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Exact Controllability of the Lazer-McKenna Suspension Bridge Equation

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EXACT CONTROLLABILITY OF THE LAZER-MCKENNA SUSPENSION BRIDGE EQUATION

by

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ABSTRACT

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by

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It is well known that suspension bridges may display certain oscillations under external aerodynamic forces. Since the collapse of the Tacoma Narrows suspension bridge in 1940, suspension bridge models have been studied by many researchers. Based upon the fundamental nonlinearity in suspension bridges that the stays connecting the supporting cables and the roadbed resist expansion, but do not resist compression, new models describing oscillations in suspension bridges have been developed by Lazer and McKenna [Lazer and McKenna (1990)]. Except for a paper by Leiva [Leiva (2005)], there have been very few work on controls of the Lazer-McKenna suspension bridge models in the existing literature. In this dissertation, I use the Hilbert Uniqueness Method and the Leray-Schauder's degree theory to study two exact controllability problems of the Lazer-McKenna suspension bridge equation.

The first problem is to study the exact controllability of the single Lazer-McKenna suspension bridge equation with a locally distributed control. Unlike most of the existing literatures on exact controllability of nonlinear systems where the nonlinearity was always assumed to be C^1-smooth, the nonlinearity in the Lazer-McKenna suspension bridge equation is not C^1-smooth, which makes the exact controllability problem challenging to study. It is proved that the control system is exactly controllable. The key step is to establish an observability inequality of the auxiliary linear control problem. The proof of such an inequality relies on deriving a Carleman estimate.

The second problem studied in this dissertation is the exact controllability problem of the single Lazer-McKenna suspension bridge equation with a piezoelectric bending actuator. It is proved that the control system is exactly controllable when the location of the actuator is carefully chosen. The proof of exact controllability is based upon establishing an Ingham inequality for nonharmonic Fourier series.

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CHAPTER 1

INTRODUCTION

Suspension bridge is a common type of civil engineering structures. The history of suspension bridges can be dated back to the eighth century when the ancient Chinese constructed suspension bridges by laying planks between pairs of iron chains, essentially providing a flexible deck resting on cables. Similar bridges were built in various parts of the world during subsequent centuries. The modern era of suspension bridges did not begin until 1808 when an American engineer named James Finley patented a system for suspending a rigid deck from a bridge's cables. A suspension bridge consists of a horizontal roadbed; two pylons; two main cables suspended between two pylons; and vertical stays connecting the roadbed and main cables. Since any load applied to the roadbed is transformed into tensions in main cables, both main cables must be anchored at each end of the suspension bridge. A configuration of suspension bridge is given in Figure 1.1.

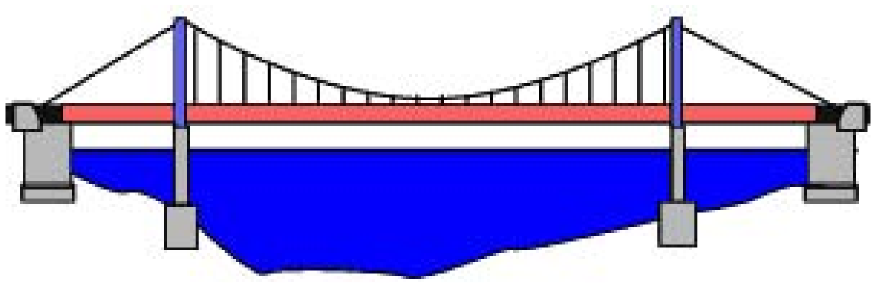

Figure 1.1. Suspension bridge configuration

Suspension bridges usually have longer center spans and lighter weights than other types of bridges. They have better performances in withstanding earthquake movements than those heavier and more rigid bridges. Moreover, suspension bridges have less construction costs than other bridges. Except for installation of initial temporary cables, the waterway can remain open while a suspension bridge is built above. The recent developments of high-strength cables and girders have led the construction of suspension bridges into a new era. According to the Wikipedia [Wikipedia (a)], among the ten world's longest suspension bridges, nine of them were built in last two decades. The Akashi Kaikyo suspension bridge in Japan (Figure 1.2), completed in 1998, has the longest center span (6532 ft) among all existing suspension bridges in the world.

Figure 1.2. The Akashi Kaikyo suspension bridge [Wikipedia (a)]

Since suspension bridges are relatively light and flexible, they are all susceptible to external aerodynamic loads such as wind. The increase in center-span length

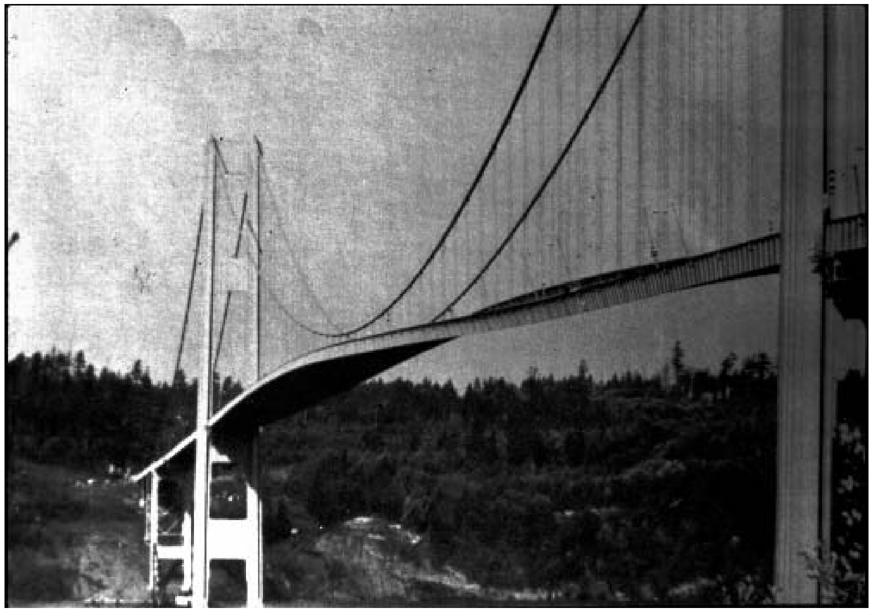

Figure 1.3. Swaying and buckling of the first Tacoma Narrows suspension bridge in relatively mild windy condition [Wikipedia (b)]

and pylon height raises many concerns about the dynamic behavior of long-span suspension bridges. It is well known that a long-span suspension bridge may display certain dangerous oscillations under extreme dynamic loads, and the large amplitude oscillations could cause fatigue or failure of this type of bridges. The collapse of the first Tacoma Narrows suspension bridge is one of the most striking examples. The first Tacoma Narrows suspension bridge had a center span of 2800 ft, and was opened on July 1, 1940. Shortly after the construction was finished, it was discovered that the bridge would sway and buckle dangerously in relatively mild windy conditions in the area (Figure 1.3). It collapsed on November 7, 1940, due to a wind blowing at a speed of 42 mph (Figure 1.4).

The report by the Federal Works Agency [Amann et al. (1941)] determined that the collapse of the first Tacoma Narrows suspension bridge was due to a never-before-

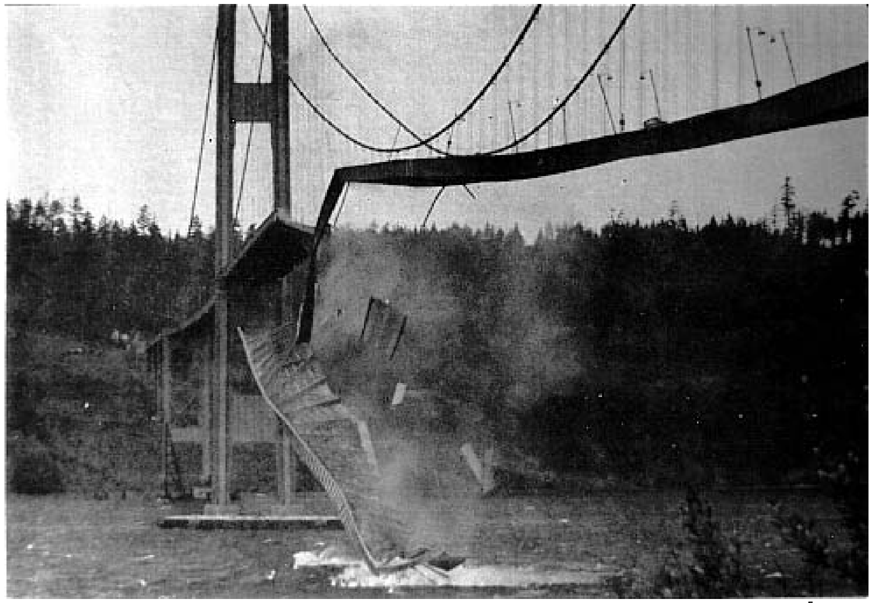

Figure 1.4. Collapse of the first Tacoma Narrows suspension bridge [Wikipedia (b)]

seen twisting mode: coupled torsional and longitudinal oscillations. However, the report was inconclusive to the precise causes of the bridge failure. Nevertheless, the report has created a widespread demand for a comprehensive investigation of dynamic oscillation problems in suspension bridges in order to understand the causes of such destructive oscillations, and to develop design techniques to prevent their recurrence in future. A systematic study of the mathematical theory of suspension bridges appears to be initiated by Bleich, McCullough, Rosecrans and Vincent [Bleich et al. (1950)] in 1950. From then on, extensive studies of dynamics of suspension bridges were carried out [Abdel-Ghaffer (1982); Pittel and Jakubovic (1969); Scanlan (1978b,a); Selberg (1961); Wiles (1960)]. In the early studies of the failure of the Tacoma Narrows suspension bridge, linear suspension bridge models were derived and analyzed, and the causes of the bridge failure was attributed to resonance. However, this explanation contradicts to the findings reported by the Federal Works Agency.

The following paragraph is quoted from the FWA report [Amann et al. (1941)] on the collapse of the first Tacoma Narrows suspension bridge.

> *It is very improbable that resonance with alternating vortices plays an important role in the oscillations of suspension bridges. First, it was found that there is no sharp correlation between wind velocity and oscillation frequency, as is required in the case of resonance with vortices whose frequency depends on the wind velocity.... It seems that it is more correct to say that the vortex formation and frequency is determined by the oscillation of the structure than that the oscillatory motion is induced by the vortex formation.*

Based upon the observation of the fundamental nonlinearity in suspension bridges that the stays connecting the supporting cables and the roadbed resist expansion, but do not resist compression, new models describing oscillations in suspension bridges have been developed by Lazer and McKenna in 1990 [Lazer and McKenna (1990)]. The new models are described by systems of coupled nonlinear partial differential equations.

Consider a simplified suspension bridge configuration which consists of a main cable, a horizontal roadbed and stays connecting the roadbed to the main cable, see Figure 1.5. The pylons and side-spans are not considered. The main cable is modeled by the vibrating string with both ends being fixed. The horizontal roadbed is modeled by the vibrating beam with both ends being simply supported. The stays are modeled by the one-sided springs which resist expansion but do not resist compression. Let

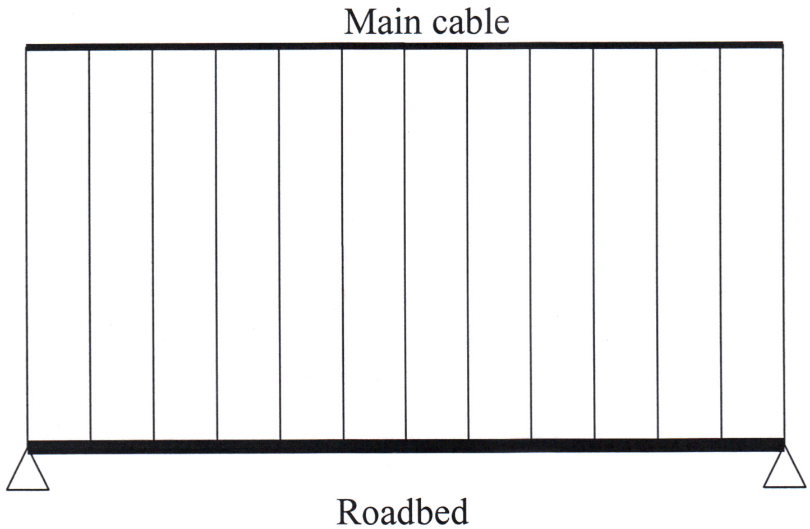

Figure 1.5. A simplified suspension bridge configuration

$u(x,t)$ denote the downward deflection of the main cable, and $w(x,t)$ denote the downward deflection of the roadbed. The following suspension bridge model was proposed by Lazer and McKenna [Lazer and McKenna (1990)] in 1990,

$$\begin{cases} m_c u_{tt} - Q u_{xx} - K(w-u)^+ = m_c g + f_1(x,t), \\ m_b w_{tt} + EI w_{xxxx} + K(w-u)^+ = m_b g + f_2(x,t) \\ u(0,t) = u(L,t) = 0, \\ w(0,t) = w(L,t) = 0, \\ w_{xx}(0,t) = w_{xx}(L,t) = 0, \end{cases} \tag{1.1}$$

where $(w-u)^+ = \max\{w-u,\ 0\}$; L is the roadbed of length L; m_c and m_b are the mass densities of the cable and the roadbed, respectively; Q is the tensile strength constant of the cable; EI is the roadbed flexural rigidity; K is the Hooke's constant of the stays; f_1 and f_2 represent the external periodic aerodynamic forces.

Assume further that the main cable is immovable, one then obtains the well-known Lazer-McKenna suspension bridge equation [Lazer and McKenna (1990)],

$$\begin{cases} m_b w_{tt} + EI w_{xxxx} + K w^+ = m_b g + f(x,t), \\ w(0,t) = w(L,t) = 0, \\ w_{xx}(0,t) = w_{xx}(L,t) = 0. \end{cases} \tag{1.2}$$

The Lazer-McKenna suspension bridge systems (1.1) and (1.2) were obtained by ignoring the torsional oscillation of the roadbed due to its relatively small scale of amplitude. Based upon the Lazer-McKenna suspension bridge models, Ahmed [Ahmed (2004)] proposed in 2004 the following general suspension bridge model describing both torsional and longitudinal oscillations by considering a simplified suspension bridge configuration which consists of two parallel main cables, a horizontal roadbed and stays connecting both sides of the roadbed to the main cables. Let $u(x,t)$ and $v(x,t)$ denote the downward deflections of two main cables, $w(x,t)$ denote the downward deflection of the roadbed, $\theta(x,t)$ denote the angular deflection of the roadbed from the horizontal plane about the centerline of the roadbed.

$$\begin{cases} m_c u_{tt} - Q u_{xx} - F_1(u,w,\theta) = m_c g + f_1(x,t), \\ m_c v_{tt} - Q v_{xx} - F_2(v,w,\theta) = m_c g + f_2(x,t), \\ m_b w_{tt} + EI w_{xxxx} + F_1(u,w,\theta) + F_2(v,w,\theta) = m_b g + f_3(x,t), \\ \mathcal{I}\theta_{tt} + \alpha\theta_{xx} + \ell\cos\theta[F_1(u,w,\theta) - F_2(v,w,\theta)] = f_4(x,t), \end{cases} \tag{1.3}$$

where

$$\begin{cases} F_1(u,v,w,\theta) = K(w + \ell\sin\theta - u)^+, \\ F_2(u,v,w,\theta) = K(w - \ell\sin\theta - v)^+, \end{cases} \tag{1.4}$$

and $\xi^+ = \max\{\xi,\ 0\}$ for $\xi \in \mathbb{R}$. The boundary conditions are given as

$$\begin{cases} u(0,t) = u(L,t) = 0, \quad v(0,t) = v(L,t) = 0, \\ w(0,t) = w(L,t) = 0, \quad w_{xx}(0,t) = w_{xx}(L,t) = 0, \\ \theta(0,t) = \theta(L,t) = 0. \end{cases} \tag{1.5}$$

In this general suspension bridge model, 2ℓ and L are the width and length of the roadbed, respectively, $\mathcal{I} = 2m\ell^2$ is the moment of inertia about the roadbed centerline, $\alpha = 2\ell^2 EI$, $F_1(u,v,w,\theta)$ and $F_2(u,v,w,\theta)$ are the restoring forces of the stays, and f_i, $1 \le i \le 4$, represent nonconservative aerodynamic forces.

The Lazer-McKenna suspension bridge equation (1.2) has been studied extensively in the literature. By using the variational methods and the critical point theory, multiple large amplitude periodic oscillations have been found theoretically and numerically by Lazer, McKenna and their collaborators (see [Choi et al. (1993a, 1991, 1993b); Glover et al. (1989); Humphreys (1997); Humphreys and McKenna (1999); Lazer and McKenna (1990, 1987)] and references therein). By using the variational methods, the existence of large amplitude periodic oscillations in the Lazer-McKenna suspension bridge system (1.1) has been studied by Ding [Ding (2001, 2002b,a,c, 2003)]. However, the general suspension bridge model (1.3)-(1.5) for torsional and longitudinal oscillations has not been studied in depth yet in the existing literature.

It has been known that a narrow and very flexible suspension bridge has a low degree of internal damping, which makes it susceptible to detrimental jitter from vibrations and flutters. Active controls are required to suppress oscillations quickly in bridge structures. A classical method to study the exact controllability of a linear control system is the Hilbert Uniqueness Method (HUM) introduced by Lions [Lions (1988)]. A key step of this method is to establish a related observability inequality. When adopting this method for an semilinear control system, an auxiliary linear system needs to be introduced to prove the observability inequality. One then applies the Leray-Schauder degree theory to derive the exact controllability of the nonlinear control system. The controllability for linear and semilinear wave equations has been studied extensively by many researchers (see [Zuazua (1993); Li and Z (2000); Zhang (2000)] and references therein). In last two decades, there are many literatures discussing the exact controllability of the linear plate equations (see [Tucsnak (1996); Crépeau and Prieu (2001); Zhang (2001); Fu (2012)], etc). The recent progresses on the study of controllability/observability of linear and semilinear wave equations can be found in a review paper by Hongheng Li, Qi Lü and Xu Zhang [LI et al. (2010)].

However except a paper by Ahmed and Harbi [Ahmed and Harbi (1998)] (1998) on the stabilization of the Lazer-McKenna suspension bridge equation (1.2) with a linear internal damping and a paper by Leiva [Leiva (2005)] on exact controllability of the Lazer-McKenna suspension bridge equation (1.2) with a distributed control over the whole roadbed, there has been very little discussion of control of suspension bridges in the existing literature. The objective of this Dissertation is to study the exact controllability of the Lazer-McKenna suspension bridge equation (1.2). We study two different types of controls on the roadbed: locally distributed control and piezoelectric actuator. Both control problems are motivated by the recent developments and applications of smart materials. For example, the piezoelectric actuators have gained wide acceptance for active control in many structural systems due to their light weight, better bending moment control and low power consumption.

We study first in this Dissertation the exact controllability of the Lazer-McKenna suspension bridge equation with the locally distributed control. Consider

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial^4 w}{\partial x^4} + Kw^+ = \chi_\omega(x)u(x,t), \quad x \in \Omega,\ t > 0, \\ w(0,t) = w(\pi,t) = 0,\ \ t > 0, \\ \dfrac{\partial^2 w}{\partial x^2}(0,t) = \dfrac{\partial^2 w}{\partial x^2}(\pi,t) = 0,\ \ t > 0, \\ w(x,0) = w^0(x), \quad \dfrac{\partial w}{\partial t}(x,0) = w^1(x),\ \ x \in \Omega, \end{cases} \tag{1.6}$$

where $w^+ = \max\{0, w\}$, $\omega = (a,b) \subseteq (0,\pi) = \Omega$, $\chi_\omega(x)$ is the characteristic function of ω, (w^0, w^1) are given initial data in appropriate spaces, and $u(x,t)$ is the control. System (1.6) is exactly L^2-controllable at time T if there exists control $u \in L^2(\omega \times$

$(0,T))$ such that the solution of the system satisfies the final state condition

$$w(x,T)=\frac{\partial w}{\partial t}(x,T)=0, \quad x\in\Omega.$$

Unlike most of the existing literatures on exact controllability of nonlinear systems where the nonlinearity was always assumed to be C^1-smooth, the nonlinearity $w\mapsto w^+$ in (1.6) is not C^1-smooth, which makes the exact controllability problem of (1.6) challenging to study. We study this problem by using the Hilbert Uniqueness Method and the Leray-Schauder's degree theory. The key step is to establish an observability inequality of the auxiliary linear control problem of (1.6). The proof of such an inequality relies on establishing a Carleman estimate.

We then study the exact controllability of the Lazer-McKenna suspension bridge equation with a piezoelectric bending actuator. Consider

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}+\dfrac{\partial^4 w}{\partial x^4}+Kw^+=u(t)\dfrac{d}{dx}\left[\delta_b(x)-\delta_a(x)\right], \quad x\in\Omega,\ t>0,\\ w(0,t)=w(\pi,t)=0,\ \ t>0,\\ \dfrac{\partial^2 w}{\partial x^2}(0,t)=\dfrac{\partial^2 w}{\partial x^2}(\pi,t)=0,\ \ t>0,\\ w(x,0)=w^0(x), \quad \dfrac{\partial w}{\partial t}(x,0)=w^1(x),\ \ x\in\Omega, \end{cases} \tag{1.7}$$

where $0\le a<b\le\pi$, $\delta_a(x)$ and $\delta_b(x)$ are the Dirac delta functions, and $u(t)$ is the control. System (1.7) is exactly L^2-controllable at time T if there exists control $u\in L^2(0,T)$ such that the solution of the system satisfies the final state condition

$$w(x,T)=\frac{\partial w}{\partial t}(x,T)=0, \quad x\in\Omega.$$

Tucsnak [Tucsnak (1996)] has studied the linear case of (1.7) when the nonlinear term Kw^+ is dropped, and showed that the linear case is exact controllable only if $\dfrac{a+b}{2\pi}\in A$ and $\dfrac{b-a}{2\pi}\in A$, where A is the set of irrational numbers in $(0,1)$ such that

$\rho \in A$ if and only if there exists a constant $C > 0$ such that $\min_{n\in\mathbb{Z}} |q\rho - n| \geq \dfrac{C}{q}$ for any $q \in \mathbb{N}$. By using the Hilbert Uniqueness Method and the Banach Contraction Mapping, we obtain the similar controllability result. The proof of the observability inequality of the auxiliary linear control problem of (1.7) is based upon establishing an Ingham inequality for nonharmonic Fourier series.

The organization of this Dissertation is as follows. In Chapter 2, we first introduce and discuss the nonharmonic Fourier series. Then we prove an Ingham inequality when the gap goes to infinity, which will be used in the discussion of the exact controllability of the suspension bridge control system with a piezoelectric actuator. In Chapter 3, we discuss the exact controllability of the suspension bridge control system with a local distributed control (1.6), and prove that the system is exactly L^2-controllable. A Carleman estimate will be established and used to prove the related observability inequality. In Chapter 4, we discuss the exact controllability of the suspension bridge control system with a piezoelectric bending actuator (1.7), and prove that the system is exactly L^2-controllable. Some further discussions of the controllability of the Lazer-McKenna suspension bridge equation and future work are given in Chapter 5.

CHAPTER 2

NONHARMONIC FOURIER SERIES

2.1 Introduction

For any $f \in L^1[-\pi, \pi]$, it is well known that $f(x)$ can be expressed by the Fourier series

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx$, $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx dx$, $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx dx$. The Fourier theorem [Fourier (1822)] indicates that the Fourier series converges to $f(x)$ pointwisely on $[-\pi, \pi]$. If $f(x)$ is piecewise continuous on $[-\pi, \pi]$, then

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \begin{cases} f(x) & \text{if } f \text{ is cont. at } x; \\ \dfrac{1}{2}[f(x-) + f(x+)] & \text{if } f \text{ has jump} \\ & \text{discontinuity at } x. \end{cases}$$

By using the Euler Formula, $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{aligned} \cos nx &= \frac{1}{2}(e^{inx} + e^{-inx}), \\ \sin nx &= \frac{1}{2i}(e^{inx} - e^{-inx}), \end{aligned}$$

the Fourier series can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{inx}dx$. Furthermore, when $f \in L^2[-\pi, \pi]$, by the Bessel's inequality, we have

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\|f\|_2^2.$$

In fact, the Parseval's Theorem gives

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\|f\|_2^2. \tag{2.1}$$

and $\{e^{inx}\}_{n=-\infty}^{\infty}$ forms a complete orthogonal basis of $L^2[-\pi, \pi]$.

Now let $\{\lambda_n\} \subset \mathbb{R}$ satisfying $\lambda_{n+1} > \lambda_n$, define nonharmonic Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n t},$$

where $\{c_n\} \subset \mathbb{C}$. The theory of nonharmonic Fourier series is concerned with the completeness and expansion properties of $\{e^{i\lambda_n t}\}$ in $L^p[-\pi, \pi]$. Paley and Wiener [Paley and Wiener (1934)] first introduced and studied the nonharmonic Fourier series. The study of nonharmonic Fourier series is closely related to the study of entire function.

A natural question is: if $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n t} \in L^2(I)$, do there exist $C_1, C_2 > 0$ such that

$$C_1 \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \left\| \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n t} \right\|_{L^2(I)}^2 \leq C_2 \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{2.2}$$

A.E. Ingham [Ingham (1936)] showed that if $\lambda_n - \lambda_{n-1} \geq \gamma > 0$ and $T = \dfrac{\pi + \epsilon}{\gamma} > \dfrac{\pi}{\gamma}$

$$C_1 \sum_{n=N}^{N'} |c_n|^2 \leq \left\| \sum_{n=N}^{N'} c_n e^{i\lambda_n t} \right\|_{L^2(-T,T)}^2 \leq C_2 \sum_{n=N}^{N'} |c_n|^2.$$

Further results have been given by many researchers ([Zygmund (1959); Ball and Slemrod (1979); Beurling (1989); Haraux (1989); Young (2001)]).

J. M. Ball and M. Slemrod showed if $\underline{\lim}_{n\to\infty}(\lambda_n - \lambda_{n-1}) \geq \gamma > 0$ and $T > \dfrac{2\pi}{\gamma}$, (2.2) is true for $I = (0, T)$. They also pointed out in a remark that the Ingham inequality is true for any $|I| > 0$ if the gap of $\{\lambda_n\}$ goes to infinity. They referred the proof of such a claim should be similar to the study of Lacunary series by A. Zygmund in his book "Trigonometric series" [Zygmund (1959)].

