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Interpolating Between Multiplicities and F-thresholds

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Abstract

We define a family of functions, called s-multiplicity for each $s > 0$, that interpolates between Hilbert-Samuel multiplicity and Hilbert-Kunz multiplicity by comparing powers of ideals to the Frobenius powers of ideals. The function is continuous in s, and its value is equal to Hilbert-Samuel multiplicity for small values of s and is equal to Hilbert-Kunz multiplicity for large values of s. We prove that it has an associativity formula generalizing the associativity formulas for Hilbert-Samuel and Hilbert-Kunz multiplicity. We also define a family of closures, called s-closures, such that if two ideals have the same s-closure then they have the same s-multiplicity, and the converse holds under mild conditions. We describe methods for computing the F-threshold, the s-multiplicity, and the s-closure of monomial ideals in toric rings using the geometry of the cone defining the ring.

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Ultimately, my knowledge, my work, and my success are gifts from God. He is the one who has provided me with all blessings, and it is for Him that I pursue excellence. Without Him, I cannot truly succeed. With Him, I cannot truly fail. Soli Deo Gloria.

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1 Introduction

Hilbert-Samuel multiplicity gives us information about the asymptotic behavior of the powers of an ideal. Given a local ring $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal I of R, one studies the modules R/I^n for $n \in \mathbb{N}$. In particular, one considers the Hilbert function $n \mapsto \lambda(R/I^n)$, where $\lambda(M)$ is the length of the R-module M. The Hilbert function is eventually polynomial, i.e. there exists a polynomial $P(n)$ with rational coefficients such that $P(n) = \lambda(R/I^n)$ for $n \gg 0$. We call $P(n)$ the Hilbert-Samuel polynomial of I. The degree of $P(n)$ is equal to the dimension of R, and furthermore the leading coefficient of $P(n)$ is of the form $\frac{e(I)}{d!}$, where $e(I)$ is a positive integer and $d = \dim R$. This number $e(I)$ is called the Hilbert-Samuel multiplicity of I.

We can compute the Hilbert-Samuel multiplicity as a limit, which allows us to use some analytic methods in its study. For a d-dimensional local ring $(R, \mathfrak{m})$ and $\mathfrak{m}$-primary ideal I of R, the Hilbert-Samuel multiplicity of I is

$$e(I) = \lim_{n \to \infty} \frac{d! \cdot \lambda(R/I^n)}{n^d}.$$

Hilbert-Samuel multiplicity is also relevant in the study of integral closure. If $(R, \mathfrak{m})$ is a local ring and I and J are $\mathfrak{m}$-primary ideals that have the same integral closure, then I and J have the same Hilbert-Samuel multiplicity. On the other hand, a theorem of Rees states that under mild conditions, if $I \subseteq J$ and $e(I) = e(J)$, then I and J have the same integral closure.

When we work in a local ring $(R, \mathfrak{m})$ with positive characteristic p, we can define a similar number attached to an $\mathfrak{m}$-primary ideal I using the Frobenius powers instead of the ordinary powers. In particular, we can study the Hilbert-Kunz multiplicity of an ideal I, defined by

$$e_{HK}(I) = \lim_{e \to \infty} \frac{\lambda\big(R/I^{[p^e]}\big)}{p^{ed}},$$

where d is the dimension of the ring and $I^{[p^e]}$ is the ideal generated by the p^e-th powers of the generators of I. The Hilbert-Kunz multiplicity is related to tight closure, as defined by Hochster and Huneke [4], similarly to how Hilbert-Samuel multiplicity is related to integral closure. Precisely, if I and J are $\mathfrak{m}$-primary ideals in a local ring $(R, \mathfrak{m})$ that have the same tight closure, then they have the same Hilbert-Kunz multiplicity, and the converse holds under mild conditions.

In this thesis we define a parameterized family of multiplicities that interpolate between Hilbert-Samuel and Hilbert-Kunz multiplicities. This family is parameterized by a positive real number s, and we call it the s-multiplicity.

Definition 1.0.1 (Definitions 3.3.2 and 3.5.1)**.** Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and of dimension d, and let I be an $\mathfrak{m}$-primary ideal of R. For each real number $s > 0$, the *s-multiplicity of I* is

$$e_s(I) := \lim_{e \to \infty} \frac{\lambda\big(R/(I^{\lceil sp^e \rceil} + I^{[p^e]})\big)}{p^{ed}\mathcal{H}_s(d)},$$

where $\mathcal{H}_s(d) = \displaystyle\sum_{i=0}^{\lfloor s \rfloor} \frac{(-1)^i}{d!}\binom{d}{i}(s-i)^d$.

We establish many properties of the function $e_s(I)$, the most important of which are summarized below, and which show that the s-multiplicity is a good choice of interpolation between Hilbert-Samuel and Hilbert-Kunz multiplicity.

Theorem 1.0.2. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and of dimension d, and let I be an $\mathfrak{m}$-primary ideal of R.*

1. *(Theorem 3.3.1) The s-multiplicity $e_s(I)$ exists for all $s > 0$.*

2. *(Corollary 3.5.2) For $s \leq 1$, $e_s(I) = e(I)$, and for $s \geq d$, $e_s(I) = e_{HK}(I)$.*

3. *(Corollary 3.5.2) If R is regular, then $e_s(\mathfrak{m}) = 1$ for all s.*

4. *(Corollary 3.5.4) For each s, $e_s(I)$ has an associativity formula generalizing the ones for Hilbert-Samuel and Hilbert-Kunz multiplicity.*

5. *(Corollary 3.5.3)* $e_s(I)$ *is Lipschitz continuous in* s.

The first four parts of this theorem show that the s-multiplicity fulfills the properties that we would want out of a function designed to translate between the Hilbert-Samuel and Hilbert-Kunz multiplicities. Since we define the s-multiplicity using a limit, Item 1 is necessary in order for us to know that the function we're studying is well-defined. Its proof is fairly technical, though Lemma 3.2.1, which details the generators of finite length modules, is of independent interest. Item 2 shows that the s-multiplicity does indeed interpolate between the Hilbert-Samuel and Hilbert-Kunz multiplicities, capturing both of their behaviors at different points on its domain. Item 3 shows that the s-multiplicity behaves like the Hilbert-Samuel and Hilbert-Kunz multiplicities when computing it at the maximal ideal in a regular local ring, while Item 4 often reduces the problem of computing the s-multiplicity to the domain case, just as the associativity formulas for the other multiplicities do.

Item 5 deals with the intermediate values of s between those described in Item 2. It shows that the behavior of the s-multiplicity, as a function of s, cannot be too pathological. In fact, the derivative with respect to s exists almost everywhere and is bounded. This result is essential to establishing the associativity formula mentioned above.

Understanding the way in which the s-multiplicity interpolates between the Hilbert-Samuel and Hilbert-Kunz multiplicities requires understanding how ordinary powers and Frobenius powers of ideals interact with each other. The F-threshold is a number attached to a pair of ideals I and J in a ring of positive characteristic. Roughly speaking, the F-threshold of I with respect to J is the infimum of those numbers s such that $I^{sp^e} \subseteq J^{[p^e]}$ for all sufficiently large e. When the ring is regular local, the F-threshold with respect to the maximal ideal is equal to another measure called the F-pure threshold, which in turn is related to the log canonical threshold, the latter two of which can be used to describe the singularities of the geometry of the ring.

It naturally arises that we wish to consider a dual notion to the F-threshold, that is, the supremum of those numbers s such that $J^{[p^e]} \subseteq I^{sp^e}$ for all sufficiently large e. In this thesis, we call this number the F-limbus, and prove some properties of it that help us understand the s-multiplicity.

As mentioned previously, the Hilbert-Samuel multiplicity is related to integral closure in the following way: Suppose I and J are $\mathfrak{m}$-primary ideals in a local ring $(R, \mathfrak{m})$. If $\overline{I} = \overline{J}$, then $e(I) = e(J)$, where $\overline{I}$ indicates the integral closure of I. Similarly, if $I^* = J^*$, then $e_{HK}(I) = e_{HK}(J)$, where I^* indicates the tight closure of I. We construct a family of closures, called s-closures, by combining the definitions of integral and tight closures. We use the notation I^{cl_s} to denote the s-closure of I, and define it as follows An element $x \in R$ is in the weak s-closure of I, denoted $I^{\mathrm{w.cl}_s}$, if there exists an element $c \in R$, not in any minimal prime, such that $cx^{p^e} \in I^{\lceil sp^e \rceil} + I^{[p^e]}$ for all sufficiently large e. We define the s-closure of I to be the ideal at which the increasing chain $I \subseteq I^{\mathrm{w.cl}_s} \subseteq \left(I^{\mathrm{w.cl}_s}\right)^{\mathrm{w.cl}_s} \subseteq \cdots$ stabilizes.

The s-closure translates between integral closure and tight closure as s increases. In particular, $I^{\mathrm{cl}_1} = \overline{I}$ and $I^{\mathrm{cl}_s} = I^*$ when s is large. If $s < s'$, then $I^{\mathrm{cl}_s} \supseteq I^{\mathrm{cl}_{s'}}$. In addition, there are often infinitely many distinct s-closures between integral closure and tight closure. Our main result related to s-closures is the following theorem, which describes precisely the relationship between s-multiplicity and s-closure.

Theorem 1.0.3 (Theorem 4.2.1)**.** *If I and J are $\mathfrak{m}$-primary ideals in a local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ and $I^{\mathrm{cl}_s} = J^{\mathrm{cl}_s}$, then $e_s(I) = e_s(J)$. Furthermore, if $I \subseteq J$ and R is an F-finite complete domain, the converse holds.*

In the final section of this thesis, we provide efficient methods of computing the F-threshold, F-limbus, s-multiplicities, and s-closures for monomials ideals in affine semigroup rings. In each case, we use the geometry of the cone defining the semigroup to realize these numbers as ratios of lengths or volumes in Euclidean space.

Some results of in this thesis have appeared in the literature as the author's paper [12].

2 Rings of Positive Characteristic

Of primary concern for us are ideals and modules over rings of characteristic p, where $p > 0$ is a positive prime number. If $x, y \in R$ and R is a ring of characteristic p, we have that $(x+y)^p = x^p + y^p$, since p divides $\binom{p}{i}$ for $1 \leq i \leq p-1$. In particular, this means the *Frobenius endomorphism* $F : R \to R$, defined by $F(x) = x^p$, is a ring homomorphism, and that $(x+y)^q = x^q + y^q$, where $q = p^e$ for any $e \in \mathbb{N}$.

2.1 Frobenius Powers of Ideals

In a ring of positive characteristic one often considers a new type of function on ideals that behaves like ordinary powers in certain ways but has distinct properties of its own.

Definition 2.1.1. Let I be an ideal of a ring R of characteristic p, and let $q = p^e$ for some $e \in \mathbb{N}$. The ideal $I^{[q]} := (f^q \mid f \in I)$ is called the *q-th Frobenius power* (or *q-th bracket power*) of I.

At first glance, it is unclear why the number q in Definition 2.1.1 should depend on the characteristic of R. However, this dependence is quite important.

Proposition 2.1.2. *Let I be an ideal of a ring R. For any $n \in \mathbb{N}$, $n! \cdot I^n \subseteq (f^n \mid f \in I)$.*

Proof. Let $f_1, \ldots, f_n \in I$. By the inclusion-exclusion principle,

$$\begin{aligned}(f_1 + \cdots + f_n)^n &= \sum_{\sum a_j = n} \binom{n}{a_1, \ldots, a_n} f_1^{a_1} \cdots f_n^{a_n} \\ &= n! \cdot f_1 \cdots f_n + \sum_{i=1}^{n} \sum_{\substack{\sum a_j = n \\ a_i = 0}} \binom{n}{a_1, \ldots, a_n} f_1^{a_1} \cdots f_n^{a_n} \\ &\quad - \sum_{i_1 < i_2} \sum_{\substack{\sum a_j = n \\ a_{i_1} = a_{i_2} = 0}} \binom{n}{a_1, \ldots, a_n} f_1^{a_1} \cdots f_n^{a_n} + \cdots + (-1)^n \sum_{j=1}^{n} f_j^n.\end{aligned}$$

Each term after the first in the sum above is of the form

$$(f_1 + \cdots + \widehat{f_{i_1}} + \cdots + \widehat{f_{i_2}} + \cdots + \widehat{f_{i_k}} + \cdots + f_n)^n,$$

which shows that $n! \cdot f_1 \cdots f_n \in (f^n \mid f \in I)$. $\square$

Corollary 2.1.3. *Let I be an ideal of a ring R and let $n \in \mathbb{N}$. If $n!$ is invertible in R then $(f^n \mid f \in I) = I^n$.*

Proof. The inclusion "$\subseteq$" is obvious. By Proposition 2.1.2, $I^n \subseteq \frac{1}{n!} \cdot (n! \cdot I) \subseteq (f^n \mid f \in I)$, which proves the other direction. $\square$

Corollary 2.1.3 shows that defining a "bracket power" only returns the ordinary powers in the case that R contains a field of characteristic 0. On the other hand, working in rings of positive characteristic and taking bracket powers with powers of the characteristic allows us to simplify the description and computation of them significantly.

Proposition 2.1.4. *Let I be an ideal of a ring R of characteristic p, and let q be a power of p. If $I = (f_1, \ldots, f_n)$ for some $f_i \in R$, then $I^{[q]} = (f_1^q, \ldots, f_n^q)$.*

Proof. Clearly, $(f_1^q, \ldots, f_n^q) \subseteq I^{[q]}$. For the other inclusion, let $f \in I$. There exist $r_1, \ldots, r_n \in R$ such that $f = r_1 f_1 + \cdots + r_n f_n$. Therefore

$$f^q = (r_1 f_1 + \cdots + r_n f_n)^q = r_1^q f_1^q + \cdots + r_n^q f_n^q \in (f_1^q, \ldots f_n^q).$$

Hence $I^{[q]} \subseteq (f_1^q, \ldots f_n^q)$. $\square$

The ideals $I^{[q]}$ and I^q are very different in general. For example, if $I = (x, y) \subseteq k[x, y]$, where k is a field of characteristic 3, then $I^3 = (x^3, x^2y, xy^2, y^3)$ but $I^{[3]} = (x^3, y^3)$. Nevertheless, we will often exploit various containment relationships between ordinary and bracket powers.

Lemma 2.1.5. *Let $I = (f_1, \ldots, f_m)$ be an ideal of a ring R of characteristic $p > 0$. For every power q of p, $I^{m(q-1)+1} \subseteq I^{[q]} \subseteq I^q$.*

Proof. The inclusion $I^{[q]} \subseteq I^q$ is immediate. For the other inclusion, the ideal $I^{m(q-1)+1}$ is generated by elements of the form $f_1^{a_1} \cdots f_m^{a_m}$ with $\sum a_i = m(q-1)+1$. Therefore $a_i \geq q$ for some i, and so $f_1^{a_1} \cdots f_m^{a_m} \in I^{[q]}$. □

Bracket powers and ordinary powers of ideals also interact in predicable ways.

Lemma 2.1.6. *Let I be an ideal of a ring R of characterstic $p > 0$. For every $n \in \mathbb{N}$, and q a power of p, $(I^n)^{[q]} = \left(I^{[q]}\right)^n$.*

Proof. Let $f_1, \ldots, f_m$ be a set of generators for I. We have that

$$(I^n)^{[q]} = \left((f_1^{a_1} \cdots f_m^{a_m})^q \mid \sum_i a_i = n \right) = \left((f_1^q)^{a_1} \cdots (f_m^q)^{a_m} \mid \sum_i a_i = n \right) = \left(I^{[q]}\right)^n. \quad \square$$

Also, bracket powers interact with each other analogously to ordinary powers. We use the notation F^e for the eth iterate of the function F.

Lemma 2.1.7. *Let I be an ideal of a ring R of characterstic $p > 0$. For every pair of powers q, q' of p, $\left(I^{[q]}\right)^{[q']} = I^{[qq']}$.*

Proof. Let $q = p^e$ and $q = p^{e'}$. We have that $I^{[q]} = F^e(I)R$, and therefore $(I^{[q]})^{[q']} = F^{e'}(F^e(I)R)R = F^{e+e'}(I)R = I^{[qq']}$. □

2.2 p^{-e}-LINEAR MAPS

The fact that the Frobenius map and its iterates are ring homomorphisms means that we can consider R as a module over itself with an action induced by the Frobenius map. To make notation easier, we write the module with the new action as $F^e_* R$, where $e \in \mathbb{N}$. As a set, we write $F^e_* R = \{F^e_* x \mid x \in R\}$, and as an additive group it is isomorphic to R. That is, for $x, y \in R$, we have that $F^e_* x + F^e_* y = F^e_*(x+y)$.

The distinctive property of $F_*^e R$ is the action of R upon it. To be precise, for $x \in R$ and $F_*^e y \in F_*^e R$, we have that $x \cdot F_*^e y = F_*^e(x^{p^e} y)$. The bracket powers defined in the previous section can be characterized using this new structure as well. If I is an ideal of R, then $I \cdot F_*^e R = F_*^e I^{[p^e]}$.

A common technique when studying rings of postive characteristic is to study maps, i.e. R-module homomorphisms, into and out of $F_*^e R$. Of particular interest to us will be the p^{-e}-linear maps.

Definition 2.2.1. Let R be a ring of positive characteristic, and let $e \in \mathbb{N}$. We call an R-module homomorphism $\varphi : F_*^e R \to R$ a *p^{-e}-linear map.*

The p^{-e} linear maps on R have certain basic but useful properties that we use in a later section.

Lemma 2.2.2. *Let R be a ring of characteristic $p > 0$, $e \in \mathbb{N}$, $x, y \in R$, I and J be ideals of R, and $\varphi : F_*^e R \to R$ be a p^{-e}-linear map. The following hold.*

1. $x \cdot \varphi(F_*^e y) = \varphi(F_*^e(x^{p^e} y))$.

2. $I \cdot \varphi(F_*^e J) = \varphi(F_*^e(I^{[p^e]} J))$.

3. *If $I \subseteq J^{[p^e]}$, then $\varphi(F_*^e I) \subseteq J$.*

Proof. The first item combines the definition of the R-action on $F_*^e R$ with the condition that φ be an R-module homomorphism:

$$x \cdot \varphi(F_*^e y) = \varphi(x \cdot F_*^e y) = \varphi(F_*^e(x^{p^e} y)).$$

The second item is similar, once we note that the additivity of φ is also guaranteed since it is an R-module homomorphism. If $I = (f_1, \ldots, f_m)$ and $J = (g_1, \ldots, g_n)$, then

$$I \cdot \varphi(F^e_* J) = \sum_{i=1}^{m} f_i \varphi\left(F^e_*\left(\sum_{j=1}^{n} g_i R\right)\right) = \sum_{i=1}^{m} \varphi\left(F^e_*\left(\sum_{j=1}^{n} f_i^{p^e} g_i R\right)\right)$$
$$= \varphi\left(F^e_*\left(\sum_{i=1}^{m}\sum_{j=1}^{n} f_i^{p^e} g_i R\right)\right) = \varphi(F^e_*(I^{[p^e]}J)).$$

The third item is a consequence of the second:

$$\varphi(F^e_* I) \subseteq \varphi(F^e_* J^{[p^e]}) = J \cdot \varphi(F^e_* R) \subseteq J.$$

□

2.3 The F-threshold

An immediate consequence of Lemma 2.1.5 is the fact that for any power q of p, if $I \subseteq \sqrt{J}$, then there exists some n such that $I^n \subseteq J^{[q]}$. Conversely, if $J \subseteq \sqrt{I}$, then there exists some n such that $J^{[q]} \subseteq I^n$. These relationships inspire the definition of two numerical values relating I and J together. The following definition was proposed by Mustaţă, Takagi, and Watanabe, and the existence of the limit was proved by De Stefani, Núñez-Betancourt, and Pérez.

Definition 2.3.1. ([7]) Let I and J be ideals of a ring R with characteristic $p > 0$. For q a power of p, let $\nu_I^J(q) = \sup\left\{n \in \mathbb{N} \mid I^n \not\subseteq J^{[q]}\right\}$. The *$F$-threshold of I with respect to J*, when it exists, is given by $c^J(I) := \lim_{q\to\infty} \dfrac{\nu_I^J(q)}{q}$.

Theorem 2.3.2. *([1]) Suppose I and J are as in Definition 2.3.1. The F-threshold $c^J(I)$ exists.*

For us, the primary application of the F-threshold will be the following observation: if $s > c^J(I)$, then for all q sufficiently large, $I^{\lceil sq\rceil} \subseteq J^{[q]}$.

