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Smoothness of Defining Functions and the Diederich-Fornæss Index

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Smoothness of Defining Functions and the Diederich-Fornæss Index

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by

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Abstract

Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain, and let $M \subset \partial\Omega$ be a complex submanifold with rectifiable boundary. In 2017, Harrington studied the equation $d_M A = \tilde{\alpha}$ on M, where $\tilde{\alpha}$ is D'Angelo's 1-form and A is real. In this thesis, we will study a non-pseudoconvex example in which M has a non-rectifiable boundary. In spite of the lack of topological obstructions on the boundary, there are no continuous solutions to $d_M A = \tilde{\alpha}$.

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Dedication

This Dissertation is dedicated to my support system at the University of Arkansas and University of the Bahamas. I would have never started nor finished my dissertation if it were not for the love and care you all have shown me.

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1 Introduction

In part one of this thesis, we will begin by building on the foundations of complex analysis of a single variable. Let $\phi(x) = x \sin x^{-3}$ when $x \neq 0$ and $\phi(x) = 0$ when $x = 0$. This curve lies inside a simply-connected domain $\mathcal{O} \subseteq \mathbb{R}^2$ and $\mathcal{O}_x\{(x_1, x_2) : |x_2 - \phi(x_1)| \leq |x_1|^{12}\}$ with a non-rectifiable boundary. Therefore, our main result in section 2 will be the following theorem:

Theorem 1.1. *There exists an open set $\mathcal{O} \subseteq \mathbb{R}^2$, and a harmonic function $h \in C^2(\overline{\mathcal{O}})$ but the harmonic conjugate of h is not continuous on $\overline{\mathcal{O}}_x$.*

To do this, we will straighten the curve $y = \phi(x)$ and show that the hypothesis of the Cauchy-Kovalevsky Theorem 2.5 (see [4] and [16]) are satisfied for suitable Cauchy conditions. We want to show that a solution exists using the Cauchy-Kovalevsky Theorem along with the Cauchy Estimates. We want the function $u(y_1, y_2)$ (see (2.36)), which is derived from the pullback of our harmonic function h after straightening $y = \phi(x)$ to satisfy the Cauchy conditions (2.26) in Theorem 2.5. Together with the Cauchy estimates, this will imply that u is a C^2 solution, but we will see that the harmonic conjugate of h is not continuous.

In the section 3, we then turn to complex analysis of several variables. First we want to define a domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - e^{ih(z_2)}|^2 < g(z_2)\}$ where $h(z_2)$ is real-valued and $g(z_2) \geq 0$. This domain was inspired by Diederich and Fornæss' Worm Domain [5] . It is necessary for Ω to be C^2 and bounded. Therefore, the defining function $\rho(z_1, z_2) = |z_1 - e^{ih(z_2)}|^2 - g(z_2)$ must satisfy:

i. ρ is C^2 on a neighborhood of $\overline{\Omega}$,

ii. $\Omega = \{(z_1, z_2) \in \rho(z_1, z_2) < 0\}$,

iii $\nabla\rho \neq 0$ on $\partial\Omega$.

In order to ensure Ω has the necessary properties, we require that $g(z_2)$ satisfy:

i. $g(z_2) \leq 0$ outside a compact subset of $\mathbb{C}$,

ii. $\nabla g(z_2) \neq 0$ if $g(z_2) = 0$.

We also want to study when our defining function defines a Levi pseudoconvex domain Ω. This occurs when the Levi-form is positive semi-definite [5]. The Levi-form is given by

$$i\partial\overline{\partial}\rho(t,t)(p) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k$$

for all $t \in T_p^{1,0}(\partial\Omega) = \left\{ t = (t_1, ..., t_n) \in \mathbb{C}^n : \sum_{j=1}^{n} t_j \left(\frac{\partial \rho}{\partial z_j} \right)(p) = 0 \right\}$ where $T_p^{1,0}(\partial\Omega)$ is the space of type $(1,0)$ vector fields which are tangent to the boundary at the point p. In particular, if the Levi-form is positive definite, then the domain is called strongly pseudoconvex. The geometry of pseudoconvex domains is an important part of the study of Several Complex Variables (see [5] and [22]). Next, we will introduce an analytic disc $M \subset \partial\Omega$ that is biholomorphic to the domain $\mathcal{O}$ constructed in part one. The Levi form vanishes on M and so $\partial\Omega$ is at best weakly pseudoconvex on M. On M, we also want to study when we have a plurisubharmonic defining function in Lemma 3.4.

In 1942, Lelong [17] and Oka [20] were the first to define plurisubharmonic functions. A model example of a plurisubharmonic function in their paper is the logarithm of the modulus of a holomorphic mapping.

This dissertation work was motivated by Harrington's paper, "The Diederich-Fornæss Index and Good Vector Fields" (see [12] Remark 2.6) and Liu [18]. Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain, and let $M \subset \partial\Omega$ be a complex submanifold with rectifiable boundary. They studied the equation $d_M A = \tilde{\alpha}$ on M, where $\tilde{\alpha}$ is D'Angelo's 1-form and A is real. In this thesis, we will study a non-pseudoconvex example in which M has a non-rectifiable boundary. Unfortunately, we were unable to construct a pseudoconvex domain because the proof falls apart at the boundary in Lemma 3.5. We built an example where the equation $d_M A = \tilde{\alpha}$, and where A is the harmonic conjugate of h. But since the harmonic conjugate of h is discontinuous, there is no real continuous solution to $d_M A = \tilde{\alpha}$. Therefore, we were able to come up with the following:

Theorem 1.2. *There exists a C^2 bounded domain $\Omega \subset \mathbb{C}^2$ such that for some $M_2 \subset \mathbb{C}$, $M = \{(0, z_2) : z_2 \in M_2\} \subset \partial\Omega$ and on M,*

$$\tilde{\alpha} \equiv i \frac{\partial h(z_2)}{\partial \overline{z_2}} d\overline{z}_2 \mod dz_1, dz_2, d\overline{z}_1,$$

where h is a real harmonic function on M_2 but the harmonic conjugate of h is not continuous on M_2.

The D'Angelo 1−form $\tilde{\alpha}$ ([6] and [22]) on M will be defined in Section 3.1.1. By studying the equation $d_M A = \tilde{\alpha}$, we see that $\tilde{\alpha}$ is not exact on M because it does not have a real continuous solution A on M. D'Angelo's useful 1−form $\tilde{\alpha}$ is a geometric invariant that encodes information about the existence of plurisubharmonic defining functions and the Diederich-Fornæss index. We will demonstrate this in Sections 3.1.1 and 3.1.2. The Worm domain of Diederich and Fornæss ([22] Lemma 5.20) is an example of a domain that does not admit a defining function that is plurisubharmonic on the boundary. Boas and Straube [3] have shown that this phenomenon can be understood in terms of $\tilde{\alpha}$.

In [8], Diederich and Fornæss proved that for every bounded pseudoconvex domain Ω with C^2 boundary in $\mathbb{C}^n$, there exists a defining function ρ and an exponent $0 < \eta < 1$ such that $-(-\rho)^\eta$ is strictly plurisubharmonic on Ω. The Diederich-Fornæss index is the supremum over all such exponents and it is a number that measures the strength of hyperconvexity. A domain is said to be hyperconvex if it admits a bounded plurisubharmonic exhaustion function. If $\partial\Omega$ is strictly pseudoconvex, we know that $\partial\Omega$ admits a strictly plurisubharmonic defining function, hence $\eta(\Omega) = 1$. In order for Ω to have positive exponent η, Ω must be pseudoconvex. This result of Diederich and Fornæss was generalized by Kerzman and Rosay [14] to domains with C^1 boundary. This was generalized further by Demailly [7] and Harrington [10] to domains with Lipschitz boundary. For a given bounded pseudoconvex domain in $\mathbb{C}^n$, it is difficult to compute the Diedrich-Fornæss index. However, Diederich and Fornæss show that the Diederich-Fornæss index for the Worm domain Ω_γ goes to zero as γ goes to infinity, where γ is the winding of Ω_γ. Liu [18] has explicitly computed the Diederich-Fornæss index for Ω_γ. A few recent results due to Fornæss and Herbig [9] showed that a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function that is plurisubharmonic on the boundary has Diederich-Fornæss index 1. In further work, we would like to show that M can be embedded in the boundary of a pseudoconvex domain Ω with the same $\tilde{\alpha}\big|_M$. We would also like to show that h (and hence Ω) can be constructed to be a C^k function for some $k > 2$. Our ultimate goal is to construct an example for which the Diederich-Fornæss Index can be improved by considering non-smooth defining functions.

2 Part 1- Complex Analysis in One Variable With PDEs

2.1 Drill Bit Domain

Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} 0 & x = 0 \\ x \sin x^{-3} & x \neq 0. \end{cases}$$

Then we define the hypersurface $\Gamma_x = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \phi(x_1)\}$ on $\mathcal{O}_x$, an open set such that $\Gamma_x \backslash \{0\} \subset \mathcal{O}_x \backslash \{0\}$ and $\mathcal{O}_x = \{(x_1, x_2) : |x_2 + \phi(x_1)| < |x_1|^{12}\}$.

Theorem 2.1. *There exists an open set $\mathcal{O}_x \subseteq \mathbb{R}^2$, and a harmonic function $h \in C^2(\overline{\mathcal{O}}_x)$ but the harmonic conjugate of h is not continuous on $\overline{\mathcal{O}}_x$.*

The rest of section 2 is proof of Theorem 2.1. First, we want to prove that there exists a harmonic function h, which is C^2 on $\overline{\mathcal{O}}_x$.

Lemma 2.2. *Let $h(x) : \mathcal{O}_x \mapsto \mathbb{R}$ be a harmonic function such that:*

$$\begin{cases} \Delta_x h(x) = 0 & on \quad \mathcal{O}_x \\ h(x) = x_1^2 & on \quad \Gamma_x \\ \dfrac{\partial h(x)}{\partial x_2} = x_1^7 & on \quad \Gamma_x, \end{cases}$$

then h is C^2 on Γ_x.

Proof. We want h to be twice differentiable with uniformly bounded second derivatives on Γ_x. We compute

$$\frac{\partial}{\partial x_1}(h(x_1, \phi(x_1))) = \frac{\partial h}{\partial x_1}(x_1, \phi(x_1)) + \frac{\partial h}{\partial x_2}(x_1, \phi(x_1)) \frac{\partial \phi(x_1)}{\partial x_1}. \tag{2.1}$$

Calculating the second derivatives,

$$\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1))) = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}(h(x_1,\phi(x_1)))\right)$$

$$= \frac{\partial^2 h}{\partial x_1^2}(x_1,\phi(x_1)) + 2\frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1,\phi(x_1))\frac{\partial \phi(x_1)}{\partial x_1} \tag{2.2}$$

$$+ \frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1))\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2 + \frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}.$$

In addition to this, we have the following partial derivative

$$\frac{\partial}{\partial x_1}\Big(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\Big) = \frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1,\phi(x_1)) + \frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1))\frac{\partial \phi(x_1)}{\partial x_1}. \tag{2.3}$$

Therefore

$$\frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1,\phi(x_1)) = \frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right) - \frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1))\frac{\partial \phi(x_1)}{\partial x_1}. \tag{2.4}$$

Substituting (2.4) into (2.2) and simplifying, we end up with the following:

$$\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1))) = \frac{\partial^2 h}{\partial x_1^2}(x_1,\phi(x_1)) + 2\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\frac{\partial \phi(x_1)}{\partial x_1}$$

$$-2\frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1))\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2 + \frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1))\Big(\frac{\partial \phi(x_1)}{\partial x_1}\Big)^2 \tag{2.5}$$

$$+ \frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}.$$

Then we collect like terms from (2.5) and we determine that

$$\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1))) = \frac{\partial^2 h}{\partial x_1^2}(x_1,\phi(x_1)) + 2\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\frac{\partial \phi(x_1)}{\partial x_1}$$
$$- \frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1))\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2 + \frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}. \tag{2.6}$$

Since h is harmonic, we obtain

$$\frac{\partial^2 h}{\partial x_1^2}(x_1,\phi(x_1)) = -\frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1)). \tag{2.7}$$

Substituting (2.7) into (2.6) and collecting like terms, we formulate the following:

$$\frac{\partial^2 h}{\partial x_1^2}(x_1,\phi(x_1))\left[1 + \left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2\right] = \frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1)))$$
$$- 2\left(\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\right)\frac{\partial \phi(x_1)}{\partial x_1} \tag{2.8}$$
$$- \frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}.$$

