5-1-2012

# Valuation of Financial Derivatives Subject to Liquidity Risk

Yanan Jiang
*University of Nevada, Las Vegas*, jiangy4@unlv.nevada.edu

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# VALUATION OF FINANCIAL DERIVATIVES SUBJECT TO LIQUIDITY RISK

by

Yanan Jiang

Bachelor of Science in Mathematics
University of Science and Arts of Oklahoma
2003

A Dissertation submitted in partial fulfillment of
the requirement for the

**Doctor of Philosophy Degree in Mathematical Sciences**

**Department of Mathematical Sciences**
**College of Sciences**
**The Graduate College**

**University of Nevada, Las Vegas**
**May 2012**

THE GRADUATE COLLEGE

We recommend the dissertation prepared under our supervision by

**Yanan Jiang**

entitled

**Valuation of Financial Derivatives Subject to Liquidity Risk**

be accepted in partial fulfillment of the requirements for the degree of

**Doctor of Philosophy in Mathematical Sciences**
Department of Mathematical Sciences

Michael Marcozzi, Committee Chair

Chih-Hsiang Ho, Committee Member

Hongtao Yang, Committee Member

Seungmook Choi, Graduate College Representative

Ronald Smith, Ph. D., Vice President for Research and Graduate Studies
and Dean of the Graduate College

**May 2012**

# ABSTRACT

**Valuation of Financial Derivatives Subject to
Liquidity Risk**

By

Yanan Jiang

Dr. Michael Marcozzi and Dr. Chih-Hsiang Ho, Examination Committee Chairs

Professors of Mathematical Sciences

University of Nevada Las Vegas

Valuation of financial derivatives subject to liquidity risk remains an open problem in finance. This dissertation focuses on the valuation of European-style call option under limited market liquidity through the dynamic management of a portfolio of assets. We investigate liquidity from three perspectives: market breadth, depth, and immediacy. We present a general framework of valuation based on the optimal realization of a performance index relative to the set of all feasible portfolio trajectories. Numerical examples are then presented and analyzed that show option price increases as the market transitions from liquid to less liquid state. Furthermore, buying and selling activities, based on our optimal trading strategy, decrease as the market becomes less liquid because the gain from more frequent rebalancing of the portfolio is not able to offset the liquidity risk.

# ACKNOWLEDGEMENT

I would like to express my sincere gratitude to my supervisors professor Michael D. Marcozzi and professor Chih-Hsiang Ho for their expert guidance and continued support throughout the course of my graduate study. In particular, I thank professor Marcozzi for his invaluable insights and inspiring discussions that made this work possible. His patience and support helped me overcome my health crisis and finish this dissertation.

I would like to thank my committee members, pofessors Hongtao Yang, and Seungmook Choi for their valuable inputs, and review of this manuscript.

Finally, I would like to thank my parents. None of this would have been possible without their love and support.

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# CHAPTER 1

# INTRODUCTION

Valuation of financial derivatives is one of the central problems in modern finance. The classical Black-Scholes model provides a framework for pricing derivatives in a market that is perfectly liquid, a term we will define in the next section. However, as we have seen from the recent financial crisis, real market often lacks liquidity. The purpose of this dissertation research is to investigate how option prices are affected by lack of liquidity, and how hedging strategy should be adjusted given different liquidity level in the market. The rest of this chapter will proceed as follows: in section 1.1, we will define what liquidity is. In section 1.2, we introduce some necessary background information. In section 1.3, we review the previous literature on asset liquidity model.

## 1.1 Defining Liquidity

Liquidity, due to its multidimensionality, is a delicate concept to define. We all know liquidity roughly refers to the ease of transacting and little price perturbation as a result of trading. However, to model liquidity mathematically, we need a more accurate definition. Many researchers have attempted to give a precise definition of liquidity. Krakovsky [104] describes liquidity as the sensitivity of the stock price to the quantity

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traded, i.e. the ratio between the change in stock price and the change in the amount of stocks traded. Schwartz [134] considers liquidity as the ability of quickly trading an asset at its fair price. Amihud and Mendelson [11], [12], [13] denote liquidity as the cost of executing a transaction quickly. Hachmeister [78] defines liquidity as the ability to buy or sell large amounts of shares quickly without negatively affecting the price. Dowd [63] refers to liquidity as the ability of a trader to execute a trade or liquidate a position with little or no cost, risk or inconvenience. Dowd [63] further explained: "Liquidity is a function of the market, and depends on such factors as the number of traders in the market, the frequency and size of the trades, the time it takes to carry out a trade, and the cost of transacting." In this dissertation research, we described asset liquidity as the ability to sell an asset efficiently, that is, in a timely manner and without loss of value, and measure liquidity from three aspects: market breadth and depth as well as immediacy (cf. [78], [96], [107], [111]).

Market breadth, also known as bid-ask spread, describes the difference between the buying price and selling price of an asset and indicates a cost of transacting [53], [76]. If the size of the bid-ask spread is small, then the market is more liquid; and vice versa. Bid-ask spread can be measured as the absolute difference between bid and ask prices or as a percentage spread [10], [131], [137], i.e.

$$\rho = S_{ask} - S_{bid}, \tag{1.1}$$

or

$$\rho = (S_{ask} - S_{bid})/((S_{ask} + S_{bid})/2), \qquad (1.2)$$

where $S_{ask}$ is the ask price and $S_{bid}$ is the bid price.

Market depth describes the size of an order needed to move the market a given amount (cf. [98], [104], [124]) and indicates a price perturbation. If the market is deep, a large order is needed to change the price, then the market is more liquid; and vice versa. Market depth can be measured as

$$L = \frac{\partial s}{\partial S}, \qquad (1.3)$$

where $L$ is the liquidity ratio (number), $s$ is the transaction size, and $S$ is the stock price. Equation 1.3 can be interpreted as the transaction size needed to drive the stock price up or down one unit.

Variants of the this market depth measure includes

- Amivest liquidity ratio [92]

$$L = \frac{\sum Ss}{\sum(\Delta S\%) \times 1000}, \qquad (1.4)$$

where

    $L$: liquidity ratio (number),

    $s$: transaction size,

    $S$: stock price,

    $\Delta S\%$: percentage change in the transaction price.

- Hui-Heubel liquidity ratio [92]

$$L = \frac{(S_{max} - S_{min})/S_{min}}{s/(N*\bar{S})}, \tag{1.5}$$

where

    $L$: liquidity ratio (number),

    $S_{max}$: highest daily stock price over last 5 days,

    $S_{min}$: lowest daily stock price over last 5 days,

    $s$: total transaction volume over last 5 days,

    $N$: number of shares outstanding,

    $\bar{S}$: average closing price of the stock over last 5 days.

- Marsh-Rock liquidity ratio [119]

$$L = \frac{\sum |\Delta S\%|}{s},$$  (1.6)

where

$L$:  liquidity ratio (number),

$\Delta S\%$:  percentage change in the transaction price,

$s$: number of transactions within a given period.

Immediacy refers to the speed with which a trade can be executed at a prescribed cost thus impacting the portfolio's hedge.

An example of a very liquid market is the foreign exchange, where there are always willing participants. The market is deep, and trades can be executed immediately. An example of an illiquid market is the real estate market, where there aren't always willing buyers. A sale can incur a huge of loss value when the market condition is not favorable, and buying or selling takes time.

## 1.2 Background Information

A financial derivative is a contract whose value depends upon the value of some underlying asset, such as a stock [139]. The contract specifies the rights and obligations between the buyer and the seller to receive or deliver future cash flows based on some future event [139].

One of the most popular types of derivative is option. Option is a contract between two parties on trading an asset at a future date [93]. There are two basic types of option: call and put. A call option gives the holder the right (not the obligation) to buy an underlying asset for a specified strike price by a certain expiration date. A put option gives the holder the right (not the obligation) to sell an underlying asset for a specified strike price by a certain expiration date. Depending upon the dates on which the option may be exercised, most options are either European or American options. A European–style option allows the holder to exercise his/her right to buy or sell only on the expiration date [93]. An American-style option, however, allows the holder to exercise his/her right to buy or sell any time before or on the expiration date [93]. Since the holder of the option receives a privilege, he/she has to pay a premium to the option writer. It is the value of this premium and how market illiquidity affects this premium that we are investigating.

As an option's expiration date is reached, the holder of the option may choose to exercise his/her rights. In the case of a call option, if the spot price is higher than the strike price, the holder of the option will buy a predetermined amount of the underlying asset and sell it on the market for an instant profit. The profit is equal to the spot price of the underlying asset minus the strike price and the option premium. On the other hand, if the spot price is lower than the strike price, the holder of the option will not exercise. Besides speculation, options are also used to protect against risk. For example, a public utility company who owns a gas-fired power plant would buy call options on natural gas to protect against high gas prices and thereby mitigate price spikes on customer's electric bill.

There are three option statuses [139]: in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM). A call option is in-the-money if the stock price is higher than the strike price, a put option is in-the-money if the strike price is higher than the stock price. An option is at-the-money if the stock price is equal to the strike price. A call option is out-of-the-money if the stock price is lower than the strike price, a put option is out-of-the-money if the strike price is below the stock price. Why are we interested in option's moneyness? Some researchers [32] have shown that market illiquidity have a different impact on option's value depending on the option's moneyness. Out-of-the-money (OTM) options are more affected by illiquidity while in-the-money (ITM) options are less affected. Why are people interested in buying or selling OTM options? OTM options have a much higher percentage gain on the same move of the underlying security

than ATM options or ITM options, and OTM options are very cheap to buy. The following is a real world example. Suppose natural gas is trading at $4.00 per MMBTU, and a public utility company wants to buy call option on natural gas to protect against high prices. The company has three choices: OTM option with strike price of $5.00 per MMBTU, ATM option with strike price of $4.00 per MMBTU, or ITM option with strike price of $3.00 per MMBTU. The prices for the contracts are $0.001, $0.50, and $1.00 respectively. If, on the expiration date, natural gas trades at $7.00, then the percentage gain for OTM option would be ($7.00-$5.00)/$0.001=200,000%, the percentage gain for ATM option would be ($7.00-$4.00)/$0.50=600%, and the percentage gain for ITM option would be ($7.00-$3.00)/$1.00=400%.

In the United States, option trading began in 1973, in the Chicago Board Options Exchange. Then almost no one could have predicted that in subsequent decades, it brought a huge impact on the practice and theory of finance. Today, the options market has become an important component of the financial markets. Option is an example of successful innovation in finance, and its development injects vitality into the field of finance.

Other popular types of derivatives are future and forward contracts. A futures contract is an agreement to exchange a specified asset or commodity at a certain date for a certain delivery price [93]. The buyer hopes that the asset price is going to increase, while the seller hopes that it will decrease. Futures contracts are highly standardized and

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specify delivery date and contract size [139]. The contract also stipulates the minimum price fluctuation or tick size and the daily price limit [93]. Futures contracts are traded on a futures exchange, and require both parties to put up an initial margin, which is designed to protect both parties against default [139]. Futures contracts do not contain the element of choice; the parties concerned are obligated to honor the contract [139]. The value a futures contract is evaluated every day, and the change in value is paid to one party by the other, so that the net profit or loss is paid gradually over the lifetime of the contract [139].

A forward contract is an agreement between two parties to buy or sell an asset on a specified date in the future for a specified price, known as the forward price [93]. Forward contracts are traded in an over-the-counter market among major financial institutions and cost nothing to enter. Unlike futures contracts, forward contracts are not standardized and can be tailored to individual needs [93]. Forward contracts do not contain the element of choice, the parties concerned are obligated to ultimately buy or sell the asset [139]. In a forward contract, profit or loss is only realized at the expiry date.

How do we value a derivative, in particular, an option? The history of option pricing can be traced back to French mathematician Bachelier [14]. In his doctoral thesis, he used Brownian motion to describe the stock price process and gave pricing formula for the European call option. Unfortunately, his model was based on unrealistic assumptions, namely: First, the assumption that the underlying stock price follows a normal distribution; second, the value of call option may be greater than the value of the

underlying stock; third, the assumption that the stock's expected return is zero. For this reason, Bachelier's thesis did not receive people's attention until 1965 when economist Paul Samuelson discovered the paper. In 1973, Black and Scholes [20] proposed the famous Black-Scholes formula. Almost at the same time, Merton [20] expanded the mathematical understanding of Black-Scholes model and the pricing formula. In 1976, Cox and Ross [41] proposed the Risk-Neutral valuation method. In 1979, Cox, Ross and Rubinstein [42] gave a simplified proof of the Black-Scholes formula using the fundamental theorem of asset pricing and proposed the binomial option pricing model.

Now we introduce the Black-Scholes model and its derivation. First let's review the basic stock price dynamics. Suppose $S$ denotes the price of a stock, then the return of the stock is given by $\frac{\Delta S}{S}$, where $\Delta S$ is a small increment in price $S$. Suppose there is no risk and expected rate of return of $S$ is $\mu$, then the return of $S$ over a time of $\Delta t$ is given as:

$$\frac{\Delta S}{S} = \mu \Delta t. \tag{1.7}$$

As the time interval becomes smaller, i.e. $\Delta t \to 0$ , we get the ordinary differential equation:

$$\frac{dS}{S} = \mu dt, \tag{1.8}$$

whose solution is given by [135]:

$$S(t) = S(0)e^{\mu t}. \tag{1.9}$$

Since in reality uncertainties always exist, a more practical model of the stock price process is obtained by adding a random term to (1.9)

$$\frac{dS}{S} = \mu dt + noise = \mu dt + \sigma \mathrm{dw(t)}, \tag{1.10}$$

where $\mathrm{dw(t)} = \varepsilon\sqrt{dt}$ is a standard Brownian motion normally distributed with mean zero and variance $dt$, and $\varepsilon$ is a random sampling from a standardized normal variable with mean zero and variance one.

Equation (1.10) can be expressed as:

$$dS(t) = \mu S(t)\mathrm{dt} + \sigma S(t)\,\mathrm{dw(t)}. \tag{1.11}$$

Equation (1.11) is referred to as the geometric Brownian motion.

Now Let's consider two financial assets in which the price per share of the bank account is denoted by $B(t)$ and that of a stock by $S(t)$. A portfolio is a pair $\big(b(t), s(t)\big)$ consisting of the number of shares of $B(t)$ and $S(t)$ held at time $t$, respectively.

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For the bank account, we suppose

$$dB(t) = rB(t)dt, \qquad (1.12)$$

where $r$ represents the risk-free rate of return.

For underlying stock, we suppose

$$dS(t) = \mu S(t)dt + \sigma S(t) \, dw(t), \qquad (1.13)$$

for drift $\mu \in R$, volatility $\sigma > 0$, and a standard Brownian motion $dw(t)$.

