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The Maximal Thurston-Bennequin Number on Grid Number n Diagrams

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The Maximal Thurston-Bennequin Number
on Grid Number n Diagrams

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Abstract

We will prove an upper bound for the Thurston-Bennequin number of Legendrian knots and links on a rectangular grid with arc index n.

$$TB(n) = CR(n) - \left\lceil \frac{n}{2} \right\rceil$$

In order to prove the bound, we will separate our work for when n is even and when n is odd. After we prove the upper bound, we will show that there are unique knots and links on each grid which achieve the upper bound. When n is even, torus links achieve the maximum, and when n is odd, torus knots achieve the maximum.

Acknowledgements

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I want to acknowledge this was conceivable only because of God. He equipped me for this path and has surrounded me with precisely what I have needed.

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Lastly, I would like to thank my husband, Dr. Ryan Thomas. I am so glad we could go through this together. To have your love, care, and understanding is a gift that I do not take for granted. My heartfelt thanks for your constant encouragement, but especially on the days that were hard and seemed impossible. I am forever grateful.

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1 Introduction

In this paper we will prove the following theorem:

Theorem 1.1. The maximum allowable Thurston-Bennequin number for any knot or link drawn on a grid with arc index n is,

$$TB(n) = CR(n) - \left\lceil \frac{n}{2} \right\rceil = \begin{cases} \left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{2}\right)^2 - \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases},$$

and there are unique knots and links which achieve the maximum, specifically the $T(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$ torus knots and links.

Peter Cromwell's work shows us that an arc presentation of a link L is an embedding of L into finitely many pages of the open-book decomposition so that every page meets L in a simple arc [4]. In other words, we have horizontal and vertical line segments where the vertical strand is the over-strand at each crossing. Again, due to Cromwell we know that the minimum number of pages used to represent a link in this way is called the arc index [4]. Lenhard Ng gives us the following definitions for a grid diagram and the arc index:

Definition 1.1. [13] For a knot or link K in S^3 a **grid diagram** is an oriented knot diagram made up of horizontal and vertical segments, where at any crossing the vertical segment crosses over the horizontal segment.

For this paper, we will only use grid diagrams where a column, or row, of the grid has a single segment in it, which we will explain in more detail after establishing some other information.

Definition 1.2. [13] The **arc index** of K, $\alpha(K)$, is the minimal arc number, or number of horizontal line segments in the diagram over all grid diagrams for K.

The arc index has a long history of being used as a tool inside proofs, dating all the way back to Hermann Brunn in 1898 who used a diagram similar to a grid diagram to find a singular point of high multiplicity [2]. In 1995, exploration into the topic revved up when Peter Cromwell showed that every link admits an arc presentation, which can be converted to a grid diagram [3]. Ian Nutt worked with Cromwell in 1996 to show that the minimal crossing number provides an upper bound on the arc index for alternating links [5]. Bae and Park continued that work in 2000 when they proved that the crossing number gives an upper bound on the arc index of non-alternating knots [1]. Then in 2010, Jin and Park determined the arc index for prime knots with twelve or fewer crossings [9]. Right around the same time, Ng provided us with the arc index and maximum Thurston-Bennequin number for all knots with at most 11 crossings[13]. We define the Thurston-Bennequin number in Definition 1.4. Another big development in the field happened in 2013 when Dynnikov and Prasolov looked at stabilizations and destabilizations of rectangular diagrams and used their work to prove the Jones' conjecture [7].

One of the more recent developments in this area of study has been the use of rectangular diagrams in Knot Floer homology. Manolescu, Ozsváth, Szabó and others have been working to develop a homology theory which uses grid diagrams to calculate the homology [12, 11]. This topic is also related to questions about 4-genus or the minimal slice genus, which is the minimal genus of a surface embedded in a 4-ball with the knot or link as the boundary in S^3.

Rectangular or grid diagrams of knots are drawn so that all of the lines are horizontal or vertical and the over-crossing strand is vertical with each column or row containing just one segment. Grid forms of knots have been used to construct 3-manifolds and are also called square-bridge presentations [Lyon, 10]. In the rectangular diagram form, a relationship to Legendrian links is easily seen. There are three diagrammatic worlds at play here. There are grid diagrams, link diagrams, and front projections of Legendrian links. We will be working within the world of grid diagrams, but the connections are easy to see. In Figure 1 we have demonstrated the relationship between grid diagrams of knots and their Legendrian form. We started with the grid diagram of the trefoil knot, then rotated it counterclockwise by forty-five degrees, and then smoothed half of the corners and turned the rest into cusps. We will come back to the relationship between grid diagrams of links and their Legendrian forms, but for more details on this part, see the work of Dynnikov and Prasolov [7].

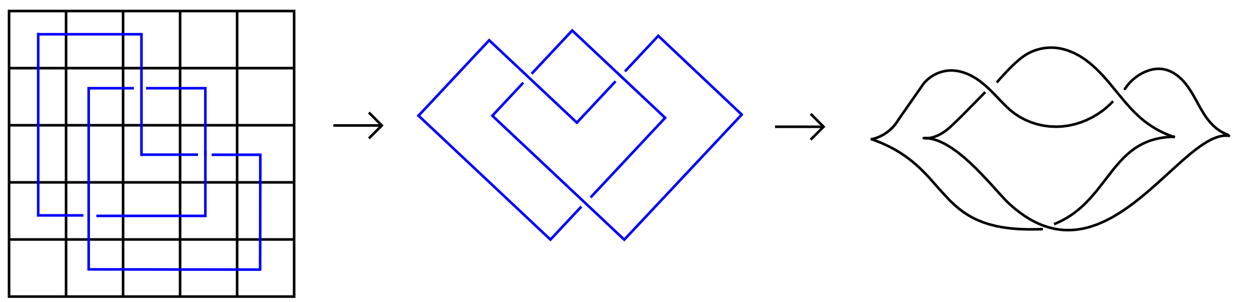

Figure 1

As stated above, every knot or link can be drawn to fit on a grid diagram. Using the setup of Lenhard Ng and Dylan Thurston, a grid diagram with grid number n is an $n \times n$ square grid with n X's and n O's placed in distinct squares with each row and each column containing exactly one X and one O [14]. For orientation, we connect X's to O's in the rows and O's to X's in the columns, recalling that the over-crossing strand is vertical. In Figure 3, we have the $T(3,2)$ torus knot, also known as the trefoil knot. For demonstration, we have drawn a grid diagram of the trefoil knot shown in Figure 2. Notice that the grid is five by five, and this grid cannot get any smaller and still fit the trefoil knot, so the arc index of the trefoil knot is 5.

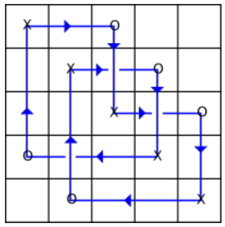

Figure 2: Grid Diagram

Figure 3: Trefoil Knot

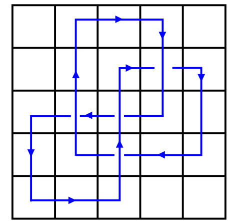

Figure 4: Grid Diagram

Now, we have two rectangular diagrams shown above for the trefoil knot with the knot diagram in the middle. In 1995, Cromwell proved that even on a minimally sized grid, which we have here, there may not be a unique grid presentation. In fact, the different diagrams are related by a finite set of allowable moves which are analogous to Reidemeister moves for smooth links [3]. To emphasize the similarities, we first discuss the smooth Reidemeister moves, then the Legendrian Reidemeister moves, and finally the grid diagram moves with some discussion on the connection to smooth and Legendrian links. Since we are working in the world of grid diagrams, the grid moves give us equivalent diagrams of the same knot or link. First we have the smooth Reidemeister moves:

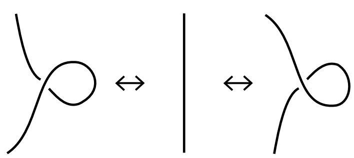

Figure 5: Type 1

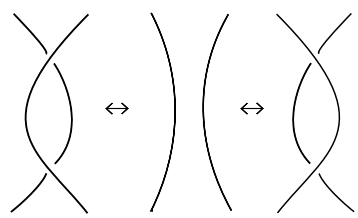

Figure 6: Type 2

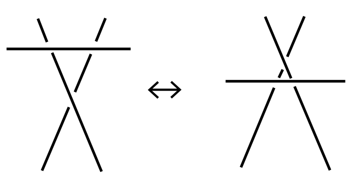

Figure 7: Type 3

Now we have the Legendrian Reidemeister moves as demonstrated by Gompf [8]:

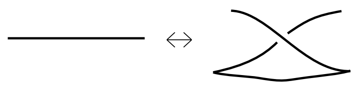

Figure 8: Type 1

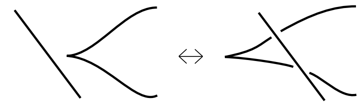

Figure 9: Type 2

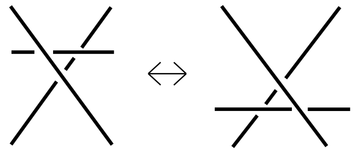

Figure 10: Type 3

The allowable moves for grid diagrams are similar and are referred to as elementary moves. They first appear in Cromwell's work, and they are further demonstrated and explained by Dynnikov and Prasolov [3, 7]. We have the following moves:

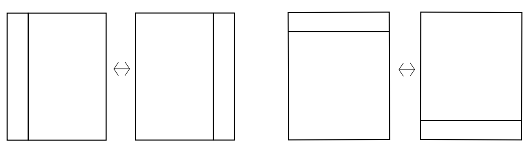

Figure 11: Permutation or Translation

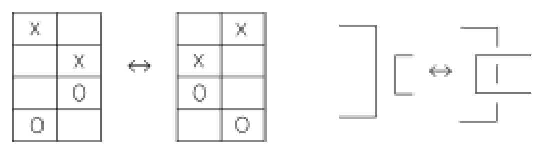

Figure 12: Commutation

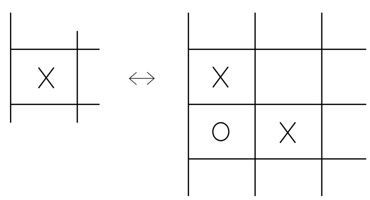

Figure 13: Stabilization or Destabilization

Now, a permutation or translation moves an entire row or column to the other side of the grid diagram as shown in Figure 11. This is a similar move to the Type 2 Reidemeister move and is the allowed move of most importance to us for the purposes of this argument. Also, notice that the commutation move for grid diagrams as shown in Figure 12 is similar to a Type 2 Reidemeister move. This move involves adding (or removing) both a positive and a negative crossing. Both the permutation and the commutation move types are smooth isotopies. The resulting diagram is of the same knot and has the same Thurston-Bennequin number.

Now, we have to be more careful with the grid stabilization. A grid stabilization is akin to the Type 1 Reidemeister move. We have only shown one of the four possible configurations in Figure 13. For the other configurations and further explanation, see Dynnikov's work on arc presentations [6]. It is important to see that this type of move creates a new corner in our grid diagram. Namely, we are allowed a southwest stablilization and a northeast stablilization, as these corners will get smoothed out after we rotate the grid diagram by forty-five degrees to convert to the Legendrian form of the link. or we end up with a Legendrian Type 1 Reidemeister move. The northwest and southeast stabilizations result in new cusps in the Legendrian form of the link and therefore change the Legendrian knot type. These moves also change the Thurston-Bennequin number of the grid [15].

As we stated above, the move we will use the most in our argument, is the *cyclic permutation*. We can also call it a *translation* to the opposite side of the diagram. In Figure 14, we have shown a single strand undergoing a cyclic permutation.

