically zero, the function $\log|f^*|$ is in $L^1(\mathbb{T})$ and

$$G_f(z)=\exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|f^*(e^{it})|\}dt$$

is an outer function in H^p. For such f, there exists an inner function M_f such that $f=M_fG_f$. Thus, for all $p>0$, every f in H^p may be factored uniquely into the product of an inner function and an outer function. Thus, by our previous statement regarding the factorization of inner functions, we have that every f in H^p may be written uniquely in the form

$$f(z)=c\cdot B(z)\cdot S_\mu(z)\cdot G_f(z)$$

for z in $\mathbb{D}$, where G_f is an outer function in H^p, see [19].) Note that any analytic self-map φ of the unit disk $\mathbb{D}$ is in H^∞. Hence, by our above discussion, φ may be factored as above and so has only a few possible forms. The function φ may be written as a Blaschke product, a singular inner function, an outer function, or it may be written as a product of these types of functions.

It is natural to ask for which φ is the composition operator C_φ bounded, compact, or closed-range on a Banach space of analytic functions on $\mathbb{D}$. We will now catalog such results for various classical Banach spaces. These results are standard in the literature and more information can be found in [22] and [27]. We begin by examining composition operators on the Hardy space H^2.

1.1 Composition Operators on the Hardy Space H^2

Littlewood's Subordination Principle (see [22]) states that if φ is an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$, then for each f in H^2, $C_\varphi(f) \in H^2$ and $||C_\varphi(f)|| \leq ||f||$.

Thus, if φ fixes the origin, then C_φ is bounded on H^2. To see that this is the case for any holomorphic self-map φ of $\mathbb{D}$, we will use $\alpha_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$, the special automorphism of $\mathbb{D}$ where $\alpha_\lambda(\lambda) = 0$, $\alpha_\lambda(0) = \lambda$, and $\alpha_\lambda^{-1} = \alpha_\lambda$. Letting $\lambda = \varphi(0)$, we consider the function $\psi = \alpha_\lambda \circ \varphi$ which is a holomorphic self-map of $\mathbb{D}$ that fixes the origin. Then, $\varphi = \alpha_\lambda^{-1} \circ \psi = \alpha_\lambda \circ \psi$ and by Littlewood's Subordination Principle, for all f in H^2 we have

$$\begin{aligned}||f \circ \varphi||^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2 d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f \circ \alpha_\lambda \circ \psi(e^{i\theta})|^2 d\theta\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\int_0^{2\pi} |f\circ\alpha_\lambda(e^{i\theta})|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 |\alpha'_\lambda(e^{it})| dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \frac{1-|\lambda|^2}{|1-\overline{\lambda}e^{it}|^2} dt \\
&\leq \frac{1-|\lambda|^2}{(1-|\lambda|)^2}\cdot\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt \\
&= \frac{1+|\lambda|}{1-|\lambda|}\cdot ||f||^2
\end{aligned}
$$

Thus we have that C_φ is bounded on H^2 for every analytic self-map φ of $\mathbb{D}$. $\square$

We may also address the question of compactness of composition operators. The *First Compactness Theorem* (page 23 in [22]) states that the composition operator C_φ is a compact operator on H^2 if $||\varphi||_\infty < 1$. In other words, C_φ is a compact composition operator if $\varphi(\mathbb{D})$ is relatively compact. The Univalent Compactness theorem (see page 39 in [22]) says that if φ is a univalent self-map of $\mathbb{D}$, then, C_φ is compact on H^2 if and only if

$$
\lim_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|} = \infty.
$$

It should be noted that necessity in this theorem does not require univalence. As this requirement for compactness deals with a difference quotient, it is reasonable to think that there may be some relationship between this condition and the derivative of φ at the boundary of the disk.

Let φ be a holomorphic self-map of $\mathbb{D}$, and let ω be a point on $\partial\mathbb{D}$. We say that φ has angular limit $\mathcal{L} = \angle\lim_{z\to\omega}\varphi(z)$ if $\varphi(z)\to\mathcal{L}$ as $z\to\omega$ through any stolz region based at ω. The map φ has an angular derivative at ω, denoted $\varphi'(\omega)$, if for some point η in $\partial\mathbb{D}$,

$$
\angle\lim_{z\to\omega}\frac{\eta-\varphi(z)}{\omega-z}
$$

exists. This suggests that the angular limit of φ at ω exists and is equal to η. Hence, if φ has an angular derivative at any point on $\partial\mathbb{D}$, then it must have an angular limit of modulus one at that point.

The *Julia-Caratheodory Theorem* clarifies the relationship between compactness and the existence of angular derivatives. This theorem states that the angular derivative $\angle\lim_{z\to\omega}\frac{\eta-\varphi(z)}{\omega-z}$ exists for some η in $\partial\mathbb{D}$ if and only if $\liminf_{z\to\omega}\frac{1-|\varphi(z)|}{1-|z|}=\delta$ for some δ, $0<\delta<\infty$. But, by the *Univalent Compactness theorem*, $\liminf_{z\to\omega}\frac{1-|\varphi(z)|}{1-|z|}<\infty$ implies that C_φ is not compact on H^2.

Next, we discuss when C_φ is compact on H^2 for arbitrary self-maps φ of $\mathbb{D}$. In other words, we want a condition for compactness on H^2 when φ is not necessarily univalent. For a function φ holomorphic on $\mathbb{D}$, the Nevanlinna Counting Funtion of φ, denoted N_φ, is defined as follows:

$$N_\varphi(w)=\begin{cases}\sum_{z\in\varphi^{-1}\{w\}}\log\frac{1}{|z|} & w\in\varphi(\mathbb{D})\\ 0 & w\notin\varphi(\mathbb{D})\end{cases}$$

.

For a function f analytic on $\mathbb{D}$, the *Littlewood-Paley Identity* (see[22]) gives that

$$||f||^2=|f(0)|^2+2\int_{\mathbb{D}}|f'(z)|^2\log\frac{1}{|z|}dA(z).$$

The *change-of-variable formula* (see [22]) states that for any analytic map φ on $\mathbb{D}$,

$$||C_\varphi(f)||_2^2=|f(\varphi(0))|_2^2+2\int_{\mathbb{D}}|f'(w)|^2N_\varphi(w)dA(w).$$

Notice that if φ is univalent, then the change-of-variable formula is just the Littlewood-Paley Identity with the substitution $w=\varphi(z)$.

Theorem 2.3 in [20] gives the following result. Suppose φ is a holomorphic self-map of $\mathbb{D}$. Then, C_φ is compact on H^2 if and only if

$$\lim_{|w|\to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0.$$

If φ is univalent, we have

$$N_\varphi(w) = \log \frac{1}{|z|} \approx 1 - |z|$$

for $|z|$ large, where $\varphi(z) = w$. Thus, in the case that φ is univalent, this theorem is the same as the Univalent Compactness Theorem stated above.

In [3], it is shown that this condition on φ involving the Nevalinna counting function is equivalent to the condition

$$\lim_{|a|\to 1^-} \int_{\mathbb{T}} \frac{1-|a|^2}{|1-\bar{a}\varphi(z)|^2} dm(z) = 0.$$

In [28], Nina Zorboska gives conditions regarding when a composition operator C_φ will be closed-range on H^2 and on $\mathbb{A}^2_\alpha$ for $\alpha > -1$. The function $\pi_\varphi(w)$, having domain $\mathbb{D} \setminus \varphi(0)$, is defined by

$$\pi_\varphi(w) = \frac{N_\varphi(w)}{\log \frac{1}{|w|}},$$

where $N_\varphi(w)$ is the Nevanlinna Counting Function as defined above. For a positive constant c, the set G_c^φ is defined by

$$G_c^\varphi = \{z : \pi_\varphi(z) > c\}.$$

Theorem 3.4 in [28] states that a composition operator C_φ will be closed-range on H^2 if and only if there exist positive constants c and δ such that

$$A(G_c^\varphi \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r)) \tag{1}$$

for all ξ in $\partial\mathbb{D}$, where $D(\xi, r) = \{z \in \mathbb{D} : |z - \xi| < r\}$. In [14], it is shown that condition (1) may be restated as follows. There exist constants $\delta_1 > 0$ and b, $0 < b < 1$, so that

$$A(G_c^{\varphi} \cap D_b(a)) > \delta_1 \cdot A(D_b(a)) \tag{2}$$

for every a in D where $D_b(a) = \{z \in \mathbb{D} : |z - a| < b(1 - |a|)\}$.