The Lacunary series was first introduced by Zygmund [Zygmund (1959)]. Let $\{n_k\} \subseteq \mathbb{N}$ and $\frac{n_{k+1}}{n_k} > q > 1$, define the Fourier series as

$$\sum_{k=1}^{\infty} (a_k \cos(n_k x) + b_k \sin(n_k x)) = \sum_{k=1}^{\infty} A_k(x).$$

Note that the Lacunary series is a special type of nonharmonic Fourier series that requires

$$\frac{n_{k+1}}{n_k} > q > 1 \Rightarrow |n_{k+1} - n_k| \to \infty \text{ as } k \to \infty.$$

Unfortunately, this cannot be applied to our problem since $\frac{n_{k+1}}{n_k} > q > 1$ is not satisfied.

For the Lazer-McKenna suspension bridge equation, we need to show (2.2) for any $|I| > 0$ if the gap of nonharmonic Fourier series goes to infinity and $\frac{\lambda_{n+1}}{\lambda_n} \to 1$. The Arne Beurling's Theorem [Beurling (1989)] may lead to the answer but the proof is profound. We will give a direct proof to show that if $\lim_{n\to\infty} (\lambda_n - \lambda_{n-1}) = \infty$, (2.2) is true for any $|I| > 0$.

2.2 Upper bound for the L^2-norm of nonharmonic Fourier series

First, we show the right half of inequality (2.2) for all nonharmonic Fourier series. Some preparation of entire functions is needed.

Definition 2.1. An entire function $f(z)$ is said to be of exponential type $\tau > 0$ if there exists a constant $C > 0$ such that

$$|f(z)| \leq Ce^{\tau|z|}, \qquad \forall z \in \mathbb{C}.$$

Theorem 2.1. *(Plancherel-Pólya) If $f(z)$ is an entire function of exponential type τ, and if for some $p > 0$,*

$$\int_{-\infty}^{\infty} |f(x)|^p dx \, < \, \infty,$$

then

$$\int_{-\infty}^{\infty} |f(x+yi)|^p dx \, \leq \, e^{p\tau|y|} \int_{-\infty}^{\infty} |f(x)|^p dx.$$

Theorem 2.2. *(Paley-Wiener)[Paley and Wiener (1934)] Let $f(z)$ be an entire function such that*

$$|f(z)| \, \leq \, Ce^{A|z|}, \qquad \forall z \in \mathbb{C}$$

for $A > 0$ and $C > 0$, and $\int_{\mathbb{R}} |f(x)|^2 dx \, < \, \infty$. Then there exists a function $\phi \in L^2[-A, A]$ such that

$$f(z) \, = \, \int_{-A}^{A} \phi(t) e^{izt} dt.$$

Theorem 2.3. *Let $f(z)$ be an entire function of exponential type τ. Suppose for some $p > 1$,*

$$\int_{-\infty}^{\infty} |f(x)|^p dx \, < \, \infty.$$

If $\{\lambda_n\}$ is an increasing sequence of real numbers such that

$$\lambda_{n+1} - \lambda_n \, \geq \, \epsilon \, > \, 0,$$

then

$$\sum_n |f(\lambda_n)|^p \, \leq \, B \int_{\mathbb{R}} |f(x)|^p dx,$$

where B is a constant that depends only on p, τ and ϵ.

Proof. Since $|f(z)|^p$ is subharmonic, we have

$$|f(z_0)|^p \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|^p d\theta, \qquad z_0 \in \mathbb{C},\ r>0.$$

Then

$$|f(z_0)|^p \le \frac{1}{\pi\delta^2}\int\int_{|z-z_0|\le\delta} |f(z)|^p dxdy, \qquad \forall z_0 \in \mathbb{C},\ \delta>0.$$

Then

$$\begin{aligned}\sum_n |f(\lambda_n)|^p &\le \frac{1}{\pi\delta^2}\sum_n \int\int_{|z|<\delta} |f(\lambda_n+z)|^p dxdy \\ &\le \frac{1}{\pi\delta^2}\sum_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f(\lambda_n+x+yi)|^p dxdy \\ &= \frac{1}{\pi\delta^2}\sum_n \int_{-\delta}^{\delta}\int_{\lambda_n-\delta}^{\lambda_n+\delta} |f(x+yi)|^p dxdy.\end{aligned}$$

Let $\delta = \frac{\epsilon}{2}$, the $\{(\lambda_n-\delta, \lambda_n+\delta)\}$ are pairwise disjoint, hence by the Plancherel-Pólya Theorem (Theorem 2.1)

$$\begin{aligned}\sum_n |f(\lambda_n)|^p &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{-\infty}^{\infty} |f(x+yi)|^p dxdy \\ &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\left[e^{p\tau|y|}\int_{-\infty}^{\infty}|f(x)|^p dx\right] dy \\ &= B\int_{-\infty}^{\infty}|f(x)|^p dx.\end{aligned}$$

□

Theorem 2.4. *If $\{\lambda_n\}$ is a separated sequence of real numbers, then for any $A>0$,*

$$\left\|\sum_n c_n e^{i\lambda_n t}\right\|_2^2 \le M\sum_n |c_n|^2 \qquad \text{in } L^2[-A,A].$$

Proof. If $\phi \in L^2[-A,A]$, then $a_n = \langle \phi, e^{i\lambda_n t}\rangle = \int_{-A}^{A}\phi(t)e^{-i\lambda_n t}dt$ is first the value of $f(\lambda_n)$ of the entire function

$$f(z) = \int_{-A}^{A}\phi(t)e^{-izt}dt.$$

By the Paley-Wiener Theorem (Theorem 2.2), f is an entire function of exponential type A, hence by Theorem 2.3

$$\sum_n |a_n|^2 = \sum_n |f(\lambda_n)|^2 \le B \int_{\mathbb{R}} |f(x)|^2 dx < \infty .$$

Define a mapping T: $L^2[-A, A] \to \ell^2$ by $T(\phi) = \{\langle \phi, e^{i\lambda_n t}\rangle\}$. Thus T has a closed graph. By the closed graph theorem, there exists a $M > 0$ such that

$$\sum_n \left|\langle \phi, e^{i\lambda_n t}\rangle\right|^2 \le M\|\phi\|_2^2.$$

Let $\phi(t) = \sum_n c_n e^{i\lambda_n t}$, where $\sum_n |c_n|^2 < \infty$. Then

$$\begin{aligned} \|\phi\|_2^4 = |\langle \phi, \phi\rangle|^2 &= \left|\sum_n \bar{c_n}\langle \phi, e^{i\lambda_n t}\rangle\right|^2 \\ &\le \sum_n |c_n|^2 \sum_n \left|\langle \phi, e^{i\lambda_n t}\rangle\right|^2 \\ &\le M\|\phi\|_2^2 \sum_n |c_n|^2. \end{aligned}$$

Then

$$\|\phi\|_2^2 \le M \sum_n |c_n|^2.$$

That is

$$\left\|\sum_n c_n e^{i\lambda_n t}\right\|_2^2 \le M \sum_n |c_n|^2.$$

□

Remark 2.1. If $\sum_n |c_n|^2 < \infty$, then $\sum_{n=-k}^{k} c_n e^{i\lambda_n t}$ converges to a function $f \in L^2[-A, A]$, and $\|f\|_2^2 \le M \sum_n |c_n|^2$. Note $A > 0$ is arbitrary.

There remains the lower bound of $\left\|\sum_n c_n e^{i\lambda_n t}\right\|_2^2$ to be estimated.

2.3 Ingham inequality - with gap going to infinity

First, a classical version of Ingham inequality is given below.

Theorem 2.5. *Let $A > 0$ be given. If $\{\lambda_n\} \subset \mathbb{R}$ is a separated sequence satisfying*

$$\lambda_{n+1} - \lambda_n \;\geq\; \gamma \;>\; \frac{\pi}{A}.$$

Then there exists a $D > 0$ such that, for any $\{c_n\} \in \ell^2$,

$$D \sum_n |c_n|^2 \;\leq\; \left\| \sum_n c_n e^{i\lambda_n t} \right\|_2^2 \qquad \text{in } L^2[-A, A]\,.$$

Proof. Let $f(t) = \sum_n c_n e^{i\lambda_n t}$, then $f \in L^2[-A, A]$. WLOG, let $A = \pi$. If $k(t)$ is any integrable function on $\mathbb{R}$, and let

$$K(x) \;=\; \int_{\mathbb{R}} k(t) e^{ixt} dt\,.$$

Then

$$\int_{-\infty}^{\infty} k(t)|f(t)|^2 dt \;=\; \int_{-\infty}^{\infty} k(t) \sum_{m,n} c_m \bar{c_n} e^{i\lambda_m t} e^{-i\lambda_n t} dt \;=\; \sum_{m,n} c_m \bar{c_n} K(\lambda_m - \lambda_n).$$

Choose

$$k(t) \;=\; \begin{cases} \cos\dfrac{t}{2}, & \text{if } |t| \leq \pi, \\ 0, & \text{if } |t| > \pi. \end{cases}$$

Then

$$K(x) \;=\; \frac{4\cos \pi x}{1 - 4x^2}.$$

Thus

$$\sum_{m,n} K(\lambda_m - \lambda_n) c_m \bar{c_n} \;=\; \int_{-\infty}^{\infty} k(t)|f(t)|^2 dt \;\leq\; \int_{-\pi}^{\pi} |f(t)|^2 dt \;=\; \|f\|_2^2.$$

Let

$$\sum_{m,n} K(\lambda_m - \lambda_n) c_m \bar{c_n} = \sum_{m=n} K(0) c_m \bar{c_n} + \sum_{m \neq n} K(\lambda_m - \lambda n) c_m \bar{c_n}$$
$$= 4 \sum_n |c_n|^2 + \sum_{m \neq n} K(\lambda_m - \lambda_n) c_m \bar{c_n}.$$

Note that K is even, and $|c_m \bar{c_n}| \leq \frac{1}{2}[|c_m|^2 + |c_n|^2]$, then

$$\left| \sum_{m \neq n} K(\lambda_m - \lambda_n) c_m \bar{c_n} \right| \leq \sum_{m \neq n} \frac{1}{2} \left[|c_m|^2 + |c_n|^2 \right] |K(\lambda_m - \lambda_n)|$$
$$= \sum_n |c_n|^2 \sum_{m,\, m \neq n} |K(\lambda_m - \lambda_n)|.$$

Note that $|\lambda_m - \lambda_n| \geq |m - n|\gamma > 1$. Thus

$$\sum_{m,\, m \neq n} |K(\lambda_m - \lambda_n)| \leq \sum_{m,\, m \neq n} \frac{4}{4(m-n)^2\gamma^2 - 1} < \frac{8}{\gamma^2} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$
$$= \frac{4}{\gamma^2} \sum_{k=1}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2k-1} \right) = \frac{4}{\gamma^2}.$$

Then

$$\sum_{m,n} K(\lambda_m - \lambda_n) c_m \bar{c_n} \geq 4 \sum_n |c_n|^2 - \frac{4}{\gamma^2} \sum_n |c_n|^2 = 4 \left(1 - \frac{1}{\gamma^2} \right) \sum_n |c_n|^2.$$

□

Remark 2.2. 1. $D = \frac{4A}{\pi} \left(1 - \frac{1}{\gamma^2} \right)$;

2. There are many generalized version of the Ingham Inequality. For example: $\{\lambda_n\} \subseteq \mathbb{C}$, $\lambda_n = a_n + b_n i$, $|b_n| \leq B > 0$. An equivalent version of the Ingham Inequality: Let $\{\lambda_n\} \subseteq \mathbb{R}$ be a separated sequence satisfying

$$|\lambda_{n+1} - \lambda_n| \geq \gamma > 0.$$

Then, for any interval $I \subset \mathbb{R}$ such that $|I| > \dfrac{2\pi}{\gamma}$, there exists $D_1 > 0$ and $D_2 > 0$ such that

$$D_1 \sum_n |c_n|^2 \leq \int_I |f(t)|^2 dt \leq D_2 \sum_n |c_n|^2$$

for any $\{c_n\} \in \ell^2$, where $f(t) = \sum_n c_n e^{i\lambda_n t}$.

Note that this theorem may not be applied if the length of interval is relatively small.

Theorem 2.6. *[Zygmund (1959)] Let* $P(x) = \sum_k A_k(x)$ *be a Lacunary series. Let* $I \subset [0, 2\pi]$ *and* $|I| > 0$.

(a) *There exists a* $\lambda > 0$ *and* $k_0 > 0$ *such that*

$$\lambda^{-1}|I|\frac{1}{2}\sum_{k \geq k_0} \left(a_k^2 + b_k^2\right) \leq \int_I |P(x)|^2 dx \leq \lambda|I|\frac{1}{2}\sum_{k \geq k_0} \left(a_k^2 + b_k^2\right).$$

(b) *If* $\sum A_k(x)$ *converges to zero on* I*, then the series vanishes identically on* I.

Example 2.1. $\sum_{n=1}^{\infty} \left(a_n \cos(n^2 x) + b_n \sin n^2 x\right)$, $\dfrac{n_{k+1}}{n_k} \to 1$ as $k \to \infty$, but $n_{k+1} - n_k = (k+1)^2 - k^2 = 2k + 1 \to \infty$ as $k \to \infty$.

Remark 2.3. Zygmund's result can not be applied to the case in the control theory.

Here let us introduce Arne Beurling's Theorem that can indicate (2.2) for non-harmonic Fourier series with gap goes to infinity.

Let $\{\lambda_n\} \subset \mathbb{R}$ be a separated sequence satisfying $\lambda_{n+1} > \lambda_n$, that is $\inf_{n \neq m} |\lambda_n - \lambda_m| \geq \alpha > 0$. Let $\Lambda = \{\lambda_n\}$. Define $n^+(r)$ to be the largest number of points from Λ to be found in an interval of length r.

Definition 2.2. The upper density of Λ is defined by

$$D^+(\Lambda) = \lim_{r\to\infty} \frac{n^+(r)}{r}.$$

Remark 2.4. The limit exists due to the subadditivity of $n^+(r)$, $n^+(r+s) \le n^+(r) + n^+(s)$, for all $r > 0$, $s > 0$.

Theorem 2.7. *(Arne Beurling)[Beurling (1989)] Let $\{\lambda_n\}$ be a separated sequence of real numbers satisfying $\lambda_{n+1} > \lambda_n$. If $|I| > 2\pi D^+$, then there exists $D_1 > 0$ and $D_2 > 0$ such that*

$$D_1 \sum_n |c_n|^2 \le \int_I |f(t)|^2 dt \le D_2 \sum_n |c_n|^2, \tag{2.3}$$

where $f(t) = \sum_n c_n e^{i\lambda_n t}$. If $|I| < 2\pi D^+$, then (2.3) does not hold.

Let $\{\lambda_n\} \subset \mathbb{R}_+$ be separated sequence satisfying $\lambda_{n+1} > \lambda_n$ and $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \infty$. For any $\gamma > 0$, there exists a $n_0 > 0$ such that

$$\lambda_{n+1} - \lambda_n \ge \gamma, \qquad \text{when } n \ge n_0.$$

Then

$$n^+(k\gamma) \le n_0 + k - 1.$$

Thus

$$\frac{n^+(k\gamma)}{k\gamma} \le \frac{n_0 + k - 1}{k\gamma}.$$

Then

$$D^+(\Lambda) = \lim_{r\to\infty} \frac{n^+(r)}{r} = \lim_{k\to\infty} \frac{n^+(k\gamma)}{k\gamma} \le \frac{1}{\gamma}.$$

Since $\gamma > 0$ is arbitrary, we have $D^+(\Lambda) = 0$. There we have the following theorem.

Theorem 2.8. *Let $\{\lambda_n\} \subset \mathbb{R}$ be a separated sequence satisfying $\lambda_{n+1} > \lambda_n$ and $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \infty$. For any $I \subset \mathbb{R}$ and $|I| > 0$, there exists $D_1 > 0$ and $D_2 > 0$ such that*

$$D_1 \sum_n |c_n|^2 \le \int_I |f(t)|^2 dt \le D_2 \sum_n |c_n|^2,$$

where $f(t) = \sum_n c_n e^{i\lambda_n t}$.

Example 2.2. For any $T > 0$,

$$D_1 \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le \int_0^T \left(\sum_{n=1}^{\infty} (a_n \cos n^2 t + b_n \sin n^2 t) \right)^2 dt \le D_2 \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

This is the expected result. To give a direct proof, an extension of A. Haraux's Theorem [Haraux (1989)] needs to be introduced first.

Theorem 2.9. *Let $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be separated sequence, that is, $\inf_{n\neq m} |\lambda_n - \lambda_m| \ge \alpha > 0$. Let $I \subset \mathbb{R}$ be an interval such that $|I| > 2\pi/\alpha$. Assume there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 \sum_{n=1}^{\infty} |c_n|^2 \le \int_I |f(t)|^2 dt \le C_2 \sum_{n=1}^{\infty} |c_n|^2,$$

where $f(t) = \sum_{n=1}^{\infty} c_n e^{i\lambda_n t}$. Let $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \notin \{\lambda_n\}_{n=1}^{\infty}$, $\mu_0 = \inf_{n\neq m,\, n,m\ge 0} |\lambda_n - \lambda_m| > 0$. Thus there exists $D_1 > 0$ and $D_2 > 0$ such that

$$D_1 \sum_{n=0}^{\infty} |c_n|^2 \le \int_I |g(t)|^2 dt \le D_2 \sum_{n=0}^{\infty} |c_n|^2, \tag{2.4}$$

where $g(t) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n t}$.

Proof. WLOG, let $I = [0, T]$ and $T > T_0$. Choose $\epsilon > 0$, such that $T - \epsilon > T_0$. Let $\{c_n\} \in \ell^2$, that is, $\sum_{n=0}^{\infty} |c_n|^2 < \infty$. Then (2.4) is equivalent to

$$D_1 \sum_{n=0}^{\infty} |c_n|^2 \le \int_0^T |g_0(t)|^2 dt \le D_2 \sum_{n=0}^{\infty} |c_n|^2,$$

where $g_0(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{i(\lambda_n - \lambda_0)t} = c_0 + \sum_{n=1}^{\infty} c_n e^{i\mu_n t}$, $\mu_n = \lambda_n - \lambda_0$.

Let

$$h(t) = \frac{1}{\epsilon} \int_0^{\epsilon} [g_0(t+\eta) - g_0(t)] d\eta, \qquad t \in [0, T-\epsilon] = [0, T'].$$

Then

$$h(t) = \sum_{n=1}^{\infty} \frac{c_n}{\epsilon} \left[\frac{e^{i\mu_n \epsilon} - 1}{i\mu_n} - \epsilon \right] e^{i\mu_n t}.$$

Note that $|\mu_n| \geq \mu_0 > 0$, then there exists a $\delta = \delta(\mu_0) > 0$ such that

$$\left| e^{i\mu_n \epsilon} - 1 \right| = 2 \left| \sin \frac{\mu_n \epsilon}{2} \right| \leq \epsilon |\mu_n| (1 - \delta).$$

Thus

$$\frac{1}{\epsilon} \left| \frac{e^{i\mu_n \epsilon} - 1}{i\mu_n} - \epsilon \right| = \left| \frac{e^{i\mu_n \epsilon} - 1}{i\mu_n \epsilon} - 1 \right| \geq 1 - \left| \frac{e^{i\mu_n \epsilon} - 1}{i\mu_n \epsilon} \right| \geq 1 - (1 - \delta) = \delta > 0.$$

Thus, by the assumption, $T' = T - \epsilon > T_0$,

$$\int_0^{T'} |h(t)|^2 dt \geq C_1 \sum_{n=1}^{\infty} \left| \frac{e^{i\mu_n \epsilon} - 1}{i\mu_n \epsilon} - 1 \right|^2 |c_n|^2 \geq C_1 \delta^2 \sum_{n=1}^{\infty} |c_n|^2.$$

Note that

$$\int_0^{T'} |h(t)|^2 dt = \int_0^{T'} \left| \frac{1}{\epsilon} \int_0^{\epsilon} [g_0(t+\eta) - g_0(t)] d\eta \right|^2 dt \leq C_3 \int_0^T |g_0(t)|^2 dt.$$

Therefore

$$\int_0^T |g_0(t)|^2 dt \geq \frac{C_1 \delta^2}{C_3} \sum_{n=1}^{\infty} |c_n|^2.$$

Note that

$$\begin{aligned} |c_0|^2 &= \frac{1}{T} \int_0^T |c_0|^2 dt \\ &\leq \frac{1}{T} \int_0^T \left[|c_0 - g_0(t)| + |g_0(t)| \right]^2 dt \\ &\leq \frac{2}{T} \left[\int_0^T |c_0 - g_0(t)|^2 dt + \int_0^T |g_0(t)|^2 dt \right]. \end{aligned}$$

Since

$$\int_0^T |c_0 - g_0(t)|^2 dt = \int_0^T \left| \sum_{n=1}^{\infty} c_n e^{i\mu_n t} \right|^2 dt$$
$$\leq C_2 \sum_{n=1}^{\infty} |c_n|^2 \leq \frac{C_2 C_3}{C_1 \delta^2} \int_0^T |g_0(t)|^2 dt.$$

We have

$$|c_0|^2 \leq \frac{2}{T} \left(1 + \frac{C_2 C_3}{C_1 \delta^2} \right) \int_0^T |g_0(t)|^2 dt.$$

Thus

$$\sum_{n=0}^{\infty} |c_n|^2 \leq \left[\frac{2}{T} \left(1 + \frac{C_2 C_3}{C_1 \delta^2} \right) + \frac{C_3}{C_1 \delta^2} \right] \int_0^T |g_0(t)|^2 dt.$$

Then

$$\int_0^T |g_0(t)|^2 dt \geq D_1 \sum_{n=0}^{\infty} |c_n|^2.$$

□

The following is the main result of this chapter.

Theorem 2.10. *Let* $\{\lambda_n\} \subset \mathbb{R}$ *be a separated sequence satisfying* $\lambda_{n+1} - \lambda_n > 0$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \infty$*. For any* $I_0 \subset \mathbb{R}$ *and* $|I_0| > 0$*, there exists* $D_1 > 0$ *and* $D_2 > 0$ *such that*

$$D_1 \sum_n |c_n|^2 \leq \int_{I_0} |f(t)|^2 dt \leq D_2 \sum_n |c_n|^2,$$

where $f(t) = \sum_n c_n e^{i\lambda_n t}$*.*

Proof. WLOG, let $\{\lambda\} \subset \mathbb{R}_+$. Since $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \infty$, there exists a $n_0 > 0$ such that

$$|\lambda_{n+1} - \lambda_n| \geq \gamma > \frac{2\pi}{|I_0|} = \gamma_0.$$

By the Theorem 2.5, for any $|I| > |I_0|$, there exists $D_1 > 0$ and $D_2 > 0$ such that

$$D_1 \sum_{n \geq n_0} |c_n|^2 \leq \int_I |f_0(t)|^2 dt \leq D_2 \sum_{n \geq n_0} |c_n|^2,$$

where $f_0(t) = \sum_{n=n_0}^{\infty} c_n e^{i\lambda_n t}$. By using Theorem 2.9 and the mathematical induction, there exists D_1' and D_2' such that

$$D_1' \sum_n |c_n|^2 \leq \int_I |f(t)|^2 dt \leq D_2' \sum_n |c_n|^2,$$

where $f(t) = \sum_n c_n e^{i\lambda_n t}$. □

CHAPTER 3

EXACT CONTROLLABILITY OF THE LAZER-MCKENNA SUSPENSION BRIDGE EQUATION WITH DISTRIBUTED CONTROL

3.1 Question and Literatures

In the past few decades, researchers focused on the analytic properties of solutions of suspension bridge models. They have been trying to explain the reasons of large amplitude oscillations and torsional oscillations that may cause the collapse of suspension bridges. A natural question is: are the oscillations controllable? In this chapter, we will prove the exact controllability of the suspension bridge equation proposed by Lazer and McKenna [Lazer and McKenna (1990)] with a distributed control. The key step is to establish a Carleman estimate for an auxiliary linear system, and similar estimation has been given for wave equation by Xu Zhang [Zhang (2000)].

Let $\Omega = (0, \pi)$, $\omega = (a, b)$ be an interval with $0 \leq a < b \leq \pi$. Consider the nonlinear system

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) + Kw^+(x,t) = u(x,t)\chi_\omega, \qquad x \in \Omega, \quad t > 0, \tag{3.1}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{3.2}$$

$$w(x,0) = w^0(x), \quad \frac{\partial w}{\partial t}(x,0) = w^1(x), \qquad x \in \Omega, \tag{3.3}$$

with $u(x,t) \in L^2(\omega \times (0,T))$ and initial data $\{w^0, w^1\} \in Y_2 \times L^2(\Omega)$, where Y_2 will be introduced next. It is known that the system (3.1)-(3.3) admits a unique solution

w satisfying the following regularity

$$w \in C([0,T]; Y_2) \bigcap C^1([0,T]; L^2(\Omega)).$$

Definition 3.1. We say system (3.1)-(3.3) is exactly L^2-controllable at time $T > 0$ if there exists control $u \in L^2(\omega \times (0,T))$ such that the solution of the system satisfies the final state condition

$$w(x,T) = \frac{\partial w}{\partial t}(x,T) = 0, \qquad x \in \Omega. \tag{3.4}$$

We will show in this chapter system (3.1)-(3.3) is exactly L^2-controllable at time T.

3.2 Notation and Lemmas

Let $\Omega = (0, \pi)$ in this chapter. Now introduce the function spaces $(Y_\alpha)_{\alpha \in \mathbb{R}}$.

Definition 3.2. For $\alpha \in \mathbb{R}$, define $Y_\alpha = \overline{\left\{ \sum_{n=1}^{\infty} a_n \sin(nx) \,\middle|\, \sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty \right\}}$, and for any $f = \sum_{n=1}^{\infty} a_n \sin(nx) \in Y_\alpha$, $\|f\|_{Y_\alpha}^2 = \sum_{n=1}^{\infty} n^{2\alpha} a_n^2$.