Example 2.3.3. Let $R = k[x_1, \ldots, x_d]$, let $I = (x_1^{a_1}, \ldots, x_d^{a_d})$, $a_i > 0$ and let $J = (x_1^{b_1}, \ldots, x_d^{b_d})$, $b_i > 0$. For q a power of $p = \mathrm{char}(k)$, $J^{[q]} = (x_1^{qb_1}, \ldots, x_d^{qb_d})$, and for

$n \in \mathbb{N}$, $I^n = (x_1^{n_1 a_1} \cdots x_d^{n_d a_d} \mid n_i \in \mathbb{N}, \sum n_i = n)$. Now $I^n \not\subseteq J^{[q]}$ if and only if there exist n_i with $\sum n_i = n$ and for each i, $n_i a_i < q b_i$, i.e. $n_i \leq \lceil q b_i / a_i \rceil - 1$. Therefore,

$$\nu_I^J(q) = \sum_{i=1}^{d} \left(\left\lceil \frac{q b_i}{a_i} \right\rceil - 1 \right) = \sum_{i=1}^{d} \left\lceil \frac{q b_i}{a_i} \right\rceil - d.$$

Hence,

$$c^J(I) = \lim_{q \to \infty} \frac{\sum_{i=1}^{d} \left\lceil \frac{q b_i}{a_i} \right\rceil - d}{q} = \sum_{i=1}^{d} \frac{b_i}{a_i}$$

Example 2.3.4. Let $R = k[x, y, z, w]/(xy - zw)$, and let $I = (x, y, z, w)$. If q is a power of $p = \operatorname{char}(k)$, then we claim that $I^{2q-1} \subseteq I^{[q]}$. The ideal I^{2q-1} is generated by elements of the form $x^a y^b z^c w^d$, with $a + b + c + d = 2q - 1$. Without loss of generality we can assume that $a \leq b$ and $c \leq d$. Since $a + b + c + d = 2q - 1$, we must have that either $a + d \geq q$ or $b + c \geq q$. If $a + d \geq q$, then

$$x^a y^b z^c w^d = y^{b-a} z^{a+c} w^{a+d} \in I^{[q]}.$$

If $b + c \geq q$, then

$$x^a y^b z^c w^d = x^{a+c} y^{b+c} w^{d-c} \in I^{[q]}.$$

Therefore $I^{2q-1} \subseteq I^{[q]}$.

Now we claim that $x^{q-1} y^{q-1} \notin I^{[q]}$. We can give R an $\mathbb{N}^2$-graded ring structure by setting $\deg x = \deg z = (1, 0)$ and $\deg y = \deg w = (0, 1)$. Under this grading, $I^{[q]}$ is a homogeneous ideal generated by elements of degree $(q, 0)$ and $(0, q)$. Therefore, any element of $I^{[q]}$ is a sum of terms of degree (t, u) with either $t \geq q$ or $u \geq q$. Hence $x^{q-1} y^{q-1}$, which is homogeneous of degree $(q - 1, q - 1)$, is not an element of $I^{[q]}$. Therefore $I^{2q-2} \not\subseteq I^{[q]}$.

We have now shown that $\nu_I^I(q) = 2q - 2$, and therefore $c^I(I) = \lim\limits_{q \to \infty} \dfrac{\nu_I^I(q)}{q} = 2$.

Example 2.3.5. Let $n \geq 1$ and $R = k[x, y, z]/(xy - z^{n+1})$, let $I = (x^5 z, x z^4)$ and $J = (x^3 z^2)$. We claim that $c^J(I) = \max\{2, \frac{3n+5}{n+5}\}$. Let q be a power of p and set

$N = \max\{2q, \lceil \frac{3n+5}{n+5} q \rceil\}$.

Fix a generator of I^N, which is of the form $x^{5i+(N-i)} z^{i+4(N-i)} = x^{4i+N} z^{4N-3i}$ for some $0 \le i \le N$. Since $N \ge 2q$, we have that $4N - 3i \ge N \ge 2q$. Set $j \in \mathbb{N}$ the greatest integer such that $4N - 3i - (n+1)j \ge 2q$. In this case we have that

$$\begin{aligned} 4i + N + j &> 4i + N + \frac{4N - 3i - 2q}{n+1} - 1 \\ &= \frac{4i(n+1) + (n+5)N - 3i - 2q - (n+1)}{n+1} \\ &\ge \frac{4i(n+1) - 3i + (3n+5)q - 2q - (n+1)}{n+1} \\ &= \frac{i(4n+1) + 3(n+1)q - (n+1)}{n+1} \\ &\ge 3q - 1. \end{aligned}$$

Since $4i + N + j$ is an integer strictly greater than $3q - 1$, it is at least $3q$. Therefore, we have that $x^{4i+N} z^{4N-3i} = x^{4i+N+j} y^j z^{4N-3i-(n+1)j} \in (x^{3q} z^{2q}) = J^{[q]}$. Hence $I^N \subseteq (x^{3q} z^{2q}) = J^{[q]}$, and so $\nu_I^J(q) \le N - 1$.

Now consider the ideal I^{N-1}. We have that either $N = 2q$ or $N = \lceil \frac{3n+5}{n+5} q \rceil$. Suppose $N = 2q$. We may give R an $\mathbb{N}$-graded ring structure by setting $\deg x = 0$, $\deg y = n+1$, and $\deg z = 1$. Given this grading, $J^{[q]} = (x^{3q} z^{2q})$ is generated by an element of degree $2q$. However, the ideal I^{N-1} contains $x^{5(N-1)} z^{N-1}$, an element of degree $2q - 1$. Therefore $I^{N-1} \not\subseteq J^{[q]}$.

Now suppose that $N = \lceil \frac{3n+5}{n+5} q \rceil$. we may give R an $\mathbb{N}$-graded ring structure by setting $\deg x = n+1$, $\deg y = 0$, and $\deg z = 1$. Given this grading, $J^{[q]} = (x^{3q} z^{2q})$ is generated by an element of degree $3q(n+1) + 2q = (3n+5)q$. However, the ideal I^{N-1} contains $x^{N-1} z^{4(N-1)}$, an element of degree

$$(n+5)(N-1) = (n+5)\left(\left\lceil \frac{3n+5}{n+5} q \right\rceil - 1\right) < (n+5) \cdot \frac{3n+5}{n+5} q = (3n+5)q.$$

Therefore $I^{N-1} \not\subseteq J^{[q]}$.

Hence we have shown that $\nu_I^J(q) = \max\{2q, \lceil \frac{3n+5}{n+5} q \rceil\} - 1$, and therefore $c^J(I) = \max\{2, \lceil \frac{3n+5}{n+5} \rceil\}$.

Some basic properties of F-thresholds are summarized in the following result.

Proposition 2.3.6. *Let I, I', J, J' be ideals of a ring R of characteristic $p > 0$. The following hold.*

(i) If $I \not\subseteq \sqrt{J}$ then $c^J(I) = \infty$, and if $J = R$ then $c^J(I) = -\infty$.

(ii) If $I \subseteq \sqrt{J} \neq R$ then $0 \leq c^J(I) < \infty$.

(iii) If $I \subseteq I'$ and $J \supseteq J'$, then $c^J(I) \leq c^{J'}(I')$.

(iv) If S is an R-algebra, then $c^{JS}(IS) \leq c^J(I)$. If futhermore S is faithfully flat over R, then equality holds.

(v) If W is a multiplicative system of R such that $W \cap \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R/J^{[q]})} \mathfrak{p} = \emptyset$ for infinitely many powers q of p, then $c^J(I) = c^{W^{-1}J}(W^{-1}I)$.

(vi) If I' is a reduction of I, then $c^J(I) = c^J(I')$.

(vii) If $I^n \subseteq J^{[q]}$ and I has a reduction generated by m elements, then $c^J(I) \leq \frac{mn}{q}$.

(viii) If $(R, \mathfrak{m})$ is a local ring and $I^n \subseteq J^{[q]}$, then $c^J(I) \leq \frac{n \cdot \dim R}{q}$.

Proof. Throughout, let q stand for a power of p.

For part (i), note that if $I \not\subseteq \sqrt{J}$, then $\nu_I^J(q) = \infty$ for all q and so $c_J(I) = \infty$. If $J = R$ then $\nu_I^J(q) = -\infty$ for all q and so $c^J(I) = -\infty$.

For part (ii), for all q, $0 \leq \nu_I^J(q)$ and so $c^J(I) \geq 0$. The proof of the existence of $c^J(I)$ in [1, Theorem 3.4] shows also that $c^J(I) < \infty$ in this case.

For part (iii), the inequality $\nu_I^J(q) \leq \nu_{I'}^{J'}(q)$ is immediate for all q from the definition, and so $c^J(I) \leq c^{J'}(I')$

For part (iv), note that if $I^n \subseteq J^{[q]}$, then $(IS)^n = I^n S \subseteq J^{[q]}S \subseteq (JS)^{[q]}$, so $\nu_{IS}^{JS}(q) \leq \nu_I^J(q)$ for all q. Suppose S is faithfully flat over R and $(IS)^n \subseteq (JS)^{[q]}$. By the flatness of S we have that

$$0 = \frac{(IS)^n + (JS)^{[q]}}{(JS)^{[q]}} \cong \frac{I^n + J^{[q]}}{J^{[q]}} \otimes_R S$$

and by the faithful flatness of S we have that $(I^n + J^{[q]})/J^{[q]} = 0$, hence $I^n \subseteq J^{[q]}$. Thus $\nu_I^J(q) \leq \nu_{IS}^{JS}(q)$ for all q.

For part (v), let q be such that $W \cap \bigcup_{\mathfrak{p} \in \operatorname{Ass}(R/J^{[q]})} \mathfrak{p} = \emptyset$. If $(W^{-1}I)^n \subseteq (W^{-1}J)^{[q]}$, then there exists $w \in W$ such that $wI^n \subseteq J^{[q]}$. Since w is not a zerodivisor on $R/J^{[q]}$, $I^n \subseteq J^{[q]}$. Hence $\nu_I^J(q) \leq \nu_{W^{-1}I}^{W^{-1}J}(q)$, and so $c^J(I) \leq c^{W^{-1}J}(W^{-1}I)$. Part (iv) gives the other inequality.

For part (vi), let w be the reduction number of I with respect to I'. We have that $I^{\nu_{I'}^J(q)+1+w} = I^w (I')^{\nu_{I'}^J(q)+1} \subseteq J^{[q]}$. Combining this with part (iii), we have that $\nu_{I'}^J(q) \leq \nu_I^J(q) \leq \nu_{I'}^J(q) + w$. Dividing all terms by q and taking the limit as q goes to infinity gives the result.

For part (vii), first note that by part (vi) we may assume that I is generated by m elements. Second, by Lemma 2.1.5, for any power q' of p we have that $I^{m(q'-1)+1} \subseteq I^{[q']}$, and then by Lemma 2.1.6, we have that

$$I^{n(m(q'-1)+1)} = \left(I^{m(q'-1)+1}\right)^n \subseteq \left(I^{[q']}\right)^n = (I^n)^{[q']} \subseteq J^{[qq']}.$$

Therefore $\nu_I^J(qq') \leq n(m(q'-1)+1) - 1$.

For part (viii), the extension $R \subseteq S = R[X]_{\mathfrak{m}R[X]}$ is faithfully flat. Since S has infinite residue field, IS has a reduction generated by at most $\dim S = \dim R$ elements. Therefore, the conclusion follows by parts (iv) and (vii). □

2.4 The F-limbus

A sort of dual notion to the F-threshold can be obtained by reversing the noncontainment condition in Theorem 2.3.2. So far, this limit does not appear in the literature, and so its existence and basic properties will be proved below. In particular, the proof if its existence is very similar to the proof of the existence of the F-threshold in [1].

Definition 2.4.1. Let I and J be ideals of a ring R with characteristic $p > 0$. For q a power of p, let $\mu_I^J(q) = \inf\left\{n \in \mathbb{N} \mid J^{[q]} \not\subseteq I^n\right\}$. The *$F$-limbus of I with respect to J*, when it exists, is given by $b^J(I) := \lim\limits_{q\to\infty} \dfrac{\mu_I^J(q)}{q}$.

Theorem 2.4.2. *Suppose I and J are as in Definition 2.4.1. The F-limbus $b^J(I)$ exists.*

Proof. If $J \not\subseteq \sqrt{I}$, then $\mu_J^I(q) = 1$ for all q and so $b^J(I) = 0$. If $I = R$, then $\mu_I^J(q) = \infty$ for all q, and so $b^J(I) = \infty$. Suppose that $J \subseteq \sqrt{I} \neq R$. Let q and q' be powers of p. We have that $J^{[qq']} = \left(J^{[q']}\right)^{[q]} \subseteq \left(I^{\mu_I^J(q')-1}\right)^{[q]} \subseteq I^{q\mu_I^J(q')-q}$, and so $\mu_I^J(qq') > q\mu_I^J(q') - q$. Therefore,

$$\liminf_{q\to\infty} \frac{\mu_I^J(q)}{q} = \liminf_{q\to\infty} \frac{\mu_I^J(qq')}{qq'} \geq \liminf_{q\to\infty} \frac{\mu_I^J(q')-1}{q'} = \frac{\mu_I^J(q')-1}{q'}$$

Hence $\liminf\limits_{q\to\infty} \dfrac{\mu_I^J(q)}{q} \geq \limsup\limits_{q'\to\infty} \dfrac{\mu_I^J(q')-1}{q'} = \limsup\limits_{q'\to\infty} \dfrac{\mu_I^J(q')}{q'}$ and so the limit defining $b^J(I)$ exists. ∎

Example 2.4.3. Let $R = k[x_1, \ldots, x_d]$, let $I = (x_1^{a_1}, \ldots, x_d^{a_d})$, $a_i > 0$ and let $J = (x_1^{b_1}, \ldots, x_d^{b_d})$, $b_i > 0$. For q a power of $p = \mathrm{char}(k)$, $J = (x_1^{qb_1}, \ldots, x_d^{qb_d})$, and for $n \in \mathbb{N}$, $I^n = (x_1^{n_1a_1} \cdots x_d^{n_da_d} \mid n_i \in \mathbb{N}, \sum n_i = n)$. For any $1 \leq j \leq d$, $x_j^{qb_j} \in I^n$ if and only if $qb_j \geq na_j$, that is, $n \leq \frac{qb_j}{a_j}$. Hence $J^{[q]} \subseteq I^n$ if and only if $n \leq \min_{1\leq j\leq d}\{\frac{qb_j}{a_j}\}$, and so $\mu_I^J(q) = \lfloor \min_{1\leq j\leq d}\{\frac{qb_j}{a_j}\}\rfloor + 1$. Therefore, $b^J(I) = \min_{1\leq j\leq d}\{\frac{b_j}{a_j}\}$.

Example 2.4.4. Let $R = k[x,y,z,w]/(xy - zw)$, and let $I = (x,y,z,w)$. We always have that $I^{[q]} \subseteq I^q$, so $\mu_I^I(q) \geq q+1$. Giving R the standard grading, i.e. letting $\deg x = \deg y = \deg z = \deg w = 1$, the ideal I^{q+1} is homogeneous of degree $q+1$. Since $I^{[q]}$

is generated by homogeneous elements of degree q, it cannot be contained in I^{q+1}, and therefore $\mu_I^I(q) = q+1$. Hence $b^I(I) = 1$.

Example 2.4.5. Let $n \geq 1$ and $R = k[x,y,z]/(xy - z^{n+1})$, let $I = (x^5 z, xz^4)$ and $J = (x^3 z^2)$. We claim that $b^J(I) = \frac{17}{19}$. Let q be a power of p, and let $N \in \mathbb{N}$. We have that

$$(x^5 z)^{\lfloor \frac{10}{19} q \rfloor} (xz^4)^{\lfloor \frac{7}{19} q \rfloor} = x^{5\lfloor \frac{10}{19} q \rfloor + \lfloor \frac{7}{19} q \rfloor} z^{\lfloor \frac{10}{19} q \rfloor + 4\lfloor \frac{7}{19} q \rfloor} \in I^{\lfloor \frac{10}{19} q \rfloor + \lfloor \frac{7}{19} q \rfloor}.$$

Notice that $5\lfloor \frac{10}{19} q \rfloor + \lfloor \frac{7}{19} q \rfloor \leq 5 \cdot \frac{10}{19} q + \frac{7}{19} q = 3q$ and $\lfloor \frac{10}{19} q \rfloor + 4\lfloor \frac{7}{19} q \rfloor \leq \frac{10}{19} q + 4 \cdot \frac{7}{19} q = 2q$. Therefore,

$$J^{[q]} = (x^{3q} z^{2q}) \subseteq (x^{5\lfloor \frac{10}{19} q \rfloor + \lfloor \frac{7}{19} q \rfloor} z^{\lfloor \frac{10}{19} q \rfloor + 4\lfloor \frac{7}{19} q \rfloor}) \subseteq I^{\lfloor \frac{10}{19} q \rfloor + \lfloor \frac{7}{19} q \rfloor},$$

and hence $\mu_J^I(q) \geq \lfloor \frac{10}{19} q \rfloor + \lfloor \frac{7}{19} q \rfloor + 1$.

Now let us give R the structure of an $\mathbb{N}$-graded ring by setting $\deg x = 3$, $\deg y = 4n+1$, and $\deg z = 4$. With this grading, I is a homogeneous ideal of degree 19 and J is a homogeneous ideal of degree 17. Therefore, $J^{[q]}$ is an ideal of degree $17q$, and hence is not contained in $I^{\lfloor \frac{17}{19} q \rfloor + 1}$, which is an ideal of degree greater than $17q$. Therefore $\mu_J^I(q) \leq \lfloor \frac{17}{19} q \rfloor + 1$.

Since

$$\frac{17}{19} = \lim_{q \to \infty} \frac{1}{q} \left(\left\lfloor \frac{10}{19} q \right\rfloor + \left\lfloor \frac{7}{19} q \right\rfloor + 1 \right) \leq \lim_{q \to \infty} \frac{\mu_J^I(q)}{q} \leq \lim_{q \to \infty} \frac{1}{q} \left(\left\lfloor \frac{17}{19} q \right\rfloor + 1 \right) = \frac{17}{19},$$

we conclude that $b^J(I) = \frac{17}{19}$.

The F-limbus behaves similarly to the F-threshold, and so we can prove many analogous statements. Again, the most important property we use, which is implicit in the definition, is that if $s < b^J(I)$, then for all sufficiently large q, $J^{[q]} \subseteq I^{\lceil sq \rceil}$.

Proposition 2.4.6. *Let I, I', J, J' be ideals of a ring R of characteristic $p > 0$. The following hold.*

(i) If $J \not\subseteq \sqrt{I}$, then $b^J(I) = 0$, and if $I = R$ then $b^J(I) = \infty$.

(ii) If $I \subseteq I'$ and $J \supseteq J'$, then $b^J(I) \leq b^{J'}(I')$.

(iii) If S is an R-algebra, then $b^{JS}(IS) \geq b^J(I)$. If furthermore S is faithfully flat over R, then equality holds.

(iv) If W is a multiplicative system of R such that $W \cap \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R/I^n)} \mathfrak{p} = \emptyset$ for all $n \in \mathbb{N}$, then $b^J(I) = b^{W^{-1}J}(W^{-1}I)$.

(v) If I' is a reduction of I, then $b^J(I) = b^J(I')$.

(vi) If $J^{[q]} \subseteq I^n$, then $b^J(I) \geq \frac{n}{q}$.

Proof. Throught the proof, let q stand for a power of p.

Part (i) is just the first two lines of the proof of Theorem 2.4.2.

For part (ii), the inequality $\mu_I^J(q) \leq \mu_{I'}^{J'}(q)$ is immediate for all q from the definition, and so $b^J(I) \leq b^{J'}(I')$.

For part (iii), note that if $J^{[q]} \subseteq I^n$, then $(JS)^{[q]} = J^{[q]}S \subseteq I^n S = (IS)^n$, so $\mu_{IS}^{JS}(q) \geq \mu_I^J(q)$ for all q. Suppose S is faithfully flat over R and $(JS)^{[q]} \subseteq (IS)^n$. By the flatness of S we have that

$$0 = \frac{(JS)^{[q]} + (IS)^n}{(IS)^n} \cong \frac{J^{[q]} + I^n}{I^n} \otimes_R S,$$

and by the faithful flatness of S we have that $(J^{[q]} + I^n)/I^n = 0$, hence $J^{[q]} \subseteq I^n$. Thus $\mu_I^J(q) \geq \mu_{IS}^{JS}(q)$ for all q.