A good reference for the following discussion is [28]. Suppose φ is a univalent function such that C_φ does not have closed-range on H^2. Let ψ be a holormorphic self-map of $\mathbb{D}$ such that $\psi(\mathbb{D})$ is contained in $\varphi(\mathbb{D})$. Let ω be the self-map of $\mathbb{D}$ defined by $\omega := \varphi^{-1} \circ \psi$. Then, $\psi = \varphi \circ \omega$. Let $\{f_n\}$ be a sequence of functions in H^2 such that $||f_n||_{H^2} = 1$ and $||C_\varphi f_n||_{H^2} \to 0$. Then,

$$||C_\psi f_n||_{H^2} = ||f_n \circ \varphi \circ \omega||_{H^2} \leq ||C_\omega||_{H^2} ||f_n \circ \varphi||_{H^2} \to 0.$$

Hence, C_ψ will not be closed-range on H^2.

Example 1 in [28] states that if there exists a point $\xi \in \mathbb{T}$ and a neighborhood N_ξ about the point ξ such that $N_\xi \cap \varphi(\mathbb{D}) = \emptyset$, then C_φ will not be closed-range on H^2. To see that this is the case, choose a Euclidean disk $D(\xi, r)$ to be contained in N_ξ. Then, for all $z \in D(\xi, r)$, we have that $\gamma_\varphi(z) = 0$ and so the set G_c^{φ} is empty for all $c > 0$. Hence, for any $c > 0$, $A(G_c^{\varphi} \cap D(\xi, r)) = 0$ and condition (1) above will not be satisfied.

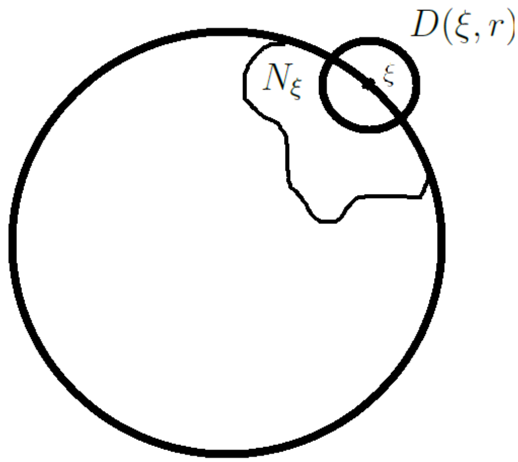

Figure 2: Example

Another example described in [28] is the following. A composition operator C_φ will not be closed-range on H^2 if there is a disk D_1 that is tangent to ∂D such that $D_1 \cap \varphi(\mathbb{D}) = \emptyset$. In this case, for any choice of b, we can choose a point a in $\mathbb{D}$ close enough to the boundary of the disk so $D_b(a)$ will be contained entirely in the disk D_1. Then, similar to the previous example, we have that $\gamma_\varphi(z) = 0$ on $D_b(a)$. Hence, $A(G_c^\varphi \cap D_b(a)) = 0$ for any $c > 0$ and so condition (2) above is not satisfied.

In [28], Zorboska also remarks that the composition operator C_φ will not be closed-range on H^2 if $\varphi(\mathbb{D})$ is a proper subset of $\mathbb{D} \setminus [0, 1)$.

We will now introduce several other classical spaces of analytic functions in $\mathbb{D}$. For more information on the following spaces, see [27].

1.2 Composition Operators on the Bloch space $\mathcal{B}$

The *Bloch space* $\mathcal{B}$ is the space of analytic functions on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty.$$

Under the norm

$$||f||_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|,$$

$\mathcal{B}$ forms a Banach space. By Proposition 5.1 in [27], H^∞ is properly contained in $\mathcal{B}$, and $||f||_{\mathcal{B}}\leq||f||_\infty$ for all $f\in H^\infty$. The set of analytic functions in $\mathbb{D}$ having the property that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)|=0$$

is called the *little Bloch space* and is denoted by $\mathcal{B}_0$. The little Bloch space is a closed subspace of $\mathcal{B}$.

For z in $\mathbb{D}$, let $\tau_\varphi(z)=\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}$. We can apply the Schwarz-Pick lemma to get $|\tau_\varphi(z)|\leq 1$ for all z in $\mathbb{D}$. Now, for f in $\mathcal{B}$,

$$\begin{aligned}(1-|z|^2)|(f\circ\varphi)'(z)|&=(1-|z|^2)|f'(\varphi(z))||\varphi'(z)|\\&=\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}(1-|\varphi(z)|^2)|f'(\varphi(z))|\\&=|\tau_\varphi(z)|(1-|\varphi(z)|^2)|f'(\varphi(z))|\\&\leq(1-|\varphi(z)|^2)|f'(\varphi(z))|\end{aligned}$$

Thus C_φ is a bounded composition operator on $\mathcal{B}$ for every analytic self-map φ of $\mathbb{D}$.

It is shown in Theorem 2 in [17] that C_φ wll be compact on $\mathcal{B}$ if and only if for every $\varepsilon > 0$, there exists r, $0 < r < 1$, such that

$$\tau_\varphi(z) = \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} < \varepsilon$$

whenever $|\varphi(z)| > r$.

By theorem 2 in [26], for an analytic self-map φ of the unit disk $\mathbb{D}$, the composition operator C_φ is compact on $\mathcal{B}$ if and only if

$$\lim_{n\to\infty} ||\varphi^n||_{\mathcal{B}} = 0.$$

In [17], Madigan and Matheson give a similar condition for the compactness of a composition operator on $\mathcal{B}_0$. Theorem 1 in [17] states that C_φ is compact on $\mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1^-} |\tau_\varphi(z)| = 0.$$

In [11], a necessary and sufficient condition for a composition operator C_φ to be closed-range on $\mathcal{B}$ is given. Letting $\mathcal{C}$ be the closed subspace of constant functions, we have $||f||_{\mathcal{B}/\mathcal{C}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$. Theorem 0 in [11] states that C_φ will be closed-range on $\mathcal{B}$ if and only if

$$||f \circ \varphi||_{\mathcal{B}/\mathcal{C}} \geq k \cdot ||f||_{\mathcal{B}/\mathcal{C}}$$

for a constant $k > 0$.

For a subset K of $\mathbb{D}$, if there exists $k > 0$ with

$$\sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\} \leq k \cdot \sup\{(1-|z|^2)|f'(z)| : z \in K\}$$

for every function f in $\mathcal{B}$, then K is called a *sampling set* for $\mathcal{B}$. Define $F_\varepsilon := \varphi(\Lambda_\varepsilon)$ where $\Lambda_\varepsilon := \{z \in \mathbb{D} : \tau_\varphi(z) \geq \varepsilon\}$ and $\varepsilon > 0$. Theorem 1 in [11] says that a composition operator C_φ

will be closed-range on the Bloch space $\mathcal{B}$ if and only if there exists $\varepsilon > 0$ such that F_ε is a sampling set for $\mathcal{B}$. It is also shown in [11] that the set F_ε is a sampling set for $\mathcal{B}$ if it satisfies the *reverse Carleson condition*. That is, F_ε is a sampling set for $\mathcal{B}$ if there exist constants c and s with $0 < c, s < 1$, such that $A(F_\varepsilon \cap \Delta(z,s)) \geq c \cdot A(\Delta(z,s))$ for all z in the unit disk $\mathbb{D}$. Hence, by Theorem 1 and Proposition 1 in [11], if F_ε satisfies the reverse Carleson condition, then the composition operator C_φ will be closed-range on $\mathcal{B}$. If φ is univalent, then, by theorem 2 in [11], the converse of the previous statement also holds.