Simply dividing (2.8) by $1 + \left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2$, we obtain and by similar computation, we derive

$$\frac{\partial^2 h}{\partial x_1^2}(x_1,\phi(x_1)) = \frac{\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1))) - 2\left(\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\right)\frac{\partial \phi(x_1)}{\partial x_1}}{1 + \left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}$$
$$- \frac{\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}}{1 + \left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2} \tag{2.9}$$

$$\frac{\partial^2 h}{\partial x_2^2}(x_1,\phi(x_1)) = \frac{-\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1))) + 2\left(\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\right)\frac{\partial \phi(x_1)}{\partial x_1}}{1+\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}$$

$$+\frac{\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}}{1+\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}. \tag{2.10}$$

Substituting (2.10) into (2.4), we determine the following:

$$\frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1,\phi(x_1)) = \frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right) + \frac{\frac{\partial \phi(x_1)}{\partial x_1}\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1)))}{1+\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}$$

$$-\frac{2\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2 - \frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}\frac{\partial \phi(x_1)}{\partial x_1}}{1+\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}. \tag{2.11}$$

This simplifies to the following:

$$\frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1,\phi(x_1)) = \frac{\frac{\partial \phi(x_1)}{\partial x_1}\frac{\partial^2}{\partial x_1^2}(h(x_1,\phi(x_1))) - \frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\frac{\partial^2 \phi(x_1)}{\partial x_1^2}\frac{\partial \phi(x_1)}{\partial x_1}}{1+\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}$$

$$+\frac{\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_2}(x_1,\phi(x_1))\right)\left[1-\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2\right]}{1+\left(\frac{\partial \phi(x_1)}{\partial x_1}\right)^2}. \tag{2.12}$$

Now we evaluate where the first and second partial derivatives are bounded. We start with

$\frac{\partial h}{\partial x_2}(x_1, \phi(x_1))$. Since x_1^7 is bounded and

$$\frac{\partial h}{\partial x_2}(x_1, \phi(x_1)) = x_1^7,$$

$\frac{\partial h}{\partial x_2}(x_1, \phi(x_1))$ is bounded. Next, we have the following calculation:

$$\frac{\partial \phi(x_1)}{\partial x_1} = \sin x_1^{-3} - 3x_1^{-3} \cos x_1^{-3}. \tag{2.13}$$

From (2.1) and (2.13), we derive

$$\frac{\partial h}{\partial x_1}(x_1, \phi(x_1)) = 2x_1 - x_1^7 \sin x_1^{-3} + 3x_1^4 \cos x_1^{-3}. \tag{2.14}$$

According to the above partial derivative, $\frac{\partial h}{\partial x_1}(x_1, \phi(x_1))$ is bounded by $O(x_1)$ since $|\cos x_1| \leq 1$ and $|\sin x_1| \leq 1$.

Remark 1. Big-O notation uses to denote order of magnitude. We consider behavior of functions in a neighborhood of a point a. Then $O(f(x))$ denotes any function $g(x)$ such that $|g(x)| \leq C|f(x)|$ for x near a.

Next we calculate the bounds for $\frac{\partial^2 h}{\partial x_2^2}(x_1, \phi(x_1))$ and $\frac{\partial^2 h}{\partial x_1^2}(x_1, \phi(x_1))$. Note that

$$\frac{\partial^2 \phi(x_1)}{\partial x_1^2} = x_1^{-7}(-9 \sin x_1^{-3} + 6x_1^3 \cos x_1^{-3}). \tag{2.15}$$

Using (2.10), (2.13) and (2.15) and we have the following calculation,

$$\frac{\partial^2 h}{\partial x_2^2}(x_1, \phi(x_1)) \quad = \frac{2 + 12x_1^6 \sin x_1^{-3} - 36x_1^3 \cos x_1^{-3} - 9 \sin x_1^{-3}}{1 + (\sin x_1^{-3} - 3x_1^{-3} \cos x_1^{-3})^2}. \tag{2.16}$$

We decompose $\mathbb{R}^+ = V_1 \cup V_2$, where $V_1 = \{x_1 : |x_1^{-3} \cos x_1^{-3}| \leq 1\}$ and $V_2 = \{x_1 : |x_1^{-3} \cos x_1^{-3}| > 1\}$. Then on V_1,

$$\left| \frac{\partial^2 h}{\partial x_2^2}(x_1, \phi(x_1)) \right| \leq O(1),$$

and on V_2,

$$\left| \frac{\partial^2 h}{\partial x_2^2}(x_1, \phi(x_1)) \right| \leq O(x_1^9).$$

The bound here would be $O(1)$, which means $\frac{\partial^2 h}{\partial x_2^2}(x_1, \phi(x_1))$ is uniformly bounded by a constant, and since

$$\frac{\partial^2 h}{\partial x_1^2}(x_1, \phi(x_1)) = -\frac{\partial^2 h}{\partial x_2^2}(x_1, \phi(x_1)),$$

we have the same bounds. Now, we calculate the bounds for the following second mixed partial derivative from (2.12),

$$\frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1, \phi(x_1)) = \frac{2 \sin x_1^{-3} - 6x_1^{-3} \cos x_1^{-3} - (\sin x_1^{-3} - 3x_1^{-3} \cos x_1^{-3})}{1 + (\sin x_1^{-3} - 3x_1^{-3} \cos x_1^{-3})^2}$$

$$\cdot \frac{(-9 \sin x_1^{-3} + 6x_1^3 \cos x_1^{-3}) + 7x_1^6 - 7x_1^6 (\sin x_1^{-3} - 3x_1^{-3} \cos x_1^{-3})^2}{1 + (\sin x_1^{-3} - 3x_1^{-3} \cos x_1^{-3})^2}.$$

Using the same decomposition of $\mathbb{R}^+$, on V_1 we determine the following:

$$\left| \frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1, \phi(x_1)) \right| \leq O(1),$$

and on V_2,

$$\left| \frac{\partial^2 h}{\partial x_1 \partial x_2}(x_1, \phi(x_1)) \right| \leq O(x_1^3).$$

Therefore, our worst case is $O(1)$, which is uniformly bounded. With the previous calculations, we have a good bound without any discontinuities for each first and second derivatives, so h is C^2 on Γ_x. $\square$

The following lemma gives a good estimate on integration needed to find the harmonic conjugate function $C(x_1, \phi(x_1))$.

Lemma 2.3. *We have that*

$$\begin{aligned} &\Big| \int_{x_0}^{x} t^{\alpha} \sin kt^{-\beta} dt - \frac{1}{k\beta} x^{\alpha+\beta+1} \cos kx^{-\beta} \Big| < O(1), \\ \textit{and} \\ &\Big| \int_{x_0}^{x} t^{\alpha} \cos kt^{-\beta} dt - \frac{1}{k\beta} x^{\alpha+\beta+1} \sin kx^{-\beta} \Big| < O(1), \end{aligned} \tag{2.17}$$

for $\alpha \in \mathbb{R}$, $\beta > 0$, $x_0 > x > 0$ *and* $k > 0$, *satisfying* $\alpha + 2\beta + 1 > 0$.

Proof. We obtain the following:

$$\Big| \int_{x_0}^{x} t^{\alpha} \sin kt^{-\beta} dt \Big| \leq \int_{x_0}^{x} t^{\alpha} dt = \frac{x^{\alpha+1}}{\alpha+1} - \frac{x_0^{\alpha+1}}{\alpha+1}.$$

This is because $|\sin kt^{-\beta}| \leq 1$. The same goes for $\Big| \int_{x_0}^{x} t^{\alpha} \cos kt^{-\beta} \Big| \leq \int_{0}^{x} t^{\alpha} dt$ which is the principal term. Now we consider

$$\int_{x_0}^{x} t^{\alpha} \sin kt^{-\beta} dt,$$

Using integration by parts

$$\begin{bmatrix} u = t^{\alpha+\beta+1} & \dfrac{dv}{dt} = t^{-\beta-1} \sin kt^{-\beta} \\ \dfrac{du}{dt} = (\alpha+\beta+1) t^{\alpha+\beta} & v = \dfrac{1}{k\beta} \cos kt^{-\beta} \end{bmatrix},$$

we have the following:

$$\int_{x_0}^{x} t^{\alpha} \sin kt^{-\beta} dt = \frac{1}{k\beta} t^{\alpha+\beta+1} \cos kt^{-\beta} \Big|_{x_0}^{x} - \int_{x_0}^{x} \frac{(\alpha+\beta+1)}{k\beta} t^{\alpha+\beta} \cos kt^{-\beta} dt. \tag{2.18}$$

Using integration by parts again we have

$$\left[\begin{array}{ll} u = t^{\alpha+2\beta+1} & \dfrac{dv}{dt} = t^{-\beta-1}\cos kt^{-\beta} \\ \dfrac{du}{dt} = (\alpha+2\beta+1)t^{\alpha+2\beta} & v = \dfrac{1}{-k\beta}\sin kt^{-\beta} \end{array}\right],$$

then

$$\begin{aligned} -\int_{x_0}^{x} \frac{\alpha+\beta+1}{k\beta} t^{\beta+\alpha}\cos kt^{-\beta}dt =& \frac{-(\alpha+\beta+1)}{k\beta}\Big[\frac{-t^{\alpha+2\beta+1}}{k\beta}\sin kt^{-\beta}\Big|_{x_0}^{x} \\ & - \int_{x_0}^{x}\frac{-(\alpha+2\beta+1)}{k\beta}t^{\alpha+2\beta}\sin kt^{-\beta}dt\Big] \\ \le& O(x^{\alpha+2\beta+1}). \end{aligned} \tag{2.19}$$

Putting (2.18) and (2.19) together, we end up with

$$\begin{aligned} \int_{x_0}^{x} t^{\alpha}\sin kt^{-\beta}dt =& \frac{x^{(\alpha+\beta+1)}\cos kx^{-\beta}}{k\beta} + \frac{(\alpha+\beta+1)x^{\alpha+2\beta+1}}{(k\beta)^2}\sin kx^{-\beta} \\ & - \frac{x_0^{(\alpha+\beta+1)}\cos kx_0^{-\beta}}{k\beta} - \frac{(\alpha+\beta+1)x_0^{\alpha+2\beta+1}}{(k\beta)^2}\sin kx_0^{-\beta} \\ & - \frac{(\alpha+\beta+1)(\alpha+2\beta+1)}{(k\beta)^3}\int_{x_0}^{x} t^{\alpha+2\beta}\sin kt^{-\beta}dt. \end{aligned}$$

Therefore (2.17) is proven. Notice that each integration by parts adds $\beta+1$ to the principal order term. □

Next, we will calculate the bounds for our harmonic conjugate using the Cauchy-Riemann equations.

Remark 2. The Cauchy-Riemann equations for a pair of real-valued functions of two real variables $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$ and $v(x,y) : \mathbb{R}^2 \to \mathbb{R}$ defined by the two equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

2.1.1 Harmonic Conjugate

Lemma 2.4. *Let h satisfy the hypotheses of Lemma 2.2. Let C on $\mathcal{O}_x$ satisfy*

$$\frac{\partial C}{\partial x_1}(x_1, \phi(x_1)) = -\frac{\partial h}{\partial x_2}(x_1, \phi(x_1)),$$

and

$$\frac{\partial C}{\partial x_2}(x_1, \phi(x_1)) = \frac{\partial h}{\partial x_1}(x_1, \phi(x_1)).$$

We have that

$$C(x_1, \phi(x_1)) + \frac{26}{9} x_1^{-1} \cos x_1^{-3}, \tag{2.20}$$

is uniformly bounded on Γ_x.