For part (iv), if $(W^{-1}J)^{[q]} \subseteq (W^{-1}I)^n$, then there exists $w \in W$ such that $wJ^{[q]} \subseteq I^n$. Since w is not a zerodivisor on R/I^n, $J^{[q]} \subseteq I^n$. Hence $\mu_I^J(q) \geq \mu_{W^{-1}I}^{W^{-1}J}(q)$, and so $b^J(I) \geq b^{W^{-1}J}(W^{-1}I)$. Part (iii) gives us the other inequality.

For part (v), let w be the reduction number of I with respect to I'. We have that $J^{[q]} \subseteq I^{\mu_I^J(q)-1} \subseteq (I')^{\mu_I^J(q)-1-w}$. Combining this with part (ii), we have that $\mu_I^J(q) - w \leq \mu_{I'}^J(q) \leq \mu_I^J(q)$. Dividing all terms by q and taking the limit as q goes to infinity gives the result.

For part (vi), suppose $q' \geq q$. We have then that $J^{[q']} = (J^{[q]})^{[q'/q]} \subseteq (I^n)^{[q'/q]} \subseteq I^{nq'/q}$, and therefore $\mu_I^J(q') > \frac{nq'}{q}$. Therefore, $b^J(I) = \lim_{q' \to \infty} \frac{\mu_I^J(q')}{q'} \geq \frac{n}{q}$. $\square$

The F-threshold and the F-limbus are related to each other under certain conditions in the following way, assuming both of them are positive and finite.

Lemma 2.4.7. *Let I and J be ideals of a ring R with characteristic $p > 0$. If $\sqrt{I} = \sqrt{J} \neq \sqrt{0}$ and I is in the Jacobson radical of R, then $b^J(I) \leq c^J(I)$.*

Proof. For q a power of p, we have that $I^{\nu_I^J(q)+1} \subseteq J^{[q]} \subseteq I^{\mu_I^J(q)-1}$. If $\mu_I^J(q) - 1 \geq \nu_I^J(q) + 1$, then we have equality throughout and by Nakayama's Lemma, $\mu_I^J(q) - 1 = \nu_I^J(q) + 1$. Therefore, we have that $\mu_I^J(q) - 1 \leq \nu_I^J(q) + 1$, and so dividing both sides of the inequality by q and taking the limit as q goes to infinity, we prove the statement. $\square$

The F-threshold and the F-limbus measure which powers of one ideal contain, or are contained in, the bracket powers of the other, at least asymptotically. Another way of testing for whether two ideals are contained in one another is suggested by the proofs of Propositions 2.3.6(iv) and 2.4.6(iii). To be precise, the condition $I \subseteq J$ is equivalent to the condition $I + J = J$. Thus, if $s > c^J(I)$, then $I^{\lceil sq \rceil} + J^{[q]} = J^{[q]}$ for all large q, and for $s < b^J(I)$, we have that $I^{\lceil sq \rceil} + J^{[q]} = I^{\lceil sq \rceil}$. Values of s in between the two values will give ideals that seem to interpolate between I^q and $J^{[q]}$. In the next section we will exploit this relationship to compare two more numerical measures attached to ideals: the Hilbert-Samuel and the Hilbert-Kunz multiplicities.

3 Multiplicities

In this section, we discuss two measures associated to an ideal of a commutative ring, and then construct a framework that incorporates both of them. Throughout this and later sections, if R is a ring and M is a module of R, by $\lambda_R(M)$ we mean the length of the M as an R-module. When the ring R is understood we may write $\lambda(M)$ for $\lambda_R(M)$.

3.1 The Hilbert-Samuel and Hilbert-Kunz Multiplicities

Hilbert-Samuel multiplicity is a numerical measure associated to an ideal I in a commutative ring R (of any characteristic) and an R-module M. There are multiple equivalent definitions of Hilbert-Samuel multiplicity, and we will use the following.

Definition 3.1.1. Let $(R, \mathfrak{m})$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal of R, and M a finitely generated R-module. The *Hilbert-Samuel multiplicity of M with respect to I* is defined to be

$$e(I; M) = \lim_{n\to\infty} \frac{d! \cdot \lambda(M/I^n M)}{n^d}.$$

We often write $e(I)$ for $e(I; R)$.

Many properties of the Hilbert-Samuel multiplicity are well known. The properties most important to us are the following:

- If I and J are ideals that have the same integral closure, then $e(I) = e(J)$.
- If $I \subseteq J$ and R is formally equidimensional, then the converse to the previous item holds [10].
- The Hilbert-Samuel multiplicity is always a positive integer.
- If $(R, \mathfrak{m})$ is regular, then $e(\mathfrak{m}) = 1$.

- If R is formally equidimensional the converse to the previous item holds [8, Theorem 40.6].

When the ring R is of positive characteristic, we may construct a similar limit using bracket powers instead of ordinary powers. Doing so gives us the Hilbert-Kunz multiplicity.

Definition 3.1.2. Let $(R, \mathfrak{m})$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal of R, and M a finitely generated R-module. The *Hilbert-Kunz multiplicity of M with respect to I* is defined to be

$$e_{HK}(I; M) = \lim_{e \to \infty} \frac{\lambda\big(M/I^{[p^e]}M\big)}{p^{ed}}.$$

We often write $e_{HK}(I)$ for $e_{HK}(I; R)$.

The Hilbert-Kunz multiplicity has some properties similar to the Hilbert-Samuel multiplicity.

- If I and J are ideals that have the same tight closure, then $e_{HK}(I) = e_{HK}(J)$.

- If $I \subseteq J$ and R is complete and equidimensional then the converse to the previous item holds [4, Theorem 8.17].

- The Hilbert-Kunz multiplicity is a real number at least 1, though unlike the Hilbert-Samuel multiplicity it need not be an integer.

- However, like the Hilbert-Samuel multiplicity, if $(R, \mathfrak{m})$ is regular, then $e_{HK}(\mathfrak{m}) = 1$.

- If R is unmixed then the converse to the previous statement holds [13, Theorem 1.5].

Out first goal is to define a function that behaves like the two multiplicities given here and interpolates between them. Our first task in that direction will be to find a general strategy for calculating, or at least bounding, the length of the R-modules in question. Since all the rings we consider contain a field, this amounts to calculating a vector space dimension.

3.2 Vector Space Generators of Finite Length Modules

In this section, we construct sets of generators for certain R-modules as vector spaces over the residue field of R. By counting these generating sets, we get upper bounds on the dimensions of the vector spaces. We begin with a technical lemma.

Lemma 3.2.1. *Let $(R, \mathfrak{m}, k)$ be a local ring containing its residue field, and let M be an R-module of finite length. Let $\{x_1, \dots, x_t\}$ be a set of generators for $\mathfrak{m}$ and $\{m_1, \dots, m_n\}$ a set of generators for M. In this case,*

(i) M is generated as a k-vector space by elements of the form $x_1^{b_1} \cdots x_t^{b_t} m_j$, where $b_1, \dots, b_t \in \mathbb{N}$ and $1 \leq j \leq n$; and

(ii) If $I = (f_1, \dots, f_m)$ is an $\mathfrak{m}$-primary ideal of R then M is generated as a k-vector space by elements of the form $f_1^{a_1} \cdots f_m^{a_m} g m_j$, where $a_1, \dots, a_m \in \mathbb{N}$, $1 \leq j \leq n$, and g is a generator of R/I as a k-vector space.

Proof. (i) By definition, M is generated as a k-vector space by elements of the form rm_j with $r \in R$ and $1 \leq j \leq n$. For each such r, we have that $r = v + \sum_{i=1}^{t} r_i x_i$ for some $v \in k$ and $r_i \in R$, since $R = k \oplus \mathfrak{m}$ as a k-vector space. For each i, we may write $r_i = v_i + \sum_{j=1}^{n} r_{ij} x_j$ with $v_i \in k$ and $r_{ij} \in R$, and so

$$r = v + \sum_{i=1}^{t} v_i x_i + \sum_{1 \leq i,j \leq t} r_{ij} x_i x_j.$$

We may repeat this process until every term either has a coefficient of the x_i's which is an element of k or has a degree in the x_i's large enough that the term annihilates M and so may be removed.

(ii) By part (i), M is generated as a k-vector space by terms of the form $f_1^{a_1} \cdots f_m^{a_m} x_1^{b_1} \cdots x_t^{b_t} m_j$ with $a_i, b_i \in \mathbb{N}$. Fix a set of k-vector space generators $\{g_i\}$ of R/I. Suppose that we have an element $\alpha = f_1^{a_1} \cdots f_m^{a_m} x_1^{b_1} \cdots x_t^{b_t} m_j \in M$ with $x_1^{b_1} \cdots x_t^{b_t} \notin \{g_i\}$.

There exist $c_i \in k$ such that $x_1^{b_1} \cdots x_t^{b_t} - \sum_i c_i g_i \in I$, and so there exist $r_1, \cdots, r_m \in R$ such that $x_1^{b_1} \cdots x_t^{b_t} - \sum_i c_i g_i = \sum_{\ell=1}^m r_\ell f_\ell$. Therefore,

$$\alpha = \sum_i c_i f_1^{a_1} \cdots f_m^{a_m} g_i m_j + \sum_{\ell=1}^{m} f_1^{a_1} \cdots f_\ell^{a_\ell+1} \cdots f_m^{a_m} r_\ell m_j.$$

We know by part (i) that $r_\ell m_j$ is a k-linear combination of terms of the form $x_1^{b'_1} \cdots x_t^{b'_t} m_{j'}$, and so we have that α is a k-linear combination of terms of the form $f_1^{a_1} \cdots f_m^{a_m} g_i m_j$ and $f_1^{a'_1} \cdots f_m^{a'_m} x_1^{b'_1} \cdots x_t^{b'_t} m_{j'}$ with $\sum_\ell a'_\ell = 1 + \sum_\ell a_\ell$. Continuing in this way, we may write α as a k-linear combination of terms either of the form $f_1^{a_1} \cdots f_m^{a_m} g_i m_j$ for some i or of the form $f_1^{a_1} \cdots f_m^{a_m} x_1^{b_1} \cdots x_t^{b_t} m_j$ with $\sum_i a_i$ arbitrarily large. Since I^n annihilates M for some n, we may throw out all the terms of the second kind, which finishes the proof. □

Bounding the lengths of the ideals we are concerned with will involve some combinatorial calculations. For convenience we introduce some notation.

Definition 3.2.2. For positive integers d and m and real number r, we set $S_d^m(r)$ to be the number of monomials in d variables with degree less than r and with degree in each variable less than m.

Certain properties of the numbers $S_d^m(r)$ are easy to see. First, if $r \geq 0$, then $S_1^m(r) = \min\{m, \lceil r \rceil\}$. Second, for $d > 1$, we have that $S_d^m(r) = \sum_{i=0}^{m-1} S_{d-1}^m(r-i)$. Indeed, if we denote one of the variables by x, then for $i = 0, 1, \ldots, m-1$, there are $S_{d-1}^m(r-i)$ monomials with degree in x exactly i, degree less than r, and degree in each variable less than m.

We occasionally use a combinatorial description of the numbers $S_m^d(r)$, which is established in the following lemma. This result appeared in a more general form as [11, Lemma 2.5], though the method of proof was different.

Lemma 3.2.3. *For positive integers d and m and real number r,*

$$S_d^m(r) = \sum_{i=0}^{d}(-1)^i \binom{d}{i}\binom{\lceil r\rceil - im - 1 + d}{d}.$$

Proof. The number of monomials in d variables, of degree less than r, where each of a given set of i variables has degree at least m is the number of monomials in d variables of degree less than $r - im$, that is, $\binom{\lceil r\rceil - im-1+d}{d}$. Thus the total number of monomials in d variables of degree less than r with degree in each variable less than m is

$$\binom{\lceil r\rceil - 1 + d}{d} - \sum_{i=1}^{d}(-1)^{i-1}\binom{d}{i}\binom{\lceil r\rceil - im - 1 - d}{d},$$

by the inclusion-exclusion principle. □

Our next lemma is a technical result on the behavior of the numbers $S_d^m(r)$ as m and r grow.

Lemma 3.2.4. *If $f, g : \mathbb{N} \to \mathbb{R}$ are functions such that $f(n) - g(n) \leq cn + o(n)$ for some $c \in \mathbb{R}$, $f(n) \geq g(n)$ for $n \gg 0$, and u is a positive integer, then*

$$\limsup_{n\to\infty} \frac{S_d^{un}(f(n)) - S_d^{un}(g(n))}{n^d} \leq u^{d-1}c.$$

Proof. We proceed by induction on d. Suppose $d = 1$, and let $n \in \mathbb{N}$ large enough that $f(n) \geq g(n)$. If $un \leq g(n)$ we have that $S_1^{un}(f(n)) - S_1^{un}(g(n)) = 0$, and if $un > g(n)$ then

$$S_1^{un}(f(n)) - S_1^{un}(g(n)) \leq \lceil f(n)\rceil - \lceil g(n)\rceil \leq f(n) - g(n) + 1.$$

Therefore

$$\limsup_{n\to\infty} \frac{S_1^{un}(f(n)) - S_1^{un}(g(n))}{n} \leq \limsup_{n\to\infty} \frac{f(n) - g(n) + 1}{n} \leq c.$$

Now if $d > 1$,

$$\begin{aligned} S_d^{un}(f(n)) - S_d^{un}(g(n)) &= \sum_{i=0}^{un-1} \left(S_{d-1}^{un}(f(n)-i) - S_{d-1}^{un}(g(n)-i)\right) \\ &\leq un \left(S_{d-1}^{un}(f(n)-i_n) - S_{d-1}^{un}(g(n)-i_n)\right) \end{aligned}$$

where i_n is the value of i with $1 \leq i \leq un-1$ that maximizes the expression $S_{d-1}^{un}(f(n)-i) - S_{d-1}^{un}(g(n)-i)$. By induction,

$$\begin{aligned} \limsup_{n\to\infty} \frac{S_d^{un}(f(n)) - S_d^{un}(g(n))}{n^d} \leq& \limsup_{n\to\infty} \frac{un\left(S_{d-1}^{un}(f(n)-i_n) - S_{d-1}^{un}(g(n)-i_n)\right)}{n^d} \\ =& u \cdot \limsup_{n\to\infty} \frac{S_{d-1}^{un}(f(n)-i_n) - S_{d-1}^{un}(g(n)-i_n)}{n^{d-1}} \\ \leq& u \cdot u^{d-2}c = u^{d-1}c. \end{aligned}$$

□

3.3 The Multiplicity-Like Function $h_s(I, J; M)$

We are ready to consider a limit which combines aspects of the limits defining the Hilbert-Samuel and Hilbert-Kunz multiplicities. The idea is to take the colengths of a sum of ideals, one of which corresponds to the increasing Frobenius powers of an ideal J, and one of which corresponds to a subsequence of the powers of another ideal I. This subsequence will be determined by a real number s. We require that both of these ideals be primary to the maximal ideal of the ring they belong to so that at the extreme values of the parameter s one of the two ideals will dominate the other. This guarantees that in the extremal cases we will get a limit related to either the Hilbert-Samuel multiplicity of I or the Hilbert-Kunz multiplicity of J.

Theorem 3.3.1. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and characteristic $p > 0$, I and J be $\mathfrak{m}$-primary ideals of R, M be a finitely generated R-module, and $s > 0$. The following limit exists.*

$$\lim_{e\to\infty} \frac{\lambda\big(M/(I^{\lceil sp^e \rceil} + J^{[p^e]})M\big)}{p^{ed}}$$

Proof. If $d = 0$, then for large enough e, $I^{\lceil sp^e \rceil} + J^{[p^e]} = 0$ and so the limit is simply $\lambda(R)$. Suppose that $d \geq 1$. If k is not infinite, we may replace R by $S = R[X]_{\mathfrak{m}R[X]}$. For any R-module N, we have $\lambda_R(N) = \lambda_S(N \otimes_R S)$, and so we may assume without loss of generality that the ring R has infinite residue field. Futhermore, since completion is a faithfully flat operation, we may assume R is complete and hence contains its residue field. Let K be a reduction of I generated by d elements $f_1, \ldots, f_d \in R$, and let w be the reduction number of I with respect to K, $x_1, \ldots, x_t \in R$ be a set of generators for the maximal ideal $\mathfrak{m}$, and $m_1, \ldots, m_n \in M$ be a set of generators of M. Let q, q' be varying powers of p.

If $q' > \frac{w+d}{s}$, then for sufficiently large q we have that

$$\left(K^{\lceil sq' \rceil} + J^{[q']}\right)^{[q]} \subseteq \left(I^{\lceil sq' \rceil} + J^{[q']}\right)^{[q]} \subseteq I^{\lceil sq'q \rceil} + J^{[q'q]} \subseteq K^{\lceil sq'q \rceil - w} + J^{[q'q]} \subseteq \left(K^{\lceil sq' \rceil - d - 1} + J^{[q']}\right)^{[q]}.$$

Therefore,

$$\lambda\left(\frac{M}{\left(K^{\lceil sq' \rceil - d - 1} + J^{[q']}\right)^{[q]} M}\right) \leq \lambda\left(\frac{M}{\left(I^{\lceil sq'q \rceil} + J^{[q'q]}\right) M}\right) \leq \lambda\left(\frac{M}{\left(K^{\lceil sq' \rceil} + J^{[q']}\right)^{[q]} M}\right).$$

If we divide the first and last terms of this inequality by q^d, then the limit as $q \to \infty$ exists by [6, Theorem 1.8]. Hence

$$\begin{aligned}
&\limsup_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{M}{\left(I^{\lceil sq'q \rceil} + J^{[q'q]}\right) M}\right) - \liminf_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{M}{\left(I^{\lceil sq'q \rceil} + J^{[q'q]}\right) M}\right) \\
&\leq \lim_{q\to\infty} \frac{1}{q^d} \left(\lambda\left(\frac{M}{\left(K^{\lceil sq' \rceil} + J^{[q']}\right)^{[q]} M}\right) - \lambda\left(\frac{M}{\left(K^{\lceil sq' \rceil - d - 1} + J^{[q']}\right)^{[q]} M}\right)\right) \\
&= \lim_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{\left(K^{\lceil sq' \rceil - d - 1} + J^{[q']}\right)^{[q]} M}{\left(K^{\lceil sq' \rceil} + J^{[q']}\right)^{[q]} M}\right).
\end{aligned}$$

Let

$$Q = \frac{\left(K^{\lceil sq' \rceil - d - 1} + J^{[q']}\right)^{[q]} M}{\left(K^{\lceil sq' \rceil} + J^{[q']}\right)^{[q]} M} \cong \frac{\left(K^{[q]}\right)^{\lceil sq' \rceil - d - 1} M}{\left(\left(K^{[q]}\right)^{\lceil sq' \rceil} + J^{[q'q]}\right) M \cap \left(K^{[q]}\right)^{\lceil sq' \rceil - d - 1} M}.$$

As an R-module, Q is generated by elements of the form $f_1^{y_1 q} \cdots f_d^{y_d q} m_\alpha$, where $\sum_i y_i = \lceil sq' \rceil - d - 1$ and $1 \le \alpha \le n$. Therefore, by Lemma 3.2.1, Q can be generated as a k-vector space by elements of the form $f_1^{y_1 q + z_1} \cdots f_d^{y_d q + z_d} g m_\alpha$ where $b_i, y_i, z_i \in \mathbb{N}$, $\sum_i y_i = \lceil sq' \rceil - d - 1$, and g is a k-vector space generator of R/K. Letting $c_i = y_i + \lfloor z_i/q \rfloor$ and $a_i = z_i - q\lfloor z_i/q \rfloor$, we have that $c_i q + a_i = y_i q + z_i$ and $a_i < q$, and so Q can be generated as a k-vector space by elements of the form $f_1^{c_1 q + a_1} \cdots f_d^{c_d q + a_d} g m_\alpha$ where $a_i, b_i, c_i \in \mathbb{N}$, $a_i < q$, $\sum_i c_i \ge \lceil sq' \rceil - d - 1$, g is a k-vector space generator of R/K, and $1 \le \alpha \le n$. Let $v \in \mathbb{N}$ such that $K^v \subseteq J$. If $\sum_i c_i \ge sq'$ or $c_i \ge vq'$ for some i, then the product above vanishes in Q. Therefore

$$\lambda(Q) \le q^d \cdot \Big(S_d^{vq'}(sq') - S_d^{vq'}(sq' - d - 1) \Big) \cdot \lambda(R/K) \cdot n.$$

From this we have that

$$\begin{aligned}
&\limsup_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{M}{(I^{\lceil sq \rceil} + J^{[q]})\, M} \right) - \liminf_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{M}{(I^{\lceil sq \rceil} + J^{[q]})\, M} \right) \\
=& \limsup_{q\to\infty} \frac{1}{(q'q)^d} \lambda\left(\frac{M}{(I^{\lceil sq'q \rceil} + J^{[q'q]})\, M} \right) - \liminf_{q\to\infty} \frac{1}{(q'q)^d} \lambda\left(\frac{M}{(I^{\lceil sq'q \rceil} + J^{[q'q]})\, M} \right) \\
\le& \lim_{q\to\infty} \frac{q^d \cdot \Big(S_d^{vq'}(sq') - S_d^{vq'}(sq' - d - 1) \Big) \cdot \lambda(R/K) \cdot n}{(q'q)^d} \\
=& \frac{\Big(S_d^{vq'}(sq') - S_d^{vq'}(sq' - d - 1) \Big) \cdot \lambda(R/K) \cdot n}{(q')^d}.
\end{aligned}$$

Since this holds for all $q' \gg 0$, and by Lemma 3.2.4,

$$\begin{aligned}
&\limsup_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{M}{(I^{\lceil sq \rceil} + J^{[q]})\, M} \right) - \liminf_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{M}{(I^{\lceil sq \rceil} + J^{[q]})\, M} \right) \\
\le& \limsup_{q'\to\infty} \frac{\Big(S_d^{vq'}(sq') - S_d^{vq'}(sq' - d - 1) \Big) \cdot \lambda(R/K) \cdot n}{(q')^d} \le 0.
\end{aligned}$$

Thus the limit exists and the theorem is proved. $\square$

With the limit shown to exist we are ready to define our multiplicity-like function.