Let φ be univalent and suppose that the composition operator C_φ is closed-range on $\mathcal{B}$. Then, for some $\varepsilon > 0$, F_ε satisfies the reverse Carleson condition and by Proposition 3 in [11], there exists $\delta > 0$ such that for every point ω in $\partial\mathbb{D}$,

$$\overline{\lim}_{\varphi(z)\to\omega} \frac{\mathrm{dist}(\varphi(\mathrm{z}), \partial(\varphi(\mathbb{D})))}{|\varphi(z) - \omega|} \geq \delta.$$

Example 1 in [11] shows that this condition is not sufficient for C_φ to be closed-range. In the second example given in [11], we let $G = \mathbb{D} \setminus [0,1)$ and φ is chosen to be the Riemann mapping onto G. By the Koebe One-Quarter Theorem, when φ is univalent, then

$$\tau_\varphi(z) \approx \frac{\mathrm{dist}(\varphi(\mathrm{z}), \partial\mathrm{G})}{1 - |\varphi(z)|}.$$

As $\varphi(z)$ approaches a point w on the boundary of the disk other than 1, this ratio approaches 1. Then, F_ε contains all of $\mathbb{D}$ except for a pseudohyperbolic neighborhood of the segment $[0,1)$. Thus, we can choose r large enough so that every point z in $\mathbb{D}$ is within pseudohyperbolic distance r of F_ε. So, there exists a constant $c > 0$ such that $A(F_\varepsilon \cap \Delta(z,r)) \geq c \cdot A(\Delta(z,r))$ for all z in the unit disk $\mathbb{D}$. Hence, F_ε satisfies the reverse Carleson condition and C_φ is closed-range.

Proposition 1 in [10] gives a necessary condition for the composition operator C_φ to be closed-range on $\mathcal{B}$. The proposition states that if C_φ is closed-range on $\mathcal{B}$ then there will exist positive constants ε and $r < 1$ so that, for all z in $\mathbb{D}$, $\rho(\varphi(\Lambda_\varepsilon), z) \leq r$. Recall that ρ denotes the pseudohyperbolic metric. Theorem 2 in the same source ([10]) also gives a sufficient condition. This theorem gives that C_φ is closed-range on $\mathcal{B}$ if for some positive constants ε and r with $r < \frac{1}{4}$, for all w in $\mathbb{D}$ there exists a point z_w in $\mathbb{D}$ so that $\rho(\varphi(z_w), w) < r$ and $|\tau_\varphi(z_w)| > \varepsilon$.

1.3 Composition Operators on the Besov space B_p

For $0 < p < \infty$, the Besov space B_p is the collection of holomorphic functions in $\mathbb{D}$ such that

$$\begin{aligned}||f||_{B_p}^p &= \int_{\mathbb{D}} |f^{(n)}(z)(1-|z|^2)^n|^p d\lambda(z) \\ &= \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np} d\lambda(z) < \infty\end{aligned}$$

for any positive integer n satisfying $np > 1$ and where

$$d\lambda(z) = \frac{1}{(1-|z|^2)^2} dA(z).$$

Under the norm $|f(0)| + ||f||_{B_p}$, B_p is a Banach space.

Theorem 5.17 in [27] gives the atomic decomposition for B_p. The theorem states that for $p > 0$, there exists a sequence $\{a_k\}$ in $\mathbb{D}$ such that for $b > \max(0, \frac{p-1}{p})$, the space B_p is comprised of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \left(\frac{1-|a_k|^2}{1-z\bar{a}_k} \right)^b$$

with $c_k \in l^p := \{\{c_k\}_{k=1}^{\infty} \subset \mathbb{C} : \sum_{k=1}^{\infty} |c_k|^p < \infty\}$.

Recall that $\alpha_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$ is the special automorphism of the disk $\mathbb{D}$ with $\alpha_\lambda(\lambda) = 0$, $\alpha_\lambda(0) = \lambda$, and $\alpha_\lambda^{-1} = \alpha_\lambda$. By Theorem D in [24], for an analytic self-map φ of $\mathbb{D}$, C_φ is a bounded operator on the Besov space B_p if and only if

$$\sup_{\lambda \in \mathbb{D}} ||C_\varphi \alpha_\lambda||_{B_p} < \infty.$$

By theorem 3.5 in [24], for $1 < p \leq q < \infty$, when φ is a holomorphic self-map of $\mathbb{D}$ then the following are equivalent:

1. $C_\varphi : B_p \to B_q$ is a compact operator.

2. $||C_\varphi \alpha_\lambda||_{B_q} \to 0$ as $|\lambda| \to 1$.

Not much is known about conditions for which φ will induce a compact composition operator on B_p for p in general.

1.4 Composition Operators on the Dirichlet Space

The Dirichlet space $\mathcal{D}$ is the set of holomorphic functions f on $\mathbb{D}$ such that

$$||f||_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

That is, if f is in $\mathcal{D}$, then its derivative is in $\mathbb{A}^2$. Note that $\mathcal{D} = B_2$, with an equivalent norm.

For $p > 0$ and μ a finite positive Borel measure, if there exists a constant $0 < c < \infty$ such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq c \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)$$

for all f in $\mathbb{A}^p_\alpha$, then μ is called a *Carleson measure* for $\mathbb{A}^p_\alpha$. As is stated in [15], an equivalent condition for μ to be a Carleson measure is

$$\sup_{z\in\mathbb{D}} \frac{\mu(\Delta(z,\eta))}{|\Delta(z,\eta)|} < \infty$$

where, again, $\Delta(z,\eta)$ denotes the pseudohyperbolic disk. We call μ a *compact (or vanishing) Carleson measure* if

$$\sup_{r<|z|<1} \frac{\mu(\Delta(z,\eta)}{|\Delta(z,\eta)|} \to 0$$

as $r \to 1$. Let n_φ denote the cardinality of the set $\varphi^{-1}(w)$. In [15], Luecking shows that a composition operator C_φ is bounded on the Dirichlet space $\mathcal{D}$ if $n_\varphi dA$ is a Carleson measure for $\mathbb{A}^p_\alpha$ for some $p > 0$. By Proposition 5.1 in [16], C_φ is compact on $\mathcal{D}$ if $n_\varphi dA$ is a compact Carleson measure. In [15], Luecking also shows that C_φ is closed-range on the Dirichlet space $\mathcal{D}$ if and only if there exists a constant $c > 1$ such that

$$\frac{1}{c}\int |f'|^2 dA \le \int |f'|^2 n_\varphi dA \le c\int |f'|^2 dA$$

for every f in $\mathcal{D}$. If this condition is satisfied, then there exists R, $0 < R < 1$ and $\delta > 0$ such that

$$\int_{\Delta(a,r)} n_\varphi dA \le \delta|\Delta(a,r)|$$

for all $z \in \mathbb{D}$ where, again, $|\Delta(a,r)|$ denotes the area of the pseudohyperbolic disk $\Delta(a,r)$. By part 2 of Corollary 2 in [11], when φ is univalent and the composition operator C_φ is bounded below on the Bloch space $\mathcal{B}$, then C_φ is also bounded below on the Dirichlet space. That is, if φ is univalent and C_φ is closed-range on $\mathcal{B}$ then it is also closed-range on $\mathcal{D}$.