To model the financial market we assume the following [93]:

1. The stock price $S(t)$ follows geometric Brownian motion, the drift $\mu$ is a known constant and the volatility $\sigma$ is a known positive constant.

2. The risk-free rate of return $r$ is a known constant.

3. There are no transaction costs or taxes.

4. There are no dividends on the underlying asset.

5. Trading can be done continuously.

6. The market is arbitrage free and liquid.

7. All securities are perfectly divisible (i.e., we can buy or sell a fraction of a share).

8.  Short selling is permitted.

We assume that the option value is a function of the stock price $S(t)$ and time $t$, i.e. $V = V(S(t), t)$. The instantaneous change in the value of the option, by Ito's Lemma, is

$$dV = \left( V_t + \mu S(t) V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS} \right) dt + \sigma S(t) V_S dw(t), \qquad (1.14)$$

where $V_t$ is the derivative of the option value with respect to time, $V_s$ is the derivative of the option value with respect to stock price, and $V_{SS}$ is the second derivative of the option value with respect to stock price. The instantaneous change in option value per unit time, $V_s$, is also referred to as the option's delta; and the change of sensitivity of option value relative to change in price stock, $V_{SS}$, is also referred to as the option's gamma.

We also assume the portfolio $\Pi$ consisting of $b(t)$ of shares of a bond $B(t)$ and $s(t)$ of shares of the stock $S(t)$, i.e.

$$\Pi = b(t)B(t) + s(t)S(t). \qquad (1.15)$$

The instantaneous change in the value of the portfolio due to the changes in security prices, by (1.12) and (1.13), is

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$$d\Pi = b(t)(rB(t)dt) + s(t)(\mu S(t)\mathrm{dt} + \sigma S(t)\,\mathrm{dw(t)})$$

$$= (b(t)rB(t) + s(t)\mu S(t))\mathrm{dt} + s(t)\,\sigma S(t)\,\mathrm{dw(t)}. \qquad (1.16)$$

In order to replicate the option with our portfolio, we set $dV = d\Pi$, that is equation (1.14)

must coincide with equation (1.15).

$$\left(V_t + \mu S(t)V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS}\right) dt + \sigma S(t)V_s dw(t) =$$

$$(b(t)rB(t) + s(t)\mu S(t))\mathrm{dt} + s(t)\,\sigma S(t)\,\mathrm{dw(t)}\,, \qquad (1.17)$$

or

$$\begin{cases} \sigma S(t)V_S = s(t)\,\sigma S(t) & (1.18) \\[2em] V_t + \mu S(t)V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS} = b(t)rB(t) + s(t)\mu S(t). & (1.19) \end{cases}$$

From (1.18), we obtain

$$s(t) = V_s. \qquad (1.20)$$

Since $V = b(t)B(t) + s(t)S(t),$ then

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$$b(t) = \frac{V - V_s S(t)}{B(t)}. \tag{1.21}$$

Substituting (1.20) and (1.21) into (1.19), we obtain

$$V_t + \mu S(t)V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS} = \frac{V - V_s S(t)}{B(t)}rB(t) + V_s \mu\, S(t),$$

or

$$V_t + \mu S(t)V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS} = rV - rV_s S(t) + V_s \mu\, S(t). \tag{1.22}$$

Simplifying (1.22), we obtain the famous Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_s - rV = 0. \tag{1.23}$$

Note we have derived the Black-Scholes equation by replicating the option with stocks and bonds. This derivation is based on Luenberg [115]. However, it is also possible to derive the Black-Scholes equation by a more standard approach based on Hull (cf. [93]). Once again we start by assuming the stock price process follows geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)\,dw(t), \tag{1.24}$$

for drift $\mu \in R$, volatility $\sigma > 0$, and a standard Brownian motion dw(t). Applying Ito's Lemma to $V = V(S(t), t)$, we obtain the instantaneous change in the value of the option

$$dV = \left(V_t + \mu S(t)V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS}\right)dt + \sigma S(t)V_s dw(t), \tag{1.25}$$

Suppose the portfolio consists of long one unit of option and short $\Delta$ units of stocks, where $\Delta = -V_s$, then the portfolio value is

$$\Pi = V(S(t), t) + \Delta S = V(S(t), t) - V_s S. \tag{1.26}$$

The instantaneous change in the value of the portfolio is

$$d\Pi = dV(S(t), t) + \Delta dS = dV(S(t), t) - V_s dS. \tag{1.27}$$

Substituting (1.24) and (1.25) into (1.27), we obtain

$$d\Pi = \left(V_t + \mu S(t)V_s + \frac{1}{2}\sigma^2 S(t)^2 V_{SS}\right)dt + \sigma S(t)V_s dw(t)$$

$$-V_s(\mu S(t)dt + \sigma S(t)\,dw(t)$$

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$$= \left(V_t + \frac{1}{2}\sigma^2 S(t)^2 V_{SS}\right) dt. \tag{1.28}$$

Thus uncertainty has been eliminated, and the portfolio is effectively riskless. By the arbitrage-free argument, the rate of return on the portfolio must be equal to the rate of return on the riskless bond, i.e.

$$d\Pi = r\Pi dt. \tag{1.29}$$

Substituting (1.28) into (1.29) and simplifying, we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial V^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.30}$$

In the case of European-style call option, the boundary conditions are [139]:

(i)   $V(S(T), T) = \max(S - K, 0)$, where $T$ is the expiration date, $S(T)$ is the stock price at time $T$, and $K$ is the strike price;

(ii)  When $S(t) = 0$, for $t < T$, $S(t)$ will stay zero for all subsequent times, thus $V(0, t) = 0$;

(iii) As $S(t)$ increases without bound, the strike price becomes less important. Thus as $S \to \infty$, $V(S(t), t) \sim S(t)$.

The Black-Scholes equation can be solved analytically by transforming it to a heat equation using a variable transformation [139]. Let $S = Ke^x, \quad t = \frac{T-\tau}{\frac{1}{2}\sigma^2}, \quad V = Kv(x,\tau)$,

we can rewrite the Black-Scholes equation as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial V^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$\Rightarrow -\frac{1}{2}\sigma^2 K\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S^2(\frac{-K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}) + rS\frac{K}{S}\frac{\partial v}{\partial x} - rKv = 0$$

$$\Rightarrow -\frac{1}{2}\sigma^2 K\frac{\partial v}{\partial \tau} - \frac{1}{2}\sigma^2 K\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 K\frac{\partial^2 v}{\partial x^2} + rK\frac{\partial v}{\partial x} - rKv = 0$$

$$\Rightarrow -\frac{1}{2}\frac{\partial v}{\partial \tau} - \frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \frac{r}{\sigma^2}\frac{\partial v}{\partial x} - \frac{r}{\sigma^2}v = 0$$

$$\Rightarrow \frac{1}{2}\frac{\partial v}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right)\frac{\partial v}{\partial x} - \frac{r}{\sigma^2}v$$

$$\Rightarrow \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v$$

$$\xRightarrow{let\frac{r}{\frac{1}{2}\sigma^2}=l} \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (l-1)\frac{\partial v}{\partial x} - lv \qquad (1.31)$$

Let $v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$, we obtain

$$\beta e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}u(x,\tau)\frac{\partial u}{\partial\tau} = \alpha^2 e^{\alpha x+\beta\tau}u(x,\tau) + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}$$

$$e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} + (l-1)\left[\alpha e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right] - le^{\alpha x+\beta\tau}u(x,\tau)$$

$$\Rightarrow \beta u + \frac{\partial u}{\partial\tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (l-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - lu$$

$$\Rightarrow \beta + \frac{\partial u}{\partial\tau} = [\alpha^2 + (l-1)\alpha - l]u + (2\alpha + l - 1)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \qquad (1.32)$$

Let $\alpha = -\frac{1}{2}(l-1)$, $\beta = \alpha^2 + (l-1)\alpha - l = -\frac{1}{4}(l+1)^2$, then equation (1.32) is

transformed into the heat equation

$$\frac{\partial u}{\partial\tau} = \frac{\partial^2 u}{\partial x^2}. \qquad (1.33)$$

The exact solution for a European call option is given by [139]:

$$V(S(t),t) = S(t)N(d_1) - Ke^{-rT}N(d_2), \qquad (1.34)$$

where

$$d_{1,2} = \frac{Ln(\frac{S(t)}{K}) \pm (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

and

$V$ = call option premium;

$S$ = current stock price;

$K$ = strike price;

$T - t$ = time to expiry;

$\sigma$ = standard deviation of the stock price;

$Ln(\cdot)$ = natural logarithm;

$N(\cdot)$ = cumulative standard normal probability distribution;

$d_1, d_2$ =standard normal random variables;

$r$ = risk-free rate of interest.

The following is a simple example of how the Black-Scholes hedging strategy works. Suppose the stock is trading at $100 per share, the price of the call option on stock is $10, and the option's delta is 0.3.  An option writer sells a call option, and the buyer of the option buy 100 shares at maturity. To construct a hedged portfolio, the seller should buy 0.3 x 100 = 30 shares of stock. If the stock price goes up $1, the option price will go up by $0.3, and then the seller has a $30 ($1 x 30 shares) gain in its stock position, and a $30 ($0.3 x 100 shares) loss in its option position. The net gain/loss is zero. On the other

hand, if the stock price goes down by $1, the option price will go down by $0.3. The net gain/loss is also zero.

## 1.3 Literature Review

One of the first market liquidity research was conducted by Demsetz. In 1968, Demsetz published a landmark paper "The cost of transacting "[53]. This work marked the birth of stock market microstructure theory. Although the paper didn't explicitly mention the term "liquidity", it was widely considered as the pioneering work in the field. The paper focuses on the bid-asked spread, and believes that the cause of the bid-spread is the imbalance between supply and demand. One of the key viewpoints is that transaction cost is the price concession needed for an immediate conversion of an asset into money. For example, if a buyer wants an immediate purchase, he/she can apply for a higher price to attract those who are not eager to sell. On the other hand, if a seller cannot find a suitable counterparty or sufficient demand, he/she can offer a lower price for immediate transaction. Other earlier works that focused on the bid-asked spread and market microstructure includes: Copeland and Galai [39], Glosten and Harris [75], Amihub and Mendelson [8], [9], [10], [11], [12], [13], Hasbrouck et al. [79], [80], [81], [82], [83], Grossman and Miller [77], Acharya and Pedersen [1], Brennan and Subrahmanyam [25], Domowitz and Wang [62], Coughenour and Saad [40], Fernando [69], Lin, Sanger and Booth [110], Holthausen, Leftwich and Mayers [88], [89], Hong and Rady [90], Lippman and McCall [111], and Chordia et al. [34], [35], [36], [37].

While most of the above mentioned works are empirical in nature, there is a large body of literature that study liquidity from the perspective of arbitrage pricing theory. These literature can be broadly classified into three categories: modeling of price impact (market depth), relying on mappings of the Black-Scholes economy; augmenting the Black-Scholes dynamics via the introduction of an explicit exogenous event (Poisson process); perturbing the volatility in the Black-Scholes model from the perspective of large investors.

Cetin, Jarrow and Protter (cf. [32]) considered illiquidity as an extra "friction" in the price dynamics and modeled the price dynamics based on affine mappings of the Black-Scholes economy. The stock price process follows geometric Brownian motion,

$$dS(t) = \mu S(t)dt + \sigma S(t) \, dw(t), \tag{1.28}$$

for drift $\mu \in R$, volatility $\sigma > 0$, and a standard Brownian motion $dw(t)$. The transaction price to be paid at time $t$ for trading $x$ shares is

$$\bar{S}(x) = e^{Lx} \, S, \quad x \in R, \quad L > 0 \tag{1.29}$$

where $L$ is a liquidity parameter. When the market is perfectly liquid, $L$ is equal to zero, the price dynamics reduces to the Black-Scholes model. As the market becomes less liquid, $L$ becomes larger. They hypothesized the existence of a stochastic supply curve

which gives a relationship between the stock prices and quantity of stocks traded. For a perfectly liquid market, the slope of the supply curve would be zero because the stock price is not affected by the quantity traded. However, when the market is not so liquid, the slope of the supply curve becomes steeper as the amount of traded assets becomes larger. This indicates a price impact due to transaction size. Their study also shows that liquidity is a significant factor of option price. Moreover, they found that illiquidity has less impact on in-the-money (ITM) options than out-of-the-money (OTM) options. They found that in-the-money (ITM) options are subject to the lowest percentage impact of illiquidity. On the contrary, the out-of-the-money (OTM) options are significantly affected by illiquidity despite OTM options are cheaper than ITM options.

Similar models include: Bakstein and Howison [15], Rogers and Singh [130], Cetin and Rogers [30], Cetin, Soner and Touzi [31], Blais [21], Blais and Protter [22], Almgren et al. [5], [6], [7], Hea and Mamaysky [84], Jarrow [97], Isaenko [95], Liu and Yong [113], and Ting, Warachkaa and Zhao [138]. The advantage of this approach is that it is easy to implement, i.e. the liquidity parameter can be estimated through regression. However, this model works where the market is essentially "steady-state", i.e. for a liquidity parameter that is fixed or has limited ranges. Thus determining hedging strategy under varied liquidity parameters remains an issue.

Another paradigm originates from the observation that illiquid market often has price spikes of traded assets. Lee and Protter [108] use Jump-diffusion model to model price

spikes in illiquid markets via the introduction of an explicit exogenous event. The stock price process follows the dynamics:

$$dS(t) = \mu S(t)dt + \sigma S(t)dw(t) + \varphi dP, \qquad (1.30)$$

for drift $\mu \in R$, volatility $\sigma > 0$, a standard Brownian motion $dw(t)$, and the increment of a Poisson process $dP$ with jump size $\varphi$. Since jumps in prices give rise to incomplete market, perfect hedging is not always possible (cf. [129]). They implemented local risk minimization strategies based on martingale decomposition. Other works that follow this approach include: Bellamy and Jeanblanc [18], Eberlein and Jacod [65], Cont and Tankov [38], El Karoui and Quenez [67], Ladde [106], Kou [100], Kou and Wang [101], [102], and Kou, Petrella and Wang [103].

The advantage of this approach is that it captures some important empirical phenomena and offers tractable solutions. However, since liquidity is implicitly embedded in the model, it fails to provide an explicit link between market liquidity and corresponding hedging strategy.

The third paradigm originates from the observation that many markets are essentially perfectly liquid from the perspective of small investors but not perfectly liquid

from the perspective of large investors. For small traders, the price does not change much in response to their trades. For large investors, their trading volume represents a large proportion of the trading activities in the market, thus it will have a significant impact on the price. The presence of large investors is an importance source of market illiquidity.