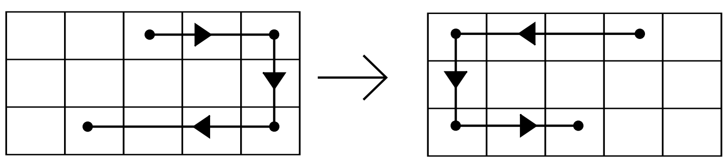

Figure 14: Cyclic Permutation

We have pulled the outer vertical strand around the diagram and then shifted the grid over to fit. In the context of a full grid, we already have an example of this type of move. If we take a look back at Figures 2 and 4, we can see this idea at play in our two diagrams of the trefoil. If we imagine talking the furthest right strand of Figure 2 and swinging it around the back of the diagram to the left side, then we have the same diagram as in Figure 4. We have translated that arc to the opposite side of the diagram, giving us a different presentation of the same knot. The idea of a cyclic permutation is an idea we will come back to several times throughout our discussion.

Our goal is to find the maximum Thurston-Bennequin number allowable on the n-grid. This is for any knot or link we can draw on the grid. We will do this in pieces. First, we will prove the bound when n is even, and show which links give us the upper bound on tb. Then, we will prove the bound when n is odd, and we will find which knots achieve the bound. Before we get to that, we establish a few definitions that we will use along the way.

Definition 1.3. The **writhe**, $wr(D)$, of a rectangular diagram, D, of a knot or a link is the algebraic crossing number with sign. That is,

$$wr(D) = \#\text{ positive crossings} - \#\text{ negative crossings}.$$

Something to note here is that reversing the orientation of a knot does not change the writhe. What would affect the writhe is switching all of the over-crossings to under-crossings in the knot or grid diagram. This would assign opposite sign to all of the crossings. For a link, however, we have to be more careful. Reversing the orientation for all of the components does not change the writhe of the link diagram. If we change the orientation of some of the components, then the crossings on the components which were reversed, change sign.

Definition 1.4. The **Thurston-Bennequin number**, $tb(D)$, of a rectangular diagram, D, of a knot or a link is $tb(D) = wr(D) - \#\text{n.w.corners}$, where $wr(D)$ is the writhe of the link.

An interesting parallel to the Legendrian knot realm is that the calculation for the tb is very similar. To calculate the tb of a Legendrian front projection of a knot, we count the writhe, and then subtract the number of left cusps in the projection. Recall, those cusps become the northwest corners after we rotate the diagram by forty-five degrees.

Now, back to the grid diagram picture. Looking at Figure 2, we have w(trefoil)$= 3 - 0$, coming from three positive crossings and zero negative crossings. There are 2 northwest corners in this diagram, and therefore, the Thurston-Bennequin number for this diagram of the trefoil is $tb = 3 - 2 = 1$. Something to note once again is that a cyclic permutation of the diagram does not change the Thurston-Bennequin number. For example, the Thurston-Bennequin Number of the diagram in Figure 4 is also equal to 1. The Thurston-Bennequin number (tb) of a knot is an invariant and can be calculated from the grid diagram of a knot or link.

Dynnikov and Prasolov showed that a destabilization, which is an allowed move preserving the topological type of the link and decreasing the arc index by one, can only increase the Thurston-Bennequin number, not decrease it. They also proved that the minimal arc index diagram, or n-grid for a link, maximizes the tb for the link [7]. Here, we will be maximizing the tb for all knots or links which cannot be further destabilized on an n by n grid. As with our example of the trefoil, the knot cannot fit on a smaller grid, so the knot cannot be further destabilized and has an arc index equal to 5. Dynnikov and Prasolov's work to prove that the minimal rectangular diagram maximizes the Thurston-Bennequin number for the corresponding Legendrian links is of interest because having that maximum tb for a knot gives lower bounds for topological knot invariants like slice-genus and concordance invariants. An interesting connection to the current research is that Ng used the maximum tb for a given knot with at most eleven crossings to determine the arc index for that knot [13]. So, this work is converse to his in that we are starting with the arc index and determining which knots or links can achieve the maximum tb on the n-grid, which we define as follows:

Definition 1.5. The **maximal n-grid Thurston-Bennequin number**, $TB(n)$, is the maximum $tb(D)$ over all rectangular diagrams D, of a knot or a link on the n-grid. That is

$$TB(n) = \max_{diagrams\ D}\{tb(D) = wr(D) - \#\text{n.w.corners}\}.$$

The difference between our definition of maximum tb and the definition Ng used, is that we are maximizing the tb on the family of knots or links on the n-grid, not for a particular knot or link.

As defined above, the writhe of a link is the algebraic crossing number and the number of northwest corners is equivalent to the number of left cusps for the Legendrian knot.

Since the writhe of a link is the algebraic crossing number, it is clear that the writhe is bounded above by the crossing number, hence $wr(L) \leqslant CR(n)$.

Definition 1.6. ***CR(n)*** is the maximum crossing number for the rectangular diagram of any link drawn on the n-grid.

Then, we have

$$tb(D) \leqslant CR(n) - \#\text{n.w.corners}.$$

This inequality leads us right into proving the bound on the Thurston-Bennequin number. First, quickly recall that we will prove the bound holds for any knot or link drawn on an n-grid, then we will show which knots or links, if any, achieve that bound on the n-grid.

2 Even Case

To prove the Thurston-Bennequin bound on any n-grid, we will begin by proving it on the n-grid when n is even. With the bound we already have on tb for a diagram D, it makes sense to look first at the crossing number, since it bounds the writhe. From the joint work of Peter

Cromwell and Ian Nutt published in 1996, we know that the maximum crossing number on an n-grid is a function of triangle numbers. In fact, we have the following proposition:

Proposition 2.1. [5] When n is even, the maximum crossing number, $CR(n)$ for any knot or link drawn on the n-grid is $2\Delta(m)$ where $\Delta(m) = \frac{1}{2}m(m+1)$ and $m = \frac{1}{2}(n-2)$.

Proof. We assume $m = \frac{1}{2}(n-2)$ and $\Delta(m) = \frac{1}{2}m(m+1)$. So we have,

$$\begin{aligned}
\Delta(m) &= \frac{1}{2}m(m+1) \\
&= \frac{1}{2}\left(\frac{1}{2}(n-2)\right)\left(\frac{1}{2}(n-2)+1\right) \\
&= \frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}\right).
\end{aligned}$$

This means the maximum crossing number allowable on an even n-grid is

$$\begin{aligned}
CR(n) &= 2\Delta(m) \\
&= 2\left(\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}\right)\right) \\
&= \left(\frac{n}{2}-1\right)\left(\frac{n}{2}\right)
\end{aligned}$$

□

Our goal is to achieve the maximum allowable tb on an n-grid, so we will first maximize the tb on a half diagram, since doubling that will give us the maximum on the full diagram by the symmetry of the grid.

Definition 2.1. The **half Thurston-Bennequin number,** $tb^{\frac{1}{2}}(D)$, of a half-rectangular diagram, D, of a knot or a link is

$$tb^{\frac{1}{2}}(D) = wr^{\frac{1}{2}}(D) - \tfrac{1}{2}(\#\text{s.e.corners}+\#\text{n.w.corners}).$$

Definition 2.2. The **maximal half Thurston-Bennequin number,** $TB^{\frac{1}{2}}(n)$, of a half-rectangular diagram, D, on an n-grid of a knot or a link is

$$TB^{\frac{1}{2}}(n) = \max_{\text{diagrams D}} \{tb^{\frac{1}{2}}(D) = wr^{\frac{1}{2}}(D) - \frac{1}{2}(\#\text{s.e.corners} + \#\text{n.w.corners})\}.$$

The number of northwest corners equals the number of southeast corners in a rectangular diagram of a knot or link, so half are represented in the half-diagram.

We will maximize the Thurston-Bennequin number on the half-diagram, then we can maximize the full diagram. We will prove the following theorem by induction.

Theorem 2.1. For an even n by n rectangular diagram, the maximum achievable Thurston-Bennequin number for any link drawn in the grid is

$$\begin{aligned} TB(n) &= CR(n) - \left\lceil \frac{n}{2} \right\rceil \\ &= \left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \left(\frac{n}{2}\right). \end{aligned}$$

Moreover, the only links which achieve this maximum are the $T\left(\frac{n}{2}, \frac{n}{2}\right)$ torus links.

Outline for proof: We will show that $tb(n) \leqslant \left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2}$ is in fact the bound on the Thurston-Bennequin number. We will show that this bound is achievable and that there is a unique knot or link on each grid which achieves it. That is to say,

$$TB(n) = \left(\tfrac{n}{2} - 1\right)\left(\tfrac{n}{2}\right) - \tfrac{n}{2} = 2TB^{\frac{1}{2}}(n).$$

To do this, we want to prove the following theorem:

Theorem 2.2. The maximum achievable $TB^{\frac{1}{2}}(n)$ for any half-diagram drawn on the $\frac{n}{2}$ by n half-grid is

$$TB^{\frac{1}{2}}(n) = \tfrac{1}{2}\left(\left(\tfrac{n}{2}-1\right)\left(\tfrac{n}{2}\right)-\tfrac{n}{2}\right).$$

We will prove this by induction on the half diagram. For the base case, we will show the maximum holds for $n = 4$ and then show some example diagrams for $n = 6$.

2.1 Base Case $n = 4$

We begin with $n = 4$ because only the unknot has an arc index of 2, as proven by Jin and Park [9]. Now, the tb of the unknot is -1, which does achieve our proposed bound, but it is not interesting. So, for $n = 4$, we want to find a maximum $tb^{\frac{1}{2}}(D)$ over all possible diagrams, D, for the lower half grid. We call this $TB^{\frac{1}{2}}(n)$, and calculate it in a similar fashion. We know

$$tb(D) = wr(D) - \tfrac{1}{2}(\#\text{n.w. corners} + \#\text{s.e. corners}).$$

For the writhe in $tb^{\frac{1}{2}}(D)$, we still count the crossings with sign, this will give us $wr^{\frac{1}{2}}(D)$. For the corners, we still count half of the southeast corners and northwest corners. So we have $tb^{\frac{1}{2}}(D) = wr^{\frac{1}{2}}(D) - \frac{1}{2}(\#\text{n.w. corners} + \#\text{s.e. corners})$. The maximum $tb^{\frac{1}{2}}(D)$ over all possible half-grid diagrams is

$$TB^{\frac{1}{2}}(n) = wr^{\frac{1}{2}}(D) - \tfrac{1}{2}\left(\#\text{s.e. corners} + \#\text{n.w.corners}\right).$$

Once we have this maximal $TB^{\frac{1}{2}}(n)$, we will have a bound for $TB(n)$. This bound comes from doubling the $TB^{\frac{1}{2}}(n)$ because rotating the lower half-diagram 180 degrees will give us a full rectangular diagram of a link with $tb(D) \leqslant 2tb^{\frac{1}{2}}(D)$, so we have $TB(n) \leqslant 2TB^{\frac{1}{2}}(n)$. One thing to notice here is that a maximal $tb^{\frac{1}{2}}(D)$ is the same for an upper half or a lower half. This is due to the symmetry of the grid.

We will draw options for the lower half-diagram fixing the bottom strand to be oriented to the right. We have included several of the more interesting options of what the lower half-diagram looks like. There are many more, but they can be grouped into categories. Either we have just one strand as in Figures 16, 17, and 20 that can be condensed or straightened out. We can have something like Figure 21, where the single strand crosses itself and can be stretched across more columns. We can also have two strands which don't interact like in Figures 15 and 24 with different orientations. We also have several options for how two strands interact in the half-grid, but we've shown the ones which do not include both a positive and a negative crossing.