1.5 Composition Operators on BMO

Let I denote an interval that is contained in $\mathbb{T}$ and let f be in $L^2(\mathbb{T})$. With $|I|$ representing the length of the interval I, the mean of the function f over I is given by

$$f_I = \frac{1}{|I|}\int_I f(\theta)d\theta.$$

The space of all functions f in $L^2(\mathbb{T})$ that have *bounded mean oscillation* is called BMO($\mathbb{T}$). A function f has bounded mean oscillation if

$$||f||_{BMO} := \sup_I \frac{1}{|I|}\int_I |f(\theta) - f_I| d\theta < \infty.$$

The space BMOA($\mathbb{T}$) is the intersection of BMO and $H^2(\mathbb{T})$. For any function f in $L^1(\mathbb{T})$ we can extend f to a function $\hat{f}$ on the disk $\mathbb{D}$ via the *Poisson extension*,

$$\hat{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2}d\theta$$

for all z in $\mathbb{D}$. Thus, BMOA($\mathbb{T}$) can be extended to BMOA($\mathbb{D}$), a space of analytic functions on $\mathbb{D}$. Recall the special automorphism of $\mathbb{D}$,

$$\alpha_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$$

where $\alpha_\lambda(\lambda) = 0$, $\alpha_\lambda(0) = \lambda$, and $\alpha_\lambda^{-1} = \alpha_\lambda$. As is stated in [23], a function f in H^2 is in BMOA if

$$||f||_* = \sup_{\lambda \in \mathbb{D}} ||f \circ \alpha_\lambda - f(\lambda)||_2 < \infty.$$

Under the norm $||f||_{BMOA} = ||f||_* + |f(0)|$, BMOA forms a Banach space.

For r in $(0,1)$, let Φ_r denote the set $\{z : 1 > |\varphi(z)| > r\}$, and for the characteristic function of Φ_r, we write $\chi_r(z)$. By Theorem 3.1 in [6], the composition operator C_φ is compact on BMOA if and only if for all $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\int_{R(I)} \chi_{\Phi_r}(z)(1-|z|^2)|f'(\varphi(z))|^2|\varphi'(z)|^2 dA(z) \leq \varepsilon|I|.$$

By Theorem 4.1 in [23], If φ is a univalent self-map of $\mathbb{D}$, then C_φ is compact on BMOA if and only if it is compact on the Bloch space $\mathcal{B}$.

By part 1 of Corollary 2 in [11], for univalent φ, if C_φ is bounded below on BMOA, then it is also bounded below on the Bloch space $\mathcal{B}$. In other words, if φ is univalent and C_φ is closed-range on BMOA, then C_φ is also closed-range on $\mathcal{B}$.

We have defined $\pi_\varphi(w)$ to be

$$\pi_\varphi(w) := \frac{N_\varphi(w)}{\log(\frac{1}{w})},$$

where $N_\varphi(w)$ is the Nevalinna counting function. Let $\pi_{\varphi,\alpha} = (\pi_\varphi(w))^\alpha$ and

$$G_c^{\varphi,\alpha} = \{z : \pi_{\varphi,\alpha+2} > c\}.$$

Theorem 4.1 in [28] states that a composition operator C_φ will be closed-range on the weighted Bergman space $\mathbb{A}^2_\alpha$, with $\alpha > -1$ if and only if there exist constants $c > 0$ and $\lambda > 0$ such that

$$A(G^{\varphi,\alpha} \cap D(\xi,r)) > \delta \cdot A(\mathbb{D} \cap D(\xi,r)).$$

Notice that not only does the condition above involve the Nevanlinna counting function, but it also depends on α. In the next section, we will give a necessary and sufficient condition for when an analytic self-map φ of $\mathbb{D}$ induces a closed-range composition operator on the weighted Bergman space $\mathbb{A}^p_\alpha$ for all p and all $\alpha > -1$. This will essentially render all the above conditions equivalent for various values of α. In [1], J. Akeroyd and P. Ghatage give a necessary and sufficient condition in the case $p = 2$ and $\alpha = -1$.

2 Closed-Range Composition Operators on Weighted Bergman Spaces

Let φ be an analytic self-map of $\mathbb{D}$. For any ε, $0 < \varepsilon < 1$, define $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$, and let $G_\varepsilon(\varphi) = G_\varepsilon = \varphi(\Omega_\varepsilon)$. Note that, in the weighted Bergman space setting, G_ε functions in much the same way that F_ε did in the Bloch space setting. The set G_ε is said to satisfy the *reverse Carleson condition* if there exists a positive constant η so that

$$\int_{G_\varepsilon} |f(z)|^p(1-|z|^2)^p dA \geq \eta \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^p dA$$

for f analytic in $\mathbb{D}$ and $\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^p dA < \infty$. As shown in [14], this is equivalent to the following condition:

($*$) There exist constants c and s with $0 < c, s < 1$, such that

$$A(G_\varepsilon \cap \Delta(z,s)) \geq c \cdot A(\Delta(z,s))$$

for all z in the unit disk $\mathbb{D}$. We will show in Theorem 2.3 that a composition operator C_φ is closed-range on $\mathbb{A}^p_\alpha$ if and only if there exists an $\varepsilon > 0$ such that G_ε satisfies condition ($*$). Recall that we defined the weighted Bergman Spaces $\mathbb{A}^p_\alpha$ by

$$\mathbb{A}^p_\alpha := \{f : f \text{ is analytic in } \mathbb{D} \text{ and } ||f||^p_{p,\alpha} = \textstyle\int_{\mathbb{D}} |f|^p dA_\alpha < \infty\}$$

where $\alpha > -1$ and $dA_\alpha := c_\alpha \cdot (1-|z|^2)^\alpha dA(z)$ where $c_\alpha = \alpha + 1$.

Define $\mathbb{A}^p_{\alpha,0}$ by

$$\mathbb{A}^p_{\alpha,0} = \{f \in \mathbb{A}^p_\alpha : f(0) = 0\},$$

a closed subspace of $\mathbb{A}^p_\alpha$. The following lemma is adapted from Lemmas 2.1, 2.2, and 2.3 in [1], and will be stated without proof.

Lemma 2.1 *Let φ be an analytic self-map of the unit disk $\mathbb{D}$, and let ψ_a be a conformal automorphism of $\mathbb{D}$. Then,*

1. *C_φ is closed-range on $\mathbb{A}^p_\alpha$ if and only if C_φ is closed-range on $\mathbb{A}^p_{\alpha,0}$*

2. *If one of C_φ, $C_{\varphi\circ\psi_a}$, or $C_{\psi_a\circ\varphi}$ is closed-range on $\mathbb{A}^p_\alpha$, then so are the other two.*

3. *If there exists $\varepsilon > 0$ such that one of $G_\varepsilon(\varphi)$, $G_\varepsilon(\varphi\circ\psi_a)$, $G_\varepsilon(\psi_a\circ\varphi)$ satisfies condition $(*)$, then there exists $\varepsilon > 0$ such that the other two also satisfy condition $(*)$.*

Lemma 2.2 *(Lemma 2.2 in [4]). Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed-range on $\mathbb{A}^p_\alpha$ then it is closed-range on $\mathbb{A}^{np}_\alpha$ for any $n \in \mathbb{N}$.*

Proof. Assume φ is not constant. Otherwise, the result is trivial. Suppose C_φ is closed-range on $\mathbb{A}^p_\alpha$. Then there exists a constant c such that $||f\circ\varphi||_{\mathbb{A}_{p,\alpha}} \geq c\cdot||f||_{\mathbb{A}_{p,\alpha}}$ for all $f \in \mathbb{A}^p_\alpha$. That is,

$$\int_{\mathbb{D}} |f\circ\varphi|^p dA_\alpha \geq c\cdot\int_{\mathbb{D}} |f|^p dA_\alpha$$

for all $f \in \mathbb{A}^p_\alpha$. Now, if $f \in \mathbb{A}^{np}_\alpha$, then $f^n \in \mathbb{A}^p_\alpha$. Thus,

$$\begin{aligned}\int_{\mathbb{D}} |f\circ\varphi|^{np} dA_\alpha &= \int_{\mathbb{D}} |f^n\circ\varphi|^p dA_\alpha \\ &\geq c\cdot\int_{\mathbb{D}} |f^n|^p dA_\alpha \\ &= c\cdot\int_{\mathbb{D}} |f|^{np} dA_\alpha.\end{aligned}$$

and so C_φ is closed-range on $\mathbb{A}^{np}_\alpha$. □

Theorem 2.3 *(1.3 in [5]) Let φ be an analytic self-map of the unit disk $\mathbb{D}$. Suppose $1 \leq p < \infty$ and $\alpha > -1$. Then, the following are equivalent:*