Remark 3.1. We have the following properties of Y_α:

- Y_α is a closed subspace of $H^\alpha(\Omega)$, for $\alpha > 0$;
- $Y_0 = L^2(\Omega)$;
- $Y_{-\alpha}$ is the dual space of Y_α, for $\alpha > 0$.

Remark 3.2. By the Sobolev Imbedding Theorem [Adams (1978)], we have $H^\alpha(\Omega) \hookrightarrow L^\infty(\Omega)$ for $\alpha > \frac{1}{2}$. Since Y_α is a closed subspace of $H^\alpha(\Omega)$, we also have $Y_\alpha \hookrightarrow L^\infty(\Omega)$ for $\alpha > \frac{1}{2}$.

Consider the following linear system

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} + a(x,t)v = b(x,t), \qquad x \in \Omega, \quad t > 0, \tag{3.5}$$

$$v(0,t) = v(\pi,t) = 0, \quad \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{3.6}$$

$$v(0) = v^0, \qquad \frac{\partial v}{\partial t}(0) = v^1, \qquad x \in \Omega, \tag{3.7}$$

where $a \in L^\infty(\Omega \times (0,T))$, $b \in L^2(\Omega \times (0,T))$ and $\{v^0, v^1\} \in Y_2 \times L^2(\Omega)$. It is straightforward to show that system (3.5)-(3.7) admits a unique solution $v \in C([0,T]; Y_2) \bigcap C^1([0,T]; L^2(\Omega))$.

For system (3.5)-(3.7), define its total energy by

$$E(t) = \frac{1}{2}\left(\left\| \frac{\partial v}{\partial t} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2_{L^2(\Omega)} \right).$$

We have the following result.

Lemma 3.1. *Let $T > 0$ be given. For every solution of* (3.5)-(3.7)*, we have*

$$E(t) \leq \left(E(0)(1 + \|a\|_\infty) + \|b\|^2_{L^2(\Omega\times(0,T))} \right) e^{\left(1+2\sqrt{\|a\|_\infty}\right)t}. \tag{3.8}$$

Proof. Let $\gamma = \|a\|_\infty$ and define the perturbed energy

$$E_\gamma(t) = E(t) + \frac{\gamma}{2}\|v\|^2_{L^2(\Omega)}.$$

Taking derivative we have

$$\frac{dE_\gamma(t)}{dt} = \int_\Omega \frac{\partial^2 v}{\partial t^2}\frac{\partial v}{\partial t} dx + \int_\Omega \frac{\partial^3 v}{\partial t \partial x^2}\frac{\partial^2 v}{\partial x^2} dx + \gamma \int_\Omega \frac{\partial v}{\partial t} v dx.$$

Applying (3.5) and integrating by parts yields

$$\frac{dE_\gamma(t)}{dt} = \int_\Omega (\gamma - a) v \frac{\partial v}{\partial t} dx + \int_\Omega b \frac{\partial v}{\partial t} dx.$$

Thus we have the following estimate

$$\begin{aligned}\frac{dE_\gamma(t)}{dt} &\le 2\gamma\|v\|_{L^2(\Omega)}\left\|\frac{\partial v}{\partial t}\right\|_{L^2(\Omega)} + \|b\|_{L^2(\Omega)}\left\|\frac{\partial v}{\partial t}\right\|_{L^2(\Omega)} \\ &\le \gamma\left(\sqrt{\gamma}\|v\|^2_{L^2(\Omega)} + \frac{1}{\sqrt{\gamma}}\left\|\frac{\partial v}{\partial t}\right\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|b\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|^2_{L^2(\Omega)} + \gamma\sqrt{\gamma}\|v\|^2_{L^2(\Omega)} + \sqrt{\gamma}\left\|\frac{\partial v}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|b\|^2_{L^2(\Omega)} \\ &\le (1+2\sqrt{\gamma})E_\gamma(t) + \frac{1}{2}\|b\|^2_{L^2(\Omega)}.\end{aligned}$$

By Gronwall's Inequality we may obtain

$$\begin{aligned}E_\gamma(t) &\le \left(E_\gamma(0) + \frac{1}{2}\int_0^t e^{-(1+2\sqrt{\gamma})s}\|b\|^2_{L^2(\Omega)}ds\right)e^{(1+2\sqrt{\gamma})t} \\ &\le \left(E_\gamma(0) + \frac{1}{2}\|b\|^2_{L^2(\Omega\times(0,T))}\right)e^{(1+2\sqrt{\gamma})t}.\end{aligned}$$

Therefore

$$\begin{aligned}E(t) \le E_\gamma(t) &\le \left(E(0) + \frac{\gamma}{2}\|v^0\|^2_{L^2(\Omega)} + \frac{1}{2}\|b\|^2_{L^2(\Omega\times(0,T))}\right)e^{(1+2\sqrt{\gamma})t} \\ &\le \left(E(0)(1+\|a\|_\infty) + \|b\|^2_{L^2(\Omega\times(0,T))}\right)e^{\left(1+2\sqrt{\|a\|_\infty}\right)t}.\end{aligned}$$

□

Lemma 3.2. *System* (3.5)-(3.7) *is time-reversible, that is, for any given* $T > 0$, *the initial state* $\left\{v(x,0), \frac{\partial v}{\partial t}(x,0)\right\}$ *can be determined uniquely from the final state* $\left\{v(x,T), \frac{\partial v}{\partial t}(x,T)\right\}$.

Proof. System (3.5)-(3.7) admits a unique solution

$$v \in C([0,T];Y_2)\bigcap C^1([0,T];L^2(\Omega))$$

Thus $\left\{v(x,T), \frac{\partial v}{\partial t}(x,T)\right\} \in Y_2 \times L^2(\Omega)$. Now consider the following system

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + a(x,t)u = b(x,t), \qquad x \in \Omega, \quad t > 0, \tag{3.9}$$

$$u(0,t) = u(\pi,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{3.10}$$

$$u(x,T) = v(x,T), \qquad \frac{\partial u}{\partial t}(x,T) = \frac{\partial v}{\partial t}(x,T), \qquad x \in \Omega. \tag{3.11}$$

We need to show $u = v$.

Let $\tau = T - t$, $\bar{u}(x,\tau) = u(x, T-\tau)$, we have system (3.9)-(3.11) is equivalent to

$$\frac{\partial^2 \bar{u}}{\partial \tau^2} + \frac{\partial^4 \bar{u}}{\partial x^4} + a(x,T-\tau)\tau u = b(x,T-\tau), \qquad x \in \Omega, \quad \tau < T, \tag{3.12}$$

$$\bar{u}(0,T-\tau) = \bar{u}(\pi,T-\tau) = 0, \ \frac{\partial^2 \bar{u}}{\partial x^2}(0,T-\tau) = \frac{\partial^2 \bar{u}}{\partial x^2}(\pi,T-\tau) = 0, \qquad \tau < T, \tag{3.13}$$

$$\bar{u}(x,0) = v(x,T), \qquad \frac{\partial \bar{u}}{\partial \tau}(x,0) = -\frac{\partial v}{\partial t}(x,T), \qquad x \in \Omega. \tag{3.14}$$

Obviously, system (3.12)-(3.14) admits a unique solution

$$\bar{u} = u \in C([0,T]; Y_2) \bigcap C^1([0,T]; L^2(\Omega)).$$

Moreover, let $w = v - u$. Thus w satisfies the following

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + a(x,t)w = 0, \qquad x \in \Omega, \quad 0 < t < T, \tag{3.15}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \quad 0 < t < T, \tag{3.16}$$

$$w(x,T) = 0, \qquad \frac{\partial w}{\partial t}(x,T) = 0, \qquad x \in \Omega. \tag{3.17}$$

Change variable as $\tau = T - t$, $\bar{w}(x,\tau) = w(x,T-\tau)$, we obtain

$$\frac{\partial^2 \bar{w}}{\partial \tau^2} + \frac{\partial^4 \bar{w}}{\partial x^4} + a(x,T-\tau)\bar{w} = 0, \qquad x \in \Omega, \quad 0 < \tau < T, \tag{3.18}$$

$$\bar{w}(0,T-\tau)=\bar{w}(\pi,T-\tau)=0,\ \frac{\partial^2\bar{w}}{\partial x^2}(0,T-\tau)=\frac{\partial^2\bar{w}}{\partial x^2}(\pi,T-\tau)=0,$$

$$0<\tau<T, \tag{3.19}$$

$$\bar{w}(x,0)=0, \qquad \frac{\partial\bar{w}}{\partial\tau}(x,0)=0, \qquad x\in\Omega. \tag{3.20}$$

Define the energy

$$E(\tau)=\frac{1}{2}\left(\left\|\frac{\partial\bar{w}}{\partial\tau}\right\|^2_{L^2(\Omega)}+\left\|\frac{\partial^2\bar{w}}{\partial x^2}\right\|^2_{L^2(\Omega)}\right), \qquad E(0)=0.$$

By Lemma 3.1 we have

$$E(\tau)\le E(0)(1+\|a\|_\infty)e^{\left(1+2\sqrt{\|a\|_\infty}\right)\tau}=0.$$

Thus the solution of system (3.18)-(3.20) satisfies

$$\frac{\partial^2\bar{w}}{\partial x^2}=0, \qquad \frac{\partial\bar{w}}{\partial\tau}=0.$$

Applying boundary and initial value conditions, we have $\bar{w}\equiv 0$. That is $w\equiv 0$. So $u\equiv v$, system (3.5)-(3.7) is time-reversible. □

Define linear operator $D:\,Y_2\to Y_{-2}$ as

$$\phi\to\frac{d^4\phi}{dx^4}.$$

$$\langle D\phi,\phi\rangle=\langle\phi,D\phi\rangle.$$

D is self-adjoint. Moreover

$$\langle D\phi,\phi\rangle=\int_0^\pi\left(\frac{d^2\phi}{dx^2}\right)^2dx\ge 0.$$

Thus D is a positive operator. Since $\langle D\phi,\phi\rangle=0$, $\dfrac{d^2\phi}{dx^2}=0$. So $\phi=c_1+c_2x$. By applying boundary value conditions we get $\phi\equiv 0$. Therefore D is invertible. We have a inverse $D^{-1} = \left(\frac{d^4}{dx^4}\right)^{-1} : Y_{-2} \to Y_2$ is also positive, and we have

$$\langle \phi, D^{-1}(\phi) \rangle = \|\phi\|^2_{Y_{-2}}.$$

Consider the following system

$$\frac{\partial^2 \phi}{\partial t^2}(x,t) + \frac{\partial^4 \phi}{\partial x^4}(x,t) + a(x,t)\phi(x,t) = 0, \qquad x \in \Omega, \quad t > 0, \quad (3.21)$$

$$\phi(0,t) = \phi(\pi,t) = 0, \quad \frac{\partial^2 \phi}{\partial x^2}(0,t) = \frac{\partial^2 \phi}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \quad (3.22)$$

$$\phi(x,0) = \phi^0(x), \quad \frac{\partial \phi}{\partial t}(x,0) = \phi^1(x), \qquad x \in \Omega, \quad (3.23)$$

with potential $a(x,t) \in L^\infty((\Omega) \times (0,T))$ and $\{\phi^0, \phi^1\} \in L^2(\Omega) \times Y_{-2}$.

Lemma 3.3. *Let ϕ be the solution of system* (3.21)-(3.23) *and* $0 \le t_1 < s_1 < s_2 < t_2 \le T$ *be given. Then there is a constant $C > 0$ depends only on $\|a\|_\infty$, t_1, t_2, s_1, s_2 such that*

$$\int_{s_1}^{s_2} \left\| \frac{\partial \phi}{\partial t} \right\|^2_{Y_{-2}} dt \le C \int_{t_1}^{t_2} \|\phi\|^2_{L^2(\Omega)} dt.$$

Proof. Let $r(t) \in C^2([t_1, t_2])$ be a positive function chosen later. By multiplying both sides of equation (3.21) by $r(t)D^{-1}(\phi)$ and integrating the resulting equation on $\Omega \times (t_1, t_2)$, we get

$$\int_{t_1}^{t_2} \int_\Omega \frac{\partial^2 \phi}{\partial t^2} r(t) D^{-1}(\phi) dx dt = \int_\Omega \frac{\partial \phi}{\partial t} r(t) D^{-1}(\phi) \Big|_{t_1}^{t_2} dx$$
$$- \int_{t_1}^{t_2} \int_\Omega \frac{\partial \phi}{\partial t} r(t) D^{-1}\left(\frac{\partial \phi}{\partial t}\right) dx dt - \int_{t_1}^{t_2} \int_\Omega \frac{\partial \phi}{\partial t} r'(t) D^{-1}(\phi) dx dt,$$
$$\int_{t_1}^{t_2} \int_\Omega \frac{\partial^4 \phi}{\partial x^4} r(t) D^{-1}(\phi) dx dt = \int_{t_1}^{t_2} \int_\Omega \frac{\partial^2 \phi}{\partial x^2} r(t) D^{-1}\left(\frac{\partial^2 \phi}{\partial x^2}\right) dx dt.$$

This is

$$\int_{t_1}^{t_2} r(t) \left\| \frac{\partial \phi}{\partial t} \right\|^2_{Y_{-2}} dt = \int_{t_1}^{t_2} r(t) \|\phi\|^2_{L^2(\Omega)} dt + \int_{t_1}^{t_2} \int_\Omega r(t) a(x,t) \phi D^{-1}(\phi) dx dt$$

$$- \int_{t_1}^{t_2} \int_\Omega \frac{\partial \phi}{\partial t} r'(t) D^{-1}(\phi) dx dt \; + \; \int_\Omega \frac{\partial \phi}{\partial t} r(t) D^{-1}(\phi) \Big|_{t_1}^{t_2} dx.$$

By choosing $r(t)$ such that $r(t_1) = r(t_2) = r'(t_1) = r'(t_2) = 0$, $r(t) = 1 \; \forall \, t \in [s_1, s_2]$ with $t_1 < s_1 < s_2 < t_2$, $\dfrac{|r''|^2}{r} \in L^\infty(t_1, t_2)$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\partial \phi}{\partial t} r(t)(D^{-1}\phi) \Big|_{t_1}^{t_2} dx = 0, \\
&\int_{t_1}^{t_2} \int_\Omega r(t) a(x,t) \phi D^{-1}(\phi) dx dt \le \|a\|_\infty \int_{t_1}^{t_2} \|\phi\|_{Y_{-2}}^2 dt \le \|a\|_\infty \int_{t_1}^{t_2} \|\phi\|_{L^2(\Omega)}^2 dt, \\
&- \int_{t_1}^{t_2} \int_\Omega \frac{\partial \phi}{\partial t} r'(t) D^{-1}(\phi) dx dt = -\frac{1}{2} \int_{t_1}^{t_2} r'(t) \frac{d}{dt} \|\phi\|_{Y_{-2}}^2 dt \\
&\qquad = \frac{1}{2} \int_{t_1}^{t_2} r''(t) \|\phi\|_{Y_{-2}}^2 dt \le C_1 \int_{t_1}^{t_2} \|\phi\|_{L^2(\Omega)}^2 dt,
\end{aligned}$$

where $C_1 = \max\limits_{t \in [t_1, t_2]} \dfrac{|r''(t)|}{2r(t)}$. Therefore, above all yield

$$\int_{s_1}^{s_2} \left\| \frac{\partial \phi}{\partial t} \right\|_{Y_{-2}}^2 dt \le C \int_{t_1}^{t_2} \|\phi\|_{L^2(\Omega)}^2 dt,$$

where $C = 1 + \|a\|_\infty + C_1$. □

Lemma 3.4. *Let $\phi(\cdot)$ be the solution of system* (3.21)-(3.23). *Define*

$$\bar{E}(t) \; \triangleq \; \frac{1}{2} \left(\|\phi\|_{L^2(\Omega)}^2 + \left\| \frac{\partial \phi}{\partial t} \right\|_{Y_{-2}}^2 \right).$$

Then there exists positive constant $A > 0$ such that

$$\bar{E}(t) \; \le \; A \bar{E}(s) (1 + \|a\|_\infty^2)^2 e^{\left(2 + 4\sqrt{\|a\|_\infty}\right)T}, \qquad \forall \, t, \; s \in [0, T].$$

Proof. Decompose the solution $\phi(x,t)$ of system (3.21)-(3.23) as

$$\phi(x,t) \; = \; p(x,t) + q(x,t),$$

such that respectively $p(x,t)$ and $q(x,t)$ are the solutions of

$$\frac{\partial^2 p}{\partial t^2}(x,t) + \frac{\partial^4 p}{\partial x^4}(x,t) = 0, \qquad x\in\Omega,\ t>0, \tag{3.24}$$

$$p(0,t) = p(\pi,t) = 0, \quad \frac{\partial^2 p}{\partial x^2}(0,t) = \frac{\partial^2 p}{\partial x^2}(\pi,t) = 0, \quad t>0, \tag{3.25}$$

$$p(x,0) = \phi^0(x), \qquad \frac{\partial p}{\partial t}(x,0) = \phi^1(x), \qquad x\in\Omega, \tag{3.26}$$

and

$$\frac{\partial^2 q}{\partial t^2}(x,t) + \frac{\partial^4 q}{\partial x^4}(x,t) + a(x,t)q(x,t) = -a(x,t)p(x,t), \quad x\in\Omega,\ t>0. \tag{3.27}$$

$$q(0,t) = q(\pi,t) = 0, \quad \frac{\partial^2 q}{\partial x^2}(0,t) = \frac{\partial^2 q}{\partial x^2}(\pi,t) = 0, \qquad t>0. \tag{3.28}$$

$$q(x,0) = 0, \qquad \frac{\partial q}{\partial t}(x,0) = 0, \qquad x\in\Omega. \tag{3.29}$$

Solve system (3.24)-(3.26), we can get

$$\begin{aligned} p(x,t) = \sum_{n=1}^{\infty}\Bigg(&\sqrt{\frac{2}{\pi}}\int_0^\pi \phi^0(y)\sin ny dy \cos n^2 t \\ &+ \sqrt{\frac{2}{\pi}}\int_0^\pi \frac{1}{n^2}\phi^1(y)\sin ny dy \sin n^2 t\Bigg)\cdot \sin nx, \\ p'(x,t) = \sum_{n=1}^{\infty}\Bigg(&- n^2\sqrt{\frac{2}{\pi}}\int_0^\pi \phi^0(y)\sin ny dy \sin n^2 t \\ &+ \sqrt{\frac{2}{\pi}}\int_0^\pi \phi^1(y)\sin ny dy \cos n^2 t\Bigg)\cdot \sin nx. \end{aligned}$$

Easily we have

$$\begin{aligned} \|p\|^2_{L^2(\Omega)} =& \sum_{n=1}^{\infty}\left(\sqrt{\frac{2}{\pi}}\int_0^\pi \phi^0(y)\sin ny dy \cos n^2 t + \sqrt{\frac{2}{\pi}}\int_0^\pi \frac{1}{n^2}\phi^1(y)\sin ny dy \sin n^2 t\right)^2 \\ \le& 2\sum_{n=1}^{\infty}\Bigg[\left(\sqrt{\frac{2}{\pi}}\int_0^\pi \phi^0(y)\sin ny dy \cos n^2 t\right)^2 \\ &+ \left(\sqrt{\frac{2}{\pi}}\int_0^\pi \frac{1}{n^2}\phi^1(y)\sin ny dy \sin n^2 t\right)^2\Bigg] \\ \le& 2\left(\|\phi^0\|^2_{L^2(\Omega)} + \|\phi^1\|^2_{Y_{-2}}\right) = 4\bar{E}(0), \end{aligned}$$

$$\begin{aligned}\left\|\frac{\partial p}{\partial t}\right\|_{Y_{-2}}^2 =& \sum_{n=1}^{\infty}\frac{1}{n^4}\left(-n^2\sqrt{\frac{2}{\pi}}\int_0^{\pi}\phi^0(y)\sin ny dy \sin n^2 t\right. \\ &\left. + \sqrt{\frac{2}{\pi}}\int_0^{\pi}\phi^1(y)\sin ny dy \cos n^2 t\right)^2 \\ \leq& \, 2\sum_{n=1}^{\infty}\left[\left(\sqrt{\frac{2}{\pi}}\int_0^{\pi}\phi^0(y)\sin ny dy \sin n^2 t\right)^2\right. \\ &\left. + \frac{1}{n^4}\left(\sqrt{\frac{2}{\pi}}\int_0^{\pi}\phi^1(y)\sin ny dy \cos n^2 t\right)^2\right] \\ \leq& \, 2\left(\|\phi^0\|_{L^2(\Omega)}^2 + \|\phi^1\|_{Y_{-2}}^2\right) = 4\bar{E}(0).\end{aligned}$$

Thus

$$\|p\|_{L^2(\Omega)}^2 + \left\|\frac{\partial p}{\partial t}\right\|_{Y_{-2}}^2 \leq 8\bar{E}(0), \qquad \forall\, t \in [0,T].$$

Apply Lemma 3.1 to system (3.27)-(3.29) with $v = q$, $v^0 = v^1 = 0$ and $b = -ap$. We obtain

$$\begin{aligned}\|q\|_{Y_2}^2 + \left\|\frac{\partial q}{\partial t}\right\|_{L^2(\Omega)}^2 &\leq \|ap\|_{L^2(\Omega\times(0,T))}^2 e^{\left(1+2\sqrt{\|a\|_\infty}\right)t} \\ &\leq \|a\|^2\|p\|_{L^2(\Omega\times(0,T))}^2 e^{\left(1+2\sqrt{\|a\|_\infty}\right)t} \\ &\leq 4T\|a\|_\infty^2\bar{E}(0)e^{\left(1+2\sqrt{\|a\|_\infty}\right)t}.\end{aligned}$$

Choose $A^* = \max\{8, 4T\}$, we get

$$\bar{E}(t) \leq A^*\bar{E}(0)(1+\|a\|_\infty^2)e^{\left(1+2\sqrt{\|a\|_\infty}\right)t}, \qquad \forall\, t \in [0,T].$$

By the time-reversibility (Lemma 3.2) of system (3.5)-(3.7) we have

$$\bar{E}(0) \leq A^*\bar{E}(s)(1+\|a\|_\infty^2)e^{\left(1+2\sqrt{\|a\|_\infty}\right)s}, \qquad \forall\, s \in [0,T].$$

Therefore

$$\bar{E}(t) \le (A^*)^2 \bar{E}(s)(1+\|a\|_\infty^2)^2 e^{\left(1+2\sqrt{\|a\|_\infty}\right)(t+s)}$$
$$\le A\bar{E}(s)(1+\|a\|_\infty^2)^2 e^{\left(2+4\sqrt{\|a\|_\infty}\right)T}.$$

□

3.3 A Carleman estimate

To prove the observability inequality, we need to establish a Carleman estimate for the differential operator

$$\mathcal{L}w = w_{tt} + w_{xxxx}$$

on $Q = [0,T] \times \Omega$ where $\Omega = (0,\pi)$ and $T > 2\pi$ be given.

Since $T > 2\pi$, one can choose a $x_0 \in \mathbb{R} \setminus \Omega$ such that $T > 2R_1 > 2\pi$ where $R_1 = \max_{x\in\Omega} |x - x_0|$. Let $R_0 = \min_{x\in\Omega} |x - x_0|$, then $R_1 > R_0 > 0$. Since $T > 2R_1$, there exist constants $c \in (0, R_0/2)$ and $\alpha \in (0,1)$ such that

$$R_1^2 < c^2 + \alpha\left(\frac{T}{2}\right)^2. \tag{3.30}$$

In fact, one can choose $c = \epsilon R_1$ with ϵ being sufficiently small, and

$$\frac{4R_1^2(1-\epsilon^2)}{T^2} < \alpha < \frac{4R_1^2}{T^2}. \tag{3.31}$$

Define

$$a(t,s,x) = \frac{1}{2}\left[|x-x_0|^2 - \alpha\left(t-\frac{T}{2}\right)^2 - \alpha\left(s-\frac{T}{2}\right)^2\right],$$
$$\theta = e^{\lambda a(t,s,x)}. \tag{3.32}$$

Then

$$a_{xxx} = a_{xxxx} = 0, \quad a_{ts} = 0, \quad a_{tx} = 0, \quad a_{sx} = 0. \tag{3.33}$$

Let

$$\mathcal{P}u = u_{tt} + u_{ss} + u_{xxxx}.$$

We establish first the following inequality which play an essential role to derive the Carleman estimate.