Proof. Let $x_0 > 0$. Substituting $\frac{\partial h}{\partial x_2}(t, \phi(t)) = t_1^7$ along with $\frac{\partial}{\partial x_1}(h(x_1, \phi(x_1)))$ we have the following using (2.1):

$$
\begin{aligned}
C(x_1,\phi(x_1)) =& C(x_0,\phi(x_0)) + \int_{x_0}^{x_1} \frac{d}{dt}(C(t,\phi(t))dt \\
=& C(x_0,\phi(x_0)) + \int_{x_0}^{x_1} \Big(-\frac{\partial h}{\partial x_2}(t,\phi(t)) + \frac{\partial h}{\partial x_1}(t,\phi(t))\frac{\partial \phi(t)}{\partial t}\Big)dt \\
=& C(x_0,\phi(x_0)) + \int_{x_0}^{x_1} \Big(-\frac{\partial h}{\partial x_2}(t,\phi(t)) + \Big[\frac{\partial}{\partial t}(h(t,\phi(t))) \\
& -\frac{\partial h}{\partial x_2}(t,\phi(t))\frac{\partial \phi(t)}{\partial t}\Big]\frac{\partial \phi(t)}{\partial t}\Big)dt.
\end{aligned} \tag{2.21}
$$

After simplifying, we obtain

$$
\begin{aligned}
C(x_1,\phi(x_1)) =& C(x_0,\phi(x_0)) + \int_{x_0}^{x_1} \Big(-\frac{\partial h}{\partial x_2}(t,\phi(t))\Big[1+\Big(\frac{\partial \phi(t)}{\partial t}\Big)^2\Big] \\
& + \frac{\partial}{\partial t}h(t,\phi(t))\frac{\partial \phi(t)}{\partial t}\Big)dt \\
=& C(x_0,\phi(x_0)) + \int_{x_0}^{x_1} \Big(-t^7\Big[1+\Big(\frac{\partial \phi(t)}{\partial t}\Big)^2\Big] + \frac{\partial}{\partial t}h(t,\phi(t))\frac{\partial \phi(t)}{\partial t}\Big)dt.
\end{aligned} \tag{2.22}
$$

Using integration by parts, where

$$
\begin{bmatrix}
u = \dfrac{\partial \phi(t)}{\partial t} & \dfrac{dv}{dt} = \dfrac{\partial}{\partial t}h(t,\phi(t)) \\
\dfrac{du}{dt} = \dfrac{\partial^2 \phi(t)}{\partial t^2} & v = h(t,\phi(t))
\end{bmatrix},
$$

we obtain the following:

$$C(x_1,\phi(x_1)) = C(x_0,\phi(x_0)) + \int_{x_0}^{x_1}\Big(-t^7\Big[1+\Big(\frac{\partial\phi(t)}{\partial t}\Big)^2\Big] - h(t,\phi(t))\frac{\partial^2\phi(t)}{\partial t^2}\Big)dt$$

$$+ h(x_1,\phi(x_1))\frac{\partial\phi(x_1)}{\partial x_1} - h(x_0,\phi(x_0))\frac{\partial\phi(x_0)}{\partial x_1}.$$

Then we have:

$$\begin{aligned} C(x_1,\phi(x_1)) = C(x_0,\phi(x_0)) + \int_{x_0}^{x_1}\Big(&-t^7(1+(\sin t^{-3} - 3t^{-3}\cos t^{-3})^2) \\ &- t^2(t^{-7}(-9\sin t^{-3} + 6t^3\cos t^{-3}))\Big)dt + x_1^2(\sin x_1^{-3} - 3x_1^{-3}\cos x_1^{-3}) \\ &- h(x_0,\phi(x_0))\frac{\partial\phi(x_0)}{\partial x_1} \\ &\le O(1) - 3x_1^{-1}\cos x_1^{-3} + \frac{1}{9}x_1^{-1}\cos x_1^{-3}. \end{aligned} \tag{2.23}$$

Here, we use the technique of integration in Lemma 2.3 to estimate $\int_{x_0}^{x} 9t^{-5}\sin t^{-3}dt$, so we end up with (2.20) where we have a singularity for our harmonic conjugate. □

2.1.2 Straightening the Boundary

Our goal is to "flatten" Γ_x by finding a smooth mapping that straightens out Γ_x near some points $x^0 \in \Gamma_x$. We need this in order to use the following version of the Cauchy-Kovalevsky Theorem (9.4.5) in Hörmander in [13]. This presentation of Cauchy-Kovalevsky Theorem would give a precise bounds on the solution.

Theorem 2.5. *Assume that the coefficients in the differential equation*

$$\sum_{|\alpha_1|\leq 2} a^{\alpha_1} D^{\alpha_1} u = f, \tag{2.24}$$

are analytic in $\Omega_{R,\delta_1} = \{z \in \mathbb{C}^2; |z_1| < R$ *and* $|z_2| < \delta_1 R\}$ *and that the coefficient* $a^{(0,2)}$ *is equal to 1. If*

$$2(2^2 e)^2 \sum_{\alpha_1 \neq \beta_1} R^{2-|\alpha_1|} \delta_1^{2-\alpha_n} |a^{\alpha_1}(z)| \leq 1, \quad z \in \Omega_{R,\delta_1}, \tag{2.25}$$

and f *is bounded and analytic on* Ω_{R,δ_1}*, then (2.24) has a unique analytic solution in* $\Omega_{R/2,\delta_1}$ *satisfying the Cauchy boundary conditions*

$$D_2^j u = 0 \quad \textit{when } z_2 = 0, j < 2. \tag{2.26}$$

For u *we have the estimate*

$$\sup_{\Omega_{R/2,\delta_1}} |u| \leq 2(R\delta_1)^2 \sup_{\Omega_{R,\delta_1}} |f|. \tag{2.27}$$

To straighten the boundary of the Drill Bit Domain $\phi(x)$, we need a change of coordinates near a point on Γ_x. Therefore, to flatten out the boundary, we consider the following hypersurface

$$\Gamma_y = \{(y_1, y_2) : y_2 = 0\}.$$

Let $\mathcal{O}_y = \{(y_1, y_2) : |y_2| < |y_1|^{12}\}$ and with a change of coordinates.

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 - \phi(x_1). \end{cases} \tag{2.28}$$

The inverse change of coordinates from (2.28) is given by

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 + \phi(y_1). \end{cases} \tag{2.29}$$

We define the Laplacian in terms of x_1:

$$\Delta_{x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_2}\right). \tag{2.30}$$

To find the partials $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$, we obtain the following:

$$\frac{\partial}{\partial x_1} = \frac{\partial y_1}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_1}\frac{\partial}{\partial y_2},$$

which is equal to

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1} - \phi'(x_1)\frac{\partial}{\partial y_2}.$$

Then, using the appropriate substitution from (2.29), we have

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1} - \phi'(y_1)\frac{\partial}{\partial y_2}. \tag{2.31}$$

In addition, we compute

$$\frac{\partial}{\partial x_2} = \frac{\partial y_1}{\partial x_2}\frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_2}\frac{\partial}{\partial y_2},$$

which simplifies to

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y_2}. \tag{2.32}$$

Then, combining (2.31) with the first term of (2.30), we can evaluate the following:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right) = \left(\frac{\partial}{\partial y_1} - \phi'(y_1)\frac{\partial}{\partial y_2}\right)\left(\frac{\partial}{\partial y_1} - \phi'(y_1)\frac{\partial}{\partial y_2}\right). \tag{2.33}$$

Using the Leibniz Rule on (2.33), we have the following:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right) =\frac{\partial^2}{\partial y_1^2} - \phi''(y_1)\frac{\partial}{\partial y_2} - \phi'(y_1)\frac{\partial^2}{\partial y_1 \partial y_2} - \phi'(y_1)\frac{\partial^2}{\partial y_2 \partial y_1} + (\phi'(y_1))^2\frac{\partial^2}{\partial y_2^2}.$$

Evaluation of the last term of (2.30) using (2.32), gives us

$$\frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial^2}{\partial y_2^2}.$$

Then we have

$$\Delta_x = \frac{\partial^2}{\partial y_1^2} - \phi''(y_1)\frac{\partial}{\partial y_2} - 2\phi'(y_1)\frac{\partial^2}{\partial y_1 \partial y_2} + (1 + (\phi'(y_1))^2)\frac{\partial^2}{\partial y_2^2}.$$

Moreover, factoring out $1 + \phi'(y_1)^2$, we end up with

$$\Delta_x =(1 + \phi'(y_1)^2)\left[\frac{\partial^2}{\partial y_2^2} + \frac{1}{(1 + \phi'(y_1)^2)} \cdot \left[\frac{\partial^2}{\partial y_1^2} - 2\phi'(y_1)\frac{\partial^2}{\partial y_1 \partial y_2} - \phi''(y_1)\frac{\partial}{\partial y_2}\right]\right].$$

This gives us the differential operator

$$L_y =\frac{1}{(1 + (\phi'(y_1))^2)}\frac{\partial^2}{\partial y_1^2} - \frac{\phi''(y_1)}{(1 + (\phi'(y_1))^2)}\frac{\partial}{\partial y_2} - \frac{2\phi'(y_1)}{(1 + (\phi'(y_1))^2)}\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2^2}. \tag{2.34}$$

Then using (2.24) to identify the coefficients from Theorem 2.5, we obtain the following:

$$\begin{cases} a^{(0,1)}(y_1) = \dfrac{-\phi''(y_1)}{(1 + (\phi'(y_1))^2)} \\ a^{(0,2)}(y_1) = 1 \\ a^{(1,1)}(y_1) = \dfrac{-2\phi'(y_1)}{(1 + (\phi'(y_1))^2)} \\ a^{(2,0)}(y_1) = \dfrac{1}{(1 + (\phi'(y_1))^2)}. \end{cases} \tag{2.35}$$

We also need to modify our function h so that the Cauchy boundary conditions match those in Theorem 2.5. Ideally, we want to ensure that a unique solution exists. Therefore, we define

$$u(y_1, y_2) = h(y_1, y_2 + \phi(y_1)) - y_1^2 - y_2 y_1^7. \tag{2.36}$$

Then we deduce Cauchy boundary conditions from (2.26) using (2.36)

$$\begin{cases} u(y_1, 0) = h(y_1, \phi(y_1)) - y_1^2 = 0 \\ \dfrac{\partial}{\partial y_2} u(y_1, y_2)\Big|_{y_2=0} = \dfrac{\partial}{\partial y_2} h(y_1, \phi(y_1)) - y_1^7 = 0. \end{cases} \tag{2.37}$$

Then $\Delta_x h = 0$ implies

$$L_y u = -L_y(y_1^2 + y_2 y_1^7).$$

So if we set $f = -L_y(y_1^2 + y_2 y_1^7)$, then we have the following calculation:

$$f = -\frac{1}{(1 + (\phi'(y_1))^2)} \Big[-2 + 14\phi'(y_1) y_1^6 + \phi''(y_1) y_1^7 \Big]. \tag{2.38}$$

Therefore, we wish to solve the following partial differential equation from (2.26):

$$\begin{cases} L_y u = f & \text{on} \quad \mathcal{O}_y \\ u = 0 & \text{on} \quad \Gamma_y \\ \dfrac{\partial u}{\partial y_2} = 0 & \text{on} \quad \Gamma_y. \end{cases} \tag{2.39}$$

Set $\Omega_{p_1,R,\delta_1} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - p_1| < R, |z_2| < \delta_1 R\}$ for $p_1 \in \mathbb{R}$, and fix $R > 0$ and $\delta_1 > 0$ such that $(y_1, y_2) \in \mathcal{O}_y$ whenever $|y_1 - p_1| < R$ and $|y_2| < \delta_1 R$. This implies $R < |p_1|^{12}$. To use Theorem 2.5, we need to find a bound on f, but first we need to estimate $|\Re\phi'(z_1)|$ and $|\Re\phi''(z_1)|$. Our goal is to show $\mathcal{O}_y$ is sufficiently small so that $\phi(y_1) \approx \phi(z_1)$, where $\phi(z_1) = z_1 \sin {z_1}^{-3}$ and $\phi'(z_1) = \sin z_1^{-3} - 3z_1^{-3} \cos z_1^{-3}$.

Remark 3. With Hörmander's Cauchy-Kovalesky Theorem, we need to complexify $\phi'(y_1)$ and $\phi''(y_1)$ and work with complex coordinates.

Now let $z_1 = a + ib$. As before $|z_1 - p_1| < R$. This implies that $a > |p_1| - R$ and $b < R$ by the Triangle Inequality. Then we have the following:

$$\begin{aligned} b &< |p_1|^{12}, \\ a &> |p_1| - |p_1|^{12}. \end{aligned} \tag{2.40}$$

This then implies that

$$b < O(a^{12}).$$

Remember our goal is that we need $\mathcal{O}_y$ to be sufficiently small. Therefore, we need b to be bounded by $O(a^{12})$. Next we need to find the estimate of $\phi'(z_1)$. However, first we need the following computation:

$$\begin{aligned} z_1^{-\tau} =& (a + ib)^{-\tau} \\ =& a^{-\tau}\left(1 + i\frac{b}{a}\right)^{-\tau}, \end{aligned} \tag{2.41}$$

where $\tau \in \mathbb{R}$. We want to expand (2.41) using the Taylor expansion method.

Remark 4. Taylor expansion states if $U \subset \mathbb{R}^n$ is open, $x \in U$ and $f : U \to \mathbb{R}$ is C^2 then $f(x + h) = f(x) + Df(x)(h) + O(|h|)$ as $h \to 0$.

