5-2013

# Hardy Space Properties of the Cauchy Kernel Function for a Strictly Convex Planar Domain

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# Hardy Space Properties of the Cauchy Kernel Function for a Strictly Convex Planar Domain

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

By

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Bachelor of Science in Mathematics, 2006

May 2013
University of Arkansas

## ABSTRACT

This work is based on a paper by Edgar Lee Stout, where it is shown that for every strictly pseudoconvex domain $D$ of class $C^2$ in $\mathbb{C}^N$, the Henkin-Ramírez Kernel Function belongs to the Smirnov class, $E^q(D)$, for every $q \in (0, N)$.

The main objective of this dissertation is to show an analogous result for the Cauchy Kernel Function and for any strictly convex bounded domain in the complex plane. Namely, we show that for any strictly convex bounded $D \subset \mathbb{C}$ of class $C^2$ if we fix $\zeta$ in the boundary of $D$ and consider the Cauchy Kernel Function

$$\mathcal{K}(\zeta, z) = \frac{1}{2\pi i}\frac{1}{\zeta - z}$$

as a function of $z$, then the Cauchy Kernel Function belongs to the Smirnov class $E^q(D)$ for every $q \in (0, 1)$.

This dissertation is approved for recommendation
to the Graduate Council.

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## ACKNOWLEDGEMENTS

First and foremost I would like to express my sincere gratitude to my advisor, Dr. Lanzani for sharing her time and attention with me during a couple of very busy semesters, for her excellent guidance and patience during the completion of this dissertation, as well as for her continuous and infectious enthusiasm without which, finishing this work would have been impossible.

I would like to thank the members of my committee, Dr. Harrington and Dr. Luecking. Dr. Harrington for his constant sunny disposition, for his careful and dedicated reading of this dissertation and for the valuable input offered in the improvement of this work. To Dr. Luecking for the very valuable contribution of ideas to continue my research.

To Chaim, for his friendship over the last few years. Without him, this experience wouldn't have been possible.

To the staff of the Math Department Office: Their daily support throughout this time has really made a difference.

To my friends in Fayetteville, for giving us a home away from home.

A special mention to Pablo for helping me with the pictures in this dissertation and for sharing his life with me.

# DEDICATION

To my mother, a continuous source of inspiration.

To my father, who taught me how to smile.

## CONTENTS

# INTRODUCTION

This work is based on a paper by Edgar Lee Stout, [16], where it is shown that for every strictly pseudoconvex domain $D$ of class $C^2$ in $\mathbb{C}^N$, the Henkin-Ramírez Kernel Function belongs to the Smirnov class, $E^q(D)$, for every $q \in (0, N)$.

The main objective of this dissertation is to show an analogous result for the Cauchy Kernel Function and for any strictly convex bounded domain in the complex plane. Namely, we show that for any strictly convex bounded $D \subset \mathbb{C}$ of class $C^2$ if we fix $\zeta$ in the boundary of $D$ and consider the Cauchy Kernel Function

$$\mathcal{K}(\zeta, z) = \frac{1}{2\pi i}\frac{1}{\zeta - z} \tag{0.1}$$

as a function of $z$, then the Cauchy Kernel Function belongs to the Smirnov class $E^q(D)$ for every $q \in (0, 1)$.

It is important to point out that this work is influenced by Stout's paper not only in terms on the nature of the result we desire to obtain but also in terms of the method used to accomplish it. In his paper Stout uses a local convexification of the domain, this allows him to locally relate the Henkin-Ramírez Kernel Function of the domain $D$ with the corresponding Kernel Function of a ball, for which he proved the desired result directly. In this dissertation we work with a strictly convex, bounded domain which allows us to proceed by a similar argument, but in this case a global one.

An essential factor for determining the methods used to prove our result, was our interest to develop a method that would allow us to extend this finding to the setting of several variables in the case of a bounded, convex domain $D \subset \mathbb{C}^N$ and to the Cauchy-Leray Kernel Function, which is the higher dimensional analog of the classical Cauchy Kernel Function (0.1). While the higher dimensional Cauchy-Leray Kernel will be a topic for future investigation, the need to pursue an approach in the complex 1-dimension setting that could be later extended to higher dimension was the main reason for our choice to avoid using conformal mapping in the present work.

This dissertation is organized as follows. The first chapter is devoted to some preliminaries in $\mathbb{R}^2$, including the definition of two function spaces, the *Hardy Space* defined as in [8] and the *Smirnov Class*. Both of these spaces are often referred to as the $H^q$ space. In this work we make the distinction between both spaces and at the end of the chapter we give a detailed proof of the fact that for a bounded domain of class $C^2$ in the complex plane, the Hardy space is contained in the Smirnov Class for any $q \in (0, 1)$. Results exploring the relationship between these two spaces are known in the literature, for example, in [11] Lanzani showed that for any bounded simply connected domain with Lipschitz boundary and any $q \geq 1$ the Smirnov Class coincides with the Hardy Space.

In the second chapter we concentrate in two specific examples of a strictly convex domain, namely, a disc and an ellipse. Both of these cases will play a key role in the pursuit of our objective.

In the case of the disc, we begin by exploring the instance of the unit disc centered at zero. We quote results by Stout, [16] and Duren [4] that establish that the Cauchy Kernel is in both the Smirnov and Hardy spaces of any disc. It is worth noting that Duren uses a third definition of $H^q$ space with harmonic marjorants, however, Stein shows in [15] that for a domain of class $C^2$ Duren's definition of Hardy space is equivalent to the Smirnov Class, and since we will be working exclusively with $C^2$ smooth curves here, we will focus on the Smirnov Class.

The disc will play a key roll on the proofs to follow, as an "osculating model domain" that we will employ in a spirit similar to Stout.

The instance of the ellipse was used as a "toy example" to understand the main difficulties when working with a strictly convex, bounded domain.

In the final chapter of this thesis we extend our result to strictly convex bounded domains of class $C^2$ by adapting the methods used in the ellipse case.

Chapter 1

# PRELIMINARIES

## 1.1 Preliminaries in $\mathbb{R}^2$.

**Definition 1.1.** Let $D$ be a bounded domain $\mathbb{R}^2$, we say that $D$ *is of class* $C^k$, $k \in \mathbb{Z}^+$ if and only if there exists a neighborhood $U$ of $\bar{D}$ and a function $\rho \in U(\bar{D}) \to \mathbb{R}$ such that

1. $\rho \in C^k(U(\bar{D}))$
2. $D = \{z \,|\, \rho(z) < 0\}$
3. $\partial D = \{z \,|\, \rho(z) = 0\}$
4. $\nabla\rho(z) \neq 0$ for every z $\in \partial D$

$\rho$ is called a *defining function* for D.

See Chapter 2 of [13]

**Lemma 1.2.** *If $\rho_1, \rho_2$ are two defining functions for a domain $D$ of class $C^k$ with $k \geq 1$, then there exists $h \in C^{k-1}(U(\bar{D}))$, $h : U(\bar{D}) \to \mathbb{R}$ so that $h > 0$ on $U(\bar{D})$ and*

$$\rho_2(z) = h(z)\rho_1(z) \tag{1.1}$$

*for all $z \in U(\bar{D})$. Furthermore,*

$$\nabla\rho_2(\zeta) = h(\zeta)\nabla\rho_1(\zeta) \tag{1.2}$$

*for all $\zeta \in U \cap \partial D$.*

Both this Lemma and its proof can be found in Chapter 2 of [13].

**Definition 1.3.** Let $A$ be a non-empty subset of $\mathbb{R}^2$ and $x \in \mathbb{R}^2$, define the *distance from $x$ to the set $A$*, by

$$d(x, A) = \inf\{|x - a| \mid a \in A\} \tag{1.3}$$

See Chapter 2 of [3].

**Lemma 1.4.** *If $D$ has a $C^k$-smooth, $k \geq 2$ defining function $\rho$ and $p \in \partial D$, then there is a neighborhood $U$ of $p$ such that for all $z \in U$ there is a unique point $z' \in \partial D \cap U$ with*

$$|z - z'| = dist(z, \partial D) \tag{1.4}$$

*Moreover, the function $g_p$ that assigns $z'$ to every $z \in U$ is of class $C^{k-1}$.*

This Lemma and its proof can be found in Chapter 1 of [10].

**Lemma 1.5.** *If $D$ is a bounded domain of class $C^k$ with defining function $\rho$, then there exists an open neighborhood $U$ of $\partial D$ and a function $g : U \to \partial D$ such that $g$ is of class $C^{k-1}$ and*

$$|z - g(z)| = dist(z, \partial D) \tag{1.5}$$

*Proof.* By Lemma 1.4 for every $p \in \partial D$ there exists $U_p$ an open neighborhood of $p$ and a function $g_p : U_p \to \partial D$ such that $g_p$ is of class $C^{k-1}$ and

$$|z - g_p(z)| = dist(z, \partial D). \tag{1.6}$$

Consider $\mathcal{G} = \bigcup_{p \in \partial D} U_p$, then clearly $\mathcal{G}$ is an open cover of $\partial D$ and since $\partial D$ is compact we have that there is a collection $\{p_1, \cdots, p_n\} \in \partial D$ such that $U = \bigcup_{i=1}^n U_{p_i}$ is an open cover for $\partial D$.

Let $\phi_1, \cdots, \phi_n$ be a smooth partition of unity associated with the covering $\{U_{p_i}\}$, then it suffices to define,

$$g(z) = \sum_{i=1}^{n} \phi_i(z) g_{p_i}(z) \tag{1.7}$$

for any $z \in U$. □

**Lemma 1.6.** *Let $f$ be a differentiable function in a region of $\mathbb{R}^2$ that contains a smooth curve $C$ given by $\rho(x,y) = 0$. Assume that $f$ has a local extreme value (relative to values of $f$ on $C$) at a point $P = (a,b)$ on $C$, then $\nabla f(a,b)$ is orthogonal to the line tangent to $C$ at $p$. Assuming $\nabla\rho(a,b) \neq 0$, it follows that there is a real number $\lambda$ (called the Lagrange Multiplier) such that*

$$\nabla f(a,b) = \lambda \nabla \rho(a,b) \tag{1.8}$$

Both this Lemma and its proof can be found in Chapter 12 of [2].

**Definition 1.7.** Let $a$ be a nonzero real number then, we define the *function sign of $a$* as

$$sgn(a) = \frac{a}{|a|} \tag{1.9}$$

and we will take $sgn(0) = 0$.

See [6], Chapter 1.

The following Lemma and the outline of its proof can be found in Chapter 1 of [10]

**Lemma 1.8.** *If $D$ is a bounded domain with defining function $\rho$ of class $C^k$, $k \geq 2$, then the signed distance function*

$$\tilde{\rho}(z) = sgn(\rho(z))dist(z, \partial D) \tag{1.10}$$

*is a $C^k$ function in a neighborhood $U$ of $\partial D$, where $U$ is as in Lemma 1.5 and $sgn(\rho(z))$ is as in definition 1.7.*

*Proof.* Let $g$ and $U$ be the function and neighborhood of $\partial D$ defined in Lemma 1.5. For a fixed $z \in U$, consider $f(\zeta) = |z - \zeta|^2$, then by Lemma 1.6 where $C = \partial D$ we know that if $\zeta_0 \in \partial D$ is a local extrema for $f$ on $\partial D$ we have that

$$\nabla f(\zeta_0) = \lambda \nabla \rho(\zeta_0) \tag{1.11}$$

Now, if $z = (x, y)$ and $\zeta = (\zeta_1, \zeta_2)$, then

$$\frac{\partial f}{\partial \zeta_1} = 2(x - \zeta_1)(-1) = -2(x - \zeta_1) \tag{1.12}$$

and

$$\frac{\partial f}{\partial \zeta_2} = 2(y - \zeta_2)(-1) = -2(y - \zeta_2) \tag{1.13}$$

and so,

$$\nabla f(\zeta) = -2(x - \zeta_1, y - \zeta_2) = -2(z - \zeta) \tag{1.14}$$

Substituting equation (1.14) in equation (1.11) we have that if $\zeta_0 \in \partial D$ is a local extrema for $f$ then,

$$-2(z - \zeta_0) = \lambda \nabla \rho(\zeta_0) \tag{1.15}$$

But by definition of $g(z)$, for any point $z \in U$ $g(z)$ must be a minimum of $|z - \zeta|^2$, hence we can substitute $g(z)$ for $\zeta_0$ on equation (1.15). And so,

$$-2(z - g(z)) = \lambda \nabla \rho(g(z)). \tag{1.16}$$

Call $M = -\lambda/2$, then the previous equation can be rewritten as

$$z - g(z) = M \nabla \rho(g(z)) \tag{1.17}$$

Also,

$$\begin{aligned} (\tilde{\rho}(z))^2 &= (sgn(\rho(z))^2 (dist(z, \partial D))^2 = dist(z, \partial D)^2 \\ &= |z - g(z)|^2 = (x - g_1(z))^2 + (y - g_2(z))^2 \end{aligned} \tag{1.18}$$

where $z = (x, y)$ and $g(z) = (g_1(z), g_2(z))$.