Definition 3.3.2. Let $(R, \mathfrak{m})$ be a local ring of dimension d and characteristic $p > 0$, I and J be $\mathfrak{m}$-primary ideals of R, and M be a finitely generated R-module. For $s > 0$, we set

$$h_s(I, J; M) = \lim_{e\to\infty} \frac{\lambda\big(M/(I^{\lceil sp^e\rceil} + J^{[p^e]})M\big)}{p^{ed}}.$$

We often write $h_s(I, J)$ for $h_s(I, J; R)$, $h_s(I; M)$ for $h_s(I, I; M)$, $h_s(I)$ for $h_s(I; R)$, and $h_s(M)$ for $h_s(\mathfrak{m}; M)$. If we wish to emphasize the ring R, we write $h_s^R(I, J; M)$ or a similarly decorated variant.

We next establish some properties of $h_s(I, J; M)$. We use the next result repeatedly throughout the thesis, often without explicit reference.

Proposition 3.3.3. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and characteristic $p > 0$, I and J be $\mathfrak{m}$-primary ideals of R, and M be a finitely generated R-module. The following statements hold.*

(i) $h_s(I, J; M) \le \min\{\frac{s^d}{d!}e(I; M), e_{HK}(J; M)\}$.

(ii) *If* $\dim M < d$ *then* $h_s(I, J; M) = 0$.

(iii) *If* $s' \ge s$ *then* $h_{s'}(I, J; M) \ge h_s(I, J; M)$.

(iv) *If I' and J' are ideals of R such that $I \subseteq I'$ and $J \subseteq J'$, then* $h_s(I', J'; M) \le h_s(I, J; M)$.

(v) *If I' is an ideal of R with the same integral closure as I, then* $h_s(I', J; M) = h_s(I, J; M)$.

(vi) *If J' is an ideal of R with the same tight closure as J, then* $h_s(I, J'; M) = h_s(I, J; M)$.

Proof. Throughout the proof, let q stand for a power of p.

(i) For all q we have that $I^{\lceil sq\rceil} + J^{[q]} \supseteq I^{\lceil sq\rceil}$, hence

$$\lim_{q\to\infty} \frac{\lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big)}{q^d} \leq \lim_{q\to\infty} \frac{\lambda\big(M/I^{\lceil sq\rceil}M\big)}{\lceil sq\rceil^d} \cdot \frac{\lceil sq\rceil^d}{q^d} = \frac{s^d}{d!} e(I;M).$$

Furthermore, for all q we have that $I^{\lceil sq\rceil} + J^{[q]} \supseteq J^{[q]}$, hence

$$\lim_{q\to\infty} \frac{\lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big)}{q^d} \leq \lim_{q\to\infty} \frac{\lambda\big(M/J^{[q]}M\big)}{q^d} = e_{HK}(J;M).$$

(ii) By [6, Lemma 1.2], $e_{HK}(J;M) = 0$ for any M with $\dim M < d$, and so part (i) gives us the result.

(iii) For all q we have that $I^{\lceil sq\rceil} + J^{[q]} \supseteq I^{\lceil s'q\rceil} + J^{[q]}$, hence

$$\lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big) \leq \lambda\Big(M/(I^{\lceil s'q\rceil} + J^{[q]})M\Big).$$

(iv) For all q we have that $I'^{\lceil sq\rceil} + J'^{[q]} \supseteq I^{\lceil sq\rceil} + J^{[q]}$, hence

$$\lambda\Big(M/(I'^{\lceil sq\rceil} + J'^{[q]})M\Big) \leq \lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big).$$

(v) It suffices to prove the case where $I' = \overline{I}$, the integral closure of I. If $s > 0$, then we have that, by part (iv) and [5, Proposition 11.2.1],

$$\begin{aligned} 0 \leq h_s(I,J;M) - h_s\big(\overline{I},J;M\big) &= \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{\overline{I}^{\lceil sq\rceil} + J^{[q]}}{I^{\lceil sq\rceil} + J^{[q]}}\right) \\ &\leq \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{\overline{I}^{\lceil sq\rceil}}{I^{\lceil sq\rceil}}\right) = \frac{s^d}{d!}\left(e(I) - e(\overline{I})\right) = 0. \end{aligned}$$

(vi) It suffices to prove the case where $J = J^*$, the tight closure of J. We have that, by part (iv) and [4, Theorem 8.17],

$$\begin{aligned} 0 &\le h_s(I,J;M) - h_s(I,J^*;M) \\ &= \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{I^{\lceil sq\rceil} + (J^*)^{[q]}}{I^{\lceil sq\rceil} + J^{[q]}}\right) \le \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{(J^*)^{[q]}}{J^{[q]}}\right) = e_{HK}(J) - e_{HK}(J^*) = 0. \qquad \square \end{aligned}$$

We will see in the next section that for small values of s, the function $h_s(I,J;M)$ is related to the Hilbert-Samuel multiplicity, and for large values it is related to the Hilbert-Kunz multiplicity. What happens for values in between these two extremes, however, is mostly unknown. The following result shows that the behavior of this function cannot be too pathological.

Theorem 3.3.4. *Let $(R,\mathfrak{m})$ be a local ring of characteristic $p>0$, I and J be $\mathfrak{m}$-primary ideals of R, and M be a finitely generated R-module. The function $h_s(I,J;M)$ is Lipschitz continuous.*

Proof. Let $\delta > 0$. The function $h_s(I,J;M)$ is increasing by Proposition 3.3.3(iii), so we need only bound $h_{s+\delta}(I,J;M) - h_s(I,J;M)$ above in terms of δ.

Let $d = \dim R$. If $d=0$, then $h_{s+\delta}(I,J;M) = h_s(I,J;M) = \lambda(M)$, so 0 is a Lipschitz constant for $h_s(I,J;M)$. Suppose $d \ge 1$. We may assume that $R/\mathfrak{m}$ is infinite, and so we may assume that I is generated by d elements by replacing it with a minimal reduction by Proposition 3.3.3(v). Let $I = (f_1,\ldots,f_d)$, let $\mathfrak{m} = (x_1,\ldots,x_t)$, let $v \in \mathbb{N}$ such that $I^v \subseteq J$, and let $m_1,\ldots,m_n$ be a set of generators for M. Let q stand for a power of p. We have that

$$\begin{aligned} h_{s+\delta}(I,J;M) - h_s(I,J;M) &= \lim_{q\to\infty} \frac{1}{q^d}\left(\lambda\big(M/(I^{\lceil (s+\delta)q\rceil} + J^{[q]})M\big) - \lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big)\right) \\ &= \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{(I^{\lceil sq\rceil} + J^{[q]})M}{(I^{\lceil (s+\delta)q\rceil} + J^{[q]})M}\right) \\ &= \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{I^{\lceil sq\rceil}M}{(I^{\lceil (s+\delta)q\rceil} + J^{[q]})M \cap I^{\lceil sq\rceil}M}\right). \end{aligned}$$

The quotient module in the last line is generated as a k-vector space by elements of the form $f_1^{a_1}\cdots f_d^{a_d} g m_\alpha$, where $\sum_i a_i \geq sq$, g is a k-vector space generator of R/I, and $1 \leq \alpha \leq n$. However, if $\sum_i a_i \geq (s+\delta)q$ or $a_i \geq vq$ for some i, then the corresponding product vanishes. Therefore,

$$\lambda\left(\frac{I^{\lceil sq\rceil}M}{(I^{\lceil (s+\delta)q\rceil}+J^{[q]})M\cap I^{\lceil sq\rceil}M}\right) \leq (S_d^{vq}((s+\delta)q)-S_d^{vq}(sq))\cdot\lambda(R/I)\cdot n,$$

and so, by Lemma 3.2.4,

$$\begin{aligned} h_{s+\delta}(I,J;M)-h_s(I,J;M) &\leq \limsup_{q\to\infty}\frac{(S_d^{vq}((s+\delta)q)-S_d^{vq}(sq))\cdot\lambda(R/I)\cdot n}{q^d} \\ &\leq \delta\cdot v^{d-1}\cdot\lambda(R/I)\cdot n. \end{aligned}$$

Hence $v^{d-1}\cdot\lambda(R/I)\cdot n$ is a Lipschitz constant for $h_s(I,J;M)$. $\square$

Our most important application of Theorem 3.3.4 is the next result, which proves that $h_s(I,J;M)$ is additive on short exact sequences. A direct consequence of this will be the associativity formula for h_s.

Theorem 3.3.5. *Let $(R,\mathfrak{m})$ be a local ring of characteristic $p>0$ and I and J be $\mathfrak{m}$-primary ideals of R. If $0\to M'\to M\to M''\to 0$ is a short exact sequence of finitely generated R-modules, then $h_s(I,J;M)=h_s(I,J;M')+h_s(I,J;M'')$.*

Proof. Let $d=\dim R$, let m be the minimal number of generators of I, and let q and q' be powers of p. We have that $I^{\lceil (s+m/q)qq'\rceil}+J^{[qq']}\subseteq \left(I^{\lceil sq\rceil}+J^{[q]}\right)^{[q']}\subseteq I^{\lceil sqq'\rceil}+J^{[qq']}$. Therefore, by [6, Theorem 1.6], we have that

$$\begin{aligned} &\lambda\left(\frac{M'}{(I^{\lceil sqq'\rceil}+J^{[qq']})M'}\right)+\lambda\left(\frac{M''}{(I^{\lceil sqq'\rceil}+J^{[qq']})M''}\right) \\ &\leq \lambda\left(\frac{M'}{(I^{\lceil sq\rceil}+J^{[q]})^{[q']}M'}\right)+\lambda\left(\frac{M''}{(I^{\lceil sq\rceil}+J^{[q]})^{[q']}M''}\right) \\ &=\lambda\left(\frac{M}{(I^{\lceil sq\rceil}+J^{[q]})^{[q']}M}\right)+O((q')^{d-1})\leq\ \lambda\left(\frac{M}{(I^{\lceil (s+m/q)qq'\rceil}+J^{[qq']})M}\right)+O((q')^{d-1}). \end{aligned}$$

Dividing by $(qq')^d$ and taking the limit as $q' \to \infty$, we obtain that

$$h_s(I,J;M') + h_s(I,J;M'') \le h_{s+m/q}(I,J;M)\,.$$

This holds for all q, and so $h_s(I,J;M') + h_s(I,J;M'') \le h_s(I,J;M)$ since by Theorem 3.3.4, $h_s(I,J;M)$ is continuous in s.

For the other inequality, note that for any q, the sequence

$$\frac{M'}{(I^{\lceil sq\rceil} + J^{[q]})M'} \to \frac{M}{(I^{\lceil sq\rceil} + J^{[q]})M} \to \frac{M''}{(I^{\lceil sq\rceil} + J^{[q]})M''} \to 0$$

is exact, whence

$$\lambda\left(\frac{M'}{(I^{\lceil sq\rceil} + J^{[q]})M'}\right) + \lambda\left(\frac{M''}{(I^{\lceil sq\rceil} + J^{[q]})M''}\right) \ge \lambda\left(\frac{M}{(I^{\lceil sq\rceil} + J^{[q]})M}\right).$$

Therefore $h_s(I,J;M') + h_s(I,J;M'') \ge h_s(I,J;M)$. $\square$

The additivity of $h_s(I,J;M)$ on short exact sequences is exactly what we need to prove the associativity formula for h_s. This proof follows the proof in [8, Theorem 23.5] for the associativity formula for Hilbert-Samuel multiplicity.

Theorem 3.3.6 (The associativity formula)**.** *Let $(R,\mathfrak{m})$ be a local ring of characteristic $p > 0$, I and J be $\mathfrak{m}$-primary ideals of R, and M be a finitely generated R-module. We have that*

$$h_s^R(I,J;M) = \sum_{\mathfrak{p}\in \operatorname{Assh} R} h_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\,\lambda_{R_\mathfrak{p}}(M_\mathfrak{p})\,,$$

where $\operatorname{Assh} R = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = \dim R\}$.

Proof. We proceed by induction on $\sigma(M) = \sum_{\mathfrak{p}\in\operatorname{Assh} R} \lambda_{R_\mathfrak{p}}(M_\mathfrak{p})$. If $\sigma(M) = 0$, then $\dim M < \dim R$ and so $h_s^R(I,J;M) = 0$.

Now suppose that $\sigma(M) \ge 1$ and fix $\mathfrak{q} \in \operatorname{Assh} R$ such that $\lambda_{R_\mathfrak{q}}(M_\mathfrak{q}) \ge 1$. We have that

$\mathfrak{q} = (0 :_R x)$ for some $x \in M$ and so we have an exact sequence

$$0 \to R/\mathfrak{q} \to M \to M/Rx \to 0.$$

We have that $\sigma(M/Rx) = \sigma(M) - 1$ and so by induction,

$$\begin{aligned}
h_s^R(I, J; M/Rx) &= \sum_{\mathfrak{p} \in \operatorname{Assh} R} h_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\, \lambda_{R_\mathfrak{p}}((M/Rx)_\mathfrak{p}) \\
&= \sum_{\mathfrak{p} \in \operatorname{Assh} R} h_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\, \lambda_{R_\mathfrak{p}}(M_\mathfrak{p}) - h_s^{R/\mathfrak{q}}(I(R/\mathfrak{q}), J(R/\mathfrak{q}))\,.
\end{aligned}$$

Therefore, it suffices to show that $h_s^R(I, J; R/\mathfrak{q}) = h_s^{R/\mathfrak{q}}(I(R/\mathfrak{q}), J(R/\mathfrak{q}))$ since then by Theorem 3.3.5 we will have the desired formula. This, however, is an easy computation. Letting q stand for a power of p, we have that

$$\begin{aligned}
h_s^R(I, J; R/\mathfrak{q}) &= \lim_{q \to \infty} \frac{1}{q^d} \lambda_R\left(\frac{R/\mathfrak{q}}{(I^{\lceil sq \rceil} + J^{[q]})R/\mathfrak{q}}\right) \\
&= \lim_{q \to \infty} \frac{1}{q^d} \lambda_{R/\mathfrak{q}}\left(\frac{R/\mathfrak{q}}{(I(R/\mathfrak{q}))^{\lceil sq \rceil} + (J(R/\mathfrak{q}))^{[q]}}\right) = h_s^{R/\mathfrak{q}}(I(R/\mathfrak{q}), J(R/\mathfrak{q}))\,. \quad \square
\end{aligned}$$

To finish out this section, we notice that the function $h_s(I, J; M)$ is closely related to the F-threshold and the F-limbus from the previous section.

Lemma 3.3.7. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and characteristic $p > 0$, I and J be $\mathfrak{m}$-primary ideals of R, and M be a finitely generated R-module.*

1. *If $s \leq b^J(I)$ then $h_s(I, J; M) = \frac{s^d}{d!} e(I; M)$.*

2. *If $s \geq c^J(I)$ then $h_s(I, J; M) = e_{HK}(J; M)$.*

Proof. Let q stand for a power of p.

If $s < b^J(I)$, then for infinitely many q, $J^{[q]} \subseteq I^{\lceil sq \rceil}$. Therefore

$$\begin{aligned} h_s(I, J; M) &= \lim_{q\to\infty} \frac{\lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big)}{q^d} \\ &= \lim_{q\to\infty} \frac{\lambda\big(M/I^{\lceil sq\rceil}M\big)}{q^d} = \lim_{q\to\infty} \frac{\lambda\big(M/I^{\lceil sq\rceil}M\big)}{(\lceil sq\rceil)^d} \cdot \frac{(\lceil sq\rceil)^d}{q^d} = \frac{e(I;M)s^d}{d!}. \end{aligned}$$

If $s > c^J(I)$, then for infinitely many q, $I^{\lceil sq \rceil} \subseteq J^{[q]}$. Therefore

$$h_s(I, J; M) = \lim_{q\to\infty} \frac{\lambda\big(M/(I^{\lceil sq\rceil} + J^{[q]})M\big)}{q^d} = \lim_{q\to\infty} \frac{\lambda\big(M/J^{[q]}M\big)}{q^d} = e_{HK}(J;M).$$

The continuity of $h_s(I, J; M)$ gives the cases $s = b^J(I)$ and $s = c^J(I)$. □

3.4 The Normalizing Factor $\mathcal{H}_s(d)$

One of the most important and useful properties of the Hilbert-Samuel and Hilbert-Kunz multiplicities are their behavior in regular rings. In particular, if $(R, \mathfrak{m})$ is a regular local ring, then $e(\mathfrak{m}) = e_{HK}(\mathfrak{m}) = 1$. In order to properly define a function that interpolates between these two multiplicities, we need to understand the behavior of $h_s(\mathfrak{m})$ when $(R, \mathfrak{m})$ is a regular local ring.

Proposition 3.4.1. *If k is a field of characteristic $p > 0$ and $R = k[[x_1, \ldots, x_d]]$, then*

$$h_s(\mathfrak{m}) = \sum_{i=0}^{\lfloor s\rfloor} \frac{(-1)^i}{d!}\binom{d}{i}(s-i)^d.$$

Proof. Let $\mathfrak{m} = (x_1, \ldots, x_d)$. If $d = 0$, then $\mathfrak{m} = 0$, and so $h_s(R) = 1 = \sum_{i=0}^{\lfloor s\rfloor}(-1)^i\binom{0}{i}(s-i)^0$. If $d \geq 1$, then for any power q of p, $k[[x_1, \ldots, x_d]]/(\mathfrak{m}^{\lceil sq\rceil} + \mathfrak{m}^{[q]})$ is generated as a k-vector space by all monomials in the x_i with degree less than sq and with the exponent on each x_i less than q. The number of such monomials is precisely $S_d^q(sq)$. Therefore, by Lemma 3.2.3 we have that

$$\begin{aligned} h_s(\mathfrak{m}) &= \lim_{q\to\infty} \frac{S_d^q(sq)}{q^d} \\ &= \sum_{i=0}^{d} (-1)^i \binom{d}{i} \lim_{q\to\infty} \frac{1}{q^d} \binom{\lceil sq \rceil - iq - 1 + d}{d} = \sum_{i=0}^{\lfloor s \rfloor} \frac{(-1)^i}{d!} \binom{d}{i} (s-i)^d. \end{aligned} \qquad \square$$

This function will serve as a normalizing factor, and fills the same role that the factor of $d!$ does in the definition of Hilbert-Samuel multiplicity. We will want to use various properties of this function, so here we define notation for it and prove some of them.

Definition 3.4.2. Let $d \in \mathbb{N}$ and $s \in \mathbb{R}$. We set

$$\mathcal{H}_s(d) = \sum_{i=0}^{\lfloor s \rfloor} \frac{(-1)^i}{d!} \binom{d}{i} (s-i)^d.$$

Note that if $s < 0$, then $\mathcal{H}_s(d) = 0$.

Example 3.4.3. We begin our analysis of the functions $\mathcal{H}_s(d)$ by computing several of them. Note that we only show the values for $s \geq 0$.

$$\mathcal{H}_s(0) = 1$$

$$\mathcal{H}_s(1) = \begin{cases} s & \text{if } 0 \leq s < 1 \\ 1 & \text{if } s \geq 1 \end{cases}$$

$$\mathcal{H}_s(2) = \begin{cases} \frac{1}{2}s^2 & \text{if } 0 \leq s < 1 \\ \frac{1}{2}s^2 - (s-1)^2 & \text{if } 1 \leq s < 2 \\ 1 & \text{if } s \geq 2 \end{cases}$$

$$\mathcal{H}_s(3) = \begin{cases} \frac{1}{6}s^3 & \text{if } 0 \le s < 1 \\ \frac{1}{6}s^3 - \frac{1}{2}(s-1)^3 & \text{if } 1 \le s < 2 \\ \frac{1}{6}s^3 - \frac{1}{2}(s-1)^3 + \frac{1}{2}(s-2)^3 & \text{if } 2 \le s < 3 \\ 1 & \text{if } s \ge 3 \end{cases}$$

Certain properties of $\mathcal{H}_s(d)$ are suggested by the above examples, and are confirmed in the next lemma.

