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On the Representation of Inverse Semigroups by Difunctional Relations

Nathan Bloomfield
University of Arkansas, Fayetteville

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On the Representation of Inverse Semigroups by Difunctional Relations

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by

Nathan E. Bloomfield
Drury University
Bachelor of Arts in Mathematics, 2007
University of Arkansas
Master of Science in Mathematics, 2011

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University of Arkansas

Abstract

A semigroup S is called *inverse* if for each $s \in S$, there exists a unique $t \in S$ such that $sts = s$ and $tst = t$. A relation $\sigma \subseteq X \times Y$ is called *full* if for all $x \in X$ and $y \in Y$ there exist $x' \in X$ and $y' \in Y$ such that (x, y') and (x', y) are in σ, and is called *difunctional* if σ satisfies the equation $\sigma\sigma^{-1}\sigma = \sigma$. Inverse semigroups were introduced by Wagner and Preston in 1952 [55] and 1954 [38], respectively, and difunctional relations were introduced by Riguet in 1948 [39]. Schein showed in 1965 [45] that every inverse semigroup is isomorphic to an inverse semigroup of full difunctional relations and proposed the following question: given an inverse semigroup S, can we describe *all* of its representations by full difunctional relations? We demonstrate that each such representation may be constructed using only S itself.

It so happens that the full difunctional relations on a set X are essentially the bijections among its quotients. This observation invites us to consider Schein's question as fundamentally a problem of *symmetry*, as we explain. By Cayley's Theorem, groups are naturally represented by permutations, and more generally, every permutation representation of a group can be constructed using representations induced by its subgroups. Analogously, by the Wagner-Preston Theorem, inverse semigroups are naturally represented by one-to-one partial mappings, and every representation of an inverse semigroup can be constructed using representations induced by certain of its inverse subsemigroups. From a universal algebraic point of view the permutations and one-to-one partial functions on a set X are the automorphisms (global symmetries) of X and the isomorphisms among subsets (local symmetries) of X, respectively. Inspired by the interpretation of difunctional relations as isomorphisms among quotients, or *colocal* symmetries, we introduce a class of partial algebras which we call *inverse magmoids*. We then show that these algebras include all inverse semigroups and groupoids and play a role among difunctional relations analogous to that played by groups among permutations and of inverse semigroups among one-to-one partial functions.

This dissertation is approved for recommendation to the Graduate Council.

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Dr. Boris Schein

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Dedication

For Violet and Lucy; you amaze me every day. And for Stacie, my dearest friend and companion and the mother of my children, without whose unending love and encouragement I would be nothing.

Contents

Summary of Main Results

1. A class of partial algebras which we call *inverse magmoids* generalizes the classes of inverse semigroups, groupoids, and posets under meet, and includes the set $\mathsf{Dif}(X)$ of full difunctional relations on a fixed set X under composition. Moreover, every inverse magmoid can be (weakly) embedded in $\mathsf{Dif}(X)$ for some set X.
2. Given an inverse magmoid M, a family $\mathcal{M} = \{M_x\}_{x\in X}$ of strong inverse submagmoids of M which contain the idempotents, and a coset system H for $\mathcal{M}$, we can construct a difunctional representation of M which we say is *induced* by H.
3. Every difunctional representation of an inverse magmoid M is obtained by inflating a representation of the form $\sum_I \varphi_i$ where each φ_i is induced by a coset system.

1 Introduction

Recall that a *group* is a set G with an associative binary operation $\cdot$ and an element $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$ and given $g \in G$, there exists an element $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$. Two fundamental results regarding groups are that (i) the set $\mathsf{Sym}(X)$ of permutations on a set X is a group under composition and (ii) every group is isomorphic to a group of permutations. This second result is named in honor of its discoverer Arthur Cayley, who demonstrated that every group is in bijective correspondence with a set of permutations of itself (as a set) in 1854 [6]. This was refined to an injective homomorphism by Camille Jordan in 1870 [26, p.60]. In modern language Cayley's Theorem may be stated as follows.

Theorem 1.1 (Cayley)**.** If G is a group, then the map $\varphi : G \to \mathsf{Sym}(G)$ given by $\varphi(g)(a) = ag^{-1}$ is an injective group homomorphism.

Permutations are concrete, computationally useful objects, while axiomatic groups are usually not. On the other hand, it is typically more pleasant to prove theorems of sweeping generality using an axiomatic approach. We thus have two vantage points from which to view groups, each having its own merits, and Cayley's Theorem allows us to move between the two with no loss of generality. We exploit this equivalence between the concrete and the abstract to great effect, particularly when we generalize from permutations on *sets* to automorphisms on algebras in other varieties[1] such as, say, lattices or vector spaces.

We can interpret Cayley's Theorem in a slightly different way. Given a group G and a set X, a *permutation representation* of G on X is a group homomorphism $\varphi : G \to \mathsf{Sym}(X)$. If φ is injective, we say the representation is *faithful*. In this light Cayley's theorem asserts

[1]With apologies to any algebraic geometers in the audience, for the next few dozen pages the word *algebra* will mean *universal algebra*, viz., a set equipped with some finitary operations. A *variety* is a class of algebras which all satisfy a given set of universally quantified equations. More on universal algebra can be found in Grätzer [21].

that every group has a faithful permutation representation. We can study permutation representations as a class of structures of their own interest, complete with an appropriate notion of isomorphism: two representations φ and ψ of G on X and Y, respectively, are called *isomorphic*, denoted $\varphi \cong \psi$, if there is a bijective map $\theta : X \to Y$ such that for all $g \in G$ we have $\psi(g) = \theta \circ \varphi(g) \circ \theta^{-1}$. Given a particular group G we might reasonably ask: what are *all* of its permutation representations, up to isomorphism? For example, each subgroup induces a permutation representation as follows.

Proposition 1.2. Let G be a group and $H \leq G$ a subgroup. Define a relation σ_H on G by $\sigma_H = \{(a, b) \mid ab^{-1} \in H\}$.

(i) The relation σ_H is an equivalence on G, and if $a, b, c \in G$ such that $a\ \sigma_H\ b$, then $ac\ \sigma_H\ bc$ (such relations are sometimes called *right congruences*). The classes of σ_H are precisely the subsets of the form Ha with $a \in G$, which we call the *cosets* of H.

(ii) For each $g \in G$, the relation $\varphi(g) = \{(Ha, Hag^{-1}) \mid a \in G\}$ on G/σ_H is a permutation.

(iii) The mapping $\varphi : G \to \mathsf{Sym}(G/\sigma_H)$ is a permutation representation of G, called the *coset representation* induced by H.

In fact Cayley's Theorem essentially concerns the coset representation induced by the trivial subgroup $H = 1$. More generally, we can think of coset representations as the basic pieces from which all other permutation representations are constructed, as outlined in the following well-known result (cf. Hall [23, §5.3]).

Proposition 1.3. Let G be a group.

(i) If $\varphi_i : G \to \mathsf{Sym}(X_i)$ is a family of permutation representations of G indexed by a set I and $\coprod_I X_i = \bigcup_I (X_i \times \{i\})$ is the disjoint union of the sets X_i, then the map $\sum_I \varphi_i : G \to \mathsf{Sym}(\coprod_I X_i)$ given by $(\sum_I \varphi_i)(g)(x, k) = (\varphi_k(g)(x), k)$ is a permutation representation of G.

(ii) If $\varphi : G \to \mathsf{Sym}(X)$ is a permutation representation and ε the relation on X such that $x\ \varepsilon\ y$ precisely when $y = \varphi(g)(x)$ for some $g \in G$, then (1) ε is an equivalence,

(2) if C is an ε-class of X, then the restriction $\varphi_C : G \to \mathsf{Sym}(C)$ of $\varphi(g)$ to C is a permutation representation of G, and (3) $\varphi \cong \sum_{C \in X/\varepsilon} \varphi_C$ via $\theta : X \to \bigsqcup_{A \in X/\varepsilon} A$ given by $\theta(x) = (x, [x]_\varepsilon)$.

(iii) A representation $\varphi : G \to \mathsf{Sym}(X)$ is called *transitive* if for all $x, y \in X$, there exists $g \in G$ such that $\varphi(g)(x) = y$. If φ is transitive then there is a subgroup $H \leq G$ such that $\varphi \cong \varphi_H$, where φ_H is the coset representation of G induced by H. Specifically, we may choose $H_x = \{g \in G \mid \varphi(g)(x) = x\}$ for any $x \in X$, and the isomorphism is then $\theta_x = \{(y, H_x a) \mid y \in X, a \in G, \varphi(a^{-1})(x) = y\}$.

(iv) In particular, the representations φ_C in part (ii) are transitive. So every permutation representation of G is isomorphic to $\sum_I \varphi_{H_i}$, where $\{H_i\}_I$ is a family of subgroups indexed by a set I and φ_H is the coset representation induced by H.

Together these results demonstrate that permutations form a natural class of representation objects for groups because (i) every group has a faithful representation by permutations and (ii) every permutation representation of a given group G can be explicitly built, in an easily described manner, using only the subgroups of G and basic constructions on sets.

It is natural, then, to think of groups as encapsulating the notion of *global symmetry*; that is, the permutations of an object which preserve its structure (whatever that means). Indeed historically the axiomatic definition of groups (by Dyck [54]) came after the interpretation of permutations as symmetries. But of course there are situations where an object may have interesting structure which is not detected by its global symmetries. As a simple example, the symmetric group $\mathsf{Sym}(4)$ on four objects and the quaternion group Q_8 have the same automorphism group, namely $\mathsf{Sym}(4)$, and so are indistinguishable from this perspective. A more visually dramatic example is the isometry groups of the square and of the figure known commonly as the Sierpiński carpet, shown in Figure 1. Introduced by Wacław Sierpiński in 1916 [50], this fractal is obtained from a solid square by removing the middle ninth and recursing on the remaining eight smaller solid squares. Intuitively, a Sierpiński carpet is vastly more self-similar than a square, and we would like for our mathematical notion of symmetry to reflect this. However these figures have the same isometry group, namely the dihedral group of order 8.

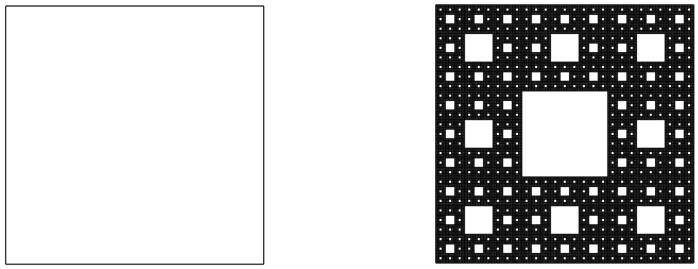

Figure 1: A square and a Sierpiński carpet

Examples such as these seem to expose a limitation in the group-centric abstraction of symmetry. A natural refinement is to consider isomorphisms among substructures of an object; some authors have called these *partial* [31] or *internal* [62] symmetries, in contrast with *total* or *external* symmetries. We will refer to an isomorphism between subsets of a set (or more generally subobjects of an object) as a *local symmetry*.