Frey et al. [71], [72], [73] study market illiquidity due to the influence of large traders; the stock price process follows the dynamics:

$$dS(t) = \mu S(t)dt + \frac{\sigma}{1 - L\lambda V_{ss}} S(t)d\mathrm{w}(t), \qquad (1.31)$$

where $L$ is a liquidity parameter (market depth), $\lambda$ is a parameter describing the asymmetry of liquidity, i.e. the asymmetric relationship between moneyness and liquidity, and $V_{ss}$ represents the value of gamma, i.e. the second derivative of the option value with respect to stock price. The model argues that, unlike the volatility in the Black-Scholes model, the volatility is not a constant. Instead, the volatility term is dominated by three main parameters $L$, $\lambda$, and $V_{ss}$. Depending upon the values of these parameters, a large trader can adopt different trading strategies.

Other related works includes: Esser and Moench [68], Bank and Baum [16], Cvitanic and Ma [46], Cuoco and Cvitanic [43], Schonbucher and Wilmott [133], Kabanov and Safarian [91], Longstaff [114], Boyle and Vorst [23], Palmer [126], Moulton [121],

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Bordag [23], Gennotte and Leland [74], Broadie, Cvitanic and Soner [26], Platen and Schweizer [128], Sircar and Papanicolaou [136].

# CHAPTER 2

# VALUE PROBLEM

We consider in this chapter the determination of the fair price of European style call option in a market with limited liquidity. We assume that in this market one asset is perfectly liquid, namely bond (bank account); the other asset is subjects to limited liquidity, namely stock. In section 2.1, we develop the model of the economy incorporating market breadth as a transaction costs related to the bid-ask spread, market depth as a price perturbation dependent upon the trading strategy, and immediacy via transaction rates. Since the market is made incomplete by the limited market breadth and depth, the Black-Scholes delta hedging argument no longer applies. Instead, we follow the value-maximizing strategy of a dynamically evolving portfolio of assets proposed by Hodges and Neuberger [86] and Davis et al. [49], [50], [51], [52]. Other authors who have employed value-maximizing strategy includes Munk [122], [123], Zakamouline [140], Damgaard [47], [48], Monoyios [120], Forsyth [70], Barles and Soner [17], Bertsimas and Lo [19], Henderson and Hobson [88], and Hugonnier, Kramkov, and Schachermayer [91]. As such, our model is a generalization of the more familiar Black-Scholes framework. In section 2.2, we characterize the indexed value function as the unique solution to the ultraparabolic Hamilton-Jacobi equation. Valuation of the model is described in section 2.3.

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## 2.1 Model of the economy

We consider two financial assets in which the price per share of the bank account is denoted by $B(t)$ and that of a stock by $S(t)$. A portfolio is a pair $(b(t), s(t))$ consisting of the number of shares of $B(t)$ and $S(t)$ held at time $t$, respectively. The value of the portfolio or wealth $W(t)$ is

$$W(t) = b(t)B(t) + s(t)S(t). \tag{2.1}$$

We make the further distinction between shares which are bought and sold, such that

$$b(t) = b_+(t) - b_-(t) \text{ and } s(t) = s_+(t) - s_-(t),$$

where $\big(b_+(t),\ b_-(t)\big)$ denote shares of bonds bought ("+") and sold ("-"), respectively; and $\big(s_+(t),\ s_-(t)\big)$ denote shares of stocks bought ("+") and sold ("-"), respectively.

We rebalance the portfolio through the trading strategy $\big(\overline{\beta}(t), \overline{\varsigma}(t)\big)$,

$$\overline{\beta}(t) = \big(\beta_-(t), \beta_+(t)\big) \text{ and } \overline{\varsigma}(t) = \big(\varsigma_-(t), \varsigma_+(t)\big),$$

denoting the respective rates at which shares in the portfolio are bought and sold. In particular, we relate the strategy to the portfolio via the dynamics

$$\frac{db_+(t)}{dt} = \beta_+(t); \qquad \frac{db_-(t)}{dt} = \beta_-(t), \qquad\qquad (2.2a)$$

$$\frac{ds_+(t)}{dt} = \varsigma_+(t); \qquad \frac{ds_-(t)}{dt} = \varsigma_-(t), \qquad\qquad (2.2b)$$

such that $0 \leq \beta_\pm(t) \leq \gamma$ and $0 \leq \varsigma_\pm(t) \leq \gamma$, for some immediacy $\gamma < \infty$. We note that a larger $\gamma$ would indicate greater asset liquidity.

In a realistic market setting where there is limited liquidity, there are always bid-ask spreads, and thus transaction cost. Let $\rho$ denote the bid-ask spread, given by

$$d\rho(t) = \lambda\rho(t)dt + \sigma_1\rho(t)dw_1(t) + \sigma_2\rho(t)dw_2(t), \qquad\qquad (2.3)$$

where $\lambda$, $\sigma_1$, and $\sigma_2$ are positive constants, and $dw_1(t)$ and $dw_2(t)$ are Wiener processes. We take the market breadth to be $\mathcal{B} \cdot \rho$, that is, an affine mapping of the bid-ask spread scaled by the breadth constant $\mathcal{B} > 0$. In particular, we note that the smaller the value of B, the greater the liquidity of the asset. Since we withdraw an amount of wealth for buying stocks from the bank account and add an amount of wealth gained from selling stocks to the bank account, the cost of a refinancing the portfolio is

$$\beta(t)B(t) + \varsigma(t)S(t) = -\varsigma_+(t)\mathcal{B}\rho(t), \qquad\qquad (2.4)$$

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where

$$\beta(t) = \beta_+(t) - \beta_-(t) \text{ and } \varsigma(t) = \varsigma_+(t) - \varsigma_-(t).$$

In terms of the limited market depth, we have the following behavior. We suppose that perturbations to the price of the stock in a market lacking depth is given incrementally by

$$\frac{\Delta S(t)}{S(t)} = \mathcal{D} \cdot \Delta s(t),$$

or

$$\frac{dS(t)}{S(t)} = \mathcal{D} \cdot ds(t),$$

where $\mathcal{D} > 0$. Again, the smaller the depth constant $\mathcal{D}$, the greater the asset liquidity. The overall stock price process can be described as having two components: stock price process under perfectly liquid condition, which follows the classical Black Scholes model; and price perturbation component due to limited market depth, i.e.

$$\Delta S = \Delta S_{liquid} + \Delta S_{perturbation},$$

or infinitesimally as

$$dS = dS_{liquid} + dS_{perturbation}. \tag{2.5}$$

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As such, the dynamics of the asset in a market with limited depth is given as follows

$$dS(t) = \mu S(t)dt + \sigma_2 S(t)dw_2(t) + \sigma_3 S(t)dw_3(t) + \mathcal{D}S(t)ds(t)$$

or from (2.2b),

$$dS(t) = [\mu + \mathcal{D}\varsigma(t)]S(t)dt + \sigma_2 S(t)dw_2(t) + \sigma_3 S(t)dw_3(t), \qquad (2.6)$$

where the drift $\mu \in$ R, and volatilities $\sigma_2$ and $\sigma_3$ are positive, $dw_3(t)$ is a Wiener process, and $dw_2(t)$ provides a correlation between the dynamics of the stock price and the bid-ask spread. We will show in chapter 4 that the different sizes of price perturbation due to limited market depth can result in significant changes in the option values as well as trading strategies.

For the bank account, we suppose

$$dB(t) = rB(t)dt, \qquad (2.7)$$

where $r > 0$ represents the risk-free rate of return.

Finally, for the portfolio velocity, i.e. the change in wealth during a small time interval $dt$, we differentiate (2.1),

$$dW(t) = \beta(t)B(t) + rb(t)B(t) + \varsigma(t)S(t) + \{[\mu + \mathcal{D}\varsigma(t)]S(t)\}dt \qquad (2.8)$$

$$+\sigma_2 s(t)S(t)dw_2(t) + \sigma_3 s(t)S(t)dw_3(t).$$

Since the first term and the third term of (2.8) add up to $-\varsigma_+(t)\mathcal{B}\rho(t)$ by (2.4), we have

$$dW(t) = rb(t)B(t) - \varsigma_+(t)\mathcal{B}\rho(t) + \{[\mu + \mathcal{D}\varsigma(t)]S(t)\}dt \qquad (2.9)$$

$$+\sigma_2 s(t)S(t)dw_2(t) + \sigma_3 s(t)S(t)dw_3(t).$$

By adding and subtracting the term $rs(t)S(t)$, and some simple algebra, we obtain

$$dW(t) = \{rW(t) + [(\mu - r)s(t) - L^{\bar{\varsigma}}(\rho)(t)]S(t)\}dt \qquad (2.10)$$

$$+\sigma_2 s(t)S(t)dw_2(t) + \sigma_3 s(t)S(t)dw_3(t),$$

where the asset liquidity is given by

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$$L^{\bar{s}}(\rho) = \mathcal{B}\,\varsigma_+\rho - \mathcal{D}\,\varsigma. \tag{2.11}$$

We note that the asset liquidity manifests as a component of the drift of the portfolio's velocity. Moreover, in a perfectly liquid market, i.e. $L^{\bar{s}}(\rho) = 0$, our model reduces to the Black-Scholes model.

## 2.2 Hamilton-Jacobi equation

We may assume that the trading of stocks and bonds is done continuously over a finite time interval $[0, \tilde{T}]$, such that $\tilde{T} > T$, where $T$ denotes the contract life of the option. Summarizing section 2.1, the dynamics of the economy are specified by the ultradiffusion process

$$ds_+(t) = \varsigma_+(t; \bar{s})dt \tag{2.12a}$$

$$ds_-(t) = \varsigma_-(t; \bar{s})dt \tag{2.12b}$$

$$d\rho(t) = \lambda\rho(t)dt + \sigma_1\rho(t)dw_1(t) + \sigma_2\rho(t)dw_2(t) \tag{2.12c}$$

$$dS(t) = [\mu + \mathcal{D}\varsigma(t)]S(t)dt + \sigma_2 S(t)dw_2(t) + \sigma_3 S(t)dw_3(t) \tag{2.12d}$$

$$dW(t) = \{\mu W(t) + [(\mu - r)s(t) - L^{\bar{\varsigma}}(\rho)(t)]S(t)\}dt \tag{2.12e}$$

$$+\sigma_2 s(t)S(t)dw_2(t) + \sigma_3 s(t)S(t)dw_3(t)$$

for $t \in (0, \tilde{T})$, such that $s_+(0) = s_+$, $s_-(0) = s_-$, $\rho(0) = \rho$, $S(0) = S$, $W(0) = W$, and $\bar{s} = (s_+, s_-)$. Here, the initial conditions $s_\pm$ are temporal, while $\rho$, $S$ and $W$ are variables

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of state. Note that we have a choice as to the initial composition of the portfolio, and as such, $s_+$ and $s_-$ are indices of the system, whereas the trader's efforts to rebalance the portfolio in the form of buying and selling shares $\varsigma_\pm$ constitutes a system control. The system control is the optimal trading strategy that affects the wealth process to achieve the desired objective, i.e. the maximization of expected excess wealth. The system control depends on not only the time $t$, but also the wealth process $W(t)$, the stock price process $S(t)$, and the dynamics of the bid-ask spread $\rho(t)$.

Let $\bar{x} = (\rho, S, W)$ and $\bar{t} = (t, s_+, s_-)$, then along with the process (2.10), we consider the expected value of the optimization criteria

$$J_{\bar{x}}(T; \ \bar{s}) = E_{\bar{x}; \bar{s}}[W(t) - \psi(S(T))], \tag{2.13a}$$

and the indexed value function

$$V(\bar{x}, \bar{t}) = \max_{\bar{\varsigma} \in [0, \gamma]^2} J_{\bar{x}}(T; \ \bar{s}), \tag{2.13b}$$

such that $\psi(S)$ is the pay-off of the option. The $\bar{s}$-indexed value function attempts to maximize the expected excess wealth, indexed upon the choice of the initial portfolio $\bar{s}$.

We next seek to determine the indexed value function $V(\bar{x}, \bar{t})$. To this end, let $\Omega =$

$\mathbb{R} \times (0, \infty)^2$ and $Q = (0, \tilde{T}) \times (0, \infty)^2$. The indexed value function $V(\bar{x}, \bar{t})$, by the principle of optimal stochastic control, is the unique solution of the ultraparabolic Hamilton-Jacobi equation

$$\frac{\partial V}{\partial t} + \mathcal{A}V + \mathcal{H}^{\bar{\varsigma}^*}(\bar{t})V = 0 \ \text{a.e. in} \ \ \Omega \times Q, \tag{2.14a}$$

for terminal condition

$$V|_{\{t=T\}} = W - \psi(S) \ \text{in} \ \bar{\Omega} \times \partial Q, \tag{2.14b}$$

and boundary data

$$V|_{\partial Q} = W - \psi(S) \ \text{in} \ \partial\Omega \times Q, \tag{2.14c}$$

Such that the optimal feedback control law $\bar{\varsigma}^*(t; \bar{s}) = (\varsigma_-^*(t; \bar{s}), \varsigma_+^*(t; \bar{s})$ satisfies

$$\mathcal{D}\varsigma^* S \frac{\partial V}{\partial S} - L^{\bar{\varsigma}}(\rho)(t) S \frac{\partial V}{\partial W} + \varsigma_+^*(t; \bar{s}) \frac{\partial V}{\partial s^+} + \varsigma_-^*(t; \bar{s}) \frac{\partial V}{\partial s^-} = \tag{2.15}$$

$$\max_{\varsigma \pm (t; \bar{s}) \in [0, \gamma]} \left\{ \mathcal{D}\varsigma S \frac{\partial V}{\partial S} - L^{\bar{\varsigma}}(\rho)(t) S \frac{\partial V}{\partial W} + \varsigma_+(t; \bar{s}) \frac{\partial V}{\partial s^+} + \varsigma_-(t; \bar{s}) \frac{\partial V}{\partial s^-} \right\},$$

where

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$$\mathcal{A}V = \frac{1}{2}\{(\sigma_1^2 + \sigma_2^2)\rho^2 \frac{\partial^2 V}{\partial \rho^2} + \sigma_2^2 \rho S \frac{\partial^2 V}{\partial \rho \partial s} + s\sigma_2^2 \rho S \frac{\partial^2 V}{\partial \rho \partial W}$$

$$+\sigma_2^2 \rho S \frac{\partial^2 V}{\partial s \partial \rho} + (\sigma_2^2 + \sigma_3^2)S^2 \frac{\partial^2 V}{\partial s^2} + s(\sigma_2^2 + \sigma_3^2)\frac{\partial^2 V}{\partial s \partial W}$$

$$+s\sigma_2^2 \rho S \frac{\partial^2 V}{\partial W \partial \rho} + s(\sigma_2^2 + \sigma_3^2)\frac{\partial^2 V}{\partial W \partial s} + s^2(\sigma_2^2 + \sigma_3^2)S^2 \frac{\partial^2 V}{\partial W^2}\},$$

$$\mathcal{H}^{\bar{\varsigma}^*}(\bar{t}) = \max_{\overline{\varsigma}(t;\bar{s}) \in [0,\gamma]^2} \left\{ L^{\bar{\varsigma}}(\bar{t})V + \varsigma_+(t;\bar{s})\frac{\partial V}{\partial s^+} + \varsigma_-(t;\bar{s})\frac{\partial V}{\partial s^-} \right\},$$

and

$$L^{\bar{\varsigma}}(\bar{t})V = \lambda\rho\frac{\partial V}{\partial \rho} + (\mu + \mathcal{D}\varsigma)S\frac{\partial V}{\partial s}$$

$$+\{rW + [(\mu - r)s(t) - L^{\bar{\varsigma}}(\rho)(t)]S\}\frac{\partial V}{\partial W}.$$

## 2.3 Valuation of the model

As an option writer, we form a portfolio in order to hedge the option and liquidate the portfolio at time $T$ to meet the pay-off of the option. The pay-off is $\psi(S(T)) = \max(S(T) - E, 0)$ for a European call option, where $\psi(S(T))$ denotes the pay-off, $S(T)$ denotes the stock price at expiry, and E denotes the strike price. To determining the fair price of an option is equivalent to finding the expected minimum wealth necessary to meet the pay-off of the option. We will achieve this goal by doing the following.