Lemma 3.4.4. *The functions $\mathcal{H}_s(d)$ have the following properties.*

(i) If $d \ge 1$, then $\mathcal{H}_s(d) = \int_{s-1}^{s} \mathcal{H}_t(d-1)\,\mathrm{d}t$.

(ii) $\mathcal{H}_s(d)$ is nondecreasing.

(iii) $\mathcal{H}_s(d)$ is a Lipschitz continuous function of s on the interval $(0, \infty)$.

(iv) If $s \ge d$, then $\mathcal{H}_s(d) = 1$.

(v) If $0 < s \le 1$, then $\mathcal{H}_s(d) = s^d/d!$.

Proof. (i) This is clear for $d = 1$, so suppose that $d \ge 2$. Let q and q' be varying powers of p. We have that

$$\begin{aligned} \mathcal{H}_s(d) = \lim_{q\to\infty} \frac{S_d^{qq'}(sqq')}{(qq')^d} &= \lim_{q\to\infty} \frac{\sum_{i=0}^{qq'-1} S_{d-1}^{qq'}(sqq'-i)}{(qq')^d} \\ &\le \lim_{q\to\infty} \frac{q\sum_{i=0}^{q'-1} S_{d-1}^{qq'}(sqq'-qi)}{(qq')^d} \\ &= \frac{1}{q'} \sum_{i=0}^{q'-1} \lim_{q\to\infty} \frac{S_{d-1}^{qq'}\left((s-i/q')qq'\right)}{(qq')^{d-1}} = \frac{1}{q'} \sum_{i=0}^{q'-1} \mathcal{H}_{s-i/q'}(d-1) \end{aligned}$$

Since the above holds for all q', we have that

$$\mathcal{H}_s(d) \le \lim_{q'\to\infty} \frac{1}{q'} \sum_{i=0}^{q'-1} \mathcal{H}_{s-i/q'}(d-1) = \int_{s-1}^{s} \mathcal{H}_t(d-1)\,\mathrm{d}t.$$

A similar argument, only using the inequality

$$\sum_{i=0}^{qq'-1} S_{d-1}^{qq'}(sqq' - i) \geq q \sum_{i=1}^{q'} S_{d-1}^{qq'}(sqq' - qi)$$

in the second line, shows that $\mathcal{H}_s(d) \geq \int_{s-1}^{s} \mathcal{H}_t(d-1)\,\mathrm{d}t$.

(ii) This is by inspection for $d = 0$. For $d \geq 1$, let $\delta > 0$, so by induction

$$\mathcal{H}_{s+\delta}(d) - \mathcal{H}_s(d) = \int_{s-1}^{s} \mathcal{H}_{t+\delta}(d-1) - \mathcal{H}_t(d-1)\,\mathrm{d}t \geq 0.$$

(iii) This is trivial for $d = 0$. For $d \geq 1$, we will actually show that the functions $\mathcal{H}_s(d)$ are Lipschitz continuous with Lipschitz constants at most 1 on the entire real line. This can be seen for $d = 1$ by Example 3.4.3, so suppose $d \geq 2$, $s \in \mathbb{R}$, and $0 < \delta < 1$. By induction,

$$\mathcal{H}_{s+\delta}(d) - \mathcal{H}_s(d) = \int_{s-1}^{s} \mathcal{H}_{t+\delta}(d-1) - \mathcal{H}_t(d-1)\,\mathrm{d}t \leq \int_{s-1}^{s} \delta\ \mathrm{d}t = \delta.$$

(iv) This statement is true for $d = 0$ by inspection. Assume that $d \geq 1$ and $\mathcal{H}_s(d-1) = 1$ for $s \geq d-1$. For $s \geq d$, we have that

$$\mathcal{H}_s(d) = \int_{s-1}^{s} \mathcal{H}_t(d-1)\,\mathrm{d}t = \int_{s-1}^{s} 1\ \mathrm{d}t = 1$$

and the result follows by induction.

(v) This is clear from the definition. $\square$

3.5 s-Multiplicity

With all the results from the previous sections in hand, we are ready to define our main object of study in this section.

Definition 3.5.1. Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, I and J be $\mathfrak{m}$-primary ideals of R, M be a finitely generated R-module, and $s > 0$. The *s-multiplicity of M with* *respect to the pair* (I, J) is defined to be

$$e_s(I, J; M) = \frac{h_s(I, J; M)}{\mathcal{H}_s(d)}.$$

We write $e_s(I, J)$ for $e_s(I, J; R)$, $e_s(I; M)$ for $e_s(I, I; M)$, $e_s(I)$ for $e_s(I; R)$, and $e_s(M)$ for $e_s(\mathfrak{m}; M)$. If we wish to emphasize the ring R, we will write $e_s^R(I, J; M)$ or a similarly decorated variant.

Many properties of the $h_s(I, J; M)$ immediately imply similar properties for the s-multiplicity. Some of these properties are listed in the next three corollaries. The first corollary makes explicit the interpolating properties of the s-multiplicity, while the second contains some auxiliary results listed for completeness. The third is the associativity formula for s-multiplicity.

Corollary 3.5.2. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d *and characteristic* $p > 0$*,* I *and* J *be* $\mathfrak{m}$*-primary ideals of* R*, and* M *be a finitely generated* R*-module.*

(i) If $s \leq \min\{1, b^J(I)\}$*, then* $e_s(I, J; M) = e(I; M)$*.*

(ii) If $s \geq \max\{d, c^J(I)\}$*, then* $e_s(I, J; M) = e_{HK}(J; M)$*.*

(iii) If R *is a regular ring, then* $e_s(R) = 1$ *for all* s*.*

Proof. Statements (i) and (ii) simply combine Lemma 3.3.7 and Lemma 3.4.4. For statement (iii), we may assume without loss of generality that R is complete with residue field k, in which case $R \cong k[[x_1, \ldots, x_d]]$. The result then follows from Definition 3.5.1 and Proposition 3.4.1. □

Corollary 3.5.3. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d *and characteristic* $p > 0$*,* I *and* J *be* $\mathfrak{m}$*-primary ideals of* R*, and* M *be a finitely generated* R*-module. The following statements hold for all* $s > 0$*.*

(i) $e_s(I, J; M)$ *is a Lipschitz continuous function of* s*.*

(ii) $e_s(I, J; M) \leq e_{HK}(J; M)/\mathcal{H}_s(d)$.

(iii) *If* $\dim M < d$ *then* $e_s(I, J; M) = 0$.

(iv) *If* I' *and* J' *are* $\mathfrak{m}$*-primary ideals of* R *such that* $I \subseteq I'$ *and* $J \subseteq J'$*, then* $e_s(I', J'; M) \leq e_s(I, J; M)$.

(v) *If* I' *is an* $\mathfrak{m}$*-primary ideal of* R *with the same integral closure as* I*, then* $e_s(I', J; M) = e_s(I, J; M)$.

(vi) *If* J' *is an* $\mathfrak{m}$*-primary ideal of* R *with the same tight closure as* J*, then* $e_s(I, J'; M) = e_s(I, J; M)$.

(vii) *If* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of finitely generated* R*-modules, then* $e_s(I, J; M) = e_s(I, J; M') + e_s(I, J; M'')$.

Proof. (i) We have that $e_s(I, J; M)$ is constant, hence Lipschitz continuous, on $(0, \min\{1, b^J(I)\}]$. By Lemma 3.4.4, $\mathcal{H}_s(d)$ is Lipschitz continuous and bounded away from 0 on $[\min\{1, b^J(I)\}, \infty)$ and by Theorem 3.3.4, $h_s(I, J; M)$ is Lipschitz continuous, and so $e_s(I, J; M)$ is Lipschitz continuous on $[\min\{1, b^J(I)\}, \infty)$. Thus $e_s(I, J; M)$ is Lipschitz continuous.

Parts (ii)-(vi) follow from Proposition 3.3.3. Part (vii) follows from Theorem 3.3.5. □

The following corollary now follows directly from Theorem 3.3.6.

Corollary 3.5.4 (Associativity formula for s-multiplicity)**.** *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$*,* I *and* J *be* $\mathfrak{m}$*-primary ideals of* R*, and* M *be a finitely generated* R*-module. We have that*

$$e_s^R(I, J; M) = \sum_{\mathfrak{p} \in \operatorname{Assh} R} e_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\, \lambda_{R_\mathfrak{p}}(M_\mathfrak{p})$$

where $\operatorname{Assh} R = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = \dim R\}$.

Proof. For any $\mathfrak{p} \in \operatorname{Assh} R$, $\dim R/\mathfrak{p} = d$, and so

$$e_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p})) = \frac{h_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))}{\mathcal{H}_s(d)}.$$

By Theorem 3.3.6, we have that

$$h_s^R(I, J; M) = \sum_{\mathfrak{p} \in \operatorname{Assh} R} h_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\, \lambda_{R_\mathfrak{p}}(M_\mathfrak{p}).$$

Therefore, dividing each term of this equation by $\mathcal{H}_s(d)$ proves the result. □

An immediate application of Corollary 3.5.4 is the following result, which shows that the s-multiplicity of a module is in many cases determined by the s-multiplicity of the ring itself.

Proposition 3.5.5. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, let I and J be $\mathfrak{m}$-primary ideals of R, and let M be a finitely generated R-module. If $M_\mathfrak{p}$ is free of constant rank r for every $\mathfrak{p} \in \operatorname{Assh} R$, in particular if R is a domain, then $e_s(I, J; M) = e_s(I, J) \cdot r$.*

Proof. By the associativity formula, we have that

$$\begin{aligned} e_s^R(I, J; M) &= \sum_{\mathfrak{p} \in \operatorname{Assh} R} e_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\, \lambda_{R_\mathfrak{p}}(M_\mathfrak{p}) \\ &= \sum_{\mathfrak{p} \in \operatorname{Assh} R} e_s^{R/\mathfrak{p}}(I(R/\mathfrak{p}), J(R/\mathfrak{p})) \cdot r = e_s^R(I, J) \cdot r \end{aligned}$$

□

The problem of finding general bounds for the value of the s-multiplicity seems to be difficult, but we have a few results along those lines.

Proposition 3.5.6. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings of dimension d and characteristic $p > 0$ such that $\mathfrak{m}S$ is $\mathfrak{n}$-primary, let I and J be $\mathfrak{m}$-primary ideals of R, and let M be a finitely generated R-module. In this case,*

$$e_s^S(IS, JS; M \otimes_R S) \leq e_s^R(I, J; M) \cdot \lambda_S(S/\mathfrak{m}S)$$

and we have equality if φ is a flat ring homomorphism.

Proof. For any R-module N of finite length, we have that

$$\lambda_S(N \otimes_R S) \le \lambda_R(N) \cdot \lambda_S(S/\mathfrak{m}S)\,.$$

Thus, for any $s > 0$ and q a power of p we have that

$$\begin{aligned}\lambda_S\left(\frac{M \otimes_R S}{((IS)^{\lceil sq \rceil} + (JS)^{[q]})(M \otimes_R S)}\right) &= \lambda_S\left(\frac{M}{(I^{\lceil sq \rceil} + J^{[q]})M} \otimes_R S\right) \\ &\le \lambda_R\left(\frac{M}{(I^{\lceil sq \rceil} + J^{[q]})M}\right) \cdot \lambda_S(S/\mathfrak{m}S)\,.\end{aligned}$$

Dividing both sides by q^d and taking the limit as q goes to infinity gives us that

$$h_s^S(IS, JS; M \otimes_R S) \le h_s^R(I, J; M) \cdot \lambda_S(S/\mathfrak{m}S)\,,$$

and dividing both sides by $\mathcal{H}_s(d)$ gives us the result for s-multiplicity.

If φ is a flat ring homomorphism, then for any R-module N we have that $\lambda_S(N \otimes_R S) = \lambda_R(N) \cdot \lambda_S(S/\mathfrak{m}S)$ and so we have equality everywhere. □

Corollary 3.5.7. *If $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and I is an ideal generated by a system of parameters in R, then $e_s(I) \le \lambda(R/I)$. Furthermore, equality holds if R is Cohen-Macaulay.*

Proof. We may assume that R is complete. Let $d = \dim R$, let $x_1, \ldots, x_d$ be a system of parameters generating I, and let $S = k[[x_1, \ldots, x_d]] \subseteq R$. Now by Proposition 3.5.6 and Corollary 3.5.2(iii), $e_s^R(I) \le e_s^S((x_1, \ldots, x_d))\, \lambda_R(R/I) = \lambda_R(R/I)$. Furthermore, if R is Cohen-Macaulay, then R is a free S-module, hence is flat over S, so equality holds. □

4 Closures Related to s-Multiplicity

The s-multiplicity is related to closures, just as the Hilbert-Samuel and Hilbert-Kunz multiplicities are. We see this already in the guise of Proposition 3.3.3 and Corollary 3.5.3 with respect to integral and tight closure. The natural question to ask at this point is whether there are closures that are similarly related to the various s-multiplicities. In this section we define these closures and show that in sufficiently nice rings, we get a strong connection between the closure operators and the s-multiplicity. We use the notation R° to stand for the complement of the union of the minimal primes of R.

4.1 The s-Closure

We can take a guess as to an appropriate kind of closure to relate to s-multiplicity by looking at integral closure and tight closure. Integral closure, like Hilbert-Samuel multiplicity, has many equivalent definitions, but the most relevant to us is the following.

Definition 4.1.1. Let I be an ideal of a ring R. An element x is in $\overline{I}$, the *integral closure of I*, if there exists $c \in R^\circ$ such that for infinitely many $n \in \mathbb{N}$, $cx^n \in I^n$.

Tight closure has fewer common definitions, but has one that closely matches the form of the previous definition.

Definition 4.1.2. Let I be an ideal of a ring R of positive characteristic p. An element $x \in R$ is in I^*, the *tight closure of I*, if there exists $c \in R^\circ$ such that for all sufficiently large powers q of p, $cx^q \in I^{[q]}$.

Given these two definitions, it is natural for us to try to define a closure in a similar way, using the sums of ideals that translate between I^q and $I^{[q]}$.

Definition 4.1.3. Let R be a ring of characteristic $p > 0$, I be an ideal of R, and $s \geq 1$ be a real number. An element $x \in R$ is said to be in the *weak s-closure of I* if there exists $c \in R^\circ$ such that for all $q \gg 0$, where q is a power of p, $cx^q \in I^{\lceil sq \rceil} + I^{[q]}$. We denote the set of all x in the weak s-closure of I by $I^{\mathrm{w.cl}_s}$.

Remark 4.1.4. If I is of positive height, then $x \in I^{\mathrm{w.cl}_s}$ if and only if there exists $c \in R^\circ$ such that $cx^q \in I^{\lceil sq \rceil} + I^{[q]}$ for all $q \geq 1$. To see this, suppose that there exists $c' \in R^\circ$ and q' such that $c'x^q \in I^{\lceil sq \rceil} + I^{[q]}$ for $q > q'$. Since I is of positive height, there exists $c'' \in (I^{\lceil sq' \rceil} + I^{[q']}) \cap R^\circ$. Setting $c = c'c''$, we have that $c \in R^\circ$ and $cx^q \in I^{\lceil sq \rceil} + I^{[q]}$ for all $q \geq 1$.

For a given ideal I, $I^{\mathrm{w.cl}_s}$ is clearly an ideal containing I. However, it is not clear that the weak s-closure is idempotent; that is, it is not clear that $\left(I^{\mathrm{w.cl}_s}\right)^{\mathrm{w.cl}_s} = I^{\mathrm{w.cl}_s}$. If the ring is noetherian, we can construct an idempotent operation out of the weak s-closure by iterating the operation until the chain of ideals stabilizes.

Definition 4.1.5. Let R be a ring of characteristic $p > 0$, let I be an ideal of R, and let $s \geq 1$ be a real number. The *s-closure of I* is defined to be the ideal at which the following chain of ideals stabilizes:

$$I \subseteq I^{\mathrm{w.cl}_s} \subseteq \left(I^{\mathrm{w.cl}_s}\right)^{\mathrm{w.cl}_s} \subseteq \left(\left(I^{\mathrm{w.cl}_s}\right)^{\mathrm{w.cl}_s}\right)^{\mathrm{w.cl}_s} \subseteq \cdots .$$

We denote this ideal by I^{cl_s}.

Notice that, for $s = 1$, the s-closure is integral closure, and for $s > c^I(I)$, the s-closure is tight closure. Furthermore, if $s \leq s'$, then $I^{\mathrm{cl}_s} \supseteq I^{\mathrm{cl}_{s'}}$ for all ideals I. Thus the s-closure interpolates monotonically between integral closure and tight closure as s increases. One should note that new closures do in fact arise.

Example 4.1.6. Let $R = k[x, y]$, where k is a field of characteristic $p > 0$. If $I = (x^3, y^3)$, then

$$I^{\mathrm{cl}_s} = \begin{cases} (x, y)^3 & \text{if } s = 1 \\ (x^3, x^2y^2, y^3) & \text{if } 1 < s \leq \frac{4}{3} \\ (x^3, y^3) & \text{if } s > \frac{4}{3}. \end{cases}$$

In particular, if $1 < s \leq \frac{4}{3}$, then $I = I^* \subsetneq I^{\mathrm{cl}_s} \subsetneq \overline{I} = (x, y)^3$.

Example 4.1.6 demonstrates that in some cases, an ideal I will only have finitely many distinct s-closures for various values of s; in fact, this will occur whenever R is local and I is primary to the maximal ideal. However, even in regular rings there can be infinitely many distinct s-closures.

Example 4.1.7. Let $R = k[x, y]$, where k is a field of characteristic $p > 0$. Let $1 \leq s < s' \leq 2$. Choose $n \in \mathbb{N}$ such that $n > 2/(s' - s)$, and let $I = (x^{2n}, y^{2n})$. We have that $x^{\lceil sn \rceil} y^{\lceil sn \rceil} \in I^{\mathrm{w.cl}_s}$, since for any power q of p,

$$2 \left\lfloor \frac{2n + \lceil sn \rceil q}{2n} \right\rfloor \geq 2 \left\lfloor 1 + \frac{s}{2} q \right\rfloor \geq sq,$$

and so $x^{2n} y^{2n} (x^{\lceil sn \rceil} y^{\lceil sn \rceil})^q \in (x^{2n}, y^{2n})^{\lceil sq \rceil}$. However, $x^{\lceil sn \rceil} y^{\lceil sn \rceil} \notin I^{\mathrm{w.cl}_{s'}}$, since for any $a \in \mathbb{N}$, if $q \geq a$, then we have that

$$2 \left\lfloor \frac{a + \lceil sn \rceil q}{2n} \right\rfloor \leq \frac{a + (sn + 1)q}{n} \leq \frac{(sn + 2)q}{n} = sq + \frac{2q}{n} < sq + (s' - s)q = s'q$$

and so $x^a y^a (x^{\lceil sn \rceil} y^{\lceil sn \rceil})^q \notin (x^{2n}, y^{2n})^{\lceil s'q \rceil}$. Thus $I^{\mathrm{w.cl}_s} \neq I^{\mathrm{w.cl}_{s'}}$, and hence $I^{\mathrm{cl}_s} \neq I^{\mathrm{cl}_{s'}}$ by Theorem 4.2.1. Thus we find that there are infinitely many distinct s-closures on R, one for every real number in the interval $[1, 2]$.

4.2 s-Closure and s-Multiplicity

If I and I' have the same integral closure, then $e(I) = e(I')$, while if I and I' have the same tight closure, then $e_{HK}(I) = e_{HK}(I')$. Our main theorem in this section is a similar result for s-multiplicity and s-closure.