Theorem 3.1. *Let $u \in C^4([0,T]\times[0,T]\times\Omega)$. There exist constants $\lambda_0 = \lambda_0(\Omega, T, x_0) > 0$ and $C_0 = C_0(\Omega, T, x_0, \lambda_0) > 0$ such that the following inequality holds*

$$\begin{aligned}\theta^2|\mathcal{P}u| &= \theta^2|u_{ss} + u_{tt} + u_{xxxx}|^2\\ &\geq C_0(\lambda^7 v^2 + \lambda^3 v_t^2 + \lambda^3 v_s^2 + \lambda^5 v_x^2 + \lambda^3 v_{xx}^2 + \lambda v_{xt}^2 + \lambda v_{xs}^2 + \lambda v_{xxx}^2)\\ &\quad + (M_3)_t + (M_4)_s + (M_5)_x, \qquad \forall \lambda \geq \lambda_0,\end{aligned} \tag{3.34}$$

where M_3, M_4, M_5 are given by

$$\begin{aligned}M_3 &= -2\lambda a_t v_t^2 + 2\lambda a_t v_s^2 - 2\lambda a_t v_{xx}^2 + 2\lambda a_t C_1 v_x^2 - 2\lambda a_t A_1 v^2\\ &\quad -4\lambda a_s v_s v_t - 8\lambda a_x v_{xxx} v_t + 2B_1 v_t v_x + 18\lambda^4 a_t a_x^2 a_{xx} v^2 - 18\lambda^3 a_x^2 a_{xx} v v_t,\\ M_4 &= -2\lambda a_s v_s^2 + 2\lambda a_s v_t^2 - 2\lambda a_s v_{xx}^2 + 2\lambda a_s C_1 v_x^2 - 2\lambda a_s A_1 v^2\\ &\quad -4\lambda a_t v_t v_s - 8\lambda a_x v_{xxx} v_s + 2B_1 v_s v_x + 18\lambda^4 a_s a_x^2 a_{xx} v^2 - 18\lambda^3 a_x^2 a_{xx} v v_s,\\ M_5 &= -4\lambda(a_t v_t v_{xxx} - a_t v_{tx} v_{xx}) - 4\lambda a_t C_1 v_t v_x - 4\lambda(a_s v_s v_{xxx} - a_s v_{sx} v_{xx})\\ &\quad -4\lambda a_s C_1 v_s v_x + 8\lambda a_x v_{xxt} v_t - 8\lambda a_{xx} v_{xt} v_t - 4\lambda a_x v_{xt}^2 + 8\lambda a_x v_{xxs} v_s\\ &\quad -8\lambda a_{xx} v_{xs} v_s - 4\lambda a_x v_{xs}^2 - 4\lambda a_x v_{xxx}^2 - 4\lambda a_x C_1 v_{xx}^2 + 4\lambda a_x A_1 v_x^2\\ &\quad -8\lambda a_x A_1 v v_{xx} + 8\lambda(a_x A_1)_x v v_x - 4\lambda(a_x A_1)_{xx} v^2 - B_1 v_t^2 - B_1 v_s^2\end{aligned}$$

$$+2B_1v_xv_{xxx} + B_{1xx}v_x^2 - 2B_{1x}v_xv_{xx} - B_1v_{xx}^2 + B_1C_1v_x^2$$

$$+A_1B_1v^2 + 36\lambda^4a_xa_x^2a_{xx}v_x^2 - 72\lambda^4a_x^3a_{xx}v_{xx}v + 72\lambda^4a_x^3a_{xx}v_xv$$

$$-36\lambda^4[(a_x^3a_{xx})_{xx}v^2]_x - 9\lambda^3B_1a_x^2a_{xx}v^2 - 18C_1\lambda^3a_x^2a_{xx}v_xv$$

$$-18\lambda^3a_x^2a_{xx}v_{xxx}v + 36\lambda^3a_xa_{xx}^2v_{xx}v + 18\lambda^3a_x^2a_{xx}v_{xx}v_x$$

$$-36\lambda^3a_{xx}^3v_xv - 36\lambda^3a_xa_{xx}^2v_x^2.$$

Proof. Let $v = \theta u$, then $u = \dfrac{1}{\theta}v = e^{-\lambda a(x,t,s)}v$. Then

$$\begin{aligned}
u_t &= \frac{1}{\theta}[v_t - \lambda a_tv],\\
u_{tt} &= \frac{1}{\theta}[v_{tt} - 2\lambda a_tv_t + (-\lambda a_{tt} + \lambda^2a_t^2)v],\\
u_s &= \frac{1}{\theta}[v_s - \lambda a_sv],\\
u_{ss} &= \frac{1}{\theta}[v_{ss} - 2\lambda a_sv_s + (-\lambda a_{ss} + \lambda^2a_s^2)v],\\
u_x &= \frac{1}{\theta}[v_x - \lambda a_xv],\\
u_{xx} &= \frac{1}{\theta}[v_{xx} - 2\lambda a_xv_x + (-\lambda a_{xx} + \lambda^2a_x^2)v],\\
u_{xxx} &= \frac{1}{\theta}[v_{xxx} - 3\lambda a_xv_{xx} + (3\lambda^2a_x^2 - 3\lambda a_{xx})v_x + (-\lambda^3a_x^3 + 3\lambda^2a_xa_{xx} - \lambda a_{xxx})v],\\
u_{xxxx} &= \frac{1}{\theta}[v_{xxxx} - 4\lambda a_xv_{xxx} + (6\lambda^2a_x^2 - 6\lambda a_{xx})v_{xx} + (-4\lambda^3a_x^3 + 12\lambda^2a_xa_{xx}\\
&\quad -4\lambda a_{xxx})v_x + (\lambda^4a_x^4 - 6\lambda^3a_x^2a_{xx} + 3\lambda^2a_{xx}^2 + 4\lambda^2a_xa_{xxx} - \lambda a_{xxxx})v].
\end{aligned}$$

Then we have

$$\theta^2|\mathcal{P}u|^2 = \theta^2(u_{ss} + u_{tt} + u_{xxxx})^2$$

$$= \big\{[v_{tt} - 2\lambda a_tv_t + (-\lambda a_{tt} + \lambda^2a_t^2)v] + [v_{ss} - 2\lambda a_sv_s + (-\lambda a_{ss} + \lambda^2a_s^2)v]$$

$$+[v_{xxxx} - 4\lambda a_sv_{xxx} + (6\lambda^2a_x^2 - 6\lambda a_{xx})v_{xx} + (-4\lambda^3a_x^3 + 12\lambda^2a_xa_{xx})v_x$$

$$+(\lambda^4a_x^4 - 6\lambda^3a_x^2a_{xx} + 3\lambda^2a_{xx}^2)v]\big\}^2$$

$$
\begin{aligned}
&= \big\{v_{tt} - 2\lambda a_t v_t + v_{ss} - 2\lambda a_s v_s + v_{xxxx} - 4\lambda a_x v_{xxx} + (6\lambda^2 a_x^2 - 6\lambda a_{xx}) v_{xx} \\
&\quad + (-4\lambda^3 a_x^3 + 12\lambda^2 a_x a_{xx}) v_x + [\lambda^4 a_x^4 - 6\lambda^3 a_x^2 a_{xx} + 3\lambda^2 a_{xx}^2 + (-\lambda a_{tt} + \lambda^2 a_t^2) \\
&\quad + (-\lambda a_{ss} + \lambda^2 a_s^2)] v\big\}^2 \\
&= (I_1 + I_2 + I_3)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= -2\lambda a_t v_t - 2\lambda a_s v_s - 4\lambda a_x v_{xxx} + B_1 v_x, \\
I_2 &= v_{tt} + v_{ss} + v_{xxxx} + C_1 v_{xx} + A_1 v, \\
I_3 &= -9\lambda^3 a_x^3 a_{xx} v, \\
A_1 &= \lambda^4 a_x^4 + 3\lambda^3 a_x^2 a_{xx} + 3\lambda^2 a_{xx}^2 - \lambda a_{tt} + \lambda^2 a_t^2 - \lambda a_{ss} + \lambda^2 a_s^2, \\
B_1 &= -4\lambda^3 a_x^3 + 12\lambda^2 a_x a_{xx}, \\
C_1 &= 6\lambda^2 a_x^2 - 6\lambda a_{xx}.
\end{aligned}
$$

Thus

$$
\theta^2 |\mathcal{P}u|^2 = (I_1 + I_2 + I_3)^2 \geq 2I_1 I_2 + 2I_1 I_3 + 2I_2 I_3. \tag{3.35}
$$

We first compute

$$
2I_1 I_2 = 2(-2\lambda a_t v_t - 2\lambda a_s v_s - 4\lambda a_x v_{xxx} + B_1 v_x)(v_{tt} + v_{ss} + v_{xxxx} + C_1 v_{xx} + A_1 v). \tag{3.36}
$$

By using (3.33) , we have

$$
\begin{aligned}
-4\lambda a_t v_t v_{tt} &= -2\lambda a_t (v_t^2)_t = 2\lambda a_{tt} v_t^2 - 2\lambda (a_t v_t^2)_t, \\
-4\lambda a_t v_t v_{ss} &= -2\lambda [a_{tt} v_s^2 - (a_t v_s^2)_t + (2a_t v_t v_s)_s], \\
-4\lambda a_t v_t v_{xxxx} &= 2\lambda [-2(a_t v_t v_{xxx} - a_t v_{tx} v_{xx})_x - (a_t v_{xx}^2)_t + a_{tt} v_{xx}^2],
\end{aligned}
$$

$$-4\lambda a_t v_t C_1 v_{xx} = -2\lambda[-2(a_t C_1)_x v_t v_x + (a_t C_1)_t v_x^2 - (a_t C_1 v_x^2)_t + (2a_t C_1 v_t v_x)_x],$$

$$-4\lambda a_t A_1 v_t v = 2\lambda[(a_t A_1)_t v^2 - (a_t A_1 v^2)_t],$$

$$-4\lambda a_s v_s v_{tt} = -2\lambda[a_{ss} v_t^2 - (a_s v_t^2)_s + (2a_s v_s v_t)_t],$$

$$-4\lambda a_s v_s v_{ss} = 2\lambda a_{ss} v_s^2 - 2\lambda(a_s v_s^2)_s,$$

$$-4\lambda a_s v_s v_{xxxx} = 2\lambda[-2(a_s v_s v_{xxx} - a_s v_{sx} v_{xx})_x - (a_s v_{xx}^2)_s + a_{ss} v_{xx}^2],$$

$$-4\lambda a_s C_1 v_s v_{xx} = -2\lambda[-2(a_s C_1)_x v_s v_x + (a_s C_1)_s v_x^2 - (a_s C_1 v_x^2)_s + (2a_s C_1 v_s v_x)_x],$$

$$-4\lambda a_s A_1 v_s v = 2\lambda[(a_s A_1)_s v^2 - (a_s A_1 v^2)_s],$$

$$-8\lambda a_x v_{xxx} v_{tt} = -2\lambda[4(a_x v_{xxx} v_t)_t - 4(a_x v_{xxt} v_t)_x + 4(a_{xx} v_{xt} v_t)_x$$

$$+2(a_x v_{xt}^2)_x - 6a_{xx}(v_{xt})^2],$$

$$-8\lambda a_x v_{xxx} v_{ss} = -2\lambda[4(a_x v_{xxx} v_s)_s - 4(a_x v_{xxs} v_s)_x + 4(a_{xx} v_{xs} v_s)_x$$

$$+2(a_x v_{xs}^2)_x - 6a_{xx}(v_{xs})^2],$$

$$-8\lambda a_x v_{xxx} v_{xxxx} = -4\lambda[(a_x v_{xxx}^2)_x - a_{xx} v_{xxx}^2],$$

$$-8\lambda a_x v_{xxx} C_1 v_{xx} = 4\lambda[(a_x C_1)_x v_{xx}^2 - (a_x C_1 v_{xx}^2)_x],$$

$$-8\lambda a_x v_{xxx} A_1 v = -4\lambda\big\{3(a_x A_1)_x v_x^2 - (a_x A_1)_{xxx} v^2 - (a_x A_1 v_x^2)_x$$

$$+2(a_x A_1 v v_{xx})_x - 2[(a_x A_1)_x v v_x]_x + [(a_x A_1)_{xx} v^2]_x\big\},$$

$$2B_1 v_x v_{tt} = -2B_{1t} v_t v_x + B_{1x} v_t^2 - (B_1 v_t^2)_x + 2(B_1 v_t v_x)_t,$$

$$2B_1 v_x v_{ss} = -2B_{1s} v_s v_x + B_{1x} v_s^2 - (B_1 v_s^2)_x + 2(B_1 v_s v_x)_s,$$

$$2B_1 v_x v_{xxxx} = 2(B_1 v_x v_{xxx})_x + 3B_{1x} v_{xx}^2 + (B_{1xx} v_x^2)_x - B_{1xxx} v_x^2$$

$$-2(B_{1x} v_x v_{xx})_x - (B_1 v_{xx}^2)_x,$$

$$2B_1 v_x C_1 v_{xx} = (B_1 C_1 v_x^2)_x - (B_1 C_1)_x v_x^2,$$

$$2B_1 v_x A_1 v = -(A_1 B_1)_x v^2 + (A_1 B_1 v^2)_x.$$

Next we compute $2I_1I_3$,

$$2I_1I_3 = 2(-2\lambda a_t v_t - 2\lambda a_s v_s - 4\lambda a_x v_{xxx} + B_1 v_x)(-9\lambda^3 a_x^2 a_{xx} v). \tag{3.37}$$

By using (3.33) , we have

$$-4\lambda a_t(-9\lambda^3 a_x^2 a_{xx}) v_t v = -2\lambda\Big\{ -9[\lambda^3 a_t a_x^2 a_{xx} v^2]_t + 9[\lambda^3 a_t a_x^2 a_{xx}]_t v^2 \Big\},$$

$$-4\lambda a_s(-9\lambda^3 a_x^2 a_{xx}) v_s v = -2\lambda\Big\{ -9[\lambda^3 a_s a_x^2 a_{xx} v^2]_s + 9[\lambda^3 a_s a_x^2 a_{xx}]_t v^2 \Big\},$$

$$\begin{aligned} -8\lambda a_x(-9\lambda^3 a_x^2 a_{xx}) v_{xxx} v = -4\lambda\Big\{ & [-9\lambda^3 a_x^3 a_{xx}]_{xxx} v^2 + 27[\lambda^3 a_x^3 a_{xx}]_x v_x^2 \\ & -9[\lambda^3 a_x^3 a_{xx} v_x^2]_x + 18[\lambda^3 a_x^3 a_{xx} v_{xx} v]_x \\ & -18[\lambda^3 a_x^3 a_{xx} v_x v]_x + 9[(\lambda^3 a_x^3 a_{xx})_{xx} v^2]_{xx} \Big\}, \end{aligned}$$

$$2B_1(-9\lambda^3 a_x^2 a_{xx}) v_x v = [B_1(-9\lambda^3 a_x^2 a_{xx}) v^2]_x - [B_1(-9\lambda^3 a_x^2 a_{xx})]_x v^2.$$

Next compute $2I_2I_3$,

$$2I_2I_3 = 2(v_{tt} + v_{ss} + v_{xxxx} + C_1 v_{xx} + A_1 v)(-9\lambda^3 a_x^2 a_{xx} v). \tag{3.38}$$

By using (3.33) , we have

$$2(-9\lambda^3 a_x^2 a_{xx}) v_{tt} v = -18[\lambda^3 a_x^2 a_{xx} v v_t]_t + 18(\lambda^3 a_x^2 a_{xx})_t v v_t + 18\lambda^3 a_x^2 a_{xx} v_t^2,$$

$$2(-9\lambda^3 a_x^2 a_{xx}) v_{ss} v = -18[\lambda^3 a_x^2 a_{xx}) v v_s]_s + 18(\lambda^3 a_x^2 a_{xx})_s v v_s + 18\lambda^3 a_x^2 a_{xx} v_s^2,$$

$$2C_1(-9\lambda^3 a_x^2 a_{xx}) v_{xx} v = -18[\lambda^3 C_1 a_x^2 a_{xx} v v_x]_x + 18[\lambda^3 C_1 a_x^2 a_{xx}]_x v v_x + 18\lambda^3 C_1 a_x^2 a_{xx} v_x^2,$$

$$2A_1(-9\lambda^3 a_x^2 a_{xx}) v^2 = -18\lambda^3 A_1 a_x^2 a_{xx} v^2,$$

$$\begin{aligned} 2(-9\lambda^3 a_x^2 a_{xx}) v_{xxxx} v = & -18(\lambda^3 a_x^2 a_{xx} v_{xxx} v)_x + 18(2\lambda^3 a_x a_{xx}^2 v_{xx} v)_x \\ & +18(\lambda^3 a_x^2 a_{xx} v_{xx} v_x)_x - 18(2\lambda^3 a_{xx}^3 v_x v)_x - 18(2\lambda^3 a_x a_{xx}^2 v_x^2)_x \\ & -18\lambda^3 a_x^2 a_{xx} v_{xx}^2 + 72\lambda^3 a_{xx}^3 v_x^2. \end{aligned}$$

Thus, it follows from (3.35)-(3.38) that

$$\theta^2|\mathcal{P}u|^2 \geq M_1 + M_2 + (M_3)_t + (M_4)_s + (M_5)_x, \tag{3.39}$$

where

$$\begin{aligned}
M_1 =\,& 2\lambda a_{tt}v_t^2 - 2\lambda a_{tt}v_s^2 + 2\lambda a_{tt}v_{xx}^2 - 2\lambda(a_tC_1)_tv_x^2 + 2\lambda(a_tA_1)_tv^2 \\
& -2\lambda a_{ss}v_t^2 + 2\lambda a_{ss}v_s^2 + 2\lambda a_{ss}v_{xx}^2 - 2\lambda(a_sC_1)_sv_x^2 + 2\lambda(a_sA_1)_sv^2 \\
& +12\lambda a_{xx}(v_{xt})^2 + 12\lambda a_{xx}(v_{xs})^2 + 4\lambda a_{xx}v_{xxx}^2 + 4\lambda(a_xC_1)_xv_{xx}^2 \\
& -12\lambda(a_xA_1)_xv_x^2 + 4\lambda(a_xA_1)_{xxx}v^2 + B_{1x}v_t^2 + B_{1x}v_s^2 + 3B_{1x}v_{xx}^2 \\
& -B_{1xxx}v_x^2 - (B_1C_1)_xv_x^2 - (A_1B_1)_xv^2 + 2\lambda[a_t(-9\lambda^3a_x^2a_{xx})]_tv^2 \\
& +2\lambda[a_s(-9\lambda^3a_x^2a_{xx})]_sv^2 - 4\lambda[a_x(-9\lambda^3a_x^2a_{xx})]_{xxx}v^2 \\
& +12\lambda[a_x(-9\lambda^3a_x^2a_{xx})]_xv_x^2 - [B_1(-9\lambda^3a_x^2a_{xx})]_xv^2 \\
& -2(-9\lambda^3a_x^2a_{xx})v_t^2 - 2(-9\lambda^3a_x^2a_{xx})v_s^2 - 18\lambda^3a_x^2a_{xx}v_{xx}^2 \\
& -2C_1(-9\lambda^3a_x^2a_{xx})v_x^2 + 2A_1(-9\lambda^3a_x^2a_{xx})v^2 + 72\lambda^3a_{xx}^3v_x^2, \\
M_2 =\,& 4\lambda(a_tC_1)_xv_tv_x + 4\lambda(a_sC_1)_xv_sv_x - 2B_{1t}v_tv_x - 2B_{1s}v_sv_x \\
& -4(-9\lambda^3a_x^2a_{xx})_tvv_t - 4(-9\lambda^3a_x^2a_{xx})_svv_s \\
& -4[C_1(-9\lambda^3a_x^2a_{xx})]_xvv_x, \\
M_3 =\,& -2\lambda a_tv_t^2 + 2\lambda a_tv_s^2 - 2\lambda a_tv_{xx}^2 + 2\lambda a_tC_1v_x^2 - 2\lambda a_tA_1v^2 \\
& -4\lambda a_sv_sv_t - 8\lambda a_xv_{xxx}v_t + 2B_1v_tv_x - 2\lambda a_t(-9\lambda^3a_x^2a_{xx})v^2 \\
& +2(-9\lambda^3a_x^2a_{xx})vv_t, \\
M_4 =\,& -2\lambda a_sv_s^2 + 2\lambda a_sv_t^2 - 2\lambda a_sv_{xx}^2 + 2\lambda a_sC_1v_x^2 - 2\lambda a_sA_1v^2 \\
& -4\lambda a_tv_tv_s - 8\lambda a_xv_{xxx}v_s + 2B_1v_sv_x - 2\lambda a_s(-9\lambda^3a_x^2a_{xx})v^2
\end{aligned}$$

$$+2(-9\lambda^3 a_x^2 a_{xx}) v v_s,$$

$$
\begin{aligned}
M_5 = & -4\lambda(a_t v_t v_{xxx} - a_t v_{tx} v_{xx}) - 4\lambda a_t C_1 v_t v_x - 4\lambda(a_s v_s v_{xxx} - a_s v_{sx} v_{xx}) \\
& -4\lambda a_s C_1 v_s v_x + 8\lambda a_x v_{xxt} v_t - 8\lambda a_{xx} v_{xt} v_t - 4\lambda a_x v_{xt}^2 \\
& +8\lambda a_x v_{xxs} v_s - 8\lambda a_{xx} v_{xs} v_s - 4\lambda a_x v_{xs}^2 - 4\lambda a_x v_{xxx}^2 \\
& -4\lambda a_x C_1 v_{xx}^2 + 4\lambda a_x A_1 v_x^2 - 8\lambda a_x A_1 v v_{xx} + 8\lambda (a_x A_1)_x v v_x \\
& -4\lambda (a_x A_1)_{xx} v^2 - B_1 v_t^2 - B_1 v_s^2 + 2B_1 v_x v_{xxx} + B_{1xx} v_x^2 - 2B_{1x} v_x v_{xx} \\
& -B_1 v_{xx}^2 + B_1 C_1 v_x^2 + A_1 B_1 v^2 - 4\lambda a_x(-9\lambda^3 a_x^2 a_{xx}) v_x^2 \\
& +8\lambda a_x(-9\lambda^3 a_x^2 a_{xx}) v_{xx} v - 8\lambda a_x(-9\lambda^3 a_x^2 a_{xx}) v_x v \\
& +4\lambda[(a_x(-9\lambda^3 a_x^2 a_{xx}))_{xx} v^2]_x + B_1(-9\lambda^3 a_x^2 a_{xx}) v^2 \\
& +2C_1(-9\lambda^3 a_x^2 a_{xx}) v_x v - 18\lambda^3 a_x^2 a_{xx} v_{xxx} v + 36\lambda^3 a_x a_{xx}^2 v_{xx} v \\
& +18\lambda^3 a_x^2 a_{xx} v_{xx} v_x - 36\lambda^3 a_{xx}^3 v_x v - 36\lambda^3 a_x a_{xx}^2 v_x^2.
\end{aligned}
$$

Rewrite M_1 as

$$
\begin{aligned}
M_1 = & D_1 v^2 + D_2 v_t^2 + D_3 v_s^2 + D_4 v_x^2 + D_5 v_{xx}^2 \\
& +12\lambda a_{xx}(v_{xt})^2 + 12\lambda a_{xx}(v_{xs})^2 + 4\lambda a_{xx} v_{xxx}^2,
\end{aligned}
\tag{3.40}
$$

where

$$
\begin{aligned}
D_1 = & 2\lambda(a_t A_1)_t + 2\lambda(a_s A_1)_s + 4\lambda(a_x A_1)_{xxx} - (A_1 B_1)_x \\
& +2\lambda[a_t(-9\lambda^3 a_x^2 a_{xx})]_t + 2\lambda[a_s(-9\lambda^3 a_x^2 a_{xx})]_s \\
& -4\lambda[a_x(-9\lambda^3 a_x^2 a_{xx})]_{xxx} - [B_1(-9\lambda^3 a_x^2 a_{xx})]_x, \\
& +2A_1(-9\lambda^3 a_x^2 a_{xx}), \\
D_2 = & 2\lambda a_{tt} - 2\lambda a_{ss} + B_{1x} - 2(-9\lambda^3 a_x^2 a_{xx}),
\end{aligned}
$$

$$D_3 = 2\lambda a_{ss} - 2\lambda a_{tt} + B_{1x} - 2(-9\lambda^3 a_x^2 a_{xx}),$$

$$D_4 = -2\lambda(a_t C_1)_t - 2\lambda(a_s C_1)_s - 12\lambda(a_x A_1)_x - B_{1xxx} - (B_1 C_1)_x$$

$$+12\lambda[a_x(-9\lambda^3 a_x^2 a_{xx})]_x - 2C_1(-9\lambda^3 a_x^2 a_{xx}) + 72\lambda^3 a_{xx}^3,$$

$$D_5 = 2\lambda a_{tt} + 2\lambda a_{ss} + 4\lambda(a_x C_1)_x + 3B_{1x} - 18\lambda^3 a_x^2 a_{xx}.$$

Rewrite M_2 as

$$M_2 = [4\lambda(a_t C_1)_x - 2B_{1t}]v_t v_x + [4\lambda(a_s C_1)_x - 2B_{1s}]v_s v_x$$

$$+[-2(-9\lambda^3 a_x^2 a_{xx})_t]vv_t + [-2(-9\lambda^3 a_x^2 a_{xx})_s]vv_s$$

$$+[-2(C_1(-9\lambda^3 a_x^2 a_{xx}))_x]vv_x.$$

By (3.32), we have

$$a_x = x - x_0, \qquad a_t = -\alpha\left(t - \frac{T}{2}\right), \qquad a_s = -\alpha\left(s - \frac{T}{2}\right),$$

$$a_{xx} = 1, \qquad a_{tt} = -\alpha, \qquad a_{ss} = -\alpha,$$

$$A_1 = \lambda^4(x - x_0)^4 + 3\lambda^3(x - x_0)^2 + 3\lambda^2 + 2\lambda\alpha$$

$$+\lambda^2\alpha^2\left(t - \frac{T}{2}\right)^2 + \lambda^2\alpha^2\left(s - \frac{T}{2}\right)^2,$$

$$C_1 = 6\lambda^2(x - x_0)^2 - 6\lambda,$$

$$B_1 = -4\lambda^3(x - x_0)^3 + 12\lambda^2(x - x_0).$$

Since $|x - x_0| > R_0$ on Ω, we have the following asymptotes,

$$D_1 = 10\lambda^7(x - x_0)^6 + O(\lambda^6),$$

$$D_2 = 6\lambda^3(x - x_0)^2 + O(\lambda^2),$$

$$D_3 = 6\lambda^3(x - x_0)^2 + O(\lambda^2),$$

$$D_4 = 60\lambda^5(x-x_0)^4 + O(\lambda^4),$$

$$D_5 = 42\lambda^3(x-x_0)^2 + O(\lambda^2).$$

Thus it follows from (3.40) that

$$\begin{aligned} M_1 = &\left[10\lambda^7(x-x_0)^6 + O(\lambda^6)\right] v^2 + \left[6\lambda^3(x-x_0)^2 + O(\lambda^2)\right]\left(v_t^2 + v_s^2\right) \\ &+ \left[60\lambda^5(x-x_0)^4 + O(\lambda^4)\right] v_x^2 + \left(42\lambda^3(x-x_0)^2 + O(\lambda^2)\right) v_{xx}^2 \\ &+12\lambda(v_{xt})^2 + 12\lambda(v_{xs})^2 + 4\lambda v_{xxx}^2. \end{aligned}$$

Note that

$$\begin{aligned} M_2 = &[4\lambda(a_tC_1)_x]v_tv_x + [2\lambda(a_sC_1)_x]v_sv_x + [-4(C_1(-9\lambda^3a_x^2a_{xx}))_x]vv_x \\ = &\left[-48\alpha\lambda^3\left(t-\frac{T}{2}\right)(x-x_0)\right]v_tv_x + \left[-48\alpha\lambda^3\left(s-\frac{T}{2}\right)(x-x_0)\right]v_sv_x \\ &+[108\lambda^5(x-x_0)^4 - 108\lambda^4(x-x_0)^2]_x vv_x \\ = &\left[-48\alpha\lambda^3\left(t-\frac{T}{2}\right)(x-x_0)\right]v_tv_x + \left[-48\alpha\lambda^3\left(s-\frac{T}{2}\right)(x-x_0)\right]v_sv_x \\ &+[432\lambda^5(x-x_0)^3 + O(\lambda^4)]vv_x \\ \geq &-12\alpha\lambda^3T[(x-x_0)^2v_t^2 + v_x^2] - 12\alpha\lambda^3T[(x-x_0)^2v_s^2 + v_x^2] \\ &-\lambda^5[1296(x-x_0)^2v^2 + 36(x-x_0)^4v_x^2]. \end{aligned}$$

Thus

$$\begin{aligned} M_1 + M_2 \geq &[10\lambda^7(x-x_0)^6 + O(\lambda^6)]v^2 \\ &+[(6-12\alpha T)\lambda^3(x-x_0)^2](v_t^2 + v_s^2) \\ &+[24\lambda^5(x-x_0)^4 + O(\lambda^4)]v_x^2 \\ &+[42\lambda^3(x-x_0)^2 + O(\lambda^2)]v_{xx}^2 + 12\lambda(v_{xt})^2 + 12\lambda(v_{xs})^2 + 4\lambda v_{xxx}^2. \end{aligned}$$

By (3.31), αT is small when T is sufficiently large. Then there exists a $C_0 > 0$, depending on x_0, Ω and T, such that

$$M_1 + M_2 \geq C_0(\lambda^7 v^2 + \lambda^3 v_t^2 + \lambda^3 v_s^2 + \lambda^5 v_x^2 + \lambda^3 v_{xx}^2 + \lambda v_{xt}^2 + \lambda v_{xs}^2 + \lambda v_{xxx}^2).$$

Thus, (3.34) follows from (3.39). $\square$

With Theorem 3.1, we are ready to introduce and prove a Carleman estimate for the beam equation with a potential,

$$\begin{cases} w_{tt} + w_{xxxx} = qw, \quad x \in \Omega, \ t > 0, \\ w(t,0) = w(t,\pi) = 0, \ \ t > 0, \\ w_{xx}(t,0) = w_{xx}(t,\pi) = 0, \ \ t > 0, \end{cases} \tag{3.41}$$

where $q \in L^\infty(Q)$.