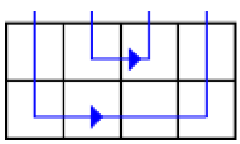

Figure 15: $tb^{\frac{1}{2}} = 0 - \frac{1}{2}(2) = -1$

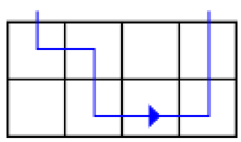

Figure 16: $tb^{\frac{1}{2}} = 0 - \frac{1}{2}(1) = -\frac{1}{2}$

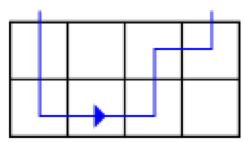

Figure 17: $tb^{\frac{1}{2}} = 0 - \frac{1}{2}(3) = -\frac{3}{2}$

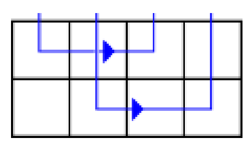

Figure 18: $tb^{\frac{1}{2}} = 1 - \frac{1}{2}(2) = 0$

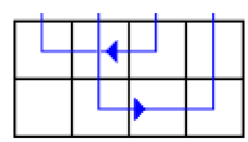

Figure 19: $tb^{\frac{1}{2}} = -1 - \frac{1}{2}(2) = -2$

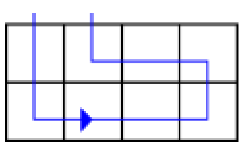

Figure 20: $tb^{\frac{1}{2}} = 0 - \frac{1}{2}(1) = -\frac{1}{2}$

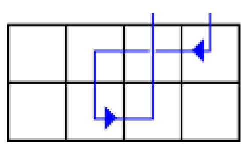

Figure 21: $tb^{\frac{1}{2}} = 1 - \frac{1}{2}(3) = -\frac{1}{2}$

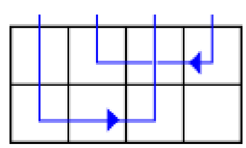

Figure 22: $tb^{\frac{1}{2}} = 1 - \frac{1}{2}(2) = 0$

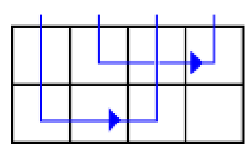

Figure 23: $tb^{\frac{1}{2}} = -1 - \frac{1}{2}(2) = -2$

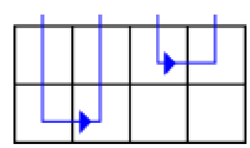

Figure 24: $tb^{\frac{1}{2}} = 0 - \frac{1}{2}(2) = -1$

Hence, we have several options for the lower half-diagram, even limiting ourselves to the diagrams with the bottom row oriented to the right, but only two of them give us the maximal $tb^{\frac{1}{2}}(D)$ of 0. Thus, we have $TB^{\frac{1}{2}}(4) = 0$, and therefore, $TB(4) \leqslant 2TB^{\frac{1}{2}}(4) = 0$. The half-diagrams which give us our maximum are Figures 18 and 22. Now, we will use those half-diagrams to maximize the full diagram, which will come from rotating a copy of the maximal half-diagram by 180 degrees, but we've demonstrated here why that has to be true.

From the half diagrams in Figures 18 and 22, we have the following options for complete diagrams. Only two options give us $tb(D) = 2tb^{\frac{1}{2}}(D)$. So we have

$$TB(4) = 2TB^{\frac{1}{2}}(4) = 0,$$

and we have two diagrams shown in Figures 25 and 28 that achieve that maximum.

First, the complete diagrams from Figure 18:

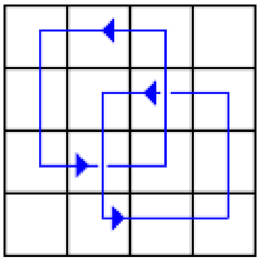

Figure 25:
$tb = 2 - 2 = 0$
$= 2tb^{\frac{1}{2}}(4)$

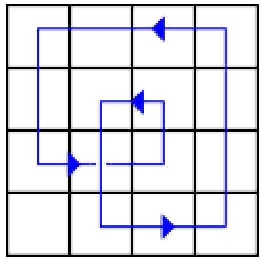

Figure 26:
$tb = 1 - 2 = -1$
$\leqslant 2tb^{\frac{1}{2}}(4)$

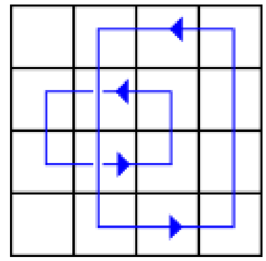

Figure 27:
$tb = 0 - 2 = -2$
$\leqslant 2tb^{\frac{1}{2}}(4)$

Now, the complete diagrams from Figure 22

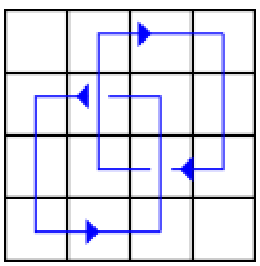

Figure 28:
$tb = 2 - 2 = 0$
$= 2tb^{\frac{1}{2}}(4)$

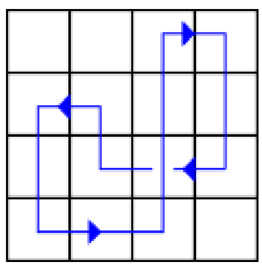

Figure 29:
$tb = 1 - 2 = -1$
$\leqslant 2tb^{\frac{1}{2}}(4)$

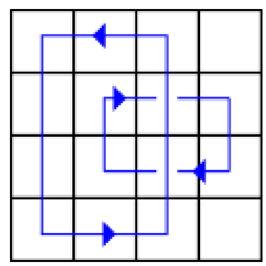

Figure 30:
$tb = 0 - 2 = -2$
$\leqslant 2tb^{\frac{1}{2}}(4)$

From all six of those diagrams, we have just two maximal diagrams which we have displayed side-by-side below in order to see the similarities. Notice that the two maximal diagrams are in fact two copies of the maximal lower half, with one copy rotated by π and glued in place to create a full diagram.

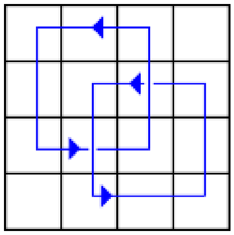

Figure 31: Maximal Full Diagram from Figure 18

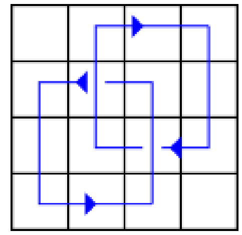

Figure 32: Maximal Full Diagram from Figure 22

Now, the two rectangular representations of links are similar. Actually, Figure 31 is simply Figure 32 with the top loop rotated around the outside of the diagram. Imagine grabbing the outside vertical strand and swinging it around the back, giving us Figure 32 via a cyclic permutation of 31. The two rectangular diagrams are representations of the $T(2,2)$ torus link, also called the Hopf link.

So for $n = 4$ we have shown,

$$\begin{aligned}
0 = TB(4) &\leqslant 2TB^{\frac{1}{2}}(4) \\
&= 2\left(wr^{\frac{1}{2}}(L) - \frac{1}{2}(\#\text{ s.e. corners} + \#\text{n.w. corners})\right) \\
&= 2\left(\frac{1}{2}CR(4) - \frac{1}{2}(\#\text{ s.e. corners} + \#\text{n.w. corners})\right) \\
&= CR(4) - \#\text{ s.e .corners} \\
&= CR(4) - \#\text{ n.w. corners} \\
&= 2 - \frac{n}{2} \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

2.2 Examples for $n = 6$

We will now present some examples for $n = 6$ and check that

$$3 = TB(6) \leqslant 2TB^{\frac{1}{2}}(6) = CR(6) - \tfrac{n}{2} = 6 - 3 = 3.$$

When we move to $n = 6$ we can use the half-diagrams from $n = 4$. We are adding in one row and two columns to the previous half diagrams. We will show in Section 2.3.1 why we can only consider adding the new strands to the half-diagrams from $n = 4$ which achieved $TB^{\frac{1}{2}}(4)$. After drawing our options, we are left with three half-diagrams that give us $TB^{\frac{1}{2}}(6) = \frac{3}{2}$. We have the following maximal half-diagrams for $n = 6$:

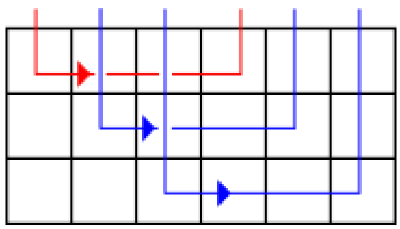

Figure 33: $tb^{\frac{1}{2}} = 3 - \frac{1}{2}(3) = \frac{3}{2}$

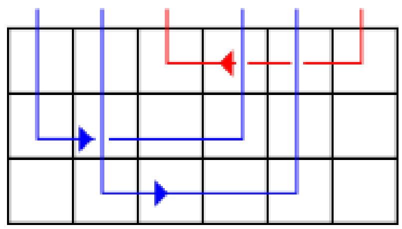

Figure 34: $tb^{\frac{1}{2}} = 3 - \frac{1}{2}(3) = \frac{3}{2}$

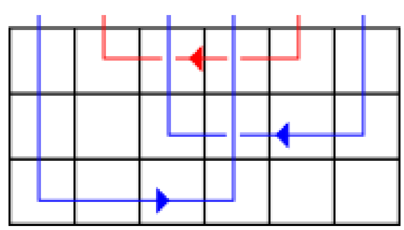

Figure 35: $tb^{\frac{1}{2}} = 3 - \frac{1}{2}(3) = \frac{3}{2}$

Now by Theorem 2.1, the maximal achievable Thurston-Bennequin number for the full 6 by 6 diagram is

$$\left(\tfrac{6}{2} - 1\right)\left(\tfrac{6}{2}\right) - \left(\tfrac{6}{2}\right) = (2)(3) - 3 = 3 = 2\left(\tfrac{3}{2}\right) = 2TB^{\frac{1}{2}}(6).$$

We can achieve the maximum on the full diagram by rotating the maximal lower half-diagram by π and connecting the strands. If the diagram does not match up after rotating, we may need to apply one or more cyclic permutations to one of the half-diagrams to get the two to match. When we do this, we get three complete maximal diagrams for $n = 6$. The lower half in all three figures is the same as the maximal half-diagrams we have above in Figures 33, 34, and 35. The upper half of each of these diagrams is the lower half rotated by π (Figure 36), then permuted for Figures 37 and 38, and attached in place.

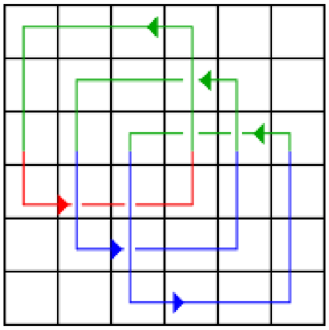

Figure 36:
$tb = 6 - 3 = 3 = 2tb^{\frac{1}{2}}(6)$

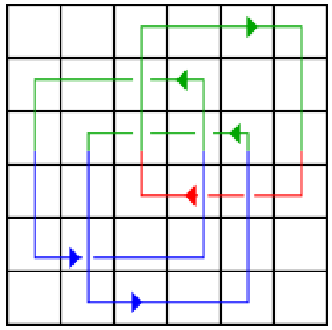

Figure 37:
$tb = 6 - 3 = 3 = 2tb^{\frac{1}{2}}(6)$

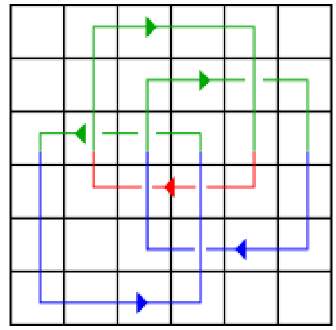

Figure 38:
$tb = 6 - 3 = 3 = 2tb^{\frac{1}{2}}(6)$

After completing the maximal half-diagrams,with the bottom strand oriented to the right, we are left with three diagrams that give us $tb(D) = 2TB^{\frac{1}{2}}(6) = TB(6)$. All three diagrams are rectangular representations of the same link, namely, the $T(3,3)$ torus link. Again, to move from Figure 36 to Figure 37, we can grab the leftmost vertical strand and swing it around the outside of the diagram. Then we can do the same thing to cyclically permute our diagram from Figure 37 to Figure 38.