Multiplying through by $a^{-\tau}$, we get

$$\begin{aligned} z_1^{-\tau} =& a^{-\tau}\left(1 - \tau i\frac{b}{a} + O\left(\left|\frac{b}{a}\right|^2\right)\right) \\ =& a^{-\tau} - \tau iba^{-\tau-1} + O(|a|^{22-\tau}). \end{aligned} \tag{2.42}$$

Let $\tau = 3$ in (2.42). Then we have the following estimate

$$\begin{aligned} z_1^{-3} &= a^{-3}\left(1 - 3i\frac{b}{a} + O\big(|a|^{22}\big)\right) \\ &= a^{-3} - 3iba^{-4} + O(|a|^{19}). \end{aligned} \tag{2.43}$$

Recall Euler's formulas:

$$\begin{aligned} \sin(u+iv) &= \sin u \cosh v + i \cos u \sinh v, \\ \cos(u+iv) &= \cos u \cosh v + i \sin u \sinh v. \end{aligned} \tag{2.44}$$

Evaluating $\cos(a+ib)$ and $\sin(a+ib)$ from (2.44) and (2.43), we have the following estimates

$$\sin(z_1^{-3}) = \sin a^{-3} \cosh\left(\frac{-3b}{a^4}\right) + i \cos a^{-3} \sinh\left(\frac{-3b}{a^4}\right) + O(|a|^{19}), \tag{2.45}$$

and

$$\cos(z_1^{-3}) = \cos a^{-3} \cosh\left(\frac{-3b}{a^4}\right) + i \sin a^{-3} \sinh\left(\frac{-3b}{a^4}\right) + O(|a|^{19}). \tag{2.46}$$

Combining (2.43),(2.46) and (2.45), we have the following estimate for $\phi'(z_1)$:

$$\begin{aligned} \phi'(z_1) = \sin a^{-3} \cosh\left(\frac{-3b}{a^4}\right) &- 3a^{-3} \cos a^{-3} \cosh\left(\frac{-3b}{a^4}\right) \\ &- 3ba^{-4} \sin a^{-3} \sinh\left(\frac{-3b}{a^4}\right) - i\left[\cos a^{-3} \sinh\left(\frac{-3b}{a^4}\right)\right. \\ &\left.+ 3a^{-3} \sin a^{-3} \sinh\left(\frac{-3b}{a^4}\right) - 3ba^{-4} \cos a^{-3} \cosh\left(\frac{-3b}{a^4}\right)\right] \\ &+ O(|a|^{19}). \end{aligned} \tag{2.47}$$

Since $\left|\frac{b}{a^4}\right| < O(|a|^8)$, this implies that

$$\begin{cases} \left|\sinh\left(\frac{-3b}{a^4}\right)\right| \le O(|a|^8) \\ \\ \left|\cosh\left(\frac{-3b}{a^4}\right) - 1\right| \le O(|a|^{16}). \end{cases} \tag{2.48}$$

Now combining (2.47) and (2.48), we have:

$$\phi'(z_1) = \sin a^{-3} - 3a^{-3}\cos a^{-3} + O(|a|^5). \tag{2.49}$$

Hence $\phi'(z_1) \approx \phi'(\Re z_1)$ as $a \to 0$. Next, we calculate

$$\phi''(z_1) = z_1^{-7}(-9\sin z_1^{-3} + 6z_1^3\cos z_1^{-3}). \tag{2.50}$$

Using $\tau = -3$ and $\tau = 7$ from (2.42), we have the following estimates:

$$\begin{aligned} z_1^3 &= a^3\left(1 + 3i\frac{b}{a}\right)^3 = a^3 + 3iba^2 + O(|a|^{25}), \\ z_1^{-7} &= a^{-7}\left(1 - 7i\frac{b}{a}\right)^{-7} = a^{-7} - 7iba^{-8} + O(|a|^{15}). \end{aligned} \tag{2.51}$$

We now have an estimate for $\phi''(z_1)$ using (2.50) and (2.51):

$$\begin{aligned} \phi''(z_1) = (a^{-7} - 7iba^{-8})\Bigg[&-9\left(\sin a^{-3}\cosh\left(\frac{-3b}{a^4}\right) + i\cos a^{-3}\sinh\left(\frac{-3b}{a^4}\right)\right) \\ &+ 6(a^3 + 3iba^2)\left(\cos a^{-3}\cosh\left(\frac{-3b}{a^4}\right) + i\sin a^{-3}\sinh\left(\frac{-3b}{a^4}\right)\right)\Bigg] \\ &+ O(|a|^{15}). \end{aligned} \tag{2.52}$$

Simplifying (2.52), we end up with the following:

$$
\begin{aligned}
\phi''(z_1) = & -9a^{-7}\sin a^{-3}\cosh\left(\frac{-3b}{a^4}\right) - 63a^{-8}b\cos a^{-3}\sinh\left(\frac{-3b}{a^4}\right) \\
& + 6a^{-6}(a^2 - 21b^2)\cos a^{-3}\cosh\left(\frac{-3b}{a^4}\right) - 24ba^{-5}\sin a^{-3}\sinh\left(\frac{-3b}{a^4}\right) \\
& + i\bigg(- 9a^{-7}\cos a^{-3}\sinh\left(\frac{-3b}{a^4}\right) - 63a^{-8}b\sin a^{-3}\cosh\left(\frac{-3b}{a^4}\right) \\
& + 6a^{-6}(a^2 - 21b^2)\sin a^{-3}\sinh\left(\frac{-3b}{a^4}\right) + 24ba^{-5}\cos a^{-3}\cosh\left(\frac{-3b}{a^4}\right)\bigg) \\
& + O(|a|^{15}).
\end{aligned} \tag{2.53}
$$

Using (2.48), this implies the following estimate:

$$
\phi''(z_1) = -9a^{-7}\sin a^{-3} + 6a^{-4}\cos a^{-3} + O(|a|). \tag{2.54}
$$

Hence, $\phi''(z_1) \approx \phi''(\Re z_1)$ as $a \to 0$.

Next, we want to ensure that the coefficients in (2.24) and (2.35) are analytic in Ω_{p_1,R,δ_1}.

Lemma 2.6. *There exist $\delta_1 > 0$ sufficiently small such that (2.25) holds when $z \in \Omega_{p_1,R,\delta_1}$, where $\Omega_{p_1,R,\delta_1} = \{z \in \mathbb{C}^2; |z_1 - p_1| < R$ and $|z_2| < \delta_1 R\}$.*

Proof. By (2.35), it suffices to show

$$
2(2^2e)^2\left[\delta^2\left|\frac{1}{1+(\phi'(z_1))^2}\right| + \delta\left|\frac{2\phi'(z_1)}{1+(\phi'(z_1))^2}\right| + R\delta\left|\frac{\phi''(z_1)}{1+(\phi'(z_1))^2}\right|\right] \le 1. \tag{2.55}
$$

We rewrite (2.55) as

$$
2(2^2e)^2\left[\delta^2(\mathrm{I}) + \delta(\mathrm{J}) + R\delta(\mathrm{K})\right] \le 1,
$$

where we show

$$\begin{aligned} I =& \left| \frac{1}{1+(\phi'(z_1))^2} \right| \le O(1); \\ J =& \left| \frac{2\phi'(z_1)}{1+(\phi'(z_1))^2} \right| \le O(1); \\ K =& \left| \frac{\phi''(z_1)}{1+(\phi'(z_1))^2} \right| \le O(|z_1|^{-7}). \end{aligned} \tag{2.56}$$

Recall that

$$\phi'(z_1) \approx \phi'(y_1) = \sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3}.$$

Then for I,

$$\frac{1}{1+(\phi'(y_1))^2} = \frac{1}{1+(\sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3})^2} \le 1, \tag{2.57}$$

so

$$\frac{1}{1+(\phi'(z_1))^2} \le O(1). \tag{2.58}$$

For J, we have

$$\frac{2\phi'(y_1)}{1+(\phi'(y_1))^2} = \frac{\sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3}}{1+(\sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3})^2}. \tag{2.59}$$

By Cauchy's Inequality:

$$2\phi'(y_1) \le 1 + (\phi'(y_1))^2,$$

So

$$\frac{2\phi'(y_1)}{1+(\phi'(y_1))^2} \le 1,$$

and hence

$$J \le O(1).$$

For K, we have the following from (2.50) and (2.57):

$$\frac{\phi''(y_1)}{1+(\phi'(y_1))^2} = \frac{y_1^{-7}(-9\sin y_1^{-3} + 6y_1^3 \cos y_1^{-3})}{1+(\sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3})^2}. \tag{2.60}$$

We decompose $\mathbb{R}^+ = U_1 \cup k_1$, where $k_1 = \{y : |y_1^{-3} \cos y_1^{-3}| \le 1\}$ and

$U_1 = \{y : |y_1^{-3} \cos y_1^{-3}| > 1\}$. Then on k_1, we have

$$K \leq O(|y_1|^{-7}),$$

and on U_1, we have the following estimate:

$$|\sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3}| \geq 2|y_1^{-3} \cos y_1^{-3}|$$
$$1 + |\sin y_1^{-3} - 3y_1^{-3} \cos y_1^{-3}|^2 \geq 1 + 4|y_1^{-3} \cos y_1^{-3}|^2.$$

Then

$$K \leq O(|y_1|^2).$$

Therefore $K \leq O(|y_1|^{-7})$ is the worst bound on $\mathbb{R}^+$. □

Here the coefficients are analytic in Ω_{p_1,R,δ_1} and we now use Lemma 2.6 to calculate if $f(y_1, y_2)$ is bounded and analytic.

Lemma 2.7. *Let $f(y_1, y_2)$ be defined by (2.38). Then*

$$f(y_1, y_2) \leq O(1), \tag{2.61}$$

on $\mathcal{O}_y$.

Proof. Recall that

$$f(y_1, y_2) = \frac{-2 + 14\phi'(y_1)y_1^6 + \phi''(y_1)y_1^7}{(1 + \phi(y_1)^2)},$$

from (2.38). Then using the coefficients from (2.44) and the notation and estimates from Lemma 2.6, we have the following estimate:

$$\begin{aligned} |f(y_1, y_2)| &\leq O(|I| + |J|y_1^6 + |K|y_1^7), \\ &\lesssim O(1) + O(|y_1|^6) + O(1), \qquad (2.62) \\ &\lesssim O(1). \end{aligned}$$

Therefore, f is bounded and analytic on $\mathcal{O}_y$. $\square$

The last lemma in part one, we now estimate u from (2.36) using Cauchy Estimates.

Lemma 2.8. *Assume $p_1 \in \mathbb{R}$ and $R < O(|p_1|^{12})$ from (2.40), then $u \in C^2(\Omega_{p_1,R,\delta_1})$ from (2.36), where $\Omega_{p_1,R,\delta_1} = \{z \in \mathbb{C}^2 : |z_1 - p| < \delta_1 R$ and $|z_2| < R\}$ with uniform bounds on the second derivatives, hence $u \in C^2(\overline{\mathcal{O}}_y)$.*

Proof. Notice we have $|z_1 - p_1| < R$ and $z_1 = a + ib$. Then $|z_1 - p_1| < R$ is equivalent to $(a - p_1)^2 + b^2 < R^2$. Using the Cauchy Kovalevsky Theorem (2.27) and $f(y_1, y_2) \leq O(1)$ from (2.38), we now have an estimate for u from (2.36):

$$\begin{aligned}\sup_{\Omega_{p_1,R,\delta_1}} |u| &\lesssim R^2 \sup_{\Omega_{p_1,R,\delta_1}} |f| \\ &\lesssim O(|p_1|^{24}).\end{aligned}$$

Using the Cauchy Estimates on the Drill Bit Domain, we have the following on Ω_{p_1,R,δ_1}:

$$\begin{aligned}\sup_{\Omega_{p_1,R,\delta_1}} |D^3 u| &\lesssim \frac{R^2 \sup_{\Omega_{p_1,R,\delta_1}} |f|}{r^3} \\ &\lesssim O(|p_1|^{-12}),\end{aligned}$$

where $r \approx R$. Since the second derivative of u is uniformly bounded on Γ_y, the third derivative tells us how fast the second derivative changes. Therefore

$$\sup_{\Omega_{p_1,R,\delta_1}} |D^2 u| \lesssim \sup_{\Gamma_y} |D^2 u| + \tilde{R} \sup_{\Gamma_y} |D^3 u|.$$

If $\tilde{R} \leq O(|p_1|^{12})$, then $|D^2 u|$ is uniformly bounded and hence $u \in C^2(\overline{\mathcal{O}}_y)$. $\square$

Finally, we define h on $\mathcal{O}_x$ to be the pull-back of u from $\mathcal{O}_y$, and obtain a solution to (2.39). This concludes the proof of our main theorem, Theorem 2.1.

3 Part 2- Complex Analysis in Several Variables with PDEs

3.1 Drill Bit Domain in Several Complex Variables

Let

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - e^{ih(z_2)}|^2 < g(z_2)\}, \tag{3.1}$$

where $h(z_2)$ and $g(z_2)$ are real-valued functions in $C^2(\mathbb{C})$. Then a defining function for Ω is

$$\rho(z_1, z_2) = |z_1 - e^{ih(z_2)}|^2 - g(z_2). \tag{3.2}$$

We parameterize $\partial\Omega$ by $\partial\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = e^{ih(z_2)} + \sqrt{g(z_2)}e^{i\theta}, \theta \in \mathbb{R}\}$ where we have a disc centered at $e^{ih(z_2)}$ with radius $\sqrt{g(z_2)}$.