Next we introduce the following notations.

Let $T_i = T/2 - \epsilon_i T$ and $T_i' = T/2 + \epsilon_i T$ for $i = 0, 1$ with $0 < \epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \frac{1}{2}$. Let $\mathcal{Q} = (0,T) \times (0,T) \times \Omega$ and $\mathcal{Q}^\omega = (0,T) \times (0,T) \times \omega$. Let $\mathcal{Q}_i = (T_i, T_i') \times (T_i, T_i') \times \Omega$ for $i = 0.1, 2, 3$. Let x_0, c and α be defined as before and satisfy (3.30). Let $a(t,s,x)$ and θ be defined in (3.32). For any $b > 0$, define

$$\mathcal{Q}(b) = \{(t,s,x) \in \mathbb{R} \times \mathbb{R} \times \Omega \mid a(t,s,x) > b^2\}.$$

By choosing $\epsilon_1 \in (0, 1/2)$ to be sufficiently close to $1/2$, we then have $\mathcal{Q}(c) \subset \mathcal{Q}_1$. Note that $\{T/2\} \times \{T/2\} \times \Omega \subset \mathcal{Q}(c)$. For any small $\epsilon > 0$, there exists $\epsilon_0 \in (0, \epsilon_1)$ such that

$$\mathcal{Q}_0 \subset \mathcal{Q}(c + 2\epsilon) \subset \mathcal{Q}(c + \epsilon) \subset \mathcal{Q}(c) \subset \mathcal{Q}_1. \tag{3.42}$$

Introduce the following transformation which plays a critical role in the sequel,

$$u(t,s,x) = \int_s^t w(\tau, x) d\tau, \tag{3.43}$$

where w is the weak solution of (3.41). Then u satisfies

$$\begin{cases} u_{tt} + u_{ss} + u_{xxxx} = \displaystyle\int_s^t q(\tau, x) u_t(\tau, s, x) d\tau, \\ u(t,s,0) = 0, \quad u(t,s,\pi) = 0, \\ u_{xx}(t,s,0) = 0, \quad u_{xx}(t,s,\pi) = 0. \end{cases} \tag{3.44}$$

We will prove the following Carleman estimate.

Theorem 3.2. *Let $\omega = (a,b) \subset (0,\pi) = \Omega$. Assume $T > 2\pi$ and $q \in L^\infty(Q)$. Suppose $u \in C^4(\mathcal{Q})$ satisfy (3.44). Then there exists a $\lambda^* > 0$, depending only on Ω, T and q, such that*

$$\begin{aligned} &\int_{\mathcal{Q}_0} \left(\lambda^7 u^2 + \lambda^3 u_t^2 + \lambda^3 u_s^2 + \lambda^5 u_x^2 + \lambda^3 u_{xx}^2 + \lambda u_{xt}^2 + \lambda u_{xs}^2 + \lambda u_{xxx}^2\right) dt ds dx \\ &\le D_1 e^{D_2 \lambda} \int_{\mathcal{Q}^\omega} (u_t^2 + u_s^2) dt ds dx \\ &\qquad + D_3 \int_{\mathcal{Q}_1} [u^2 + u_t^2 + u_s^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2] dt ds dx, \quad \lambda \ge \lambda^*, \end{aligned} \tag{3.45}$$

where $D_1 > 0$, $D_2 > 0$ and $D_3 > 0$ are constants depending on Ω, T and ω.

Proof. The proof of (3.45) is lengthy, and can be divided into several steps.

Step 1. Suppose $x_0 < 0$. Let $a < l_1 < b$ with l_1 being very close to b, and $\omega_1 = (0, l_1)$ and $\omega_2 = (0, b)$. Then $\omega_1 \subset \omega_2 \subset \Omega$ and $\omega \subset \omega_2$.

Define $\mu \in C^\infty(\Omega; [0,1])$ by

$$\mu(x) = \begin{cases} 1, & x \in \omega_1; \\ 0, & x \in \Omega \setminus \omega_2. \end{cases} \tag{3.46}$$

Define $\chi \in C^\infty(\mathcal{Q}; [0,1])$ by

$$\chi(t,s,x) = \begin{cases} 1, & (t,s,x) \in \mathcal{Q}(c + 2\epsilon); \\ 0, & (t,s,x) \in \mathcal{Q} \setminus \mathcal{Q}(c + \epsilon). \end{cases} \tag{3.47}$$

Let $y(t,s,x) = \chi(t,s,x)\mu(x)u(t,s,x)$. Then $y(t,s,x)$ satisfies

$$\begin{cases} y_{tt} + y_{ss} + y_{xxxx} = F(t,s,x), \\ y(t,s,0) = 0, \quad y_{xx}(t,s,0) = 0, \\ y(t,s,b) = 0, \quad y_x(t,s,b) = 0, \\ y_{xx}(t,s,b) = 0, \quad y_{xxx}(t,s,b) = 0, \end{cases} \tag{3.48}$$

where

$$F(t,s,x) = \chi\mu \int_s^t q(\tau,x)u_t(\tau,s,x)d\tau + (\mathcal{P}(\mu\chi))u + 2\mu\chi_t u_t + 2\mu\chi_s u_s$$
$$+4(\mu\chi)_{xxx}u_x + 6(\mu\chi)_{xx}u_{xx} + 4(\mu\chi)_x u_{xxx}.$$

Let

$$v(t,s,x) = e^{\lambda a(t,s,x)}y(t,s,x) = \theta y(t,s,x).$$

By applying Theorem 3.1 and integrating (3.34) over $\mathcal{Q}$, we have the following inequality,

$$\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \left(\lambda^7 v^2 + \lambda^3 v_t^2 + \lambda^3 v_s^2 + \lambda^5 v_x^2 + \lambda^3 v_{xx}^2 + \lambda v_{xt}^2 + \lambda v_{xs}^2 + \lambda v_{xxx}^2\right) dtdsdx$$
$$\leq C_0 \left[\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 |F(t,s,x)| dtdsdx - \int_{T_1}^{T_1'}\int_{T_1}^{T_1'} \left[M_5\Big|_{x=0}^{x=b}\right] dtds\right], \quad \lambda \geq \lambda_0, \tag{3.49}$$

where $C_0 > 0$ is a constant depending only on Ω, T and x_0; and M_5 is defined in Theorem 3.1. Note that, by (3.42) and (3.47),

$$\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \left[(M_3)_t + (M_4)_s\right] dtdsdx = 0.$$

Step 2. We now estimate

$$\int_{T_1}^{T_1'}\int_{T_1}^{T_1'} \left[M_5\Big|_{x=0}^{x=b}\right] dtds.$$

Note that M_5 can be rewritten as

$$M_5(t,s,x) = L_1(t,s,x) + L_2(t,s,x), \tag{3.50}$$

where L_1 and L_2 are given by

$$
\begin{aligned}
L_1 = & -4\lambda a_t v_t v_{xxx} - 4\lambda a_t C_1 v_t v_x - 4\lambda a_s v_s v_{xxx} - 4\lambda a_s C_1 v_s v_x \\
& +8\lambda a_x v_{xxt} v_t - 8\lambda a_{xx} v_{xt} v_t + 8\lambda a_x v_{xxs} v_s - 8\lambda a_{xx} v_{xs} v_s \\
& -8\lambda a_x A_1 v v_{xx} + 8\lambda (a_x A_1)_x - 4\lambda (a_x A_1)_{xx} v^2 - B_1 v_t^2 - B_1 v_s^2 + A_1 B_1 v^2 \\
& +8\lambda a_x (-9\lambda^3 a_x^2 a_{xx}) v_{xx} v - 8\lambda a_x (-9\lambda^3 a_x^2 a_{xx}) v_x v \\
& +4\lambda [(a_x(-9\lambda^3 a_x^2 a_{xx}))_{xx} v^2]_x + B_1(-9\lambda^3 a_x^2 a_{xx}) v^2 \\
& +2C_1(-9\lambda^3 a_x^2 a_{xx}) v_x v - 18\lambda^3 a_x^2 a_{xx} v_{xxx} v \\
& +36\lambda^3 a_x a_{xx}^2 v_{xx} v - 36\lambda^3 a_{xx}^3 v_x v, \\
L_2 = & -4a_x v_{xxx}^2 - 4\lambda a_x C_1 v_{xx}^2 + 4\lambda a_x A_1 v_x^2 + 2B_1 v_x v_{xxx} + B_{1xx} v_x^2 \\
& -2B_{1x} v_x v_{xx} - B_1 v_{xx}^2 + B_1 C_1 v_x^2 - 4\lambda a_x (-9\lambda^3 a_x^2 a_{xx}) v_x^2 \\
& +4\lambda a_t v_{tx} v_{xx} + 4\lambda a_s v_{sx} v_{xx} - 4\lambda a_x v_{xt}^2 - 4\lambda a_x v_{xs}^2 \\
& +18\lambda^3 a_x^2 a_{xx} v_{xx} v_x - 36\lambda^3 a_x a_{xx}^2 v_x^2.
\end{aligned}
$$

By the boundary conditions in (3.48), we have

$$
L_1(t,s,x)\Big|_{x=0}^{x=b} = 0, \quad L_2(t,s,b) = 0. \tag{3.51}
$$

Thus we only need to look at

$$
L_2(t,s,x)\Big|_{x=0}^{x=b} = -L_2(t,s,0). \tag{3.52}
$$

Note that

$$
\begin{aligned}
L_2\Big|_{x=0} = & \Big\{(-4\lambda a_x) v_{xxx}^2 + (-4\lambda a_x C_1 - B_1) v_{xx}^2 + (2B_1) v_x v_{xxx} \\
& +[4\lambda a_x A_1 + B_{1xx} + B_1 C_1 - 4\lambda a_x(-9\lambda^3 a_x^2 a_{xx})] v_x^2 - 2B_1 v_x v_{xx}
\end{aligned}
$$

$$+4\lambda a_t v_{tx} v_{xx} + 4\lambda a_s v_{sx} v_{xx} - 4\lambda a_x v_{xt}^2 - 4\lambda a_x v_{xs}^2$$

$$+18\lambda^3 a_x^2 a_{xx} v_{xx} v_x - 36\lambda^3 a_x a_{xx}^2 v_x^2 \Big\}\Big|_{x=0}.$$

Since $v_{xx}(t,s,0) = 2\lambda a_x(t,s,0) v_x(t,s,0)$, we have

$$\begin{aligned} L_2\Big|_{x=0} = & \Big\{(-4\lambda a_x) v_{xxx}^2 + 2B_1 v_x v_{xxx} + [4\lambda^2 a_x^2(-4\lambda a_x C_1 - B_1) + 36\lambda^4 a_x^3 a_{xx} \\ & +4\lambda a_x A_1 + B_{1xx} + B_1 C_1 - 4\lambda a_x(-9\lambda^3 a_x^2 a_{xx}) - 4\lambda a_x B_1 - 36\lambda^3 a_x a_{xx}^2] v_x^2 \\ & +8\lambda^2 a_t a_x v_{tx} v_x + 8\lambda^2 a_s a_x v_{sx} v_x - 4\lambda a_x v_{xt}^2 - 4\lambda a_x v_{xs}^2 \Big\}\Big|_{x=0} \\ = & \Big\{ -4\lambda(x-x_0) v_{xxx}^2 - [100\lambda^5 (x-x_0)^5 + O(\lambda^4)] v_x^2 - 8\alpha\lambda^2 t(x-x_0) v_x v_{tx} \\ & -8\alpha\lambda^2 s(x-x_0) v_x v_{sx} - 4\lambda(x-x_0) v_{xt}^2 - 4\lambda(x-x_0) v_{xs}^2 \\ & +2(-4\lambda^3(x-x_0)^3 + 12\lambda^2(x-x_0)) v_x v_{xxx} \Big\}\Big|_{x=0}. \end{aligned}$$

Then

$$\begin{aligned} L_2(t,s,0) = & \Big\{ -(x-x_0)\Big[4\lambda v_{xxx}^2 + [100\lambda^5(x-x_0)^4 + O(\lambda^4)] v_x^2 + 8\alpha\lambda^2 t v_x v_{tx} \\ & +8\alpha\lambda^2 s v_x v_{xs} + 4\lambda v_{xt}^2 + 4\lambda v_{xs}^2 + 2[4\lambda^3(x-x_0)^2 - 12\lambda^2] v_x v_{xxx}\Big]\Big\}\Big|_{x=0}. \end{aligned}$$

Note that

$$\begin{aligned} & 8\alpha\lambda^2\left(t - \frac{T}{2}\right) v_x v_{tx} + 4\lambda v_{xt}^2 \\ & \qquad = \left(2\sqrt{\lambda} v_{xt} + 2\alpha\lambda^{\frac{3}{2}}\left(t-\frac{T}{2}\right) v_x\right)^2 - 4\alpha^2\lambda^3\left(t-\frac{T}{2}\right)^2 v_x^2, \\ & 8\alpha\lambda^2\left(s - \frac{T}{2}\right) v_x v_{sx} + 4\lambda v_{xs}^2 \\ & \qquad = \left(2\sqrt{\lambda} v_{xs} + 2\alpha\lambda^{\frac{3}{2}}\left(s-\frac{T}{2}\right) v_x\right)^2 - 4\alpha^2\lambda^3\left(s-\frac{T}{2}\right)^2 v_x^2, \\ & 8\lambda^3(x-x_0)^2 v_x v_{xxx} + 4\lambda v_{xxx}^2 = \left[2\sqrt{\lambda} v_{xxx} + 2\lambda^{\frac{5}{2}}(x-x_0)^2 v_x\right]^2 - 4\lambda^5(x-x_0)^4 v_x^2. \end{aligned}$$

Since $x_0 < 0$, we have $L_2(t,s,0) \le 0$ for λ is sufficiently large, that is, $\lambda \ge \lambda_1$ for some $\lambda_1 > \lambda_0$. Therefore, by (3.50)-(3.52), we have

$$\int_{T_1}^{T_1'} \int_{T_1}^{T_1'} \left[M_5 \Big|_{x=0}^{x=b} \right] dtds \geq 0.$$

Hence, (3.49) becomes

$$\begin{aligned} &\int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \left(\lambda^7 v^2 + \lambda^3 v_t^2 + \lambda^3 v_s^2 + \lambda^5 v_x^2 + \lambda^3 v_{xx}^2 + \lambda v_{xt}^2 + \lambda v_{xs}^2 + \lambda v_{xxx}^2 \right) dtdsdx \\ &\leq C_0 \int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 |F(t,s,x)|^2 dtdsdx, \quad \lambda \geq \lambda_1. \end{aligned} \tag{3.53}$$

Note that $v(t,s,x) = \theta y(t,s,x)$. By using the Hölder inequality, we have

$$\begin{aligned} &\int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 \left(\lambda^7 y^2 + \lambda^3 y_t^2 + \lambda^3 y_s^2 + \lambda^5 y_x^2 + \lambda^3 y_{xx}^2 + \lambda y_{xt}^2 + \lambda y_{xs}^2 + \lambda y_{xxx}^2 \right) dtdsdx \\ &\leq C_1 \int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 |F(t,s,x)|^2 dtdsdx \quad \lambda \geq \lambda_1, \end{aligned} \tag{3.54}$$

where $C_1 > 0$ is a constant depending only on Ω, x_0 and T.

Step 3. Next, we estimate

$$\int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 |F(t,s,x)|^2 dtdsdx.$$

Note that

$$\begin{aligned} &\int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 |F(t,s,x)|^2 dtdsdx \\ &= \int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 \Big[\mu\chi \int_s^t q(\tau,x) u_t(\tau,s,x) d\tau + (\mathcal{P}(\mu\chi))u + 2\mu\chi_t u_t + 2\mu\chi_s u_s \\ &\quad + 4(\mu\chi)_{xxx} u_x + 6(\mu\chi)_{xx} u_{xx} + 4(\mu\chi)_x u_{xxx} \Big]^2 dtdsdx \\ &\leq 2 \int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 \Big| \mu\chi \int_s^t q(\tau,x) u_t(\tau,s,x) d\tau \Big|^2 dtdsdx + 2 \int_{\mathcal{Q}_1 \bigcap \{x \in \omega_2\}} \theta^2 \Big[(\mathcal{P}(\mu\chi))u \\ &\quad + 2\mu\chi_t u_t + 2\mu\chi_s u_s + 4(\mu\chi)_{xxx} u_x + 6(\mu\chi)_{xx} u_{xx} + 4(\mu\chi)_x u_{xxx} \Big]^2 dtdsdx \\ &= 2I_1 + 2I_2. \end{aligned} \tag{3.55}$$

We then estimate I_1 and I_2 separately.

$$\begin{aligned}
I_1 \leq& \int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_s^t q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
=& \int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_s^{T/2} q(\tau,x)u_t(\tau,s,x)d\tau + \int_{T/2}^t q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
\leq& 2\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^s q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
&+2\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^t q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
=& 2\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^s q(\tau,x)u_s(t,\tau,x)d\tau\right)^2 dtdsdx \\
&+2\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^t q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx,
\end{aligned}$$

where we have used $u_t(\tau,s,x) = -u_s(t,\tau,x)$ by (3.43). Note that $r = \|q\|_{L^\infty(\mathcal{Q})}$. Then

$$\begin{aligned}
&\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^t q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
&= \int_{\omega_2}\int_{T_1}^{T_1'}\int_{T_1}^{T/2} \theta^2 \left(\int_t^{T/2} q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
&\quad + \int_{\omega_2}\int_{T_1}^{T_1'}\int_{T/2}^{T_1'} \theta^2 \left(\int_{T/2}^t q(\tau,x)u_t(\tau,s,x)d\tau\right)^2 dtdsdx \\
&\leq \frac{T}{2}\int_{\omega_2}\int_{T_1}^{T_1'}\int_{T_1}^{T/2} \theta^2 \int_t^{T/2} \left(q(\tau,x)u_t(\tau,s,x)\right)^2 d\tau dtdsdx \\
&\quad +\frac{T}{2}\int_{\omega_2}\int_{T_1}^{T_1'}\int_{T/2}^{T_1'} \theta^2 \int_{T/2}^t \left(q(\tau,x)u_t(\tau,s,x)\right)^2 d\tau dtdsdx \\
&\leq \frac{Tr^2}{2}\int_{\omega_2}\int_{T_1}^{T_1'}\int_{T_1}^{T/2} \theta^2 \int_t^{T/2} u_t^2(\tau,s,x)d\tau dtdsdx \\
&\quad +\frac{Tr^2}{2}\int_{\omega_2}\int_{T_1}^{T_1'}\int_{T/2}^{T_1'} \theta^2 \int_{T/2}^t u_t^2(\tau,s,x)d\tau dtdsdx \\
&\leq \frac{Tr^2}{2}\int_{\omega_2}\int_{T_1}^{T_1'}\int_{T_1}^{T/2}\int_t^{T/2} \theta^2 u_t^2(\tau,s,x)d\tau dtdsdx \\
&\quad +\frac{Tr^2}{2}\int_{\omega_2}\int_{T_1}^{T_1'}\int_{T/2}^{T_1'}\int_{T/2}^t \theta^2 u_t^2(\tau,s,x)d\tau dtdsdx,
\end{aligned}$$

where we have used $0 < \theta(t,s,x) \leq \theta(\tau,s,x)$ when $T_1 \leq t \leq \tau \leq T/2$ or $T/2 \leq \tau \leq t \leq T_1'$. Then, by interchanging the integration order of t and τ, we have

$$\begin{aligned}
&\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^{t} q(\tau,x)u_t(\tau,s,x)d\tau \right)^2 dtdsdx \\
&\leq \frac{T^2r^2}{4} \int_{\omega_2} \int_{T_1}^{T_1'} \int_{T_1}^{T/2} \theta^2 u_t^2(\tau,s,x)d\tau dsdx \\
&\quad + \frac{T^2r^2}{4} \int_{\omega_2} \int_{T_1}^{T_1'} \int_{T/2}^{T_1'} \theta^2 u_t^2(\tau,s,x)d\tau dsdx, \\
&= \frac{T^2r^2}{4} \int_{\omega_2} \int_{T_1}^{T_1'} \int_{T_1}^{T_1'} \theta^2 u_t^2(t,s,x)dtdsdx \\
&= \frac{T^2r^2}{4} \int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 u_t^2 dtdsdx.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \left(\int_{T/2}^{s} q(\tau,x)u_s(t,\tau,x)d\tau \right)^2 dtdsdx \\
\leq \frac{T^2r^2}{4} \int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 u_s^2 dtdsdx.
\end{aligned}$$

Thus, by (3.30) and (3.42), we have

$$\begin{aligned}
I_1 &\leq \frac{T^2r^2}{2} \int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2(u_t^2+u_s^2)dtdsdx \\
&= \frac{T^2r^2}{2} \left[\int_{\mathcal{Q}_1 \bigcap \{x\in\omega\}} \theta^2(u_t^2+u_s^2)dtdsdx + \int_{T_1}^{T_1'} \int_{T_1}^{T_1'} \int_0^a \theta^2(u_t^2+u_s^2)dtdsdx \right] \\
&\leq \frac{T^2r^2}{2} \left[\int_{\mathcal{Q}^\omega} \theta^2(u_t^2+u_s^2)dtdsdx + e^{\lambda(c+2\epsilon)^2} \int_{T_1}^{T_1'} \int_{T_1}^{T_1'} \int_0^a (u_t^2+u_s^2)dtdsdx \right] \\
&\leq \frac{T^2r^2}{2} \left[e^{C_2\lambda} \int_{\mathcal{Q}^\omega} (u_t^2+u_s^2)dtdsdx + e^{\lambda(c+2\epsilon)^2} \int_{\mathcal{Q}_1} (u_t^2+u_s^2)dtdsdx \right], \qquad (3.56)
\end{aligned}$$

where C_2 is a constant depending on Ω, ω, T and x_0. By using (3.46) and (3.47), we estimate I_2 which is defined in (3.55).

$$I_2 = \int_{\mathcal{Q}_1 \bigcap \{x\in\omega_2\}} \theta^2 \Big[(\mathcal{P}(\mu\chi))u + 2\mu\chi_t u_t + 2\mu\chi_s u_s + 4(\mu\chi)_{xxx}u_x$$

$$+6(\mu\chi)_{xx}u_{xx}+4(\mu\chi)_x u_{xxx}\Big]^2 dtdsdx$$

$$= \int_{(\mathcal{Q}(c+\epsilon)\setminus\mathcal{Q}(c+2\epsilon))\bigcap\{x\in\omega_2\}} \theta^2\Big[(\mathcal{P}(\mu\chi))u+2\mu\chi_t u_t+2\mu\chi_s u_s+4(\mu\chi)_{xxx}u_x$$

$$+6(\mu\chi)_{xx}u_{xx}+4(\mu\chi)_x u_{xxx}\Big]^2 dtdsdx$$

$$\le C_3\int_{(\mathcal{Q}(c+\epsilon)\setminus\mathcal{Q}(c+2\epsilon))\bigcap\{x\in\omega_2\}} \theta^2[u^2+u_t^2+u_s^2+u_x^2+u_{xx}^2+u_{xxx}^2]dtdsdx$$

$$\le C_3 e^{\lambda(c+2\epsilon)^2}\int_{(\mathcal{Q}(c+\epsilon)\setminus\mathcal{Q}(c+2\epsilon))\bigcap\{x\in\omega_2\}}[u^2+u_t^2+u_s^2+u_x^2+u_{xx}^2+u_{xxx}^2]dtdsdx$$

$$\le C_3 e^{\lambda(c+2\epsilon)^2}\int_{\mathcal{Q}_1}[u^2+u_t^2+u_s^2+u_x^2+u_{xx}^2+u_{xxx}^2]dtdsdx, \tag{3.57}$$

where $C_3>0$ depends on l_2, Ω, ω.