So for $n = 6$ we have shown,

$$\begin{aligned}
3 = TB(6) \leqslant 2TB^{\frac{1}{2}}(6)& \\
&= 2\left(wr^{\frac{1}{2}}(L) - \frac{1}{2}(\#\text{ s.e. corners} + \#\text{n.w.corners})\right) \\
&= 2\left(\frac{1}{2}CR(6) - \frac{1}{2}(\#\text{ s.e. corners} + \#\text{n.w.corners})\right) \\
&= CR(6) - \#\text{ n.w. corners} \\
&= 6 - \frac{6}{2} \\
&= 3
\end{aligned}$$

We have also shown that the $T(3,3)$ torus link achieves that maximum.

2.3 Proving Theorem 2.1

We want to show that $TB(n) \leqslant CR(n) - \lceil \frac{n}{2} \rceil$. We will do this by proving the bound on the half grid by induction, then use it to prove the bound on the full n-grid.

2.3.1 Proof of Theorem 2.2

Proof. We want to show that $tb^{\frac{1}{2}}(D) \leqslant \frac{1}{2}\left(CR(D) - \frac{n}{2}\right) = \frac{1}{2}\left(\left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2}\right)$ for any rectangular diagram D on an n-grid. Hence, we will show

$$TB^{\frac{1}{2}}(n) = \frac{1}{2}\left(CR(n) - \frac{n}{2}\right) = \frac{1}{2}\left(\left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2}\right)$$

We want to show the inequality holds first, then show that it is realized. In fact, we will prove that the only maximal half-diagram on the n-grid is a cyclic permutation of a diagram like the one shown below in Figure 39.

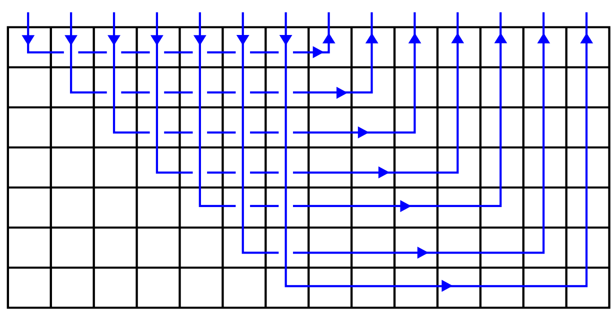

Figure 39: Here we have 7 nested stands on the $n = 14$ half-grid.
This can be expanded for $\frac{n}{2}$ nested strands on the n half-grid.

Notice that the vertical strands in the left half of Figure 39 are all coming into the grid, and the vertical strands in the right half of the figure are all leaving the grid. We can cyclically permute this diagram to change that, but this is the simplest way to see how these half-diagrams build. Recall, from the work we did above in Section 2.1, we have proven the base case when $n = 4$.

Now for the induction, suppose we have an $\frac{n}{2}$ by n half-diagram. We assume for an $\frac{n-2}{2}$ by $n-2$ half-grid that

$$\begin{aligned} TB^{\frac{1}{2}}(n-2) &= \frac{1}{2}\left(CR(n-2) - \frac{n-2}{2}\right) \\ &= \frac{1}{2}\left[\left(\frac{n-2}{2} - 1\right)\left(\frac{n-2}{2}\right) - \frac{n-2}{2}\right] \\ &= \frac{n^2 - 8n + 12}{8}. \end{aligned}$$

To proceed, we want to remove the top horizontal strand of the bigger diagram, D which is $\frac{n}{2}$ by n. Now, D has $\frac{n}{2}$ rows, so it also has $\frac{n}{2}$ X's and $\frac{n}{2}$ O's in the half-diagram. When we remove the top strand, we lose an X and an O, giving us $\frac{n-2}{2}$ X's and $\frac{n-2}{2}$ O's. After deleting empty columns, we find that D' has at most $n-2$ columns filled, and thus D' is an $\frac{n-2}{2}$ by $n-2$ half-grid. Recall that we assumed the bound for this size half-diagram.

When we add in the additional horizontal strand to move to D, there are four options for the layout of the top strand.

Figure 40: 1

Figure 41: 2

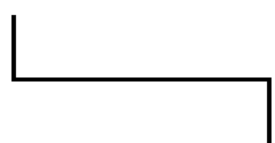

Figure 42: 3

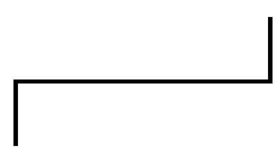

Figure 43: 4

We have $n-2$ columns and therefore at most $n-2$ vertical strands. Half of those strands are oriented up and half of them are oriented down. Thus, the maximum increase possible for the writhe is $\frac{n-2}{2}$, since the new strand can cross one set of those vertical strands in the half-diagram positively.

If the top row of D takes the shape of options 2 or 3, shown in Figures 41 and 42, then our increase in writhe is strictly less than $\frac{n-2}{2}$, that is $\Delta(\text{writhe}) < \frac{n-2}{2}$. Then, our new diagram can never achieve the sharpness of the bound. We have,

$$\begin{aligned}
tb^{\frac{1}{2}}(D) &= tb^{\frac{1}{2}}(D') + \Delta(\text{writhe}) - \frac{1}{2}(\text{new n.w + s.e. corners}) \\
&\leqslant TB^{\frac{1}{2}}(n-2) + \Delta(\text{writhe}) - \frac{1}{2}(\text{new n.w + s.e. corners}) \\
&\leqslant TB^{\frac{1}{2}}(n-2) + \frac{n-4}{2} - \frac{1}{2}(\text{new n.w + s.e. corners}), \text{ since } \Delta(\text{writhe}) < \frac{n-2}{2} \\
&= \frac{1}{2}\left[\left(\frac{n-2}{2}-1\right)\left(\frac{n-2}{2}\right) - \left(\frac{n-2}{2}\right)\right] + \frac{n-4}{2} - \frac{1}{2}(\text{new n.w + s.e. corners}) \\
&= \frac{n^2-8n+12}{8} + \frac{n-4}{2} - \frac{1}{2}(\text{new n.w + s.e. corners}) \\
&= \frac{n^2-4n-4}{8} - \frac{1}{2}(\text{new n.w + s.e. corners}) \\
&< \frac{n^2-4n}{8}, \text{ with 0 or 1 new n.w. or s.e. corners} \\
&= TB^{\frac{1}{2}}(n)
\end{aligned}$$

So, our *tb* fits within our proposed bound. Thus, if the change in writhe is less than $\frac{n-2}{2}$, then we are done. We have satisfied the bound, however the bound cannot be sharp.

However, if we actually increase the writhe by $\frac{n-2}{2}$, the new strand has crossed $\frac{n-2}{2}$ strands positively. This can happen if the top strand of D is either option 1 or option 4, shown in Figures 40 and 43. If, however, the top strand does take the shape shown in Figure 43, we must also consider the fact that we have picked up two additional northwest or southeast corners.

This gives,

$$
\begin{aligned}
tb^{\frac{1}{2}}(D) &= tb^{\frac{1}{2}}(D') + \Delta(\text{writhe}) - \frac{1}{2}(\text{new n.w. or s.e. corners}) \\
&\leqslant TB^{\frac{1}{2}}(n-2) + \Delta(\text{writhe}) - \frac{1}{2}, \text{1 or 2 new corners} \\
&= TB^{\frac{1}{2}}(n-2) + \frac{n-2}{2} - \frac{1}{2}, \text{ since } \Delta(\text{writhe}) = \frac{n-2}{2} \\
&= \frac{1}{2}\left[\left(\frac{n-2}{2} - 1\right)\left(\frac{n-2}{2}\right) - \left(\frac{n-2}{2}\right)\right] + \frac{n-2}{2} - \frac{1}{2} \\
&= \frac{n^2 - 8n + 12}{8} + \frac{n-2}{2} - \frac{1}{2} \\
&= \frac{n^2 - 4n}{8} \\
&= TB^{\frac{1}{2}}(n)
\end{aligned}
$$

Thus, either type of diagram fits within our proposed bound. In fact, in this situation, the bound can be sharp as long as we only pick up one new northwest or southeast corner, which restricts us to the top row having only one new corner as in Figure 40. So, $TB^{\frac{1}{2}}(n) = TB^{\frac{1}{2}}(n-2) + \frac{n-2}{2} - \frac{1}{2}(1)$. This is due to the writhe increasing by $\frac{n-2}{2}$, and in order to increase the writhe and get the best increase in $tb^{\frac{1}{2}}$, we have to pick up one southeast corner decreasing our new $tb^{\frac{1}{2}}$ by $\frac{1}{2}$.

Thus,

$$
\begin{aligned}
TB^{\frac{1}{2}}(n) &\leqslant TB^{\frac{1}{2}}(n-2) + \left(\frac{n-2}{2} - \frac{1}{2}\right) \\
&= \frac{1}{2}\left[\left(\frac{n-2}{2} - 1\right)\left(\frac{n-2}{2}\right) - \frac{n-2}{2}\right] + \frac{n-2}{2} - \frac{1}{2} \\
&= \frac{n^2 - 8n + 12}{8} + \frac{n-2}{2} - \frac{1}{2} \\
&= \frac{n^2 - 8n + 12 + 4n - 8 - 4}{8} \\
&= \frac{n^2 - 4n}{8} \\
&= \frac{1}{2}\left(\frac{n^2}{4} - n\right) \\
&= \frac{1}{2}\left(\frac{n^2}{4} - \frac{n}{2} - \frac{n}{2}\right) \\
&= \frac{1}{2}\left[\left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2}\right] \\
&= \frac{1}{2}\left[CR(n) - \frac{n}{2}\right].
\end{aligned}
$$

This completes the proof of Theorem 2.2. □

Remark: We have shown that in order to achieve the maximal diagram, D, the diagram with the top row removed, D', must also be maximal. We can now state and prove the following corollary:

Corollary 2.1. The maximal $TB(n)$ for any knot or link on an even n-grid is bounded above. Namely,

$$TB(n) \leqslant CR(n) - \tfrac{n}{2}.$$

Proof. We have, that $TB^{\frac{1}{2}}(n) \leqslant \frac{1}{2}\left[CR(n) - \frac{n}{2}\right]$. Thus,

$$\begin{aligned} TB(n) &\leqslant 2TB^{\frac{1}{2}}(n) \\ &\leqslant (2)\frac{1}{2}\left[CR(n) - \frac{n}{2}\right] \\ &= CR(n) - \frac{n}{2}. \end{aligned}$$

□

2.3.2 Proof of Theorem 2.1

Proof. Now that we have proven the bound, we want to construct the maximal diagrams. We want the maximum tb over all diagrams, D, which will give us $TB(n)$. We first cut our grid in half to maximize the $tb^{\frac{1}{2}}$ in the half-grid. Then, we look at the possible half-diagrams and determine which one gives us the maximal $tb^{\frac{1}{2}}(D) = TB^{\frac{1}{2}}(n)$.

So, imagine we have the lowest strand of a diagram, D, and that it is oriented to the right. We want to add a row/strand above it and get the best net change in $tb^{\frac{1}{2}}$. We have examples in Figures 44, 45, 46, and 47 that show us how to fit a strand in to an existing diagram. We have demonstrated the ways this new strand can be connected or completely separate from the previous strand and how this affects $tb^{\frac{1}{2}}(D)$.

For the net change in $tb^{\frac{1}{2}}$, we are counting the crossings and the northwest and southeast corners other than the bottom southeast corner. We added a strand to a diagram with the bottom strand oriented to the right, giving us that corner in all diagrams.