1. C_φ is closed-range on $\mathbb{A}^p_\alpha$.

2. There exists $\varepsilon > 0$ such that $G_\varepsilon = \varphi(\Omega_\varepsilon)$ satisfies condition $(*)$.

Proof. By Lemma 2.1 we may assume that $\varphi(0) = 0$ and we may also restrict our attention to C_φ on $\mathbb{A}^p_{\alpha,0}$. By the proof of Theorem 4.28 in [27], there is a constant $C > 1$ such that

$$\frac{1}{C}||f||_{p,\alpha} \leq \left\{\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p dA_\alpha(z)\right\}^{\frac{1}{p}} \leq C||f||_{p,\alpha}$$

for all f in $\mathbb{A}^p_{\alpha,0}$. We will denote this by

$$||f||^p_{p,\alpha} \approx \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p dA_\alpha(z).$$

We will first show that (2) implies (1). Note that this argument can also be found in the proof of Theorem 2.3 in [4]. Suppose that for some $\varepsilon > 0$, G_ε satisfies condition $(*)$. First consider the case that $1 \leq p < 2$. By the Schwarz-Pick lemma (Lemma 1.2 in [8]), $0 \leq \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1$ for all z in $\mathbb{D}$. Hence, for all z in Ω_ε, $\varepsilon|\varphi'(z)| < 1$, and thus, since $1 \leq p < 2$, we have $\varepsilon^2|\varphi'(z)|^2 \leq \varepsilon^p|\varphi'(z)|^p$. Hence, $\varepsilon^{2-p} < |\varphi'(z)|^{p-2}$. Then,

$$\begin{aligned}
||f\circ\varphi||^p_{p,\alpha} &\approx \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^p(1-|z|^2)^p dA_\alpha(z)\\
&\geq \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z)\\
&= \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^{p-2}|\varphi'(z)|^2(1-|z|^2)^{p+\alpha}dA(z)\\
&\geq \varepsilon^{2-p}\int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^2(1-|z|^2)^{p+\alpha}dA(z)\\
&\geq \varepsilon^{\alpha+2}\int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^2(1-|\varphi(z)|^2)^{p+\alpha}dA(z)\\
&= \varepsilon^{\alpha+2}\sum_n \int_{\Omega_\varepsilon\bigcap R_n} |f'(\varphi(z))|^p|\varphi'(z)|^2(1-|\varphi(z)|^2)^{p+\alpha}dA(z)
\end{aligned}$$

where $\mathcal{Z} := \{z \in \mathbb{D} : \varphi'(z) = 0\}$ and $\{R_n\}$ is a partition of $\mathbb{D} \setminus \mathcal{Z}$ into at most countably many polar rectangles so that φ is univalent on R_n for all n. Let $S_n = \varphi(\Omega_\varepsilon \bigcap R_n)$ and let ψ_n denote the inverse of $\varphi|_{R_n}$. Then, letting $z = \psi_n(w)$, we have

$$\begin{aligned}
&\varepsilon^{\alpha+2} \sum_n \int_{\Omega_\varepsilon \bigcap R_n} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha} dA(z) \\
&\qquad = \varepsilon^{\alpha+2} \sum_n \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} \chi_{S_n}(w) dA(w) \\
&\qquad = \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} (\sum_n \chi_{S_n}(w)) dA(w) \\
&\qquad \geq \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w)
\end{aligned}$$

Since G_ε satisfies condition $(*)$, we have

$$\begin{aligned}
\varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) &\geq \eta \varepsilon^{\alpha+2} \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) \\
&\approx \int_{\mathbb{D}} |f(w)|^p dA_\alpha(w) \\
&= ||f||_{p,\alpha}^p
\end{aligned}$$

Hence, for $1 \leq p < 2$, we have that C_φ is closed-range on $\mathbb{A}_\alpha^p$. We may apply Lemma 2.2 to see that C_φ is closed-range on $\mathbb{A}_\alpha^p$ for $1 \leq p < \infty$ and, thus, (1) is satisfied.

We will now show that (1) implies (2) by means of the contrapositive, as is also shown in the proof of Theorem 1.3 in [5]. Suppose that condition $(*)$ is not satisfied. Then there does not exist $\varepsilon > 0$ such that G_ε satisfies the reverse Carleson condition. In other words, for any $\varepsilon > 0$, there does not exist a positive constant η such that

$$\int_{G_\varepsilon} |f(z)|^p(1-|z|^2)^p dA \geq \eta \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^p dA$$

for all f in $\mathbb{A}^p_{\alpha,0}$. So, we can find a sequence $\{f_k\}_{k=1}^\infty \subset \mathbb{A}^p_{\alpha,0}$ such that

$$\int_{\mathbb{D}} |f_k'(w)|^p(1-|w|^2)^p dA_\alpha(w) = 1$$

for all k, but

$$\int_{G_k} |f_k'(w)|^p(1-|w|^2)^p dA_\alpha(w) \to 0$$

as $k \to \infty$, where $\Omega_k = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \frac{1}{k}\}$ and $G_k = \varphi(\Omega_k)$.

First suppose that $p \geq 3$. Note that for all $z \in \Omega_{j+1} \setminus \Omega_j$, we have $\frac{1}{j} \geq \frac{1-|z|^2}{1-|\varphi(z)|^2} > \frac{1}{j+1}$ and, by the Schwarz-Pick Lemma, $0 \leq \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1$ for all z in $\mathbb{D}$. Thus, on $\Omega_{j+1} \setminus \Omega_j$,

$$|\varphi'(z)|^{p-2} \leq (j+1)^{p-2}.$$

Also, on $\Omega_{j+1} \setminus \Omega_j$,

$$(1-|z|^2)^{p+\alpha-1} \approx \frac{1}{(j+1)^{p+\alpha-1}} \cdot (1-|\varphi(z)|^2)^{p+\alpha-1}.$$

Now, $\mathbb{D}$ is equal to the pairwise disjoint union $\Omega_k \cup (\cup_{j=k}^\infty \Omega_{j+1} \setminus \Omega_j)$. So,

$$\begin{aligned}
||f_k \circ \varphi||^p_{p,\alpha} &\approx \int_{\mathbb{D}} |f_k'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^p dA_\alpha(z) \\
&= \int_{\Omega_k} |f_k'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^p dA_\alpha(z) \\
&\qquad + \sum_{j=k}^\infty \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^p dA_\alpha(z)
\end{aligned}$$