Both Viktor Wagner [55, 56] and Gordon Preston [37] studied the set $\mathsf{SymInv}(X)$ of all one-to-one partial mappings from a set to itself. A *one-to-one partial mapping* from X to Y is a subset $\varphi \subseteq X \times Y$ which is well-defined (if $(x, y_1), (x, y_2) \in \varphi$ then $y_1 = y_2$) and one-to-one (if $(x_1, y), (x_2, y) \in \varphi$ then $x_1 = x_2$). This set is prototypical among a class of semigroups S having the property that for every element $x \in S$, there is a unique element $y \in S$ such that $xyx = x$ and $yxy = y$. Such semigroups are called *inverse*, and by analogy with $\mathsf{Sym}(X)$ the set $\mathsf{SymInv}(X)$ is called the *symmetric inverse semigroup* on X. By 'prototypical' we mean that (i) $\mathsf{SymInv}(X)$ is itself an inverse semigroup and (ii) every other inverse semigroup S can be embedded in $\mathsf{SymInv}(S)$. This second result is called the Wagner-Preston Theorem in honor of its co-discoverers, who published the result on opposite sides of the Iron Curtain in 1952 [55] and 1954 [38], respectively.

Theorem 1.4 (Wagner-Preston)**.** If S is an inverse semigroup, then $\varphi : S \to \mathsf{SymInv}(S)$ given by $\varphi(s)(t) = ts^{-1}$ if $t \in Ss^{-1}s$ and undefined otherwise is an injective homomorphism of inverse semigroups.

A proof can be found in Howie [24, §5.1] or in Lawson [31, §1.5], which is also an excellent source of historical notes. Every group is also an inverse semigroup, and in this case we see that Wagner-Preston is a direct generalization of Cayley's Theorem. In 1962 [48], Boris Schein[2] characterized the semigroups which can be embedded in an inverse semigroup; as a corollary of this work we can deduce that analogues of 1.2 and 1.3 also hold for inverse semigroups and one-to-one partial maps. We describe the proof here with just enough detail to demonstrate the similarity to Theorem 1.3. A homomorphism $\varphi : S \to \mathsf{SymInv}(X)$ is called a representation of S by one-to-one partial functions on X. Two representations φ and ψ of S on X and Y, respectively, are called *isomorphic* if there is a bijection $\theta : X \to Y$ such that for all $s \in S$ we have $\psi(s) = \theta \circ \varphi(s) \circ \theta^{-1}$. Every inverse semigroup S comes equipped with a 'natural' partial order relation; given $s, t \in S$ we say that $s \leq t$ if there is an idempotent $e \in S$ such that $s = et$. Given a subset $H \subseteq S$, the *up-closure* of H is defined to be $H^{\uparrow} = \{s \mid h \leq s \text{ for some } h \in H\}$ and we say a subset is *up-closed* if $H^{\uparrow} = H$. Every inverse subsemigroup which is up-closed under the natural partial order induces a representation by one-to-one partial mappings as follows.

Theorem 1.5 (Schein [48])**.** Let S be an inverse semigroup. Given an up-closed inverse subsemigroup $H \subseteq S$, define a relation σ_H on S by $\sigma_H = \{(s,t) \mid st^{-1} \in H\}$.

(i) The relation σ_H is a partial equivalence (that is, symmetric and transitive) on S which is reflexive precisely on the set $D_H = \{s \mid ss^{-1} \in H\}$, and if $a, b, c \in S$ with $a \; \sigma_H \; b$ then either $ac \; \sigma_H \; bc$ or neither ac nor bc are in D_H (such relations are sometimes called *partial right congruences*). The classes of σ_H are precisely the sets $(Hs)^{\uparrow}$ with $s \in S^1$.

(ii) The relation $\varphi(s) = \{(A, (As^{-1})^{\uparrow}) \mid A, (As^{-1})^{\uparrow} \in S/\sigma_H\}$ is a one-to-one partial mapping on S/σ_H.

(iii) The map $\varphi : S \to \mathsf{SymInv}(S/\sigma_H)$ is a representation of S by one-to-one partial mappings, called the *principal representation* induced by H.

[2]Schein's original paper is in Russian; treatments in English may be found in volume II of Clifford and Preston's book [7, §§7.2-7.3] and in Howie's monograph [24, §5.8].

The inverse subsemigroups of a group are precisely its subgroups, and every subgroup is up-closed simply because the natural partial order on a group is trivial. In this case the σ_H in Theorem 1.5 is the same as the σ_H in Theorem 1.2. We can think of the principal representations of S as the basic pieces from which all other one-to-one partial representations are constructed.

Theorem 1.6 (Schein [48])**.** Let S be an inverse semigroup.

(i) If $\varphi_i : S \to \mathsf{SymInv}(X_i)$ is a family (indexed by I) of representations of S by one-to-one partial mappings, then the map $\sum_I \varphi_i : S \to \mathsf{SymInv}(\coprod_I X_i)$ given by $(\sum_I \varphi_i)(x, k) = (\varphi_k(x), k)$ if $x \in \mathsf{dom}\ \varphi_k$ and undefined otherwise is a representation of S by one-to-one partial mappings.

(ii) A representation $\varphi : S \to \mathsf{SymInv}(X)$ is called *effective* if for every $x \in X$, there exists an $s \in S$ and $y \in X$ such that $\varphi(s)(x) = y$. Any representation which is not effective may be 'cut down' to an effective representation by tossing out those elements of X which are not in the domain of any $\varphi(s)$. If φ is effective, then (1) the relation ε on X given by $x\ \varepsilon\ y$ precisely when there exists $s \in S$ such that $\varphi(s)(x) = y$ is an equivalence, (2) if C is a ε-class of X then then map $\varphi_C : S \to \mathsf{SymInv}(C)$ such that $\varphi_C(s)$ is the restriction of $\varphi(s)$ to C is an effective representation of S, and (3) $\varphi \cong \sum_{C \in X/\varepsilon} \varphi_C$.

(iii) A representation $\varphi : S \to \mathsf{SymInv}(X)$ is called *transitive* if for all $x, y \in X$, there exists $s \in S$ such that $\varphi(s)(x) = y$. If φ is effective and transitive, then there is an up-closed inverse subsemigroup $H \subseteq S$ such that $\varphi \cong \varphi_H$.

(iv) In particular, the representations φ_C in (ii) are transitive. So every effective representation of S by one-to-one partial mappings is isomorphic to a representation of the form $\sum_I \varphi_{H_i}$, where $\{H_i\}_I$ is a family of down-closed inverse subsemigroups of S indexed by a set I and φ_H is the principal representation induced by H.

As we have described them, groups and inverse semigroups have certain features in common. In both cases, we have (i) an axiomatic class of algebras with (ii) a family of concrete instances, such that (iii) every abstract instance can be embedded in a concrete instance and (iv) every such embedding of an abstract instance X can be described using only the 'pieces' of X. Most importantly, we have (v) an interpretation of the axioms as encapsulating some kind of symmetry. With groups, the interpretation is *global symmetry*, and with inverse semigroups, *local symmetry*. The dual notion to local symmetry, what we call *colocal symmetry*, is an isomorphism among quotient objects. Are the colocal symmetries of an object the interpretation of some axiomatic class, in the above sense? Our primary goal is to resolve this question in the affirmative.

In Chapter 2 we discuss difunctional relations, our basic computational objects and the colocal analogues of permutations and one-to-one partial transformations. We deduce some of the properties such relations share and use these as axioms to define a class of partial algebras which we call *inverse magmoids*. In Chapter 3 we consider the representations of an inverse magmoid by difunctional relations and prove colocal analogues of Theorems 1.4, 1.5, and 1.6, showing that every inverse magmoid M can be represented in an inverse magmoid of difunctional relations and that every such representation is isomorphic to a representation constructed using only the structure of M itself.

2 Difunctional Relations and Inverse Magmoids

A *relation* σ is any subset $\sigma \subseteq X \times Y$, where X and Y are sets. We will write $x \ \sigma \ y$ to mean $(x, y) \in \sigma$, a convention which enables shorthand such as $x \ \sigma \ y \ \tau \ z$ if σ and τ are relations. If $\sigma \subseteq X \times X$ we say σ is a relation *on* X. Any given set X has some distinguished relations: the *diagonal* relation $\Delta_X = \{(x, x) \mid x \in X\}$ and the *entire* relation $\nabla_X = X \times X$.

If σ and τ are relations, their *composite* is $\sigma\tau = \{(x, z) \mid x \ \sigma \ y \ \tau \ z \text{ for some } y\}$. The *converse* of σ is $\sigma^{-1} = \{(y, x) \mid (x, y) \in \sigma\}$. Certainly we have that $(\sigma\tau)\omega = \sigma(\tau\omega)$, $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$, and $(\sigma^{-1})^{-1} = \sigma$ for all relations σ, τ, and ω. Moreover, if $\sigma \subseteq \tau$ then both $\sigma\omega \subseteq \tau\omega$ and $\sigma^{-1} \subseteq \tau^{-1}$. If $\sigma \subseteq X \times Y$ is nonempty, then $\Delta_X\sigma = \sigma = \sigma\Delta_Y$.

A relation $\sigma \subseteq X \times Y$ is called *total* if $\Delta_X \subseteq \sigma\sigma^{-1}$, *onto* if $\Delta_Y \subseteq \sigma^{-1}\sigma$, *well-defined* if $\sigma^{-1}\sigma \subseteq \Delta_Y$, and *one-to-one* if $\sigma\sigma^{-1} \subseteq \Delta_X$. A relation which is both total and well-defined is called a *function* or *map*, and functions which are also onto or one-to-one are called *surjective* or *injective*, respectively. A function which is both injective and surjective is called *bijective*. We will typically say $f : X \to Y$ rather than the clunkier "$f \subseteq X \times Y$ is a function"; in this case X and Y are called the *domain* and *codomain* of f, respectively. If $f : X \to Y$ and $x \in X$, then the unique $y \in Y$ such that $x \ f \ y$ is called the *image* of f *at* x and denoted $f(x)$; the *image* of f is the set of all $f(x)$ where $x \in X$. By convention we compose functions from right to left using an explicit operation $\circ$, so that if α and β are functions then $\beta \circ \alpha = \alpha\beta$ as sets.