Differentiating the previous equation with respect to $x$ we obtain,

$$
\begin{aligned}
2\tilde{\rho}(z)\frac{\partial\tilde{\rho}}{\partial x} =& 2(x-g_1(z))\left(1-\frac{\partial g_1}{\partial x}\right)+2(y-g(z))\left(-\frac{\partial g_2}{\partial x}\right) \\
=& 2(x-g_1(z))-2(x-g_1(z))\frac{\partial g_1}{\partial x}-2(y-g_2(z))\frac{\partial g_2}{\partial x}
\end{aligned}
\tag{1.19}
$$

On the other hand, by looking at each coordinate of equation (1.17) we know that,

$$
x-g_1(z)=M\frac{\partial\rho}{\partial\zeta_1}(g(z)) \tag{1.20}
$$

and

$$
y-g_2(z)=M\frac{\partial\rho}{\partial\zeta_2}(g(z)) \tag{1.21}
$$

Substituting equation (1.20) on equation (1.19) we have that,

$$
\begin{aligned}
2\tilde{\rho}(z)\frac{\partial\tilde{\rho}}{\partial x} =& 2M\frac{\partial\rho}{\partial\zeta_1}(g(z))-2M\frac{\partial\rho}{\partial\zeta_1}(g(z))\frac{\partial g_1}{\partial x}-2M\frac{\partial\rho}{\partial\zeta_2}(g(z))\frac{\partial g_2}{\partial x} \\
=& 2M\frac{\partial\rho}{\partial\zeta_1}(g(z))-2M\left[\frac{\partial\rho}{\partial\zeta_1}(g(z))\frac{\partial g_1}{\partial x}+\frac{\partial\rho}{\partial\zeta_2}(g(z))\frac{\partial g_2}{\partial x}\right] \\
=& 2M\frac{\partial\rho}{\partial\zeta_1}(g(z))-2M\frac{\partial(\rho\circ g(z))}{\partial x}
\end{aligned}
\tag{1.22}
$$

But for all $z\in U$, $g(z)\in\partial D$ and so $\rho(g(z))=0$, hence

$$
\frac{\partial(\rho\circ g(z))}{\partial x}=0 \quad \text{and} \quad \frac{\partial(\rho\circ g(z))}{\partial y}=0 \tag{1.23}
$$

Substituting this value on equation (1.22) we get,

$$
\tilde{\rho}(z)\frac{\partial\tilde{\rho}}{\partial x}=M\frac{\partial\rho}{\partial\zeta_1}(g(z)). \tag{1.24}
$$

Similarly, we can show that

$$\tilde{\rho}(z)\frac{\partial\tilde{\rho}}{\partial y} = M\frac{\partial\rho}{\partial\zeta_2}(g(z)) \tag{1.25}$$

Combining equations (1.24) and (1.25) we obtain,

$$\tilde{\rho}(z)\nabla\tilde{\rho}(z) = M\nabla\rho(g(z)) \tag{1.26}$$

Also, taking absolute value of equation (1.17) we see that,

$$|z - g(z)| = |M||\nabla\rho(g(z))| \tag{1.27}$$

On the other hand,

$$\tilde{\rho}(z) = sgn(\rho(z))dist(z, \partial D) = sgn(\rho(z))|z - g(z)| \tag{1.28}$$

Combining these last two equations,

$$\tilde{\rho}(z) = sgn(\rho(z))|M||\nabla\rho(g(z))| \tag{1.29}$$

Substituting equation (1.29) in equation (1.26) we have that,

$$sgn(\rho(z))|M||\nabla\rho(g(z))|\nabla\tilde{\rho}(z) = M\nabla\rho(g(z)) \tag{1.30}$$

Now, assume that $z \in U \setminus \partial D$, then $sign(\rho(z)) \neq 0$, also observe that since $g(z) \in \partial D$ and $\rho$ is a defining function for $D$, we know that $\nabla\rho(g(z))$ is also different from zero. Hence we may divide by those quantities. Then from equation (1.30) we get,

$$\nabla\tilde{\rho}(z) = \frac{M}{|M|}\frac{1}{sgn(\rho(z))}\frac{\nabla\rho(g(z))}{|\nabla\rho(g(z))|} \tag{1.31}$$
$$= \frac{sgn(M)}{sgn(\rho(z))}\frac{\nabla\rho(g(z))}{|\nabla\rho(g(z))|}$$

for all $z \in U \setminus \partial D$.

Now, we can assume without loss of generality that $\partial D$ is positively oriented and that for all $\zeta \in \partial D$, $\nabla\rho(\zeta)$ is the outer normal to $\partial D$ at $\zeta$. Hence if $M$ satisfies equation (1.17) and $z \in U \cap D$, then $M$ must be negative while if $z \in U \setminus \overline{D}$ then $M > 0$. Therefore if $z \in U \setminus \partial D$ we have that

$$\frac{sgn(M)}{sgn(\rho(z))} = 1 \tag{1.32}$$

And so, for all $z \in U \setminus \partial D$ we have that,

$$\nabla\tilde{\rho}(z) = \frac{\nabla\rho(g(z))}{|\nabla\rho(g(z))|} \tag{1.33}$$

Notice that since $\rho$ is a defining function for $D$ the right hand side of this equality is also well defined on the boundary of $D$, hence the equality must hold on $\partial D$ as well. □

This result was published in 1981 by Krantz and Parks in [9] and later on in 1984 Foote gave a coordinate free proof, this can be found in [5].

**Definition 1.9.** For a domain of class $C^k$ and for each defining function $\rho$ of $D$, we define *the set* $D_\varepsilon(\rho)$ as

$$D_\varepsilon(\rho) = \{z \in \mathbb{R}^2 \mid \rho(z) < -\varepsilon\} \tag{1.34}$$

See [15] Chapter 1.

**Lemma 1.10.** *Let $D$ be bounded domain with $\rho$ a $C^k -$ smooth defining function, $k \geq 2$ and* *consider $V$ an open neighborhood of $\partial D$,*

$$V = \{z \in \mathbb{R}^2 \mid dist(z, \partial D) < \varepsilon_0\} \tag{1.35}$$

*with $\varepsilon_0$ small enough so that $V \subset U$ where $U$ is as in Lemma 1.5 and Lemma 1.8. Then for*

$$D_\varepsilon(\tilde{\rho}) = \{z \in \mathbb{R}^2 \mid \tilde{\rho}(z) < -\varepsilon\} \tag{1.36}$$

*where $\tilde{\rho}$ is the signed distance function defined in Lemma 1.8, we have that*

1. *$\partial D_\varepsilon(\tilde{\rho}) \subset V \cap D$ and $D_\varepsilon(\tilde{\rho}) \subset D$.*
2. *$\bigcup_\varepsilon D_\varepsilon(\tilde{\rho}) = D$*
3. *$D_\varepsilon(\tilde{\rho})$ is a domain of class $C^k$ and $\tilde{\rho} + \varepsilon$ is a defining function for $D_\varepsilon(\tilde{\rho})$*

*for all $0 < \varepsilon < \varepsilon_0$.*

*Proof.* 1. Let $0 < \varepsilon < \varepsilon_0$ and $z \in \partial D_\varepsilon(\tilde{\rho})$, then $sgn(\rho(z))dist(z, \partial D) = -\varepsilon$ which implies that $\rho(z) < 0$ and $dist(z, \partial D) = \varepsilon < \varepsilon_0$, then $z \in D$ and $z \in V$.

For the second statement, let $z \in D_\varepsilon(\tilde{\rho})$ then $\tilde{\rho}(z) < -\varepsilon$, so in particular $\rho(z) < 0$, hence $z \in D$.

2. Let $0 < \varepsilon_2 < \varepsilon_1$, and let $z \in D_{\varepsilon_1}(\tilde{\rho})$, then $\tilde{\rho}(z) < -\varepsilon_1 < -\varepsilon_2$, hence $D_{\varepsilon_1}(\tilde{\rho}) \subset D_{\varepsilon_2}(\tilde{\rho})$, and since each $D_\varepsilon(\tilde{\rho}) \subset D$, then $\bigcup_\varepsilon D_\varepsilon(\tilde{\rho}) \subseteq D$.

Now, let $z \in D$, then $\rho(z) < 0$, and $D$ open implies that there exist $\varepsilon > 0$ such that $\mathbb{D}_\varepsilon(z) \subset D$ and so $dist(z, \partial D) > \varepsilon$, therefore $sgn(\rho(z))dist(z, \partial D) < -\varepsilon$, hence $z \in D_\varepsilon(\tilde{\rho})$, and so $\bigcup_\varepsilon D_\varepsilon(\tilde{\rho}) = D$.

3. If $0 < \varepsilon < \varepsilon_0$, then by Lemma 1.8 $\tilde{\rho}$ is a $C^k$ function on $V$, and so $\tilde{\rho} + \varepsilon$ is $C^k$ on $V$ too.

Now, by definition

$$D_\varepsilon(\tilde{\rho}) = \{z \in \mathbb{R}^2 \mid \tilde{\rho}(z) < -\varepsilon\} = \{z \in \mathbb{R}^2 \mid \tilde{\rho}(z) + \varepsilon < 0\}. \tag{1.37}$$

Finally, by equation (1.33) we know that for all $z \in V$

$$\nabla\tilde{\rho}(z) = \frac{\nabla\rho(g(z))}{|\nabla\rho(g(z))|} \tag{1.38}$$

where $g(z)$ is as in Lemma 1.5, and since for all $z \in D$ we know that $g(z) \in \partial D$ and $\rho$ is a defining function for $D$ then $\nabla\rho(g(z)) \neq 0$ and so for all $z \in \partial D_\varepsilon$, $\nabla\tilde{\rho}(z)$ is well defined and different from zero.

□

**Lemma 1.11.** *Let $D$ be a bounded domain of class $C^k$, $k \geq 2$ with defining function $\rho$. Let $\tilde{\rho}$ be the signed distance function for $D$ as defined in Lemma 1.8. Then for all $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that*

$$D_\varepsilon(\rho) \subset D_{\varepsilon'}(\tilde{\rho}). \tag{1.39}$$

*Conversely for all $\alpha > 0$ there exists $\alpha' > 0$ such that*

$$D_\alpha(\tilde{\rho}) \subset D_{\alpha'}(\rho) \tag{1.40}$$

*Proof.* Let $\varepsilon > 0$ and take $z \in D_\varepsilon(\rho)$ then $\rho(z) < -\varepsilon < 0$ and so $z \in D$. But $D$ open implies that there exists $\varepsilon' > 0$ such that $\mathbb{D}_{\varepsilon'}(z) \subset D$, and so $dist(z, \partial D) > \varepsilon'$, therefore $sgn(\rho(z))dist(z, \partial D) < -\varepsilon'$, hence $z \in D_{\varepsilon'}(\tilde{\rho})$. Therefore $D_\varepsilon(\rho) \subset D_{\varepsilon'}(\tilde{\rho})$.

Now let $\alpha > 0$ and take $z \in D_\alpha(\tilde{\rho})$, then by definition $sgn(\rho(z))dist(z, \partial D) < -\alpha$ and since $dist(z, \partial D) \geq 0$ for all $z$ we have that $\rho(z) < 0$ and $dist(z, \partial D) > \alpha$ in particular $\rho(z) < 0$ implies that there exists $\alpha' > 0$ such that $\rho(z) < -\alpha < 0$, and so $z \in D_{\alpha'}(\rho)$. Therefore $D_\alpha(\tilde{\rho}) \subset D_{\alpha'}(\rho)$. □

**Lemma 1.12.** *Let $\gamma_1$ and $\gamma_2$ be two closed curves in $\mathbb{R}^2$ and $\Lambda : \gamma_1 \to \gamma_2$ a homeomorphism between $\gamma_1$ and $\gamma_2$. Then*

$$\int\limits_{z\in\gamma_2} f(z)d\sigma(z) = \int\limits_{w\in\gamma_1} f(\Lambda(w))|\lambda'(w))|d\sigma(w) \tag{1.41}$$

*for every $f$ integrable on $\gamma_2$*

See [14].

### 1.2 Function Spaces

**Definition 1.13.** Let $f$ be a continuously differentiable function on a domain $D \subset \mathbb{C}$. We say that $f(z)$ *is analytic in* $D$ if and only if

$$\frac{\partial f}{\partial \bar{z}}(z) = 0 \qquad \text{for all } z \in D. \tag{1.42}$$

where $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$.

See [3].

We will denote the *space of analytic functions on* $D$ by $\mathcal{O}(D)$.

**Definition 1.14.** Let $D$ be a $C^k$ domain with defining function $\rho$ we define the *Smirnov* class of $D$ to be

$$E^q(D) = \left\{ f \in \mathcal{O}(D) : \sup_{\varepsilon > 0} \int\limits_{\partial D_\varepsilon(\rho)} |f(z)|^q \, d\sigma(z) \leq C < \infty \right\} \tag{1.43}$$

where $D_\varepsilon(\rho)$ is as in definition 1.9.

See [4] and [15].

It is clear from the definition that for any $D$ bounded, if $0 < p < q < \infty$, then $E^q(D) \subset E^p(D)$.

The following definition and two lemmas show that the Smirnov class of a domain $D$ is independent of the choice of defining function.

**Definition 1.15.** Let $D$ be a domain and let $u : D \to \mathbb{R}$ be a continuous function we say that $u$ *is subharmonic* if whenever $\mathbb{D}_r(z) \subset D$,

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta \tag{1.44}$$

See [3].

**Lemma 1.16.** *If $f(z)$ is an analytic function in a domain $D$ and $q > 0$, then $|f(z)|^q$ is subharmonic in $D$.*

The proof of the Lemma can be found in [4].

**Lemma 1.17.** *Let $\rho_1$ and $\rho_2$ be two defining functions for a domain $D$ and $f$ an analytic function in $D$, then*

$$\sup_{\varepsilon > 0} \int_{\partial D_\varepsilon(\rho_1)} |f(z)|^q \, d\sigma_1(z) < \infty \tag{1.45}$$

*if and only if*

$$\sup_{\varepsilon > 0} \int_{\partial D_\varepsilon(\rho_2)} |f(z)|^q \, d\sigma(z)_2 < \infty \tag{1.46}$$

*for all $q > 0$.*

The proof of the previous Lemma can be found in Chapter 1 of [15], in the special case when $u = |f(z)|^q$.

**Definition 1.18.** Let $D \subset \mathbb{R}^2$ be a bounded domain of class $C^k$, $k \geq 1$ and $p$ a point in $\partial D$. We define the *non-tangential approach region* as

$$\Gamma_\alpha(p) = \{z \in D \mid |z - p| \leq (1 + \alpha) dist(z, \partial D)\} \tag{1.47}$$

See [15].

**Lemma 1.19.** *For any bounded domain of class $C^k$, $k \geq 1$ there exists a positive number $\alpha = \alpha(D)$ with the property that every point $p \in \partial D$ admits a non-empty non-tangential approach region $\Gamma_\alpha(p)$.*

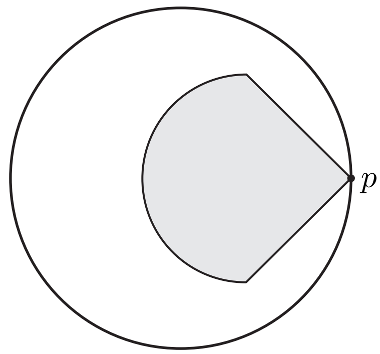


Figure 1.1: Non-tangential approach region $\Gamma_\alpha(p)$ in the case where the domain D is a disc

See [11].

**Definition 1.20.** Let $D \subset \mathbb{R}^2$ be a bounded domain of class $C^k$, $k \geq 1$ and $f$ a function defined in $D$. The *Nontangential Maximal function of* $f$, $f^*$, is defined as follows,

$$f^*(p) = \sup_{\zeta \in \Gamma_\alpha(p)} |f(\zeta)| \quad \text{a.e.,} \quad p \in \partial D \tag{1.48}$$

See [11].

**Definition 1.21.** Let $D$ be a domain of class $C^k$, $k \geq 1$, we define the *Hardy* space of $D$ as

$$H^q(D) = \{f \in \mathcal{O}(D) \mid f^* \in L^q(\partial D)\}. \tag{1.49}$$

with $0 \leq q \leq \infty$.

**Lemma 1.22.** *Suppose $D$ is a bounded domain of class $C^2$ then $H^q(D) \subseteq E^q(D)$.*

*Proof.* Let $\rho(z)$ be a defining function for $D$, then by definition $\nabla\rho(\zeta) \neq 0$ for all $\zeta \in \partial D$, so the normal direction is well defined for all $\zeta \in \partial D$.

We can assume without loss of generality that $\partial D$ is positively oriented, so $\nabla\rho(\zeta)$ defines the outer normal direction to $\partial D$ at $\zeta$.

We will denote

$$n(\zeta) = -\frac{\nabla\rho(\zeta)}{|\nabla\rho(\zeta)|} \tag{1.50}$$

to the inner unit normal vector to $\partial D$ at $\zeta$.