Then, by the corollary on page 188 of [22], we have

$$\int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z)$$
$$= \int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 |\varphi'(z)|^{p-2} (1-|z|^2)^{p+\alpha-1} (1-|z|^2) dA(z)$$

$$\leq \int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \tfrac{1}{|z|} dA(z)$$

$$\leq \int_{G_k} |f_k'(w)|^p (1-|w|^2)^{p+\alpha-1} N_\varphi(w) dA(w)$$

$$\leq \int_{G_k} |f_k'(w)|^p (1-|w|^2)^{p+\alpha} dA(w)$$

$$= \int_{G_k} |f_k'(w)|^p (1-|w|^2)^p dA_\alpha(w) \to 0$$

as $k \to \infty$. Then, again using the corollary on page 188 of [22], we have

$$\sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z)$$
$$\approx \sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 |\varphi'(z)|^{p-2} (\frac{1}{(j+1)^{p+\alpha-1}}) (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$\leq \sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 ((j+1)^{p-2}) (\frac{1}{(j+1)^{p+\alpha-1}}) (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$\leq \frac{1}{k^{\alpha+1}} \sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$= \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}\setminus\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$\leq \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$= \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^{p+\alpha-1} N_\varphi(w) dA(w)$$
$$\leq \frac{c}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^{p+\alpha} dA$$

$$= \frac{c}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^p dA_\alpha \to 0$$

as $k \to \infty$. Thus, $||f_k \circ \varphi||_{p,\alpha} \to 0$ as $k \to \infty$, even though $||f_k||_{p,\alpha} = 1$ for all k. Hence, it must be that C_φ is not closed-range on $\mathbb{A}^p_{\alpha,0}$ for $p \geq 3$. By the contrapositive of the previous lemma then, it must be that C_φ is not closed-range on $\mathbb{A}^p_{\alpha,0}$ for any $p \geq 1$. Thus, by means of the contrapositive of what we have just shown, our proof is complete. □

Corollary 2.4 *If φ is univalent and C_φ is closed-range on the weighted Bergman space $\mathbb{A}^p_\alpha$, then C_φ is closed-range on the Hardy space H^2.*

Proof. If C_φ is closed-range on any weighted Bergman space $\mathbb{A}^p_\alpha$, then, by Theorem 2.3, C_φ is closed-range on $\mathbb{A}^2$. Then, by Corollary 4.3 in [28], C_φ is closed-range on H^2.

3 Examples

3.1 An Outer Function

We note that an example of the same type as the following was developed concurrently by P. Ghatage. We also note that φ in the following example is a purely outer function. J. Akeroyd and P. Ghatage discuss the case when φ is a singular inner function in [1].

Let ψ be a conformal mapping of the unit disk $\mathbb{D}$ onto the semi-annulus

$$S = \{re^{i\theta} : \frac{1}{2} < r < 1, 0 < \theta < \pi\}.$$

By Theorem 13.2.3 in [12], since $\mathbb{D}$ and S are bounded domains in $\mathbb{C}$, each bounded by a single Jordan curve, then ψ extends to a homeomorphism from $\bar{\mathbb{D}}$ onto $\bar{S}$.

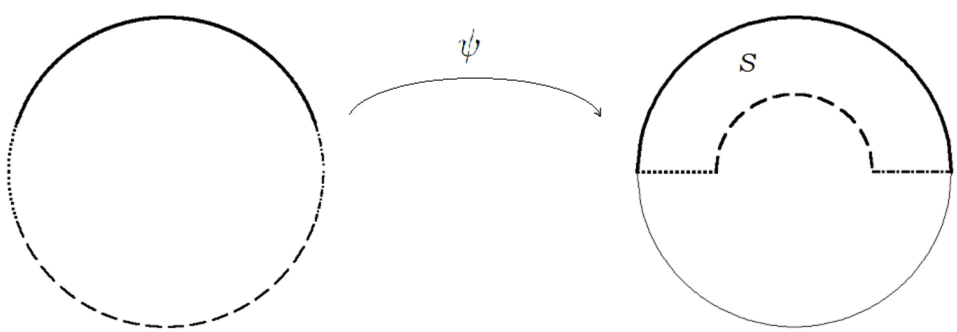

Figure 3: $\psi : \mathbb{D} \to S = \{re^{i\theta} : \frac{1}{2} < r < 1, 0 < \theta < \pi\}$

By the Schwarz-Pick Theorem (see [12]), for any z in $\mathbb{D}$,

$$\frac{1-|z|^2}{1-|\psi(z)|^2} \leq \frac{1}{|\psi'(z)|}.$$

We let φ be the analytic self-map of $\mathbb{D}$ given by $\varphi(z) := (\psi(z))^{2+\delta}$, $\delta \geq 0$. First, suppose that $\delta = 0$. Then, φ maps the unit disk $\mathbb{D}$ to the set

$$S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\} \setminus (\frac{1}{4}, 1).$$

As in the case of the square root function on the upper half plane, $|\psi'(z)|$ grows without bound for the points that ψ maps to the corner points of ∂S.

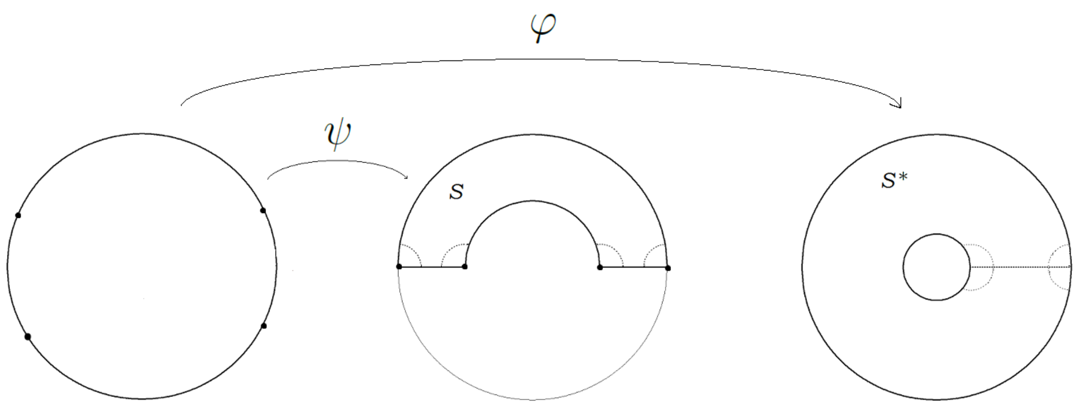

Figure 4: $\varphi : \mathbb{D} \to \mathrm{S}^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\} \setminus (\frac{1}{4}, 1)$

Let ξ be a point in $\mathbb{T}$ such that $\psi(\xi)$ is a corner point of ∂S. Then, for any $\varepsilon > 0$, we can find a region R about ξ, such that, for all z in R,

$$\frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|} < \varepsilon.$$

Since $|\psi(z)| < 1$, for all z in R we have

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} = \frac{1 - |z|^2}{1 - |\psi(z)|^4} < \frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|} < \varepsilon.$$

Thus, these points are not contained in

$$\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \geq \varepsilon\}$$

and, hence, the image of this set R under φ is not contained in G_ε. To see that φ does not induce a closed-range composition operator in this case, we need to show that the condition $(*)$ is not satisfied. To this end, let $\{p_n\}$ be a sequence of points in $[0, 1)$ converging to 1 and consider the sequence of pseudohyperbolic disks, $\Delta(p_n, r)$, of radius r with center p_n. Each pseudohyperbolic disk $\Delta(p_n, r)$ is a Euclidean disk with radius

$$q_n = \frac{r(1 - p_n^2)}{1 - p_n^2 r^2}$$

and center

$$c_n = \frac{p_n(1 - r^2)}{1 - p_n^2 r^2}.$$

The Euclidean distance from the point 1 to the boundary of $\Delta(p_n, r)$ is given by $\frac{(1-p_n)(1-r)}{1+p_n r}$ for each n. Notice that the ratio of this distance to the Euclidean radius of each k is $\frac{r(1+p_n)}{1-r}$, which approaches the ratio $\frac{2r}{1-r}$ as $p_n \to 1$. Hence the sequence of pseudohyperbolic disks, $\Delta(p_n, r)$ is approaching the unit circle $\mathbb{T}$ nontangentially. In other words, we can find a stolz region which contains each of the disks.