A relation σ on X is called *reflexive* if $\Delta_X \subseteq \sigma$, *symmetric* if $\sigma^{-1} \subseteq \sigma$, *antisymmetric* if $\sigma \cap \sigma^{-1} \subseteq \Delta_X$, *transitive* if $\sigma\sigma \subseteq \sigma$, an *equivalence* if it is simultaneously reflexive, symmetric, and transitive, and a *partial order* if it is simultaneously reflexive, antisymmetric and transitive. If ε is an equivalence on X and $x \in X$, the set $[x]_\varepsilon = \{y \in X \mid x \ \varepsilon \ y\}$ is called the ε-class of x. The ε-classes of elements in X form a partition of X which we denote X/ε. If ε is an equivalence on X then the map $\pi_\varepsilon : X \to X/\varepsilon$ given by $\pi_\varepsilon(x) = [x]_\varepsilon$ is called the natural projection of X onto X/ε and is surjective. If X is a set and σ a partial order on X, we say the pair (X, σ) is a *poset*.

If $f : X \to Y$, then the relation $\mathsf{ker}\ f = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is an equivalence on X. In this case the relation $F \subseteq X/(\mathsf{ker}\, f) \times Y$ given by $\Phi = \{([x], \varphi(x)) \mid x \in X\}$ is an injective function, and in fact is the unique function $X/(\mathsf{ker}\, f) \to Y$ with the property that $F \circ \pi_{\mathsf{ker}\, f} = f$. We will refer to this result as the First Isomorphism Theorem for sets.

A relation $\sigma \subseteq X \times Y$ is called *full* if it is both total and onto. If σ satisfies $\sigma\sigma^{-1}\sigma = \sigma$ we say it is *difunctional*, and if $Y = X$ we say σ is difunctional *on* X. Clearly σ is difunctional if and only if σ^{-1} is difunctional. In addition, the containment $\sigma \subseteq \sigma\sigma^{-1}\sigma$ holds for any relation, so to show that σ is difunctional it suffices to show that $\sigma\sigma^{-1}\sigma \subseteq \sigma$. Jaques Riguet introduced difunctional relations in 1948 [39] and explored them further in his dissertation in 1951 [40]; since then they have seen use in computer science and the theory of databases [25, 49, 5], though we will not address these applications here. Given a set X we denote by $\mathsf{Dif}(X)$ the set of all full difunctional relations on X.

From a universal algebraic point of view a set is an algebra with no operations. And so, to generalize, given an algebra $\mathcal{X}$ of variety $\mathcal{V}$ we will let $\mathsf{Dif}_{\mathcal{V}}(\mathcal{X})$ denote the full difunctional relations on the carrier of $\mathcal{X}$ which are also $\mathcal{V}$-subalgebras of $\mathcal{X} \times \mathcal{X}$. Many of our proofs involving $\mathsf{Dif}(X)$ with X a set will generalize immediately to $\mathsf{Dif}_{\mathcal{V}}(\mathcal{X})$ with $\mathcal{X}$ a $\mathcal{V}$-algebra. However, in the interest of clarity we will focus our attention on difunctional relations on sets and to relegate the generalization to other varieties to corollaries. We will see that $\mathsf{Dif}_{\mathcal{V}}(X)$ is not closed under composition in general; however, it is worth noting that this is true for some values of $\mathcal{V}$ such as the variety of groups or of k-vector spaces.

As a set, $\mathsf{Dif}(X)$ essentially consists of the isomorphisms among *quotients* of X. Many basic facts about difunctional relations were first proved by Riguet [39] and included in Wagner's monograph on relation algebras [58]; apparently neither of these documents has been translated to English.

Theorem 2.1 (Riguet [39], Wagner [58])**.** Let X be a set and let σ be a full relation on X. Then σ is difunctional if and only if there exist unique equivalence relations λ and ρ on X and a unique bijection $\theta : X/\lambda \to X/\rho$ such that $\sigma = \pi_\lambda \theta \pi_\rho^{-1}$.

Theorem 2.1 is the historical justification of the name 'difunctional'; such relations have the form $\alpha\beta^{-1}$ where α and β are functions having the same image.

Corollary 2.2.

(i) $\mathsf{Dif}(X)$ contains all permutations and equivalence relations on X.

(ii) (Wagner [58, 3.6]) If σ is a full relation on a set X, then σ is difunctional precisely when there exist partitions $\mathcal{A} = \{A_i\}_I$ and $\mathcal{B} = \{B_i\}_I$ of X, indexed by a set I, such that $\sigma = \bigcup_I A_i \times B_i$. Moreover, $\mathcal{A}$ and $\mathcal{B}$ are unique.

(iii) If $\mathcal{X}$ is an algebra of variety $\mathcal{V}$ and σ a full relation on $\mathcal{X}$, then σ is difunctional if and only if there exist unique congruences λ and ρ on $\mathcal{X}$ and a unique isomorphism $\theta : \mathcal{X}/\lambda \to \mathcal{X}/\rho$ such that $\sigma = \pi_\lambda \theta \pi_\rho^{-1}$.

In the remainder of this section we will consider the elements of $\mathsf{Dif}(X)$ in more detail. First, we give some examples to demonstrate that many of the properties we look for in group-like structures do *not* hold in $(\mathsf{Dif}(X), \circ)$, at least over the variety of sets.

Example 2.3.

(i) Let $\sigma = (\{1,2\} \times \{1\}) \cup (\{3\} \times \{2,3\})$. Then $\sigma \in \mathsf{Dif}(X)$ but $\sigma^2 \notin \mathsf{Dif}(X)$.

(ii) Now letting $\tau = (\{1\} \times \{2\}) \cup (\{2,3\} \times \{1,3\})$ and $\omega = (\{1,3\} \times \{1\}) \cup (\{2\} \times \{2,3\})$, we have that $\sigma\tau$ and $\tau\omega$ are difunctional, but $\sigma\tau\omega$ is not.

(iii) If ε, δ, and η are the equivalences whose classes are $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, and $\{\{1,2,3\},\{4\}\}$, respectively, then $(\varepsilon\delta)\eta$ is difunctional but $\delta\eta$ is not.

(iv) The equivalence relations ε and δ whose classes are $\{\{1,2\},\{3\}\}$ and $\{\{1\},\{2,3\}\}$, respectively, are noncommuting idempotents.

So lots of 'bad' things can happen in $\mathsf{Dif}(X)$. Example (ii) is especially disturbing; usually the most interesting algebras with a binary operation may also be viewed as categories, with the elements acting as maps and multiplication as composition. This example demonstrates that whatever structure $\mathsf{Dif}(X)$ has, it doesn't behave like a category, at least in the usual sense. All is not lost, however; we can precisely describe the idempotents in $\mathsf{Dif}(X)$ (that is, relations σ such that $\sigma\sigma = \sigma$) and characterize the pairs (σ, τ) such that $\sigma\tau$ is again difunctional.

Proposition 2.4. Let $\varepsilon, \delta, \sigma, \tau \in \mathsf{Dif}(X)$.

(i) The following are equivalent: (a) ε is idempotent, (b) ε is reflexive, (c) ε is transitive, and (d) ε is an equivalence. [58]

(ii) If ε and δ are idempotent, then the following are equivalent: (a) $\varepsilon\delta$ is difunctional, (b) $\varepsilon\delta$ is idempotent, (c) $\varepsilon\delta$ is an equivalence, (d) $\varepsilon\delta$ is symmetric, and (e) ε and δ commute. [45]

(iii) The composite $\sigma\tau$ is in $\mathsf{Dif}(X)$ if and only if $\sigma^{-1}\sigma$ and $\tau\tau^{-1}$ commute.

Recall that Theorem 2.1 characterizes the difunctional relations on X as precisely those relations of the form $\alpha\beta^{-1}$ where $\alpha, \beta : X \to Y$ are surjective functions on X having the same codomain. With 2.4(iii), this allows us to give the following explicit characterization of the difunctional composite of difunctional relations.

Proposition 2.5. Let σ and τ be difunctional relations on X; say $\sigma = \alpha_1\beta_1^{-1}$ and $\tau = \alpha_2\beta_2^{-1}$, where $\alpha_1, \beta_1 : X \to Y_1$ and $\alpha_2, \beta_2 : X \to Y_2$ are surjective. Note that $\sigma^{-1}\sigma = \mathsf{ker}\ \beta_1$ and $\tau\tau^{-1} = \mathsf{ker}\ \alpha_2$. If $\sigma\tau$ is difunctional, then $\omega = (\mathsf{ker}\ \beta_1)(\mathsf{ker}\ \alpha_2)$ is an equivalence. By the First Isomorphism Theorem for sets, there exist unique mappings $B_1 : Y_1 \to X/\omega$ and $A_2 : Y_2 \to X/\omega$ such that $B_1 \circ \beta_1 = \pi_\omega = A_2 \circ \alpha_2$; that is, unique B_1 and A_2 such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[dr, "\alpha_1"'] & & X \arrow[dl, "\beta_1"] \arrow[dd, "\pi_\omega"] \arrow[dr, "\alpha_2"] & & X \arrow[dl, "\beta_2"] \\
& Y_1 \arrow[dr, "B_1"'] & & Y_2 \arrow[dl, "A_2"] & \\
& & X/\omega & &
\end{tikzcd}$$

Then $\sigma\tau = \alpha_1 B_1 A_2^{-1}\beta_2^{-1}$.

To summarize, composition is a partial binary operation and conversion a unary operation on $\mathsf{Dif}(X)$ which satisfy the following properties: (i) if $\sigma\tau$ and $\tau\omega$ are difunctional, then if either of $(\sigma\tau)\omega$ or $\sigma(\tau\omega)$ is difunctional, then so is the other, and the two are equal; (ii) $(\sigma^{-1})^{-1} = \sigma$; (iii) $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$ are difunctional; (iv) if $\sigma\tau$ is difunctional, then $\tau^{-1}\sigma^{-1}$ is difunctional and is equal to $(\sigma\tau)^{-1}$; (v) $\sigma(\sigma^{-1}\sigma)$ is difunctional and equals σ; and (vi) $\sigma\tau$ is difunctional if and only if $(\sigma^{-1}\sigma)(\tau\tau^{-1})$ and $(\tau\tau^{-1})(\sigma^{-1}\sigma)$ are difunctional and equal.