First, we want to maximize the expected excess wealth of the portfolio, i.e. the value of the portfolio at time $T$ minus the pay-off of the option $W(T) - \psi(S(T))$. This will ensure that at any time $t \in [0, \tilde{T}]$, we have sufficient funds to meet our obligation without losing money. Since the stock price is stochastic, the quantity $W(T) - \psi(S(T))$ at any given time $t$ is random, we take the expectation of the excess wealth of the portfolio. The maximum will then be taken with respect to our control parameter $\overline{\varsigma}$ (cf. (2.2b)). Parameter $\overline{\varsigma}$ represents the control over the buying/selling of assets. For simplicity, we have considered the "bang-bang" type control. ("Bang-bang" refers to the fact that the optimal action for $\varsigma_+$ and $\varsigma_-$ is either 0 or $\gamma$, the control is either "buy" or "sell".) To this end, we introduce the conditional value function $V$ which is dependent on the initial portfolio distribution, such that,

$$V(S, W, s_+, s_-, t) = \max_{\overline{\varsigma} \in [0,\gamma]^2} \mathrm{E}\big[W(T) - \psi(S(T))\big] , \qquad (2.16)$$

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where $\psi(S(T))$ is the pay-off of the option, and $\gamma > 0$. The conditional value function $V$ attempts to maximize the expected excess wealth, indexed upon the choice of the initial portfolio $\bar{s}$ (cf. (2.12b))

Next, since the fair price of the option would be the minimum amount of initial stock holding that is needed to replicate the pay-off, we introduce the concept of performance index, i.e. the minimum wealth needed to meet our obligation. To this end, let the performance index be given by

$$\widetilde{U}(S, t; \bar{s}) = min\{W | V(S, W, s_+, s_-, t) \geq 0\}, \qquad (2.17)$$

Since the conditional value function depends on the initial portfolio distribution, we take the minimum of the performance index among all feasible initial portfolio distributions relative to a terminal constraint

$$U(S, t) = min\{\widetilde{U}(S, t; \bar{s}) | \bar{s} \epsilon F_{(\bar{x}, T)}\}, \qquad (2.18)$$

The terminal constraint is constructed such that if the stock price $S$ at time $T$ is higher than the strike price E, and the buyer of the option decides to exercise the option, we will deliver one share of stock to him/her. To this end, let

$$g_-(S, W) = 0,$$

and

$$g_+(S, W) = \begin{cases} 1, & if\ S > E \\ 0, & if\ S \leq E \end{cases}$$

where E is the strike price of the option. We then suppose the terminal set $M(\bar{x}, T)$ is specified by

$$M(\bar{x}, T) = \{(g_-(S, W),\ g_+(S, W) | (S, W) \in \Omega)\},\ \text{for}\ \Omega = \mathbb{R} \times (0, \infty)$$

The value function $U(S, t)$ represents the expected minimum wealth necessary at time $t$ required to meet the pay-off $\psi(S)$ at time $T$ and as such is the fair price of the European option.

Note that in contrast to the Black-Scholes model, which is derived under complete market, our model does not care whether the market is complete or not. If the market is complete, in this particular context perfectly liquid, our model reduces to the Black-Scholes model. For an incomplete market, such as a market that has limited liquidity, our

model offers a feasible way to price an option. Therefore our model can be regarded as a generalization of the Black-Scholes option pricing model.

# CHAPTER 3

# NUMERICAL SIMULATION

In this chapter, we show the numerical results from our model based on the methods introduced in Marcozzi [117], [118]. For computation expedience, we consider immediacy and depth effects only; that is, we suppose $\lambda = \sigma_1 = \sigma_2 = 0$. Moreover, we set $\mu = 0.10$ per year, $\sigma_3^2 = (0.3)^2$ per year and $r = 0.05$ per year, an option life of $T = 1 \times 10^{-4}$ years, and a strike price for the European call option of $E = \$1$. The mesh utilized $h = \Delta S = \Delta W = 5 \times 10^{-4}$, $\Delta T = 1 \times 10^{-6}$, and $\Delta \rho = \Delta \varsigma = 1 \times 10^{-2}$. The computational domain was $[0.99, 1.01]^2 \times [0.0, 0.4]^2 \times [0, 10^{-4}]$. All the code is written in FORTRAN and the computations are conducted on an Intel Core i7 3.40 GHz CPU with 12 GB RAM.

Case 1: We first consider the case of a liquid market featuring relative depth and little immediacy friction ( $\mathcal{D} = 0.001$ and $\gamma = 0.5$). Figure 1 gives the relationship between the option prices and the stock prices at various time until expiration. Figure 2 shows, according to our trading strategy, how many shares of stocks we should hold as the stock price evolves. Figure 3 and Figure 4 show the optimal time to buy and sell according to our trading strategy. Table 1 reports option prices at different time until expiration with reference to option's moneyness; and Table 2 provides summary statistics of option prices for case 1.
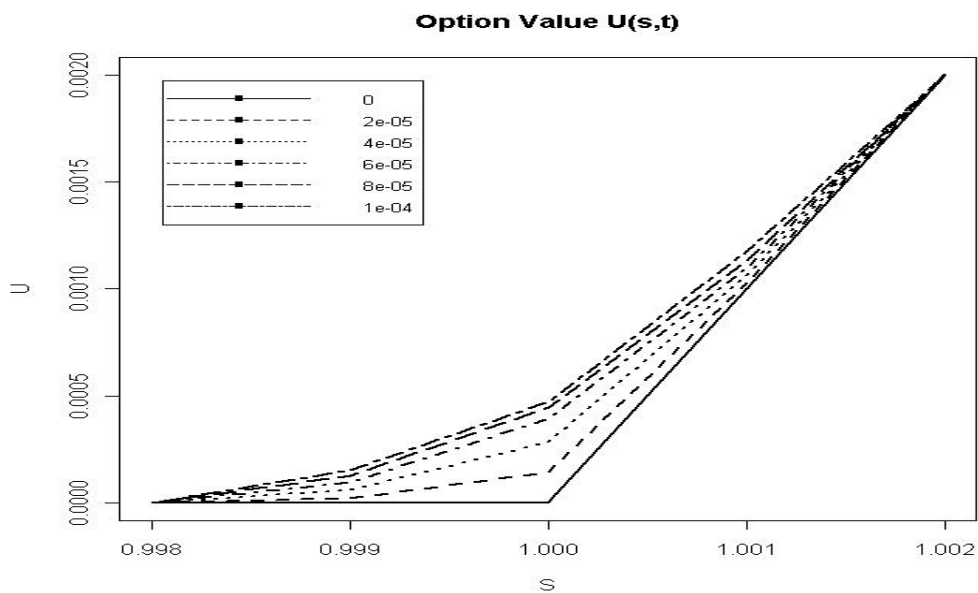
**Figure 1. Option price** $U(s, t)$ **in case 1 (** $\mathcal{D} = 0.001$ **and** $\gamma = 0.5$**).**
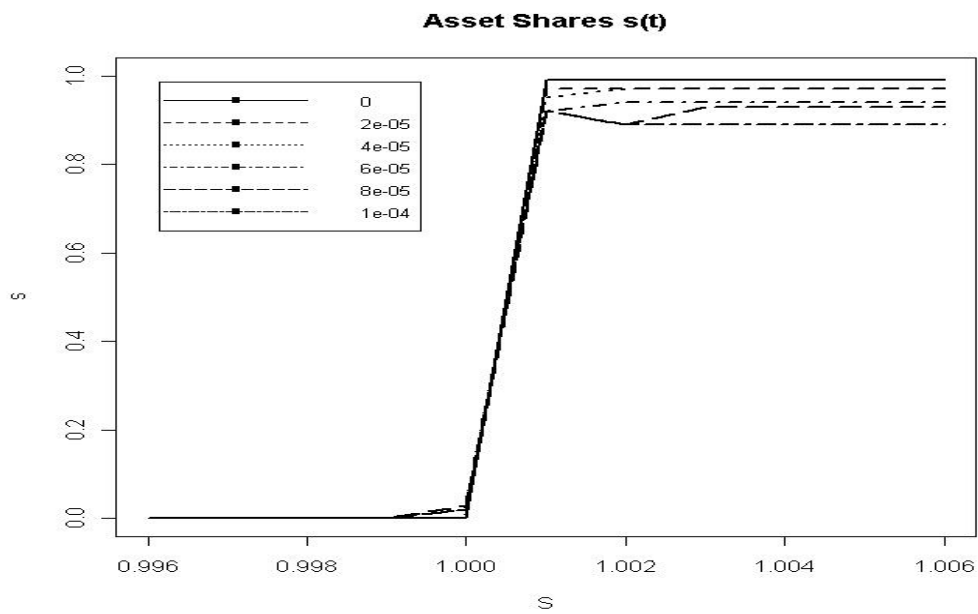


**Figure 2.** **Asset shares** $s(t)$ **in case 1 (** $\mathcal{D} = 0.001$ **and** $\gamma = 0.5$**) .**
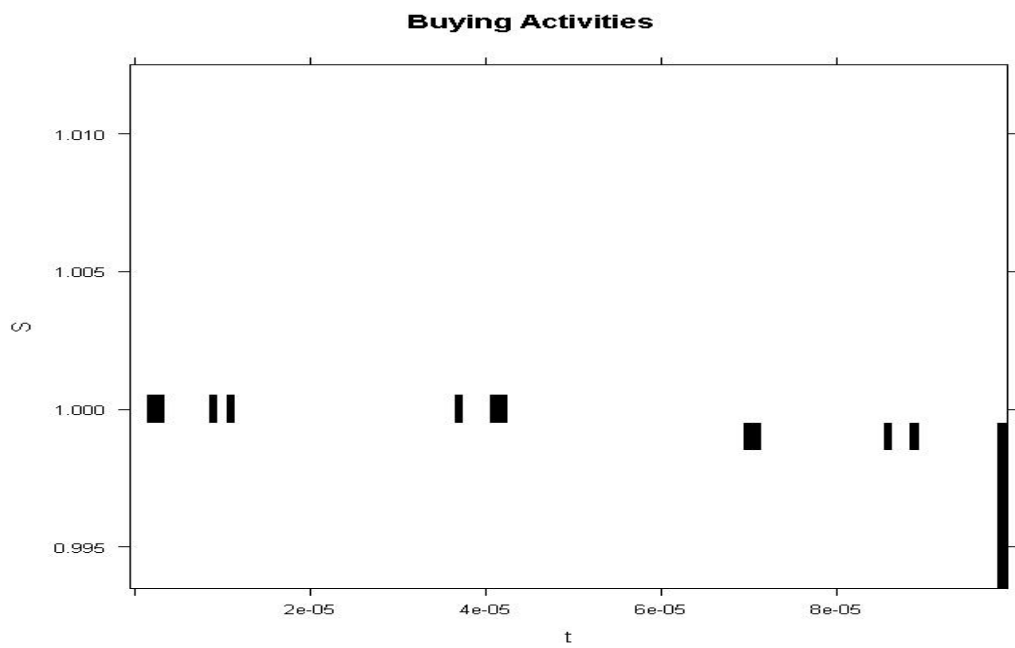
**Figure 3. Buying activities $\varsigma_+(t)$ in case 1 ( $\mathcal{D} = 0.001$ and $\gamma = 0.5$) .**
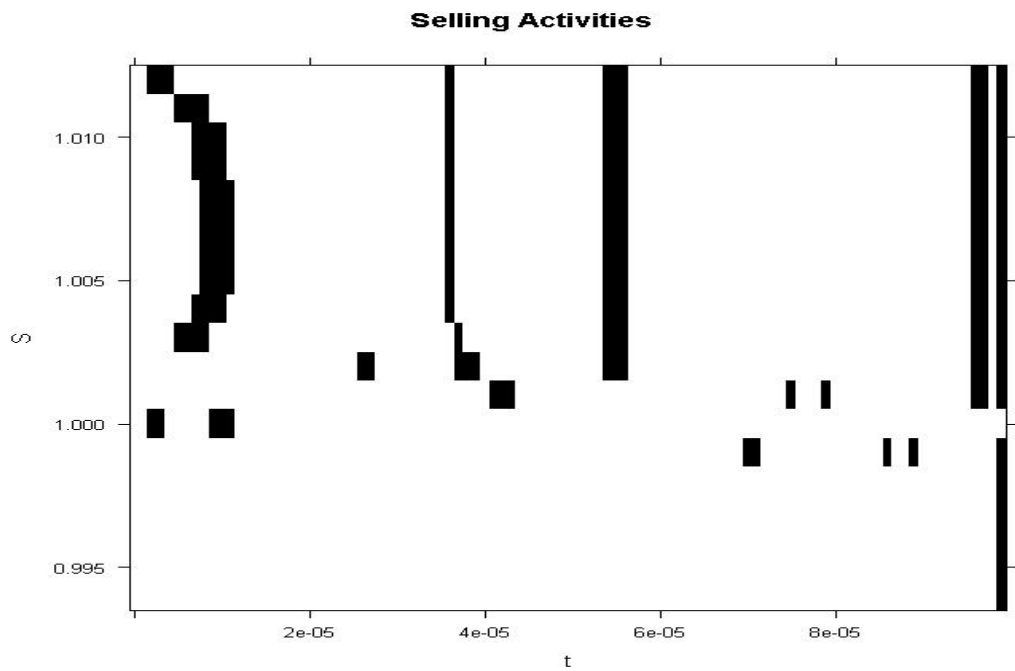


**Figure 4. Selling activities $\varsigma_-(t)$ in case 1 ( $\mathcal{D} = 0.001$ and $\gamma = 0.5$) .**

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## Table 1:  Option Prices relative to Option's Moneyness

(D=0.001 and $\gamma = 0.5$)

| Time until Expiration (in years) | Option Moneyness | Option Price (in dollars) |
|---|---|---|
| 0.0001 | In-the-money | .000999 |
| | At-the-money | .000004 |
| | Out-of-the-money | .000000 |
| 0.00008 | In-the-money | .001000 |
| | At-the-money | .000118 |
| | Out-of-the-money | .000000 |
| 0.00006 | In-the-money | .001019 |
| | At-the-money | .000242 |
| | Out-of-the-money | .000017 |
| 0.00004 | In-the-money | .001041 |
| | At-the-money | .000336 |
| | Out-of-the-money | .000041 |
| 0.00002 | In-the-money | .001068 |
| | At-the-money | .000382 |
| | Out-of-the-money | .000063 |
| 0.00000 | In-the-money | .001105 |
| | At-the-money | .000402 |
| | Out-of-the-money | .000085 |

**Table 2:  Summary Statistics of Option Prices for Case 1**

|  | Mean | Median | SD | Min | Max |
|---|---|---|---|---|---|
| In-the-money | 0.001038 | 0.001030 | 0.000042 | 0.000999 | 0.001105 |
| At-the-money | 0.000247 | 0.000289 | 0.000158 | 0.000004 | 0.000402 |
| Out-of-the-money | 0.000034 | 0.000029 | 0.000035 | 0.000000 | 0.000085 |

Case 2: We maintain a relative deep market while limiting the transaction rate through the immediacy ($D = 0.001$ and $\gamma = 0.005$). Figure 5 through Figure 8 give the graphs on option prices versus stock prices, asset shares, buying and selling activities, respectively. Table 3 and Table 4 report option prices with reference to option's moneyness and summary statistics of option prices for case 2, respectively.