Theorem 4.2.1. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and let I and J be $\mathfrak{m}$-primary ideals of R with $I \subseteq J$. If $J \subseteq I^{\mathrm{cl}_s}$, then $e_s(J) = e_s(I)$. If R is an F-finite complete domain, then the converse holds and $I^{\mathrm{cl}_s} = I^{\mathrm{w.cl}_s}$.*

Proof. Let $d = \dim R$. Suppose that $x \in I^{\mathrm{w.cl}_s}$, so that there exists $c \in R^\circ$ such that for all

$q \gg 0$, where q is a power of p, we have that $cx^q \in I^{\lceil sq \rceil} + I^{[q]} \subseteq I^q$. Hence x is in the integral closure of I and so $h_s((I,x),(I,x)) = h_s(I,(I,x))$ by Proposition 3.3.3(v). Now for large q, c annihilates $\frac{I^{\lceil sq \rceil}+(I,x)^{[q]}}{I^{\lceil sq \rceil}+I^{[q]}}$. Let $S = R/cR$, so that for $q \gg 0$,

$$\lambda_R\left(\frac{I^{\lceil sq \rceil} + (I,x)^{[q]}}{I^{\lceil sq \rceil} + I^{[q]}}\right) = \lambda_S\left(\frac{I^{\lceil sp^e \rceil} + (I,x)^{[p^e]}}{I^{\lceil sq \rceil} + I^{[q]}} \otimes S\right) = \lambda_S\left(\frac{(IS)^{\lceil sq \rceil} + ((I,x)S)^{[q]}}{(IS)^{\lceil sq \rceil} + (IS)^{[q]}}\right).$$

So, since $\dim S = d-1$,

$$\begin{aligned} h_s(I,I) - h_s(I,(I,x)) &= \lim_{q\to\infty} \frac{1}{q^d}\lambda_R\left(\frac{I^{\lceil sq \rceil} + (I,x)^{[q]}}{I^{\lceil sq \rceil} + I^{[q]}}\right) \\ &= \lim_{q\to\infty} \frac{1}{q^d}\lambda_S\left(\frac{(IS)^{\lceil sq \rceil} + ((I,x)S)^{[q]}}{(IS)^{\lceil sq \rceil} + (IS)^{[q]}}\right) \\ &= \left(\lim_{q\to\infty} \frac{1}{q}\right) \cdot \left(\lim_{q\to\infty} \frac{1}{q^{d-1}}\lambda_S\left(\frac{(IS)^{\lceil sq \rceil} + ((I,x)S)^{[q]}}{(IS)^{\lceil sq \rceil} + (IS)^{[q]}}\right)\right) \\ &= 0 \cdot (h_s^S(IS,IS) - h_s^S(IS,(I,x)S)) = 0. \end{aligned}$$

Therefore $h_s((I,x)) = h_s(I)$ for any $x \in I^{\mathrm{w.cl}_s}$, hence $h_s\big(I^{\mathrm{w.cl}_s}\big) = h_s(I)$. By induction, $h_s\big(I^{\mathrm{cl}_s}\big) = h_s(I)$, hence $h_s(J) = h_s(I)$ and so $e_s(J) = e_s(I)$.

Now suppose that R is an F-finite complete domain and $x \in R$ such that $e_s((I,x)) = e_s(I)$. In this case $h_s((I,x)) = h_s(I)$, and so $h_s(I,(I,x)) = h_s(I,I)$, and therefore

$$0 = \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{I^{\lceil sq \rceil} + (I,x)^{[q]}}{I^{\lceil sq \rceil} + I^{[q]}}\right) = \lim_{q\to\infty} \frac{1}{q^d}\lambda\big(R/\left(\left(I^{\lceil sq \rceil} + I^{[q]}\right) :_R x^q\right)\big).$$

Let $\psi \in \mathrm{Hom}_R\left(F_*R, R\right)$ be a nonzero p^{-1}-linear map and let $\varphi(-) = \psi\left(F_*(f_1^{p-1}\cdots f_n^{p-1})\cdot -\right)$, where $f_1,\ldots,f_n$ is a generating set for I. We have that

$$\begin{aligned} \varphi\left(F_*\left(\left(I^{\lceil spq \rceil} + I^{[pq]}\right) :_R x^{pq}\right)\right)\cdot x^q &\subseteq \varphi\left(F_*\left(I^{\lceil spq \rceil} + I^{[pq]}\right)\right) \\ &\subseteq \psi\left(F_*\left(f_1^{p-1}\cdots f_n^{p-1} I^{\lceil spq \rceil}\right)\right) + I^{[q]}. \end{aligned}$$

If $a_1, \ldots, a_n \in \mathbb{N}$ with $a_1 + \cdots + a_n \geq spq$, then

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i + p - 1}{p} \right\rfloor \geq \sum_{i=1}^{n} \frac{a_i}{p} \geq sq$$

and so $f_1^{p-1} \cdots f_n^{p-1} I^{\lceil spq \rceil} \subseteq \left(I^{\lceil sq \rceil}\right)^{[p]}$. Therefore $\psi\left(F_*\left(f_1^{p-1} \cdots f_n^{p-1} I^{\lceil spq \rceil}\right)\right) \subseteq I^{\lceil sq \rceil}$ and so

$$\varphi\left(F_*\left(\left(I^{\lceil spq \rceil} + I^{[pq]}\right) :_R x^{pq}\right)\right) \cdot x^q \subseteq I^{\lceil sq \rceil} + I^{[q]},$$

that is,

$$\varphi\left(F_*\left(\left(I^{\lceil spq \rceil} + I^{[pq]}\right) :_R x^{pq}\right)\right) \subseteq \left(\left(I^{\lceil sq \rceil} + I^{[q]}\right) :_R x^q\right).$$

Since this holds for all q, by [9, Theorem 5.5], we must have that $\bigcap_{q \geq 0} \left(\left(I^{\lceil sq \rceil} + I^{[q]}\right) :_R x^q\right) \neq 0$, that is, there is some $0 \neq c \in R$ such that for all q, $cx^q \subseteq I^{\lceil sq \rceil} + I^{[q]}$. Therefore $x \in I^{\mathrm{w.cl}_s}$.

Thus we have that if R is an F-finite complete domain and $h_s((I, x)) = h_s(I)$, then $x \in I^{\mathrm{w.cl}_s}$. Therefore if $h_s(J) = h_s(I)$ then $J \subseteq I^{\mathrm{w.cl}_s} \subseteq I^{\mathrm{cl}_s}$. Furthermore, in this case, if $x \in I^{\mathrm{cl}_s}$, then $h_s((I, x)) = h_s(I)$ and hence $x \in I^{\mathrm{w.cl}_s}$. Therefore $I^{\mathrm{cl}_s} = I^{\mathrm{w.cl}_s}$. □

This theorem shows that the s-multiplicity and the s-closures are intimately related.

Remark 4.2.2. The domain hypothesis in the backwards direction of Theorem 4.2.1 is difficult to remove. In the case of both integral closure and tight closure, the relationship with the corresponding multiplicity was established for domains first, and then afterward expanded to the more general cases. These two arguments used different techniques. In the case of integral closure and Hilbert-Samuel multiplicity, a different definition of the closure was used which involves an "equation of integral dependence", an object that we do not have for s-closure. In the case of tight closure and Hilbert-Kunz multiplicity, the existence of "stable test elements" for tight closure was relied upon, which again we do not have for s-closure. So far, there is no clear way to replace the domain hypothesis with a weaker one.

5 Toric Rings

In this section we will study the F-threshold, F-limbus, s-muliplicity, and s-closure in toric rings, also called semigroup rings. These rings can be realized as subrings of Laurent polynomial rings generated by monomials. When we study the s-multiplicity of these toric rings we will complete the ring at the maximal homogneous ideal in order to work in the local setting.

Definition 5.0.1. Let k be a field. By a *toric ring of dimension d over k*, or simply *toric ring*, we will mean the ring $k[S]$, where $S = \sigma^\vee \cap \mathbb{Z}^d$. Here $\sigma^\vee$ is a cone in $\mathbb{R}^d$ not containing any line through the origin, and S inherits the semigroup structure of $\mathbb{Z}^d$. Furthermore, we will require that the cone $\sigma^\vee$ be rational, that is, $\sigma^\vee = \operatorname{cone}(v_1, \ldots, v_n)$ for some $v_1, \ldots, v_n \in \mathbb{Z}^d$, and of full dimension, that is, the $\mathbb{R}$-span of $\sigma^\vee$ is all of $\mathbb{R}^d$. We will denote the monomial elements of $k[S]$ by x^v for $v \in S$, and if $\sigma^\vee = \operatorname{cone}(v_1, \ldots, v_n)$, we may write $k[x^{v_1}, \ldots, x^{v_n}]$ for $k[S]$.

The ideals of toric rings that are of most interest to us are the monomial ideals. For a monomial ideal $I \subseteq k[S]$, we let $\operatorname{Exp} I := \{v \in S \mid x^v \in I\}$ be the *exponent set of I*. Also of interest to us will be the *convex hull of I*, also called the Newton polytope of I, which we denote by $\operatorname{Hull} I$ and which is the convex hull of $\operatorname{Exp} I$ in $\mathbb{R}^d$, that is, $\operatorname{Hull} I$ is the smallest convex set in $\mathbb{R}^d$ containing $\operatorname{Exp} I$. We will also sometimes use the descriptions given in the following lemma:

Lemma 5.0.2. *Let $I = (x^{u_1}, \ldots, x^{u_n})$ be a monomial ideal in the toric ring $k[S]$, where $S = \sigma^\vee = \operatorname{cone}(t_1, \ldots, t_m)$. We have that $\operatorname{Exp} I = \{u_1, \ldots, u_n\} + S$, and $\operatorname{Hull} I = \{\sum_i a_i u_i \mid a_i \in \mathbb{R}_{\geq 0}, \sum_i a_i = 1\} + \sigma^\vee$.*

Proof. A monomial x^v is in I if and only if it is a multiple of x^{u_i} for some i, which occurs precisely when there exists $w \in S$ such that $x^v = x^w x^{u_i}$, that is, $v = u_i + w \in u_i + S$. This shows the first statement.

For the second statement, if $v \in \operatorname{Hull} I$ then there are elements $v_1, \ldots, v_r \in \operatorname{Exp} I$ and $b_j \in \mathbb{R}_{\geq 0}$ such that $v = \sum_j b_j v_j$ and $\sum_j b_j = 1$. Therefore, by the first part, for each j there exists $1 \leq i_j \leq n$ and $w_j \in S$ such that $v_j = u_{i_j} + w_j$. Hence
$v = \sum_j b_j(u_{i_j} + w_j) = \sum_j b_j u_{i_j} + \sum_j b_j w_j \in \{\sum_i a_i u_i \mid a_i \in \mathbb{R}_{\geq 0}, \sum_i a_i = 1\} + \sigma^\vee$.

Now we claim that for each i, $u_i + \sigma^\vee \subseteq \operatorname{Hull} I$. First, take a cone generator t_j of $\sigma^\vee$, let $\alpha > 0$, and consider $u_i + \alpha t_j$. Let $\beta = \alpha - \lfloor \alpha \rfloor$. We have that $u_i + \lfloor \alpha \rfloor t_j \in \operatorname{Exp} I$ and $u_i + \lceil \alpha \rceil t_j \in \operatorname{Exp} I$, and therefore

$$u_i + \alpha t_j = (1 - \beta)\left(u_i + \lfloor \alpha \rfloor t_j\right) + \beta\left(u_i + \lceil \alpha \rceil t_j\right) \in \operatorname{Hull} I.$$

Now if $w \in \sigma^\vee \setminus \{0\}$, then there exist $w_j \geq 0$ such that $w = \sum_j w_j t_j$. Let $W = \sum_j w_j$, and observe that

$$u_i + w = u_i + \sum_j w_j t_j = \sum_j \frac{w_j u_i}{W} + \sum_j w_j t_j = \sum_j \frac{w_j}{W}\left(u_i + W t_j\right).$$

By the previous paragraph, each $u_i + W t_j \in \operatorname{Hull} I$, and also $\sum_j \frac{w_j}{W} = 1$, and so $u_i + w \in \operatorname{Hull} I$.

Now suppose that $a_i \in \mathbb{R}_{\geq 0}$ with $\sum_i a_i = 1$ and $w \in \sigma^\vee$, and consider $v = \sum_i a_i u_i + w$. For each i, let $w_i = w/a_i$ if $a_i > 0$ and $w_i = 0$ if $a_i = 0$. Now each $w_i \in \sigma^\vee$, and so by the previous paragraph, $v = \sum_i a_i u_i + w = \sum_i a_i(u_i + w_i) \in \operatorname{Hull} I$, $\square$

Example 5.0.3. Consider the toric ring $R = k[x, x^2y, x^3y^2]$. We can visualize this ring using the shaded cone in Figure 5.1. The lattice points in the shaded region to monomials in R. For instance. the point $(1, 0)$ corresponds to the monomial x, the point $(2, 1)$ corresponds to the monomial x^2y, and the point $(3, 2)$ corresponds to the monomial x^3y^2.

Consider the monomial ideal $I = (x^4, x^4y^2, x^6y^4)$. The generators of I correspond to the points $(4, 0)$, $(4, 2)$, and $(6, 4)$ in $\mathbb{Z}^d$, which are the red dots in Figures 5.2 and 5.3.

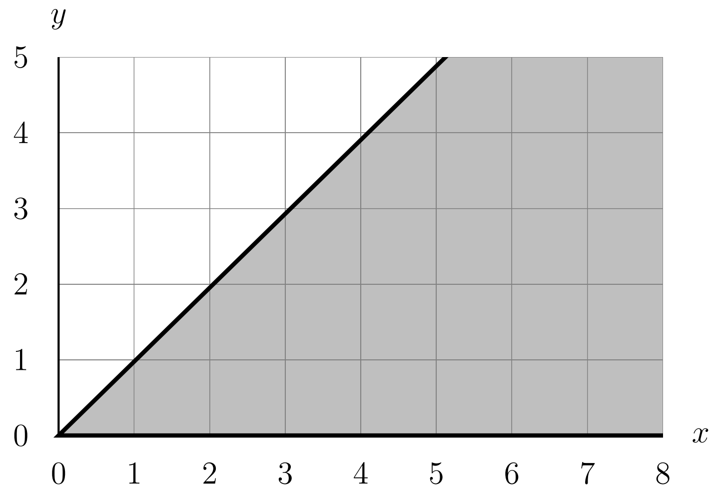


Figure 5.1

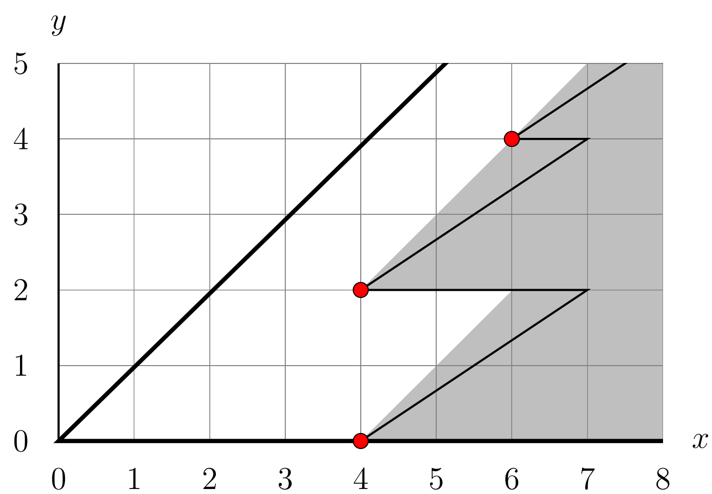


Figure 5.2

Figure 5.2 illustrates $\operatorname{Exp} I$. From every lattice point corresponding to a generator of I, we draw a copy of the shaded cone from the first figure. The lattice pointsin the shaded region of Figure 5.2 correspond to the monomials that are in I. Figure 5.3 illustrates $\operatorname{Hull} I$, which is the smallest convex set containing all the lattice points in $\operatorname{Exp} I$.

Our next result is a lemma that describes $\operatorname{Exp} I^m$ and $\operatorname{Exp} I^{[q]}$ for monomial ideals in toric rings and will be essential to the rest of the section.

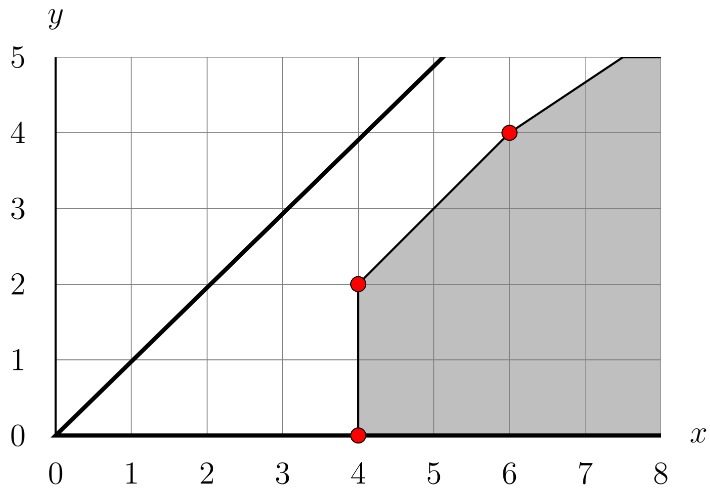


Figure 5.3

Lemma 5.0.4. *Let $R = k[S]$ be a positive characteristic toric ring of dimension d, where $S = \sigma^\vee \cap \mathbb{Z}^d$, and I be an ideal of R generated by n monomials. For any $m \in \mathbb{N}_{\geq 1}$ and q a power of p,*

$$(m+n)\operatorname{Hull} I \subseteq \operatorname{Exp} I^m + \sigma^\vee \subseteq m \operatorname{Hull} I \quad \text{and} \quad \operatorname{Exp} I^{[q]} + \sigma^\vee = q \operatorname{Exp} I + \sigma^\vee.$$

Proof. Let $x^{u_1}, \ldots, x^{u_n}$ be a set of monomial generators for I.

If $v \in (m+n)\operatorname{Hull} I$ then there exist $a_i \in \mathbb{R}_{\geq 0}$ such that $\sum_i a_i = 1$ and $v \in (m+n)\sum_i a_i u_i + \sigma^\vee$. For each $1 \leq i \leq n$, let $b_i = \lfloor (m+n)a_i \rfloor$. Since each $u_i \in \sigma^\vee$, we have that $(m+n)a_i u_i \in b_i u_i + \sigma^\vee$, and so

$$v \in (m+n)\sum_i a_i u_i + \sigma^\vee \subseteq \sum_i b_i u_i + \sigma^\vee.$$

Since $\sum_i b_i \geq (m+n)\sum_i a_i - n \geq m$, we have that $x^{\sum_i b_i u_i} \in I^m$ and so $v \in \operatorname{Exp} I^m + \sigma^\vee$. This shows the first inclusion in the first statement.

If $v \in \operatorname{Exp} I^m + \sigma^\vee$, then there exist $a_i \in \mathbb{N}$ such that $v \in \sum_i a_i u_i + \sigma^\vee$ and $\sum_i a_i = m$. Therefore $v \in m \cdot \sum_i \frac{a_i}{m} u_i + \sigma^\vee$, and since $\sum_i \frac{a_i}{m} = 1$, we have that $\sum_i \frac{a_i}{m} u_i + \sigma^\vee \subseteq \operatorname{Hull} I$.

This shows that $\operatorname{Exp} I^m + \sigma^\vee \subseteq m \operatorname{Hull} I$.

For the second statement, we have that

$$\operatorname{Exp} I^{[q]} + \sigma^\vee = \operatorname{Exp}(x^{qu_1}, \ldots, x^{qu_n}) + \sigma^\vee = \bigcup_i (qu_i + \sigma^\vee) = q \bigcup_i (u_i + \sigma^\vee) = q \operatorname{Exp} I + \sigma^\vee. \quad \square$$

5.1 The F-threshold and F-limbus in Toric Rings

The geometric way in which we may describe monomial ideals in toric rings gives us a method for computing certain numerical invariants. In particular, we may use Lemma 5.0.4 to measure the F-threshold and F-limbus for two monomial ideals I and J in toric rings. The theorem below was proved in [3, Theorem 3.3] for the F-threshold, though we include a proof of our own.

Theorem 5.1.1. *Let I and J be monomial ideals in a positive characteristic toric ring $R = k[S]$. We have that*

$$c^J(I) = \inf \{ s \mid s \operatorname{Hull} I \subseteq \operatorname{Exp} J + \sigma^\vee \}$$

and

$$b^J(I) = \sup \{ s \mid \operatorname{Exp} J \subseteq s \operatorname{Hull} I \}.$$

Proof. Suppose $s > c^J(I)$, and let n be the size of a generating set for I. For $q \gg 0$, $I^{\lceil sq \rceil} \subseteq J^{[q]}$, and so by Lemma 5.0.4,

$$(\lceil sq \rceil + n) \operatorname{Hull} I \subseteq \operatorname{Exp} I^{\lceil sq \rceil} + \sigma^\vee \subseteq \operatorname{Exp} J^{[q]} + \sigma^\vee = q \operatorname{Exp} J + \sigma^\vee.$$

Dividing the first and last terms in the inequality by q gives us that $\frac{\lceil sq \rceil + n}{q} \operatorname{Hull} I \subseteq \operatorname{Exp} J + \sigma^\vee$, and since this holds for all large q, we have that $s \operatorname{Hull} I \subseteq \operatorname{Exp} J + \sigma^\vee$.