By Lemma 1.8 there is a neighborhood $V$ of $D$ such that $\tilde{\rho}(z) = sgn(\rho(z))dist(z, \partial D)$ is of class $C^k$ on $V$.

Let $\varepsilon_0$ be small enough so that for all $0 < \varepsilon < \varepsilon_0$

$$\partial D_\varepsilon(\tilde{\rho}) = \{z \in \mathbb{R}^2 \mid \tilde{\rho}(z) = -\varepsilon\} \subset V \cap D \tag{1.51}$$

See Lemma 1.10.

For such $\varepsilon_0$ there exists $N_0 \in \mathbb{N}$ such that $1/N_0 < \varepsilon_0$. For all $j \in \mathbb{N}$ consider

$$D_{\frac{1}{N_0+j}}(\tilde{\rho}) = \left\{ z \in \mathbb{R}^2 \mid \tilde{\rho}(z) < -\frac{1}{N_0 + j} \right\} \tag{1.52}$$

and define $\Lambda_j : \partial D \to \partial D_{\frac{1}{N_0+j}}(\tilde{\rho})$ by

$$\Lambda_j(\zeta) = \frac{1}{N_0 + j} n(\zeta) + \zeta \tag{1.53}$$

Now, $D$ of class $C^2$ means $\nabla\rho(\zeta)$ is of class $C^1$ and hence $\Lambda_j(\zeta)$ is itself of class $C^1$ on $\partial D$.

Also, for all $\zeta \in \partial D$ and for all $j \in \mathbb{N}$,

$$\begin{aligned} |\zeta - \Lambda_j(\zeta)| &= \left| \zeta - \left( \frac{1}{N_0 + j} n(\zeta) + \zeta \right) \right| = \left| \zeta - \frac{1}{N_0 + j} n(\zeta) - \zeta \right| = \left| -\frac{1}{N_0 + j} n(\zeta) \right| \\ &= \frac{1}{N_0 + j} |n(\zeta)| = \frac{1}{N_0 + j} \end{aligned} \tag{1.54}$$

and so, for all $j \in \mathbb{N}$,

$$\sup_{\zeta \in \partial D} |\zeta - \Lambda_j(\zeta)| = \frac{1}{N_0 + j} \tag{1.55}$$

and then,

$$\lim_{j \to \infty} \sup_{\zeta \in \partial D} |\zeta - \Lambda_j(\zeta)| = \lim_{j \to \infty} \frac{1}{N_0 + j} = 0 \tag{1.56}$$

Note that since $\partial D$ is compact this convergence is uniform.

Now, $\Lambda_j(\zeta) \in \partial D_{1/N_0+j}$ so, $\tilde{\rho}(\Lambda_j(\zeta)) = -1/(N_0+j)$, then $dist(\Lambda_j(\zeta), \partial D) = 1/(N_0+j) = |\zeta - \Lambda_j(\zeta)|$. So for all $\alpha$, we have that,

$$|\zeta - \Lambda_j(\zeta)| \leq (1+\alpha) dist(\Lambda_j(\zeta), \partial D) \tag{1.57}$$

and so, $\Lambda_j(\zeta) \in \Gamma_\alpha(\zeta)$ for all $\zeta \in \partial D$.

Hence, if $f \in \mathcal{O}(D)$ is such that $f \in H^q(D)$. We have that,

$$|f(\Lambda_j(\zeta))| \leq \sup_{w \in \Gamma_\alpha(\zeta)} |f(w)| \tag{1.58}$$

and then $|f(\Lambda_j(\zeta))| \leq f^*(\zeta)$.

Call $h(\zeta) = |\nabla \rho(\zeta)|$, then $h(\zeta) \neq 0$ for all $\zeta \in \partial D$. We can write $\Lambda_j(\zeta)$ as

$$\Lambda_j(\zeta) = (F_1^j(\zeta), F_2^j(\zeta)) \tag{1.59}$$

where

$$F_k^j(\zeta) = -\frac{1}{N_0+j}\frac{1}{h(\zeta)}\frac{\partial \rho}{\partial x_k} + x_k \tag{1.60}$$

with $k = 1, 2$.

and since $\rho$ is of class $C^2$, we may compute $\dfrac{\partial}{\partial x}\left[F_k^j(\zeta)\right]$ and $\dfrac{\partial}{\partial y}\left[F_k^j(\zeta)\right]$ where $\zeta = (x, y)$.

$$\frac{\partial}{\partial x}\left[F_1^j(\zeta)\right] = -\frac{1}{N_0+j}\left(\frac{1}{h(\zeta)}\frac{\partial^2 \rho}{\partial x}(\zeta) - \frac{1}{h^2(\zeta)}h'(\zeta)\frac{\partial \rho}{\partial x}(\zeta)\right) + 1 \tag{1.61}$$

and

$$\frac{\partial}{\partial x}\left[F_2^j(\zeta)\right] = -\frac{1}{N_0+j}\left(\frac{1}{h(\zeta)}\frac{\partial^2 \rho}{\partial y \partial x}(\zeta) - \frac{1}{h^2(\zeta)}h'(\zeta)\frac{\partial \rho}{\partial x}(\zeta)\right) \tag{1.62}$$

and then,

$$\lim_{j\to\infty}\frac{\partial}{\partial x}\left[F_1^j(\zeta)\right]=\lim_{j\to\infty}\left[-\frac{1}{N_0+j}\left(\frac{1}{h(\zeta)}\frac{\partial^2\rho}{\partial x}(\zeta)-\frac{1}{h^2(\zeta)}h'(\zeta)\frac{\partial\rho}{\partial x}(\zeta)\right)+1\right] \quad (1.63)$$
$$=1$$

while

$$\lim_{j\to\infty}\frac{\partial}{\partial x}\left[F_2^j(\zeta)\right]=\lim_{j\to\infty}\left[-\frac{1}{N_0+j}\left(\frac{1}{h(\zeta)}\frac{\partial^2\rho}{\partial y\partial x}(\zeta)-\frac{1}{h^2(\zeta)}h'(\zeta)\frac{\partial\rho}{\partial x}(\zeta)\right)\right] \quad (1.64)$$
$$=0$$

Similarly,

$$\lim_{j\to\infty}\frac{\partial}{\partial y}\left[F_1^j(\zeta)\right]=0 \quad (1.65)$$

and

$$\lim_{j\to\infty}\frac{\partial}{\partial y}\left[F_2^j(\zeta)\right]=1 \quad (1.66)$$

So if, $J_{\Lambda_j}(\zeta)$ denotes the jacobian matrix of $\Lambda_j(\zeta)$, we have that

$$\lim_{j\to\infty}J_{\Lambda_j}(\zeta)=\lim_{j\to\infty}\begin{pmatrix}\frac{\partial F_1^j}{\partial x} & \frac{\partial F_1^j}{\partial y}\\ \frac{\partial F_2^j}{\partial x} & \frac{\partial F_2^j}{\partial y}\end{pmatrix}=\begin{pmatrix}\lim_{j\to\infty}\frac{\partial F_1^j}{\partial x} & \lim_{j\to\infty}\frac{\partial F_1^j}{\partial y}\\ \lim_{j\to\infty}\frac{\partial F_2^j}{\partial x} & \lim_{j\to\infty}\frac{\partial F_2^j}{\partial y}\end{pmatrix} \quad (1.67)$$
$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Hence, $\lim_{j\to\infty}det(J_{\Lambda_j}(\zeta))=1$, and again since $\partial D$ is compact this convergence is uniform on $\partial D$. Then $det(J_{\Lambda_j}(\zeta))$ converges to 1 on $L^q(\partial D)$, for all $0\leq q\leq\infty$.

So finally, by Lemma 1.12 if $f\in H^q(D)$, we have

$$\int\limits_{z\in\partial D_{1/(N_0+j)}(\tilde{\rho})}|f(z)|^q d\sigma_j(z)=\int\limits_{\zeta\in\partial D}|f(\Lambda_j(\zeta))|^q det(J_{\Lambda_j}(\zeta))d\sigma(\zeta) \quad (1.68)$$

and since $det(J_{\lambda_j}(\zeta)) \to 1$ on $L^q$, then $det(J_{\Lambda_j}(\zeta))$ is bounded by some constant $M > 1$. Therefore,

$$\int_{\zeta \in \partial D} |f(\Lambda_j(\zeta))|^q det(J_{\Lambda_j}(\zeta) d\sigma(\zeta) \leq \int_{\zeta \in \partial D} M f^*(\zeta)^q d\sigma(\zeta) = M \int_{\zeta \in \partial D} f^*(\zeta)^q d\sigma(\zeta) \tag{1.69}$$
$$\leq C < \infty$$

Hence,

$$\sup_{j \in \mathbb{N}} \int_{z \in \partial D_{1/(N_0+j)}(\tilde{\rho})} |f(z)|^q d\sigma_j(z) \leq \int_{\zeta \in \partial D} f^*(\zeta)^q d\sigma(\zeta) < C. \tag{1.70}$$

Then,

$$\sup_{0<\varepsilon<\varepsilon_0} \int_{z \in \partial D_\varepsilon(\tilde{\rho})} |f(z)|^q \, d\sigma_\varepsilon(z) \leq \int_{\zeta \in \partial D} f^*(\zeta) d\sigma(\zeta) < C \tag{1.71}$$

and by Lemma 1.17 we have that

$$\sup_{0<\varepsilon<\varepsilon_0} \int_{z \in \partial D_\varepsilon(\rho)} |f(z)|^q \, d\sigma_\varepsilon(z) < \infty \tag{1.72}$$

Therefore, $f \in E^q(D)$. □

Chapter 2

# TWO MODEL DOMAINS: THE DISC AND THE ELLIPSE

## 2.1 The Cauchy Kernel function for a planar domain

**Definition 2.1.** Let $\zeta, z \in \mathbb{C}$, we define the *Cauchy Kernel* as

$$C(\zeta, z) = \frac{1}{2\pi i}\frac{d\zeta}{\zeta - z} \tag{2.1}$$

and for each $\zeta$ fixed in $\mathbb{C}$ we call the scalar part of the Cauchy Leray Kernel viewed as a function of $z$:

$$\mathcal{K}(\zeta, z) = \frac{1}{2\pi i}\frac{1}{\zeta - z} \tag{2.2}$$

with $\zeta, z \in \mathbb{C}$ and $\zeta \neq z$ the *Cauchy Kernel function*.

## 2.2 The Cauchy Kernel function of a disc

The object of this section is to show that the Cauchy Kernel function belongs to $H^q$ and $E^q$ of any disc in the complex plane provided that $q \in (0, 1)$, as well as to revisit some classical results for both spaces of a disc.

We begin with the case when the disc is the unit disc centered at the origin. We will denote by $\mathbb{D}_r(p)$ the disc of radius $r$ centered at $p$ and $\mathbb{D}$ to the unit disc centered at 0.

**Lemma 2.2.** *If $f(z) = \sum a_n z^n \in E^1(\mathbb{D})$, then $a_n \to 0$ as $n \to \infty$.*

This Lemma and its proof can be found in Chapter 3 of [4].

**Lemma 2.3.** *Let $f(z) = \sum a_n z^n$ be in $\mathcal{O}(\mathbb{D})$ then $f \in E^2(\mathbb{D})$ if and only if $\sum |a_k|^2 < \infty$.*

The proof of Lemma 2.3 can be found in Chapter 1 of [4].

**Lemma 2.4.** *For each fixed $\zeta_0 \in \partial\mathbb{D}$, we have that the Cauchy Kernel function ( 2.2) belongs to $E^q(\mathbb{D})$ if and only if $0 < q < 1$. Moreover, it belongs to $E^q(\mathbb{D})$ uniformly in $\zeta$. That is there is a constant $C$ such that*

$$\sup_{\varepsilon>0} \int_{\partial D_\varepsilon} \frac{1}{|\zeta - z|^q} \, d\sigma(z) < C \tag{2.3}$$

*for every $\zeta \in \partial\mathbb{D}$.*

Here $D_\varepsilon$ is as in definition 1.9.

*Proof.* The proof of the first statement is as in [16]. We first show that the Cauchy Kernel Function is in $E^q(\mathbb{D})$ if $0 < q < 1$.

We begin considering the case in which $\zeta_0 = 1$.

Let $f(z) = \mathcal{K}(1, z) = \frac{1}{1-z}$.

Take $f(z)^{q/2} = \dfrac{1}{(1-z)^{q/2}}$ and consider its power series expansion about zero.

$$\begin{aligned} f(z)^{\frac{q}{2}} = \frac{1}{(1-z)^{q/2}} &= \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}q}{k} z^k \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(-\frac{q}{2}+1)}{\Gamma(k+1)\Gamma(-\frac{q}{2}-k+1)} z^k \end{aligned} \tag{2.4}$$

where $\Gamma$ denotes the Gamma function.

For more about the Gamma Function, see [1].

Now, $k$ is an integer, so $\Gamma(k+1) = k!$. Therefore

$$f(z)^{\frac{q}{2}} = \frac{1}{(1-z)^{q/2}} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(-\frac{q}{2}+1)}{k!\Gamma(-\frac{q}{2}-k+1)} z^k \tag{2.5}$$

Also, observe that

$$\begin{aligned}\Gamma(-\frac{q}{2}+1) &= \Gamma((-\frac{q}{2}-k+1)+k) \\ &= -\frac{q}{2}(-\frac{q}{2}-1)(-\frac{q}{2}-2)\cdots(-\frac{q}{2}-k+1)\Gamma(-\frac{q}{2}-k+1)\end{aligned} \tag{2.6}$$

Hence,

$$\begin{aligned}\binom{-\frac{1}{2}q}{k} &= \frac{\Gamma(-\frac{q}{2}+1)}{k!\Gamma(-\frac{q}{2}-k+1)} \\ &= \frac{-\frac{q}{2}(-\frac{q}{2}-1)(-\frac{q}{2}-2)\cdots(-\frac{q}{2}-k+1)\Gamma(-\frac{q}{2}-k+1)}{k!\Gamma(-\frac{q}{2}-k+1)} \\ &= \frac{-\frac{q}{2}(\frac{q}{2}+1)(\frac{q}{2}+2)\cdots(\frac{q}{2}+k-1)}{k!}\end{aligned} \tag{2.7}$$

Then,

$$f(z)^{q/2} = \sum_{k=0}^{\infty} a_k z^k \tag{2.8}$$

where

$$a_k = \frac{(-1)^{k+1}\frac{q}{2}(\frac{q}{2}+1)(\frac{q}{2}+2)\cdots(\frac{q}{2}+k-1)}{k!} \tag{2.9}$$

Then,

$$\sum_{k=0}^{\infty} |a_k|^2 = \frac{|\frac{q}{2}|^2}{1} + \frac{|\frac{q}{2}|^2|\frac{q}{2}+1|^2}{1\cdot 2\cdot 1\cdot 2} + \frac{|\frac{q}{2}|^2|\frac{q}{2}+1|^2|\frac{q}{2}+2|^2}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}\cdots \tag{2.10}$$

which is the hypergeometric series $F(q/2, q/2, 1, 1) - 1$, see [17], and $F(q/2, q/2, 1, 1)$ converges if $Re(q/2 + q/2 - 1) < 0$, that is when $q < 1$.