**Option Value U(S,t)**
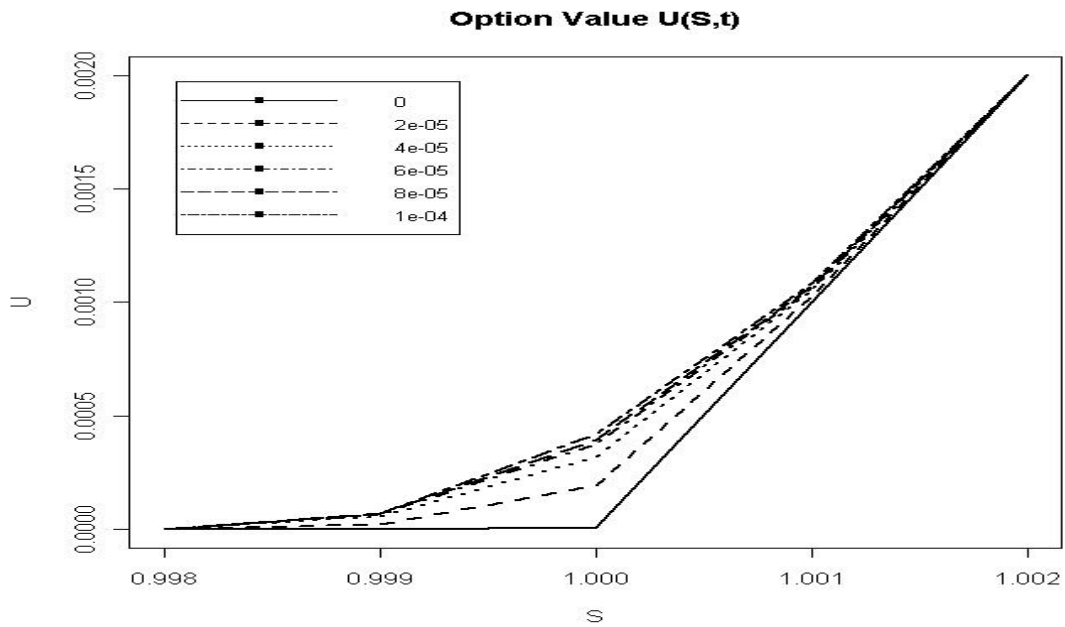


**Figure 5. Option price** $U(s,t)$ **in case 2 (** $\mathcal{D} = 0.001$ **and** $\gamma = 0.005$ **) .**
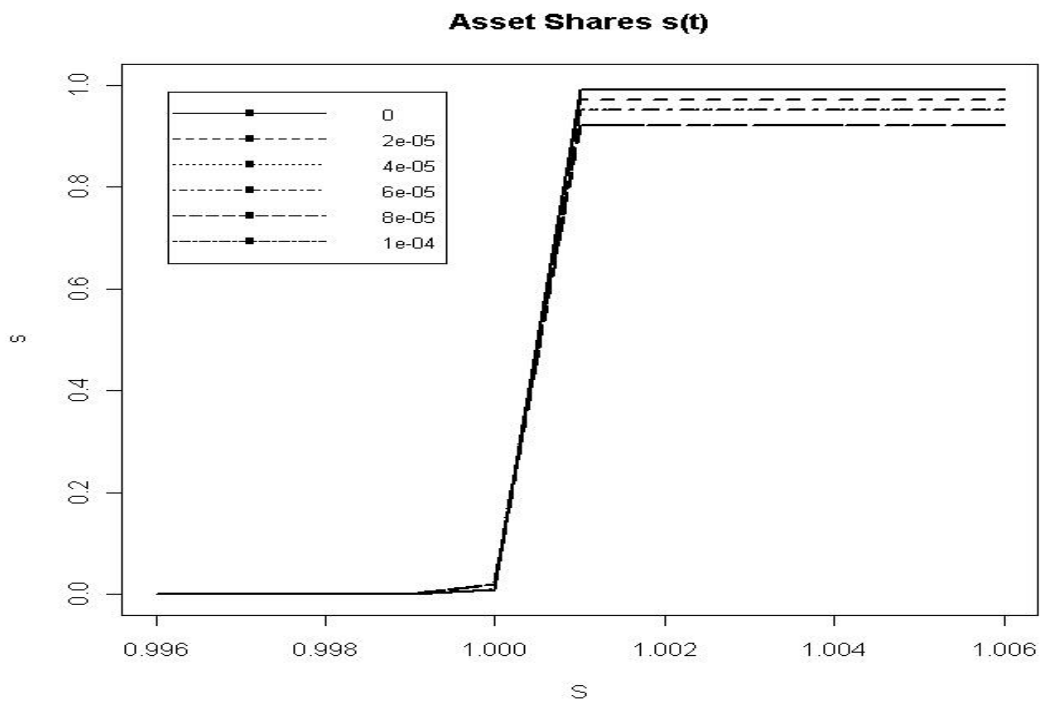
**Asset Shares s(t)**



**Figure 6. Asset shares** $s(t)$ **in case 2 (** $\mathcal{D} = 0.001$ **and** $\gamma = 0.005$ **) .**
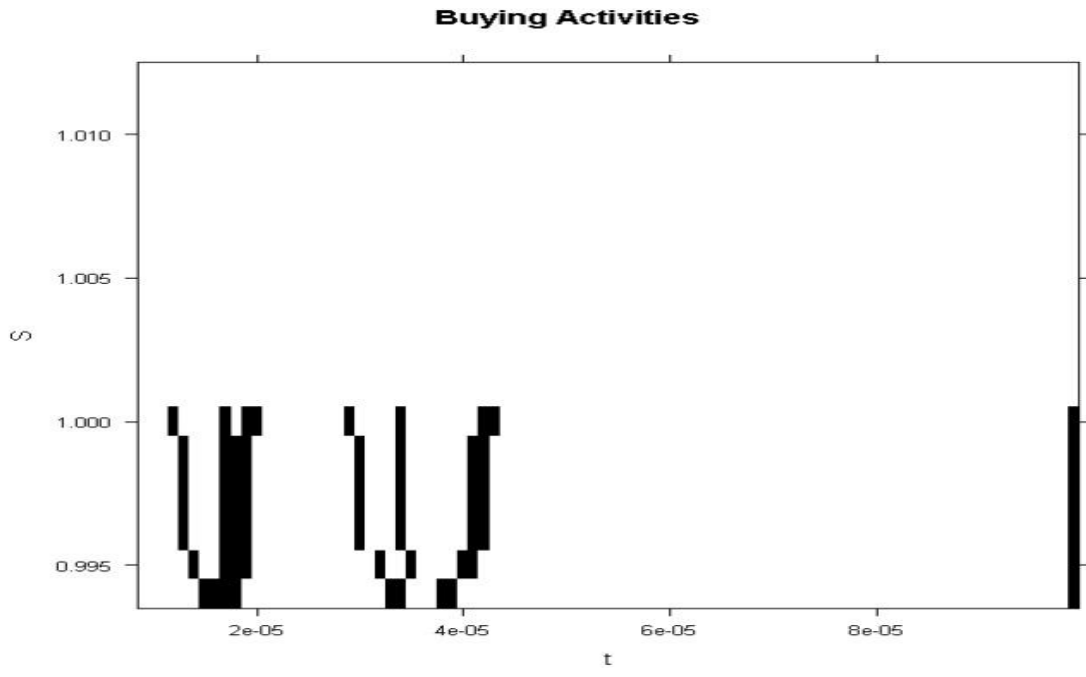
**Figure 7. Buying activities $\varsigma_+(t)$ in case 2 ( $\mathcal{D} = 0.001$ and $\gamma = 0.005$) .**
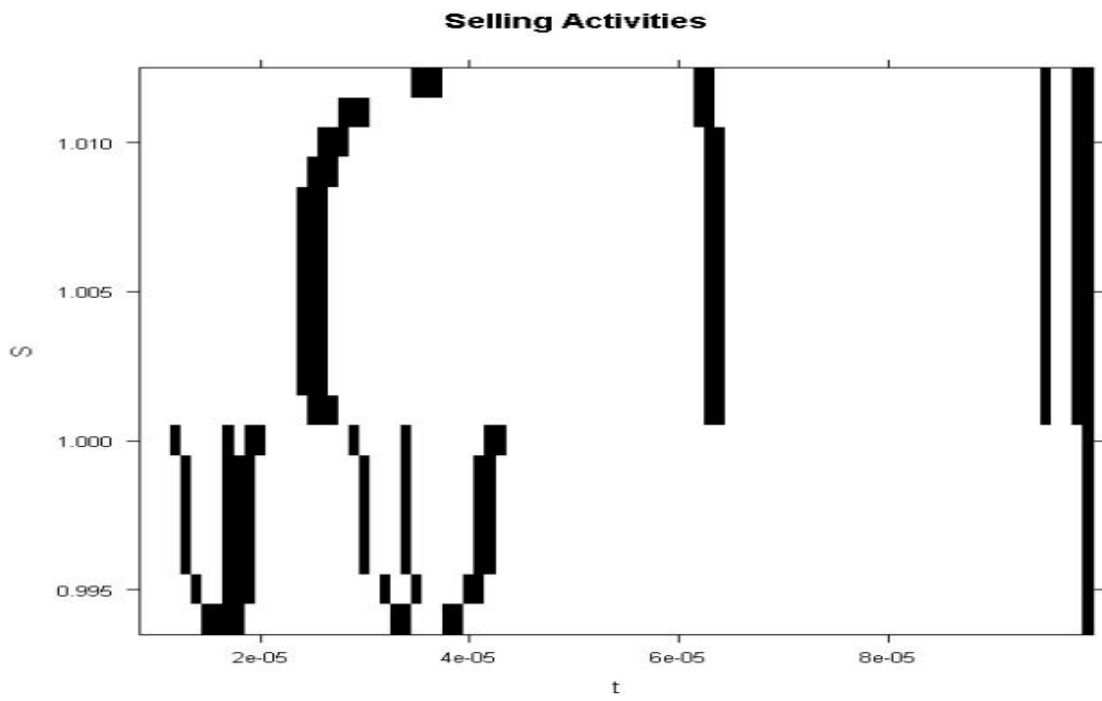


**Figure 8. Selling activities $\varsigma_-(t)$ in case 2 ( $\mathcal{D} = 0.001$ and $\gamma = 0.005$) .**

**Table 3: Option Prices relative to Option's Moneyness**

(D = 0.001 and $\gamma$ =0.005)

| Time until Expiration (in years) | Option Moneyness | Option Price (in dollars) |
|---|---|---|
| 0.0001 | In-the-money | .000100 |
| | At-the-money | .000007 |
| | Out-of-the-money | .000001 |
| 0.00008 | In-the-money | .001041 |
| | At-the-money | .000157 |
| | Out-of-the-money | .000024 |
| 0.00006 | In-the-money | .001105 |
| | At-the-money | .000278 |
| | Out-of-the-money | .000062 |
| 0.00004 | In-the-money | .001174 |
| | At-the-money | .000372 |
| | Out-of-the-money | .000106 |
| 0.00002 | In-the-money | .001240 |
| | At-the-money | .000453 |
| | Out-of-the-money | .000151 |
| 0.00000 | In-the-money | .001274 |
| | At-the-money | .000497 |
| | Out-of-the-money | .000179 |

**Table 4: Summary Statistics of Option Prices for Case 2**

| | Mean | Median | SD | Min | Max |
|---|---|---|---|---|---|
| In-the-money | 0.000989 | 0.001139 | 0.000044 | 0.000100 | 0.001274 |
| At-the-money | 0.000294 | 0.000325 | 0.000186 | 0.000007 | 0.000497 |
| Out-of-the-money | 0.000087 | 0.000084 | 0.000071 | 0.000001 | 0.000179 |

Case 3: We look at the market with limited depth while transaction rate provides little friction ($D = 0.05$ and $\gamma = 0.5$). Figure 9 through Figure 12 give the graphs on option prices versus stock prices, asset shares, buying and selling activities, respectively. Table 5 and Table 6 report option prices with reference to option's moneyness and summary statistics of option prices for case 3, respectively.