Remark 5. Ω is inspired by the Worm Domain of Diederich and Fornæss. They produced a smoothly bounded domain known as the Worm domain [5] that is pseudoconvex but does not have a plurisubharmonic defining function. In this case, $h(z_2) = \log|z_2|^2 + \pi$ and $g(z_2) = 1 - \eta(\log|z_2|^2)$ where

i. $\eta(x) \geq 0$, η is even and convex

ii. $\eta^{-1}(0) = I_{\beta - \frac{\pi}{2}}$, where $I_{\beta-\frac{\pi}{2}} = \left[-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}\right]$

iii. there exists an $a > 0$ such that $\eta(x) > 1$ if either $x < -a$ or $x > a$

iv. $\eta'(x) \neq 0$ if $\eta(x) = 1$

Proposition 3.1. *The domain $\Omega \subset \mathbb{C}^2$ is C^2 and bounded if and only if $g(z_2)$ has the following properties:*

i. $g(z_2) \leq 0$ outside a compact subset of $\mathbb{C}$, *ii.* $\nabla g(z_2) \neq 0$ *if* $g(z_2) = 0$.

Proof. Clearly, Ω is bounded since $g(z_2)$ has compact support (i). For domains to be C^2, we need to show that ρ is C^2 by checking the gradient on the boundary (ii). Observe

$$\begin{aligned} \rho(z_1, z_2) &= |z_1 - e^{ih(z_2)}|^2 - g(z_2) \\ &= (z_1 - e^{ih(z_2)})(\overline{z_1} - e^{-ih(z_2)}) - g(z_2) \\ &= z_1\overline{z_1} - z_1 e^{-ih(z_2)} - \overline{z_1} e^{ih(z_2)} + 1 - g(z_2). \end{aligned} \tag{3.3}$$

Next we do a few simple calculations:

$$\frac{\partial \rho}{\partial z_1} = \overline{z_1} - e^{-ih(z_2)}, \tag{3.4}$$

and

$$\frac{\partial \rho}{\partial z_2} = i z_1 e^{-ih(z_2)} \frac{\partial h(z_2)}{\partial z_2} - i\overline{z_1} e^{ih(z_2)} \frac{\partial h(z_2)}{\partial z_2} - \frac{\partial g(z_2)}{\partial z_2}. \tag{3.5}$$

Now assume for contradiction that $\nabla \rho(p) = 0$ for some $p \in \partial\Omega$. If $\frac{\partial \rho}{\partial z_1}(p) = 0$ at some boundary point, we get

$$\frac{\partial \rho}{\partial z_1} = \overline{z_1} - e^{-ih(z_2)} = 0.$$

Substituting $\overline{z_1} = e^{-ih(z_2)} + \sqrt{g(z_2)} e^{-i\theta}$ into $\frac{\partial \rho}{\partial z_1}$, we derive the following:

$$\begin{aligned} \frac{\partial \rho}{\partial z_1} &= e^{-ih(z_2)} + \sqrt{g(z_2)} e^{-i\theta} - e^{-ih(z_2)} \\ &= \sqrt{g(z_2)} e^{-i\theta}, \end{aligned} \tag{3.6}$$

so $g(p)=0$. Also, substituting z_1 and $\overline{z_1}$ in $\frac{\partial\rho}{\partial z_2}$, we obtain

$$\begin{aligned}\frac{\partial\rho}{\partial z_2}(z_1,\overline{z_1}) &= i(e^{ih(z_2)}+\sqrt{g(z_2)}e^{i\theta})e^{-ih(z_2)}\frac{\partial h(z_2)}{\partial z_2} \\ &\quad - i(e^{-ih(z_2)}+\sqrt{g(z_2)}e^{i\theta})e^{ih(z_2)}\frac{\partial h(z_2)}{\partial z_2} - \frac{\partial g(z_2)}{\partial z_2} \\ &= i\sqrt{g(z_2)}e^{i(\theta-h(z_2))}\frac{\partial h(z_2)}{\partial z_2} - i\sqrt{g(z_2)}e^{-i(\theta-h(z_2))}\frac{\partial h(z_2)}{\partial z_2} \\ &\quad - \frac{\partial g(z_2)}{\partial z_2}.\end{aligned} \tag{3.7}$$

Since $\sqrt{g(p)}=0$, this implies that $-\frac{\partial g(p)}{\partial z_2}=0$, which is a contradiction, and so this proves that Ω is C^2. □

Definition 3.1. Let Ω be a bounded domain in $\mathbb{C}^2$ and let ρ be a C^2 defining function for Ω. Ω is called Levi pseudoconvex at $p\in\partial\Omega$ if the Levi form satisfies

$$i\partial\overline{\partial}\rho(t,t)(p)=\sum_{j,k=1}^{2}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(z)t_j\bar{t}_k\geq 0,$$

for all $t\in T_p^{1,0}(\partial\Omega)=\left\{t=(t_1,t_2)\in\mathbb{C}^2:\sum_{j=1}^{2}t_j\left(\frac{\partial\rho}{\partial z_j}\right)(p)=0\right\}$. $T_p^{1,0}(\partial\Omega)$ is the space of type $(1,0)$ vector fields which are tangent to the boundary at the point p. The domain Ω is said to be strictly Levi pseudoconvex at p if the Levi form is strictly positive definite for all such $t\neq 0$.

Proposition 3.2. *Ω is a C^2 bounded pseudoconvex domain in $\mathbb{C}^2$ with C^2-boundary if and* *only if* $g(z_2)$ *and* $h(z_2)$ *satisfy hypotheses of Proposition 3.1 and*

$$2g(z_2)\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2 + g^2(z_2)\frac{\partial^2 \log g(z_2)}{\partial z_2\partial\overline{z_2}} - 2\sqrt{g(z_2)}\left|i\overline{\frac{\partial g(z_2)}{\partial\overline{z_2}}}\frac{\partial h(z_2)}{\partial z_2}\right.$$
$$\left. - g(z_2)\left(\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2 + i\frac{\partial^2 h(z_2)}{\partial z_2\partial\overline{z_2}}\right)\right| \geq 0, \tag{3.8}$$

whenever $g(z_2) \geq 0$.

Proof. We first calculate the second derivatives of (3.2). On $\partial\Omega$,

$$z_1 = e^{ih(z_2)} + \sqrt{g(z_2)}e^{i\theta},$$

for $\theta \in \mathbb{R}$. Then $\partial\Omega$

$$\begin{aligned}
(i) \quad & \frac{\partial^2\rho}{\partial z_1\partial\overline{z_1}} = 1 \\
(ii) \quad & \frac{\partial^2\rho}{\partial z_2\partial\overline{z_1}} = -ie^{ih(z_2)}\frac{\partial h(z_2)}{\partial z_2} \\
(iii) \quad & \frac{\partial^2\rho}{\partial z_1\partial\overline{z_2}} = ie^{-ih(z_2)}\frac{\partial h(z_2)}{\partial\overline{z_2}} \\
(iv) \quad & \frac{\partial^2\rho}{\partial z_2\partial\overline{z_2}} = 2\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2 + 2\sqrt{g(z_2)}Re\left[e^{i(\theta-h(z_2))}\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2\right] \\
& \qquad + \sqrt{g(z_2)}e^{i(\theta-h(z_2))}i\frac{\partial^2 h(z_2)}{\partial z_2\partial\overline{z_2}} - \sqrt{g(z_2)}e^{-i(\theta-h(z_2))}i\frac{\partial^2 h(z_2)}{\partial z_2\partial\overline{z_2}} \\
& \qquad - \frac{\partial^2 g(z_2)}{\partial z_2\partial\overline{z_2}}.
\end{aligned} \tag{3.9}$$

Moreover, $T^{1,0}(\partial\Omega)$ is spanned by

$$L = \frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial z_2}. \tag{3.10}$$

Using (3.4) and (3.5), we calculate L

$$
\begin{aligned}
L =&\left(i\sqrt{g(z_2)}e^{i(\theta-h(z_2))}\frac{\partial h(z_2)}{\partial z_2} - i\sqrt{g(z_2)}e^{-i(\theta-h(z_2))}\frac{\partial h(z_2)}{\partial z_2} - \frac{\partial g(z_2)}{\partial z_2}\right)\frac{\partial}{\partial z_1} \\
&- \sqrt{g(z_2)}e^{-i\theta}\frac{\partial}{\partial z_2}.
\end{aligned} \tag{3.11}
$$

Then the Levi-form is

$$
\begin{aligned}
\mathcal{L}(L,\overline{L}) =& 2g(z_2)\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2 + \left|\frac{\partial g(z_2)}{\partial z_2}\right|^2 - g(z_2)\frac{\partial^2 g(z_2)}{\partial z_2 \partial \overline{z_2}} \\
&- 2\sqrt{g(z_2)}Re\left[ie^{i\theta}e^{-ih(z_2)}\frac{\partial h(z_2)}{\partial z_2}\frac{\partial \overline{g(z_2)}}{\partial \overline{z_2}}\right] - g^{3/2}(z_2)e^{-i(\theta-h(z_2))}i\frac{\partial^2 h(z_2)}{\partial z_2 \partial \overline{z_2}} \\
&+ g^{3/2}(z_2)e^{i(\theta-h(z_2))}i\frac{\partial^2 h(z_2)}{\partial z_2 \partial \overline{z_2}} + 2g^{3/2}Re\left[e^{i(\theta-h(z_2))}\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2\right].
\end{aligned} \tag{3.12}
$$

If we minimize (3.12) with respect to θ, the Levi form is bounded below by (3.8). Therefore $\mathcal{L}(L,\overline{L}) \geq 0$ if and only if (3.8) holds whenever $g(z_2) \geq 0$. □

Next, we will introduce an analytic disc $M \subset \partial\Omega$ that is biholomorphic to $\mathcal{O}_x$ in Lemma 2.2.

Lemma 3.3. *Let an analytic disc* $M = \{(z_1, z_2) : (0, z_2) \in \partial\Omega\}$ *be biholomorphic to* $\mathcal{O}_x$ *and suppose the interior of* M *is nonempty. If* $h(z_2)$ *is harmonic on* M*, then the Levi form vanishes on* M*.*

Proof. Since $z_1 = 0$, this implies that $g(z_2) = |e^{ih(z_2)}|^2 = 1$, when $(0, z_2) \in M$. We observe now that since g is constant on a set with nonempty interior,

$$\begin{cases} \dfrac{\partial g(z_2)}{\partial z_2} = 0 \\ \\ \dfrac{\partial^2 g(z_2)}{\partial z_2 \partial \overline{z_2}} = 0. \end{cases}$$

when $(0, z_2) \in M$. Since $h(z_2)$ is harmonic, $\dfrac{\partial^2 h(z_2)}{\partial z_2 \partial \overline{z_2}} = 0$. Then by (3.12), we obtain

$$\mathcal{L}(L, \overline{L}) = 2\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2 - 2\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2 = 0.$$

□

Note also that $e^{i\theta} = -e^{-ih(z_2)}$ on M. Next, let $\tilde{\rho}$ be a defining function for Ω. We wish to determine if the defining function $\tilde{\rho}$ is plurisubharmonic on M.

Definition 3.2. For a domain $\Omega \subset \mathbb{C}^n$, a function $\rho \in C^2$ is plurisubharmonic if

$$i\partial\overline{\partial}\rho(t, t)(z) = \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k \geq 0,$$

for all $t = (t_1, t_2)$ and for all $z \in \Omega$.

ρ is a strictly plurisubharmonic function if

$$i\partial\overline{\partial}\rho(t, t)(z) = \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k > 0,$$

for all $t \in \mathbb{C}^2, t \neq 0$.

Remark 6. The simplest example of a strictly pseudoconvex domain is a ball $B(p, R)$. The corresponding function $\rho = |z-p|^2 - R$ is a strictly plurisubharmonic defining function for $B(p,R) = \{z \in C^2 : \rho(z) < 0\}$.