Now suppose that $s \operatorname{Hull} I \subseteq \operatorname{Exp} J + \sigma^\vee$. For $q \geq 1$, we have that

$$\operatorname{Exp} I^{\lceil sq \rceil} \subseteq \lceil sq \rceil \operatorname{Hull} I \subseteq sq \operatorname{Hull} I \subseteq q\left(\operatorname{Exp} J + \sigma^\vee\right) = q \operatorname{Exp} J + \sigma^\vee = \operatorname{Exp} J^{[q]} + \sigma^\vee.$$

Therefore $\operatorname{Exp} I^{\lceil sq \rceil} \subseteq \left(\operatorname{Exp} J^{[q]} + \sigma^\vee\right) \cap \mathbb{Z}^d = \operatorname{Exp} J^{[q]}$, and hence $I^{\lceil sq \rceil} \subseteq J^{[q]}$. Therefore $s \geq c^J(I)$. Hence $c^J(I) = \inf\{s \mid s \operatorname{Hull} I \subseteq \operatorname{Exp} J + \sigma^\vee\}$.

Similarly, suppose that $s < b^J(I)$. For $q \gg 0$, $J^{[q]} \subseteq I^{\lceil sq \rceil}$, and so

$$\operatorname{Exp} J \subseteq \frac{1}{q}\left(q \operatorname{Exp} J + \sigma^\vee\right) \subseteq \frac{1}{q}\left(\operatorname{Exp} J^{[q]} + \sigma^\vee\right) \subseteq \frac{1}{q}\left(\operatorname{Exp} I^{\lceil sq \rceil} + \sigma^\vee\right) \subseteq \frac{1}{q}\lceil sq \rceil \operatorname{Hull} I.$$

Since this holds for all large q, we have that $\operatorname{Exp} J \subseteq s \operatorname{Hull} I$.

Now suppose that $\operatorname{Exp} J \subseteq s \operatorname{Hull} I$. For $q \geq 1$, we have that

$$\operatorname{Exp} J^{[q]} \subseteq q \operatorname{Exp} J + \sigma^\vee \subseteq sq \operatorname{Hull} I \subseteq \lfloor sq \rfloor \operatorname{Hull} I \subseteq \operatorname{Exp} I^{\lfloor sq \rfloor - n} + \sigma^\vee,$$

which shows that $J^{[q]} \subseteq I^{\lfloor sq \rfloor - n}$. Therefore, for every q, $\mu_I^J(q) > \lfloor sq \rfloor - n$, and therefore $s \leq b^J(I)$. Hence $b^J(I) = \sup\{s \mid \operatorname{Exp} J \subseteq s \operatorname{Hull} I\}$. $\square$

We begin by studying Examples 2.3.5 and 2.4.5 again in this context.

Example 5.1.2. We take $R = k[x, y, z]/(xy - z^{n+1})$, and note that we may identify R with $k[X, Y, X^{-1}Y^{n+1}]$, with the ring map sending $x \mapsto X$, $y \mapsto X^{-1}Y^{n+1}$, $z \mapsto Y$ realizing the isomorphism. Under this identification we have that $I = (x^5 z, xz^4) \mapsto (X^5 Y, XY^4)$ and $J = (x^3 z^2) \mapsto (X^3 Y^2)$. We can represent this information in Figure 5.4, in which the red points correspond to the monomial generators of I and the blue point corresponds to the monomial generator of J. The dashed line indicates the boundary of $\operatorname{Hull} I$.

We can use the geometry in this figure to calculate the F-threshold using Theorem 5.1.1. According to the theorem, the F-threshold is the minimum s such that $s \operatorname{Hull} I \subseteq \operatorname{Exp} J + \sigma^\vee$. The current situation is simpler than the general one since J is principal, which means that $\operatorname{Exp} J + \sigma^\vee$ is itself convex, which means we only need to find

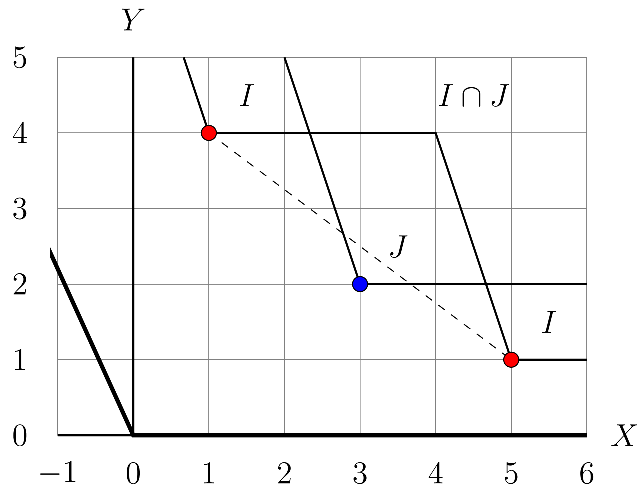


Figure 5.4

the minimum s that will send the points corresponding to the generators of I into $\operatorname{Exp} J + \sigma^{\vee}$. We illustrate the situation in Figure 5.5.

Drawing lines from the origin through the points $(1,4)$ and $(5,1)$, which correspond to the generators of I, we can find the points on those lines closest to the origin which lie in $\operatorname{Exp} J + \sigma^{\vee}$. From this we can determine that $2 \cdot (5,1) \in \operatorname{Exp} J + \sigma^{\vee}$ and $\frac{3n+5}{n+5} \cdot (1,4) \in \operatorname{Exp} J + \sigma^{\vee}$. Thus the F-threshold of I with respect to J is the maximum of these two values.

The calculation of the F-limbus is similar. In this calculation we need to find the maximum s such that $\operatorname{Exp} J \subseteq s \operatorname{Hull} I$. We illustrate this situation in Figure 5.6. In this case we can draw a line through the origin and the point corresponding to the generator of J and find where it intersects with $\operatorname{Hull} I$. We can then take the sum of the coefficients of $(3,2)$ and divide by the sum of the coefficients of the intersection $\left(\frac{57}{17}, \frac{38}{17}\right)$ to obtain that $b^J(I) = 5 \cdot \frac{17}{95} = \frac{17}{19}$.

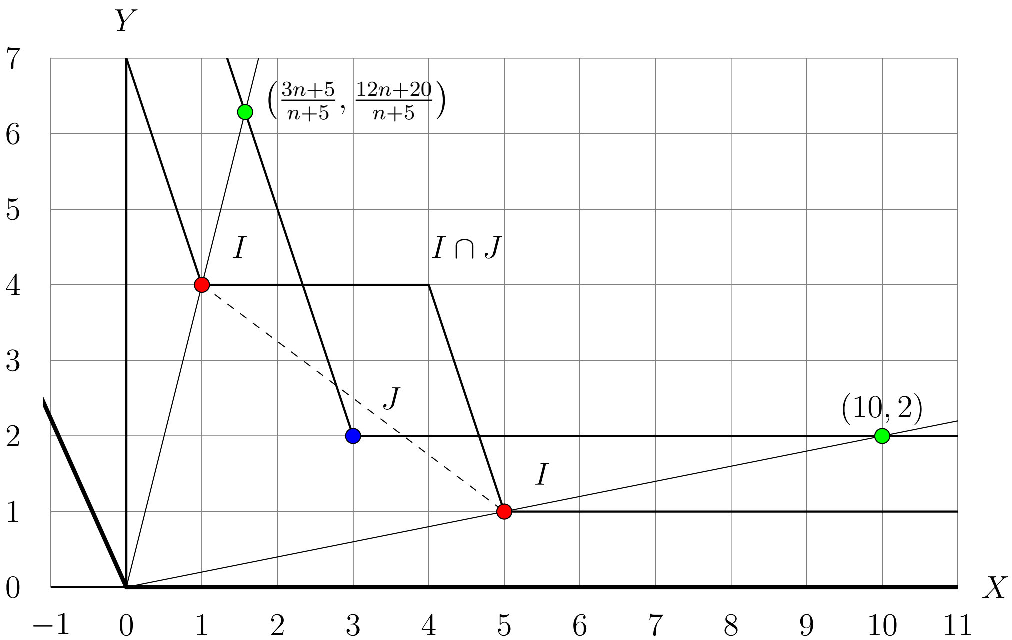


Figure 5.5

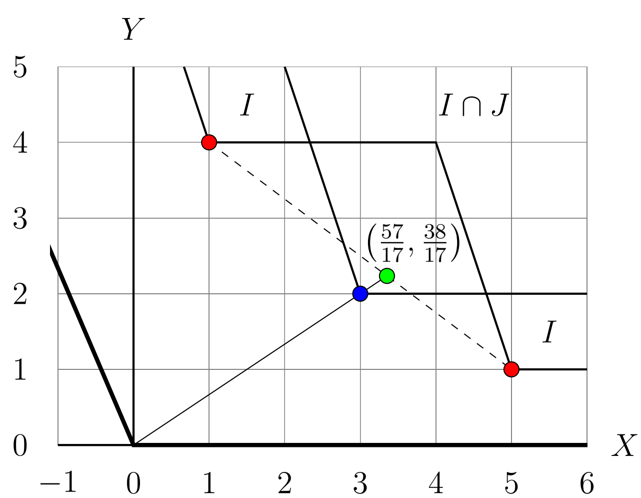


Figure 5.6

5.2 s-Multiplicity in Toric Rings

Next we consider the s-multiplicity of semigroup rings. Since we only defined the s-multiplicity for local rings, we will consider the completion of a semigroup ring $k[S]$ at its unique maximal homogeneous ideal. The ring we obtain is the power series ring $k[[S]]$ in the monomials that generate $k[S]$ as a k-algebra.

In this section we relate the s-multiplicity function for toric rings to certain volumes in Euclidean space, and use this to compute the s-multiplicity for a few toric rings. See [2] for a more general treatment of the correspondence between limits in positive characteristic and volumes in real space.

Theorem 5.2.1. *Let $(R, \mathfrak{m}) = (k[[S]], (S))$ be the completion of a normal toric ring of dimension d over a field k of characteristic $p > 0$, where $S = \sigma^\vee \cap \mathbb{Z}^d$, and I and J be $\mathfrak{m}$-primary monomial ideals of R. We have that*

$$h_s(I, J) = \operatorname{vol}\left(\sigma^\vee \setminus (s \operatorname{Hull} I \cup (\operatorname{Exp} J + \sigma^\vee)\right),$$

where $\operatorname{vol}(-)$ *is the standard Euclidean volume in* $\mathbb{R}^d$.

Proof. Let q be a power of p, and let n be the size of a generating set for I. The length of $R/(I^{\lceil sq \rceil} + J^{[q]})$ is precisely the size of the set

$$V_q := \left\{ v \in S \mid x^v \notin I^{\lceil sq \rceil} + J^{[q]} \right\} = \left\{ v \in S \mid v \notin \operatorname{Exp} I^{\lceil sq \rceil} \cup \operatorname{Exp} J^{[q]} \right\}.$$

From Lemma 5.0.4, we have that

$$(\sigma^\vee \setminus (\lceil sq \rceil \operatorname{Hull} I \cup q \operatorname{Exp} J + \sigma^\vee)) \cap \mathbb{Z}^d \subseteq V_q \subseteq (\sigma^\vee \setminus ((\lceil sq \rceil + n) \operatorname{Hull} I \cup q \operatorname{Exp} J + \sigma^\vee)) \cap \mathbb{Z}^d.$$

Since the volume of $\sigma^\vee \setminus (s\,\mathrm{Hull}\, I \cup \mathrm{Exp}\, J + \sigma^\vee)$ is equal to the volume of its interior,

$$\begin{aligned}
&\mathrm{vol}\left(\sigma^\vee \setminus (s\,\mathrm{Hull}\, I \cup \mathrm{Exp}\, J + \sigma^\vee)\right)\\
&= \lim_{q\to\infty} \frac{1}{q^d} \left| (\sigma^\vee \setminus (s\,\mathrm{Hull}\, I \cup \mathrm{Exp}\, J + \sigma^\vee)) \cap \left(\frac{1}{q}\mathbb{Z}\right)^d \right|\\
&= \lim_{q\to\infty} \frac{1}{q^d} \left| (\sigma^\vee \setminus (sq\,\mathrm{Hull}\, I \cup q\,\mathrm{Exp}\, J + \sigma^\vee)) \cap \mathbb{Z}^d \right|\\
&\leq \lim_{q\to\infty} \frac{1}{q^d} |V_q|\\
&\leq \lim_{q\to\infty} \frac{1}{q^d} \left| (\sigma^\vee \setminus ((\lceil sq+n \rceil)\,\mathrm{Hull}\, I \cup q\,\mathrm{Exp}\, J + \sigma^\vee)) \cap \mathbb{Z}^d \right|\\
&= \lim_{q\to\infty} \frac{1}{q^d} \left| (\sigma^\vee \setminus ((\lceil sq+n \rceil / q)\,\mathrm{Hull}\, I \cup \mathrm{Exp}\, J + \sigma^\vee)) \cap \left(\frac{1}{q}\mathbb{Z}\right)^d \right|\\
&= \mathrm{vol}\left(\sigma^\vee \setminus (s\,\mathrm{Hull}\, I \cup \mathrm{Exp}\, J + \sigma^\vee)\right),
\end{aligned}$$

and so we have equality throughout. Since $h_s(I, J) = \lim_{q\to\infty} \frac{1}{q^d}|V_q|$, the theorem is proved. □

Theorem 5.2.1 allows us to calculate the s-multiplicity of toric rings. We compute two examples.

Example 5.2.2 (A_n Singularities)**.** Let $n \in \mathbb{N}$, $n \geq 1$, and let

$$A_n = k[[x, y, z]]/(xy - z^{n+1}) \cong k[[X, Y, X^{-1}Y^{n+1}]].$$

The geometry of this toric ring is illustrated in Figure 5.7 in the case $n = 2$, though our calculations will be for general n. The shaded region corresponds to the cone $\sigma^\vee$, and the lattice points $(1, 0)$, $(0, 1)$, and $(-1, n+1)$ correspond to X, Y, and $X^{-1}Y^{n+1}$, respectively. We wish to calculate $e_s(A_n)$, so we need to calculate $\mathrm{Hull}\,\mathfrak{m}$ and $\mathrm{Exp}\,\mathfrak{m} + \sigma^\vee$ where $\mathfrak{m} = (X, Y, X^{-1}Y^{n+1})$. These are illustrated in Figure 5.8.

There are three situations to consider: $s \leq 1$, $1 \leq s \leq 2 - \frac{1}{n+1}$, and $s \geq 2 - \frac{1}{n+1}$. When $s \leq 1$, $s\,\mathrm{Hull}\,\mathfrak{m} \cup \mathrm{Exp}\,\mathfrak{m} + \sigma^\vee$ is illustrated is Figure 5.9a, and from this we can compute

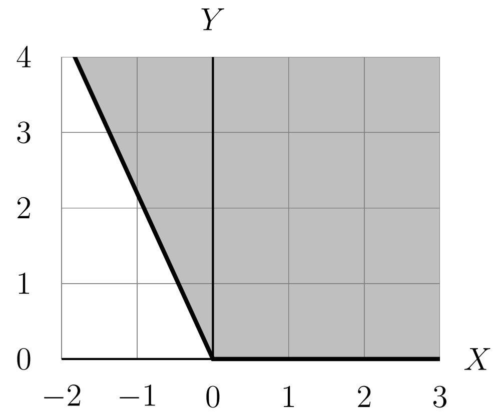


Figure 5.7

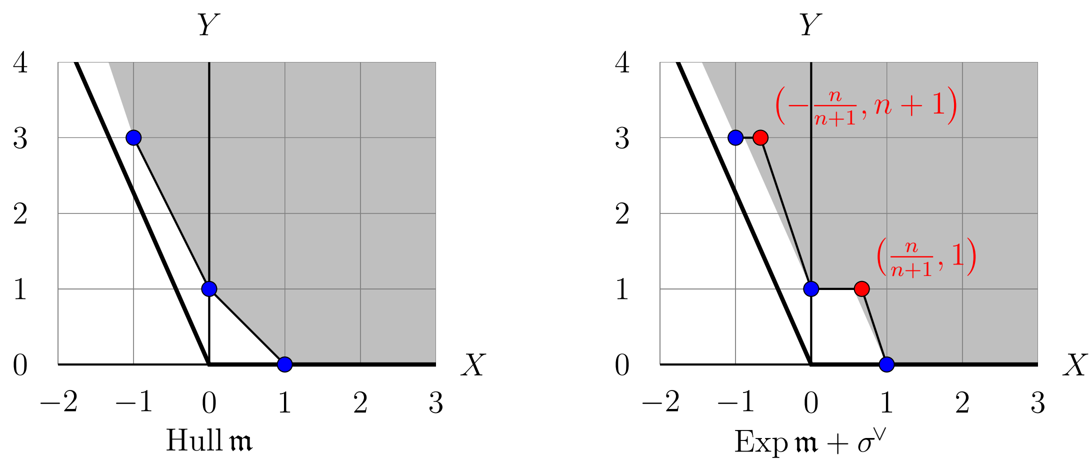


Figure 5.8

$h_s(A_n) = s^2$ for $s \leq 1$.

Now suppose that $1 \leq s \leq 2 - \frac{1}{n+1}$. The picture now becomes Figure 5.9b. Calculating the area of the unshaded region in $\sigma^\vee$ gives

$$h_s(A_n) = -\frac{n+1}{n}(s-1)^2 + 2(s-1) + 1$$

when $1 \leq s \leq 2 - \frac{1}{n+1}$.

Now consider the case when $s \geq 2 - \frac{1}{n+1}$. In this case the picture becomes Figure 5.9c and so we compute $h_s(A_n) = 2 - \frac{1}{n+1}$ when $s \geq 2 - \frac{1}{n+1}$. With this, we can write down the

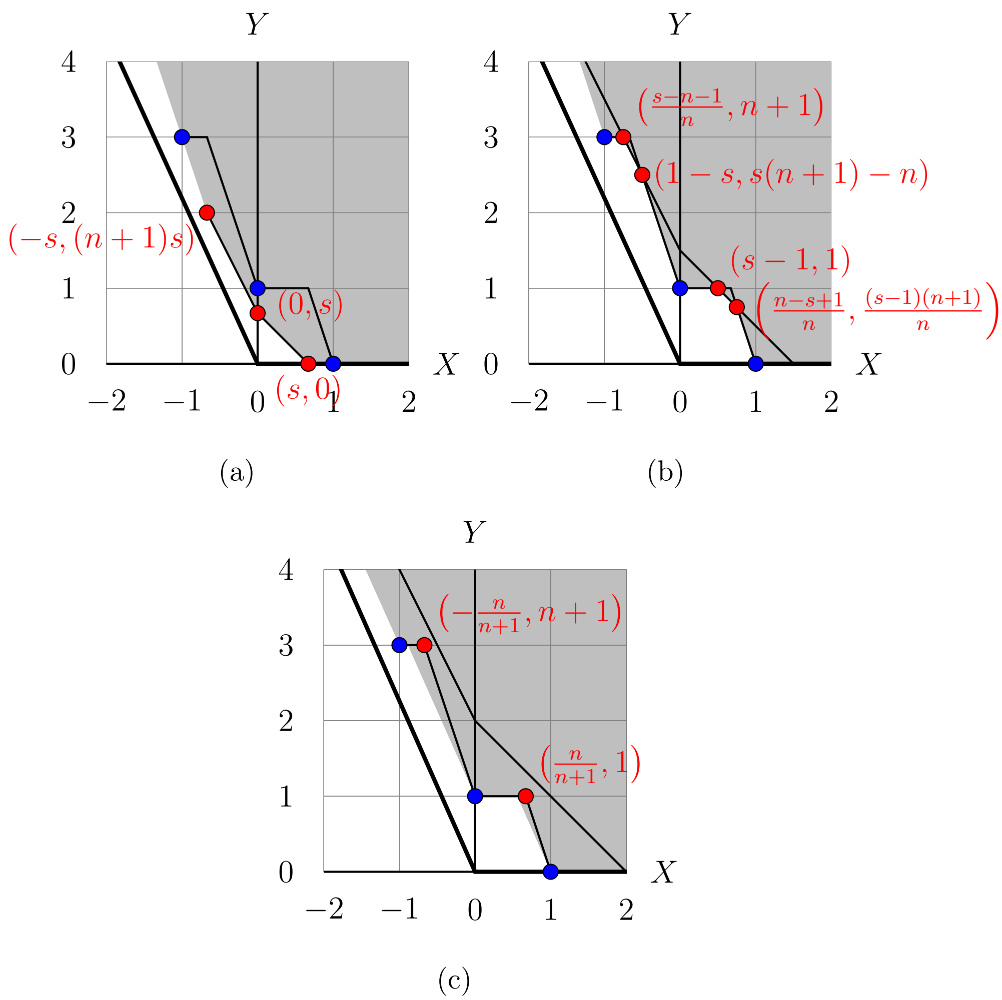


Figure 5.9

s-multiplicity for the A_n singularities.

$$e_s(A_n) = \begin{cases} 2 & \text{if } 0 < s < 1 \\ \frac{-\frac{n+1}{n}(s-1)^2+2(s-1)+1}{\frac{1}{2}s^2-(s-1)^2} & \text{if } 1 \leq s < 2 - \frac{1}{n+1} \\ \frac{2-\frac{1}{n+1}}{\frac{1}{2}s^2-(s-1)^2} & \text{if } 2 - \frac{1}{n+1} \leq s < 2 \\ 2 - \frac{1}{n+1} & \text{if } s \geq 2. \end{cases}$$

Example 5.2.3. Let k be a field, and consider the ring $V_n = k[[x, xy, \ldots, xy^n]]$. The geometry of this ring is illustrated in Figure 5.10 below; the shaded region corresponds to

$\sigma^\vee$ and for $0 \leq a \leq n$, the lattice points $(1, a)$ corresponds to the monomial xy^a.