Step 4. By using (3.55)-(3.57), it follows from (3.54) that

$$\int_{\mathcal{Q}_1\bigcap\{x\in\omega_2\}}\theta^2\left(\lambda^7y^2+\lambda^3y_t^2+\lambda^3y_s^2+\lambda^5y_x^2+\lambda^3y_{xx}^2+\lambda y_{xt}^2+\lambda y_{xs}^2+\lambda y_{xxx}^2\right)dtdsdx$$

$$\le C_4e^{C_2\lambda}\int_{\mathcal{Q}^\omega}(u_t^2+u_s^2)dtdsdx$$

$$+C_5e^{\lambda(c+2\epsilon)^2}\int_{\mathcal{Q}_1}[u^2+u_t^2+u_s^2+u_x^2+u_{xx}^2+u_{xxx}^2]dtdsdx,\quad \lambda\ge\lambda_1, \tag{3.58}$$

where C_4 and C_5 are constants depending on Ω, ω and T and r. By (3.42), (3.46) and (3.47), we have

$$\int_{\mathcal{Q}_1\bigcap\{x\in\omega_2\}}\theta^2\left(\lambda^7y^2+\lambda^3y_t^2+\lambda^3y_s^2+\lambda^5y_x^2+\lambda^3y_{xx}^2+\lambda y_{xt}^2+\lambda y_{xs}^2+\lambda y_{xxx}^2\right)dtdsdx$$

$$=\int_{\mathcal{Q}(c+\epsilon)\bigcap\{x\in\omega_2\}}\theta^2\big(\lambda^7y^2+\lambda^3y_t^2+\lambda^3y_s^2+\lambda^5y_x^2$$

$$+\lambda^3y_{xx}^2+\lambda y_{xt}^2+\lambda y_{xs}^2+\lambda y_{xxx}^2\big)dtdsdx$$

$$\ge e^{\lambda(c+2\epsilon)^2}\int_{\mathcal{Q}(c+2\epsilon)\bigcap\{x\in\omega_1\}}\big(\lambda^7u^2+\lambda^3u_t^2+\lambda^3u_s^2+\lambda^5u_x^2$$

$$+\lambda^3u_{xx}^2+\lambda u_{xt}^2+\lambda u_{xs}^2+\lambda u_{xxx}^2\big)dtdsdx$$

$$\ge e^{\lambda(c+2\epsilon)^2}\int_{\mathcal{Q}_0\bigcap\{x\in\omega_1\}}\big(\lambda^7u^2+\lambda^3u_t^2+\lambda^3u_s^2+\lambda^5u_x^2$$

$$+\lambda^3u_{xx}^2+\lambda u_{xt}^2+\lambda u_{xs}^2+\lambda u_{xxx}^2\big)dtdsdx.$$

Thus it follows from (3.58) that

$$\int_{\mathcal{Q}_0 \bigcap \{x\in\omega_1\}} \left(\lambda^7 u^2 + \lambda^3 u_t^2 + \lambda^3 u_s^2 + \lambda^5 u_x^2 + \lambda^3 u_{xx}^2 + \lambda u_{xt}^2 + \lambda u_{xs}^2 + \lambda u_{xxx}^2\right) dtdsdx$$
$$\leq C_4 e^{C_2\lambda} \int_{\mathcal{Q}^\omega} (u_t^2 + u_s^2) dtdsdx$$
$$+C_5 \int_{\mathcal{Q}_1} [u^2 + u_t^2 + u_s^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2] dtdsdx, \quad \lambda \geq \lambda_1. \tag{3.59}$$

Step 5. Now, suppose $x_0 > \pi$. Let $a < l_2 < l_1 < b$ with l_2 being very close to a, and $\omega_3 = (l_2, \pi)$ and $\omega_4 = (a, \pi)$. Then $\omega_3 \subset \omega_4 \subset \Omega$ and $\omega \subset \omega_4$. By repeat the procedure similar to Steps 1-4, we obtain that, there exists a $\lambda_2 > 0$, depending on Ω, ω, x_0 and T, such that

$$\int_{\mathcal{Q}_0 \bigcap \{x\in\omega_3\}} \left(\lambda^7 u^2 + \lambda^3 u_t^2 + \lambda^3 u_s^2 + \lambda^5 u_x^2 + \lambda^3 u_{xx}^2 + \lambda u_{xt}^2 + \lambda u_{xs}^2 + \lambda u_{xxx}^2\right) dtdsdx$$
$$\leq C_6 e^{C_7\lambda} \int_{\mathcal{Q}^\omega} (u_t^2 + u_s^2) dtdsdx$$
$$+C_8 \int_{\mathcal{Q}_1} [u^2 + u_t^2 + u_s^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2] dtdsdx, \quad \lambda \geq \lambda_2, \tag{3.60}$$

where C_6, C_7 and C_8 are constants depending on Ω, ω, x_0 and T. By letting $\lambda^* = \max\{\lambda_1, \lambda_2\} > 0$, $C_9 = \max\{C_2, C_7\}$, $C_{10} = 2\max\{C_4, C_6\}$, $C_{11} = 2\max\{C_5, C_8\}$, it follows from (3.59) and (3.60) that

$$\int_{\mathcal{Q}_0} \left(\lambda^7 u^2 + \lambda^3 u_t^2 + \lambda^3 u_s^2 + \lambda^5 u_x^2 + \lambda^3 u_{xx}^2 + \lambda u_{xt}^2 + \lambda u_{xs}^2 + \lambda u_{xxx}^2\right) dtdsdx$$
$$\leq C_{10} e^{C_9\lambda} \int_{\mathcal{Q}^\omega} (u_t^2 + u_s^2) dtdsdx$$
$$+C_{11} \int_{\mathcal{Q}_1} [u^2 + u_t^2 + u_s^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2] dtdsdx, \quad \lambda \geq \lambda^*. \tag{3.61}$$

□

3.4 Observability Inequality

Now we are ready to show the observability inequality which is the key step of proving exact controllability. Consider

$$\begin{cases} w_{tt} + w_{xxxx} = qw, \quad x \in \Omega,\ t > 0, \\ w(t,0) = w(t,\pi) = 0,\ \ t > 0, \\ w_{xx}(t,0) = w_{xx}(t,\pi) = 0,\ \ t > 0, \end{cases} \tag{3.62}$$

where $q \in L^\infty(Q)$.

Theorem 3.3. *Let $\omega = (a,b) \subseteq \Omega$ and $0 < t_1 < t_2 < T$. For every solution $w \in C(0,T;L^2(\Omega))$ satisfies (3.62), there exists a constant $C > 0$ such that for $T > 2\pi$*

$$\int_{t_1}^{t_2}\int_\Omega w^2 dxdt \le C\int_{Q^\omega} w^2 dtdx,$$

where C depends only on $r = \|q\|_\infty$, T, t_1, t_2, ω and Ω.

To prove this theorem, we need the following lemmas.

Lemma 3.5. *Let $w \in C^4(Q)$ be a solution of (3.62). Define a new energy function $G(t)$ by*

$$G(t) = \frac{1}{2}\int_\Omega [w^2 + w_t^2 + w_{xx}^2]dx.$$

Then

$$G(t) \le G(s)e^{2T(1+r)}, \qquad \forall\, t,\ s \in [0,T],$$

where $r = \|q\|_\infty$.

Proof. By integration by parts, we have

$$\begin{aligned} \frac{dG}{dt} &= \int_\Omega [ww_t + w_t w_{tt} + w_{xx}w_{xxt}]dx \\ &= \int_\Omega [ww_t + w_t w_{tt} + w_{xxxx}w_t]dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega [ww_t + q(x,t)w_t w]dx \\
&\le (1+r) \int_\Omega (w^2 + w_t^2)dx \\
&\le (1+r)G(t).
\end{aligned}
$$

Thus, by the Gronwall's inequality, we have

$$
G(t) \le e^{(1+r)t}G(0), \qquad t \in [0,T].
$$

By the time reversibility, we then have

$$
G(0) \le e^{(1+r)s}G(s), \qquad \forall\, s \in [0,T]. \tag{3.63}
$$

Thus

$$
G(t) \le e^{2(1+r)T}G(s), \qquad \forall\, t,\, s \in [0,T]. \tag{3.64}
$$

□

Lemma 3.6. *Let $w \in C^4(Q)$ be a solution of* (3.62). *Let $0 < t_1 < s_1 < s_2 < t_2 < T$ be given. Then there exists a constant $C > 0$ such that*

$$
\int_{s_1}^{s_2} \int_\Omega w_t^2 dxdt \le C \int_{t_1}^{t_2} \int_\Omega [w^2 + w_{xx}^2]dxdt,
$$

where C depends only on $r = \|q\|_\infty$, T, t_1, t_2, s_1 and s_2.

Proof. Define $\mu \in C^\infty(\mathbb{R}; [0,1])$ by

$$
\mu(t) = \begin{cases} 1, & s_1 \le t \le s_2, \\ 0, & \mathbb{R}\backslash(t_1, t_2). \end{cases}
$$

By multiplying (3.62) by $\mu(t)w$ and integrating over $(t_1, t_2) \times \Omega$ we have

$$
\int_{t_1}^{t_2} \int_\Omega \mu(t) w[w_{tt} + w_{xxxx}]dxdt = \int_{t_1}^{t_2} \int_\Omega q\mu(t)w^2 dxdt.
$$

Then, by integration by parts, we have

$$\int_{t_1}^{t_2}\int_\Omega [\mu'(t)ww_t - \mu(t)w_t^2 + \mu(t)w_{xx}^2]dxdt = \int_{t_1}^{t_2}\int_\Omega q\mu(t)w^2dxdt.$$

Then

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \mu(t)w_t^2dtdx &= \int_{t_1}^{t_2}\int_\Omega [q\mu(t)w_{xx}^2 - \mu(t)w^2 - \mu'(t)ww_t]dxdt\\ &= \int_{t_1}^{t_2}\int_\Omega \left[q\mu(t)w_{xx}^2 - \mu(t)w^2 + \frac{1}{2}\mu''(t)w^2\right]dxdt\\ &\leq C\int_{t_1}^{t_2}\int_\Omega [w^2 + w_{xx}^2]dxdt,\end{aligned}$$

where C depend on r and T. By the definition of $\mu(t)$, we have

$$\int_{s_1}^{s_2}\int_\Omega w_t^2dxdt \leq C\int_{t_1}^{t_2}\int_\Omega [w^2 + w_{xx}^2]dxdt.$$

□

Proof of Theorem 3.3.

Let $w(x,t) \in C^4(Q)$ be the solution of system (3.62) with initial data $\{w^0, w^1\}$ $\in Y_4 \times Y_2$. Let $u(t,s,x) = \int_s^t w(\tau,x)d\tau$. u satisfies (3.44). By Theorem 3.2, we have

$$\begin{aligned}&\int_{\mathcal{Q}_0}\left(\lambda^7u^2 + \lambda^3u_t^2 + \lambda^3u_s^2 + \lambda^5u_x^2 + \lambda^3u_{xx}^2 + \lambda u_{xt}^2 + \lambda u_{xs}^2 + \lambda u_{xxx}^2\right)dtdsdx\\ &\leq D_1e^{D_2\lambda}\int_{\mathcal{Q}^\omega}(u_t^2 + u_s^2)dtdsdx\\ &\qquad + D_3\int_{\mathcal{Q}_1}[u^2 + u_t^2 + u_s^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2]dtdsdx, \quad \lambda \geq \lambda^*.\end{aligned}$$

Next, we estimate $\int_{\mathcal{Q}_1} u_{xxx}^2dtdsdx$. Let $\epsilon_2 > 0$ satisfies $0 < \epsilon_1 < \epsilon_2 < \frac{1}{2}$, $T_2 = \frac{T}{2} - \epsilon_2T$, $T_2' = \frac{T}{2} + \epsilon_2T$, then $0 < T_2 < T_1 < T_1' < T_2' < T$. Define

$$h(t,s) = (t - T_2)(T_2' - t)(s - T_2)(T_2' - s).$$

By multiplying (3.44) by hu_{xx}, and integrating both sides of the equation over $\mathcal{Q}_2$, we then have

$$\begin{aligned}\int_{\mathcal{Q}_2} hu_{xxx}^2 dtdsdx = &\int_{\mathcal{Q}_2} [h_t u_{tx} u_x + h_s u_{sx} u_x + hu_{tx}^2 + hu_{sx}^2] dtdsdx \\ &- \int_{\mathcal{Q}_2} hu_{xx} [\int_s^t q(\tau, x) u_t(\tau, s, x) d\tau] dtdsdx \\ \leq &C_{12} \int_{\mathcal{Q}_2} [u_t^2 + u_s^2 + u_x^2 + u_{tx}^2 + u_{sx}^2 + u_{xx}^2] dtdsdx.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathcal{Q}_1} u_{xxx}^2 dtdsdx \leq &\frac{1}{(\epsilon_2 - \epsilon_1)^4 T^4} \int_{\mathcal{Q}_1} hu_{xxx}^2 dtdsdx \\ \leq &\frac{1}{(\epsilon_2 - \epsilon_1)^4 T^4} \int_{\mathcal{Q}_2} hu_{xxx}^2 dtdsdx \\ \leq &\frac{C_{12}}{(\epsilon_2 - \epsilon_1)^4 T^4} \int_{\mathcal{Q}_2} [u_t^2 + u_s^2 + u_x^2 + u_{tx}^2 + u_{sx}^2 + u_{xx}^2] dtdsdx. \quad (3.65)\end{aligned}$$

Then it follows from (3.61) that

$$\begin{aligned}&\int_{\mathcal{Q}_0} \left(\lambda^7 u^2 + \lambda^3 u_t^2 + \lambda^3 u_s^2 + \lambda^5 u_x^2 + \lambda^3 u_{xx}^2 + \lambda u_{xt}^2 + \lambda u_{xs}^2 + \lambda u_{xxx}^2\right) dtdsdx \\ \leq &C_{10} e^{C_9 \lambda} \int_{\mathcal{Q}^\omega} (u_t^2 + u_s^2) dtdsdx \\ &+C_{13} \int_{\mathcal{Q}_2} [u^2 + u_t^2 + u_s^2 + u_x^2 + u_{xt}^2 + u_{xs}^2 + u_{xx}^2] dtdsdx, \quad \lambda \geq \lambda^*. \quad (3.66)\end{aligned}$$

By the definition of u, denote $Q^\omega = (0, T) \times \omega$, we then have

$$\begin{aligned}&\int_{Q_0} \left(\lambda^3 w^2 + \lambda^5 w_x^2 + \lambda^3 w_{xx}^2 + \lambda w_{xxx}^2\right) dtdx \\ \leq &C_{10} e^{C_9 \lambda} \int_{Q^\omega} w^2 dtdx + C_{14} \int_{Q_2} [w^2 + w_x^2 + w_{xx}^2] dtdsdx, \quad \lambda \geq \lambda^*, \quad (3.67)\end{aligned}$$

where $C_{13} > 0$ and $C_{14} > 0$ are constants depending on Ω and T.

By the interpolation inequality (Lemma 4.10 in [Adams (1978)]), then exist constants $C_{15} > 0$ and $C_{16} > 0$ such that

$$\int_{Q_2} w_x^2 dtdx \leq C_{15} \int_{Q_2} w^2 dtdx + C_{16} \int_{Q_2} w_{xx}^2 dtdx. \quad (3.68)$$

Then

$$\int_{Q_0} \left(\lambda^3 w^2 + \lambda^5 w_x^2 + \lambda^3 w_{xx}^2 + \lambda w_{xxx}^2\right) dtdx$$
$$\le C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + C_{17} \int_{Q_2} [w^2 + w_{xx}^2] dtdsdx, \quad \lambda \ge \lambda^*.$$

Then

$$\lambda^3 \int_{Q_0} \left(w^2 + w_t^2 + w_{xx}^2\right) dtdx$$
$$\le C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + \lambda^3 \int_{Q_0} w_t^2 dxdt$$
$$+ C_{17} \int_{Q_2} [w^2 + w_{xx}^2] dtdsdx, \quad \lambda \ge \lambda^*. \tag{3.69}$$

By Lemma 3.6, we have

$$\begin{aligned}\lambda^3 \int_{Q_0} (w^2 + w_t^2 + w_{xx}^2) dtdx &\le C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + C\lambda^3 \int_{Q_2} (w^2 + w_{xx}^2) dxdt \\ &\quad + C_{17} \int_{Q_2} (w^2 + w_{xx}^2) dxdt \\ &= C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + C_{18} \int_{Q_2} (w^2 + w_{xx}^2) \\ &\le C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + C_{18} \int_{Q_2} (w^2 + w_t^2 + w_{xx}^2) dxdt,\end{aligned}$$

where $C_{18} = C\lambda^3 + C_{17}$, and C is the constant is the constant in Lemma 3.6.

Thus

$$2\lambda^3 \int_{T_0}^{T_0'} G(t) dt \le C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + 2C_{18} \int_{T_2}^{T_2'} G(t) dt,$$

where $G(t)$ is defined in Lemma 3.5. By Lemma 3.5, we have

$$2\lambda^3 (T_0' - T_0) e^{-2(1+r)T} G(s)$$
$$\le \quad C_{10} e^{C_9\lambda} \int_{Q^\omega} w^2 dtdx + 2C_{18} (T_2' - T_2) e^{2(1+r)T} G(s), \ \forall\, s \in [0, T].$$

Then by chooseing λ large enough, say $\lambda > \lambda_3$, we then have

$$G(s) \le \frac{C_{10} e^{C_9\lambda}}{2\lambda^3 (T_0' - T_0) e^{-2(1+r)T} - 2C_{18} (T_2' - T_2) e^{2(1+r)T}} \int_{Q^\omega} w^2 dtdx = C_{19} \int_{Q^\omega} w^2 dtdx.$$

Then for any $0 < t_1 < t_2 < T$,

$$\int_{t_1}^{t_2} G(t)dt \leq TC_{19} \int_{Q^\omega} w^2 dtdx,$$

that is,

$$\int_{t_1}^{t_2} \int_\Omega (w^2 + w_t^2 + w_{xx}^2) dxdt \leq TC_{19} \int_{Q^\omega} w^2 dtdx.$$

Thus

$$\int_{t_1}^{t_2} \int_\Omega w^2 dxdt \leq TC_{19} \int_{Q^\omega} w^2 dtdx.$$

For every $w \in C(0, T; L^2(\Omega))$, choose $\{(w_n^0(x), w_n^1(x))\}_{n=1}^\infty \subset Y_4 \times Y_2$ such that, $w_n(x,t)$, the solution of system (3.62) with initial data $(w_n^0(x), w_n^1(x))$, converges to w in $C(0, T; L^2(\Omega))$ as $n \to \infty$. Moreover

$$\int_{t_1}^{t_2} \int_\Omega w_n^2 dxdt \leq TC_{19} \int_{Q^\omega} w_n^2 dtdx \qquad \forall\, n \geq 1.$$

Therefore, by the density argument, we have for every $w \in C(0, T; L^2(\Omega))$ satisfies (3.62), there exists some constant $C > 0$ such that

$$\int_{t_1}^{t_2} \int_\Omega w^2 dxdt \leq C \int_{Q^\omega} w^2 dtdx.$$

□

The following is the observability inequality.

Theorem 3.4. *Let $\omega = (a, b) \subseteq \Omega$. There exists a constant $C > 0$ such that for all $T > 2\pi$ and every solution ϕ of system* (3.21)-(3.23) *with initial data $\{\phi^0, \phi^1\} \in L^2(\Omega) \times Y_{-2}$, we have*

$$\|\{\phi^0, \phi^1\}\|^2 \leq C \int_{\omega \times (0,T)} |\phi|^2 dxdt,$$

where $\|\{\phi^0,\phi^1\}\|^2 = \left(\|\phi^0\|^2_{L^2(\Omega)} + \|\phi^1\|^2_{Y_{-2}}\right)$ *and* C *depends only on* $\|a\|_\infty$, T, ω *and* Ω.

Proof. Since $\phi \in C(0,T;L^2(\Omega))$ and $T > 2\pi$. Thus by Theorem 3.3, we have

$$\int_{t_1}^{t_2} \|\phi\|^2_{L^2(\Omega)} dt \le C\|\phi\|^2_{L^2(\omega\times(0,T))}. \tag{3.70}$$

Choose $0 < t_1 < s_1 < s_2 < t_2 < T$. By Lemma 3.3, we get for some constant $C^* > 0$

$$\begin{aligned} 2\int_{t_1}^{t_2} \|\phi\|^2_{L^2(\Omega)} dt &\ge \int_{t_1}^{t_2} \|\phi\|^2_{L^2(\Omega)} dt + \int_{t_1}^{t_2} \|\phi\|^2_{L^2(\Omega)} dt \\ &\ge \int_{t_1}^{t_2} \|\phi\|^2_{L^2(\Omega)} dt + C^* \int_{s_1}^{s_2} \left\|\frac{\partial\phi}{\partial t}\right\|^2_{Y_{-2}} dt \\ &\ge C^{**}\int_{s_1}^{s_2} \bar{E}(t)dt. \end{aligned} \tag{3.71}$$

Combining (3.70) and (3.71), we have, for some constant $C_1 > 0$

$$\int_{s_1}^{s_2} \bar{E}(t)dt \le C_1 e^{4T^2\lambda}\|\phi\|^2_{L^2(\omega\times(0,T))}.$$

By Lemma 3.4, we have

$$\int_{s_1}^{s_2} \bar{E}(t)dt \ge \frac{1}{T}(1+\|a\|^2_\infty)^{-2} e^{-\left(2+4\sqrt{\|a\|_\infty}\right)T}\bar{E}(0).$$

Therefore, we conclude that for some constant $C_2 > 0$

$$\bar{E}(0) \le C_2(1+\|a\|^2_\infty)^2 e^{\left(2+4\sqrt{\|a\|_\infty}\right)T}\|\phi\|^2_{L^2(\omega\times(0,T))}.$$

This is

$$\|\{\phi^0,\phi^1\}\|^2 \le 2C_2(1+\|a\|^2_\infty)^2 e^{\left(2+4\sqrt{\|a\|_\infty}\right)T}\int_{\omega\times(0,T)} |\phi|^2 dxdt.$$

□

3.5 Main Theorem

Consider the nonlinear system

$$w''(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) + Kw^+(x,t) = u(x,t)\chi_\omega, \qquad x \in \Omega, \quad t > 0, \tag{3.72}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{3.73}$$

$$w(x,0) = w^0(x), \quad \frac{\partial w}{\partial t}(x,0) = w^1(x), \qquad x \in \Omega, \tag{3.74}$$

with $u(x,t) \in L^2(\omega \times (0,T))$ and initial data $\{w^0, w^1\} \in Y_2 \times L^2(\Omega)$. We have the following main theorem.

Theorem 3.5. *Let $T > 2\pi$, then for all initial data $\{w^0, w^1\} \in Y_2 \times L^2(\Omega)$, system* (3.72)-(3.74) *is exactly L^2-controllable on* $[0,T]$.

Proof. To show that system (3.72)-(3.74) is exactly controllable, we look at the following problem. For any given $\xi \in L^\infty(\Omega \times (0,T))$, find a control $u = u(x,t;\xi)$ such that the solution $w = w(x,t;\xi)$ of

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) + Kg(\xi)w(x,t) = u(x,t)\chi_\omega, \qquad x \in \Omega, \quad t > 0, \tag{3.75}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \tag{3.76}$$

$$w(x,0) = w^0(x), \quad \frac{\partial w}{\partial t}(x,0) = w^1(x), \qquad x \in \Omega, \tag{3.77}$$

satisfies

$$w(x,T) = 0, \qquad \frac{\partial w}{\partial t}(x,T) = 0, \tag{3.78}$$

where here g is the Heavy-side function.

By using the HUM to show that such control u exists, we need to solve the following systems

$$\frac{\partial^2\phi}{\partial t^2}(x,t)+\frac{\partial^4\phi}{\partial x^4}(x,t)+Kg(\xi)\phi(x,t)=0, \qquad x\in\Omega, \quad t>0, \tag{3.79}$$

$$\phi(0,t)=\phi(\pi,t)=0, \quad \frac{\partial^2\phi}{\partial x^2}(0,t)=\frac{\partial^2\phi}{\partial x^2}(\pi,t)=0, \qquad t>0, \tag{3.80}$$

$$\phi(x,0)=\phi^0(x), \quad \frac{\partial\phi}{\partial t}(x,0)=\phi^1(x), \qquad x\in\Omega, \tag{3.81}$$

$$\frac{\partial^2\psi}{\partial t^2}(x,t)+\frac{\partial^4\psi}{\partial x^4}(x,t)+Kg(\xi)\psi(x,t)=\phi(x,t)\chi_\omega, \qquad x\in\Omega, \quad t>0, \tag{3.82}$$

$$\psi(0,t)=\psi(\pi,t)=0, \quad \frac{\partial^2\psi}{\partial x^2}(0,t)=\frac{\partial^2\psi}{\partial x^2}(\pi,t)=0, \qquad t>0, \tag{3.83}$$

$$\psi(x,T)=\frac{\partial\psi}{\partial t}(x,T)=0, \qquad x\in\Omega. \tag{3.84}$$

Define the linear operator

$$\Lambda_\xi : \, L^2(\Omega)\times Y_{-2}\to L^2(\Omega)\times Y_2$$

by

$$\Lambda_\xi\{\phi^0,\phi^1\}=\left\{-\frac{\partial\psi}{\partial t}(x,0),\psi(x,0)\right\}.$$

Λ_ξ is continuous. Then the problem reduces to show the existence of $\{\phi^0,\phi^1\}\in L^2(\Omega)\times Y_{-2}$ such that

$$\Lambda_\xi\{\phi^0,\phi^1\}=\{-w^1(x),w^0(x)\}, \tag{3.85}$$

that is, ψ, the solution of system (3.82)-(3.84), satisfies

$$\psi(x,0)=w^0(x), \qquad \frac{\partial\psi}{\partial t}(x,0)=w^1(x).$$

Multiplying (3.82) by ϕ and integrating by parts yields

$$\langle\Lambda_\xi\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\rangle=\int_{\omega\times(0,T)}|\phi|^2dxdt, \qquad \forall\ \{\phi^0,\phi^1\}\in L^2(\Omega)\times Y_{-2}.$$

Apply Theorem 3.4 to system (3.79)-(3.81), we have

$$\int_{\omega \times (0,T)} |\phi|^2 dxdt \geq C \|\{\phi^0, \phi^1\}\|^2. \tag{3.86}$$

Therefore, Λ_ξ is an isomorphism from $L^2(\Omega) \times Y_{-2}$ to $L^2(\Omega) \times Y_2$. Then, equation (3.85) has a unique solution $\{\phi^0, \phi^1\} = \{\phi^0(x;\xi), \phi^1(x;\xi)\}$. And with this solution we can solve for ϕ.