So by Lemma 2.3 $f(z)^{q/2} \in E^2(\partial\mathbb{D}, d\sigma)$ if $q < 1$. Which implies that $f(z) \in E^q(\mathbb{D})$ if $q < 1$.

On the other hand observe that for $q = 1$ we have that

$$f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k. \tag{2.11}$$

Therefore $a_k = 1$ for every $k$, and then by Lemma 2.2 we know that,

$$f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \tag{2.12}$$

cannot be in $E^1(\mathbb{D})$.

Also, we know that for any domain $D$, we have that $E^q(D) \subset E^s(D)$ for $s < q$. Hence $E^q(\mathbb{D}) \subset E^1(\mathbb{D})$ for all $q > 1$ and since $f(z) \notin E^1(\mathbb{D})$ then $f(z) \notin E^q(\mathbb{D})$ for every $q > 1$.

For the second assertion we notice that the fact that $f(z)$ belongs to $E^q(\mathbb{D})$ for $0 < q < 1$ implies that

$$\int\limits_{\partial D_\varepsilon} \frac{1}{|1-z|^q}\, d\sigma(z) < M_0 \tag{2.13}$$

for all $\varepsilon > 0$.

Fix $\zeta \neq 0 \in \partial\mathbb{D}$. Then we can express $\zeta = e^{i\theta}$, with $\theta \in (0, 2\pi)$ and so,

$$\int\limits_{\partial\mathbb{D}_\varepsilon} \frac{1}{|\zeta - w|^q}\, d\sigma(w) = \int\limits_{\partial\mathbb{D}_\varepsilon} \frac{1}{|e^{i\theta} - w|^q}\, d\sigma(w) = \int\limits_{\partial\mathbb{D}_\varepsilon} \frac{1}{|e^{i\theta}|^q |1 - \frac{w}{e^{i\theta}}|^q}\, d\sigma(w) = \int\limits_{\partial\mathbb{D}_\varepsilon} \frac{1}{|1 - \frac{w}{e^{i\theta}}|^q}\, d\sigma(w) \tag{2.14}$$

Let $z = w/e^{i\theta}$. Then $|w| = |z/e^{i\theta}| = |z|$, so $w \in \partial\mathbb{D}_\varepsilon$ if and only if $z \in \partial\mathbb{D}_\varepsilon$, and $dw = dz/e^{i\theta}$, so $|dw| = |dz|$ as well.

Hence,

$$\int\limits_{\partial\mathbb{D}_\varepsilon} \frac{1}{|\zeta - w|^q}\, d\sigma(w) = \int\limits_{\partial\mathbb{D}_\varepsilon} \frac{1}{|1-z|^q}\, d\sigma(z) < M_0 \tag{2.15}$$

and so $M_0$ is independent of the choice of $\zeta$, therefore $\mathcal{K}(\zeta, z)$ as a function of $z$ is in $E^q(\mathbb{D})$ uniformly for any $\zeta \in \partial\mathbb{D}$. □

The following two corollaries establish that the Cauchy Kernel Function is in $E^q$ of any disc.

**Corollary 2.5.** *$\mathcal{K}(\zeta, z)$ belongs to $E^q(\mathbb{D}_R(0))$ for $q \in (0, 1)$ for every fixed $R \in \mathbb{R}$.*

*Proof.* Fix as in proof of (2.15), $\zeta \in \partial\mathbb{D}_R(0)$ and $z \in \mathbb{D}_R(0)$. Again we can consider the particular case when $\zeta = R$. Then

$$\lim_{r\to R}\int \frac{1}{|R-z|^q}\,d\sigma(z) = \int \frac{1}{|R|^q|1-\frac{z}{R}|^q}\,d\sigma(z) \tag{2.16}$$

Take $w = \frac{z}{R}$. Observe that this is well defined since given that $R \in \partial\mathbb{D}_R(0)$ and $z \in \mathbb{D}_R(0)$ we have that $R \neq z$. Also $w \in \mathbb{D}_1(0)$

Now, if $w = \frac{z}{R}$ then $dw = \frac{1}{R}dz$ and if $r \to R$ then $\frac{r}{R} \to 1$.

And so we have by equation (2.16) that

$$\lim_{r\to R}\int \frac{1}{|R-z|^q}\,d\sigma(z) = \lim_{\frac{r}{R}\to 1}\int \frac{1}{|1-w|^q}\,d\sigma(w) < \infty \tag{2.17}$$

if $q \in (0,1)$ by Lemma 2.4. □

**Corollary 2.6.** *$\mathcal{K}(\zeta, z)$ belongs to $E^q(\mathbb{D}_R(z_0))$ with $q \in (0,1)$ for all fixed $R \in \mathbb{R}$ and all $z_0 \in \mathbb{C}$.*

*Proof.* Let $\zeta \in \partial\mathbb{D}_R(z_0)$ and $z \in \mathbb{D}_R(z_0)$. Once again we can consider without loss of generality $\zeta = z_0 + R$.

Take $w = z - z_0$ then $z = w + z_0$ and $dw = dz$ and clearly $w \in \mathbb{D}_R(0)$. Hence,

$$\begin{aligned}\lim_{r\to R}\int_{\partial\mathbb{D}_r(z_0)} \frac{1}{|z_0+R-z|^q}\,d\sigma(z) &= \lim_{r\to R}\int_{\partial\mathbb{D}_R(0)} \frac{1}{|z_0+R-(w+z_0)|^q}\,d\sigma(w) \\ &= \lim_{r\to R}\int_{\partial\mathbb{D}_r(0)} \frac{1}{|R-w|^q}\,d\sigma(w) < \infty.\end{aligned} \tag{2.18}$$

□

We end this section with a Lemma that explores the relation between the Smirnov Class with the Hardy Space of a disc.

**Lemma 2.7.** *For any disc $\mathbb{D}_R(z_0) \subset \mathbb{C}$ and for any $q > 0$ we have that $H^q(\mathbb{D}_R(z_0)) = E^q(\mathbb{D}_R(z_0))$.*

For the proof of previous Lemma see Chapter 10 of [4].

Observe that as a Corollary of the past Lemma and Lemma 2.4 we can conclude that the Cauchy Kernel Function is in $H^q(\mathbb{D}_R(z_0))$ for any ball in the complex plane if and only if and only if $q \in (0, 1)$ and then, we have that the following statement is true.

For any fixed $\zeta \in \partial\mathbb{D}$

$$\int_{z\in\partial\mathbb{D}} [K^*(\zeta, z)]^q \, d\sigma(z) < C < \infty \tag{2.19}$$

if and only if $q \in (0, 1)$

## 2.3 The Cauchy Kernel function of an Ellipse

The goal of this section is to show that for any ellipse in the complex plane the Cauchy Kernel Function is both in the Smirnov Class and the Hardy space of its interior for any $q \in (0, 1)$. In addition we show that if $q \geq 1$ then $K(\zeta, z) \notin H^q$ of the interior of the ellipse.

Like mentioned in the introduction we will do this by relating the Cauchy Kernel Function of the ellipse with the one of a disc, for which we already have the result.

We begin by considering the case where the ellipse is centered at the origin.

Let $\mathcal{E}$ be the interior of any ellipse centered at 0, then we can parametrize its boundary, $\partial\mathcal{E}$, as

$$\partial\mathcal{E} = a\cos t + ib\sin t, \text{ where } t \in [0, 2\pi) \tag{2.20}$$

with $a > 0, b > 0$. We assume that $a \neq b$ since the case where $a = b$ is a circle, and that case was dealt with in the previous section.

Without loss of generality we will assume for the rest of the section that $a > b$.

Throughout this section we will denote $R = a^2/b$.

We want to relate the Cauchy Kernel Function of the ellipse with the Cauchy Kernel Function of a disc, for that we need to construct for every fixed point $p \in \partial\mathcal{E}$ a circumference with radius $R$ such that this circumference is tangent to $\partial\mathcal{E}$ at $p$ and $\overline{\mathcal{E}} \subset \mathbb{D}_R$.

Notice that if we parametrize $\partial\mathcal{E}$ as in equation (2.20) then, any fixed point $p$ has tangent vector $\tau(t) = -a\sin t + ib\cos t$. Since we want the circumference to be tangent to the ellipse, then we know that the center to this circumference lies in the line generated by the normal vector to $\partial\mathcal{E}$ at $p$ on the inside of $\mathcal{E}$, and since the given parametrization has counterclockwise direction, we need to rotate $\tau(z)$ by $\pi/2$. Then the unit inner normal vector is given by

$$\frac{i\tau(t_0)}{|\tau(t_0)|} = \frac{-b\cos t_0 - ia\sin t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin t_0}} \tag{2.21}$$

Translating this vector to $p$ and making the distance between $p$ and the center of the disc $R$, we see that the center of the circumference we are looking for a fixed point $p = a\cos t_0 + ib\sin t_0$ is given by,

$$c(z) = a\cos t_0 - \frac{Rb\cos t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} + i\left(b\sin t_0 - \frac{Ra\sin t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}}\right) \tag{2.22}$$

**Lemma 2.8.** *For each fixed $p \in \partial\mathcal{E}$ we have,*

$$\overline{\mathcal{E}} \setminus \{p\} \subset \mathbb{D}_R(c(p)) \tag{2.23}$$

*Furthermore $p \in \partial\mathbb{D}_R(c(p))$ and $\partial\mathcal{E}$ and $\partial\mathbb{D}_R(c(p))$ are tangent at $p$.*

*Proof.* We begin with the proof of the first statement.

Let $p \in \partial\mathcal{E}$, then $p$ can be expressed as $p = a\cos t_0 + ib\sin t_0$ with $t_0$ fixed in $[0, 2\pi)$.

Observe that to prove that $\overline{\mathcal{E}} \subset \mathbb{D}_R(c(p))$, it is enough to prove that $\partial\mathcal{E} \subset \mathbb{D}_R(c(p))$.

Let $z$ be a point in $\partial\mathcal{E}$, then $z = a\cos t + ib\sin t$ for some $t \in [0, 2\pi)$. We need to show that $|z - c(p)|^2 < R^2$. But,

$$|z-c(p)|^2 = \left(a\cos t + \frac{bR\cos t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} - a\cos t_0\right)^2 + \left(b\sin t + \frac{aR\sin t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} - b\sin t_0\right)^2 \quad (2.24)$$

and so,

$$\begin{aligned} |z-c(p)|^2 &= a^2\cos^2 t + b^2\sin^2 t + \frac{2abR\cos t_0\cos t}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} - 2a^2\cos t_0\cos t \\ &+ \frac{2abR\sin t_0\sin t}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} - 2b^2\sin t_0\sin t + R^2 - \frac{2abR}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} \\ &+ a^2\cos^2 t_0 + b^2\sin^2 t_0 \\ &= (a\cos t - a\cos t_0)^2 + (b\sin t - b\sin t_0)^2 + R^2 \\ &+ \frac{2abR}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}}\left[\cos(t_0 - t) - 1\right] \end{aligned} \quad (2.25)$$

so finally,

$$|z-c(p)|^2 = |z-p|^2 + R^2 + \frac{2abR}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}}[cos(t_0 - t) - 1] \quad (2.26)$$

Hence, it is enough to prove that,

$$|z-p|^2 \leq \frac{2abR}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}}[1 - \cos(t_0 - t)] \quad (2.27)$$

Notice that $b < a$ implies that

$$b \leq \sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0} \leq a \quad (2.28)$$

and then,

$$2b \leq \frac{2ab}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} \leq 2a \quad (2.29)$$

So,

$$2a^2 \leq \frac{2abR}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} \tag{2.30}$$

Therefore it suffices to prove that,

$$|z-p|^2 \leq 2a^2[1-\cos(t_0-t)] \tag{2.31}$$

Now,

$$\begin{aligned} |z-p|^2 =& (a\cos t_0 - a\cos t)^2 + (b\sin t_0 - b\sin t)^2 \\ =& a^2(cost_0 - \cos t)^2 + b^2(\sin t_0 - \sin t)^2 \end{aligned} \tag{2.32}$$

and then,

$$|z-p|^2 \leq a^2\left[(\cos t_0 - \cos t)^2 + (\sin t_0 - \sin t)^2\right] \tag{2.33}$$

But $(\cos t_0 - \cos t)^2 + (\sin t_0 - \sin t)^2 = |S-Q|^2$ where $S$ and $Q$ are two points in the unit circle, and we know,

$$\begin{aligned} |S-Q|^2 =& |S|^2 + |Q|^2 - 2Re(S\bar{Q}) \\ =& |S|^2 + |Q|^2 - 2\cos(t_0-t) = 2 - 2\cos(t_0-t) \\ =& 2[1-\cos(t-t_0)] \end{aligned} \tag{2.34}$$

Combining equations (2.33) and (2.34) we get,

$$|z-p|^2 \leq 2a^2[1-\cos(t_0-t)] \tag{2.35}$$

Which is what we wanted to prove.