Several authors (notably Leech [32] and FitzGerald [18]) have discussed full difunctional relations as dual partial symmetries, typically as a dual (in the categorical sense) of the symmetric inverse semigroup. However, the nonclosure of composition on $\mathsf{Dif}(X)$ complicates matters. One way to handle this complication is to 'fix' composition so that it becomes total and makes $\mathsf{Dif}(X)$ into an inverse semigroup. Indeed this can be done; define $\bullet$ on $\mathsf{Dif}(X)$ by $\sigma \bullet \tau = \bigcap\{\omega \in \mathsf{Dif}(X) \mid \sigma \circ \tau \subseteq \omega\}$. As is shown by FitzGerald in [14] and Bredikhin in [4], now $(\mathsf{Dif}(X), \bullet)$ is an inverse semigroup and $\bullet$ extends $\circ$ in the sense that if $\sigma \circ \tau$ is difunctional then $\sigma \bullet \tau = \sigma \circ \tau$. Extending composition in this way is quite natural and leads to some interesting mathematics; cf. [15, 16, 18, 17, 11, 10, 9, 30, 34, 8, 12].

Another way to handle the nonclosure of composition on $\mathsf{Dif}(X)$ is to wear the hair shirt, so to speak, and accept the fact that composition does not behave nicely. This is the point of view we will take. There are practical reasons to prefer plain composition $\circ$ over the extended composition $\bullet$; notably, from a computational point of view, it is more difficult in general to compute a difunctional closure than a composite. In addition there are *a priori* model-theoretic differences, as the theory modeled by relation composition $\circ$ is finitely axiomatizable [46] while the closure $\bullet$ is not definable in first-order logic. From a more philosophical perspective, as Schein argues in [43, 41] (and elsewhere), relation composition is a fundamental binary operation in algebra, and relation algebras (even partial algebras) are frequently interesting. In the sequel we will define a class of partial algebras which attempt to capture the 'essential nature' of $\mathsf{Dif}(X)$ under composition, and would like to avoid imposing unnecessary structure on these algebras. Most saliently we will consider

$\mathsf{Dif}(X)$ as a partial algebra because our results do not require otherwise.

We use the equational laws satisfied by $\mathsf{Dif}(X)$ to define a class of partial algebras.

Definition 2.6. Let M be a set, $\cdot$ a partial binary operation on M, and $^{-1}$ a unary operation on M. The pair $(M, \cdot)$ is called a *magmoid.*

- A magmoid $(M, \cdot)$ is called *quasiassociative* if for all $s, t, u \in M$ we have the following: (M1) if $s \cdot t$ and $t \cdot u$ exist, then if either of $(s \cdot t) \cdot u$ or $s \cdot (t \cdot u)$ exist, then so does the other, and the two are equal.
- If $(M, \cdot)$ is a quasiassociative magmoid, we say that $(M, \cdot, ^{-1})$ is an *involuted magmoid* if in addition we have the following for all $s, t \in M$: (M2a) $(s^{-1})^{-1} = s$, (M2b) $s \cdot s^{-1}$ and $s^{-1} \cdot s$ exist, and (M2c) if $s \cdot t$ exists, then $t^{-1} \cdot s^{-1}$ exists and is equal to $(s \cdot t)^{-1}$.
- An involuted magmoid is called *inverse* if in addition we have the following for all elements $s, t \in M$: (M3a) $s \cdot (s^{-1} \cdot s)$ exists and equals s and (M3b) $s \cdot t$ exists if and only if $(s^{-1} \cdot s) \cdot (t \cdot t^{-1})$ and $(t \cdot t^{-1}) \cdot (s^{-1} \cdot s)$ exist and are equal.

We will refer to $\cdot$ as the *partial product* and $^{-1}$ as *inversion.* A magmoid element s is called *idempotent* if $s \cdot s$ exists and equals s; an inverse magmoid in which every element is idempotent is called a *semilattoid.* We say that two elements s and t *commute* if both $s \cdot t$ and $t \cdot s$ exist and the two are equal.

We have several examples of inverse magmoids: $\mathsf{Dif}(X)$, of course, but also every inverse semigroup and every groupoid is an inverse magmoid, as is every poset under the partial operation "greatest lower bound" (which we call the inverse magmoid *induced by* the poset). Recall that a semigroup S is called *inverse* if for every $s \in S$, there exists a unique element $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$, and that a *poset* is a set P equipped with a relation $\leq$ which is reflexive, antisymmetric, and transitive. A *groupoid* is a small category in which every morphism is invertible. Perhaps then it would be better to call these the inverse 'semigroupoid' axioms; however this term is already in use, first by Wagner explicitly in [60, 61] and in spirit in [57, 59], by his student Pavlovskiĭ [35, 36], and more recently by others [27, 13], denoting a category with some identity morphisms removed.

Inverse semigroups were introduced by Wagner in 1952 and Preston in 1954 as an axiomatization of the algebra of one-to-one partial maps on a set under composition and conversion; there are several other equivalent definitions, some of which appear in [33] and [42]. Groupoids were introduced (as partial algebras) in 1926 [1, 2] by Heinrich Brandt, who was interested in extending the work of Gauss on quadratic forms [29, 19]. We will think of inverse magmoids as simultaneously generalizing the classes of inverse semigroups and groupoids. Many properties which hold in any inverse semigroup generalize more or less immediately to any inverse magmoid. We will merely state those properties which will be needed; proofs are straightforward and can be found in most texts on inverse semigroups.

Proposition 2.7. If M is an inverse magmoid with $s, t \in M$, then (i) $s \cdot s^{-1}$ and $s^{-1} \cdot s$ are idempotent, (ii) $(s \cdot s^{-1})^{-1} = s \cdot s^{-1}$, (iii) $(s \cdot s^{-1}) \cdot s$ exists and equals s, and (iv) $s \cdot t$ exists if and only if $s \cdot (t \cdot t^{-1})$ exists if and only if $(s^{-1} \cdot s) \cdot t$ exists.

Inverse magmoids behave very much like inverse semigroups. This is to be expected, because our inspirational example, $\mathsf{Dif}(X)$, may always be embedded in an inverse semigroup. That is not to say that inverse magmoids are subsumed by inverse semigroups; while every inverse magmoid can be embedded in an inverse semigroup, neither the inverse semigroup nor the embedding is unique in general. Also, it is not known if such an embedding can be achieved without appending new elements to M, though we can think of an inverse magmoid as an inverse semigroup from which some information has been tossed out.

Proposition 2.8. Let $e \in M$ be idempotent. Then we have the following: (i) e^{-1} is idempotent, (ii) $e^{-1} = e$, (iii) $e^{-1} \cdot e = e \cdot e^{-1} = e$, (iv) if $s \in M$ such that $s \cdot e$ exists, then $e \cdot s^{-1}$ exists, and $s \cdot (e \cdot s^{-1})$ exists and is idempotent, (v) if f is idempotent and $e \cdot f$ exists, then $f \cdot e$ exists and equals $e \cdot f$, and (vi) if f is idempotent and $e \cdot f$ exists, then $e \cdot f$ is idempotent.

Corollary 2.9. Given an inverse magmoid M the set $\mathsf{E}(M)$ of idempotents in M is a semilattoid, called the *semilattoid of idempotents* of M.

We define a relation κ on $\mathsf{E}(M)$ by $e \ \kappa \ f$ if and only if $e \cdot f$ exists. Certainly κ is both reflexive and symmetric; we will say M is κ-transitive if κ is also transitive. Evidently M is an inverse semigroup precisely when $\kappa = \nabla_{\mathsf{E}(M)}$ and a groupoid precisely when $\kappa = \Delta_{\mathsf{E}(M)}$, and M is a semilattoid precisely when $\mathsf{E}(M) = M$. We can think of inverse magmoids as simultaneously generalizing inverse semigroups and groupoids, with these two subclasses consisting of somehow extreme examples. Analogously, an inverse semigroup S is a group precisely when $\mathsf{E}(S)$ contains only one element and a semilattice (i.e. commutative semigroup in which every element is idempotent) precisely when $\mathsf{E}(S)$ is all of S, and a groupoid G is a group precisely when $\mathsf{E}(G)$ contains only one element and a small discrete category (i.e. category whose class of objects is a set and having no morphisms other than the identities) precisely when $\mathsf{E}(G) = G$. Small discrete categories are not tremendously interesting, though it is of note that a set-indexed categorical product (coproduct) is the limit (colimit) of a functor from a small discrete category. A given semilattoid is not necessarily induced by a poset (for example, there are five posets with three elements but six semilattoids) but can be embedded in a semilattoid so induced. We can visualize the relationships among these classes of partial algebras as in Figure 2, using arrows to indicate containment.

$$\begin{tikzcd}
 & \text{Inverse semigroups} \arrow[rr] & & \text{Inverse magmoids} \\
\text{Semilattices} \arrow[ur] \arrow[rr] & & \text{Semilattoids (Posets)} \arrow[ur] & \\
 & \text{Groups} \arrow[uu] \arrow[rr] & & \text{Groupoids} \arrow[uu] \\
1 \arrow[uu] \arrow[ur] \arrow[rr] & & \text{Small discrete categories} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Figure 2: Relationships among certain classes of inverse magmoids

As Example 2.3 shows, in general the inverse magmoid $\mathsf{Dif}(X)$ is not an inverse semigroup, groupoid, or semilattoid. Upon noting that inverse magmoids generalize inverse semigroups and groupoids our instinct is to try to generalize some of the basic tools used to study those structures. This will be our aim for the remainder of this chapter.

There is a 'natural' partial order relation on an inverse semigroup; we say $s \leq t$ precisely when there is an idempotent e such that $s = et$. This relation was first defined by Wagner [56] (who also found several equivalent definitions) and has proven to be an indispensable tool in the study of inverse semigroups. If we naïvely carry this notion over to inverse magmoids, it turns out that many of the useful properties the natural partial order enjoys on an inverse semigroup still hold, and the proofs generalize easily.

Proposition 2.10. Let M be an inverse magmoid, and let $s, t \in M$. Then the following are equivalent. (i) There exists an idempotent $e \in M$ such that $e \cdot t$ exists and equals s. (ii) There exists an idempotent $f \in M$ such that $t \cdot f$ exists and equals s. (iii) There exists an idempotent $e \in M$ such that $e \cdot t^{-1}$ exists and equals s^{-1}. (iv) $t \cdot s^{-1}$ exists and $(t \cdot s^{-1}) \cdot s = s$. (v) $s^{-1} \cdot t$ exists and $s \cdot (s^{-1} \cdot t) = s$. If any of these statements hold we say $s \preccurlyeq t$. Moreover, $\preccurlyeq$ is a partial order, which we call *natural*.

If M is an inverse semigroup, groupoid, or a semilattoid $(P, \wedge)$ induced by a poset, then the natural order is simply $\leq$, equality, or the order on P, respectively. The natural order on $\mathsf{Dif}(X)$ is the $\supseteq$ relation. Using the natural order we can show that inverses in an inverse magmoid are unique and (generalizing a result of Liber [33] on inverse semigroups) that the set of $\preccurlyeq$-minimal elements forms a subgroupoid.