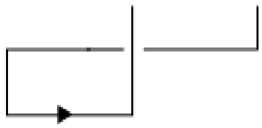

Figure 44: net change of 0

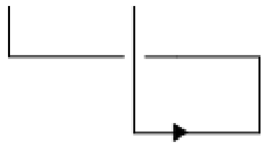

Figure 45: net change of -1

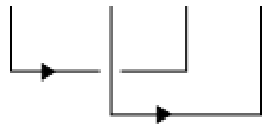

Figure 46: net change of $\frac{1}{2}$

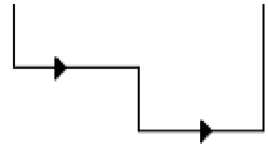

Figure 47: net change of 0

So, Figure 46 gives us the best increase in $tb^{\frac{1}{2}}(D)$ and is the only configuration (up to cyclic permutation) that allows us to continue the sharpness of the bound on $tb^{\frac{1}{2}}$ as we increase n, and therefore, increase the number of strands in the half-diagram. As we move to the next even grid, we will continue to get the best net increase in $tb^{\frac{1}{2}}(D)$, if we nest the strands as shown in Figure 46, again up to cyclic permutation. Hence, each strand we add in will cross all of the previous strands positively and give us that new southeast corner. This forces the vertical strands (in one of the permutations of the diagram) to be grouped where the strands coming into the diagram are all together, and the strands leaving the half-diagram are all together. This construction allows the new horizontal strand to cross all the previous strands positively. Thus for each even step we will pick up a new southeast corner, giving us a diagram resembling Figure 39. Since we constructed the lower half-diagram so as to maximize the Thurston-Bennequin number and that rotating the lower-half and gluing it in place for the top half gives us the $T\left(\frac{n}{2}, \frac{n}{2}\right)$ torus link, we know that this is the only construction for a rectangular diagram of a knot or link which achieves the maximum. Again, these maximal diagrams can be cyclically permuted and the link is not changed, nor is the Thurston-Bennequin number.

To summarize, we have shown that the maximum Thurston-Bennequin number allowable for any knot or link drawn on an n by n grid for even n, is

$$TB(n) = CR(n) - \left\lceil \frac{n}{2} \right\rceil = \left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2}$$

and that the only diagrams which achieve this maximum are equivalent diagrams for the $T\left(\frac{n}{2}, \frac{n}{2}\right)$ torus links. $\square$

3 Odd Case

Now we want to prove the bound holds when n is odd. We will do this in a similar fashion. First, we will show the bound holds by breaking up the argument into half-diagram bounds. Then we will show which diagrams achieve the bound.

We begin again with the work of Peter Cromwell and Ian Nutt. We know that the maximum crossing number on an n-grid (where n is odd) is a function of triangle numbers. In fact, we have the following proposition:

Proposition 3.1. [5] When n is odd, the maximum crossing number, $CR(n)$ for any knot or link drawn on the n-grid is $\Delta(m) + \Delta(m-1)$ where $\Delta(m) = \frac{1}{2}m(m+1)$ and $m = \frac{1}{2}(n-1)$.

Proof. Now for an odd arc index n, we have $m = \frac{1}{2}(n-1)$ and $\Delta(m) = \frac{1}{2}m(m+1)$. Then,

$$\begin{aligned}
\Delta(m) + \Delta(m-1) &= \frac{1}{2}m(m+1) + \frac{1}{2}(m-1)m \\
&= \frac{1}{2}m[m+1+m-1] \\
&= \frac{1}{2}m(2m)
\end{aligned}$$

Thus, the maximum allowable crossing number for any knot or link drawn on an n-grid is

$$\begin{aligned} CR(n) &= \Delta(m) + \Delta(m-1) \\ &= \frac{1}{2}m(2m) \\ &= m^2 \\ &= \left(\frac{1}{2}(n-1)\right)^2 \\ &= \left(\frac{n-1}{2}\right)^2 \end{aligned}$$

$\square$

Now, we are trying to maximize the Thurston-Bennequin number for any knot diagram drawn on an (odd) n-grid. For the even case, we maximized the lower half grid and found $TB^{\frac{1}{2}}(n)$. We then proved $TB(n) = 2TB^{\frac{1}{2}}(n)$. For the odd case, we will do something similar. First, we establish some definitions and conventions.

For the duration of this work, the upper half is the smaller half and the lower half is the bigger half. That is, for n, the upper half is a $\frac{n-1}{2}$ by n grid and the lower half is a $\frac{n+1}{2}$ by n grid.

Definition 3.1. The **upper Thurston-Bennequin number,** $tb^{\textbf{ upper}}(D)$, of a half-rectangular diagram, D, of a knot or a link is

$$tb^{\text{ upper}}(D) = wr^{\text{ upper}}(D) - \tfrac{1}{2}(\#\text{s.e.corners}+\#\text{n.w.corners}),$$

where $wr^{\text{ upper}}(D)$ is the writhe of the upper half-diagram.

Definition 3.2. The **maximum upper Thurston-Bennequin number,** $TB^{\textbf{ upper}}(n)$, over all half-rectangular diagrams, D, of a knot or a link on an n-grid is

$$TB^{\text{ upper}}(n) = \max_{\text{half-diagrams D}} \{tb^{\text{ upper}}(D)\}.$$

Definition 3.3. The **lower Thurston-Bennequin number,** $tb^{\textbf{ lower}}(D)$, of a half-rectangular diagram, D, of a knot or a link is

$$tb^{\text{ lower}}(D) = wr^{\text{ lower}}(D) - \tfrac{1}{2}(\#\text{s.e.corners}+\#\text{n.w.corners}),$$

where $wr^{\text{ lower}}(D)$ is the writhe of the lower half-diagram.

Definition 3.4. The **maximum lower Thurston-Bennequin number,** $TB^{\textbf{ lower}}(n)$, over all half-rectangular diagrams, D, of a knot or a link on an n-grid is

$$TB^{\text{ lower}}(n) = \max_{\text{half-diagrams D}} \{tb^{\text{ lower}}(D)\}.$$

We will maximize each half and then connect them to maximize the full diagram. We will prove the following theorem by induction.

Theorem 3.1. For an odd n by n rectangular diagram, the maximum achievable Thurston-Bennequin number for any link drawn in the grid is

$$TB(n) = \left(\tfrac{n-1}{2}\right)^2 - \left\lceil \tfrac{n}{2} \right\rceil$$

and $T\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ torus knots are the only links which achieve the maximum.

Outline of Proof: We want to show that $tb(D) \leqslant CR(n) - \left\lceil \frac{n}{2} \right\rceil = \left(\frac{n-1}{2}\right)^2 - \left\lceil \frac{n}{2} \right\rceil$ for any knot or link, D, drawn on an n-grid for odd n. That is, we want to show that

$$TB(n) = CR(n) - \left\lceil \tfrac{n}{2} \right\rceil = \left(\tfrac{n-1}{2}\right)^2 - \left\lceil \tfrac{n}{2} \right\rceil.$$

We will maximize $tb^{\text{ upper}}(D)$ and $tb^{\text{ lower}}(D)$ over all possible knot diagrams D by proving the following theorems:

Theorem 3.2. The maximum achievable $TB^{\text{ upper}}(n)$ for any half-diagram drawn on the $\frac{n-1}{2}$ by n half-grid is

$$TB^{\text{ upper}}(n) = TB^{\frac{1}{2}}(n-1),$$

and there are unique half-diagrams which achieve the bound.

Theorem 3.3. The maximum achievable $TB^{\text{ lower}}(n)$ for any half-diagram drawn on the $\frac{n+1}{2}$ by n half-grid is

$$TB^{\text{ lower}}(n) = \tfrac{1}{2}\left[CR(n+1) - \tfrac{n+3}{2}\right] = \tfrac{1}{2}CR(n+1) - \tfrac{n+3}{4},$$

and there are unique half-diagrams which achieve the bound.

Once we have the bounds for the half-diagrams, we will have

$$TB(n) \leqslant TB^{\text{ upper}}(n) + TB^{\text{ lower}}(n),$$

which proves Theorem 3.1. We are going to prove this by induction, so we will show that the bounds hold for the base case when $n = 5$, and then we can proceed with induction.

3.1 Base Case, $n = 5$

We begin with $n = 5$ because there are no knots which have an arc index of 3, as proven by Jin and Park [9]. Our fist step is to maximize the upper half. For the upper half we have a 2 by 5 grid, therefore we have only two rows and hence, only 2 horizontal strands. We have actually already maximized the tb for a half diagram with at most two strands in the even case. Recall our base case $n = 4$. We had a maximum $TB^{\frac{1}{2}}(4) = 0$, as shown again in Figures

48 and 49 which require the two strands to interact so as to maximize the $tb^{\frac{1}{2}}$. In Figure 50, since we moved from a 2 by 4 grid to a 2 by 5 grid, we added in an empty column. We have also reflected the diagram to move to the upper 2 by 5 grid, and achieve the maximal $tb^{\text{ upper}}(D)$.

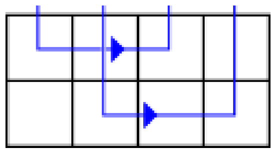

Figure 48: maximal lower half for $n = 4$

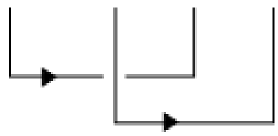

Figure 49: maximal two strands without grid

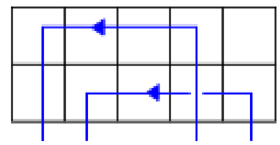

Figure 50: maximal upper half for $n = 5$

Notice that in Figure 50, the middle column is empty, but this was just a matter of choice. Any of the columns can be empty, but we may have move things around to make it work with the lower half. This gives us one piece of our overall bound, since we have

$$0 = TB^{\text{ upper}}(5) = \mathrm{TB}^{\frac{1}{2}}(4) = \frac{1}{2}\left[CR(4) - \frac{4}{2}\right] = \frac{1}{2}\left[\left(\frac{4}{2} - 1\right)\left(\frac{4}{2}\right) - \frac{4}{2}\right] = 0.$$

Now for the lower half-diagram. We want to maximize $tb^{\text{ lower}}(D)$ for any half-diagram on a 3 by 5 grid. We have 3 rows for this half-diagram. If we only fill up two of the rows, though, the maximum $tb^{\text{ lower}}(5)$ that we can reach is zero. So, we want to fill up all three rows and see if we can increase the $tb^{\text{ lower}}$. As in the base case for n even, there are a lot of possibilities here, but they can again be grouped. We have shown below some of the categories of diagrams. Notice that we have not included here any options with all three strands separated. This is because we only have five columns, but three separate strands would have the half-diagram occupying six columns.