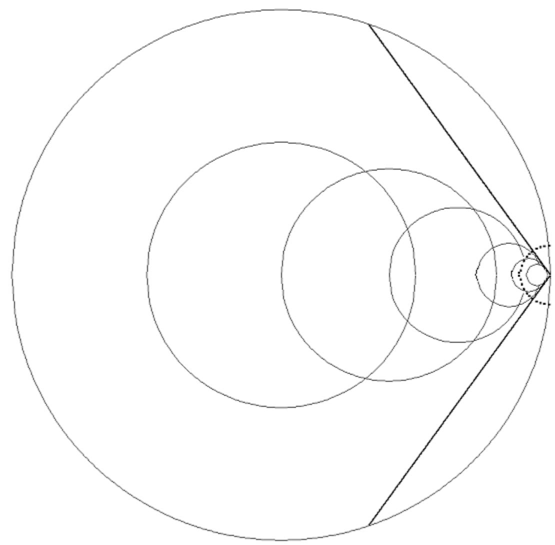

Figure 5: The Sequence $\Delta(p_n, r)$ approaches $\partial\mathbb{D}$ nontangentially.

Then, for any $\varepsilon > 0$ and any r in $(0, 1)$, there exists an N in $\mathbb{N}$ with the preimage of $\Delta(p_n, r)$ under φ contained in R whenever $n > N$. Therefore, $\Delta(p_n, r) \bigcap G_\varepsilon = \emptyset$ and condition $(*)$ fails.

Now, suppose that $\delta > 0$. Then, $\varphi(z) := (\psi(z))^{2+\delta}$ maps the unit disk $\mathbb{D}$ to the outer annulus $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$. We will see that φ now maps a region of points contained in Ω_ε to an outer annulus, and hence, condition $(*)$ will be satisfied. In fact, provided that $\varepsilon > 0$ is sufficiently small, we will see that the entire annulus $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$ is contained in G_ε. We define θ_ε to be the smallest angle such that $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$ is contained in the image of Ω_ε under ψ. For the given δ, we can choose ε small enough such that $\theta_\varepsilon < \frac{\delta\pi}{4}$. We let γ_1 denote the set of points $\{re^{i\theta} : 0 < r < 1\}$ and γ_2 denote the set of points $\{re^{i(\pi-\theta)} : 0 < r < 1\}$.

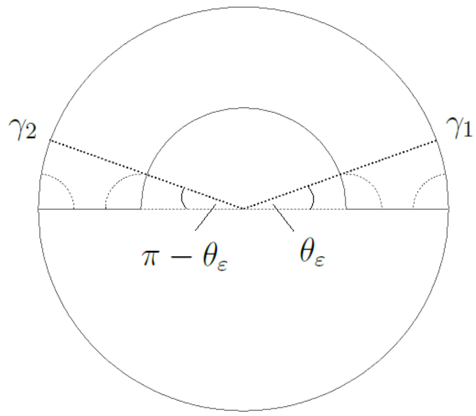

Figure 6: The set of points $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$ is contained in the image of Ω_ε under ψ.

Now, under φ, points in γ_1 are mapped to points along the radius from 0 to $e^{i(2\theta_\varepsilon+\delta\theta_\varepsilon)}$. Similarly, points in γ_2 are mapped to points along the radius from 0 to $e^{i(2+\delta)(\pi-\theta_\varepsilon)}$. Since $\theta_\varepsilon < \frac{\delta\pi}{4}$, we have that the reference angle $\delta\pi - \theta_\varepsilon(2+\delta)$ is greater than the angle $(2+\delta)\theta_\varepsilon$.

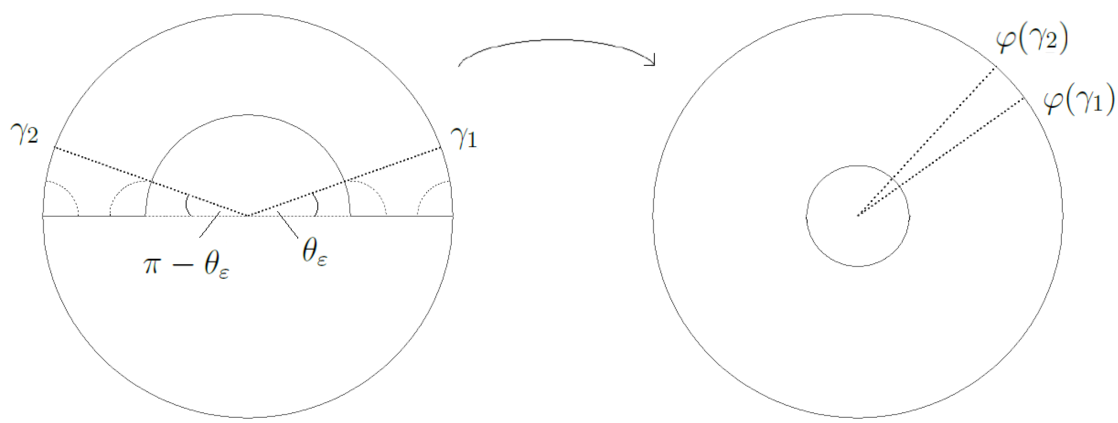

Figure 7: G_ε contains the entire outer annulus S^*.

Hence, the image under φ of the set $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$ overlaps itself and we have that G_ε contains the entire outer annulus $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$. Thus, condition $(*)$ is satisfied and φ induces a closed-range composition operator when $\delta > 0$.

3.2 Frostman Blaschke Products

Note that the following Frostman Blaschke product example appears in [4]. Remember that we define a Blaschke product B to be a function of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a_n}}$$

where $z \in \mathbb{D}$, $\{a_n\}$ is a sequence of points in $\mathbb{D}$ with the property that $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$, and $\frac{|a_n|}{a_n}$ is taken to be 1 if $a_n = 0$. For ζ in $\mathbb{T}$, define $g_B(\zeta)$ by

$$g_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|}.$$

By a theorem of Frostman, see [7], the Blaschke product B has a unimodular nontangential boundary value at ζ in $\mathbb{T}$ exactly when $g_B(\zeta) = \sum_n \frac{1-|a_n|^2}{|\zeta - a_n|} < \infty$. If $g_B(\zeta)$ converges for every ζ in $\mathbb{T}$, then we call B a *Frostman Blaschke Product*. In other words, B is a Frostman Blaschke product if B has unimodular nontangential boundary values at every point ζ in $\mathbb{T}$. Denote the set of accumulation points of the sequence $\{a_n\}$ in $\mathbb{T}$ by σ_B. By Theorem 1 in [18], if B is a Frostman Blaschke product, then σ_B is nowhere dense in $\mathbb{T}$. For ζ in $\mathbb{T}$, we define $h_B(\zeta)$ by

$$h_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.$$

By page 183 in [22], B has an angular derivative at ζ in $\mathbb{T}$ exactly when $h_B(\zeta) < \infty$. Hence, a Frostman Blaschke product B will have an angular derivative at every point ζ in the dense open set $\mathbb{T} \setminus \sigma_B$, but not necessarily at points in σ_B. Let $\{I_\nu\}$ be the collection of subarcs of $\mathbb{T} \setminus \sigma_B$, and for each ν, define ω_ν to be the (possibly infinite) number of radians through which B wraps I_ν. If, for some ν_0, $B(I_{\nu_0}) = \mathbb{T}$, then the proof of Lemma 3.1 in [1] gives us that

$$\omega_{\nu_0} = \int_{I_{\nu_0}} h_B(\zeta)|d\zeta| > 2\pi.$$

Therefore, by Theorem 3.4 in [1] and Theorem 2.3, we have that C_φ is closed-range on every weighted Bergman space $\mathbb{A}^p_\alpha$. Suppose, then, that such a ν_0 does not exist. In that case, for every ν, $B(I_\nu)$ is an open subarc of $\mathbb{T}$. If $\bigcup_\nu B(I_\nu) = \mathbb{T}$, then, since $\mathbb{T}$ is compact, there exists an integer $N > 0$ so that $\bigcup_{\nu=1}^N B(I_\nu) = \mathbb{T}$. Then, there is a compact subset K of $\bigcup_{\nu=1}^N I_\nu$ such that $B(K) = \mathbb{T}$. If $\varepsilon > 0$ is small enough, K will be contained in the closure of $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|B(z)|^2} > \varepsilon\}$. Hence, by Theorem 2.3, the composition operator C_B will be closed-range on every weighted Bergman space $\mathbb{A}^p_\alpha$. Thus, a sufficient condition for C_B to be closed-range on each of the weighted Bergman spaces is that $\bigcup_\nu B(I_\nu) = \mathbb{T}$.