Green's relations, noted first by Suškevič [51, 52, 53] (see also [20]) and reintroduced by Green in 1951 [22], are fundamental tools of semigroup theory. In an inverse semigroup, elements a and b are L-related if $a^{-1}a = b^{-1}b$, are R-related if $aa^{-1} = bb^{-1}$, and are H-related if they are both L and R related. Green's relations on an inverse magmoid enjoy several expected properties; every L and every R class contains a unique idempotent, two elements in the same L class lie in R classes having the same cardinality (and vice versa, a result known as Green's Lemma), and $\mathsf{LR} = \mathsf{RL}$. A generalization of Green's Lemma to sets with a partial binary operation (what we have called magmoids) was also considered by Kapp [28].

Next we discuss the action of an inverse magmoid on a set; Proposition 2.12 in particular is used heavily in the proof of Theorem 3.2.

Definition 2.11. Let M be an inverse magmoid, X a set, and $\cdot$ a partial function from $X \times M$ to X. We say that M *acts on* X if the following hold: (i) $x \cdot (s \cdot s^{-1})$ exists if and only if $x \cdot s$ exists, and (ii) if $x \cdot s$ and $s \cdot t$ exist, then if either $(x \cdot s) \cdot t$ or $x \cdot (s \cdot t)$ exists, then so does the other, and the two are equal.

We say an action is *faithful* if whenever $s, t \in M$ such that $s \cdot x$ exists if and only if $t \cdot x$ exists and in fact $s \cdot x = t \cdot x$ for all such x, then $s = t$. We have several examples of actions; for instance, any inverse magmoid acts on itself by right multiplication (this is a restatement of (M1) and 2.7(iv)). Using the natural order we see that this action is faithful.

If M acts on a set X, then we can sensibly multiply subsets of X by elements of M.

Proposition 2.12. Let M be an inverse magmoid acting on a set X. We define a setwise product on the powerset of X by $As = \{a \cdot s \mid a \in A \text{ and } a \cdot s \text{ exists}\}$. This product has the following properties for all $A, B \subseteq X$ and $s, t \in M$: (i) if $A \subseteq B$, then $As \subseteq Bs$; (ii) $(As)s^{-1} = A(s \cdot s^{-1})$; (iii) if $x \in X$ such that $x \cdot s$ exists, then $(x \cdot s) \cdot (s^{-1} \cdot s)$ exists and equals $x \cdot s$; (iv) $(A \cap Xs)(s^{-1} \cdot s) = A \cap Xs$; (v) if $s \cdot t$ exists, then $X(s \cdot t) \subseteq Xt$; (vi) if $x \in Xs$, then $x \cdot s^{-1}$ exists; and (vii) $(A \cup B)s = As \cup Bs$.

We conclude this chapter with a brief discussion about homomorphisms, submagmoids, and one-sided congruences. On a partial algebra there are multiple competing versions of these concepts (cf. Grätzer [21, ch2]).

Definition 2.13. Let M and N be inverse magmoids. A map $\varphi : M \to N$ is called a *homomorphism* if for all $s, t \in M$, if $s \cdot t$ exists in M, then $\varphi(s) \cdot \varphi(t)$ exists in N and equals $\varphi(s \cdot t)$. We say φ is *strong* if in addition whenever $\varphi(s) \cdot \varphi(t)$ exists, $s \cdot t$ also exists. A homomorphism which is also injective is called an *embedding*, and a strong homomorphism which is also bijective is called an *isomorphism*.

If is clear that the identity map is a (strong) homomorphism, and that the composite of (strong) homomorphisms is a (strong) homomorphism. Thus the classes of inverse magmoids and their (strong) homomorphisms form a category which we denote InvMag; it is also clear that InvMag contains the category of inverse semigroups and their homomorphisms as a full subcategory. Indeed our Dif operator is functorial on the category $\mathsf{Alg}_{\mathcal{V}}^{\mathsf{epi}}$ of $\mathcal{V}$-algebras with surjective algebra homomorphisms. Thus if $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic as $\mathcal{V}$-algebras then $\mathsf{Dif}_{\mathcal{V}}(\mathcal{X})$ and $\mathsf{Dif}_{\mathcal{V}}(\mathcal{Y})$ are isomorphic as inverse magmoids (as expected). As for inverse semigroup homomorphisms [31, p.30], preservation of the partial product implies preservation of inverses, idempotents, and the natural order.

Theorem 2.14. If $\varphi : M \to N$ is a homomorphism of inverse magmoids, then we have the following: (i) if $e \in M$ is idempotent, then $\varphi(e) \in N$ is idempotent, (ii) $\varphi(s^{-1}) = \varphi(s)^{-1}$, and (iii) if $s \preccurlyeq t$ then $\varphi(s) \preccurlyeq \varphi(t)$.

A subset of an inverse magmoid M which is closed under the partial product and inversion operations is again an inverse magmoid, which we call a *strong inverse submagmoid* of M. Given inverse magmoids N and M with $N \subseteq M$ such that the operations on N are contained in those on M, we might call N a *weak* inverse submagmoid of M. The difference between a weak submagmoid and a strong submagmoid of M is that the multiplication table of a weak submagmoid might have 'forgotten' some of the products among its elements which exist in M, while a strong submagmoid is required to have all the products it can. For example, $(\mathsf{Dif}(X), \circ)$ is a weak inverse submagmoid of $(\mathsf{Dif}(X), \bullet)$. Presently we are interested exclusively in strong submagmoids.

We will now briefly discuss strong one-sided congruences.

Definition 2.15. An equivalence ρ on M is called a *strong right congruence* if whenever $s \ \rho \ t$ and $s \cdot u$ exists, then $t \cdot u$ also exists and $(s \cdot u) \ \rho \ (t \cdot u)$.

First, we show that if ρ is a strong right congruence on M, then M acts on the quotient set M/ρ as one might expect.

Proposition 2.16. Let ρ be a strong right congruence on an inverse magmoid M and define a relation $\mu \subseteq (M/\rho \times M) \times M/\rho$ by $\mu = \{(([x], s), [x \cdot s]) \mid x \cdot s \text{ exists}\}$. Then μ is well-defined and gives an action of M on M/ρ.

Proof. First, note that if $(([x], s), [x \cdot s]) \in \mu$ and $x \; \rho \; y$, then since ρ is strong we have that $y \cdot s$ exists, so that $(([y], s), [y \cdot s]) \in \mu$. If $(([x], s), [x \cdot s])$ and $(([y], s), [y \cdot s])$ are in μ with $x \; \rho \; y$, then since ρ is a strong right congruence, $(x \cdot s) \; \rho \; (y \cdot s)$, and thus $[x \cdot s] = [y \cdot s]$, so μ is well-defined. Now $\mu([x], s)$ exists if and only if $x \cdot s$ exists, if and only if $x \cdot (s \cdot s^{-1})$ exists by 2.7(iv), if and only if $\mu([x], s \cdot s^{-1})$ exists. Now suppose $\mu([x], s)$ and $s \cdot t$ exist; in particular, $x \cdot s$ exists. If $\mu(\mu([x], s), t) = \mu([x \cdot s], t)$ exists, then $(x \cdot s) \cdot t$ exists and equals $x \cdot (s \cdot t)$, so $\mu([x], s \cdot t)$ exists. That is, we have

$$\mu(\mu([x], s), t) = \mu([x \cdot s], t) = [(x \cdot s) \cdot t] = [x \cdot (s \cdot t)] = \mu([x], s \cdot t).$$

Conversely, suppose $\mu([x], s \cdot t)$ exists. Then $x \cdot (s \cdot t)$ exists and equals $(x \cdot s) \cdot t$, and we have $\mu([x], s \cdot t) = [x \cdot (s \cdot t)] = [(x \cdot s) \cdot t] = \mu(\mu([x], s), t)$ as needed. □

Strong inverse submagmoids containing $\mathsf{E}(M)$ induce a class of strong right congruences.

Proposition 2.17. Let M be an inverse magmoid and let $H \subseteq M$ be an inverse submagmoid which contains $\mathsf{E}(M)$. Then we have the following.

(i) H is down-closed under the natural partial order.

(ii) The relation σ_H on M given by $\sigma_H = \{(s, t) \mid s^{-1} \cdot s = t^{-1} \cdot t \text{ and } s \cdot t^{-1} \in H\}$ is a strong right congruence.

(iii) Every σ_H-class is of the form As, where $A \subseteq M$ is an L-class contained in H.

Proof. (i) If $s \preccurlyeq t$ with $t \in H$, then we have $s = e \cdot t$ for some idempotent e. Since H contains all idempotents and is a strong submagmoid, $s \in H$. (ii) For all $s \in M$, we certainly have $s^{-1} \cdot s = s^{-1} \cdot s$ and that $s \cdot s^{-1} \in H$ since $\mathsf{E}(M) \subseteq H$. So σ_H is reflexive. If $s \; \sigma_H \; t$, then $s^{-1} \cdot s = t^{-1} \cdot t$ and $s \cdot t^{-1} \in H$. Since H is closed under inversion, $(s \cdot t^{-1})^{-1} = t \cdot s^{-1} \in H$, and of course $t^{-1} \cdot t = s^{-1} \cdot s$. So $t \ \sigma_H \ s$, and thus σ_H is symmetric. Now suppose $s \ \sigma_H \ t$ and $t \ \sigma_H \ u$. Then we have $s^{-1} \cdot s = t^{-1} \cdot t = u^{-1} \cdot u$ and $s \cdot t^{-1}, t \cdot u^{-1} \in H$. Now $s \cdot u^{-1}$ exists, and moreover we have

$$s \cdot u^{-1} = (s \cdot (s^{-1} \cdot s)) \cdot u^{-1} = (s \cdot (t^{-1} \cdot t)) \cdot u^{-1} = (s \cdot t^{-1}) \cdot (t \cdot u^{-1}) \in H$$

since H is closed under the partial product. So $s \ \sigma_H \ u$, and thus σ_H is transitive. Finally, suppose $s \ \sigma_H \ t$ and that $s \cdot u$ exists. Now $s^{-1} \cdot s$ and $u \cdot u^{-1}$ commute, and since $s^{-1} \cdot s = t^{-1} \cdot t$, in fact $t^{-1} \cdot t$ and $u \cdot u^{-1}$ commute, so that $t \cdot u$ exists. Moreover, we have

$$\begin{aligned} (s \cdot u)^{-1} \cdot (s \cdot u) &= (u^{-1} \cdot s^{-1}) \cdot (s \cdot u) = (u^{-1} \cdot (s^{-1} \cdot s)) \cdot u \\ &= (u^{-1} \cdot (t^{-1} \cdot t)) \cdot u = (t \cdot u)^{-1} \cdot (t \cdot u) \end{aligned}$$

and $(s \cdot u) \cdot (t \cdot u)^{-1} = (s \cdot (u \cdot u^{-1})) \cdot t^{-1} \preccurlyeq s \cdot t^{-1} \in H$. Since H is down-closed we have $(s \cdot u) \ \sigma_H \ (t \cdot u)$ as desired.