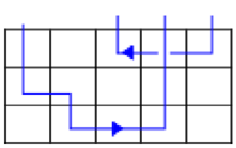

Figure 51: $tb^{\text{ lower}} = 1 - \frac{1}{2}(2)$ $=0$

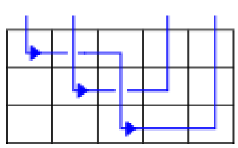

Figure 52: $tb^{\text{ lower}} = 2 - \frac{1}{2}(2)$ $=1$

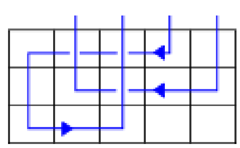

Figure 53: $tb^{\text{ lower}} = 3 - \frac{1}{2}(4)$ $=1$

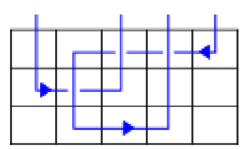

Figure 54: $tb^{\text{ lower}} = 3 - \frac{1}{2}(4)$ $=1$

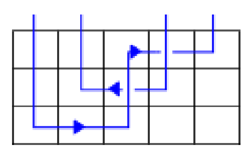

Figure 55: $tb^{\text{ lower}} = 2 - \frac{1}{2}(4)$ $=0$

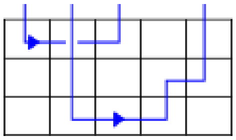

Figure 56: $tb^{\text{ lower}} = 1 - \frac{1}{2}(4)$ =-1

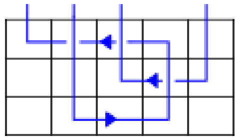

Figure 57: $tb^{\text{ lower}} = 1 - \frac{1}{2}(2)$ =0

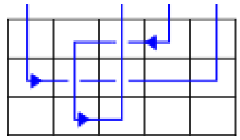

Figure 58: $tb^{\text{ lower}} = 1 - \frac{1}{2}(4)$ =-1

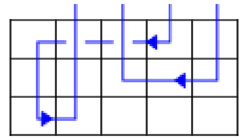

Figure 59: $tb^{\text{ lower}} = 2 - \frac{1}{2}(4)$ =0

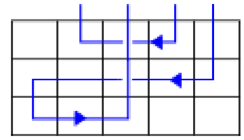

Figure 60: $tb^{\text{ lower}} = 2 - \frac{1}{2}(4)$ =0

From these diagrams we have the maximum $tb^{\text{ lower}}(D) = 1$, and we have three diagrams which achieve that maximum- Figures 52, 53, and 54. Notice that Figure 53 is a cyclic permutation of Figure 54 where the leftmost strand in Figure 54 is pulled around the outside of the diagram to become the furthest right strand. Something a little harder to see, is that Figure 52 is a cyclic permutation of Figure 54. To get from Figure 52 to Figure 54, imagine taking the left most vertical strand and dragging it around the diagram and landing on the far right side. Going from Figure 52 to Figure 54, we pick up a crossing and two southeast/northwest corners, but the $tb^{\text{ lower}}$ remains the same.

We have shown

$$\begin{aligned}
1 = TB^{\text{ lower}}(5) &= \frac{1}{2}\left[CR(5+1) - \frac{5+3}{2}\right] \\
&= \frac{1}{2}\left[\left(\frac{5+1}{2} - 1\right)\left(\frac{5+1}{2}\right) - 4\right] \\
&= \frac{1}{2}[(2)(3) - 4] \\
&= \frac{1}{2}(2) \\
&= 1
\end{aligned}$$

We will now show the maximum upper and lower half diagrams side-by-side. Figures 61 (up to choice of an empty column) and 62 and all of their cyclic permutations are the maximal half diagrams.

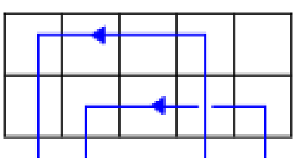

Figure 61: Maximal Upper Half Diagram for $n = 5$

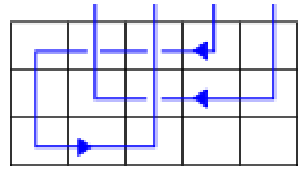

Figure 62: Maximal Lower Half Diagram for $n = 5$

These, and all of their cyclic permutations, are the maximal half-diagrams for $n = 5$. To maximize the full diagram, we can permute and glue these half-diagrams together to get the following maximal diagrams. Notice that we had to choose a different empty column for the upper half in order to glue it to the maximal lower half. So the maximal upper half has $TB^{\text{ upper}}(5) = 0$ and the maximal lower half has $TB^{\text{ lower}}(5) = 1$. Notice that the diagrams in Figures 63 and 64 are cyclic permutations of each other, and they both have a Thurston-Bennequin number equal to one.

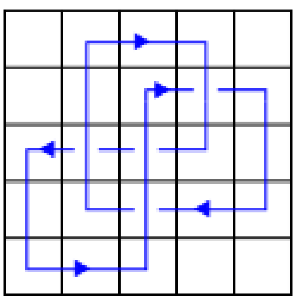

Figure 63: A Maximal Full Diagram for $n = 5$

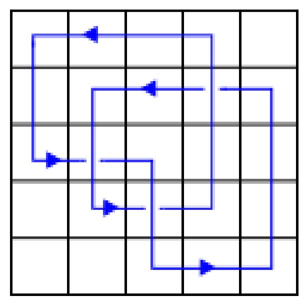

Figure 64: A Maximal Full Diagram for $n = 5$

Below we have shown the moves it requires to get from one maximal diagram to the other. Notice that we reversed the orientation of the knot in Figure 63. Doing this allows for us to clearly see that the two diagrams are representations of the same knot, and it does not change the Thurston-Bennequin number of the knot.

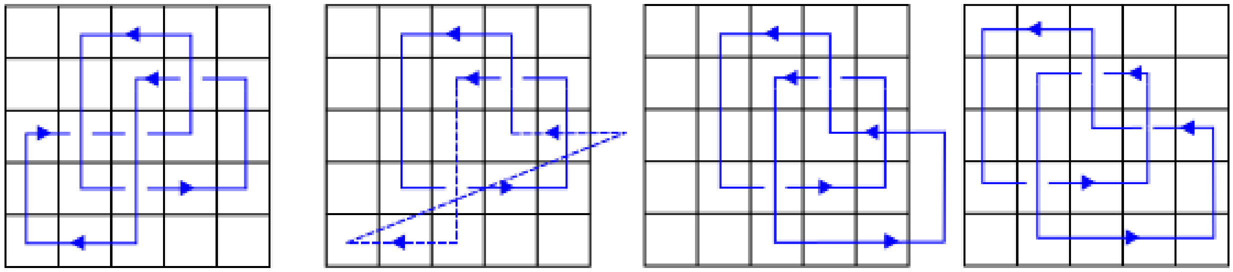

Figure 65: Permutations to get from Figure 63 to Figure 64, but moving right to left will get us from Figure 64 to Figure 63

We have shown for $n = 5$ the following:

$$\begin{aligned}
1 = TB(5) &\leqslant TB^{\text{upper}}(5) + TB^{\text{lower}}(5) \\
&= 0 + 1 \\
&= 1 \\
&= 4 - 3 \\
&= \left(\frac{5-1}{2}\right)^2 - \left\lceil \frac{5}{2} \right\rceil \\
&= \left(\frac{n-1}{2}\right)^2 - \left\lceil \frac{n}{2} \right\rceil \\
&= CR(n) - \left\lceil \frac{n}{2} \right\rceil
\end{aligned}$$

We have also shown that the $T(3, 2)$ torus knot, also called the trefoil, achieves the maximum tb on the 5-grid. This completes the base case.

3.2 Proving Theorem 3.1

We want to show that $TB(k) \leqslant CR(k) - \left\lceil \frac{k}{2} \right\rceil$ for $k = n + 2$. We use k here for simplicity instead of writing $n-2$. Again we will start by finding the maximum tb on the half-grids. We will prove the upper and lower bounds separately.

3.2.1 Proof of Theorem 3.2

Proof. We want to show that

$$TB^{\text{ upper}}(k) \leqslant TB^{\frac{1}{2}}(k-1) = \tfrac{k^2-6k+5}{8}$$

That is, the upper half has the same bound as the half-grid in the even case below. For a k-grid, we are working with a $\frac{k-1}{2}$ by k upper half-grid. So we have $\frac{k-1}{2}$ horizontal strands to maximize tb, which is the same scenario that we have in the even $k-1$ half-grid which we proved in Section 2.3.2. Hence, our half-grid will look the same with the addition of an empty column. Therefore we have

$$TB^{\text{ upper}}(k) = TB^{\frac{1}{2}}(k-1) = \tfrac{k^2-6k+5}{8}$$

and thus, we have the bound for the upper half-diagram. We also know that in order for the half-diagram to be maximal, the diagram must look like a maximal even diagram with an empty column. We have below a maximal diagram in Figure 66.

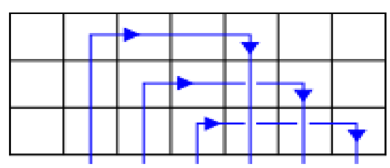

Figure 66: This is a maximal upper half-diagram for $n = 7$.

Notice, we chose the leftmost column to be the empty column. This is a matter of choice for the upper half-grid, but we chose it here so that we can glue it (easily) to the maximal lower half-diagram we'll find in the following argument. $\square$

3.2.2 Proof of Theorem 3.3

Proof. We want to show that

$$TB^{\text{ lower}}(k) = \frac{1}{2}\left[CR(k+1) - \frac{k+3}{2}\right] = \frac{1}{2}CR(k+1) - \frac{k+3}{4} = \frac{k^2-2k-7}{8}$$

for $k = n+2$. Now we want to maximize the lower half-diagram, which is a $\frac{k+1}{2}$ by k half-grid. This means we have one more row than we have in the top half of the diagram. This leaves us with four options: case 1, the top row has a southwest and a southeast corner; case 2, the top row has a northwest and a northeast corner; case 3, the top row has a southwest and a northeast corner; case 4, the top row has a northwest and a southeast corner. To help visualize the four options for the top row, we have the following figures.

Figure 67: Case 1

Figure 68: Case 2

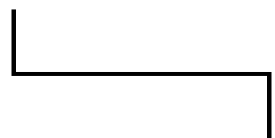

Figure 69: Case 3

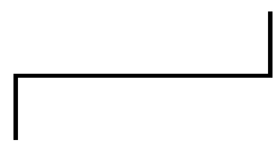

Figure 70: Case 4

We also assume that we have

$$TB^{\text{ lower}}(n) = \frac{1}{2}CR(n+1) - \frac{n+3}{4} = \frac{n^2-2n-7}{8}.$$

That is,

$$TB^{\text{ lower}}(k-2) = \frac{1}{2}CR\left((k-2)+1\right) - \frac{(k-2)+3}{4} = \frac{(k-2)^2-2(k-2)-7}{8}.$$

So we know the maximal *tb* for the odd case a step below. For this proof, we will induct on a smaller diagram with a bound we already know, and then find the maximal increase to get the bound on the larger diagram.

Case 1: Suppose that the top row in our lower half-diagram, D, has two south corners, that is, one southwest and one southeast corner as shown in Figure 67. If we remove the top strand, we are left with a $\frac{k-1}{2}$ by $k-2$ half-diagram, call it D'. Clearly we lose the row that we removed, but it is harder to see why we lose two columns. We lose two columns because the structure of rectangular diagrams prevents anything from occupying the columns underneath the two south corners. So, when we remove those, we lose the two columns in the lower half-grid. We have demonstrated this idea on the half grid for $k = 7$ in Figure 71.

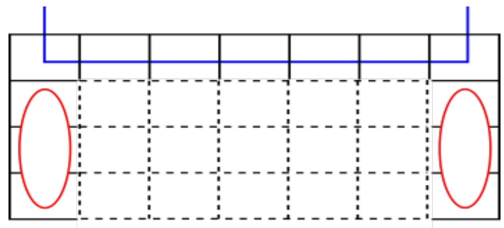

Figure 71: The dotted region shows us the grid we're left with once we remove the top row of the diagram.

Because we lost the row and two columns, we are actually now in the half-grid for the odd case a step below. Then, the maximum increase we can have moving from D' to D is $\frac{k-3}{2} - \frac{1}{2}$. This is because we have $k-2$ columns and therefore, $k-2$ possible vertical strands leaving the diagram, which is an odd number. However, we need an even number of strands leaving the half-grid, so we have at most $k-3$ vertical strands leaving the half-grid, leaving us with an empty column. Further, half of those strands are up strands and half of them are down strands, meaning that at most half can cross the horizontal strand we removed in a positive way. Hence, the writhe can increase by at most $\frac{k-3}{2}$ when we move from D' to D. Also, in this case, we have a southeast corner present in the top row, so we must subtract $\frac{1}{2}(1) = \frac{1}{2}$.