For the proof of the second statement we start by showing that $p \in \partial\mathbb{D}_R(c(p))$. Substituting

$p = a\cos t_0 + ib\sin t_0$ for $z$ in equation 2.24 we get,

$$|p-c(p)|^2 = \left(a\cos t_0 + \frac{bR\cos t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} - a\cos t_0\right)^2 \tag{2.36}$$
$$+ \left(b\sin t_0 + \frac{aR\sin t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} - b\sin t_0\right)^2$$
$$= \left(\frac{bR\cos t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}}\right)^2 + \left(\frac{aR\sin t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}}\right)^2$$
$$= R^2$$

Therefore $|p-c(p)| = R$, and then $p \in \partial\mathbb{D}_R(c(p))$. Now, to show that $\partial\mathbb{D}_R(c(p))$ and $\partial\mathcal{E}$ are tangent, we parametrize $\partial\mathbb{D}_R(c(p))$ as

$$\partial\mathbb{D}_R(c(p)) = \gamma(s) = x(s) + iy(s) = R\cos(s) + c_1(p) + i(R\sin(s) + c_2(p)) \tag{2.37}$$
$$= \left(R\cos(s) + a\cos(t_0) - \frac{Rb\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right)$$
$$+ i\left(R\sin(s) + b\sin(t_0) - \frac{aR\sin(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right)$$

Then there is $s_0 \in [0, 2\pi)$ such that

$$p = \left(R\cos(s_0) + a\cos(t_0) - \frac{Rb\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right) \tag{2.38}$$
$$+ i\left(R\sin(s_0) + b\sin(t_0) - \frac{aR\sin(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right)$$

On the other hand $p \in \partial\mathcal{E}$ and so $p = a\cos(t_0) + ib\sin(t_0)$. We need to find $s_0$ that satisfies,

$$a\cos(t_0) = \frac{a^2}{b}\cos(s_0) + a\cos(t_0) - \frac{a^2\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}} \tag{2.39}$$

and

$$b\sin(t_0) = \frac{a^2}{b}\sin(s_0) + b\sin(t_0) - \frac{a^3\sin(t_0)}{b\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}} \tag{2.40}$$

Observe that since the Ellipse is symmetric with respect to both the $x$-axis and the $y$-axis, we can assume that $0 \leq t_0 \leq \pi/2$, and then, from (2.39) we can conclude that

$$s_0 = \cos^{-1}\left(\frac{b\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right) \tag{2.41}$$

Using the definition of trigonometric functions in a right triangle and Pythagoras theorem is easy to show that this choice of $s_0$ satisfies equation (2.40) and that

$$s_0 = \cos^{-1}\left(\frac{b\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right) = \sin^{-1}\left(\frac{a\sin(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right) \tag{2.42}$$

Using equations (2.37) we get, that for any $z$ the tangent vector to $\mathbb{D}_R(c(p))$ at $z$ is given by,

$$T_{\partial\mathbb{D}_R(c(p))} = -R\cos(s) + iR\sin(s) \tag{2.43}$$

Considering a convenient restriction of domain for $\sin(t)$ and $\cos(t)$ and using equation (2.42) we see that the tangent vector to $\partial\mathbb{D}_r(c(p))$ at $p$ is

$$\begin{aligned} T_{\partial\mathbb{D}_R(c(p))}(p) =& - R\sin\left(\sin^{-1}\left(\frac{a\sin(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right)\right) \\ &+ iR\cos\left(\cos^{-1}\left(\frac{b\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}}\right)\right) \\ =& - R\frac{a\sin(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}} + iR\frac{b\cos(t_0)}{\sqrt{b^2\cos^2(t_0) + a^2\sin^2(t_0)}} \end{aligned} \tag{2.44}$$

While the normal vector to $\partial\mathcal{E}$ at $p$ is given by,

$$T_{\partial\mathcal{E}}(p) = -b\cos(t_0) - ia\sin(t_0) \tag{2.45}$$

Hence

$$\begin{aligned} T_{\partial\mathbb{D}_R(c(p))}(p) =& -R\frac{a\sin(t_0)}{\sqrt{b^2\cos^2(t_0)+a^2\sin^2(t_0)}} + iR\frac{b\cos(t_0)}{\sqrt{b^2\cos^2(t_0)+a^2\sin^2(t_0)}} \\ =& \frac{-R}{\sqrt{b^2\cos^2(t_0)+a^2\sin^2(t_0)}}(a\sin(t_0) - b\cos(t_0)) \\ =& \frac{-R}{\sqrt{b^2\cos^2(t_0)+a^2\sin^2(t_0)}}T_{\partial\mathcal{E}}(p) \end{aligned} \tag{2.46}$$

Hence $T_{\partial\mathbb{D}_R(c(p))}(p)$ and $T_{\partial\mathcal{E}}(p)$ are parallel, so $\partial\mathbb{D}_R(c(p))$ and $\partial\mathcal{E}$ are tangent at $p$. □

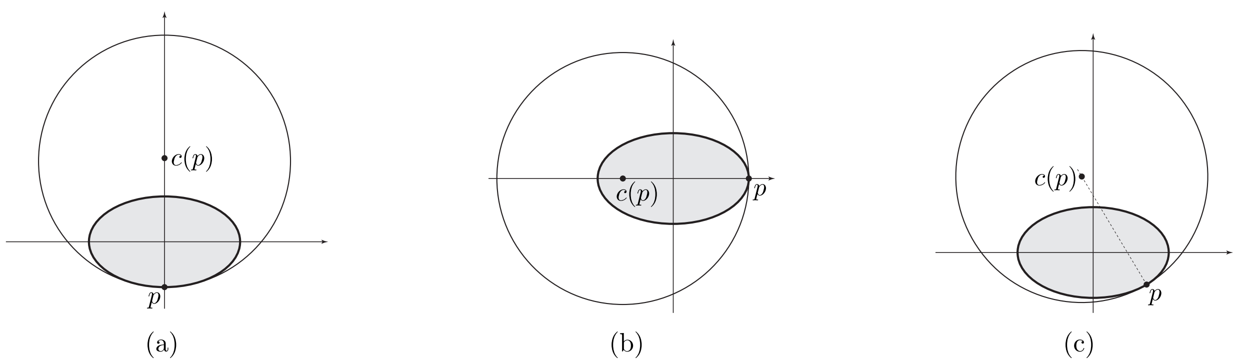


Figure 2.1: Ellipse centered at the origin and the ball $\mathbb{D}_R(c(p))$ for three different choices of $p$

**Lemma 2.9.** *Let $p \in \partial\mathcal{E}$ be an arbitrarily fixed point, $p = a\cos t_0 + ib\sin t_0$, define*

$$A_p = \partial\mathcal{E} \cap \mathbb{D}_{R/2}(p) = \{z \in \partial\mathcal{E} : |z-p| < R/2\} \tag{2.47}$$

*and the projection $\lambda_p : A_p \to \partial\mathbb{D}_R(c(p))$*

$$\lambda_p(z) = R\frac{z-c(p)}{|z-c(p)|} + c(p) \tag{2.48}$$

*then*

$$0 < b \leq |\lambda'(p)| \leq a < \infty \tag{2.49}$$

*where* $\lambda'(p)$ *denotes* $\frac{d}{dt}\lambda(z)$ *evaluated at* $t = t_0$.

*Proof.* First observe that regardless of our choice of $p$, the point $c(p)$ is not in $E$, since $|c(p) - p| = R$, and then $\lambda_p$ is a $C^\infty$ function since $|z - c(p)|$ never vanishes in $E$.

Now, to calculate $\lambda_p(p)$, we recall that $p = a\cos t_0 + ib\sin t_0$ and using equation (2.22) we obtain,

$$p - c(p) = R\frac{b\cos t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} + iR\frac{a\sin t_0}{\sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}} \tag{2.50}$$

so,

$$\begin{aligned} |p - c(p)| =& R\sqrt{\frac{b^2\cos^2 t_0}{b^2\cos^2 t_0 + a^2\sin^2 t_0} + \frac{a^2\sin^2 t_0}{b^2\cos^2 t_0 + a^2\sin^2 t_0}} \\ =& R \end{aligned} \tag{2.51}$$

Then,

$$\lambda_p(p) = R\frac{p - c(p)}{R} + c(p) = p - c(p) + c(p) = p \tag{2.52}$$

If we call $I(t_0) = \sqrt{b^2\cos^2 t_0 + a^2\sin^2 t_0}$, then we can express $c(p)$ as,

$$\begin{aligned} c(p) &= \left(\frac{-R}{I(t_0)}b\cos t_0 + a\cos t_0\right) + i\left(\frac{-R}{I(t_0)}a\sin t_0 + b\sin t_0\right) \\ &= c_1(p) + c_2(p) \end{aligned} \tag{2.53}$$

Then for any $z \in E$ with $z = a\cos t + ib\sin t = z_1 + iz_2$ we have that,

$$\begin{aligned} z - c(p) =& (z_1 - c_1(p)) + i(z_2 - c_2(p)) \\ =& \left(a\cos t + \frac{R}{I(t_0)} b\cos t_0 - a\cos t_0\right) + i\left(b\sin t + \frac{R}{I(t_0)} a\sin t_0 - b\sin t_0\right) \end{aligned} \tag{2.54}$$

and so,

$$|z - c(p)| = \sqrt{\left(a\cos t + \frac{R}{I(t_0)} b\cos t_0 - a\cos t_0\right)^2 + \left(b\sin t + \frac{R}{I(t_0)} a\sin t_0 - b\sin t_0\right)^2} \tag{2.55}$$

Call,

$$\psi(t) = \left(a\cos t + \frac{R}{I(t_0)} b\cos t_0 - a\cos t_0\right)^2 + \left(b\sin t + \frac{R}{I(t_0)} a\sin t_0 - b\sin t_0\right)^2 \tag{2.56}$$

Then we can rewrite $|z - c(p)| = [\psi(t)]^{1/2}$ hence,

$$\begin{aligned} \lambda_p(z) =& u(t) + iv(t) + c(p) \\ =& R\left(\frac{z_1 - c_1(p)}{[\psi(t)]^{1/2}}\right) + iR\left(\frac{z_1 - c_1(p)}{[\psi(t)]^{1/2}}\right) + c(p) \\ =& R\left[\frac{a\cos t + \frac{R}{I(t_0)} b\cos t_0 - a\cos t_0}{[\psi(t)]^{1/2}}\right] + iR\left[\frac{b\sin t + \frac{R}{I(t_0)} a\sin t_0 - b\sin t_0}{[\psi(t)]^{1/2}}\right] + c(p) \end{aligned} \tag{2.57}$$

Then,

$$|\lambda_p'(p))| = \sqrt{[u'(t_0)]^2 + [v'(t_0)]^2} \tag{2.58}$$

To calculate $u'(t)$ and $v'(t)$, we first observe that $R$, $I(t_0)$,$a$,$b$,$\cos t_0$ and $\sin t_0$ are constants with respect to $t$.

Using the left hand side of equation (2.57) we obtain,

$$
\begin{aligned}
u'(t) &= R\left[\frac{-a\sin t\psi(t)^{1/2} - \frac{1}{2}\psi(t)^{-1/2}\psi'(t)(a\cos t + \frac{R}{I(t_0)}b\cos t_0 - a\cos t_0)}{\psi(t)}\right] \\
&= R\left[\frac{\psi(t)^{-1/2}[-a\sin t\psi(t) - \frac{1}{2}\psi'(t)(a\cos t + \frac{R}{I(t_0)}b\cos t_0 - a\cos t_0)]}{\psi(t)}\right] \\
&= R\left[\frac{-a\sin t\psi(t) - \frac{1}{2}\psi'(t)(a\cos t + \frac{R}{I(t)}b\cos t_0 - a\cos t_0)}{\psi(t)^{3/2}}\right] \\
&= \frac{R}{\psi(t)^{3/2}}\left[-a\sin t\psi(t) - \frac{1}{2}\psi'(t)(a\cos t + \frac{R}{I(t_0)}b\cos t_0 - a\cos t_0)\right]
\end{aligned} \tag{2.59}
$$

Similarly,

$$
v'(t) = \frac{R}{\psi(t)^{3/2}}\left[b\cos t\psi(t) - \frac{1}{2}\psi'(t)(b\sin t + \frac{R}{I(t_0)}a\sin t_0 - b\sin t_0)\right] \tag{2.60}
$$

We need to calculate $\psi'(t)$. From equation (2.56) we have that,

$$
\begin{aligned}
\psi'(t) = 2(a\cos t + \frac{R}{I(t_0)}b\cos t_0 - a\cos t_0)(-a\sin t) \\
+ 2(b\sin t + \frac{R}{I(t_0)}a\sin t_0 - b\sin t_0)(b\cos t)
\end{aligned} \tag{2.61}
$$

Evaluating $\psi'(t)$ and $\psi(t)$ at $t_0$,

$$
\begin{aligned}
\psi'(t_0) &= 2(a\cos t_0 + \frac{R}{I(t_0)}b\cos t_0 - a\cos t_0)(-a\sin t_0) \\
&+ 2(b\sin t_0 + \frac{R}{I(t_0)}a\sin t_0 - b\sin t_0)(b\cos t_0) \\
&= 2(-a^2\cos t_0\sin t_0 - \frac{R}{I(t_0)}ab\cos t_0\sin t_0 + a^2\cos t_0\sin t_0) \\
&= 2(b^2\cos t_0\sin t_0 + \frac{R}{I(t_0)}ab\cos t_0\sin t_0 - b^2\sin t_0\cos t_0) \\
&= 0
\end{aligned} \tag{2.62}
$$

while,

$$\psi(t_0) = (a\cos t_0 + \frac{R}{I(t_0)} b\cos t_0 - a\cos t_0)^2 \tag{2.63}$$
$$+ (b\sin t_0 + \frac{R}{I(t_0)} a\sin t_0 - b\sin t_0)^2$$
$$= \frac{R^2}{I^2(t_0)} (b^2\cos^2 t_0 + a^2\sin^2 t_0)$$
$$= \frac{R^2}{I^2(t_0)} I^2(t_0) = R^2$$

Combining this calculations with formulas (2.59) and (2.60) we obtain,

$$u'(t_0) = \frac{R}{(R^2)^{3/2}} \left[ -a\sin t_0 R^2 - \frac{1}{2}(0)(a\cos t_0 + \frac{R}{I(t_0)} b\cos t_0 - a\cos t_0) \right] \tag{2.64}$$
$$= \frac{R}{R^3}(-R^2 a\sin t_0)$$
$$= -a\sin t_0$$

Similarly,

$$v'(t_0) = b\cos t_0 \tag{2.65}$$

and so,

$$|\lambda_p'(p))| = \sqrt{[u'(t_0)]^2 + [v'(t_0)]^2} = \sqrt{a^2\sin^2 t_0 + b^2\cos^2 t_0} = I(t_0) \tag{2.66}$$

and since we are assuming that $a > b$ we have,

$$b \leq |\lambda_p'(p))| \leq a \tag{2.67}$$

□

**Proposition 2.10.** *For any ellipse $\mathcal{E}$ in $\mathbb{C}$, and for any fixed $p \in \partial\mathcal{E}$, the Cauchy Kernel*

*Function*

$$\mathcal{K}(p,z) = \frac{1}{2\pi i}\frac{1}{p-z} \tag{2.68}$$

*is in $H^q(\mathcal{E})$ for any $q \in (0,1)$. Furthermore,*

$$\int\limits_{z\in\partial\mathcal{E}} \mathcal{K}^*(p,z)^q\, d\sigma(z) \leq c \int\limits_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q\, d\sigma(z) \leq C \int\limits_{\zeta\in\partial\mathbb{D}_R(c(p))} \mathcal{K}^*(p,\zeta)\, d\sigma(\zeta) \tag{2.69}$$

*Proof.* We first show that,

$$\int\limits_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q\, d\sigma(z) \tag{2.70}$$

is integrable.

To do so, we observe that from previous Lemma we have that, in particular $0 \neq |\lambda_p'(p)|$, so there exists $\rho > 0$ such that $\lambda_P$ is invertible in $\mathbb{D}_\rho(p) \cap A_p$.

Let $F$ the image of $\mathbb{D}_\rho(p) \cap A_p$ under $\lambda_p$, the we can define $\pi_p : F \to A_p$ as

$$\pi_p(\zeta) = \lambda_p^{-1}(\zeta) \tag{2.71}$$

Also by Lemma 2.9, we have that,

$$b \leq |\lambda_p'(p)| \leq a \tag{2.72}$$

and so by continuity there exists $\varepsilon > 0$ such that if

$$|z-p| < \varepsilon \quad \text{then} \quad b \leq |\lambda_p'(z)| \leq a \tag{2.73}$$

and without loss of generality we can choose $0 < \varepsilon < \rho$.