(iii) Let B be a σ_H-class, and let $s \in B$. Note that if $b \in B$, then $b^{-1} \cdot b = s^{-1} \cdot s$. In particular, $b \cdot s^{-1}$ exists for all $b \in B$. Moreover, since σ_H is a strong congruence, the set $Bs^{-1} = \{b \cdot s^{-1} \mid b \in B\}$ is contained in some σ_H-class; say A. Note that $s \cdot s^{-1} \in A$ is idempotent; since H contains all the idempotents in M and is a union of σ_H-classes, we have $A \subseteq H$. Note also that if $a \in A$, then $a \cdot s$ exists, and since $s \cdot s^{-1} \in A$ and $(s \cdot s^{-1}) \cdot s = s \in B$, we have $As \subseteq B$. Define $\varphi_{s^{-1}} : B \to A$ by $b \mapsto b \cdot s^{-1}$ and $\varphi_s : A \to B$ by $a \mapsto a \cdot s$. Now

$$(\varphi_s \circ \varphi_{s^{-1}})(b) = \varphi_s(b \cdot s^{-1}) = (b \cdot s^{-1}) \cdot s = b \cdot (s^{-1} \cdot s) = b \cdot (b^{-1} \cdot b) = b$$

and, since $a^{-1} \cdot a = (s \cdot s^{-1})^{-1} \cdot (s \cdot s^{-1}) = s \cdot s^{-1}$ for all $a \in A$,

$$(\varphi_{s^{-1}} \circ \varphi_s)(a) = \varphi_{s^{-1}}(a \cdot s) = (a \cdot s) \cdot s^{-1} = a \cdot (s \cdot s^{-1}) = a \cdot (a^{-1} \cdot a) = a.$$

Thus $\varphi_{s^{-1}}$ and φ_s are bijective, and we have $B = As$ and $A = Bs^{-1}$. $\square$

3 Difunctional Representations

So far we have defined a class of partial algebras, inverse magmoids, which generalizes some of the properties enjoyed by the set $\mathsf{Dif}(X)$ of full difunctional relations on a set X under composition and inversion. Moreover the set $\mathsf{Dif}(X)$ has a natural interpretation as the set of bijections among the quotients of X, which we call the colocal symmetries of X. In this chapter we will strengthen the analogy between the role of $\mathsf{Dif}(X)$ among inverse magmoids and that of $\mathsf{Sym}(X)$ among groups and $\mathsf{SymInv}(X)$ among inverse semigroups in the direction suggested by Cayley's Theorem and the Wagner-Preston Theorem. We begin by shifting our attention from the class of all inverse magmoids to the class of difunctional representations of a fixed inverse magmoid.

Definition 3.1. A *difunctional representation* of an inverse magmoid M in a nonempty set X is a (not necessarily strong) homomorphism $\varphi : M \to \mathsf{Dif}(X)$. If φ is injective, we say the representation is *faithful*.

Schein [44] showed that every inverse semigroup has a faithful representation by difunctional relations, and in fact his proof generalizes. In short, given an action of M on a set X we construct a homomorphic image of M in $\mathsf{Dif}(\mathcal{P}(X))$. The action of M on itself by right multiplication induces a faithful representation. An alternate embedding theorem for inverse semigroups in difunctional relations was given by Bredikhin [3].

Theorem 3.2 (Schein)**.** Let M be an inverse magmoid acting on a set X. For each $s \in M$, define a relation σ_s on the powerset $\mathcal{P}(X)$ of X by $\sigma_s = \{(A, B) \mid (A \cap Xs^{-1})s = B \cap Xs\}$. Then σ_s is a full and difunctional relation on $\mathcal{P}(X)$ and the mapping $\text{Ш} : M \to \mathsf{Dif}(\mathcal{P}(X))$ given by $\text{Ш}(s) = \sigma_s$ is a difunctional representation of M (not necessarily strong). If $X = M$ and the action is right multiplication, then Ш is faithful.

This result is not a perfect analogue of 1.1 (Cayley) and 1.4 (Wagner-Preston); a group G acting on a set X induces a permutation representation on X itself, while an inverse magmoid M acting on a set X induces a difunctional representation on $\mathcal{P}(X)$. However this generalization is not without its advantages; $\mathcal{P}(X)$ has some natural structure of its own which is also preserved by Ш. The following corollary is also generalized from a result of Schein [44] on inverse semigroups.

Corollary 3.3. The relation $Ш(s)$ preserves the following operations on $\mathcal{P}(X)$: finite union, finite intersection, complement, symmetric difference, $\varnothing$, and X. Thus every inverse magmoid has a faithful representation in $\mathsf{Dif}_{\mathcal{V}}(\mathcal{X})$ for some algebra $\mathcal{X}$ where $\mathcal{V}$ is the variety of sets $(\mathcal{P}(\mathcal{X}))$, groups, abelian groups, $\mathbb{Z}/(2)$-vector spaces, $(\mathcal{P}(\mathcal{X}), \Delta)$, rings $(\mathcal{P}(\mathcal{X}), \cap, \Delta)$, lattices, Boolean algebras, or Heyting algebras $(\mathcal{P}(\mathcal{X}), \cap, \cup)$. Moreover, if M is finite, then $\mathcal{X}$ may be chosen to be finite.

We will now turn our attention to the class of all difunctional representations of a fixed inverse magmoid M. We know that a faithful representation always exists, and our ultimate goal is to construct all of the difunctional representations of M in the spirit of Theorems 1.3 and 1.6. We begin by defining a gadget analogous to the cosets of a stabilizer under a group action.

Definition 3.4. Given $\varphi : S \to \mathsf{Dif}(X)$ and $x, y \in X$, we define $H^x_y = \{s \in S \mid x \ \varphi(s) \ y\}$.

We will refer to sets of this form as *H-sets* and think of H^x_y as the set of all $s \in M$ which 'move' x to y under φ, though this is a slight abuse as $\varphi(s)$ is not itself a function. These are tangentially related to the strong subsets of a semigroup introduced by Schein in [47].

Proposition 3.5. Let $\varphi : M \to \mathsf{Dif}(X)$ be a representation. Then we have the following.

(i) H^x_x is a strong inverse submagmoid of M and contains $\mathsf{E}(M)$.

(ii) If $y \in X$ and σ_x denotes the strong right congruence on M induced by H^x_x (cf. 2.17), then H^x_y is a union of σ_x-classes.

(iii) $s \ \sigma_x \ t$ if and only if $s^{-1} \cdot s = t^{-1} \cdot t$ and for all $y \in X$, either $s, t \in H^x_y$ or $s, t \notin H^x_y$.

Proof. (i) It is clear that $\mathsf{E}(M) \subseteq M$. If $s \in H_x^x$ and $t \preccurlyeq s$, then by 2.14(iii), $\varphi(s) \subseteq \varphi(t)$, so that $x\ \varphi(t)\ x$ and thus $t \in H_x^x$. Certainly if $s \in H_x^x$, then $s^{-1} \in H_x^x$. Finally, suppose $s, t \in H_x^x$ and that $s \cdot t$ exists; then $x\ \varphi(s)\ x\ \varphi(t)\ x$, so that $x\ \varphi(s \cdot t)\ x$ and thus $s \cdot t \in H_x^x$. (ii) Suppose $s \in H_y^x$ and $t\ \sigma_x\ s$; that is, $x\ \varphi(s)\ y$, $s^{-1} \cdot s = t^{-1} \cdot t$, and $s \cdot t^{-1} \in H_x^x$. Then $y\ \varphi((s^{-1} \cdot s) \cdot t^{-1})\ x$, so that $y\ \varphi((t^{-1} \cdot t) \cdot t^{-1})\ x$, so that $y\ \varphi(t^{-1})\ x$, and thus $x\ \varphi(t)\ y$ as desired. (iii) Suppose $s\ \sigma_x\ t$. Certainly $s^{-1} \cdot s = t^{-1} \cdot t$. Suppose $s \in H_y^x$. By part (ii), we have $t \in H_y^x$. Conversely, if $t \in H_y^x$ then so is s. Now suppose we have $s^{-1} \cdot s = t^{-1} \cdot t$ and that for all y, either $s, t \in H_y^x$ or $s, t \notin H_y^x$. Now $s \cdot t^{-1}$ exists. Say $y \in X$ such that $x\ \varphi(s)\ y$; then $x\ \varphi(t)\ y$, and so $x\ \varphi(s \cdot t^{-1})\ x$. Hence $s \cdot t^{-1} \in H_x^x$ as desired. $\square$

Next we define a class of morphisms among representations; we can think of a homomorphisms as a 'change of basis'.

Definition 3.6. Let $\varphi : M \to \mathsf{Dif}(X)$ and $\psi : M \to \mathsf{Dif}(Y)$ be representations. A mapping $\omega : X \to Y$ is called a *homomorphism* of representations (denoted $\widetilde{\omega} : \varphi \to \psi$) if for all $s \in M$ we have $\omega^{-1}\varphi(s)\omega \subseteq \psi(s)$. We say ω is *saturated* if equality holds for all s. We say that two homomorphisms $\widetilde{\omega}, \widetilde{\eta} : \varphi \to \psi$ are equal if for all $s \in M$ we have $\omega^{-1}\varphi(s)\omega = \eta^{-1}\varphi(s)\eta$.

Our definition of 'equal' is strange enough to warrant a more thorough motivation. We would like for our homomorphisms of representations to be induced by functions on the base set X, so that two representations which are the same but for a renaming of the elements are isomorphic, for example. However, it is possible that two distinct functions $\omega, \eta : X \to Y$ yield the same homomorphism in the sense that the 'pointwise images' of $\widetilde{\omega}$ and $\widetilde{\eta}$ are indistinguishable. Now $\widetilde{\omega}$ and $\widetilde{\eta}$ are not equal as functions (for instance, for the purpose of expressing a universal property), but are equal as homomorphisms. It is clear that if $\widetilde{\omega}$ is saturated, then ω is surjective. Moreover $\widetilde{1_X}$ is a saturated homomorphism, and the composite of (saturated) homomorphisms is a (saturated) homomorphism. Thus the classes of representations of a fixed inverse magmoid M on algebras of a given variety $\mathcal{V}$, together with the class of morphisms among them, form a category $\mathsf{DifRep}_{\mathcal{V}}\ M$. In this category, representations φ and ψ of M in X and Y, respectively, are isomorphic, denoted $\varphi \cong \psi$, if there exist morphisms $\widetilde{\omega} : \varphi \to \psi$ and $\widetilde{\eta} : \psi \to \varphi$ such that $\widetilde{\eta} \circ \widetilde{\omega} = \widetilde{1_X}$ and $\widetilde{\omega} \circ \widetilde{\eta} = \widetilde{1_Y}$. Clearly in this case both ω and η are saturated. For example, if $\theta : X \to Y$ is a bijection such that $\widetilde{\theta} : \varphi \to \psi$ is a saturated homomorphism, then $\widetilde{\theta}$ is an isomorphism.