Now,

$$
\begin{aligned}
tb^{\text{ lower}}(D) &\leqslant tb^{\text{ lower}}(D') + \frac{k-3}{2} - \frac{1}{2} \\
&\leqslant TB^{\text{ lower}}(k-2) + \frac{k-3}{2} - \frac{1}{2} \\
&= \frac{1}{2}CR((k-2)+1)) - \frac{(k-2)+3}{4} + \frac{k-3}{2} - \frac{1}{2} \\
&= \frac{1}{2}CR(k-1) + \frac{k-9}{4} \\
&= \frac{1}{2}\left(\frac{k-1}{2} - 1\right)\left(\frac{k-1}{2}\right) + \frac{k-9}{4} \\
&= \frac{k^2-4k+3}{8} + \frac{2k-18}{8} \\
&= \frac{k^2-2k-15}{8} \\
&< \frac{k^2-2k-7}{8} \\
&= \frac{1}{2}CR(k+1) - \frac{k+3}{4} \\
&= TB^{\text{ lower}}(k).
\end{aligned}
$$

Thus, our diagram has a tb that fits within our proposed bound, so the bound holds. We also know that this type of diagram cannot be maximal.

Case 2: Suppose that the top row of our $\frac{k+1}{2}$ by k half-diagram, D, has two north corners. That is, the top row of D has one northwest and one northeast corner as shown in Figure 68. When we remove it, we are left with a $\frac{k-1}{2}$ by k half grid-diagram, D'. Clearly, we lose the row we removed, but we do not lose any columns. This is because the columns containing the north corners in the top row, have vertical strands underneath that. An example of what this looks like is shown below in Figure 72.

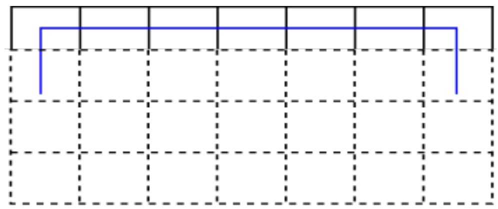

Figure 72: The dotted region is the grid that remains once we remove the top row.

Hence, we are actually in the same scenario as we were in with the upper bound. So, we have all the rows full and an empty column. There are k columns, and only $k-2$ can be occupied by vertical strands that leave the half-diagram, because our top row caps off two of the vertical strands. We also know we have an empty column, and that $k-2$ is odd, so we have at most $k-3$ exiting vertical strands. Therefore, only half of those can intersect with the boundary of our half diagram, or the top row, positively. Hence, we can increase the writhe by $\frac{k-3}{2}$ moving from D' to D. The top row has a northwest corner, so we'll decrease the tb by $\frac{1}{2}(1) = \frac{1}{2}$.

Now,

$$
\begin{aligned}
tb^{\text{ lower}}(D) &\leqslant tb^{\text{ lower}}(D') + \frac{k-3}{2} - \frac{1}{2} \\
&\leqslant TB^{\text{ upper}}(k) + \frac{k-3}{2} - \frac{1}{2} \\
&= TB^{\frac{1}{2}}(k-1) + \frac{k-4}{2} \\
&= \frac{1}{2}\left(CR(k-1) - \frac{k-1}{2}\right) + \frac{k-4}{2} \\
&= \frac{1}{2}CR(k-1) - \frac{k-1}{4} + \frac{k-4}{2} \\
&= \frac{1}{2}\left(\frac{k-1}{2} - 1\right)\left(\frac{k-1}{2}\right) + \frac{k-7}{4} \\
&= \frac{k^2-4k+3}{8} + \frac{k-7}{4} \\
&= \frac{k^2-2k-11}{8} \\
&< \frac{k^2-2k-7}{8} \\
&= \frac{1}{2}CR(k+1) - \frac{k+3}{4} \\
&= TB^{\text{ lower}}(k).
\end{aligned}
$$

Thus, we again have a tb that fits within our proposed bound, however, this still cannot be a maximal half-diagram.

Case 3: Suppose now, that we have a lower half-diagram, D, where the top row has a southwest and a northeast corner as shown in Figure 69. Now, if we remove the top row of D to get to a new half-diagram, D', we have a $\frac{k-1}{2}$ by $k-1$ half-grid, since we lose one row and one column. This idea is demonstrated in Figure 73 below.

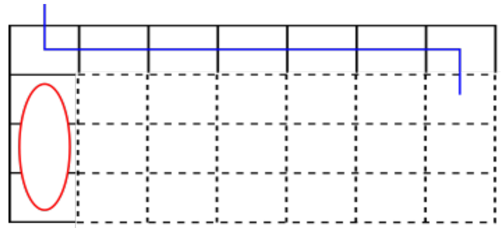

Figure 73: The dotted region is what we are left with after we remove the top row.

So, there are $k-1$ columns in the diagram. That is, we can have at most $k-1$ vertical strands in the diagram. We have a maximum of $\frac{k-1}{2}$ up strands and $\frac{k-1}{2}$ down strands. Our top row of D contains one of those exiting strands and thus, there are at most $k-1$ vertical strands exiting our smaller diagram D'. One of those strands is capped off in the top row of D, so there are $\frac{k-3}{2}$ strands left that are similarly oriented to the exiting strand from the top row of D.

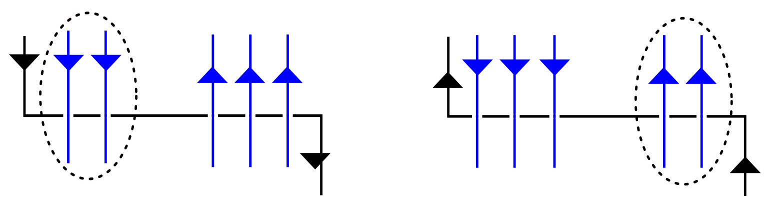

Figure 74: Notice that we have the same number of up strands and down strands leaving the half diagram and that the circled strands give us positive crossings.

The similarly oriented strands are the vertical strands leaving the diagram with the potential to cross the top row positively, and we have circled them in the figure above. Hence, we can increase the writhe by at most $\frac{k-3}{2}$ and we do not have to subtract anything from the tb for the corners when moving from D' to D. In order to realize the bound on the bigger diagram D, the strands underneath it must have realized the bound on the smaller diagram D'.

So we have,

$$
\begin{aligned}
tb^{\text{ lower}}(D) &\leqslant tb^{\text{ lower}}(D') + \frac{k-3}{2} \\
&\leqslant TB^{\frac{1}{2}}(k-1) + \frac{k-3}{2} \\
&= \frac{1}{2}\left(CR(k-1) - \frac{k-1}{2}\right) + \frac{k-3}{2} \\
&= \frac{1}{2}CR(k-1) - \frac{k-1}{4} + \frac{k-3}{2} \\
&= \frac{1}{2}\left(\frac{k-1}{2} - 1\right)\left(\frac{k-1}{2}\right) + \frac{k-5}{4} \\
&= \frac{k^2-4k+3}{8} + \frac{k-5}{4} \\
&= \frac{k^2-4k+3+2k-10}{8} \\
&= \frac{k^2-2k-7}{8} \\
&= \frac{1}{2}CR(k+1) - \frac{k+3}{4} \\
&= TB^{\text{ lower}}(k).
\end{aligned}
$$

So, not only does our bound hold in this case, but we can also realize that bound. So how can we do this? As we mentioned above, once we remove the top row we want to maximize the remaining half-diagram, which is the even half-diagram below. We then want to increase the tb by $\frac{k-3}{2}$ by picking up only positive crossings. This idea is demonstrated in the figure below for $k = 7$.

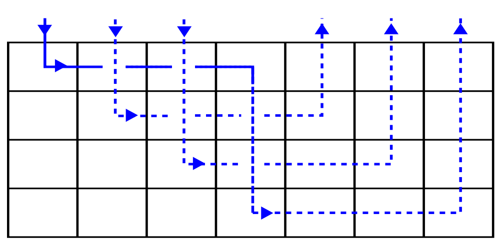

Figure 75: The dotted strands here show a maximal half-diagram when $k = 6$, which is the even case below $k = 7$.

Case 4: Suppose that the top row of our diagram, D, has a northwest and a southeast corner. When we remove the top row, we move from a $\frac{k+1}{2}$ by k half-grid, to a $\frac{k-1}{2}$ by $k-1$ half-grid. We again lose one row and one column as demonstrated in Figure 76.

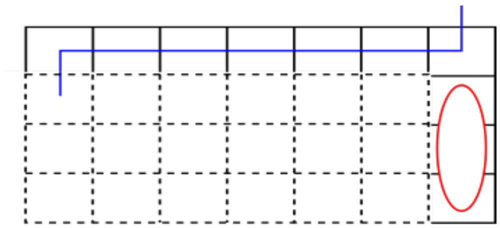

Figure 76: The dotted region shows the half-grid that is left once we remove the top row.

Therefore, we have $k-1$ columns and potential exiting vertical strands. That is, we have at most $\frac{k-1}{2}$ possible up exiting strands and $\frac{k-1}{2}$ possible down exiting strands. As in Case 3, we have $\frac{k-3}{2}$ exiting vertical strands that are oriented the same way as the exiting vertical strand coapped off in the top row of D. We also have $\frac{k-1}{2}$ strands that are oppositely oriented.

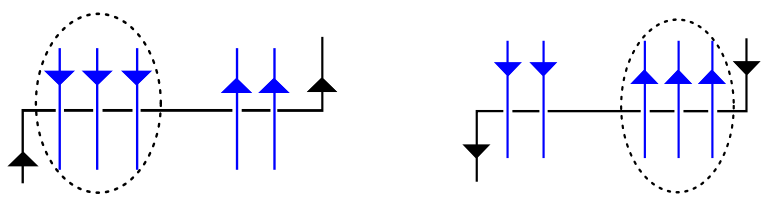

Figure 77: Notice that we have the same number of up strands and down strands leaving the half diagram and the circled strands give us positive crossings.

Those oppositely oriented strands, circled in the above figure, are the strands that give us the potential for positive crossings. So moving from D' to D we can pick up at most $\frac{k-1}{2}$ positive crossings which will increase our tb. We also pick up one southeast and one northwest corner, decreasing our tb by $2(\frac{1}{2}) = 1$.

So we have,

$$
\begin{aligned}
tb^{\text{ lower}}(D) &\leqslant tb^{\text{ lower}}(D') + \frac{k-1}{2} - 2\left(\frac{1}{2}\right) \\
&\leqslant TB^{\frac{1}{2}}(k-1) + \frac{k-1}{2} - 1 \\
&= TB^{\frac{1}{2}}(k-1) + \frac{k-3}{2} \\
&= \frac{k^2-6k+5}{8} + \frac{k-3}{2} \\
&= \frac{k^2-2k-7}{8} \\
&= \frac{1}{2}CR(k+1) - \frac{k+3}{4} \\
&= TB^{\text{ lower}}(k).
\end{aligned}
$$

So we have shown that

$$tb^{\text{ lower}}(D) \leqslant \tfrac{1}{2}CR(k+1) - \tfrac{k+3}{4} = TB^{\text{ lower}}(k),$$

and thus we have proven the bound and that this type of diagram can achieve the bound. So again, we ask, how do we do this? Once we remove the top strand, we have the even maximal diagram left in the half-grid. Again, we proved in Section 2.3.1 that in order to achieve the bound on the bigger diagram D, the strands underneath it in D' have to also achieve the bound on the smaller grid. This means we have nested strands oriented in such a way to achieve maximal *tb*. Then, we have to fit in this top horizontal strand with a southeast and a northwest corner and pick up $\frac{k-1}{2}$ positive crossings. This can only be done one way, up to cyclic permutation. An example on the 7-grid is shown below in Figure 78.