Let $A_p' = \mathbb{D}_\varepsilon(p) \cap \partial\mathcal{E}$ then, observe that for all $z \in A_p'$ we have that

$$dist(z, \partial\mathbb{D}_R) = |z-\zeta| \tag{2.74}$$

where $\zeta = \lambda_p(z)$, since $z$ is in the ray connecting $\zeta$ with $c(p)$.

And so for any $\alpha > 0$ we have that,

$$|z - \zeta| < (1+\alpha) dist(z, \partial\mathbb{D}_R(c(p))) \tag{2.75}$$

hence, for any $\alpha > 0$ and any $z \in A'_p$ we know that

$$z \in \Gamma_\alpha(\zeta) \tag{2.76}$$

where $\zeta = \lambda_p(z)$

Fix $\alpha > 0$, then by last equation and recalling that $\pi_p(p) = p$ we have that,

$$|\mathcal{K}(p, \pi_p(\zeta))| = |\mathcal{K}(p, z)| \leq \sup_{w \in \Gamma_\alpha(\zeta)} |\mathcal{K}(p, w)| \tag{2.77}$$

and by definition 1.20

$$\sup_{w \in \Gamma_\alpha(\zeta)} |\mathcal{K}(p, w)| = \mathcal{K}^*(p, \zeta) \quad \text{a.e.} \tag{2.78}$$

hence,

$$\int_{F'} |\mathcal{K}(p, \pi_p(\zeta))|^q \, d\sigma(\zeta) \leq \int_{F'} [\mathcal{K}^*(p, \zeta)]^q \, d\sigma(\zeta) \tag{2.79}$$

Where $F'$ is the image of $A'_p$ under $\lambda_p$.

And by Corollary 2.6 we know that

$$\int_{F'} [\mathcal{K}^*(p, \zeta)]^q \, d\sigma(\zeta) < C \tag{2.80}$$

Combining these two equations we have that

$$\int_{F'} |\mathcal{K}(p, \pi_p(\zeta))|^q \, d\sigma(\zeta) < C \tag{2.81}$$

On the other hand,

$$\int\limits_{\zeta\in F'} |\mathcal{K}(p,\pi_p(\zeta))|^q \, d\sigma(\zeta) = \int\limits_{z\in A'_p} |\mathcal{K}(p,\pi_p(\lambda_p(z)))|^q |\lambda'(z)| d\sigma(z) \qquad (2.82)$$

$$= \int\limits_{z\in A'_p} |\mathcal{K}(p,z)|^q |\lambda'_p(z)| d\sigma(z)$$

So,

$$\int\limits_{z\in A'_p} |\mathcal{K}(p,z)|^q |\lambda'_p(z)| d\sigma(z) < C \qquad (2.83)$$

and if $z \in A'_p$, we know that $b \le |\lambda'_p(z)| \le a$

Then,

$$\int\limits_{z\in A'_p} |\mathcal{K}(p,z)|^q d\sigma(z) \le \frac{1}{b} \int\limits_{z\in A'_p} |\mathcal{K}(p,z)|^q |\lambda'_p(z)| d\sigma(z) < \frac{1}{b} C = C_1 \qquad (2.84)$$

Now, to deal with the rest of the boundary observe that if $|z-p| > \varepsilon$, then

$$\frac{1}{|p-z|^q} < \frac{1}{\varepsilon^q}, \qquad (2.85)$$

and hence,

$$\int\limits_{z\in\partial\mathcal{E}\setminus A'_p} \frac{d\sigma(z)}{|p-z|^q} < \int\limits_{\partial\mathcal{E}\setminus A'_p} \frac{1}{\varepsilon^q} \, d\sigma < \int\limits_{\partial\mathcal{E}} \frac{1}{\varepsilon^q} \, d\sigma(z) = \frac{1}{\varepsilon^q} \, \sigma(\partial\mathcal{E}) \qquad (2.86)$$

Combining all of the previous equations we have that,

$$\begin{aligned}\int\limits_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q \, d\sigma(z) &= \int\limits_{z\in\partial\mathcal{E}\setminus A_p'} |\mathcal{K}(p,z)|^q \, d\sigma(z) + \int\limits_{z\in A_p'} |\mathcal{K}(p,z)|^q \, d\sigma(z) \\ &\leq \frac{1}{\varepsilon^q}\sigma(\partial\mathcal{E}) + \frac{1}{b}\int\limits_{z\in\ A_p'} |\mathcal{K}(p,z)|^q |\lambda_p'(z)| \, d\sigma(z) \\ &= \frac{1}{\varepsilon^q}\sigma(\partial\mathcal{E}) + \frac{1}{b}\int\limits_{\zeta\in F'} |\mathcal{K}(p,\pi_p(\zeta))|^q \, d\sigma(\zeta) \\ &\leq \frac{1}{\varepsilon^q}\sigma(\partial\mathcal{E}) + \frac{1}{b}\int\limits_{\zeta\in F'} [\mathcal{K}^*(p,\zeta)]^q \ d\sigma(\zeta) \\ &= M + \frac{1}{b}\int\limits_{\zeta\in\partial\mathbb{D}_R} [\mathcal{K}^*(p,\zeta)]^q \ d\sigma(\zeta)\end{aligned} \tag{2.87}$$

Hence,

$$\int\limits_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q \, d\sigma(z) \leq C \int\limits_{\zeta\in\partial\mathbb{D}_R(c(p))} [\mathcal{K}^*(p,\zeta)]^q \ d\sigma(\zeta) \tag{2.88}$$

and since by equation (2.19) we know that there is a constant $M_0$ such that,

$$\int\limits_{z\in\partial\mathbb{D}_R(c(p))} [K^*(\zeta,z)]^q \ d\sigma(z) < M_0 < \infty \tag{2.89}$$

we have that,

$$\int\limits_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q \, d\sigma(z) \leq C \int\limits_{\zeta\in\partial\mathbb{D}_R(c(p))} [\mathcal{K}^*(p,\zeta)]^q \ d\sigma(\zeta) \leq M_0 \tag{2.90}$$

Finally, take $w \in \Gamma_\alpha(z)$, then by definition

$$|w-z| < (1+\alpha)dist(w,\partial\mathcal{E}) \tag{2.91}$$

but $p \in \partial\mathcal{E}$ implies

$$dist(w,\partial\mathcal{E}) \leq |w-p| \tag{2.92}$$

therefore $|w-z| \leq (1+\alpha)|w-p|$, and hence

$$|p-z| \leq |p-w| + |w-z| \leq |p-w| + (1+\alpha)|p-w| = (2+\alpha)|p-w| \tag{2.93}$$

Therefore, $\dfrac{1}{|p-w|} \leq \dfrac{(2+\alpha)}{|p-z|}$ and then

$$\sup_{w\in\Gamma_\alpha(p)} |\mathcal{K}(p,w)| \leq (2+\alpha)|\mathcal{K}(p,z)| \tag{2.94}$$

and then

$$\int_{z\in\partial\mathcal{E}} [\mathcal{K}^*(p,z)]^q \; d\sigma(z) \leq (2+\alpha)^q \int_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q \, d\sigma(z) \tag{2.95}$$

Combining equation (2.90) and (2.95) we get

$$\int\limits_{z\in\partial\mathcal{E}} [\mathcal{K}^*(p,z)]^q \; d\sigma(z) \leq (2+\alpha)^q \int\limits_{z\in\partial\mathcal{E}} |\mathcal{K}(p,z)|^q \, d\sigma(z) \leq C \int\limits_{\zeta\in\partial\mathbb{D}_R(c(p))} [\mathcal{K}^*(p,\zeta)]^q \; d\sigma(\zeta) \leq M_0 \tag{2.96}$$

Hence $K(p,z) \in H^q(\mathcal{E})$ for all $q \in (0,1)$. □

**Corollary 2.11.** *For any ellipse $\mathcal{E} \in \mathbb{C}$ and for any fixed $p \in \partial\mathcal{E}$, the Cauchy Leray Kernel function.*

$$K(p,z) = \frac{1}{2\pi i}\frac{1}{p-z} \tag{2.97}$$

*is in $E^q(\mathcal{E})$ for every $q \in (0,1)$.*

*Proof.* The proof of this Corollary is an immediate consequence of previous proposition and Lemma 1.22 □

We show now that Proposition 2.10 fails if $q \geq 1$.

**Theorem 2.12.** *For any ellipse $\mathcal{E}$ in $\mathbb{C}$, and for any fixed $p \in \partial\mathcal{E}$, the Cauchy Kernel Function*

$$\mathcal{K}(p,z) = \frac{1}{2\pi i}\frac{1}{p-z} \tag{2.98}$$

*is in* $H^q(\mathcal{E})$ *if and only if* $q \in (0,1)$

*Proof.* The proof that the statement is true for any $q \in (0,1)$ is Proposition 2.10. It remains to show that if $q \geq 1$ then $K(p,z)$ is not in $H^q(\mathcal{E})$. We proceed by contradiction.

Let $1 \leq q \leq \infty$ and suppose that $\mathcal{K}(p,z) \in H^q(\mathcal{E})$, then there is a positive constant $M$ such that

$$\int\limits_{z\in\partial\mathcal{E}} [\mathcal{K}^*(p,z)]^q \; d\sigma(z) \leq M \tag{2.99}$$

Consider $\pi_p : \partial\mathbb{D}_R(c(p)) \to \partial\mathcal{E}$ to be

$$\pi_p = \lambda_p^{-1} \tag{2.100}$$

where $\lambda_p$ is defined as in equation (2.48) and recall that by equation (2.49), we have that

$$b \leq |\lambda_p'(p)| \leq a \tag{2.101}$$

So,

$$\frac{1}{a} \leq |\pi_p'(p)| \leq \frac{1}{b} \tag{2.102}$$

By continuity of $\pi_p'$ we know that there exists $\varepsilon > 0$ such that if

$$|z-p| < \delta \quad \text{then} \quad \frac{1}{2a} \leq |\pi_p'(z)| \leq \frac{1}{2b} \tag{2.103}$$

Then using the same methods that the ones used to obtain equation (2.88) we can show that

$$\int\limits_{\zeta\in\partial\mathbb{D}_r(c(p))} |\mathcal{K}(p,\zeta)|^q d\sigma(\zeta) \leq \int\limits_{z\in\partial\mathcal{E}} [\mathcal{K}^*(p,z)]^q d\sigma(z) < 2bM < C \tag{2.104}$$

And using a similar argument to the one in proposition 2.10 that for a fixed $\alpha > 0$

$$\int\limits_{\zeta\in\partial\mathbb{D}_R(c(p))} [\mathcal{K}^*(p,\zeta)]^q \; d\sigma(\zeta) \leq (2+\alpha)^q \int\limits_{\zeta\in\partial\mathbb{D}_R(c(p))} |\mathcal{K}(p,\zeta) \, d\sigma(z) \tag{2.105}$$

Combining these last two equations, we get

$$\int\limits_{\zeta \in \mathbb{D}_R(c(p))} [\mathcal{K}^*(p,\zeta)]^q \; d\sigma(\zeta) \leq (2+\alpha)^q C_1 < \infty \tag{2.106}$$

and hence $\mathcal{K}(p,\zeta) \in H^q[\mathbb{D}_R(c(p))]$ which is a contradiction by Lemma 2.4

□

Chapter 3

# $\mathbb{C}$ STRICTLY CONVEX DOMAINS

The goal of this section is to extend the results obtained in Proposition 2.10 and Theorem 2.12 for any strictly convex, bounded domain in $\mathbb{C}$.

To do so, we will use similar approach to the one used for the ellipse case, although we will use a local argument in this case. That is, we will relate the Kernel Function, $\mathcal{K}(p, z)$, of the domain $D$ to the one of a disc that locally contains our domain.

In order to be able to construct such ball, we need to first study the concept of signed curvature.

## 3.1 Signed Curvature

All of the definitions and lemmas in this section can be found in [6].

**Definition 3.1.** Let $z, w$ be in $\mathbb{C}$. We define,

$$z \cdot w = Re(z\bar{w}) \tag{3.1}$$

and

$$P(z) = iz \tag{3.2}$$

Using the definition is easy to show that $P$ and $\cdot$ have the following properties

$$P \circ P = -I \text{ ,where } I \text{ is the identity.} \tag{3.3}$$

$$P(z) \cdot P(w) = z \cdot w \tag{3.4}$$

$$P(z) \cdot z = 0 = z \cdot P(z) \tag{3.5}$$

$$P(wz) = wP(z) \tag{3.6}$$

**Definition 3.2.** Let $\alpha : (a, b) \to \mathbb{C}$ be a curve. Then the *velocity* of $\alpha$ is the function $\alpha' : (a, b) \to \mathbb{C}$. We call the function $v$ defined by $v(t) = ||\alpha'(t)||$ the *speed* of $\alpha$. The *acceleration* of $\alpha$ is $\alpha''$.

**Definition 3.3.** A curve $\gamma$ is said to be *regular* if there is a parametrization $\alpha : (a, b) \to \mathbb{C}$ that is differentiable and its velocity, $\alpha'$, is everywhere non-zero. If $||\alpha'(t)|| = 1$ for all $a < t < b$ then $\alpha$ is said to have unit speed.

**Definition 3.4.** Let $\alpha : (a, b) \to \mathbb{C}$ and $\beta : (c, d) \to \mathbb{C}$ be differentiable curves. Then $\beta$ is said to be a *positive reparametrization of* $\alpha$ provided that there exists a differentiable function $h : (c, d) \to (a, b)$ such that $h'(t) > 0$ for all $c < t < d$ and $\beta = \alpha \circ h$. Similarly $\beta$ is called a *negative reparametrization of* $\alpha$ if $h'(t) < 0$ for all $c < t < d$.

**Definition 3.5.** Let $\alpha : (a, b) \to \mathbb{C}$ be a regular parametrization for the curve $\gamma$. The *signed curvature* $sk_\alpha$ *of* $\alpha$ *at* $z = \alpha(t)$ is given by the formula,

$$sk_\alpha(t) = \frac{\alpha''(t) \cdot P(\alpha'(t))}{|\alpha'(t)|^3} \tag{3.7}$$

where $P(z)$ and $\cdot$ are as in Definition 3.1.