Definition 3.7. Let $\varphi : M \to \mathsf{Dif}(X)$ be a difunctional representation. We say φ is *deflated* if for all distinct $y, z \in X$, there exists $x \in X$ such that $H^x_y \neq H^x_z$.

We can think of a representation as being deflated if it does not contain any redundant information about X. As we show, if a representation is not deflated then there are some elements y and z which cannot be distinguished by any elements of M; in this case we might as well toss one out, or, equivalently, identify y and z.

Proposition 3.8. If φ is a deflated representation of M in X, then $\bigcap_{s \in M} \varphi(s^{-1} \cdot s) = \Delta_X$.

Proof. $(\supseteq)$ Each $\varphi(s^{-1}s)$ is an equivalence by (i) and (i), and so contains Δ_X. $(\subseteq)$ Suppose we have y and z such that $y \left(\bigcap_{s \in M} \varphi(s^{-1} \cdot s)\right) z$. Let $x \in X$, and say $s \in H^x_y$. Now $y\ \varphi(s^{-1} \cdot s)\ z$, so that $x\ \varphi(s \cdot (s^{-1} \cdot s))\ z$, and thus $s \in H^x_z$. Conversely, $H^x_z \subseteq H^x_y$, so that $H^x_y = H^x_z$ for all $x \in X$. Since φ is deflated, we have $y = z$, so that $y\ \Delta_X\ z$ as desired. $\square$

For example, if $\varphi : G \to \mathsf{Dif}(X)$ is a deflated representation of a group G, then $\varphi(g)\varphi(g)^{-1} = \Delta_X$ for all $g \in G$ and thus φ is a permutation representation. As we show, every representation has a deflated homomorphic image which is unique up to isomorphism.

Proposition 3.9. Let $\varphi : M \to \mathsf{Dif}(X)$ be a representation. Define a relation ε on X by $y\ \varepsilon\ z$ precisely when for all $x \in X$ we have $H^x_y = H^x_z$. Then we have the following.

(i) $\varepsilon = \bigcap_{s \in M} \varphi(s^{-1} \cdot s)$ is a congruence.

(ii) The relation $\delta(s) = \{([x], [y]) \mid x\ \varphi(s)\ t\}$ on X/ε is full and difunctional.

(iii) The mapping $\delta : M \to \mathsf{Dif}(X/\varepsilon)$ is a deflated representation of M.

(iv) Letting $\pi : X \to X/\varepsilon$ denote the natural projection, $\widetilde{\pi} : \varphi \to \delta$ is a saturated homomorphism.

(v) If $\psi : M \to \mathsf{Dif}(Y)$ is a deflated representation of M and $\widetilde{\eta} : \varphi \to \psi$ a saturated homomorphism, then $\psi \cong \delta$.

Proof. (i) ($\subseteq$) Suppose $y \;\varepsilon\; z$, and let $s \in M$. Since $\varphi(s)$ is full, we have $s \in H^w_y$ for some $w \in X$, so that $s \in H^w_z$. That is, $y \;\varphi(s^{-1})\; w \;\varphi(s)\; z$, so that $y \;\varphi(s^{-1} \cdot s)\; z$ for all s as desired. ($\supseteq$) Suppose $y \bigcap_{s \in S} \varphi(s^{-1} \cdot s)\; z$. Now let $x \in X$ and let $s \in H^x_y$. Now $x \;\varphi(s)\; y \;\varphi(s^{-1} \cdot s)\; z$, so that $s = s \cdot (s^{-1} \cdot s) \in H^x_z$. Similarly, $H^x_z \subseteq H^x_y$, so that $H^x_y = H^x_z$ for all x. Thus $y \;\varepsilon\; z$ as desired. In particular, ε is a congruence.

(ii) Note that if $x_1 \;\varepsilon\; x_2$, $y_1 \;\varepsilon\; y_2$, and $x_1 \;\varphi(s)\; y_1$, then $x_2 \;\varphi(s)\; y_2$, since

$$\varepsilon\varphi(s)\varepsilon \subseteq \varphi(s \cdot s^{-1})\varphi(s)\varphi(s^{-1} \cdot s) = \varphi(s).$$

Thus, if $[x_0], [y_0] \in X/\varepsilon$, then $x \;\varphi(s)\; y$ for all $x \in [x_0]$ and $y \in [y_0]$ if and only if $x_0 \;\varphi(s)\; y_0$. This enables us to define $\delta(s) = \{([x], [y]) \mid x \;\varphi(s)\; y\}$ on X/ε, confident that the choice of a representative for each ε class does not matter. Now let $[x] \in X/\varepsilon$. Since $\varphi(s)$ is full, there exist $y, z \in X$ such that $y \;\varphi(s)\; x \;\varphi(s)\; z$. Now $[y] \;\delta(s)\; [x] \;\delta(s)\; [z]$, so $\delta(s)$ is full. If $[x] \;\delta(s)\; [y] \;\delta(s)^{-1}\; [z] \;\delta(s)\; [w]$, then $x \;\varphi(s)\; y \;\varphi(s)^{-1}\; z \;\varphi(s)\; w$, so $x \;\varphi(s)\; w$, and thus $[x] \;\delta(s)\; [w]$. So $\delta(s)$ is difunctional.

(iii) Suppose $s \cdot t$ exists. Now $[x] \;\delta(s \cdot t)\; [y]$ if and only if $x \;\varphi(s \cdot t)\; y$, if and only if $x \;\varphi(s)\; z \;\varphi(t)\; y$ for some z, if and only if $[x] \;\delta(s)\; [z] \;\delta(t)\; [y]$ for some z. So δ is a representation. Now we show that δ is deflated; to this end, note that $s \in H^x_y$ if and only if $x \;\varphi(s)\; y$, if and only if $[x] \;\delta(s)\; [y]$, if and only if $s \in H^{[x]}_{[y]}$; in particular, $H^{[x]}_{[y]} = H^x_y$. Now let $[y], [z] \in X/\varepsilon$ and suppose $H^{[x]}_{[y]} = H^{[x]}_{[z]}$ for all $[x]$. Then we have $H^x_y = H^x_z$ for all x, and so $y \;\varepsilon\; z$ as desired.

(iv) If $[x] \;\pi^{-1}\; x' \;\varphi(s)\; y' \;\pi\; [y]$, we have $x \;\varepsilon\; x' \;\varphi(s)\; y' \;\varepsilon\; y$, so that $x \;\varphi(s)\; y$, and thus $[x] \;\delta(s)\; [y]$. Conversely, if $[x] \;\delta(s)\; [y]$, then $x \;\varphi(s)\; y$, and so $[x] \;\pi^{-1}\; x \;\varphi(s)\; y \;\pi\; [y]$. So we have $\pi^{-1}\varphi(s)\pi = \delta(s)$ for all $s \in M$ as desired.

(v) Suppose ψ and $\widetilde{\eta}$ exist. First, we claim that $\varepsilon \subseteq \mathsf{ker}\; \eta$. To this end, suppose $x \;\varepsilon\; y$.

Then $x\ \varepsilon(s \cdot s^{-1})\ y$ for all $s \in M$, and so $\eta(x)\ \eta^{-1}\ x\ \varepsilon(s \cdot s^{-1})\ y\ \eta\ \eta(y)$ for all s. Since $\widetilde{\eta}$ is saturated, we have $\eta(x)\ \psi(s \cdot s^{-1})\ \eta(y)$ for all s. By 3.8 we have $\eta(x) = \eta(y)$ as desired. By the First Isomorphism Theorem for sets, we have an injective mapping $\eta_\pi : X \to Y$ such that $\eta_\pi([x]) = \eta(x)$, regardless of the choice of representative. Since $\widetilde{\eta}$ is saturated, η is surjective, so that η_π is bijective. Next we claim that $\widetilde{\eta_\pi} : \delta \to \psi$ is a saturated homomorphism. To this end, note that $\pi\eta_\pi = \eta$. Then for all $s \in M$, we have $\eta_\pi^{-1}\delta(s)\eta_\pi = \eta_\pi^{-1}\pi^{-1}\varphi(s)\pi\eta_\pi = \eta^{-1}\varphi(s)\eta = \psi(s)$ as desired. Thus $\psi \cong \delta$. □

In other words, if φ is a representation on X which is not deflated then we can 'cut down' (or deflate) φ to an essentially unique deflated representation by identifying elements of X which are indistinguishable by φ. This process is also reversible, as in the following straightforward result.

Proposition 3.10. Let $\delta : M \to \mathsf{Dif}(X)$ be a deflated difunctional representation.

(i) Suppose $\theta : Y \to X$ is a surjective map. Define $\varphi_\theta(s) = \{(x, y) \mid \theta(x)\ \delta(s)\ \theta(y)\}$ on Y. Then $\varphi_\theta(s)$ is a full difunctional relation on Y and $\varphi_\theta : M \to \mathsf{Dif}(Y)$ a difunctional representation of M. Moreover, the deflation of φ_θ is isomorphic to δ. In this case, we say that φ_θ is the *inflation* of δ along θ.

(ii) If $\varphi : M \to \mathsf{Dif}(X)$ is a difunctional representation with deflation $\delta : M \to \mathsf{Dif}(X/\varepsilon)$, then there is a map $\theta : X \to X/\varepsilon$ such that $\varphi \cong \varphi_\theta$. That is, every representation is obtained by inflating a deflated representation along some map.

Next we construct sums of representations.

Proposition 3.11. Let $\varphi_i : M \to \mathsf{Dif}(X_i)$ be a family of representations indexed by a set I. Let $\coprod_I X_i$ denote the disjoint union of the sets X_i, and let $\iota_k : X_k \to \coprod_I X_i$ denote the kth canonical injection. For each $s \in M$, define $\Phi(s)$ on $\coprod_I X_i$ by $\Phi(s) = \bigcup_{i \in I} \iota_i^{-1}\varphi_i(s)\iota_i$. Then we have the following.

(i) $\Phi(s)$ is a full and difunctional relation for each $s \in M$.

(ii) The map $\Phi : M \to \mathsf{Dif}(\coprod_I X_i)$ is a difunctional representation, denoted $\sum_I \varphi_i$.

(iii) $\widetilde{\iota_k} : \varphi_k \to \sum_I \varphi_i$ is a homomorphism of representations.