**Option Value U(S,t)**
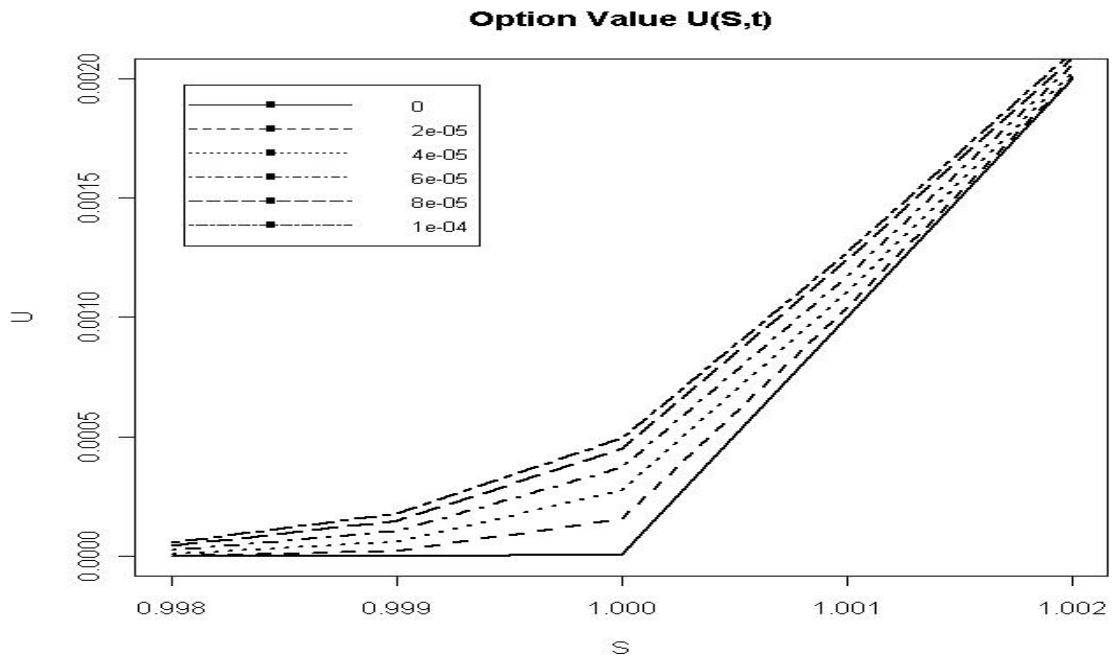
**Figure 9. Option price $U(s, t)$ in case 3 ($\mathcal{D} = 0.05$ and $\gamma = 0.5$).**
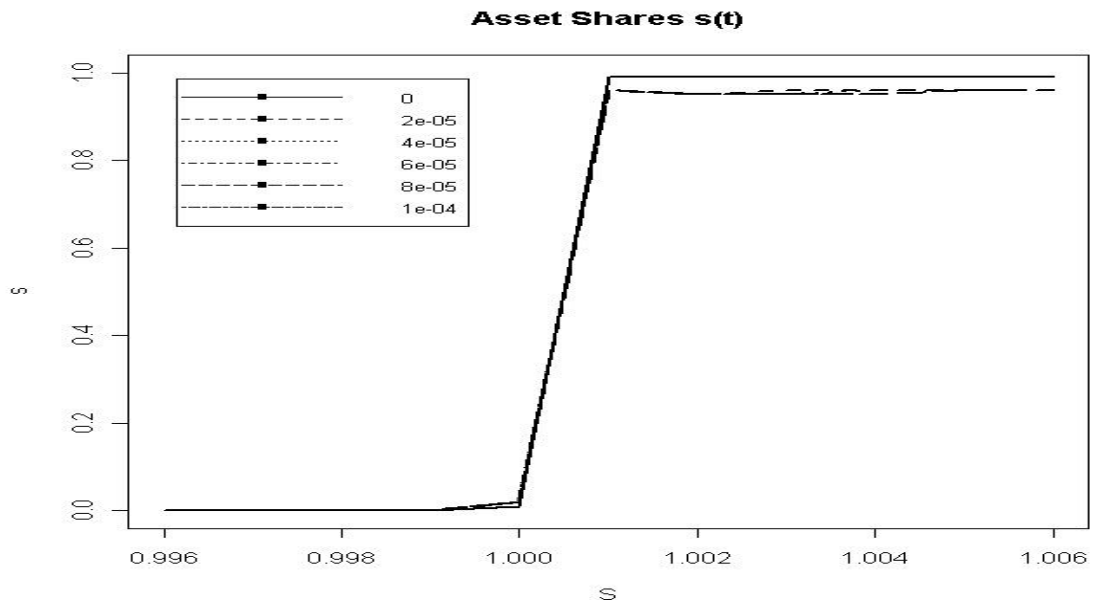
**Asset Shares s(t)**

**Figure 10. Asset shares $s(t)$ in case 3 ($\mathcal{D} = 0.05$ and $\gamma = 0.5$).**
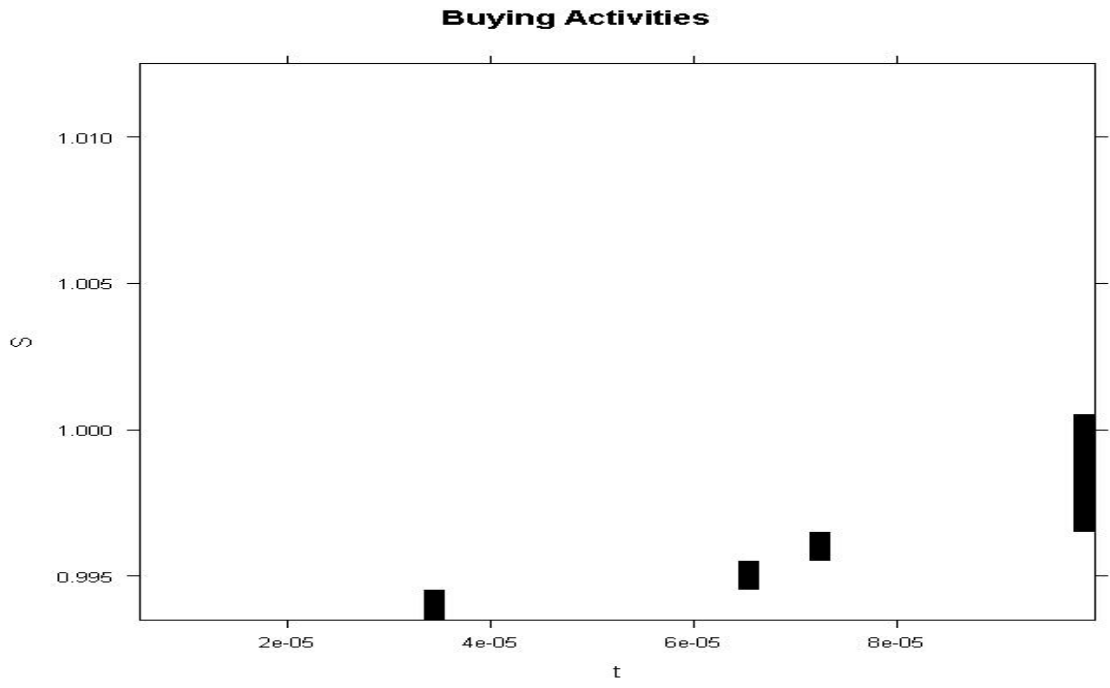
**Buying Activities**



**Figure 11. Buying activities $\varsigma_+(t)$ in case 3( $\mathcal{D} = 0.05$ and $\gamma = 0.5$) .**

**Selling Activities**
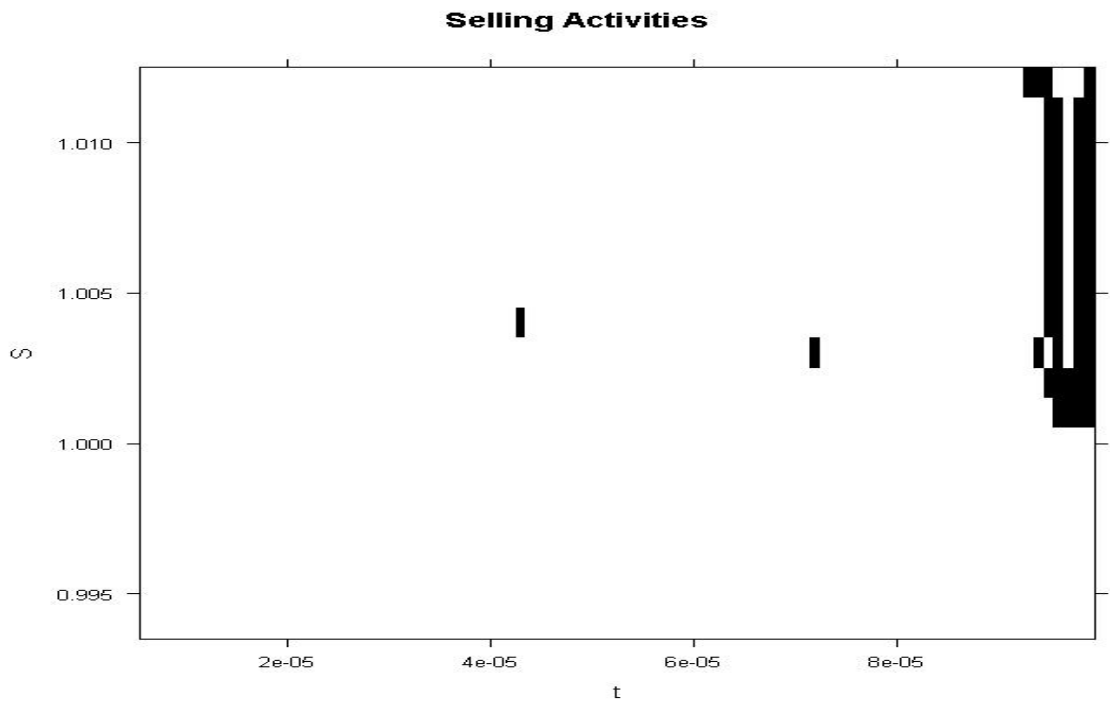


**Figure 12. Selling activities $\varsigma_-(t)$ in case 3 ( $\mathcal{D} = 0.05$ and $\gamma = 0.5$) .**

**Table 5: Option Prices relative to Option's Moneyness**

(D = 0.05 and $\gamma$ = 0.5)

| Time until Expiration (in years) | Option Moneyness | Option Price (in dollars) |
|---|---|---|
| 0.0001 | In-the-money | .000101 |
| | At-the-money | .000011 |
| | Out-of-the-money | .000001 |
| 0.00008 | In-the-money | .001044 |
| | At-the-money | .000160 |
| | Out-of-the-money | .000024 |
| 0.00006 | In-the-money | .001105 |
| | At-the-money | .000279 |
| | Out-of-the-money | .000064 |
| 0.00004 | In-the-money | .001174 |
| | At-the-money | .000377 |
| | Out-of-the-money | .000109 |
| 0.00002 | In-the-money | .001240 |
| | At-the-money | .000461 |
| | Out-of-the-money | .000157 |
| 0.00000 | In-the-money | .001296 |
| | At-the-money | .000517 |
| | Out-of-the-money | .000192 |

**Table 6: Summary Statistics of Option Prices for Case 3**

|  | Mean | Median | SD | Min | Max |
|---|---|---|---|---|---|
| In-the-money | 0.001144 | 0.001139 | 0.000112 | 0.001010 | 0.0001296 |
| At-the-money | 0.000301 | 0.000328 | 0.000191 | 0.000011 | 0.000517 |
| Out-of-the-money | 0.000091 | 0.000087 | 0.000075 | 0.000001 | 0.000192 |

Case 4: We look at the market that is relatively frozen such that there is little depth as well as excessive transaction friction ($D = 0.05$ and $\gamma = 0.005$). Figure 13 through Figure 16 give the graphs on option prices versus stock prices, asset shares, buying and selling activities, respectively. Table 7 and Table 8 report option prices with reference to option's moneyness and summary statistics of option prices for case 4, respectively.

**Option Value U(S,t)**
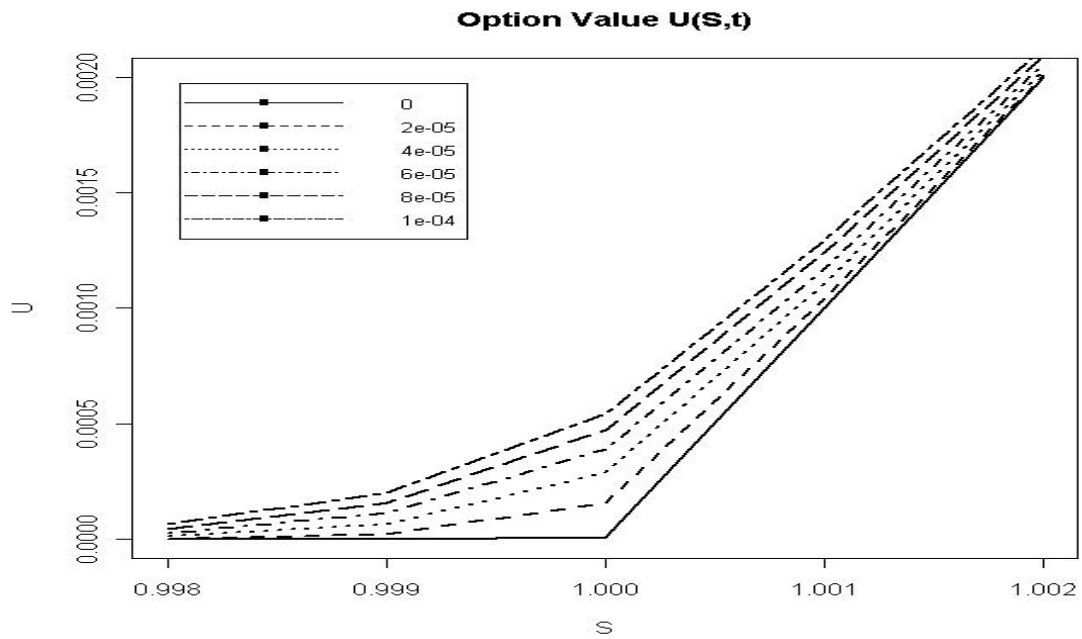


**Figure 13. Option price $U(s, t)$ in case 4 ( $\mathcal{D} = 0.05$ and $\gamma = 0.005$ ) .**
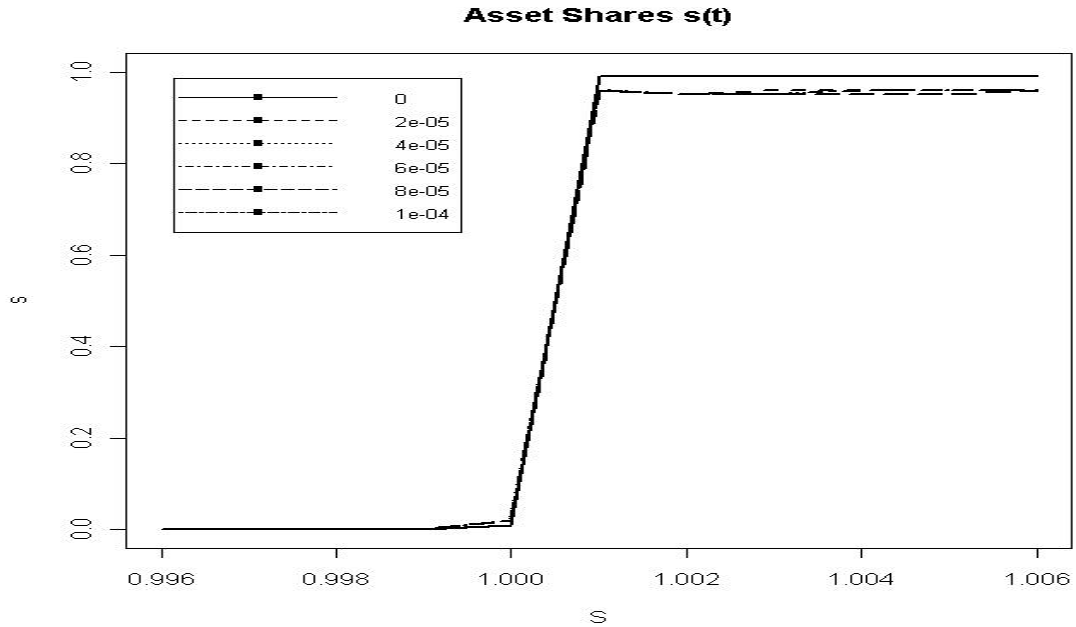
**Asset Shares s(t)**