Lemma 3.4. *Let A be a real-valued function and let ρ be defined in (3.2), then $\tilde{\rho} = \rho e^A$ is a C^2 defining function for Ω. Suppose A satisfies $\dfrac{\partial A}{\partial z_2} = i\dfrac{\partial h(z_2)}{\partial z_2}$ and $\Re\left(\dfrac{\partial A}{\partial z_1} e^{-ih(z_2)}\right) \leq \dfrac{1}{2}$ on M. Then $\tilde{\rho}$ has a positive semi-definite complex hessian on M.*

Proof. After a few calculations along with (3.9), we obtain the following:

$$\begin{aligned}
&(i) \qquad \frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_1}} = -e^A\left(e^{ih(z_2)}\frac{\partial A}{\partial z_1} + e^{-ih(z_2)}\frac{\partial A}{\partial \overline{z_1}} + 1\right)\\
&(ii) \qquad \frac{\partial^2 \tilde{\rho}}{\partial z_2 \partial \overline{z_1}} = e^A\left(e^{ih(z_2)}\frac{\partial A}{\partial z_2} - ie^{ih(z_2)}\frac{\partial h(z_2)}{\partial z_2}\right)\\
&(iii) \qquad \frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_2}} = e^A\left(-e^{ih(z_2)}\frac{\partial A}{\partial \overline{z_2}} + ie^{-ih(z_2)}\frac{\partial h(z_2)}{\partial \overline{z_2}}\right)\\
&(iv) \qquad \frac{\partial^2 \tilde{\rho}}{\partial z_2 \partial \overline{z_2}} = 0.
\end{aligned} \tag{3.13}$$

A is a function both in terms of z_1 and z_2, we need to check to see if the complex hessian matrix is positive semi-definite, so we require

$$\begin{pmatrix} \dfrac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_1}} & \dfrac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_2}} \\ \dfrac{\partial^2 \tilde{\rho}}{\partial z_2 \partial \overline{z_1}} & 0 \end{pmatrix} \geq 0.$$

Therefore, we need to calculate the determinant and the trace. First $\frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_2}} \frac{\partial^2 \tilde{\rho}}{\partial z_2 \partial \overline{z_1}} =$

$$-\left(e^A\left(-e^{ih(z_2)}\frac{\partial A}{\partial \overline{z_2}} + ie^{-ih(z_2)}\frac{\partial h(z_2)}{\partial \overline{z_2}}\right)\right)\left(e^A\left(e^{ih(z_2)}\frac{\partial A}{\partial z_2} - ie^{ih(z_2)}\frac{\partial h(z_2)}{\partial z_2}\right)\right)$$

$$= -\, e^{2A}\left(e^{2ih(z_2)}\frac{\partial A}{\partial z_2}\frac{\partial A}{\partial \overline{z_2}} + i\frac{\partial A}{\partial z_2}\frac{\partial h(z_2)}{\partial z_2} + ie^{2ih(z_2)}\frac{\partial A}{\partial \overline{z_2}}\frac{\partial h(z_2)}{\partial z_2} + \frac{\partial h(z_2)}{\partial z_2}\frac{\partial h(z_2)}{\partial \overline{z_2}}\right) \tag{3.14}$$

$$= -\, e^{2A}\left(e^{2ih(z_2)}\left|\frac{\partial A}{\partial z_2}\right|^2 + i\frac{\partial A}{\partial z_2}\frac{\partial h(z_2)}{\partial z_2} + ie^{2ih(z_2)}\frac{\partial A}{\partial \overline{z_2}}\frac{\partial h(z_2)}{\partial z_2} + \left|\frac{\partial h(z_2)}{\partial z_2}\right|^2\right).$$

Since $\frac{\partial A}{\partial z_2} = i\frac{\partial h(z_2)}{\partial z_2}$, we can make the appropriate substitution into (3.14). Then we obtain

$$\frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_2}} \frac{\partial^2 \tilde{\rho}}{\partial z_2 \partial \overline{z_1}} = -e^{2A}\left(e^{2ih(z_2)}\frac{\partial A}{\partial z_2}\frac{\partial A}{\partial \overline{z_2}} + \frac{\partial A}{\partial z_2}\frac{\partial A}{\partial \overline{z_2}} - e^{2ih(z_2)}\frac{\partial A}{\partial z_2}\frac{\partial A}{\partial \overline{z_2}} - \frac{\partial A}{\partial z_2}\frac{\partial A}{\partial \overline{z_2}}\right). \tag{3.15}$$

Then $\frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_2}} \frac{\partial^2 \tilde{\rho}}{\partial z_2 \partial \overline{z_1}} = 0$, so the determinant of the complex hessian of $\tilde{\rho}$ vanishes on M. Next, we calculate the trace. Since

$$\Re\left(\frac{\partial A}{\partial z_1}e^{-ih(z_2)}\right) \leq \frac{1}{2},$$

we obtain the following:

$$\begin{aligned}\frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z_1}} &= e^A\left(2\Re\left(-e^{-ih(z_2)}\frac{\partial A}{\partial z_1} + 1\right)\right)\\ &\geq e^A\left(-2\left(\frac{1}{2}\right) + 1\right)\\ &= 0.\end{aligned} \tag{3.16}$$

Therefore, the complex hessian of $\tilde{\rho}$ on M is positive semi-definite. □

Remark 7. Our problem lies with $\frac{\partial A}{\partial z_2} = i\frac{\partial h(z_2)}{\partial z_2}$. A satisfies this equation if and only if A

is the harmonic conjugate of h, and we have already shown that such A is necessarily discontinuous.

Remark 8. De Rham Cohomology is a cohomology theory based on differential forms on a smooth manifold. De Rham Cohomology is the cohomology of a certain chain complex, wherein each grading, the group is generated by k-forms. The boundary map for the De Rham chain complex is called the exterior derivative. Thus, the cohomology groups in each grading of the chain complex are generated by closed k-forms modulo exact k-forms. This cohomology theory contains global topological data about the manifold. For instance, the failure of the closed forms to be exact tells us topological properties of the manifold, such as holes or twists. Thus, De Rham cohomology is a way of using tangent bundles of the manifold to understand its global topology.

3.1.1 D'Angelo's useful one-form

The form we are interested in is D'Angelo's useful one-form, which is $\tilde{\alpha} = -\mathcal{L}_T \eta$ where $\mathcal{L}_T$ denotes the Lie derivative in the direction of T. Here η denote a purely imaginary, nonvanishing one-form on $\partial\Omega$ that annihilates $T^{(0,1)}(\partial\Omega) \oplus T^{(0,1)}(\partial\Omega)$ and T is a unique purely imaginary tangential vector field orthogonal to $T^{(0,1)}(\partial\Omega) \oplus T^{(0,1)}(\partial\Omega)$ and such that $\eta(T) \equiv 1$. We compute $\tilde{\alpha}(\overline{L})$ where L is a local section of $T^{(1,0)}(\partial\Omega)$. We can calculate $\tilde{\alpha}(\overline{L})$ directly in terms of the defining function ρ. The form $\tilde{\alpha}$ depends on the choice of defining function, but the cohomology class represented by $\tilde{\alpha}$ on the submanifold M does not depend on the defining function. Therefore

$$\tilde{\alpha}(\overline{L}) = \frac{1}{|\partial\rho|^2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}} \frac{\partial \rho}{\partial \overline{z_j}} \overline{w_k}, \tag{3.17}$$

where $\overline{w}_k = d\overline{z}_k(\overline{L})$, (see (5.85) in [22]).

Lemma 3.5. *For Ω defined by (3.1), and assuming $\tilde{\alpha}(\overline{L})$ is defined by (3.17), we have that*

$$\tilde{\alpha}(\overline{L}) = -e^{ih(z_2)}\overline{w_1} + i\frac{\partial h(z_2)}{\partial \overline{z_2}}\overline{w_2}. \tag{3.18}$$

Proof. Using (3.17), we obtain

$$\tilde{\alpha}(\overline{L}) = \frac{1}{|\partial\rho|^2}\left[\frac{\partial^2\rho}{\partial z_1\partial\overline{z_1}}\frac{\partial\rho}{\partial\overline{z_1}}\overline{w_1} + \frac{\partial^2\rho}{\partial z_1\partial\overline{z_2}}\frac{\partial\rho}{\partial\overline{z_1}}\overline{w_2} + \frac{\partial^2\rho}{\partial z_2\partial\overline{z_1}}\frac{\partial\rho}{\partial\overline{z_2}}\overline{w_1} + \frac{\partial^2\rho}{\partial z_2\partial\overline{z_2}}\frac{\partial\rho}{\partial\overline{z_2}}\overline{w_2}\right], \tag{3.19}$$

and where $z_1 = 0$ on M. Then, using the appropriate substitution from (3.9), we calculate $\tilde{\alpha}(\overline{L})$. Then

$$\tilde{\alpha}(\overline{L}) = \frac{1}{|-e^{ih(z_2)}|^2}\left[1(-e^{ih(z_2)})\overline{w_1} + 0 + \left(-ie^{-ih(z_2)}\frac{\partial h(z_2)}{\partial\overline{z_2}}(-e^{ih(z_2)})\right)\overline{w_2} + 0\right],$$

which simplifies to (3.18). □

According to Straube [22], when a domain admits a defining function that is plurisubharmonic at the boundary, then the form $\tilde{\alpha}$ resulting from choosing this defining function is zero on the null space of the Levi form. In particular, it vanishes when restricted to a complex submanifold in the boundary. Then its cohomology class on this submanifold is zero. However, we will build an example containing a simply-connected analytic disc in the boundary on which $\tilde{\alpha}(\overline{L}) = i\frac{\partial h(z_2)}{\partial\overline{z_2}}\overline{w_2}$, but $d_M A = \tilde{\alpha}$ has no continuous solution, so the cohomology class represented by $\tilde{\alpha}$ is non-trivial.

3.1.2 Diederich-Fornæss Index

To obtain estimates, we need to construct bounded plurisubharmonic functions using special defining functions, (see [21],[19],[2],[1],[15], and [11]). Diederich and Fornæss have shown that if $\partial\Omega$ is C^2, then Ω always has a C^2 defining function $\tilde{\rho}$ for some $0 < \eta \leq 1$, such that $-(-\tilde{\rho})^\eta$ is plurisubharmonic on Ω [8].

Proposition 3.6. *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded pseudoconvex domain defined by $\rho = |z_1 - e^{ih(z_2)}|^2 - g(z_2)$. Suppose that for some neighborhood U of M there exists a C^2, bounded real-valued function A on U satisfying*

$$-e^{4A}\eta^2(-se^A)^{2\eta-2}\left[\eta\left(-\frac{\partial^2 A}{\partial z_2 \partial \overline{z_2}} + \left|\frac{\partial A}{\partial z_2} - i\frac{\partial h(z_2)}{\partial z_2}\right|^2\right) + \frac{\partial^2 A}{\partial z_2 \partial \overline{z_2}}\right] \geq 0, \tag{3.20}$$

for some $0 < \eta < 1$. Then there exists a neighborhood $\tilde{U}$ of $\partial\Omega$ such that $-(-\rho e^A)^\eta$ is strictly plurisubharmonic on $\Omega \cap \tilde{U}$.

Proof. Let $p \in M$ and $\tilde{\rho} = e^A \rho \in C^2(M)$. If $L \in T_p^{1.0}(M)$, the complex hessian of $\tilde{\rho}$ at p is defined to be the complex hessian

$$L \mapsto (\partial\overline{\partial}\tilde{\rho})_p(L \wedge \overline{L}). \tag{3.21}$$

So let

$$\begin{cases} L_1 = \dfrac{\partial \rho}{\partial \overline{z_1}}\dfrac{\partial}{\partial z_1} + \dfrac{\partial \rho}{\partial \overline{z_2}}\dfrac{\partial}{\partial z_2} \\ L_2 = \dfrac{\partial \rho}{\partial z_1}\dfrac{\partial}{\partial z_2} - \dfrac{\partial \rho}{\partial z_2}\dfrac{\partial}{\partial z_1}. \end{cases} \tag{3.22}$$

where L_1 is the complex normal vector and L_2 is a tangent vector. The defining function $\tilde{\rho}$ is called plurisubharmonic at p if the complex hessian is positive semi-definite and is called strictly plurisubharmonic at p if the complex hessian is positive definite. Recall that

$$\rho = |z_1 - e^{ih(z_2)}|^2 - g(z_2). \tag{3.23}$$

For $-1 < s \leq 0$, fix z_2 such that $g(z_2) = 1$. Then for such a z_2, set

$$z_1 = e^{ih(z_2)} - \frac{e^{ih(z_2)}}{|e^{ih(z_2)}|}\sqrt{s+1}. \tag{3.24}$$

So we have $\rho(z_1, z_2) = s$.