**Lemma 3.6.** *Let* $\gamma$ *be a curve and* $\alpha : (a, b) \to \mathbb{C}$ *and* $\beta : (c, d) \to \mathbb{C}$ *two regular parametrizations of* $\gamma$*. Write* $\beta = \alpha \circ h$ *where* $h : (c, d) \to (a, b)$ *is differentiable, then*

$$sk_\beta(t) = sign(h'(t))sk_\alpha(h(t)) \tag{3.8}$$

Note, that since $\alpha$ and $\beta$ are regular, then $h'(t) \neq 0$ for every $t$ and so $sign(h'(t))$ is well defined for every $t$ and it will take the constant value 1 or $-1$. This shows that the signed curvature of a curve is up to sign independent of parametrization, furthermore if $\beta$ is a positive reparametrization of $\alpha$ then

$$sk_\beta(t) = sk_\alpha(h(t)) \tag{3.9}$$

*Proof.* The fact that $\beta = \alpha \circ h$ yields

$$\beta' = (\alpha' \circ h)h', \tag{3.10}$$

and

$$\beta'' = (\alpha'' \circ h)(h')^2 + (\alpha' \circ h)h'', \tag{3.11}$$

and by equation (3.6) we have that,

$$P(\beta') = P\left((\alpha \circ h)h'\right) = h'P(\alpha' \circ h). \tag{3.12}$$

So,

$$\begin{aligned} sk_\beta(t) =& \frac{\beta''(t) \cdot P(\beta'(t))}{|\beta'(t)|^3} \\ =& \frac{[(\alpha'' \circ h)(t)(h'(t))^2 + (\alpha' \circ h)(t)(h''(t)] \cdot P[(\alpha' \circ h)(t)h'(t)]}{|(\alpha' \circ h)(t)h'(t)|^3} \\ =& \frac{[\alpha''(h(t))(h'(t))^2 + \alpha'(h(t))h''(t)] \cdot h'(t)P(\alpha'(h(t))}{|\alpha'(h(t))h'(t)|^3} \\ =& \frac{\alpha''(h(t))h'(t)^3 \cdot P(\alpha'(h(t))) + [\alpha'(h(t)h''(t)h'(t)] \cdot P(\alpha'(h(t)))}{|\alpha'(h(t))|^3|h'(t)|^3} \\ =& \frac{\alpha''(h(t))h'(t)^3 \cdot P(\alpha'(h(t)))}{|\alpha'(h(t)|^3|h'(t)|^3} + \frac{[\alpha'(h(t))h''(t)h'(t)] \cdot P(\alpha'(h(t)))}{|\alpha'(h(t))|^3|h'(t)|^3} \\ =& \frac{h'(t)^3}{|h'(t)|^3}\frac{\alpha''(h(t)) \cdot P(\alpha'(h(t)))}{|\alpha'(h(t))|^3} + \frac{[\alpha'(h(t))h''(t)h'(t)] \cdot P(\alpha'(h(t)))}{|\alpha'(h(t))|^3|h'(t)|^3} \end{aligned} \tag{3.13}$$

and by equation (3.5) we know that $\alpha'(h(t)) \cdot P(\alpha'(h(t))) = 0$ and so the second term on the last equation is zero. Hence,

$$
\begin{aligned}
sk_\beta(t) =& \frac{h'(t)^3}{|h'(t)|^3}\frac{\alpha''(h(t)) \cdot P(\alpha'(h(t)))}{|\alpha'(h(t))|} \\
=& sign(h'(t)^3) sk_\alpha(h(t)) \\
=& sign(h'(t)) sk_\alpha(h(t))
\end{aligned}
\tag{3.14}
$$

□

As a consequence of this past Lemma if $z = \alpha(t)$ is a point in the curve $\gamma$ we can use without any ambiguity $sk(z)$ or $sk(t)$ to denote the signed curvature at $z$.

**Lemma 3.7.** *If $\gamma$ is a regular curve with parametrization $\alpha : (a, b) \to \mathbb{C}$ where $\alpha(t) = x(t) + iy(t)$, then the signed curvature $sk(t)$ is given by*

$$
sk(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{([x'(t)]^2 + [y'(t)]^2)^{3/2}} \tag{3.15}
$$

*Proof.*

$$
\begin{aligned}
sk(t) =& \frac{(x''(t) + iy''(t)) \cdot P\left(x'(t) + iy'(t)\right)}{|x'(t) + iy'(t)|^3} \\
=& \frac{Re\left((x''(t) + iy''(t))(-y'(t) - ix'(t))\right)}{[(x'(t))^2 + (y'(t))^2]^{3/2}} \\
=& \frac{x'(t)y''(t) - x''(t)y'(t)}{[(x'(t))^2 + (y'(t))^2]^{3/2}}.
\end{aligned}
\tag{3.16}
$$

□

**Corollary 3.8.** *Suppose now that the curve $\gamma$ is the graph of a twice differentiable function, $f(t)$, then $sk(z)$ at $z = t + if(t)$ is given by*

$$
sk(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{3/2}} \tag{3.17}
$$

*Proof.* Let $t = x(t)$ and $f(t) = y(t)$, the result follows immediately from Lemma 3.7 with this assignation of $x(t)$ and $y(t)$. ☐

Next, we will show that the signed curvature of a curve $\gamma$ is independent of the the position of $\gamma$ in the plane, i.e. that signed curvature is invariant under rotations and translations. We begin by defining several classes of transformations.

**Definition 3.9.** Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonsingular linear map. We say that

i) $A$ is *orientation preserving* if $det(A)$ is positive, or *orientation reversing* if $det(A)$ is negative.

ii) $A$ is called an *orthogonal transformation* if

$$A(p) \cdot A(q) = p \cdot q \tag{3.18}$$

iii) A *rotation* of $\mathbb{R}^2$ is an orientation preserving orthogonal transformation.

**Lemma 3.10.** *Let $B : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal transformation. Then*

$$det(B) = \pm 1 \tag{3.19}$$

**Definition 3.11.** Let $p \in \mathbb{R}^2$

i) An *affine transformation* of $\mathbb{R}^2$ is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$F(p) = A(p) + q \tag{3.20}$$

for all $p \in \mathbb{R}^2$, where $A$ is a linear transformation of $\mathbb{R}^2$. We call $A$ the *linear part* of the affine transformation $F$. An affine transformation $F$ is *orientation preserving* if $det(A) = 1$ and *orientation reversing* if $det(A) = -1$.

ii) A *translation* of $\mathbb{R}^2$ is an affine map $T_q(p) : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$T_q(p) = p + q \tag{3.21}$$

for all $p \in \mathbb{R}^2$.

iii) An *Euclidean Motion* of $\mathbb{R}^2$ is an affine transformation whose linear part is an orthogonal transformation.

iv) An *isometry* of $\mathbb{R}^2$ is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distance, that is,

$$||F(p) - F(q)|| = ||p - q|| \tag{3.22}$$

**Lemma 3.12.** *Any Euclidean motion is the composition of a translation and an orthogonal transformation.*

**Lemma 3.13.** *A map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of $\mathbb{R}^2$ if and only if it is a composition of a translation and an orthogonal transformation of $\mathbb{R}^2$. Thus the group of Euclidean motions of $\mathbb{R}^2$ coincides with the group of isometries of $\mathbb{R}^2$.*

It follows from last lemma that since an Euclidean motion preserves distance, it cannot deform a curve in the plane and hence the curvature should be invariant up to a sign under Euclidean motions. We know quote the desired result.

**Lemma 3.14.** *The signed curvature is preserved by an orientation preserving Euclidean motion of $\mathbb{R}^2$ and changes sign under an orientation reversing Euclidean motion.*

### 3.2 The Cauchy Kernel Function of a strictly convex domain of class $C^2$

All throughout this section we will be working with bounded strictly convex domains of class $C^2$. Using the concepts of the previous section we can now give a definition of what we will, in this work, understand by strictly convex domain.

**Definition 3.15.** Let $D$ be a bounded domain of class $C^2$. We say that $D$ is *strictly convex* if $sk(z) > 0$ for all $z \in \partial D$, whenever $\partial D$ is parametrized along the counterclockwise direction.

Observe that the fact that $D$ is of class $C^2$ implies by Definition 3.5 that $sk(z)$ is a continuous function on $\partial D$ and since $\partial D$ is a compact set, $sk(z)$ attains a maximum and a minimum, while the fact that $D$ is strictly convex means that the minimum is strictly greater than zero. We will denote $m_{\partial D}$ and $M_{\partial D}$ to said minimum and maximum, namely, for all $z \in \partial D$ we have that

$$0 < m_{\partial D} \leq sk(z) < M_{\partial D} < \infty \tag{3.23}$$

We now construct the osculatory ball to our domain.

**Lemma 3.16.** *For all $p \in \partial D$ there exist $R > 0$, a point $c(p)$ and a neighborhood $U$ of $p$ such that $U \subset \mathbb{D}_R(c(p))$.*

*Proof.* Fix $p \in \partial D$, then we can choose linear coordinates in $\mathbb{C}$ so that, $p$ is at the origin, the $x$-axis is the tangent line to $\partial D$ at $p$ and the $y$-axis is the normal line.

Furthermore there exist an open interval $(-a, a)$ about 0 such that $\partial D$ is locally the graph of a function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $f'(0) = 0$.

Note: For all the previous assertions about choice of coordinates see [7].

We will denote $E_p$ to the piece of boundary described above, that is,

$$E_p = \{z \in \partial D \mid z = x + if(x)\} = \{x + if(x) \mid x \in (-a, a)\} \tag{3.24}$$

Observe that $D$ strictly convex guarantees that $E_p$ is totally contained in the upper half plane, i.e. $f(x) > 0$ for all $x \in (-a, a)$.

To achieve our goal we wish to construct a function $g$ such that,

1. The image of $g$ is part of a circumference contained totally in the upper half plane.

2. The image of $g$ is tangent to $E_p$ at the origin.

3. There is an interval $(-\delta, \delta)$ such that $f(x) \geq g(x)$ for all $x \in (-\delta, \delta)$.

We proceed with the construction of such a function.

By equation (3.23) there exists a real number $m$ such that

$$0 < m < sk(z) \tag{3.25}$$

for all $z \in \partial D$, and by Lemma 3.14 we know that this inequality is satisfied for all $z \in E_p$.

Fix $\alpha \in (0,1)$, and consider $R = 1/(\alpha m)$. We define $g : (-R, R) \to \mathbb{R}$ as

$$g(x) = -\sqrt{R^2 - x^2} + R \tag{3.26}$$

and denote

$$F = \{x + ig(x) \mid (-R, R)\} \tag{3.27}$$

then clearly $F$ is the lower half of a circumference with center in $iR$ and radius $R$, and so $F$ is totally contained in the upper half plane. We now show that $F$ is tangent to $E_p$ at 0.

We begin by observing that $g(0) = 0$ and so $0 \in F \cap E_p$, Now let $T_{E_p}(x)$ denote the tangent vector to $E_p$ at $x$ and $T_F(x)$ the tangent vector to $F$ at $x$, then we have that

$$T_{E_p}(x) = 1 + f'(x) \tag{3.28}$$

while,

$$T_F(x) = 1 + g'(x) \tag{3.29}$$

but,

$$g'(x) = \frac{x}{\sqrt{R^2 - x^2}} \tag{3.30}$$

and so, $g'(0) = 0$, and by choice of coordinates we also know that $f'(0) = 0$. Hence

$$T_{E_p}(0) = 1 = T_F(0) \tag{3.31}$$

Therefore $E_p$ and $F$ are tangent at 0.

Next, we would like to construct an interval $(-\delta, \delta)$ such that $f(x) \geq g(x)$ for all $x \in (-\delta, \delta)$.

By Taylor's theorem we know that, for all $x \in (-a/2, a/2)$

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + R_2^f(x) \tag{3.32}$$

where $R_2^f(x) \in o(x^2)$ while,for all $x \in (-R/2, R/2)$

$$g(x) = g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + R_2^g(x) \tag{3.33}$$

with $R_2^g(x) \in o(x^2)$.

Take $\varepsilon = \min\{(m - m\alpha)/2, a/2, R/2\}$, and observe that $R_2^f(x)$ and $R_2^g(x)$ in $o(x^2)$ means that for such $\varepsilon$ there exists $\delta_0 > 0$ small enough so that $\delta_0 < \varepsilon$ and if $|x| < \delta_0$ then,

$$|R_2^f(x)| < \frac{\varepsilon}{4}x^2 \tag{3.34}$$

and

$$|R_2^g(x)| < \frac{\varepsilon}{4}x^2 \tag{3.35}$$

We prove now that for all $x \in (-\delta_0, \delta_0)$ with $\delta_0$ chosen as above, we have that $f(x) \geq g(x)$. First observe that $\delta_0 < a/2$ and $\delta_0 < R/2$ implies that equations (3.32) and (3.33) hold in $(-\delta_0, \delta_0)$ and then, for all $x \in (-\delta_0, \delta_0)$

$$f(x)-g(x) = f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+R_2^f(x)-\left(g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + R_2^g(x)\right) \tag{3.36}$$

But recall that $f(0) = 0 = f'(0)$ and $g(0) = 0 = g'(0)$, substituting these quantities in equation (3.36) we get,

$$f(x) - g(x) = \frac{1}{2}f''(0)x^2 + R_2^f(x) - \frac{1}{2}g''(0)x^2 - R_2^g(x) \qquad (3.37)$$
$$\frac{1}{2}x^2(f''(0) - g''(0)) + R_2^f(x) - R_2^g(x)$$

And simple calculations show that,

$$g''(x) = \frac{R^2}{(R^2 - x^2)^{3/2}} \qquad (3.38)$$

and so,

$$g''(0) = \frac{1}{R} \qquad (3.39)$$

On the other hand by formula (3.17) we have,

$$sk(0) = \frac{f''(0)}{(1 + [f'(0)]^2)^{3/2}} = f''(0) \qquad (3.40)$$

Substituting equations (3.39) and (3.40) on equation (3.37) we get,

$$f(x) - g(x) = \frac{1}{2}x^2\left(sk(0) - \frac{1}{R}\right) + (R_2^f(x) - R_2^g(x)) \qquad (3.41)$$
$$= \frac{1}{2}x^2\left(sk(0) - \alpha m\right) + (R_2^f(x) - R_2^g(x)$$

But $m$ is such that $m \leq sk(z)$ for all $z \in \partial D$, hence,

$$f(x) - g(x) = \frac{1}{2}x^2\left(sk(0) - \alpha m\right) + (R_2^f(x) - R_2^g(x)) \qquad (3.42)$$
$$\geq \frac{1}{2}x^2(m - \alpha m) + (R_2^f(x) - R_2^g(x))$$
$$> \varepsilon x^2 + (R_2^f(x) - R_2^g(x))$$

To bound the difference of the residues of the Taylor's expansion, observe that combining equations (3.34) and (3.35) we obtain,

$$R_2^f(x) - R_2^g(x) > -\frac{\varepsilon}{2}x^2 \tag{3.43}$$

Combining equations (3.42) and (3.43) we see that

$$\begin{aligned} f(x) - g(x) >& \varepsilon x^2 + (R_2^f(x) - R_2^g(x)) \\ >& \varepsilon x^2 - \frac{\varepsilon}{2}x^2 = \frac{\varepsilon}{2}x^2 > 0 \end{aligned} \tag{3.44}$$

Hence for all $x \in (-\delta_0, \delta_0)$ we have that $f(x) \geq g(x)$. □

Observe that even though we only have a local inclusion of $\partial D$ in $\mathbb{D}_R(c(p))$ this is still a global result in the sense that the radius of this disc does not depend on our choice of $p$.

We now, construct our projection $\lambda : E_p \to F$ such that $p = 0$ is invariant under $\lambda$ and such that $|\lambda'(0)|$ is bounded away from zero and infinity.