Now we have the function

$$u(x,t) = \phi(x,t;\xi)$$

is the unique control with which the solution of system (3.75)-(3.77) satisfies (3.78). Thus, for every $\xi \in L^\infty(\Omega \times (0,T))$, we have got a unique control $\phi(x,t;\xi) \in L^2(\Omega \times (0,T))$ and the solution $w \in C([0,T];Y_2) \bigcap C^1([0,T];L^2(\Omega))$. By the embedding theorem $Y_2 \hookrightarrow L^\infty(\Omega)$ we deduce that $w \in L^\infty(\Omega \times (0,T))$.

All of above, we have constructed a nonlinear operator

$$F: \quad L^\infty(\Omega \times (0,T)) \to L^\infty(\Omega \times (0,T))$$

by $F(\xi) = w$, where w is the solution of (3.75)-(3.77) with the control u defined above.

It is easy to show that F is a bounded operator. Moreover, by the compactness of embedding

$$C([0,T];Y_2) \bigcap C^1([0,T];L^2(\Omega)) \subset L^\infty(\Omega \times (0,T)),$$

we obtain that F is a continuous operator that maps bounded sets of $L^\infty(\Omega \times (0,T))$ into relatively compact sets of itself. Thus the operator $F : L^\infty(\Omega \times (0,T)) \to L^\infty(\Omega \times (0,T))$ is compact.

Next step is to show the existence of a fixed point of F. If $\xi = w \in L^\infty(\Omega \times (0,T))$ is a fixed point of F, then $\xi = w \in C([0,T]; Y_2) \bigcap C^1([0,T]; L^2(\Omega))$ and w satisfies (3.1)-(3.3) and (3.4). Therefore, when $\xi = w$ the corresponding control $u(x,t;\xi)$ is the control function we expect for system (3.1)-(3.3). Now use Leray-Schauder's degree theory to prove the existence of a fixed point of F.

Define the nonlinear operator

$$F_\epsilon : \quad [0,1] \times L^\infty(\Omega \times (0,T)) \to L^\infty(\Omega \times (0,T))$$

by

$$F_\epsilon(\xi) = F(\epsilon, \xi).$$

F_ϵ is the compact operator defined as before by replacing the nonlinearity g by ϵg in system (3.75)-(3.77). The operator F_ϵ is compact and $F_0(\xi) = F(0,\xi)$ is independent of ξ. Therefore, in order to conclude the existence of a fixed point for $F_1 = F$, it is sufficient to prove that all the solutions w of the equation

$$F(\epsilon, w) = w,$$

with $\epsilon \in [0,1]$ have an uniform bound for w in $L^\infty(\Omega \times (0,T))$. By the construction of F, the above equation is equivalent to the system

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) + \epsilon K w^+(x,t) = \phi(x,t)\chi_\omega, \qquad x \in \Omega, \quad t > 0, \tag{3.87}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{3.88}$$

$$w(x,0) = w^0(x), \quad \frac{\partial w}{\partial t}(x,0) = w^1(x), \qquad x \in \Omega, \tag{3.89}$$

$$w(x,T) = \frac{\partial w}{\partial t}(x,T) = 0, \qquad x \in \Omega, \tag{3.90}$$

$$\frac{\partial^2 \phi}{\partial t^2}(x,t) + \frac{\partial^4 \phi}{\partial x^4}(x,t) + \epsilon K g(w)\phi(x,t) = 0, \qquad x \in \Omega, \quad t > 0, \tag{3.91}$$

$$\phi(0,t) = \phi(\pi,t) = 0, \quad \frac{\partial^2 \phi}{\partial x^2}(0,t) = \frac{\partial^2 \phi}{\partial x^2}(\pi,t) = 0, \qquad t > 0. \tag{3.92}$$

By multiplying (3.87) by ϕ and integrating by parts, we obtain

$$\int_{\omega\times(0,T)} |\phi|^2 dxdt = -\langle w^1, \phi^0\rangle + \langle w^0, \phi^1\rangle.$$

By the Cauchy-Schwarz inequality, we have

$$\int_{\omega\times(0,T)} |\phi|^2 dxdt \leq C_0 \left[\|\phi^0\|_{L^2(\Omega)} + \|\phi^1\|_{Y_{-2}}\right],$$

with $C_0 = C_0\left(\|w^0\|_{Y_2}, \|w^1\|_{L^2(\Omega)}\right)$. By combining with (3.86), there exists a positive constant C^* such that

$$\|\phi^0\|_{L^2(\Omega)} + \|\phi^1\|_{Y_{-2}} \leq C^*. \tag{3.93}$$

Now apply Lemma 3.1 to the solution of system (3.87)-(3.92)

$$\frac{1}{2}\left(\left\|\frac{\partial w}{\partial t}\right\|^2_{L^2(\Omega)} + \|w\|^2_{Y_2}\right) \leq \left(E(0)(1 + K\|g\|_\infty) + \|\phi\|^2_{L^2(\omega\times(0,T))}\right) e^{\left(1+2\sqrt{K\|g\|_\infty}\right)t}.$$

Notice here $\|g\|_\infty = 1$, and $t \in (0,T)$. Using the continuity of the embedding $Y_2 \subset L^\infty(\Omega)$ we conclude that, for $B > 0$ large enough

$$\|w\|^2_{L^\infty(\Omega\times(0,T))} \leq B\left(E(0) + \int_{\omega\times(0,T)} |\phi|^2 dxdt\right) e^{BT}.$$

Since $E(0) = \frac{1}{2}\left(\|w^1\|^2_{L^2(\Omega)} + \|w^0\|^2_{Y_2}\right)$ is bounded and (3.93), we have shown that $\|w\|_\infty$ is uniformly bounded.

Thus the theorem has been proved. $\square$

CHAPTER 4

EXACT CONTROLLABILITY OF THE LAZER-MCKENNA SUSPENSION BRIDGE EQUATION WITH PIEZOELECTRIC ACTUATORS

4.1 Question and Literatures

In this chapter, we will discuss the exact controllability of the Lazer-McKenna suspension bridge equation with piezoelectric actuators. The linear plate equation with piezoelectric actuators has been studied in [Tucsnak (1996)]. More control problems of plate equation with different type of controls have been discussed by many researchers (like [Zhang (2001); Leiva (2005); Fu (2012)], etc). Does the semilinear system with piezoelectric actuators have the same regularity and exact controllability? We will prove that the semilinear system with piezoelectric actuators has the same regularity as linear system and is also exact controllable under certain conditions.

Let us consider a single nonlinear Lazer-Mckenna suspension bridge system

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + K w^+ = u(t)\frac{d}{dx}\left[\delta_b(x) - \delta_a(x)\right], \qquad x \in \Omega,\ t > 0, \tag{4.1}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \tag{4.2}$$

$$w(x,0) = w^0(x), \qquad \frac{\partial w}{\partial t}(x,0) = w^1(x), \qquad x \in \Omega, \tag{4.3}$$

where $\Omega = (0,\pi)$, $0 < a < b < \pi$ representing the locations of the actuator, and δ_μ is the Dirac mass at the point μ. The control $u : [0,T] \to \mathbb{R}$ is a function of the time variation of the voltage applied to the actuator.

Definition 4.1. We say system (4.1)-(4.3) is exactly L^2-controllable at time $T > 0$ if there exists a control $u \in L^2(0,T)$ such that the solution of the system satisfies the final state condition

$$w(x,T) = \frac{\partial w}{\partial t}(x,T) = 0, \qquad x \in \Omega. \tag{4.4}$$

We will show in this chapter system (4.1)-(4.3) is exactly L^2-controllable at time T. An auxiliary linear system will be introduced. The corresponding observability inequality will be proved by using the Ingham inequality (Theorem 2.10).

4.2 Notation and Lemmas

We use the same notation of function spaces Y_α as in Chapter 3.

Lemma 4.1. *For real-valued functions f_1 and f_2 on Ω, we have*

$$\|f_1^+ - f_2^+\|_{L^2(\Omega)} \le \|f_1 - f_2\|_{L^2(\Omega)}.$$

Moreover, for any $f \in Y_1$, we have $\|f^+\|_{Y_1} \le \|f\|_{Y_1}$, and $\|f\|_{L^2(\Omega)} \le \|f\|_{Y_1}$.

Proof. First we want to show $|f_1^+ - f_2^+| \le |f_1 - f_2|$.

If $f_1 > 0,\ f_2 > 0,\qquad |f_1^+ - f_2^+| = |f_1 - f_2|;$

If $f_1 \le 0,\ f_2 \le 0,\qquad |f_1^+ - f_2^+| = 0 \le |f_1 - f_2|;$

If $f_1 \le 0,\ f_2 > 0,\qquad |f_1^+ - f_2^+| = |f_2| \le |f_1| + |f_2| = |f_1 - f_2|;$

If $f_1 > 0,\ f_2 \le 0\qquad |f_1^+ - f_2^+| = |f_1| \le |f_1| + |f_2| = |f_1 - f_2|.$

thus

$$\|f_1^+ - f_2^+\|^2_{L^2(\Omega)} = \int_\Omega |f_1^+ - f_2^+|^2 dx \le \int_\Omega |f_1 - f_2|^2 dx = \|f_1 - f_2\|^2_{L^2(\Omega)}.$$

Note that $\|f\|_{Y_1}^2 = \sum_{n=1}^{\infty} n^2 a_n^2$, where $f = \sum_{n=1}^{\infty} a_n \sin(nx) \in Y_1$. It is straightforward to get for any $f = \sum_{n=1}^{\infty} a_n \sin(nx) \in Y_1$

$$\|f\|_{L^2(\Omega)} = \left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{\frac{1}{2}} = \|f\|_{Y_1}.$$

Now for any $f \in Y_1$, we have $f = \sum_{n=1}^{\infty} a_n \sin(nx)$ and $\sum_{n=1}^{\infty} n^2 a_n^2 < \infty$. Thus

$$\|f\|_{L^2(\Omega)}^2 = \int_0^{\pi} |f|^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} a_n^2 \leq \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 a_n^2 = \frac{\pi}{2} \|f\|_{Y_1}^2,$$

and

$$\left\|\frac{df}{dx}\right\|_{L^2(\Omega)}^2 = \int_0^{\pi} \left|\frac{df}{dx}\right|^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 a_n^2 = \frac{\pi}{2} \|f\|_{Y_1}^2.$$

Thus $f \in H_0^1(\Omega)$. Hence $Y_1 \subset H_0^1(\Omega)$.

On the other hand, for any $f \in H_0^1(\Omega) \subset L^2(\Omega)$, $f = \sum_{n=1}^{\infty} a_n \sin(nx)$ because $\{\sin(nx)\}_{n=1}^{\infty}$ is an orthogonal basis of $L^2(\Omega)$. $\frac{df}{dx} \in L^2(\Omega)$ implies $\sum_{n=1}^{\infty} n^2 a_n^2 < \infty$. Therefore $f \in Y_1$, $H_0^1(\Omega) \subset Y_1$, hence $Y_1 = H_0^1(\Omega)$. By the Pöintcare Inequality, the norm of $H_0^1(\Omega)$ can be defined by $\|f\|_{H_0^1(\Omega)} = \left(\int_0^{\pi} \left|\frac{df}{dx}\right|^2 dx\right)^{\frac{1}{2}}$. Thus $\|f\|_{H_0^1(\Omega)} = \|f\|_{Y_1}$.

For any $f \in H_0^1(\Omega)$, $f(x) = f^+(x) - f^-(x)$. So

$$\frac{df}{dx} = \frac{d}{dx}(f^+)(x) - \frac{d}{dx}(f^-)(x).$$

Then

$$\int_0^{\pi} \left|\frac{df}{dx}\right|^2 dx = \int_0^{\pi} \left|\frac{df^+}{dx}\right|^2 dx - 2\int_0^{\pi} \frac{df^+}{dx} \cdot \frac{df^-}{dx} dx + \int_0^{\pi} \left|\frac{df^-}{dx}\right|^2 dx.$$

Note that

$$\frac{df^+}{dx} \cdot \frac{df^-}{dx} = 0, \qquad \text{on } \Omega.$$

Thus

$$\int_0^\pi \left|\frac{df}{dx}\right|^2 dx = \int_0^\pi \left|\frac{df^+}{dx}\right|^2 dx + \int_0^\pi \left|\frac{df^-}{dx}\right|^2 dx.$$

Therefore

$$\|f^+\|_{H_0^1(\Omega)} = \left(\int_0^\pi \left|\frac{df^+}{dx}\right|^2 dx\right)^{\frac{1}{2}} \leq \left(\int_0^\pi \left|\frac{df}{dx}\right|^2 dx\right)^{\frac{1}{2}} = \|f\|_{H_0^1(\Omega)}.$$

By using the equivalence of $\|\cdot\|_{Y_1}$ and $\|\cdot\|_{H_0^1(\Omega)}$, we have

$$\|f^+\|_{Y_1} \leq \|f\|_{Y_1}.$$

□

We now introduce some results in diophantine approximation. For a real number λ, define

$$|||\lambda||| = \min_{n \in \mathbf{Z}} |\lambda - n|.$$

Let A denote the set of all irrationals $\lambda \in [0, 1]$ such that λ can be expressed by a continued fraction of $\{0, a_1, \cdots, a_n, \cdots\}$ with $\{a_n\}$ being bounded. Note that A is an uncountable set with zero Lebesgue measure [Cassals (1996)].

Lemma 4.2. *[Lang (1966)] A number $\lambda \in (0, 1)$ is in A if and only if there exists a constant $C > 0$ such that*

$$|||q\lambda||| \geq \frac{C}{q}$$

for any positive integer q.

Consider linear system

$$\frac{\partial^2 \tilde{w}}{\partial t^2} + \frac{\partial^4 \tilde{w}}{\partial x^4} = u(t)\frac{d}{dx}\left[\delta_b(x) - \delta_a(x)\right], \qquad x \in \Omega,\, t > 0, \tag{4.5}$$

$$\tilde{w}(0,t) = \tilde{w}(\pi,t) = 0, \quad \frac{\partial^2 \tilde{w}}{\partial x^2}(0,t) = \frac{\partial^2 \tilde{w}}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{4.6}$$

$$\tilde{w}(x,0) = \tilde{w}^0(x), \ \frac{\partial \tilde{w}}{\partial t}(x,0) = \tilde{w}^1(x), \qquad x \in \Omega. \tag{4.7}$$

Proposition 4.1. *[Tucsnak (1996)] Suppose that $\tilde{w}^0 \in Y_1$, $\tilde{w}^1 \in Y_{-1}$. Then the initial and boundary value problem* (4.5)-(4.7) *admits a unique solution having the regularity*

$$\tilde{w} \in C([0,T], Y_1) \bigcap C^1([0,T], Y_{-1}).$$

Proposition 4.2. *[Tucsnak (1996)] Suppose that $\frac{a+b}{2\pi}$ and $\frac{b-a}{2\pi}$ belong to the set A. Then all initial data $\left\{\tilde{w}(x,0), \frac{\partial \tilde{w}}{\partial t}(x,0)\right\}$ in $Y_3 \times Y_1$ are exactly L^2-controllable in (a,b) at time T, for any $T > 0$. That is, there exists $u \in L^2(0,T)$ such that the solution $\tilde{w}$ of system* (4.5)-(4.7) *satisfies the condition*

$$\tilde{w}(x,T) = \frac{\partial \tilde{w}}{\partial t}(x,T) = 0, \qquad x \in \Omega.$$

For $\tau \in [0,T]$, consider the homogeneous initial and boundary value system

$$\frac{\partial^2 \bar{V}}{\partial t^2}(x,t) + \frac{\partial^4 \bar{V}}{\partial x^4}(x,t) = 0, \qquad x \in \Omega,\, t \in (0,\tau), \tag{4.8}$$

$$\bar{V}(0,t) = \bar{V}(\pi,t) = 0, \quad \frac{\partial^2 \bar{V}}{\partial x^2}(0,t) = \frac{\partial^2 \bar{V}}{\partial x^2}(\pi,t) = 0, \quad t \in (0,\tau), \tag{4.9}$$

$$\bar{V}(x,\tau) = 0, \qquad \frac{\partial \bar{V}}{\partial t}(x,\tau) = g(x), \qquad x \in \Omega. \tag{4.10}$$

We have the following lemma shows the regularity of the solution.

Lemma 4.3. *[Tucsnak (1996)] For any $g \in Y_{-1}$, the initial and boundary value problem* (4.8)-(4.10) *admits a unique solution having the regularity*

$$\bar{V} \in C([0,T], Y_1) \bigcap C^1([0,T], Y_{-1}).$$

Moreover, for any $\sigma \in (0,\pi)$ the function $\frac{\partial \bar{V}}{\partial x}(\sigma,\cdot)$ is in $L^2(0,T)$ and there exists a constant $C > 0$ such that

$$\left\| \frac{\partial \bar{V}}{\partial x}(\sigma,\cdot) \right\|_{L^2(0,T)} \leq C\|g\|_{Y_{-1}}.$$

Lemma 4.4. *For any $u(t) \in L^2(0,T)$ and a, $b \in (0,\pi)$, there exists some constant $C_0 > 0$ such that*

$$\left\| \sum_{n=1}^{\infty} \frac{\sqrt{2}(\cos(na)-\cos(nb))}{n\sqrt{\pi}} \int_0^t u(s)\sin(n^2(t-s))ds \sin(nx) \right\|^2_{L^\infty(0,T;Y_1)} \leq C_0\|u(t)\|^2_{L^2(0,T)}.$$

Proof. For system (4.5)-(4.7), let $W(x,t) = \tilde{w}(u_1) - \tilde{w}(u_2)$ and $U(t) = u_1(t) - u_2(t)$. Clearly they satisfy the following system

$$\frac{\partial^2 W}{\partial t^2} + \frac{\partial^4 W}{\partial x^4} = U(t)\frac{d}{dx}\left[\delta_b(x) - \delta_a(x)\right], \qquad x \in \Omega,\ t > 0, \tag{4.11}$$

$$W(0,t) = W(\pi,t) = 0, \quad \frac{\partial^2 W}{\partial x^2}(0,t) = \frac{\partial^2 W}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \tag{4.12}$$

$$W(x,0) = 0, \qquad \frac{\partial W}{\partial t}(x,0) = 0, \qquad x \in \Omega. \tag{4.13}$$

Assume $g \in C_0^\infty(\Omega)$ is arbitrary, and $\bar{V}(x,t)$ is the solution of system (4.8)-(4.10). Multiply (4.11) by $\bar{V}(x,t)$ and integrate by parts we may have

$$\int_0^\pi W(x,\tau)g(x)dx = \int_0^\tau U(t)\left[\frac{\partial \bar{V}}{\partial x}(b,t) - \frac{\partial \bar{V}}{\partial t}(a,t)\right]dt.$$

By Lemma 4.3, we may obtain

$$\left|\int_0^\pi W(x,\tau)g(x)dx\right| = \left|\int_0^\tau U(t)\left[\frac{\partial \bar{V}}{\partial x}(b,t) - \frac{\partial \bar{V}}{\partial t}(a,t)\right]dt\right| \leq C\|U\|_{L^2(0,T)}\|g\|_{Y_{-1}}.$$

Thus we have $\|W(x,\cdot)\|^2_{Y_1} \leq C_0\|U\|^2_{L^2(0,T)}$, $\forall\, 0 < \tau < T$ for some constant C_0, which gives us

$$\|W(x,t)\|^2_{L^\infty(0,T;Y_1)} \leq C_0\|U\|^2_{L^2(0,T)}. \tag{4.14}$$

Let $W(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx)$. Since

$$\delta_b(x) - \delta_a(x) = \sum_{n=1}^{\infty}\left(\sqrt{\frac{2}{\pi}}n(\cos(na)-\cos(nb))\right)\sin(nx).$$

This yields for all $n \geq 1$

$$a_n''(t) + n^4 a_n(t) = U(t)\sqrt{\frac{2}{\pi}}n(\cos(na)-\cos(nb)),$$

$$a_n(0) = 0, \qquad a_n'(0) = 0.$$

It is straightforward to verify that

$$W(x,t) = \sum_{n=1}^{\infty}\frac{\sqrt{2}(\cos(na)-\cos(nb))}{n\sqrt{\pi}}\int_0^t U(s)\sin(n^2(t-s))ds\sin(nx).$$

Apply (4.14), we can get

$$\left\|\sum_{n=1}^{\infty}\frac{\sqrt{2}(\cos(na)-\cos(nb))}{n\sqrt{\pi}}\int_0^t U(s)\sin(n^2(t-s))ds\sin(nx)\right\|^2_{L^{\infty}(0,T;Y_1)} \leq C_0\|U(t)\|^2_{L^2(0,T)}.$$

Since u_1, u_2 are arbitrary in $L^2(0,T)$, we have the estimation in this lemma is true.

□

4.3 Existence and Regularity

Theorem 4.1. *If $w^0 \in Y_1$, $w^1 \in Y_{-1}$, then the initial and boundary value problem* (4.1)-(4.3) *admits a unique solution with regularity*

$$w \in C([0,T],Y_1)\bigcap C^1([0,T],Y_{-1}).$$

Proof. By Proposition 4.1, system (4.5)-(4.7) with initial condition $\tilde{w}(x,0) = w^0(x)$, $\frac{\partial \tilde{w}}{\partial t}(x,0) = w^1(x)$ admits a unique solution $\tilde{w} \in C([0,T],Y_1)\bigcap C^1([0,T],Y_{-1})$.

Now let $V = w - \tilde{w}$. Thus system (4.1)-(4.3) is equivalent to the following system:

$$\frac{\partial^2 V}{\partial t^2} + \frac{\partial^4 V}{\partial x^4} = -K\left(V + \tilde{w}\right)^+, \qquad x \in \Omega,\ t > 0, \tag{4.15}$$

$$V(0,t) = V(\pi,t) = 0, \quad \frac{\partial^2 V}{\partial x^2}(0,t) = \frac{\partial^2 V}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{4.16}$$

$$V(x,0) = 0, \qquad \frac{\partial V}{\partial t}(x,0) = 0, \qquad x \in \Omega. \tag{4.17}$$

We only need to show system (4.15)-(4.17) admits a unique solution.

It is straightforward to obtain

$$V(x,t) = \int_0^t \int_0^\pi G(x,y,t-s)(V + \tilde{w})^+ dyds, \tag{4.18}$$

where

$$G(x,y,t-s) = -\sum_{n=1}^{\infty} \frac{\sqrt{2}K}{\sqrt{\pi}n^2} \sin(nx)\sin(ny)\sin(n^2(t-s)).$$

Define a mapping F on $L^2(0,T;Y_1)$ by

$$F(v) = \int_0^t \int_0^\pi G(x,y,t-s)(v + \tilde{w})^+ dyds.$$

First we show F is a bounded contraction mapping from $L^2(0,T;Y_1)$ to $L^2(0,T;Y_1)$.

Assume $V_1,\ V_2 \in L^2(0,T;Y_1)$. Then

$$\begin{aligned}
&F(V_1)(x,t) - F(V_2)(x,t) \\
&= \int_0^t \int_0^\pi G(x,y,t-s)\left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right)(y,s)dyds \\
&= -\sum_{n=1}^{\infty}\left[\frac{\sqrt{2}K}{\sqrt{\pi}n^2}\int_0^t\int_0^\pi \sin(ny)\sin(n^2(t-s))\left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right)(y,s)dyds\right] \\
&\qquad \cdot \sin(nx),
\end{aligned}$$

$$\begin{aligned}
&\|F(V_1) - F(V_2)\|_{Y_1}^2 \\
&= \sum_{n=1}^{\infty}\frac{2K^2}{\pi n^2}\left(\int_0^t\int_0^\pi \left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right)(y,s)\sin(ny)\sin(n^2(t-s))dyds\right)^2.
\end{aligned}$$

By Hölder's inequality and Lemma 4.1

$$
\begin{aligned}
&\|F(V_1) - F(V_2)\|_{Y_1}^2 \\
&\quad \le \sum_{n=1}^{\infty} \frac{K^2}{n^2} \int_0^t \left(\sqrt{\frac{2}{\pi}} \int_0^{\pi} \left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right) \sin(ny)dy \right)^2 ds \\
&\qquad\qquad \cdot \int_0^t \sin^2(n^2(t-s))ds,
\end{aligned}
$$

$$
\begin{aligned}
&\|F(V_1) - F(V_2)\|_{L^2(0,T;Y_1)}^2 \\
&\quad \le TK^2 \int_0^T \int_0^t \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sqrt{\frac{2}{\pi}} \int_0^{\pi} \left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right) \sin(ny)dy \right)^2 dsdt \\
&\quad \le TK^2 \int_0^T \int_0^T \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sqrt{\frac{2}{\pi}} \int_0^{\pi} \left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right) \sin(ny)dy \right)^2 dsdt \\
&\quad \le T^2K^2 \int_0^T \sum_{n=1}^{\infty} \left(\sqrt{\frac{2}{\pi}} \int_0^{\pi} \left((V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right) \sin(ny)dy \right)^2 ds \\
&\quad = T^2K^2 \int_0^T \left\|(V_1+\tilde{w})^+ - (V_2+\tilde{w})^+\right\|_{L^2(\Omega)}^2 ds \\
&\quad \le T^2K^2 \int_0^T \|(V_1+\tilde{w}) - (V_2+\tilde{w})\|_{L^2(\Omega)}^2 ds \\
&\quad = T^2K^2 \int_0^T \|V_1 - V_2\|_{L^2(\Omega)}^2 ds \;\le\; T^2K^2 \int_0^T \|V_1 - V_2\|_{Y_1}^2 ds \\
&\quad = T^2K^2 \|V_1 - V_2\|_{L^2(0,T;Y_1)}^2 .
\end{aligned}
$$

Thus when T is small enough, F is a contraction mapping in $L^2(0,T;Y_1)$. By the Banach Contraction Mapping Principle, (4.18) admits a unique solution.