We then have the following calculations from (3.23) and (3.24):

$$
\begin{aligned}
\frac{\partial \rho}{\partial z_1} &= \overline{z_1} - e^{-ih(z_2)} \\
&= e^{-ih(z_2)} - \frac{e^{-ih(z_2)}}{|e^{-ih(z_2)}|}\sqrt{s+1} - e^{-ih(z_2)} \\
&= -\frac{e^{-ih(z_2)}}{|e^{-ih(z_2)}|}\sqrt{s+1}\left(\frac{\sqrt{s+1}}{\sqrt{s+|e^{-ih(z_2)}|^2}}\right) \\
&= -\frac{e^{-ih(z_2)}(s+1)}{|e^{-ih(z_2)}|\sqrt{s+|e^{-ih(z_2)}|^2}}.
\end{aligned}
\tag{3.25}
$$

As $s \to 0^-$, we have the following:

$$\frac{\partial \rho}{\partial z_1} = -e^{-ih(z_2)} + O(s). \tag{3.26}$$

Then similarly, we obtain the following:

$$\frac{\partial \rho}{\partial \overline{z_1}} = -e^{ih(z_2)} + O(s). \tag{3.27}$$

Also note that

$$\frac{\partial \rho}{\partial z_2} = iz_1 e^{-ih(z_2)}\frac{\partial h(z_2)}{\partial z_2} - i\overline{z_1}e^{ih(z_2)}\frac{\partial h(z_2)}{\partial z_2}. \tag{3.28}$$

After substituting (3.24) into (3.28), we end up with $\frac{\partial \rho}{\partial z_2} = 0$. Then from (3.22), (3.1.2) (3.27) and (3.28), we have the following calculations

$$L_1 = -e^{ih(z_2)}\frac{\partial}{\partial z_1} + O(s), \tag{3.29}$$

and

$$L_2 = -e^{-ih(z_2)}\frac{\partial}{\partial z_2} + O(s). \tag{3.30}$$

We now calculate the following partial derivatives along with the tangent and normal vectors. Note that $\partial \rho = \frac{\partial \rho}{\partial z_1}dz_1 + \frac{\partial \rho}{\partial z_2}dz_2$. Thus

$$
\begin{aligned}
(i) \quad & \partial\rho(L_2) = \frac{\partial\rho}{\partial z_1}\left(-\frac{\partial\rho}{\partial z_2}\right) + \frac{\partial\rho}{\partial z_2}\left(\frac{\partial\rho}{\partial z_1}\right) \\
& \qquad = 0 \\
(ii) \quad & \partial\rho(L_1) = e^{ih(z_2)}\frac{\partial\rho}{\partial z_1} + O(s) \\
& \qquad = 1 + O(s) \\
(iii) \quad & \partial\overline{\partial}\rho(L_j \wedge \overline{L_k}) = \frac{\partial^2\rho}{\partial z_j \partial \overline{z_k}} + O(s) \ \textbf{ for } \ j = k \\
(iv) \quad & \partial A(L_1) = e^{ih(z_2)}\frac{\partial A}{\partial z_1} + O(s) \\
(v) \quad & \partial A(L_2) = e^{-ih(z_2)}\frac{\partial A}{\partial z_2} + O(s) \\
(vi) \quad & \partial\overline{\partial} A(L_j \wedge \overline{L_k}) = \frac{\partial^2 A}{\partial z_j \partial \overline{z}_k} + O(s).
\end{aligned}
\tag{3.31}
$$

Note:

$$
\overline{\partial}(-(-\rho e^A)^\eta) = -(\eta(-\rho e^A)^{\eta-1})(-e^A\overline{\partial}\rho - e^A\rho\overline{\partial}A). \tag{3.32}
$$

Next, we calculate the second partial derivative of ρ:

$$
\begin{aligned}
\partial\overline{\partial}(-(-\tilde{\rho})^\eta) = -\Big(& \eta(\eta-1)(-\rho e^A)^{\eta-2}(-e^A\partial\rho - e^h\rho\partial A) \\
& \wedge (-e^A\overline{\partial}\rho - e^A\rho\overline{\partial}A) + (\eta(-\rho e^A)^{\eta-1})(-e^A\partial A \wedge \overline{\partial}\rho - e^A\partial\overline{\partial}\rho \\
& - e^A\partial A\rho \wedge \overline{\partial}A - e^A\partial\rho \wedge \overline{\partial}A - e^A\rho\partial\overline{\partial}A\Big).
\end{aligned}
\tag{3.33}
$$

Simplifying (3.33), we obtain:

$$
\begin{aligned}
\partial\overline{\partial}(-(-\tilde{\rho})^{\eta}) = &-\Big(e^{2A}\eta(-\rho e^{A})^{\eta-2}(\eta(\partial\rho\wedge\overline{\partial}\rho)+\eta(\partial\rho\wedge\rho\overline{\partial}A)\\
&+\eta(\rho\partial A\wedge\overline{\partial}\rho)+\eta(\rho\partial A\wedge\rho\overline{\partial}A)-(\partial\rho\wedge\overline{\partial}\rho)\\
&-(\partial\rho\wedge\rho\overline{\partial}\rho)-(\rho\partial A\wedge\overline{\partial}\rho)-(\rho\partial A\rho\wedge\overline{\partial}A)+\rho\partial A\wedge\partial\rho\\
&+\rho\partial\overline{\partial}\rho+\rho\partial A\rho\wedge\overline{\partial}A+\rho\partial\rho\wedge\overline{\partial}A+\rho^2\partial\overline{\partial}A\Big)\\
= &-\Big(e^{2A}\eta(-\rho e^{A})^{\eta-2}(\eta(\partial\rho\wedge\overline{\partial}\rho)+\eta(\partial\rho\wedge\rho\overline{\partial}A)+\eta(\rho\partial A\wedge\overline{\partial}\rho)\\
&+\eta(\rho\partial A\wedge\rho\overline{\partial}A)-(\partial\rho\wedge\overline{\partial}\rho)+\rho\partial\overline{\partial}\rho+\rho^2\partial\overline{\partial}A)\Big).
\end{aligned}
\tag{3.34}
$$

Now we derive the following from (3.31) and (3.34) and substituting into (3.21):

$$
\begin{aligned}
&\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_1\wedge\overline{L_1}) =\\
&-e^{2A}\eta(-se^{A})^{\eta-2}\Big(\eta\partial\rho(L_1)\overline{\partial}\rho(\overline{L_1})+\eta\partial\rho(L_1)\rho\overline{\partial}A(\overline{L_1})+\eta\partial A(L_1)s\overline{\partial}\rho(\overline{L_1})\\
&+\eta s\partial A(L_1)s\overline{\partial}A(\overline{L_1})-\partial\rho(L_1)\overline{\partial}\rho(\overline{L_1})+s\partial\overline{\partial}\rho(L_1\wedge\overline{L_1})+s^2\partial\overline{\partial}A(L_1\wedge\overline{L_1})\Big).
\end{aligned}
\tag{3.35}
$$

All the terms with s go into the error term, therefore (3.35) simplifies to the following using substitution from (3.31):

$$
\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_1\wedge\overline{L_1}) = -e^{2A}\eta(-se^{A})^{\eta-2}(\eta-1)+O(s)^{\eta-1}. \tag{3.36}
$$

Secondly, we have the following:

$$
\begin{aligned}
\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_1\wedge\overline{L_2}) = -e^{2A}\eta(-se^{A})^{\eta-2}\Big(&\eta\partial\rho(L_1)\overline{\partial}\rho(\overline{L_2})+\eta\partial\rho(L_1)s\overline{\partial}A(\overline{L_2})\\
&+\eta\partial A(L_1)s\overline{\partial}\rho(\overline{L_2})+\eta s\partial A(L_1)s\overline{\partial}A(\overline{L_2})-\partial\rho(L_1)\overline{\partial}\rho(\overline{L_2})\\
&+s\partial\overline{\partial}\rho(L_1\wedge\overline{L_2})+s^2\partial\overline{\partial}A(L_1\wedge\overline{L_2})\Big).
\end{aligned}
\tag{3.37}
$$

Using the appropriate substitution into (3.37) and sending each term with s^2 into the error term, we obtain the following:

$$\begin{aligned}\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_1\wedge\overline{L_2}) &= -e^{2A}\eta(-se^{A})^{\eta-1}\Bigg(\eta e^{ih(z_2)}\frac{\partial A}{\partial\overline{z_2}} \\ &\quad - e^{2ih(z_2)}\Bigg(ie^{-ih(z_2)}\frac{\partial h(z_2)}{\partial\overline{z_2}}\Bigg)\Bigg) + O(s)^{\eta} \\ &= -ie^{2A}\eta(-se^{A})^{\eta-1}\Bigg(\eta e^{ih(z_2)}\frac{\partial A}{\partial\overline{z_2}} - ie^{ih(z_2)}\frac{\partial h(z_2)}{\partial\overline{z_2}}\Bigg) + O(s)^{\eta} \\ &= -e^{2A+ih(z_2)}\eta(-se^{A})^{\eta-1}\Bigg(\eta\frac{\partial A}{\partial\overline{z_2}} + i\frac{\partial h(z_2)}{\partial\overline{z_2}}\Bigg) + O(s)^{\eta}.\end{aligned} \tag{3.38}$$

Thirdly, we have

$$\begin{aligned}\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_1}) &= -e^{2A}\eta(-se^{A})^{\eta-2}\Bigg(\eta\partial\rho(L_2)\overline{\partial}\rho(\overline{L_1}) + \eta\partial\rho(L_2)s\overline{\partial}A(\overline{L_1}) \\ &\quad + \eta\partial A(L_2)s\overline{\partial}\rho(\overline{L_1}) + \eta s\partial A(L_2)s\overline{\partial}A(\overline{L_1}) \\ &\quad - \partial\rho(L_2)\overline{\partial}\rho(\overline{L_1}) + s\partial\overline{\partial}\rho(L_2\wedge\overline{L_1}) + s^2\partial\overline{\partial}A(L_2\wedge\overline{L_1})\Bigg).\end{aligned} \tag{3.39}$$

Similar to (3.38), after substitution and simplifying, we have

$$\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_1}) = -e^{2A-ih(z_2)}\eta(-se^{A})^{\eta-1}\Bigg(\eta\frac{\partial A}{\partial z_2} - i\frac{\partial h(z_2)}{\partial z_2}\Bigg) + O(s)^{\eta}. \tag{3.40}$$

Lastly, we have

$$\begin{aligned}\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_2}) &= -e^{2A}\eta(-se^{A})^{\eta-2}\Bigg(\eta\partial\rho(L_2)\overline{\partial}\rho(\overline{L_2}) + \eta\partial\rho(L_2)s\overline{\partial}A(\overline{L_2}) \\ &\quad + \eta\partial A(L_2)s\overline{\partial}\rho(\overline{L_2}) + \eta s\partial A(L_2)s\overline{\partial}h(\overline{L_2}) - \partial\rho(L_2)\overline{\partial}\rho(\overline{L_2}) \\ &\quad + s\partial\overline{\partial}\rho(L_2\wedge\overline{L_2}) + s^2\partial\overline{\partial}A(L_2\wedge\overline{L_2})\Bigg).\end{aligned} \tag{3.41}$$

After the appropriate substitution in (3.41), we have the following:

$$\begin{aligned}\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_2}) = -e^{2A}\eta(-se^{A})^{\eta-2}\Bigg(&\eta s^2\frac{\partial A}{\partial z_2}\frac{\partial A}{\partial \overline{z_2}}+O(s^3)\\ &+\frac{s^2}{2}\Bigg(e^{ih(z_2)}\Bigg(-e^{-ih(z_2)}\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2-ie^{-ih(z_2)}\frac{\partial^2 h(z_2)}{\partial z_2\partial\overline{z_2}}\Bigg)\\ &+e^{-ih(z_2)}\Bigg(-e^{ih(z_2)}\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2+e^{ih(z_2)}\frac{\partial^2 h(z_2)}{\partial z_2\partial\overline{z_2}}\Bigg)\Bigg)\\ &+O(s^3)+s^2\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}+O(s^3)\Bigg).\end{aligned}\tag{3.42}$$

Simplifying (3.42), we obtain

$$\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_2}) = -e^{2A}\eta(-se^{A})^{\eta}\left(\eta\left|\frac{\partial A}{\partial z_2}\right|^2-\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2+\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}\right)+O(s)^{\eta+1}.\tag{3.43}$$

Finding the determinant of the complex hessian using (3.36), (3.38), (3.40) and (3.43), we deduce:

$$\begin{aligned}&\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_1\wedge\overline{L_1})(i\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_2}))\\ &\quad-(\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_1\wedge\overline{L_2})\partial\overline{\partial}(-(-\tilde{\rho})^{\eta})(L_2\wedge\overline{L_1}))\\ &\quad= e^{4A}\eta^2(-se^{A})^{2\eta-2}\left((\eta-1)\left(\eta\left|\frac{\partial A}{\partial z_2}\right|^2-\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2+\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}\right)\right)+O(s)^{2\eta}\\ &\qquad-e^{4A}\eta^2(-se^{A})^{2\eta-2}\left(\eta\frac{\partial A}{\partial\overline{z_2}}-i\frac{\partial h(z_2)}{\partial\overline{z_2}}\right)\left(\eta\frac{\partial A}{\partial z_2}-i\frac{\partial h(z_2)}{\partial z_2}\right)\\ &\qquad+O(s)^{2\eta}\\ &\quad= -e^{4A}\eta^2(-se^{A})^{2\eta-2}\left[\eta\left(-\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}+\left|\frac{\partial A}{\partial z_2}\right|^2+2\Re\left(i\frac{\partial A}{\partial z_2}\frac{\partial h(z_2)}{\partial\overline{z_2}}\right)+\left|\frac{\partial h(z_2)}{\partial z_2}\right|^2\right)\right.\\ &\qquad\left.+\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}\right]+O(s)^{2\eta}\\ &\quad= -e^{4A}\eta^2(-se^{A})^{2\eta-2}\left[\eta\left(-\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}+\left|\frac{\partial A}{\partial z_2}-i\frac{\partial h(z_2)}{\partial z_2}\right|^2\right)+\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}\right]+O(s)^{2\eta}.\end{aligned}\tag{3.44}$$

Therefore, $(-(-\tilde{\rho})^{\eta}$ is strictly plurisubharmonic if and only if

$$-e^{4A}\eta^2(-se^{A})^{2\eta-2}\left[\eta\left(-\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}+\left|\frac{\partial A}{\partial z_2}-i\frac{\partial h(z_2)}{\partial z_2}\right|^2\right)+\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}\right]\geq 0. \tag{3.45}$$

□

Remark 9. If we could solve (3.45), then $\left|\frac{\partial A}{\partial z_2}-i\frac{\partial h(z_2)}{\partial z_2}\right|^2$ would vanish since $\frac{\partial A}{\partial z_2}=i\frac{\partial h(z_2)}{\partial z_2}$ and $\frac{\partial^2 A}{\partial z_2\partial\overline{z_2}}=0$. However, since A is the harmonic conjugate of of h and we constructed our example to have a discontinuous harmonic conjugate then we have no real continuous solution.