**Figure 14.  Asset shares $s(t)$ in case 4 ( $\mathcal{D} = 0.05$ and $\gamma = 0.005$ ) .**

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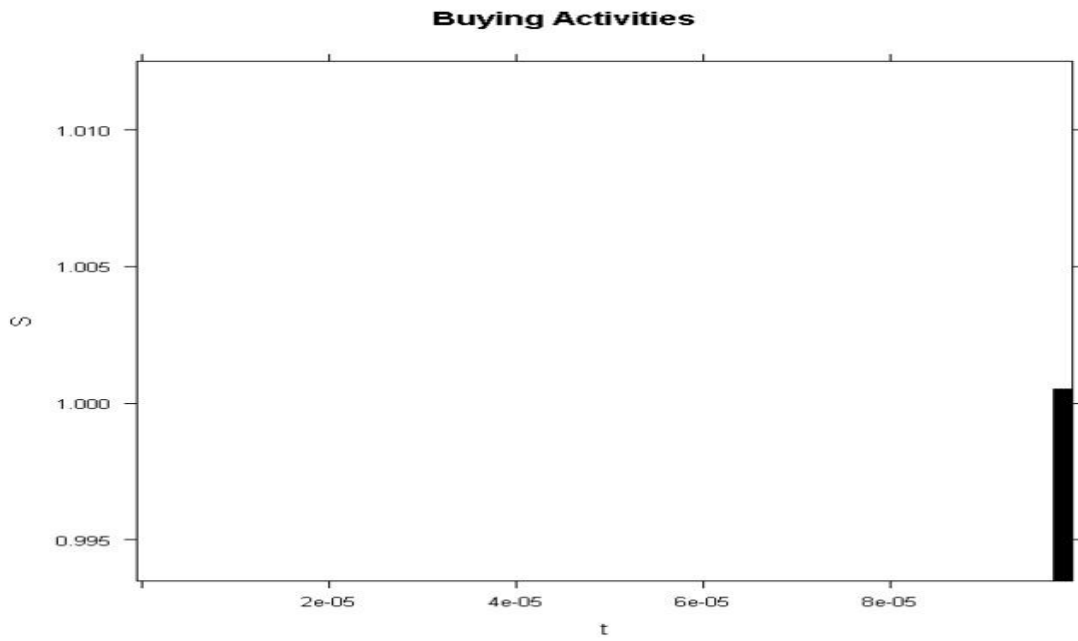
**Figure 15. Buying activities $\varsigma_+(t)$ in case 4 ( $\mathcal{D} = 0.05$ and $\gamma = 0.005$) .**
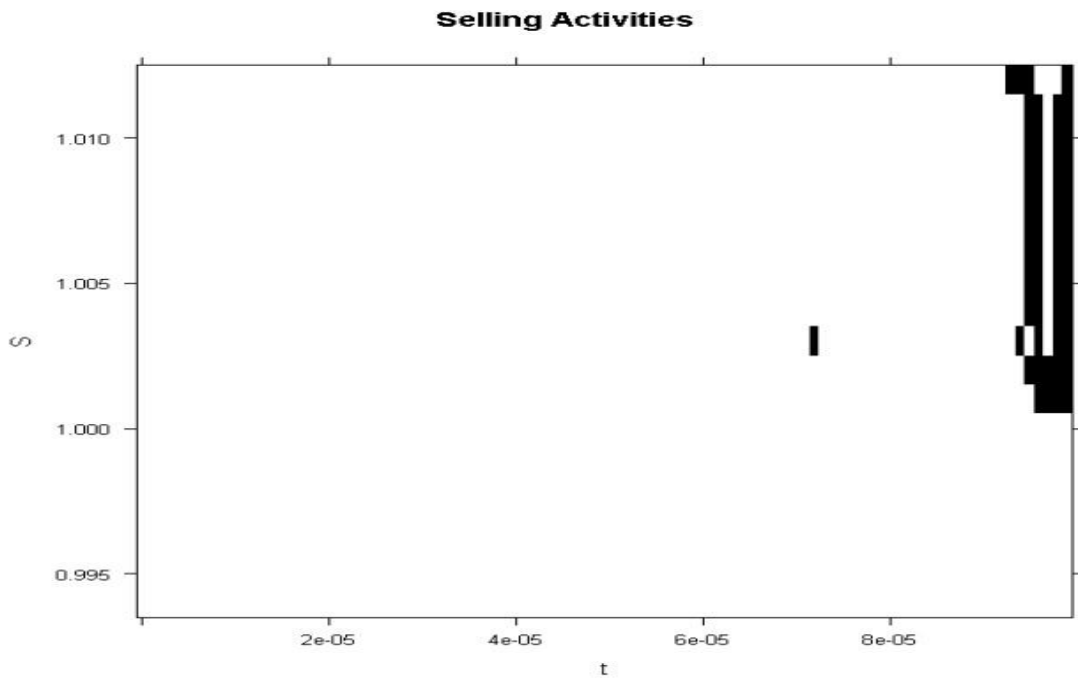


**Figure 16. Selling activities $\varsigma_-(t)$ in case 4 ($\mathcal{D} = 0.05$ and $\gamma = 0.005$) .**

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**Table 7: Option Prices relative to Option's Moneyness**

(D = 0.05 and $\gamma$ = 0.005)

| Time until Expiration (in years) | Option Moneyness | Option Price (in dollars) |
|---|---|---|
| 0.0001 | In-the-money | .000101 |
| | At-the-money | .000011 |
| | Out-of-the-money | .000001 |
| 0.00008 | In-the-money | .001044 |
| | At-the-money | .000160 |
| | Out-of-the-money | .000024 |
| 0.00006 | In-the-money | .001105 |
| | At-the-money | .000291 |
| | Out-of-the-money | .000066 |
| 0.00004 | In-the-money | .001176 |
| | At-the-money | .000395 |
| | Out-of-the-money | .000115 |
| 0.00002 | In-the-money | .001243 |
| | At-the-money | .000476 |
| | Out-of-the-money | .000161 |
| 0.00000 | In-the-money | .001296 |
| | At-the-money | .000541 |
| | Out-of-the-money | .000203 |

**Table 8: Summary Statistics of Option Prices for Case 4**

|  | Mean | Median | SD | Min | Max |
|---|---|---|---|---|---|
| In-the-money | 0.001145 | 0.001140 | 0.000112 | 0.001010 | 0.001296 |
| At-the-money | 0.000312 | 0.000343 | 0.000200 | 0.000011 | 0.000541 |
| Out-of-the-money | 0.000095 | 0.000090 | 0.000070 | 0.000001 | 0.000203 |

The first and the most obvious feature from the graphs is: buying activities are more concentrated in the lower portion of the graphs where the stock prices are low (shown in Figure 3, Figure 7, Figure 11, and Figure 15.); while selling activities are more concentrated in the upper portion of the graphs where the stock prices are high (shown in Figure 4, Figure 8, Figure 12, and Figure 16). This result is intuitive and making good economic sense because buying stocks when the prices are low and selling when prices are high is an effective way to maximize the terminal wealth in the portfolio.

The second feature we see from the graphs is: both buying and selling activities decrease as the market transitions from liquid to less liquid state. We first compare the buying activities in Figure 3, Figure 7, Figure 11, and Figure 15, we see a decreasing

trend. We then compare selling activities in Figure 4, Figure 8, Figure 12, and Figure 16, we also see a decreasing trend. This could be explained as follows. As the market becomes less and less liquid, the liquidity cost of rebalancing the portfolio becomes higher and higher. When the trading gain does not exceed the liquidity cost, our optimal trading strategy simply tells us to stop trading.

Next, an interesting question would be: how does market liquidity affects option price? The graphs and tables above give us some hints, but to see it more clearly, we plot the option prices at the time of contract expiration versus the market depth constants ($\mathcal{D} = 0.001, 0.005, 0.05$) in Figure 17; and option prices at the time of contract expiration versus the immediacy constants ($\gamma = 0.005, 0.05, 0.5$) in Figure 18. We also look at the effects on different option statuses (in-the-money for $S = 1.001$, at-the-money for $S = 1$, and out-of-the-money for $S = 0.999$)

**Option Price VS Market Depth**
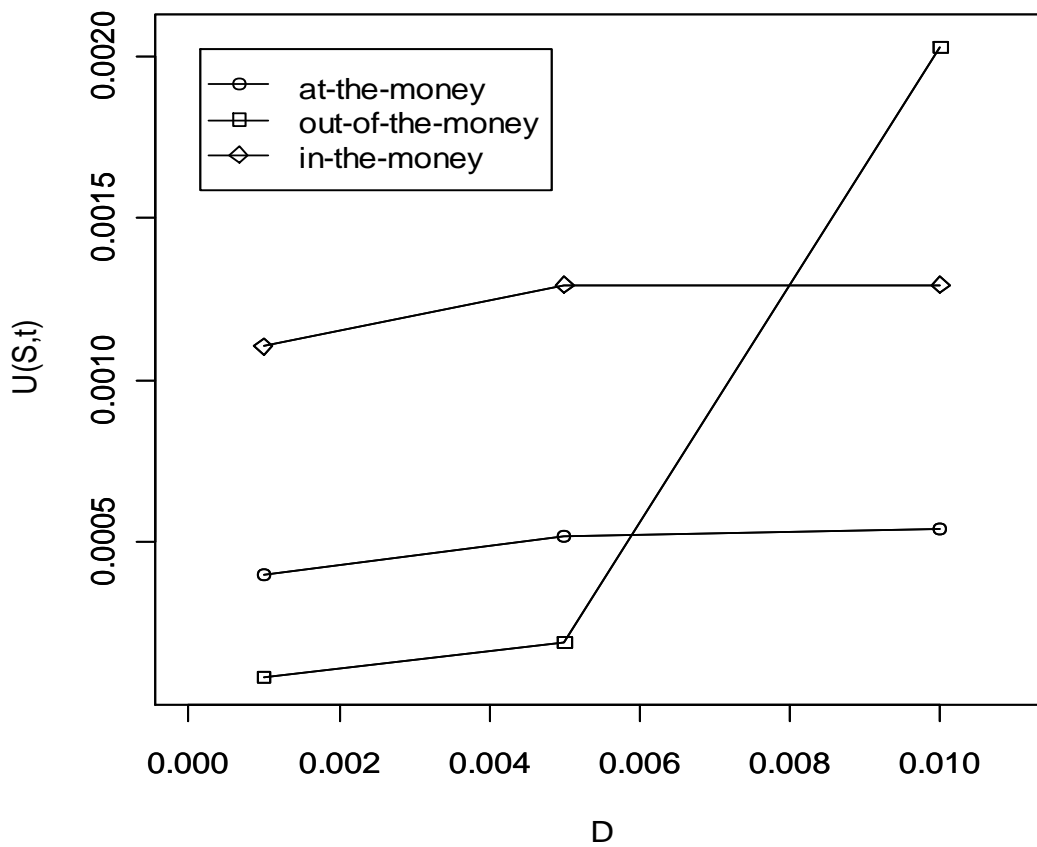


**Figure 17. Option Price $U(s, t)$ in Relation to Market Depth.**

**Option Price VS Immediacy Constant**
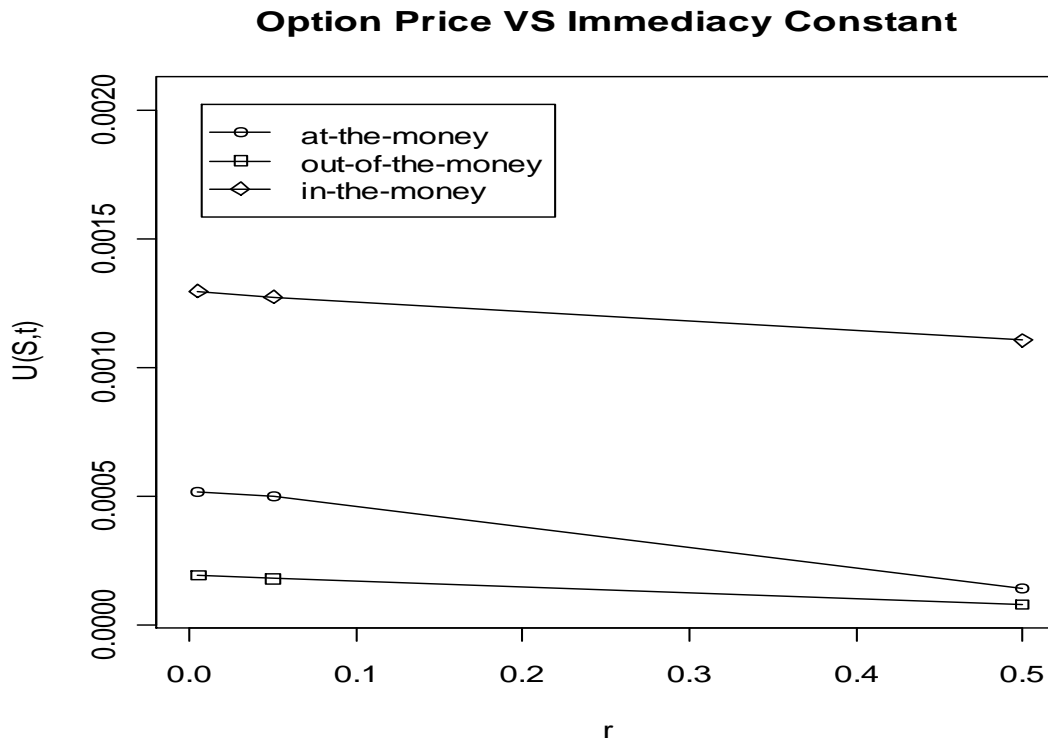


**Figure 18. Option Price $U(s,t)$ in Relation to Immediacy Constant.**

We can clearly see the option prices increase as market depth constants increase (the smaller the depth constant, the greater the asset liquidity). We also discover that the prices for in-the-money option and at-the-money option increase slightly as the market goes from liquid to less liquid, while the price for out-of-the-money option increase significantly. This finding is consistent with the results from Cetin, Jarrow, Protter and Warachka [32]. In their research, they used market depth as the primary measure of liquidity and showed that the option prices increase as the market becomes less liquid. They further showed that out-of-the-money options are subject to the highest percentage price impact among all three options.