Lemma 3.1 *(Lemma 2.5 in [4]) Let B be a Frostman Blaschke product with infinitely many zeros $\{a_n\}_{n=1}^\infty$, listed according to multiplicity. Then, for any point ζ^* in σ_B and for any $\delta > 0$, there exists a subsequence $\{a_{nk}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $|\zeta^* - a_{nk}| < \delta$ for all k and*

$$\sup_{\zeta \in \sigma_B} \frac{1-|a_{nk}|^2}{|\zeta - a_{nk}|} \to 0$$

as $k \to \infty$.

Proof. Suppose there exists $\delta, c > 0$ and $\zeta^* \in \sigma_B$ such that

$$\sup_{\zeta \in \sigma_B} \frac{1-|a_n|^2}{|\zeta - a_n|} \geq c$$

whenever $|\zeta^* - a_n| < \delta$. Since ζ^* is in σ_B, we can find ζ_1 in σ_B and $n_1 > 0$ such that $|\zeta^* - a_{n1}| < \delta$ and

$$\frac{1 - |a_{n1}|^2}{|\zeta_1 - a_{n1}|} > \frac{c}{2}.$$

So, by choosing a_{n1} close enough to ζ^*, we can make ζ_1 as close to ζ^* as we want. Since ζ_1 is in σ_B, we can find $n_2 > n_1$ so that a_{n2} is close enough to ζ_1 so that $|\zeta^* - a_{n2}| < \delta$. Then, there exists ζ_2 in σ_B so that

$$\frac{1 - |a_{n2}|^2}{|\zeta_2 - a_{n2}|} > \frac{c}{2}.$$

Thus, by choosing a_{n2} close enough to ζ_1, we can make ζ_2 to be as close to ζ_1 as we would like. Hence, for $j = 1, 2$, we can force $|\zeta^* - \zeta_2| < \delta$ and

$$\frac{1 - |a_{nj}|^2}{|\zeta_2 - a_{nj}|} > \frac{c}{2}.$$

In a similar manner, we can choose $n_3 > n_2$ so that a_{n_3} is close enough to ζ_2 to ensure that $|\zeta^* - a_{n_3}| < \delta$ and we can find ζ_3 in σ_B so that a_{n_3} is close enough to ζ^2 to ensure that $|\zeta^* - a_{n_3}| < \delta$, and we can choose ζ_3 in σ_B so that $|\zeta^* - \zeta_3| < \delta$ and

$$\frac{1 - |a_{nj}|^2}{|\zeta_3 - a_{nj}|} > \frac{c}{2}$$

for $j = 1, 2, 3$. We may then continue in this manner to find a subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and a sequence $\{\zeta_j\}_{j=1}^{\infty}$ in σ_B with the property that

$$\frac{1 - |a_{n_j}|^2}{|\zeta_J - a_{n_j}|} > \frac{c}{2},$$

where $J \in \mathbb{N}$ and $1 \leq j \leq J$. By the compactness of σ_B, $\{\zeta_j\}_{j=1}^{\infty}$ has an accumulation point, ζ_0, in σ_B, which also fulfills the condition

$$\frac{1 - |a_{n_j}|^2}{|\zeta_0 - a_{n_j}|} \geq \frac{c}{2}.$$

But this means that $g_B(\zeta_0) = \sum_n \frac{1-|a_n|^2}{|\zeta_0 - a_n|}$ diverges, and hence, B cannot be a Frostman Blaschke product. Thus, the result is proved. □

Proposition 3.2 *(Proposition 2.6 in [4].) Let B be a Frostman Blaschke product with infinitely many zeros $\{a_n\}_{n=1}^{\infty}$, listed according to multiplicity, and let $\{I_\nu\}_\nu$ be an enumeration of the components of $\mathbb{T} \setminus \sigma_B$. Let ω_ν denote the number of radians through which B wraps I_ν, which may be infinite. Then, for any point ζ^* in σ_B and any $\delta > 0$, at least one of the following hold.*

1. *There is a component I_{ν_0} of $\mathbb{T} \setminus \sigma_B$ such that $dist(\zeta^*, I_{\nu_0}) < \delta$ and $\omega_{\nu_0} = \infty$.*
2. *There are infinitely many components $\{I_{\nu_k}\}_{k=1}^{\infty}$ of $\mathbb{T} \setminus \sigma_B$ contained in $\{\zeta \in \mathbb{T} : |\zeta - \zeta^*| < \delta\}$ such that $\liminf_{k\to\infty} \omega_{\nu_k} \geq 2\pi$.*

Proof. By Lemma 3.1, we can find a sequence $\{a_{n_k}\}_{k=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ such that each a_{n_k} is as close to ζ^* as we wish. We can also find a corresponding sequence $\{I_{\nu_k}\}_{k=1}^{\infty}$ of not necessarily distinct components of $\mathbb{T} \setminus \sigma_B$ such that

$$\sup_{\zeta \in \mathbb{T} \setminus I_{\nu_k}} \frac{1 - |a_{n_k}|^2}{|\zeta - a_{n_k}|} \to 0,$$

as $k \to \infty$. Since, for $0 < a < b$, $\int_a^b \frac{1}{t^2} = \frac{b-a}{ab} < \frac{1}{a}$, we have

$$\int_{\mathbb{T} \setminus I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta| \to 0$$

and so

$$\int_{I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta| \to 2\pi$$

as $k \to \infty$. Since, by Lemma 3.1 in [1],

$$\omega_\nu \geq \sum_{\{k:\nu(k)=\nu} \int_{I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta|,$$

the proposition is proved. $\square$

If there is a ζ^* in σ_B and a $\delta > 0$ so that condition (1) in proposition 3.2 holds, then the composition operator C_B will be closed-range on $\mathbb{A}^p_\alpha$ for every p, $1 \leq p < \infty$. But then, if C_B is not closed-range on $\mathbb{A}^p_\alpha$ for every p, then it must be that condition (2) in proposition 3.2 holds. It must also be the case that $\bigcup_\nu B(I_\nu) \neq \mathbb{T}$. If condition (2) and the previous statement are both true, then, there exists ζ_0 in $\mathbb{T}$ so that for and open arc γ in $\mathbb{T}$ having nonempty intersection with φ_B, $B\left(\gamma \setminus \varphi_B\right) = \mathbb{T} \setminus \{\zeta_0\}$. This does not seem very probably and one may suspect that every Frostman Blaschke product will give rise to a closed range composition operator C_B on every $\mathbb{A}^p_\alpha$ space. Indeed, J. Akeroyd and P. Ghatage have constructed an example of a Frostman Blaschke product that does not do so. Suppose B is a Frostman Blaschke product such that C_B is not closed-range on $\mathbb{A}^p_\alpha$ for any p. Since any pertubation of a zero of B affects the image of each component under B of $\mathbb{T} \setminus \varphi_B$ unequally, then for a Blaschke product B^* obtained by shifting the location of only one of the zeros of B, C_{B^*} will be closed-range on $\mathbb{A}^p_\alpha$ for any p. Thus, if the composition operator C_B is not closed-range on $\mathbb{A}^p_\alpha$ for every p then there is a sequence of Frostman Blaschke products $\{B^*_k\}_{=1}^\infty$ so that C_{B^*} is closed-range on $\mathbb{A}^p_\alpha$ for every p.

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