Remark 10. For example, if $A=-t|z_2|^2$ for some $t>0$, then we have

$$-e^{-4t|z_2|^2}\eta^2(-se^{-t|z_2|^2})^{2\eta-2}\left[\eta\left(t+\left|-t\overline{z_2}-i\frac{\partial h(z_2)}{\partial z_2}\right|^2\right)-t\right]. \tag{3.46}$$

If we fix a constant $t>0$, then we choose η sufficiently small so that (3.46) is positive, hence $-(-\tilde{\rho})^{\eta}$ is plurisubharmonic.

3.1.3 The Construction of the Domain

Recall that:

$$\Omega=\{(z_1,z_2)\in\mathbb{C}^2:|z_1-e^{ih(z_2)}|^2<g(z_2)\},$$

where $h(z_2)$ and $g(z_2)$ are real-valued functions in $C^2(\mathbb{C})$. Then a defining function for Ω is

$$\rho(z_1,z_2)=|z_1-e^{ih(z_2)}|^2-g(z_2). \tag{3.47}$$

Using Proposition 3.1, ρ defines a bounded smooth domain if

(i.) $g(z_2)\leq 0$ outside a compact subset of $\mathbb{C}$,

(ii.) $\nabla g(z_2)\neq 0$ if $g(z_2)=0$.

Let r be a smooth function such that

$$r(x,y) = (y - x\sin x^{-3})^2 x^8 - c^2 x^{20}, \;\; x, y, c \in \mathbb{R},$$

near 0 and $\lim_{x^2+y^2 \to \infty} r(x,y) = \infty$. Next, we define $M_2 \subset \mathbb{C}$, by

$$M_2 = \{z = x + iy : r(x) \leq 0\}.$$

Then define $M \subset \partial\Omega$

$$M = \{(0, z_2) : z_2 \in M_2\}.$$

Theorem 3.7. *There exists a C^2 bounded domain $\Omega \subset \mathbb{C}^2$ such that for $M_2 \subset \mathbb{C}$, defined as above, $M = \{(0, z_2) : z_2 \in M_2\} \subset \partial\Omega$ and on M,*

$$\tilde{\alpha} \equiv i\frac{\partial h(z_2)}{\partial \overline{z_2}} d\overline{z}_2 \mod dz_1, dz_2, d\overline{z}_1,$$

where h is a real harmonic function on M_2 but the harmonic conjugate of h is not continuous on M_2.

Proof. First, we start by defining $g(z_2)$ to be the following:

$$g(z_2) = \begin{cases} 1, & r \leq 0 \\ 1 - 2e^{-\frac{c_1}{r^{c_2}}}, & r > 0, \end{cases} \tag{3.48}$$

for $c_1, c_2 > 0$ and r is our defining function. This bump function is in two real variables. We need to check that $r(x,y)$ is C^2 on M_2 so that $g(z_2)$ is C^2. So we compute the first partial derivatives in terms of r using the formula:

$$\frac{\partial r}{\partial z_2} = \frac{1}{2}\frac{\partial r}{\partial x} - \frac{i}{2}\frac{\partial r}{\partial y}. \tag{3.49}$$

From this, we derive:

$$\begin{aligned}\frac{\partial r}{\partial z_2} &= x^7(y - x\sin x^{-3})(-x\sin x^{-3} + 3x^{-2}\cos x^{-3} + 4(y - x\sin x^{-3}) - xi)\\ &\quad - 10c^2x^{19}.\\ \frac{\partial r}{\partial \overline{z_2}} &= x^7(y - x\sin x^{-3})(-x\sin x^{-3} + 3x^{-2}\cos x^{-3} + 4(y - x\sin x^{-3}) + xi)\\ &\quad - 10c^2x^{19}.\end{aligned} \tag{3.50}$$

Here, we want to find the bounds on $\frac{\partial r}{\partial z_2}$ and $\frac{\partial r}{\partial \overline{z_2}}$. Since $|\sin x^{-3}| \leq 1$ and $|\cos x^{-3}| \leq 1$, we obtain the following:

$$\begin{aligned}\left|\frac{\partial r}{\partial z_2}\right| &\leq \left|x^7(y - x)(-5x + 3x^{-2} + 4y - xi) + 10x^{19}\right|\\ &\lesssim O(x^5y).\end{aligned} \tag{3.51}$$

Similarly, we obtain the same bounds for $\frac{\partial r}{\partial \overline{z_2}}$. We now calculate the second partial derivatives using $\frac{\partial^2 r}{\partial z_2 \partial \overline{z_2}} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial \overline{z_2}}\right) + \frac{i}{2}\frac{\partial}{\partial y}\left(\frac{\partial r}{\partial \overline{z_2}}\right)$,

$$\begin{aligned}\frac{\partial^2 r}{\partial z_2 \partial \overline{z_2}} &= \frac{1}{2}\Big[x^6(y - x\sin x^{-3})\big[7(4y - 5x\sin x^{-3} + 3x^{-2}\cos x^{-3})\\ &\quad + x(-5\sin x^{-3} + 11x^{-3}\cos x^{-3} + 9x^{-6}\sin x^{-3})\big]\\ &\quad + x^7(-\sin x^{-3} + 3x^{-3}\cos x^{-3})(4y - 5x\sin x^{-3}\\ &\quad + 3x^{-2}\cos x^{-3}) + x^8 - 190c^2x^{18}\Big]\\ &= \frac{1}{2}\Big[x^6(y - x\sin x^{-3})\big[28y - 40x\sin x^{-3} + 32x^{-2}\cos x^{-3}\\ &\quad + 9x^{-5}\sin x^{-3}\big] + x^7(-\sin x^{-3} + 3x^{-3}\cos x^{-3})(4y - 5x\sin x^{-3}\\ &\quad + 3x^{-2}\cos x^{-3}) + x^8 - 190c^2x^{18}\Big].\end{aligned} \tag{3.52}$$

Again, using $|\sin x^{-3}| \leq 1$ and $|\cos x^{-3}| \leq 1$, we have $\left|\frac{\partial^2 r}{\partial z_2 \partial \overline{z_2}}\right| \lesssim O(xy)$. Now we find

$$\begin{aligned}\frac{\partial^2 r}{\partial z_2 \partial z_2} = \frac{1}{2}\Big[& x^6(y - x\sin x^{-3})\big[7(4y - 5x\sin x^{-3} + 3x^{-2}\cos x^{-3} - 2xi) \\ &+ x(-5\sin x^{-3} + 12x^{-3}\cos x^{-3} + 9x^{-6}\sin x^{-3} - 2i)\big] \\ &+ x^7(-\sin x^{-3} + 3x^{-3}\cos x^{-3})(4y - 5x\sin x^{-3} \\ &+ 3x^{-2}\cos x^{-3}) - x^8 - 190c^2x^{18}\Big].\end{aligned} \tag{3.53}$$

Therefore, again the bounds on $\left|\frac{\partial^2 r}{\partial z_2 \partial z_2}\right| \lesssim O(xy)$. Since each partial derivative has good bounds, $r(x, y)$ is C^2 on M_2, and hence $g(z_2)$ is also C^2 on M_2. □

In spite of the lack of topological obstructions on the boundary, M has a non-rectifiable boundary, so there are no real continuous solutions to $d_M A = \tilde{\alpha}$ on M.

Bibliography

[1] M. Abdulsahib and P. S. Harrington. Hartogs Domains and the Diederich Fornæss Index. *arXiv e-prints*, page arXiv:1809.02662, Sep 2018.

[2] B. Berndtsson and P. Charpentier. A Sobolev mapping property of the Bergman kernel. *Math. Z.*, 235(1):1–10, 2000.

[3] H. P. Boas and E. J. Straube. de Rham cohomology of manifolds containing the points of infinite type, and Sobolev estimates for the $\overline{\partial}$-Neumann problem. *J. Geom. Anal.*, 3(3):225–235, 1993.

[4] A. L. Cauchy. *Œuvres complètes. Series 1. Volume 7.* Cambridge Library Collection. Cambridge University Press, Cambridge, 2009. Reprint of the 1892 original.

[5] S-C. Chen and M-C. Shaw. *Partial differential equations in several complex variables*, volume 19 of *AMS/IP Studies in Advanced Mathematics.* American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

[6] J. P. D'Angelo. Finite type conditions for real hypersurfaces. *J. Differential Geom.*, 14(1):59–66 (1980), 1979.

[7] J-P. Demailly. Mesures de Monge-Ampère et mesures pluriharmoniques. *Math. Z.*, 194(4):519–564, 1987.

[8] K. Diederich and J. E. Fornæ ss. Pseudoconvex domains: existence of Stein neighborhoods. *Duke Math. J.*, 44(3):641–662, 1977.

[9] J. E. Fornæ ss and A.-K. Herbig. A note on plurisubharmonic defining functions in $\mathbb{C}^n$. *Math. Ann.*, 342(4):749–772, 2008.

[10] P. S. Harrington. The order of plurisubharmonicity on pseudoconvex domains with Lipschitz boundaries. *Math. Res. Lett.*, 15(3):485–490, 2008.

[11] P. S. Harrington. Global regularity for the $\overline{\partial}$-Neumann operator and bounded plurisubharmonic exhaustion functions. *Adv. Math.*, 228(4):2522–2551, 2011.

[12] P. S. Harrington. The Diederich-Fornæss Index and Good Vector Fields. *arXiv e-prints*, page arXiv:1705.05815, May 2017.

[13] L. Hörmander. *The analysis of linear partial differential operators. I.* Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[14] N. Kerzman and J-P. Rosay. Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut. *Math. Ann.*, 257(2):171–184, 1981.

[15] J. J. Kohn. Quantitative estimates for global regularity. In *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., pages 97–128. Birkhäuser Boston, Boston, MA, 1999.

[16] S. von Kowalevsky. Zur Theorie der partiellen Differentialgleichung. *J. Reine Angew. Math.*, 80:1–32, 1875.

[17] P. Lelong. Définition des fonctions plurisousharmoniques. *C. R. Acad. Sci. Paris*, 215:398–400, 1942.

[18] B. Liu. The Diederich-Fornæss index I: for domains of non-trivial index. *arXiv e-prints*, page arXiv:1701.00293, Jan 2017.

[19] B. Liu. The Diederich–Fornæss index and the regularities on the $\bar{\partial}$-Neumann problem. *arXiv e-prints*, page arXiv:1906.00315, Jun 2019.

[20] K. Oka. Note sur les fonctions analytiques de plusieurs variables. *Kodai Math. Sem. Rep.,*, 1(5-6):15–18, 1949. {Volume numbers not printed on issues until Vol. **7**, (1955).}.

[21] S. Pinton and G. Zampieri. The Diederich-Fornæss index and the global regularity of the $\bar{\partial}$-Neumann problem. *Math. Z.*, 276(1-2):93–113, 2014.

[22] E. J. Straube. *Lectures on the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem.* ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2010.