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We see less dramatic effects in the relationship between the option prices and immediacy constants. Option prices decrease slightly as immediacy constants increase (the larger the immediacy constant, the greater the asset liquidity). The level of price impact among all three options does not show significant difference.

Another interesting question would be: how does the composition of our hedged portfolio changes as market transitions from liquid to less liquid state? To answer that, we plot the stock holdings in relation to the market depth constants, and the stock holdings in relation to the immediacy constants. In Figure 19, we plot the changes in stock holdings as the market depth constant changes from $\mathcal{D} = 0.001$ to $\mathcal{D} = 0.05$. In Figure 20, we plot the changes in stock holdings as the immediacy constant changes from $\gamma = 0.005$ to $\gamma = 0.5$.

In Figure 19, stock holdings in our optimal hedged portfolio decrease as the market depth constant changes from $\mathcal{D} = 0.001$ to $\mathcal{D} = 0.05$ (liquid to less liquid). Similar effect is seen in Figure 20 that stock holdings in our optimal hedged portfolio increase as the immediacy constant changes from $\gamma = 0.005$ to $\gamma = 0.5$ (less liquid to more liquid).
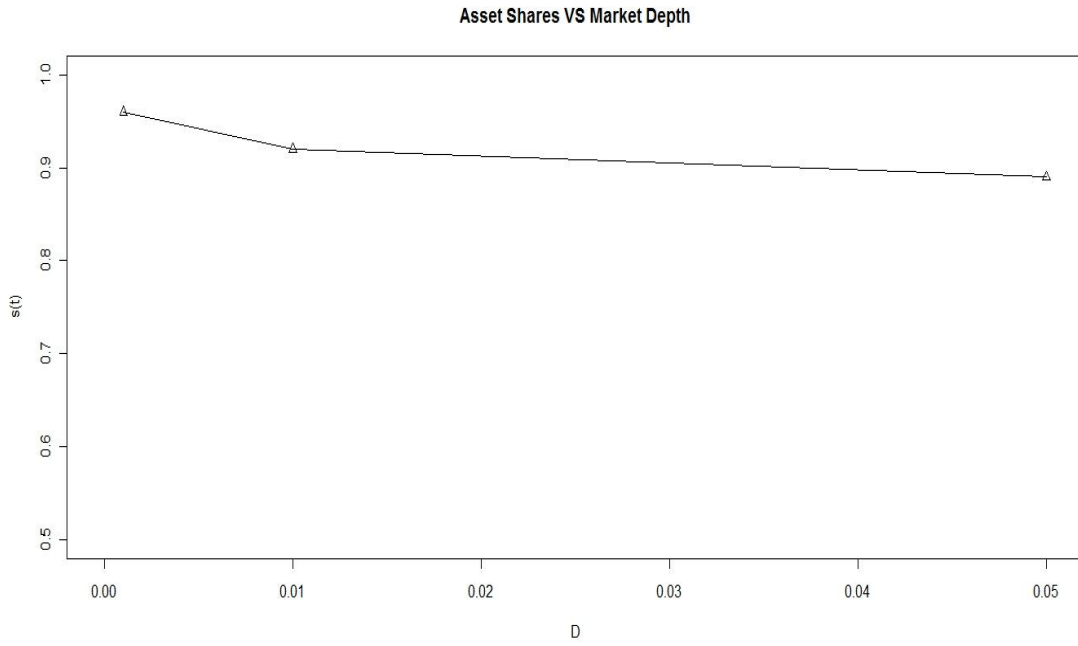
**Figure 19. Asset Shares $s(t)$ in Relation to Market Depth.**
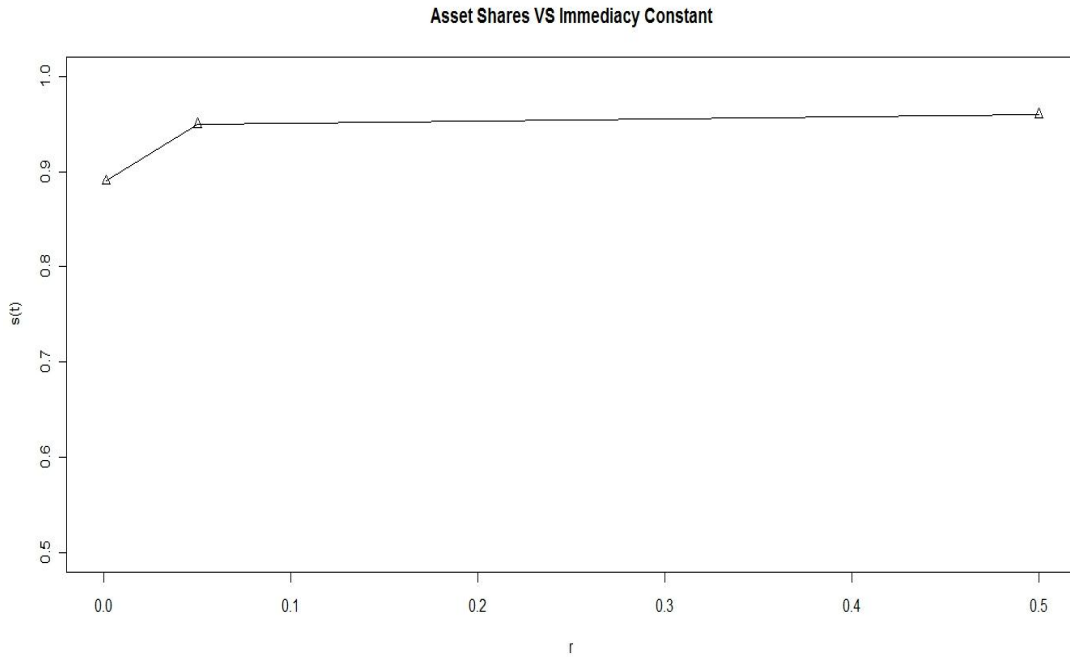


**Figure 20. Asset Shares $s(t)$ in Relation to Immediacy Constants.**

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# CHAPTER 4

# CONCLUSION AND FUTURE RESEARCH

The research in this PhD dissertation extends and generalizes the classical model of option pricing in a market with limited liquidity and offers a new paradigm of pricing options under such conditions. We defined and investigated liquidity from three perspectives: market breadth, depth, and immediacy. We then incorporated liquidity of the underlying asset into the classical Black-Scholes option pricing framework and presented model valuation based on the optimal realization of a performance index relative to the set of all feasible portfolio trajectories.

The main findings of the research are as follows:

- Market liquidity has a significant impact on option prices. Option price increases as the market transitions from liquid to less liquid state; and there is more price impact on the out-of-the-money options than in-the-money or at-the money options.

- Buying and selling activities, based on our optimal trading strategy, decrease as the market becomes less liquid. A reasonable explanation is that the gain from more frequent rebalancing of the portfolio is not able to offset the liquidity risk.

- Buying activities are more concentrated in the region of the graph where the stock price is low, while selling activities are more concentrated in the region of the graph where the stock price is high. This makes good economic sense because "buy low" and "sell high" help us achieve value maximization of our portfolio.

- Stock holdings, in our optimal hedged portfolio, decrease as the market transitions from more liquid to less liquid state. A good explanation is that, in a less liquid market, stock position is riskier than bond position.

For future research, we would like to extend our model to the pricing of derivatives in the energy markets. As evidenced in existing literatures (cf. [33], [66]), energy contracts and derivatives are traded over the counter (OTC) and often lack liquidity. Natural gas market, an evolving commodity market, frequently lacks market liquidity. We would like to incorporate our liquidity model into the commonly used models in the energy field, such as the spread option model. We offer two examples from the real world as motivation.

First, let's review what a spread option is. A spread option derives its value from the difference between the prices of two or more assets (cf. [27], [28]). Spread options can be written on all types securities including equities, bonds, currencies, as well as commodities like natural gas [135]. In some commodity markets, spread options are based on the difference between the prices of the same commodity at two different location (location spread) or at two different points of time (calendar spread) [59]. This type of option can be purchased on large exchanges, but is primarily traded in the over-the-counter market [69], [172].

We consider here a spread European call option with the payoff related to two underlying assets' price $S_1(t)$ and $S_2(t)$. The price processes for $S_1(t)$ and $S_2(t)$ follow geometric Brownian motions

$$dS_1(t) = \mu \, S_1(t)dt + \sigma_1 \, S_1(t)dw_1(t), \qquad (4.1)$$

and

$$dS_2(t) = \mu \, S_2(t)dt + \sigma_2 S_2(t) \, dw_2(t). \qquad (4.2)$$

for drifts $\mu \in R$, volatilities $\sigma_1$, $\sigma_2 > 0$, and two standard Brownian motions $dw_1(t)$ and $dw_2(t)$ with correlation $\rho$ [64]. The payoff at maturity time $T$ of this option with strike price $K$ is the amount

$$\max \ (S_2(T) - S_1(T) - K, 0).$$

The price $C$ of the option is given by: (cf. [2], [4], [28])

$$C = e^{\mu T} \mathrm{E}[(S_2(T) - S_1(T) - K)^+].$$

or

$$C = e^{\mu T} \mathrm{E}\left[\left(S_2(0)e^{(\mu - \sigma_2^2/2)T + \sigma_2 W_2(T)} - S_1(0)e^{(\mu - \sigma_1^2/2)T + \sigma_1 W_1(T)} - K\right)^+\right]. \qquad (4.3)$$

We can see that the price $C$ is given by the integral of a function of two variables with respect to a bivariate distribution [94]. Unfortunately, the price of the spread option cannot be given by a closed form formula (cf. [3], [29], [54]). An efficient numerical method is needed to price such option.

Example1. Suppose the natural gas price in January 2011 at location A is \$4.41 per MMBTU, the natural gas price in May 2011 at location B is \$4.97 per MMBTU, and the combination of injection and withdrawal cost is \$0.03 per MMBTU. We can use a natural gas storage facility to buy and inject gas when/where the price is low, i.e. January 2011 at location A, and sell gas when/where the price is high, i.e. May 2011 at location B. By

doing so, we can capture the price difference around $0.53 per MMBTU. If we hold a spread option of the price difference of January at location A and May at location B, and think of the strike price K being the combination of injection and withdrawal cost, then this spread option has payoff max $\left(S_{B,May} - S_{A,Jan} - K, 0\right)$ which is the same as $0.53 per MMBTU. This example uses locational/time spread and illustrates that we can consider the value of gas storage as a spread option. Existing literatures that have considered spread option pricing method in the valuation of gas storage includes [33], [99], [127]. We further suppose that the natural gas market has limited liquidity, we can perturb (4.1) and (4.2) with our liquidity model discussed in chapter 2, however finding an efficient numerical method to solve the model poses a challenge.

Example 2. A special type of spread option is called the spark spread option or the heat rate option. The spark spread, is defined as the difference between the electricity price and the cost of generation (cf. [55], [57], [59]). For a public utility company who owns a gas-fired power plant, the amount of natural gas that is needed to generate a given amount of electricity depends on the plant's efficiency (cf. [58], [60]) or the heat rate. Heat rate is defined as the number of British thermal units (BTU) of the natural gas required to generate 1 megawatt hour (MWh) of electricity [57]. The lower the heat rate, the more efficient the power plant. The spark spread associated with a particular heat rate is the current price of electricity minus the product of heat rate and current gas price (cf. [57], [61]), i.e.

$$\text{spark spread} = S_E(t) - H * S_G(t) \tag{4.4}$$

where $S_E(t)$ is the electricity price, $H$ is the heat rate, and $S_G(t)$ is the natural gas price.

The owner of a public utility company will dispatch a particular gas plant only if the electricity price is higher than the cost of generation for that plant. Then revenue received by the power plant can be considered as a heat rate call option. Further, the value of the plant, from a financial point of view, can be regarded as a heat rate call option as well.

Suppose the price processes for natural gas $S_G(t)$ and electricity $S_E(t)$ follow geometric Brownian motions

$$dS_G(t) = \mu_G S_G(t)dt + \sigma_G S_G(t)dw_G(t), \tag{4.5}$$

and

$$dS_E(t) = \mu_E S_E(t)dt + \sigma_E S_E(t) dw_E(t) \tag{4.6}$$

for drifts $\mu_G$ and $\mu_E \in R$, volatilities $\sigma_G$ and $\sigma_E > 0$, and two standard Brownian motions $dw_G(t)$ and $dw_E(t)$ with correlation $\rho$. The payoff at maturity time $T$ of this option with strike price $K$ is the amount

$$\max \ (S_E(T) - H * S_G(T) \ - K, 0) \, ,$$

where $H$ is the heat rate.

We further suppose that one of the assets, namely natural gas, has limited market liquidity; and the other asset, electricity has perfect market liquidity, we can perturb (4.5) with the framework we discussed in chapter 2. Again finding a feasible and efficient method to solve the model awaits future research.

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    and proportional transaction costs, *Journal of Economic Dynamics and Control*,

    30, pp. 1-25.

VITA

Graduate College

University of Nevada, Las Vegas

Yanan Jiang

Degrees:

Bachelor of Science in Mathematics, 2003

University of Science and Arts of Oklahoma, Chickasha

Publications:

*Asset Liquidity and Valuation of Derivative Securities*, J. of Computational and Applied Mathematics (accepted subject to minor revision), coauthor.

*Investment Style Analysis of Open-end Funds in China: Return-based Quantile Regression Analysis,* 2011 International Conference on E-Business and E-Government (ICEE), May 6-8, 2011, pages 1-5, coauthor.

*Barebones Particle Swarm for Multi-objective Optimization Problem*, International Journal of Innovative Computing and Applications, Volume 2, Issue 2, February 2009, pages 86-99, coauthor.

Dissertation Title:

Valuation of Financial Derivatives Subject to Liquidity Risk

Dissertation Examination Committee:

    Co-Chairperson, Dr. Michael Marcozzi, Ph.D.

    Co-Chairperson, Dr. Chih-Hsiang Ho, Ph.D.

    Committee Member, Dr. Hongtao Yang, Ph.D.

    Graduate Faculty Representative, Dr. Seungmook Choi, Ph. D.