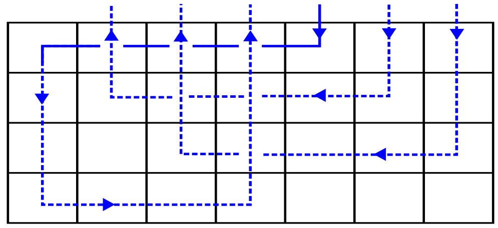

Figure 78: The dotted strands here show a maximal half-diagram when $k = 6$, which is the even case below $k = 7$.

There are permutations of this diagram which are also maximal. In fact, Figure 75 is a cyclic permutation of this diagram, with the orientation of one reversed. It requires the same moves shown in Section 2.1, Figure 65 for the base case when $n = 5$. Thus, we have shown that these diagrams, and the cyclic permutations of them are the only maximal half-diagrams on the lower half-grid when n is odd.

To summarize, in order to maintain the sharpness of the bound the new strand can either cross all of the vertical strands leaving the half-diagram positively, as demonstrated in Case 4, or it can cross all the vertical strands coming into the diagram positively, as demonstrated in Case 3. Both of these options give us maximal half-diagrams, but the thing to note is that they are equivalent half-diagrams, by a cyclic permutation. Thus, we have a unique maximal lower half-grid diagram up to cyclic permutation. $\square$

3.2.3 Proof of Theorem 3.1

Proof. Combining the upper and lower half-diagram bounds we have,

$$
\begin{aligned}
TB(k) &= TB^{\text{ upper}}(k) + TB^{\text{ lower}}(k) \\
&= TB^{\frac{1}{2}}(k-1) + \frac{1}{2}CR(k+1) - \frac{k+3}{4} \\
&= \frac{k^2-6k+5}{8} + \frac{k^2-2k-7}{8} \\
&= \frac{2k^2-8k-2}{8} \\
&= \frac{k^2-4k-1}{4} \\
&= \frac{k^2-2k+1}{4} - \frac{2k+2}{4} \\
&= \frac{k^2-2k+1}{4} - \frac{k+1}{2} \\
&= \left(\frac{k-1}{2}\right)^2 - \left\lceil\frac{k}{2}\right\rceil \\
&= CR(k) - \left\lceil\frac{k}{2}\right\rceil.
\end{aligned}
$$

We also know which diagrams can achieve that bound. We have below a maximal upper half-diagram (Figure 79) and a maximal lower half-diagram (Figure 80) for $n = 7$.

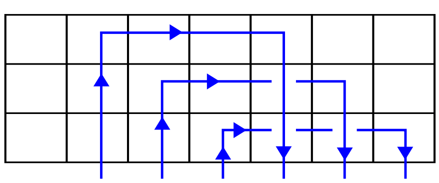

Figure 79: A maximal upper half for $n = 7$

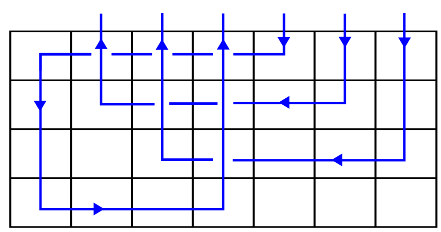

Figure 80: A maximal lower half for $n = 7$

We already know from Theorem 2.1 in Section 2.3.2 and Theorem 3.2 in Section 3.2.1 that the only maximal upper half-diagram has nested strands like we have shown in Figure 79 up to cyclic permutation. The difference here is the fact that we have an empty column, which is a matter of choice and does not affect the maximality of the half-diagram. As in the even case, the only way to maintain the sharpness of the bound is to nest the strands as we have shown in Figure 79 with each increase of n. So this is the only construction of the half-diagram which gives us a maximal $tb^{\frac{1}{2}}(D)$.

Now, a diagram similar to Figure 80 is the unique maximal lower half-diagram, up to cyclic permutation, which we proved in Section 3.2.2. Once we remove the top row of the lower half-diagram, we are on the same size grid we have in the upper half-diagram, where the appropriate nesting of strands with an empty column is uniquely maximal, with the choice of an empty column. When we add in the new strand, we have a couple of options. These options are demonstrated in Cases 3 and 4 in Section 3.2.2.

To get the unique maximal full diagram, we simply glue these maximal halves together as shown in Figure 81. Now we can cyclically permute the lower half-diagram and then glue in an upper-half diagram with the appropriate column empty. We have done this and have the result shown below in Figure 82.

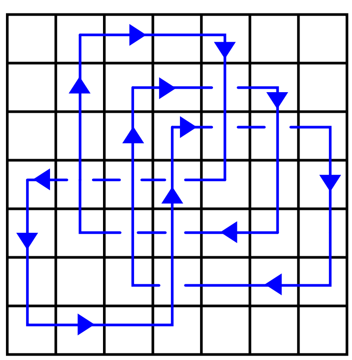

Figure 81: A maximal full diagram for $n = 7$, which is a knot diagram of the $T(3, 4)$ torus knot.

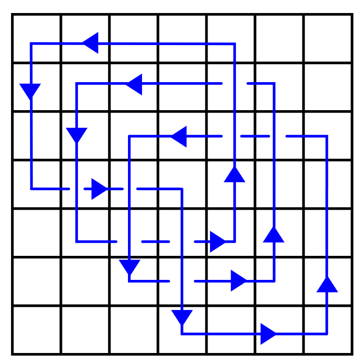

Figure 82: A maximal full diagram for $n = 7$, which is a knot diagram of the $T(3, 4)$ torus knot.

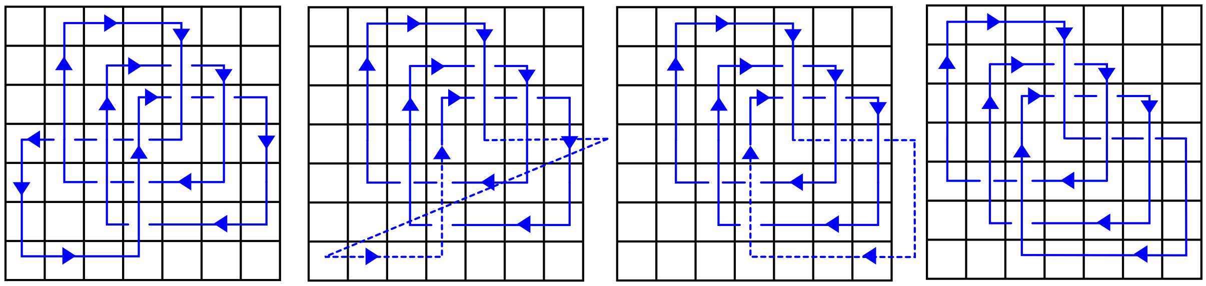

Figure 83: Here we have demonstrated the cyclic permutation we use to get to a different diagram of the same knot.

In Figure 83, we have shown for demonstration the permutation required to get from Figure 81 to Figure 82. We started with the diagram from Figure 81 and then applied a cyclic permutation. Notice that the last diagram contains the maximal lower half-diagram we have in Case 3. In this picture, though, it is the upper half, and our smaller half-diagram is the lower-half. In other words, the maximal half-diagram shown in Figure 82 is upside down in the last diagram of Figure 83. This idea can be expanded as n gets bigger, to demonstrate this idea we look at the 15-grid. We have an analogous picture to the knot-diagram in Figure 81. Both Figure 81 and Figure 84 use the permutation of the unique lower half-diagram that comes from Case 4.

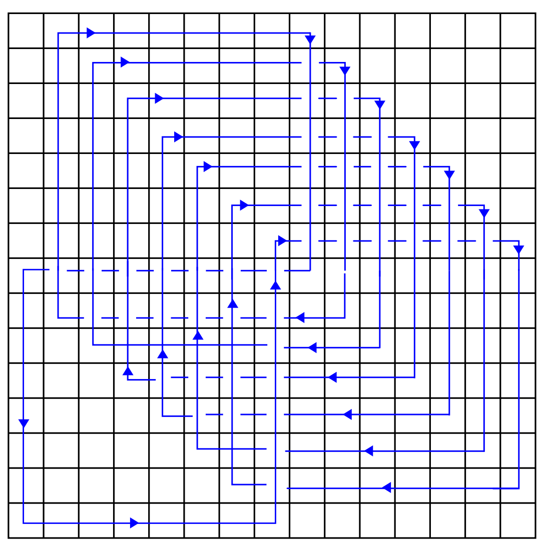

Figure 84: A maximal full diagram for $n = 15$, which is a knot diagram of the $T(7, 8)$ torus knot.

Now, we have in Figure 85 a cyclic permutation of the diagram shown above. This diagram comes from either permuting Figure 84 or using the unique lower half-diagram that comes from Case 3 and choosing the right the empty column in the upper half-diagram.

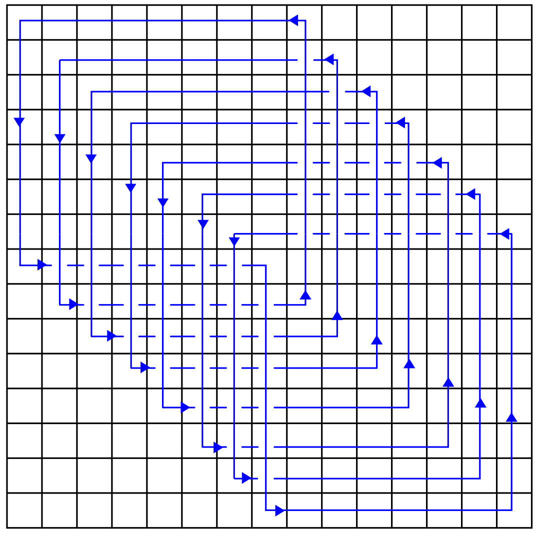

Figure 85: A maximal full diagram for $n = 15$, which is a cyclic permutation of the diagram in Figure 84, and an equivalent knot diagram of the $T(7, 8)$ torus knot.

In conclusion, we have shown that the maximal Thurston-Bennequin number allowable for any knot or link drawn on an n by n rectangular grid for odd n is,

$$TB(n) = CR(n) - \left\lceil \frac{n}{2} \right\rceil = \left(\frac{n-1}{2}\right)^2 - \frac{n+1}{2}$$

and that the only diagrams which achieve the maximum are equivalent diagrams for the $T\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$ torus knots.

$\square$

4 Conclusions and Opportunities for Further Research

Recall the goal of this work was to prove Theorem 1.1:

Theorem 1.1. The maximum allowable Thurston-Bennequin number for any knot or link drawn on a grid with arc index n is,

$$TB(n) = CR(n) - \left\lceil \tfrac{n}{2} \right\rceil = \begin{cases} \left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) - \frac{n}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{2}\right)^2 - \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases},$$

and there are unique knots and links which achieve the maximum, specifically the $T(\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor)$ torus knots and links.

Proof. This immediately follows from Theorem 2.1 and Theorem 3.1. □

There are many avenues to delve into for further research now that we have proven this theorem. One thing we would like to study is knots that are close to torus knots with arc index n. It would be interesting to see how close these knots are to achieving the maximum on the n-grid. Also, we now know from this work that torus links achieve the maximum on even n-grids. So something that would be intriguing to discover is what the maximum achievable Thurston-Bennequin number is for knots on the even grids and which knots do it. Further still, are these knots close to torus links? This can be expanded to odd n-grids too. Since the family of torus knots are in fact the family of knots that give us this optimal *tb*, then which links on the same grid get us closest to the maximal *tb*. Another interesting avenue to travel down is links of torus knots. On their perspective grids, are those links close to the optimal *tb*. If we have two torus knots or links T_1 and T_2, we have from Cromwell's work bounds on the arc index depending on how the torus knots are linked [3]. Thus, there is already some basis to continue to study the relationship between the arc index and the Thurston-Bennequin number in that way.

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