Letting $\mathfrak{m} = (x, xy, \ldots, xy^n)$, Figure 5.10 also illustrates $\operatorname{Hull}\mathfrak{m}$ and $\operatorname{Exp}\mathfrak{m} + \sigma^\vee$. Thus Figure 5.11 illustrates $s \operatorname{Hull}\mathfrak{m} \cup \operatorname{Exp}\mathfrak{m} + \sigma^\vee$ for various values of s.

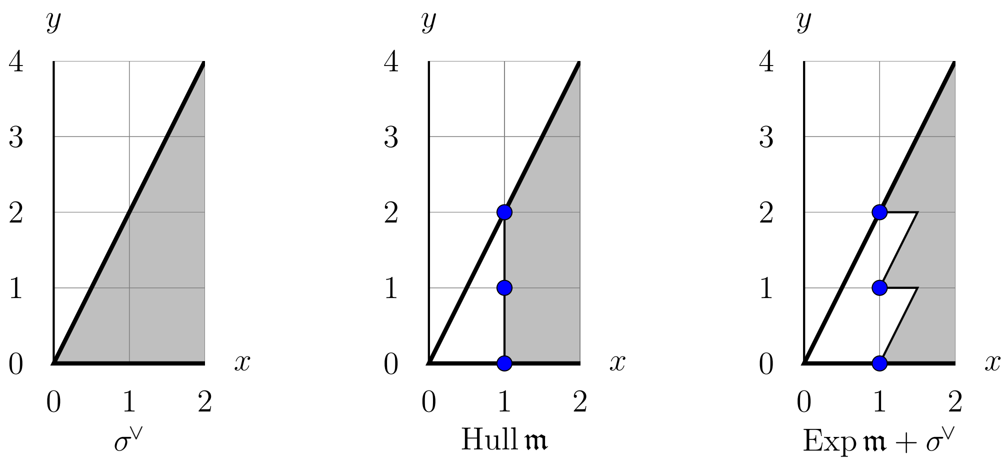
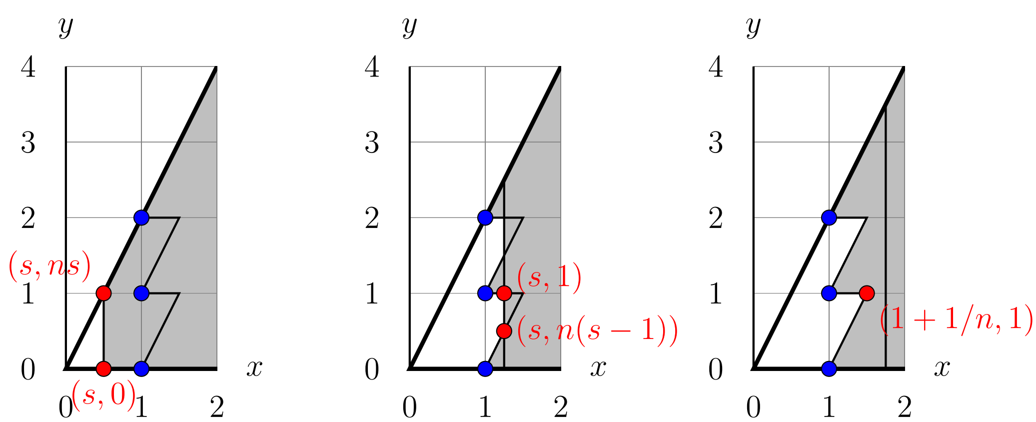


Figure 5.10



$0 \leq s \leq 1$ $\quad$ $1 \leq s \leq 1 + 1/n$ $\quad$ $s \geq 1 + 1/n$

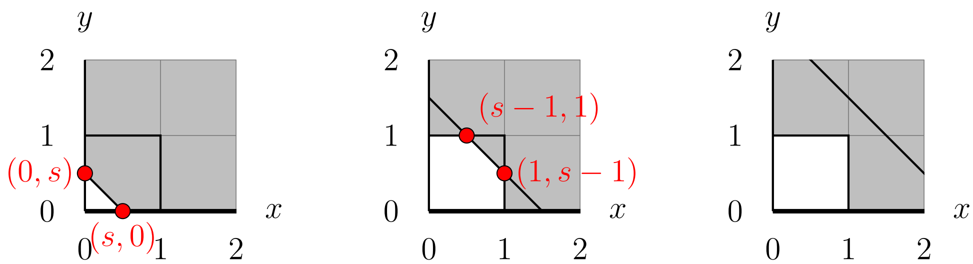
Figure 5.11

With these figures we can calculate $h_s(V_n)$ and $e_s(V_n)$:

$$h_s(V_n) = \begin{cases} \frac{ns^2}{2} & \text{if } 0 < s \leq 1 \\ -\frac{n^2}{2}(s-1)^2 + n(s-1) + \frac{n}{2} & \text{if } 1 \leq s \leq 1 + 1/n \\ \frac{n+1}{2} & \text{if } s \geq 1 + 1/n \end{cases}$$

$$e_s(V_n) = \begin{cases} n & \text{if } 0 < s < 1 \\ \frac{-n^2(s-1)^2+2n(s-1)+n}{s^2-2(s-1)^2} & \text{if } 1 \le s < 1 + 1/n \\ \frac{n+1}{s^2-2(s-1)^2} & \text{if } 1+1/n \le s < 2 \\ \frac{n+1}{2} & \text{if } s \ge 2. \end{cases}$$

Example 5.2.4. The normalizing factors $\mathcal{H}_s(d)$ can be easily visualized as areas in space in the same manner. Indeed, since $k[[x_1, \ldots, x_d]]$ is a toric ring, we simply apply the construction above to calculate $h_s((x_1, \ldots, x_d))$. For instance, when $d = 2$, we have Figure 5.12.



$0 < s \le 1$ $\quad$ $1 \le s \le 2$ $\quad$ $s \ge 2$

$\mathcal{H}_s(2) = \frac{1}{2}s^2$ $\quad$ $\mathcal{H}_s(2) = \frac{1}{2}s^2 - (s-1)^2$ $\quad$ $\mathcal{H}_s(2) = 1$

Figure 5.12

5.3 s-Closures in Toric Rings

We can consider the various s-closures we defined earlier in the specific case of toric rings. Toric rings are examples of graded rings, which have many good properties that we can exploit. Our first result using these properties is Theorem 5.3.1, which shows that homogeneous ideals have homogeneous s-closures. In the proof below, for a $\mathbb{Z}^n$-graded ring R and an element $c = \sum_{i \in \mathbb{Z}^n} c_i$, where the degree of each c_i is $i \in \mathbb{Z}^n$, we call $\operatorname{Supp}(c) := \{i \in \mathbb{Z}^n \mid c_i \neq 0\}$ the *support of* c. Furthermore, by the *diameter* of c we mean $\max\{\|i - i'\|_\infty \mid i, i' \in \operatorname{Supp}(c)\}$, where for $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $\|i\|_\infty = \max_{i \le \ell \le n} |i_\ell|$.

Theorem 5.3.1. *If R is a $\mathbb{Z}^n$-graded ring of positive characteristic, I is a homogeneous ideal of R, and $s \ge 1$, then $I^{\mathrm{w.cl}_s}$ and I^{cl_s} are homogeneous ideals. Furthermore, if*

$x \in I^{\mathrm{w.cl}_s}$, there exists a nonzero homogeneous element c such that $cx^q \in I^{\lceil sq \rceil} + I^{[q]}$ for all $q \gg 0$.

Proof. Let $x = \sum_{j \in \mathbb{Z}^n} x_j \in I^{\mathrm{w.cl}_s}$. There exists $0 \neq c = \sum_{i \in \mathbb{Z}^n} c_i \in R$ such that $cx^q \in I^{\lceil sq \rceil} + I^{[q]}$ for all $q \gg 0$.

If $c_i x_j^q \neq 0$ and $c_{i'} x_{j'}^q \neq 0$ have the same degree, then $i + qj = i' + qj'$, and so $i - i' = q(j' - j)$. If in addition q is greater than the diameter of c, we must have that $j = j'$ and $i = i'$. Therefore each nonzero homogeneous component of cx^q is $c_i x_j^q$ for some i, j. Since I is homogeneous, so is $I^{\lceil sq \rceil} + I^{[q]}$, and therefore for $q \gg 0$, we have that $c_i x_j^q \in I^{\lceil sq \rceil} + I^{[q]}$. This shows that each $x_j \in I^{\mathrm{w.cl}_s}$ and that for each c_i, $c_i x^q \in I^{\lceil sq \rceil} + I^{[q]}$ for $q \gg 0$.

Since $I^{\mathrm{w.cl}_s}$ is homogeneous, so is $\left(I^{\mathrm{w.cl}_s}\right)^{\mathrm{w.cl}_s}$, and each time we take the weak s-closure we preserve homogeneity. Since I^{cl_s} is the directed union of homogeneous ideals, it is homogeneous. $\square$

Toric rings are naturally $\mathbb{Z}^d$-graded, since they are subrings of $k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. This grading is also called the monomial grading, since all the homogeneous elements of $k[S]$ are of the form αx^v, where $\alpha \in k$ and $v \in S$. Just as there is a way to calculate the F-threshold, F-limbus, and s-multiplicity of monomial ideals using the geometry associated to the toric ring, there is also a way to calculate the s-closure of a monomial ideal in a toric ring. This method uses the same construction as the one we use to calculate the s-multiplicity in the previous section, but instead of calculating area, we find the lattice points contained in the region we define.

Theorem 5.3.2. *Let $R = k[S]$ be a toric ring, where $S = \sigma^\vee \cap \mathbb{Z}^d$ and k is a field of characteristic $p > 0$. If I is a monomial ideal of R, then I^{cl_s} is a monomial ideal of R, $I^{\mathrm{cl}_s} = I^{\mathrm{w.cl}_s}$, and $\operatorname{Exp} I^{\mathrm{cl}_s} = \operatorname{Exp} I \cup (s \operatorname{Hull} I \cap \mathbb{Z}^d)$.*

Proof. Let $x^{b_1}, \ldots, x^{b_m}$ be a set of monomial generators for I. By Theorem 5.3.1, I^{cl_s} is a monomial ideal containing I.

Let x^a be a monomial of R. If $a \in \operatorname{Exp} I$, then $x^a \in I \subseteq I^{\mathrm{w.cl}_s}$. Suppose $a \in s \operatorname{Hull} I$ and let $c = \sum_i b_i \in S$. There exist $t_1, \dots, t_m \geq 0$ such that $\sum_i t_i = s$ and $a \in \sum_i t_i b_i + \sigma^\vee$. Thus, $aq + c \in \sum_i (qt_i + 1) b_i + \sigma^\vee \subseteq \sum_i \lceil qt_i \rceil + \sigma^\vee$. Therefore, $x^c (x^a)^q = x^{aq+c} \in x^{\lceil qt_1 \rceil b_1} \cdots x^{\lceil qt_m \rceil b_m} R \subseteq I^{\sum_i \lceil qt_i \rceil} \subseteq I^{\lceil sq \rceil}$. Hence $x^a \in I^{\mathrm{w.cl}_s} \subseteq I^{\mathrm{cl}_s}$.

Now suppose that $x^a \in I^{\mathrm{w.cl}_s}$, and let $x^c \in R$ such that $x^c (x^a)^q = x^{aq+c} \in I^{\lceil sq \rceil} + I^{[q]}$ for $q \gg 0$. If $x^{aq+c} \in I^{[q]}$ for infinitely many q, then there exists i such that for infinitely many q, $x^{aq+c} \in ((x_i^b)^q)$ and therefore $aq + c \in qb_i + \sigma^\vee$. Hence, $a \in b_i - (c/q) + \sigma^\vee$ for infinitely many q and so $a \in b_i + \sigma^\vee \subseteq \operatorname{Exp} I$.

Suppose now that $x^{aq+c} \in I^{\lceil sq \rceil}$ for $q \gg 0$. Thus for all sufficiently large q there exist $a_i \in \mathbb{N}$ such that $\sum_i a_i = \lceil sq \rceil$ and $aq + c \in \sum_i a_i b_i + \sigma^\vee$. Hence $a \in -c/q + \sum_i a_i/q + \sigma^\vee$. Since $\sum_i a_i = \lceil sq \rceil$, we have that $\sum_i a_i/q = \lceil sq \rceil / q \geq s$, and so $a \in -c/q + s \operatorname{Hull} I$. Since this holds for all $q \gg 0$, $a \in s \operatorname{Hull} I$.

The above two arguments show that $\operatorname{Exp} I^{\mathrm{w.cl}_s} = \operatorname{Exp} I \cup (s \operatorname{Hull} I \cap \mathbb{Z}^d)$. Since $I^{\mathrm{w.cl}_s} \subseteq \overline{I}$, we have that $\operatorname{Hull} I^{\mathrm{w.cl}_s} \subseteq \operatorname{Hull} \overline{I} = \operatorname{Hull} I$, and so $s \operatorname{Hull} I^{\mathrm{w.cl}_s} \subseteq s \operatorname{Hull} I$. Therefore,

$$\begin{aligned} \operatorname{Exp} (I^{\mathrm{w.cl}_s})^{\mathrm{w.cl}_s} &= \operatorname{Exp} I^{\mathrm{w.cl}_s} \cup (s \operatorname{Hull} I^{\mathrm{w.cl}_s} \cap \mathbb{Z}^d) \\ &= \operatorname{Exp} I \cup (s \operatorname{Hull} I \cap \mathbb{Z}^d) \cup (s \operatorname{Hull} I^{\mathrm{w.cl}_s} \cap \mathbb{Z}^d) \\ &= \operatorname{Exp} I \cup (s \operatorname{Hull} I \cap \mathbb{Z}^d) \\ &= \operatorname{Exp}(I^{\mathrm{w.cl}_s}). \end{aligned}$$

Therefore $I^{\mathrm{w.cl}_s} = (I^{\mathrm{w.cl}_s})^{\mathrm{w.cl}_s}$ and so $I^{\mathrm{w.cl}_s} = I^{\mathrm{cl}_s}$. □

Example 5.3.3. Let $R = k[x, y]$ and $I = (x^4, y^3)$. The red points in Figure 5.13 are the points in $\operatorname{Exp} I$, while the shaded regions are $s \operatorname{Hull} I$ for various values of s.

As we vary s, the lattice points that lie outside of $\operatorname{Exp} I$ but inside $s \operatorname{Hull} I$ are additional monomials that belong to I^{cl_s}. For instance, when $1 \leq s \leq \frac{13}{12}$, the blue points in Figure 5.13a correspond to the monomials in $I^{\mathrm{cl}_s} \setminus I$. Therefore, $I^{\mathrm{cl}_s} = (x^4, x^3 y, x^2 y^2, y^3)$.

Similarly, when $\frac{13}{12} < s \leq \frac{7}{6}$, Figure 5.13b gives us that $I^{\mathrm{cl}_s} = (x^4, x^2y^2, y^3)$. When $\frac{7}{6} < s \leq \frac{17}{12}$, Figure 5.13c gives us that $I^{\mathrm{cl}_s} = (x^4, x^3y^2, y^3)$. Finally, when $s > \frac{17}{12}$, we have that $I^{\mathrm{cl}_s} = (x^4, y^3) = I$, as shown in Figure 5.13d.

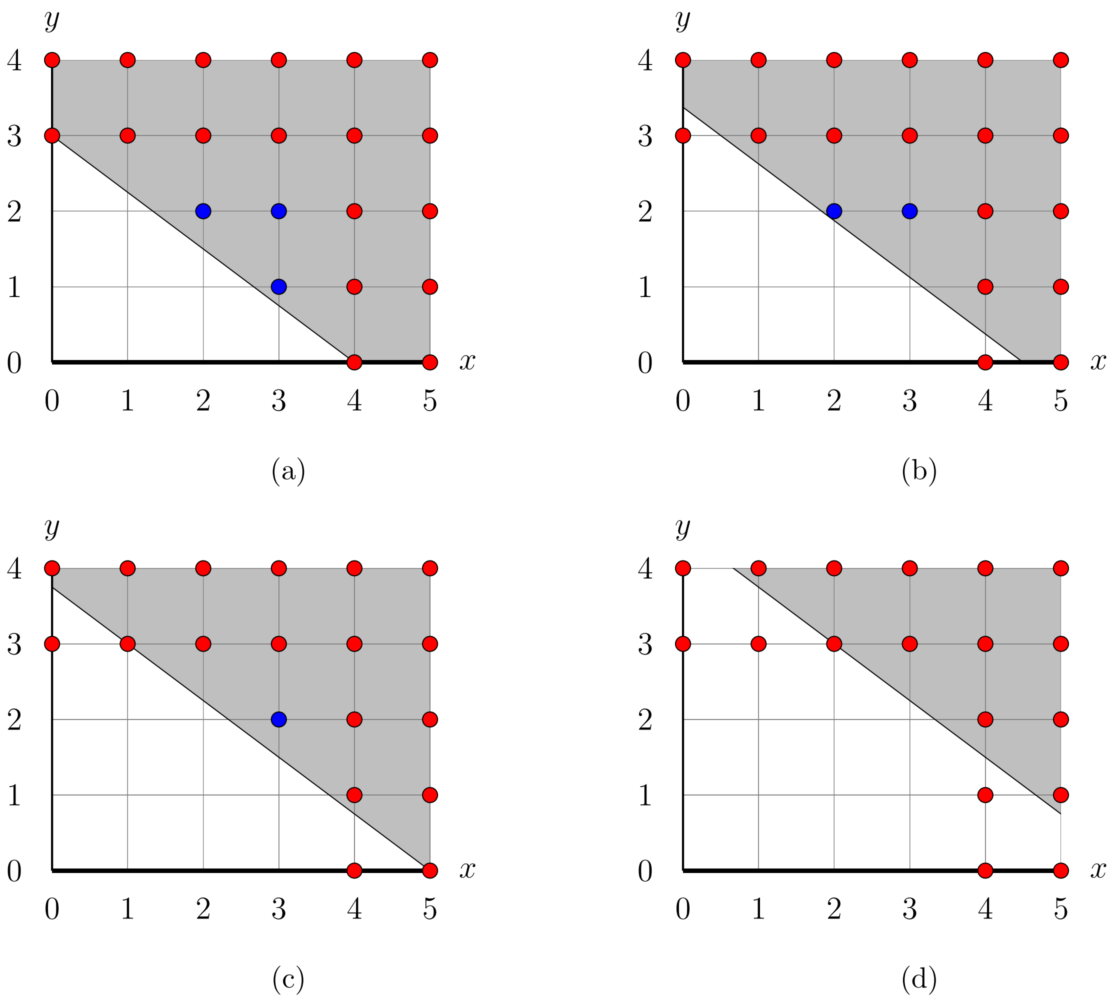


Figure 5.13

In summmary, we have that

$$I^{\mathrm{cl}_s} = \begin{cases} (x^4, x^3y, x^2y^2, y^3) = \overline{I} & 1 \leq s \leq 13/12 \\ (x^4, x^2y^2, y^3) & 13/12 < s \leq 7/6 \\ (x^4, x^3y^2, y^3) & 7/6 < s \leq 17/12 \\ (x^4, y^3) = I & s > 17/12. \end{cases}$$

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