(iv) If $\Psi : M \to \mathsf{Dif}(Y)$ is a representation and $\widetilde{\eta} : \varphi_i \to \Psi$ a family of homomorphisms indexed by I, then there exists a unique homomorphism $\widetilde{\Theta} : \sum_I \varphi_i \to \Psi$ such that $\widetilde{\eta_k} = \widetilde{\Theta} \circ \widetilde{\iota_k}$ for each $k \in I$. That is, there is a unique $\widetilde{\Theta}$ such that the following diagram commutes.

$$\begin{tikzcd}
\varphi_k \arrow[r, "\widetilde{\iota_k}"] \arrow[dr, "\widetilde{\eta_k}"'] & \sum_I \varphi_i \arrow[d, "\widetilde{\Theta}"] \\
& \Psi
\end{tikzcd}$$

Proof. (i) Let $(x, k) \in \coprod_I X_i$. Since $\varphi_k(s)$ is full, there exist elements $y, z \in X_k$ such that $y\ \varphi_k(s)\ x\ \varphi_k(s)\ z$. So $(y, k)\ \Phi(s)\ (x, i)\ \Phi(s)\ (z, i)$, and thus $\Phi(s)$ is full. Now suppose $(x, k)\ \Phi(s)\ (y, k)\ \Phi(s)^{-1}\ (z, k)\ \Phi(s)\ (w, k)$. Then $x\ \varphi_k(s)\ y\ \varphi_k(s)^{-1}\ z\ \varphi_k(s)\ w$, so that $x\ \varphi_k(s)\ w$, and thus $(x, k)\ \Phi(s)\ (w, k)$. So $\Phi(s)$ is difunctional.

(ii) Suppose $s \cdot t$ exists in M. Note that $(x, k)\ \Phi(s \cdot t)\ (y, k)$ if and only if $x\ \varphi_k(s \cdot t)\ y$, if and only if $x\ \varphi_k(s)\ z\ \varphi_k(t)\ y$ for some z, if and only if $(x, k)\ \Phi(s)\ (z, k)\ \Phi(t)\ (y, k)$. In particular, $\Phi(s)\Phi(t)$ is difunctional and equal to $\Phi(s \cdot t)$; so Φ is a representation. (iii) If $(x, k)\ (\iota_k)^{-1}\ x\ \varphi_k(s)\ y\ \iota_k\ (y, k)$, then by definition we have $(x, k)\ \Phi(s)\ (y, k)$. Thus we have $(\iota_k)^{-1}\varphi(s)\iota_k \subseteq \Phi(s)$ as desired.

(iv) Now suppose $\Psi : M \to \mathsf{Dif}(Y)$ and the family of $\widetilde{\eta_i} : \varphi_i \to \Psi$ exist. We have $\eta_i : X_i \to Y$. By the universal property of disjoint unions of sets, there exists a mapping $\Theta : \coprod_I X_i \to Y$ given by $\Theta(x, k) = \eta_k(x)$. We claim that $\widetilde{\Theta}$ is a homomorphism. Indeed, for all $s \in M$, if $\eta_k(x)\ \Theta^{-1}\ (x, k)\ \Phi(s)\ (y, k)\ \Theta\ \eta_k(y)$, then $\eta_k(x)\ \eta_k^{-1}\ x\ \varphi_k(s)\ y\ \eta_k\ \eta_k(y)$, so that $\eta_k(x)\ \Psi(s)\ \eta_k(y)$, and thus $\Theta^{-1}\Phi(s)\Theta \subseteq \Psi(s)$ as required. Moreover, note that

$$\eta_k(x)\ \eta_k^{-1}\ x\ \varphi_k(s)\ y\ \eta_k\ \eta_k(y)$$

if and only if

$$\eta_k(x)\ \Theta^{-1}\ (x, k)\ \iota_k^{-1}\ x\ \varphi_k(s)\ y\ \iota_k\ (y, k)\ \Theta\ \eta_k(y);$$

so $\eta_k^{-1}\varphi_k(s)\eta_k = \Theta^{-1}\iota_k^{-1}\varphi_k(s)\iota_k\Theta$ for all $s \in M$, and thus $\widetilde{\Theta} \circ \widetilde{\iota_k} = \widetilde{\eta_k}$.

Now if $\widetilde{\Omega} : \sum_I \varphi_i \to \Psi$ is a homomorphism such that $\widetilde{\eta_k} = \widetilde{\Omega} \circ \widetilde{\iota_k}$ for all $k \in I$, then

$$\begin{aligned} \Omega^{-1}\ (\textstyle\sum_I \varphi_i)(s)\ \Omega &= \Omega^{-1}\,(\textstyle\bigcup_I \iota_i^{-1}\varphi_i(s)\iota_i)\,\Omega &&= \textstyle\bigcup_I \Omega^{-1}\iota_i^{-1}\varphi_i(s)\iota_i\Omega \\ &= \textstyle\bigcup_I \eta_i^{-1}\varphi_i(s)\eta_i &&= \textstyle\bigcup_I \Theta^{-1}\iota_i^{-1}\varphi_i(s)\iota_i\Theta_i \\ &= \Theta^{-1}\,(\textstyle\bigcup_I \iota_i^{-1}\varphi_i(s)\iota_i)\,\Theta &&= \Theta^{-1}\ (\textstyle\sum_I \varphi_i)(s)\ \Theta, \end{aligned}$$

so that $\widetilde{\Omega} = \widetilde{\Theta}$ as desired. $\square$

This sum is the coproduct in the category of difunctional representations of M.

Definition 3.12. A representation $\varphi : M \to \mathsf{Dif}(X)$ is said to be *connected* if for all $x, y \in X$, we have a natural number k and elements $z_i \in X$ for $1 \le i \le k$ such that $z_1 = x$, $z_k = y$, and $H_{z_{i+1}}^{z_i}$ is nonempty for each $1 \le i < k$. We say that φ is *transitive* if for all $x, y \in X$, the set H_y^x is nonempty.

Note that if M is an inverse semigroup, connected and transitive are equivalent. In fact every representation is isomorphic to a sum of connected representations, and by abstract nonsense this decomposition is unique up to a permutation of the summands.

Theorem 3.13. Let $\varphi : M \to \mathsf{Dif}(X)$ be a representation. There is a family of connected representations $\varphi_i : M \to \mathsf{Dif}(X_i)$, indexed by a set I, such that $\varphi \cong \sum_I \varphi_i$. Moreover, φ is deflated if and only if the φ_i are deflated.

Proof. Define a relation ε on X by $x\ \varepsilon\ y$ if and only if $x\ \varphi(s_1)\varphi(s_2)\cdots\varphi(s_k)\ y$ for some elements $s_i \in M$, $1 \le i \le k$. (Note that we do not demand that any products exist among the s_i.) Clearly ε is an equivalence on X. Let $I = X/\varepsilon$, and define $\varphi_A(s) = \varphi(s) \cap (A \times A)$ for each $A \in I$. Evidently each φ_A is a representation of M in A, and in fact $\varphi \cong \sum_{A \in I} \varphi_A$ via the map $\Omega : X \to \coprod_I A$ given by $x \mapsto (x, [x]_\varepsilon)$. $\square$

We now consider families of subsets in an inverse magmoid which behave like the H-sets of a representation.

Definition 3.14. Let M be an inverse magmoid and let $\mathcal{M} = \{M_x\}_{x \in X}$ be a family of strong inverse submagmoids of M, each containing $\mathsf{E}(M)$. A *coset system* for $\mathcal{M}$ is a mapping $H : X \times X \to \mathcal{P}(M)$ such that the following hold.

(i) For each $x \in X$, the sets $H(x, y)$ cover M.

(ii) For each $x \in X$ we have $H(x, x) = M_x$.

(iii) For all $x, y \in X$ and $s, t \in M$ such that $s \cdot t$ exists, $s \cdot t \in H(x, y)$ if and only if $s \in H(x, z)$ and $t \in H(z, y)$ for some $z \in X$.

(iv) For all $x, y \in X$, $H(x, y)^{-1} = H(y, x)$.

Clearly if φ is a representation of M in X, then $H(x, y) = H^x_y$ is a coset system which we say is *induced* by φ. Conversely, given a coset system, we may construct a representation.

Proposition 3.15. Let M be an inverse magmoid and H a coset system of M indexed by X. For each $s \in M$, the relation $\varphi_H(s) = \{(y, z) \mid s \in H(y, z)\}$ is a full difunctional relation on $\mathcal{H}$. Moreover, $\varphi_H : M \to \mathsf{Dif}(X)$ is a representation of M, and under this representation, $H(x, y) = H^x_y$.

In other words, the coset systems on M correspond to the H-sets of representations of M. In particular, we may thus speak of a coset system as being deflated, connected, or transitive. Moreover, every deflated representation is isomorphic to the representation induced by its H-sets.

Proposition 3.16. If $\varphi : M \to \mathsf{Dif}(X)$ is a deflated representation then $\varphi \cong \varphi_H$, where H is the coset system induced by φ.

Proof. We have $x\ \varphi(s)\ y$ if and only if $s \in H(x, y)$, if and only if $s \in H^x_y$, if and only if $x\ \varphi(s)\ y$; thus the identity on X is an isomorphism $\varphi \to \varphi_H$. □

Combining these results, we have that every representation of an inverse magmoid M may be obtained by inflating representations of the form $\sum_{i \in I} \varphi_{H_i}$, where the H_i are deflated and connected coset systems on M. This result is almost analogous to 1.3 and 1.6; every difunctional representation of M can be constructed using only the structure of M itself. However this proof is nonconstructive in the sense that we do not have an explicit description of the coset systems for a given $\mathcal{M}$.

Specifically, given a family $\mathcal{M}$ of strong inverse submagmoids, each containing $\mathsf{E}(M)$, can we construct the coset systems for $\mathcal{M}$? We do have some partial results; for example, $H(x, y)$ has the form $\bigcup_{A \in \mathcal{A}} As_A$, where $\mathcal{A}$ is a set of $\sigma_{H(x,x)}$-classes, and $H(x, y)$ is down-closed. Certianly $H(x, y) \subseteq \{s \mid s^{-1}H(x, x)s \subseteq H(y, y)\}$. These are open problems and a clear avenue for future work.

Conclusion

The difunctional relations $\mathsf{Dif}(X)$ on a set X are essentially the isomorphisms among quotients of X, and thus have a clear interpretation as the colocal (dual partial) symmetries of X. Under relative composition and conversion, $\mathsf{Dif}(X)$ is a concrete instance of a class of partial algebras which we have called *inverse magmoids*, and every abstract inverse magmoid M can be represented as an inverse magmoid of difunctional relations. Moreover, very such representation is isomorphic to an inflation of a sum of representations induced by coset systems on M.

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