**Lemma 3.17.** *Let*

$$E = \{z = x + if(x) \mid x \in (-\delta_0, \delta_0)\} \tag{3.45}$$

*Define* $\lambda : E \to F$ *as*

$$\lambda(z) = R\frac{z - iR}{|z - iR|} + iR \tag{3.46}$$

*Then*

$$|\lambda(0)| = 0 \quad \textit{and} \quad |\lambda'(0)| = 1 \tag{3.47}$$

*Proof.* We first observe that $iR$ is not in $E$ so

$$|z - iR| \neq 0 \quad \text{for all } z \in E, \tag{3.48}$$

and so $\lambda$ is a $C^\infty$ function on $E$. Next we evaluate $\lambda$ at 0,

$$\lambda(0) = R\frac{0 - iR}{|0 - iR|} + iR = -iR + iR = 0 \tag{3.49}$$

Therefore 0 is invariant under $\lambda$.

We now need to calculate $|\lambda'(0)|$. Observe that since any $z \in E$ can be expressed as $z = x + if(x)$ then $\lambda(z) = \lambda(x + if(x))$ is really a function of $x$, with $x \in (-\delta_0, \delta_0)$. Then we can can express $\lambda(x)$ as

$$\lambda(x) = u(x) + iv(x) = \frac{Rx}{I(x)} + i\left(\frac{Rf(x) - R^2}{I(x)} + R\right) \tag{3.50}$$

with

$$I(x) = \left[x^2 + (f(x) - R)^2\right]^{1/2} \tag{3.51}$$

and $|\lambda'(0)| = |u'(0) + iv'(0)|$. But

$$u'(x) = \frac{RI(x) - RxI'(x)}{(I(x))^2} \tag{3.52}$$

and

$$v'(x) = \frac{Rf'(x)I(x) - \left(Rf(x) - R^2\right)I'(x)}{(I(x))^2} \tag{3.53}$$

with

$$I'(x) = \frac{1}{2}\left(x^2 + (f(x) - R)^2\right)^{-1/2}(2x + 2(f(x) - R)f'(x)) \tag{3.54}$$

evaluating all of these quantities at zero we get,

$$I(0) = R \quad \text{and} \quad I'(0) = 0 \tag{3.55}$$

and so,

$$u'(0) = \frac{RI(0) - 0}{(I(0))^2} = \frac{R^2}{R^2} = 1 \tag{3.56}$$

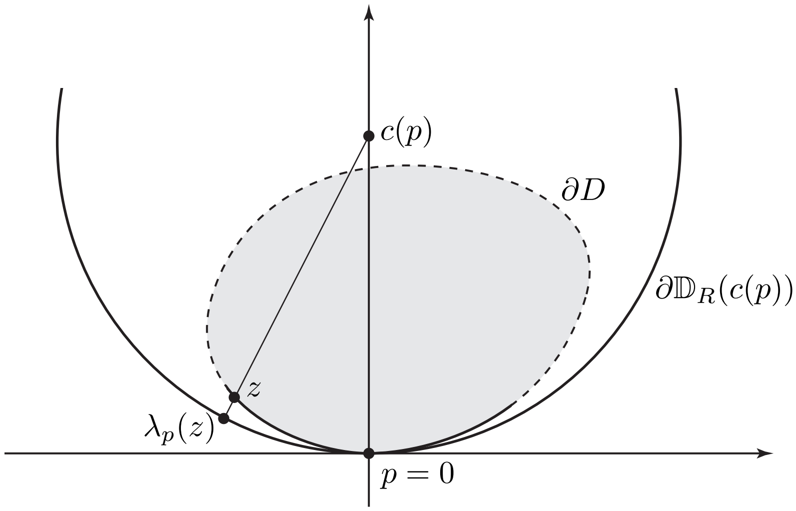


Figure 3.1: The projection $\lambda$ from $\partial D$ to $\partial\mathbb{D}_R(c(p))$

and recalling that $f'(0) = 0$ we see that,

$$v'(0) = \frac{Rf'(0)I(0) - (Rf(0) - R^2)I'(0)}{(I(0))^2} = 0 \tag{3.57}$$

and so, finally,

$$|\lambda'(0)| = |u'(0) + iv'(0)| = 1 \tag{3.58}$$

□

**Theorem 3.18.** *For any bounded strictly convex domain $D$ of class $C^2$ and fixed $p \in \partial D$, the Cauchy-Leray Kernel function*

$$\mathcal{K}(p, z) = \frac{1}{2\pi i}\frac{1}{p - z} \tag{3.59}$$

*is in $H^q(D)$ if and only if $q \in (0, 1)$.*

*Proof.* Fix $p \in \partial D$ and consider the change of coordinates as discussed in Lemma 3.16, and let $\lambda$ be the projection described in Lemma 3.17. Then, we know that $|\lambda'(0)| = 1 \neq 0$, so there is $\rho > 0$ so that $\lambda$ is one-to-one in $E' = \mathbb{D}_\rho(p) \cap E$. Let $G$ be the image of of $E'$ under $\lambda$, then we can define $\pi : G \to E'$ as $\pi(\zeta) = \lambda^{-1}(\zeta)$ for all $\zeta \in G$.

Just like in the case of Proposition 2.10 we begin by showing that

$$\int_{z \in \partial D} |\mathcal{K}(0,z)|^q d\sigma(z) < C \tag{3.60}$$

By continuity of $\lambda'$ and Lemma 3.16 we can find $\varepsilon > 0$ small enough so that $A = \mathbb{D}_\varepsilon(0) \cap \partial D$ is contained in $E'$ and for all $z \in A$ we have that,

$$\frac{1}{2} < |\lambda'(z)| < 2 \tag{3.61}$$

Now, observe that If $z \in A$ then

$$dist(z, \partial \mathbb{D}_R(iR)) = |z - \lambda(z)| = |z - \zeta| \tag{3.62}$$

and so for any $\alpha > 0$ we have that

$$|z - \lambda(z)| \leq (1+\alpha) dist(z, \partial \mathbb{D}_R(iR)) \tag{3.63}$$

then for any $\alpha > 0$ and any $z \in A$

$$z \in \Gamma_\alpha(\lambda(z)) \tag{3.64}$$

Fix $\alpha > 0$, then by last equation and using the fact that $\pi(0) = 0$ we have,

$$|\mathcal{K}(\pi(0), \pi(\zeta))| = |\mathcal{K}(0,z)| \leq \sup_{w \in \Gamma_\alpha(\zeta)} |\mathcal{K}(0,w)| \tag{3.65}$$

and since

$$\sup_{w \in \Gamma(\zeta)} |\mathcal{K}(0,w)| = \mathcal{K}^*(0,\zeta) \quad \text{a.e.} \tag{3.66}$$

hence,

$$\int_{\zeta\in F} |\mathcal{K}(0,\pi(\zeta)|^q d\sigma(\zeta) \leq \int_{\zeta\in F} [\mathcal{K}^*(0,\zeta)]^q d\sigma(\zeta) \leq \int_{\zeta\in\partial\mathbb{D}_R(iR)} [\mathcal{K}^*(0,\zeta)]^q d\sigma(\zeta) \tag{3.67}$$

and by Corollary 2.6 we know that

$$\int_{\zeta\in\partial\mathbb{D}_R(iR)} [\mathcal{K}^*(0,\zeta)]^q d\sigma(\zeta) < M \tag{3.68}$$

On the other hand, using a change of variables we know that,

$$\begin{aligned} \int_{\zeta\in F} |\mathcal{K}(0,\pi(\zeta))|^q d\sigma(\zeta) &= \int_{z\in A} |\mathcal{K}(0,\pi(\lambda(z)))|^q |\lambda'(z)| d\sigma(z) \\ &= \int_{z\in A} |\mathcal{K}(0,z)| |\lambda'(z)| d\sigma(z) \end{aligned} \tag{3.69}$$

Combinig equations (3.67) and (3.69) we have that

$$\int_{z\in A} |\mathcal{K}(0,z)|^q |\lambda'(z)| d\sigma(z) \leq \int_{\zeta\in\mathbb{D}_R(iR)} [\mathcal{K}^*(0,\zeta)]^q d\sigma(\zeta) < M \tag{3.70}$$

but if $z \in A$ then we know that $1/2 < |\lambda'(z)| < 2$ and so,

$$\int_{z\in A} |\mathcal{K}(0,z)|^q d\sigma(z) \leq 2 \int_{z\in A} |\mathcal{K}(0,z)|^q |\lambda'(z)| d\sigma(z) < 2M \tag{3.71}$$

On the other hand if $|z| \geq \varepsilon$ then

$$\frac{1}{|z|^q} \leq \frac{1}{\varepsilon^q} \tag{3.72}$$

and hence,

$$\int_{|z|\geq\varepsilon} |\mathcal{K}(0,z)|^q \, d\sigma(z) \leq \int_{|z|\geq\varepsilon} \frac{1}{\varepsilon^q} \, d\sigma(z) \leq \int_{z\in\partial D} \frac{1}{\varepsilon^q} \, d\sigma(z) = \frac{1}{\varepsilon^q}\sigma(\partial D) \tag{3.73}$$

Combining equations (3.71) and (3.73) we get,

$$\begin{aligned} \int_{z\in\partial D} |\mathcal{K}(0,z)|^q \, d\sigma(z) &= \int_{\substack{z\in\partial D \\ |z|<\varepsilon}} |\mathcal{K}(0,z)|^q \, d\sigma(z) + \int_{\substack{z\in\partial D \\ |z|\geq\varepsilon}} |\mathcal{K}(0,z)|^q \, d\sigma(z) \\ &< 2M + \frac{1}{\varepsilon^q}\sigma(\partial D) = C_1 \end{aligned} \tag{3.74}$$

Therefore,

$$\int_{z\in\partial D} |\mathcal{K}(0,z)|^q < C_1 < \infty \tag{3.75}$$

Lastly, let $z \in \partial D$ and take $w \in \Gamma_\alpha(z)$. Then by definition

$$|w - z| < (1+\alpha)dist(w, \partial D) \tag{3.76}$$

but $0 \in \partial D$ implies

$$dist(w, \partial D) \leq |w| \tag{3.77}$$

therefore, $|w - z| \leq (1+\alpha)|w|$, and hence

$$|z| \leq |w| + |w - z| \leq |w| + (1+\alpha)|w| = (2+\alpha)|w| \tag{3.78}$$

So,

$$\frac{1}{|w|} \leq \frac{2+\alpha}{|z|} \tag{3.79}$$

and then

$$\mathcal{K}^*(0,z) = \sup_{w\in\Gamma_\alpha(z)} |\mathcal{K}(0,w)| = \sup_{w\in\Gamma_\alpha(z)} \frac{1}{|w|} \tag{3.80}$$
$$\leq \frac{2+\alpha}{|z|} = (2+\alpha)|\mathcal{K}(0,z)|$$

so,

$$\int\limits_{z\in\partial D} \mathcal{K}^*(0,z)^q \, d\sigma(z) \leq (2+\alpha)^q \int\limits_{z\in\partial D} |\mathcal{K}(0,z)|^q \, d\sigma(z) \leq (2+\alpha)C_2 = C_0 < \infty \tag{3.81}$$

Hence, $\mathcal{K}$ is in $H^q(D)$ for $0 < q < 1$.

It remains to prove that $\mathcal{K}(p,z) \notin H^q(D)$ for $q \geq 1$.

We proceed by contradiction. Suppose that $\mathcal{K}(p,z) \in H^q(D)$ for some $q \geq 1$, then by definition we have that,

$$\int\limits_{z\in\partial D} [\mathcal{K}^*(0,z)]^q \, d\sigma(z) \leq C_1 \tag{3.82}$$

By Lemma 3.17 we know that $|\lambda'(0)| = 1$ and so $|\pi'(0)| = 0$, then we know that by continuity we know that there exists $\delta > 0$ such that if

$$|\zeta| < \delta \quad \text{then} \quad \frac{1}{2} < |\pi'(\zeta)| < 2 \tag{3.83}$$

with $\zeta \in F \subset \partial\mathbb{D}_R(iR)$.

The by a similar argument that the one done previously we can show that

$$\int\limits_{\substack{\zeta\in\partial\mathbb{D}_R(iR)\\ |\zeta|<\varepsilon}} |\mathcal{K}(0,\zeta)|^q d\sigma(\zeta) \leq 2 \int\limits_{\substack{\zeta\in\partial\mathbb{D}_R(iR)\\ |\zeta|<\varepsilon}} |\mathcal{K}(0,\zeta)|^q |\pi'(\zeta)| d\sigma(\zeta) \tag{3.84}$$
$$\leq 2 \int\limits_{z\in\partial D} |\mathcal{K}^*(0,z)|^q d\sigma(z)$$

and if $|\zeta| \geq \varepsilon$, then $1/|\zeta|^q \leq 1/\varepsilon^q$,and so,

$$\int_{\zeta \in \partial \mathbb{D}_R(iR)} |\mathcal{K}(0,\zeta)|^q d\sigma(\zeta) \leq \int_{\zeta \in \partial \mathbb{D}_R(iR)} \frac{1}{\varepsilon^q} d\sigma(\zeta) = \frac{1}{\varepsilon^q} \sigma(\partial \mathbb{D}_R(iR)) = C_2 \tag{3.85}$$

Therefore,

$$\begin{aligned} \int_{\zeta \in \partial \mathbb{D}_R(iR)} |\mathcal{K}(0,\zeta)|^q d\sigma(\zeta) = & \int_{\substack{\zeta \in \partial \mathbb{D}_R(iR) \\ |\zeta| < \varepsilon}} |\mathcal{K}(0,\zeta)|^q d\sigma(\zeta) + \int_{\substack{\zeta \in \partial \mathbb{D}_R(iR) \\ |\zeta| \geq \varepsilon}} |\mathcal{K}(0,\zeta)|^q d\sigma(\zeta) \\ \leq & 2C_1 + C_2 < M \end{aligned} \tag{3.86}$$

Finally, by a similar argument to the one used to obtain equation (3.79) we can show that for any $\zeta \in \partial \mathbb{D}_R(iR)$

$$[\mathcal{K}^*(0,\zeta)]^q \leq |\mathcal{K}(0,\zeta)|^q \tag{3.87}$$

and hence,

$$\int_{\zeta \in \partial \mathbb{D}_R(iR)} [\mathcal{K}^*(0,\zeta)]^q \, d\sigma(\zeta) \leq \int_{\zeta \in \partial \mathbb{D}_R(iR)} |\mathcal{K}(0,\zeta)|^q \, d\sigma(\zeta) < M \tag{3.88}$$

Therefore, $\mathcal{K}(0,\zeta) \in H^q(\mathbb{D}_R(iR))$ for some $q \geq 1$ which is a contradiction of Lemma 1.22. From which we conclude that if $q \geq 1$ then $\mathcal{K} \notin H^q(D)$. □

To conclude this work we combine 3.18 and 1.22

**Corollary 3.19.** *Let $D$ be a bounded strictly convex set in $\mathbb{C}$, and $p$ a fixed point in $\partial D$, then for every $q \in (0,1)$ we have that the Cauchy Kernel Function $\mathcal{K}(p,z)$ belongs to $E^q(D)$.*

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