For any $V \in L^2(0,T;Y_1)$,

$$
\begin{aligned}
&\|F(V)\|_{L^2(0,T;Y_1)}^2 \\
&= \int_0^T \sum_{n=1}^{\infty} \frac{2K^2}{\pi n^2} \left(\int_0^t \int_0^{\pi} (V+\tilde{w})^+(y,s) \sin(ny) \sin(n^2(t-s))dyds \right)^2 dt \\
&\le TK^2 \int_0^T \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^t \left(\sqrt{\frac{2}{\pi}} \int_0^{\pi} (V+\tilde{w})^+(y,s) \sin(ny)dy \right)^2 dsdt
\end{aligned}
$$

$$\leq TK^2 \int_0^T \int_0^T \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sqrt{\frac{2}{\pi}} \int_0^{\pi} (V+\tilde{w})^+(y,s) \sin(ny) dy \right)^2 dsdt$$

$$\leq T^2K^2 \int_0^T \left\| (V+\tilde{w})^+ \right\|_{Y_1}^2 ds \ \leq \ T^2K^2 \int_0^T \| (V+\tilde{w}) \|_{Y_1}^2 ds$$

$$= T^2K^2 \| (V+\tilde{w}) \|_{L^2(0,T;Y_1)}^2$$

$$\leq 2T^2K^2 \left(\|V\|_{L^2(0,T;Y_1)}^2 + \|\tilde{w}\|_{L^2(0,T;Y_1)}^2 \right).$$

Thus F is a bounded mapping from $L^2(0,T;Y_1)$ to $L^2(0,T;Y_1)$.

Next we show $V \in C([0,T],Y_1)$.

$$V(x,t+\tau) \ - \ V(x,t)$$

$$= -\sum_{n=1}^{\infty} \left(\int_0^{t+\tau} \int_0^{\pi} \frac{\sqrt{2}K}{\sqrt{\pi}n^2} \sin(ny) \sin(n^2(t+\tau-s))(V+\tilde{w})^+(y,s) dyds \right) \sin(nx)$$

$$+ \sum_{n=1}^{\infty} \left(\int_0^{t} \int_0^{\pi} \frac{\sqrt{2}K}{\sqrt{\pi}n^2} \sin(ny) \sin(n^2(t-s))(V+\tilde{w})^+(y,s) dyds \right) \sin(nx)$$

$$= -\sum_{n=1}^{\infty} \left(\int_0^{t} \int_0^{\pi} \frac{\sqrt{2}K}{\sqrt{\pi}n^2} \sin(ny) \left[\sin(n^2(t+\tau-s)) - \sin(n^2(t-s)) \right] \right.$$

$$\left. \cdot (V+\tilde{w})^+(y,s) dyds \right) \sin(nx)$$

$$- \sum_{n=1}^{\infty} \left(\int_t^{t+\tau} \int_0^{\pi} \frac{\sqrt{2}K}{\sqrt{\pi}n^2} \sin(ny) \sin(n^2(t+\tau-s))(V+\tilde{w})^+(y,s) dyds \right) \sin(nx).$$

Thus

$$\|V(\cdot,t+\tau) \ - \ V(\cdot,t)\|_{Y_1}^2$$

$$\leq 2K^2 \sum_{n=1}^{\infty} \left(\int_0^{t} \int_0^{\pi} \frac{\sqrt{2}}{\sqrt{\pi}n} \sin(ny) \left[\sin(n^2(t+\tau-s)) - \sin(n^2(t-s)) \right] \right.$$

$$\left. \cdot (V+\tilde{w})^+(y,s) dyds \right)^2$$

$$+ 2K^2 \sum_{n=1}^{\infty} \left(\int_t^{t+\tau} \int_0^{\pi} \frac{\sqrt{2}}{\sqrt{\pi}n} \sin(ny) \sin(n^2(t+\tau-s))(V+\tilde{w})^+(y,s) dyds \right)^2$$

$$\leq 2K^2 \sum_{n=1}^{\infty} \left[\int_0^{t} \left(\int_0^{\pi} \frac{\sqrt{2}n}{\sqrt{\pi}} (V+\tilde{w})^+(y,s) \sin(ny) dy \right)^2 ds \right.$$

$$\cdot \int_0^t \left(\frac{\sin(n^2(t+\tau-s)) - \sin(n^2(t-s))}{n^2} \right)^2 ds \Bigg]$$

$$+ 2K^2 \sum_{n=1}^{\infty} \left[\int_t^{t+\tau} \left(\int_0^{\pi} \frac{\sqrt{2}n}{\sqrt{\pi}} (V+\tilde{w})^+(y,s) \sin(ny) dy \right)^2 ds \right.$$

$$\left. \cdot \int_t^{t+\tau} \left(\frac{\sin(n^2(t+\tau-s))}{n^2} \right)^2 ds \right].$$

Notice

$$\begin{aligned}
A_n &= \int_0^t \frac{1}{n^4} [\sin(n^2(t+\tau-s)) - \sin(n^2(t-s))]^2 ds \\
&= \int_0^t \frac{1}{n^4} \left[2\cos(n^2(t+\frac{\tau}{2}-s)) \sin\left(\frac{n^2\tau}{2}\right) \right]^2 ds \\
&\le \int_0^t \frac{4}{n^4} \sin^2\left(\frac{n^2\tau}{2}\right) ds \;\le\; \int_0^t \frac{4}{n^4} \left| \sin\left(\frac{n^2\tau}{2}\right) \right| ds \\
&\le \int_0^t \frac{4}{n^4} \cdot \frac{n^2\tau}{2} ds \le \int_0^t \frac{2}{n^2} \tau ds \;\le\; \tau T, \qquad \text{for all } n \ge 1, \qquad (4.19) \\
B_n &= \int_t^{t+\tau} \left(\frac{\sin(n^2(t+\tau-s))}{n^2} \right)^2 ds \;\le\; \int_t^{t+\tau} \frac{1}{n^4} ds \\
&\le \frac{1}{n^4}\tau \;\le\; \tau, \qquad \text{for all } n \ge 1. \qquad (4.20)
\end{aligned}$$

From (4.19), (4.20) and Lemma 4.1, we obtain

$$\begin{aligned}
\|V(\cdot,t+\tau) - V(\cdot,t)\|_{Y_1}^2 \;\le\; & 2\tau T K^2 \sum_{n=1}^{\infty} \int_0^t \left(\int_0^{\pi} \frac{\sqrt{2}n}{\sqrt{\pi}} (V+\tilde{w})^+(y,s) \sin(ny) dy \right)^2 ds \\
& + 2\tau K^2 \sum_{n=1}^{\infty} \int_t^{t+\tau} \left(\int_0^{\pi} \frac{\sqrt{2}n}{\sqrt{\pi}} (V+\tilde{w})^+(y,s) \sin(ny) dy \right)^2 ds \\
=\; & 2\tau T K^2 \int_0^t \|(V+\tilde{w})^+\|_{Y_1}^2 ds \;+\; 2\tau K^2 \int_t^{t+\tau} \|(V+\tilde{w})^+\|_{Y_1}^2 ds \\
\le\; & 2\tau(TK^2 + K^2) \int_0^T \|(V+\tilde{w})^+\|_{Y_1}^2 ds \\
=\; & 2\tau(TK^2 + K^2) \|(V+\tilde{w})\|_{L^2(0,T;Y_1)}^2.
\end{aligned}$$

Therefore

$$\|V(x,t+\tau) - V(x,t)\|_{L^2(0,T;Y_1)}^2 \;=\; \int_0^T \|V(x,t+\tau) - V(x,t)\|_{Y_1}^2 dt$$

$$\leq 2\tau(T^2+T)K^2\|(V+\tilde{w})\|^2_{L^2(0,T;Y_1)}.$$

Thus

$$\lim_{\tau\to 0}\|V(x,t+\tau)-V(x,t)\|^2_{L^2(0,T;Y_1)}=0.$$

It implies $V\in C([0,T];Y_1)$, so by Proposition 4.1 we get that

$$w\in C([0,T];Y_1), \tag{4.21}$$

and this gives us

$$\frac{\partial^2 w}{\partial x^4}\in C([0,T];Y_{-3}). \tag{4.22}$$

Since w satisfies equation (4.1), from the above it follows that

$$\frac{\partial^2 w}{\partial t^2}\in L^2((0,T);Y_{-3}). \tag{4.23}$$

Using the intermediate derivative theorem [Lions and Magenes (1972)], from (4.21) and (4.23) we obtain that

$$\frac{\partial w}{\partial t}\in L^2((0,T);Y_{-1}). \tag{4.24}$$

From (4.21) to (4.24), by applying the general lifting result from [Lasiecka and Triggiani (1988)] we conclude

$$w\in C([0,T];Y_1)\bigcap C^1([0,T];Y_{-1}).$$

□

4.4 Main Theorem

Theorem 4.2. *Suppose that* $\frac{a+b}{2\pi}$ *and* $\frac{b-a}{2\pi}$ *are in set* A. *Let* T *be small enough. Then for all initial data* $\{w^0(x),\ w^1(x)\}\in Y_3\times Y_1$, *system* (4.1)-(4.3) *is exactly*

L^2-controllable.

Proof. Since we can't apply the HUM method directly to this nonlinear system, we split our system first. Let $\bar{w}$ denote the solution of the linear system

$$\frac{\partial^2 \bar{w}}{\partial t^2} + \frac{\partial^4 \bar{w}}{\partial x^4} = u(t)\frac{d}{dx}\left[\delta_b(x) - \delta_a(x)\right], \qquad x \in \Omega,\ t > 0, \tag{4.25}$$

$$\bar{w}(0,t) = \bar{w}(\pi,t) = 0, \quad \frac{\partial^2 \bar{w}}{\partial x^2}(0,t) = \frac{\partial^2 \bar{w}}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \tag{4.26}$$

$$\bar{w}(x,0) = w^0(x) - v(x,0), \ \frac{\partial \bar{w}}{\partial t}(x,0) = w^1(x) - \frac{\partial v}{\partial t}(x,0), \quad x \in \Omega, \tag{4.27}$$

$$\bar{w}(x,T) = 0, \qquad \frac{\partial \bar{w}}{\partial t}(x,T) = 0, \qquad x \in \Omega, \tag{4.28}$$

where v is the solution of nonlinear system

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} + K[v + \bar{w}]^+ = 0, \qquad x \in \Omega,\ t > 0, \tag{4.29}$$

$$v(0,t) = v(\pi,t) = 0, \quad \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(\pi,t) = 0, \quad t > 0, \tag{4.30}$$

$$v(x,T) = 0, \qquad \frac{\partial v}{\partial t}(x,T) = 0, \qquad x \in \Omega. \tag{4.31}$$

Thus $w = \bar{w} + v$ satisfies system (4.1)-(4.3) and the final state condition

$$w(x,T) = 0, \qquad \frac{\partial w}{\partial t}(x,T) = 0.$$

We only need to show the existence of $u(t)$ in system (4.25)-(4.28).

Apply the HUM method to system (4.25)-(4.27), we start from the next homogenous initial and boundary value problem

$$\frac{\partial^2 \phi}{\partial t^2}(x,t) + \frac{\partial^4 \phi}{\partial x^4}(x,t) = 0, \qquad x \in \Omega,\ t > 0, \tag{4.32}$$

$$\phi(0,t) = \phi(\pi,t) = 0, \quad \frac{\partial^2 \phi}{\partial x^2}(0,t) = \frac{\partial^2 \phi}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \tag{4.33}$$

$$\phi(x,0) = \phi^0, \qquad \frac{\partial \phi}{\partial t}(x,0) = \phi^1, \qquad x \in \Omega, \tag{4.34}$$

Given ϕ^0, ϕ^1, the above system admits a unique solution. Then we solve

$$\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^4\psi}{\partial x^4} = \left(\frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t)\right)\frac{d}{dx}\left(\delta_b(x) - \delta_a(x)\right), \; x \in \Omega, \; t > 0, \quad (4.35)$$

$$\psi(0,t) = \psi(\pi,t) = 0, \quad \frac{\partial^2\psi}{\partial x^2}(0,t) = \frac{\partial^2\psi}{\partial x^2}(\pi,t) = 0, \qquad t > 0, \quad (4.36)$$

$$\psi(x,T) = 0, \qquad \frac{\partial\psi}{\partial t}(x,T) = 0, \qquad x \in \Omega. \quad (4.37)$$

Define a mapping $\Lambda : Y_{-1} \times Y_{-3} \to Y_1 \times Y_3$ by

$$\Lambda\{\phi^0, \phi^1\} = \left\{\frac{\partial\psi}{\partial t}(x,0), -\psi(x,0)\right\}.$$

It is easy to verify

$$\int_0^T \left[\frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t)\right]^2 dt = \langle\Lambda\{\phi^0,\phi^1\}, \{\phi^0,\phi^1\}\rangle.$$

Let $\phi^0(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \in Y_{-1}$, $\phi^1(x) = \sum_{n=1}^{\infty} n^2 b_n \sin(nx) \in Y_{-3}$. Therefore

$$\|\phi^0\|^2_{Y_{-1}} = \sum_{n=1}^{\infty} n^{-2}a_n^2 < \infty, \qquad \|\phi^1\|^2_{Y_{-3}} = \sum_{n=1}^{\infty} n^{-2}b_n^2 < \infty.$$

Thus, the solution of system (4.32)-(4.34) is

$$\phi(x,t) = \sum_{n=1}^{\infty}[a_n\cos(n^2t)\sin(nx) + b_n\sin(n^2t)\sin(nx)].$$

Furthermore

$$\frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t) = \sum_{n=1}^{\infty} 2\sin\left[\frac{n(b+a)}{2}\right]\sin\left[\frac{n(b-a)}{2}\right][na_n\cos(n^2t) + nb_n\sin(n^2t)].$$

By Theorem 2.10, there exists a constant $C > 0$ such that

$$\int_0^T \left[\frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t)\right]^2 dt \geq C\sum_{n=1}^{\infty}\sin^2\left[\frac{n(b+a)}{2}\right]\sin^2\left[\frac{n(b-a)}{2}\right] \cdot [n^2a_n^2 + n^2b_n^2] \quad (4.38)$$

Since $\dfrac{a+b}{2\pi}$ and $\dfrac{b-a}{2\pi}$ belong to set A. By Lemma 4.2, there exists a constant $C^* > 0$ such that

$$\left|\sin\left[\frac{n(b\pm a)}{2}\right]\right| = \left|\sin\left\{\pi\left[\frac{n(b\pm a)}{2\pi} - p\right]\right\}\right| \geq \left|\sin\left(\frac{\pi C^*}{n}\right)\right| \geq \frac{C^*}{n}, \forall n \geq 1,$$

where p is a proper integer. Then for a constant $C_1 > 0$

$$\int_0^T \left[\frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t)\right]^2 dt \geq C_1 \sum_{n=1}^{\infty}(n^{-2}a_n^2 + n^{-2}b_n^2) \tag{4.39}$$

That is

$$\int_0^T \left[\frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t)\right]^2 dt \geq C_1\left(\|\phi^0\|_{Y_{-1}}^2 + \|\phi^1\|_{Y_{-3}}^2\right).$$

Λ is an isomorphism. Thus Λ^{-1} exists, and is also bounded. Let

$$\left\{\frac{\partial\psi}{\partial t}(x,0), -\psi(x,0)\right\} = \left\{\frac{\partial\bar{w}}{\partial t}(x,0), -\bar{w}(x,0)\right\} \in Y_3 \times Y_1.$$

Solve system (4.32)-(4.34) with initial value data

$$\{\phi^0, \phi^1\} = \Lambda^{-1}\{\bar{w}^1(x,0), -\bar{w}^0(x,0)\},$$

and choose

$$u(t) = \frac{\partial\phi}{\partial x}(b,t) - \frac{\partial\phi}{\partial x}(a,t). \tag{4.40}$$

Thus $\bar{w}(x,t) = \psi(x,t)$, and $u(t)$ defined in (4.40) is the desired control.

If given $v(x,0) = v_0^0(x) \in Y_3$ and $\dfrac{\partial v}{\partial t}(x,0) = v_0^1(x) \in Y_1$, the system (4.25)-(4.27) admits a unique solution $\bar{w}_0(x,t) = \bar{w}(v^0, v^1) \in C([0,T];Y_1)\bigcap C^1([0,T];Y_{-1})$ and the corresponding unique control $u_0(t) = u(v^0, v^1)$ satisfy (4.28). With this $\bar{w}_0$ in system (4.29)-(4.31), we may at least admit a weak solution, denoted as $v_1(x,t)$. By the result in the previous section, this solution is unique and $v_1 \in C([0,T];Y_1)\bigcap C^1([0,T];Y_{-1})$.

If $v_1^0(x) = v_1(x,0) \in Y_3$ and $v_1^1(x) = \dfrac{\partial v_1}{\partial t}(x,0) \in Y_1$, we may repeat this to get $\bar{w}_1(x,t)$ and $u_1(t)$, and then $v_2^0(x)$ and $v_2^1(x)$. By repeating this process, we have defined a mapping

$$F: \, Y_3 \times Y_1 \, \rightarrow \, Y_3 \times Y_1$$

by

$$F(\{v_n^0(x), v_n^1(x)\}) = \{v_{n+1}^0(x), v_{n+1}^1(x)\}, \quad \forall \, n \geq 0.$$

If F is a contraction mapping, by the Banach Contraction Mapping Principle there is a unique fixed point of F, denoted as $\{v^0(x), v^1(x)\}$. Let $v(x,0) = v^0(x)$ and $\dfrac{\partial v}{\partial t}(x,0) = v^1(x)$ in system (4.25)-(4.28), by Proposition 4.2 we will get a control function $u(x,t)$ that drives the solution w of system (4.1)-(4.3) to rest at $t = T$. Therefore Theorem 4.2 has been proved.

First, let's verify that given $v_n^0(x) \in Y_3$ and $v_n^1(x) \in Y_1$ will imply $v_{n+1}^0(x) \in Y_3$ and $v_{n+1}^1(x) \in Y_1$ for all $n \geq 0$. Notice that we have regularity that $\bar{w}_n \in C([0,T];Y_1)\bigcap C^1([0,T];Y_{-1})$ for all $n \geq 0$ and $v_n \in C([0,T];Y_1)\bigcap C^1([0,T];Y_{-1})$ for all $n \geq 1$.

It is straightforward to obtain from (4.29)-(4.31)

$$\begin{aligned} v_{n+1}(x,t) = & \int_t^T \int_0^\pi \sum_{m=1}^\infty \frac{\sqrt{2}K}{\sqrt{\pi}m^2} (v_{n+1} + \bar{w}_n)^+(y,s) \sin{(my)} \\ & \cdot \sin{(mx)} \sin{(m^2(t-s))} dyds, \\ \frac{\partial v_{n+1}}{\partial t}(x,t) = & \int_t^T \int_0^\pi \sum_{m=1}^\infty \frac{\sqrt{2}K}{\sqrt{\pi}} (v_{n+1} + \bar{w}_n)^+(y,s) \sin{(my)} \\ & \cdot \sin{(mx)} \cos{(m^2(t-s))} dyds. \end{aligned}$$

Therefore

$$v_{n+1}(x,0) = -\int_0^T\int_0^\pi \sum_{m=1}^\infty \frac{\sqrt{2}K}{\sqrt{\pi}m^2}(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)\sin(mx)\sin(m^2s)dyds,$$
$$\frac{\partial v_{n+1}}{\partial t}(x,0) = \int_0^T\int_0^\pi \sum_{m=1}^\infty \frac{\sqrt{2}K}{\sqrt{\pi}}(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)\sin(mx)\cos(m^2s)dyds.$$

Thus

$$\begin{aligned}
&\|v_{n+1}(x,0)\|_{Y_3}^2\\
&=\sum_{m=1}^\infty K^2m^2\left(\int_0^T\int_0^\pi\sqrt{\frac{2}{\pi}}(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)\sin(m^2s)dyds\right)^2\\
&\leq K^2\sum_{m=1}^\infty m^2\int_0^T\left(\int_0^\pi\sqrt{\frac{2}{\pi}}(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)dy\right)^2ds\int_0^T\sin^2(m^2s)ds\\
&\leq K^2T\int_0^T\sum_{m=1}^\infty m^2\left(\sqrt{\frac{2}{\pi}}\int_0^\pi(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)dy\right)^2ds\\
&=K^2T\int_0^T\|(v_{n+1}+\bar{w}_n)^+\|_{Y_1}^2ds \;\leq\; K^2T\int_0^T\|(v_{n+1}+\bar{w}_n)\|_{Y_1}^2ds\\
&=K^2T\|(v_{n+1}+\bar{w}_n)\|_{L^2(0,T;Y_1)}^2,
\end{aligned}\tag{4.41}$$

and

$$\begin{aligned}
&\left\|\frac{\partial v_{n+1}}{\partial t}(x,0)\right\|_{Y_1}^2\\
&=\sum_{m=1}^\infty K^2m^2\left(\int_0^T\int_0^\pi\sqrt{\frac{2}{\pi}}(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)\cos(m^2s)dyds\right)^2\\
&\leq K^2\sum_{m=1}^\infty m^2\int_0^T\left(\int_0^\pi\sqrt{\frac{2}{\pi}}(v_{n+1}+\bar{w}_n)^+(y,s)\sin(my)dy\right)^2ds\int_0^T\cos^2(m^2s)ds\\
&\leq K^2T\int_0^T\|(v_{n+1}+\bar{w}_n)^+\|_{Y_1}^2ds \;\leq\; K^2T\|(v_{n+1}+\bar{w}_n)\|_{L^2(0,T;Y_1)}^2.
\end{aligned}\tag{4.42}$$

Thus $v_{n+1}^0(x) \in Y_3$, $v_{n+1}^1(x) \in Y_1$.

Then by mathematical induction method $v_n^0(x) \in Y_3$ and $v_n^1(x) \in Y_1$ for all $n \geq 0$.

Therefore F is a mapping from $Y_3 \times Y_1$ to itself. Moreover, we have for all $n \geq 1$

$$\left\| F(\{v_n^0, v_n^1\}) - F(\{v_{n-1}^0, v_{n-1}^1\}) \right\|_{Y_3 \times Y_1}^2 = \|v_{n+1}^0 - v_n^0\|_{Y_3}^2 + \|v_{n+1}^1 - v_n^1\|_{Y_1}^2.$$

From (4.41) and (4.42) we obtain, for $n \geq 1$,

$$\begin{aligned}
\|v_{n+1}^0 - v_n^0\|_{Y_3}^2 &\leq K^2 T \int_0^T \|(v_{n+1} - \bar{w}_n)^+ - (v_n - \bar{w}_{n-1})^+\|_{Y_1}^2 ds \\
&\leq K^2 T \int_0^T \|(v_{n+1} - \bar{w}_n) - (v_n - \bar{w}_{n-1})\|_{Y_1}^2 ds \\
&\leq 2K^2 T \int_0^T \left(\|v_{n+1} - v_n\|_{Y_1}^2 + \|\bar{w}_n - \bar{w}_{n-1}\|_{Y_1}^2 \right) ds \\
&\leq 2K^2 T^2 \left(\|v_{n+1} - v_n\|_{L^\infty(0,T;Y_1)}^2 + \|\bar{w}_n - \bar{w}_{n-1}\|_{L^\infty(0,T;Y_1)}^2 \right), \quad (4.43) \\
\|v_{n+1}^1 - v_n^1\|_{Y_1}^2 &\leq K^2 T \int_0^T \|(v_{n+1} - \bar{w}_n)^+ - (v_n - \bar{w}_{n-1})^+\|_{Y_1}^2 ds \\
&\leq 2K^2 T^2 \left(\|v_{n+1} - v_n\|_{L^\infty(0,T;Y_1)}^2 + \|\bar{w}_n - \bar{w}_{n-1}\|_{L^\infty(0,T;Y_1)}^2 \right). \quad (4.44)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|v_{n+1} - v_n\|_{L^\infty(0,T;Y_1)}^2 \\
&= \max_{t\in[0,T]} \left| \sum_{m=1}^\infty \frac{2K^2}{\pi m^2} \left(\int_t^T \int_0^\pi [(v_{n+1} + \bar{w}_n)^+ - (v_n + \bar{w}_{n-1})^+] \sin(my) \right.\right. \\
&\qquad\qquad \left.\left. \cdot \sin(m^2(t-s)) dy ds \right)^2 \right| \\
&\leq \max_{t\in[0,T]} \left| \sum_{m=1}^\infty \frac{2K^2}{\pi m^2} \int_t^T \left(\int_0^\pi [(v_{n+1} + \bar{w}_n)^+ - (v_n + \bar{w}_{n-1})^+] \sin(my) dy \right)^2 ds \right. \\
&\qquad\qquad \left. \cdot \int_t^T \sin^2(m^2(t-s)) ds \right| \\
&\leq \max_{t\in[0,T]} \left| \int_t^T K^2 T \sum_{m=1}^\infty \left(\sqrt{\frac{2}{\pi}} \int_0^\pi [(v_{n+1} + \bar{w}_n)^+ - (v_n + \bar{w}_{n-1})^+] \sin(my) dy ds \right)^2 ds \right| \\
&\leq \max_{t\in[0,T]} \left| \int_t^T K^2 T \|(v_{n+1} + \bar{w}_n)^+ - (v_n + \bar{w}_{n-1})^+\|_{Y_0}^2 ds \right| \\
&\leq \max_{t\in[0,T]} \left| \int_t^T K^2 T \|(v_{n+1} + \bar{w}_n) - (v_n + \bar{w}_{n-1})\|_{Y_0}^2 ds \right| \\
&\leq \max_{t\in[0,T]} \left| \int_t^T K^2 T \|((v_{n+1} + \bar{w}_n) - (v_n + \bar{w}_{n-1})\|_{Y_1}^2 ds \right|
\end{aligned}$$

$$\leq K^2T^2\|((v_{n+1}+\bar{w}_n)-(v_n+\bar{w}_{n-1})\|^2_{L^\infty(0,T;Y_1)}$$

$$\leq 2K^2T^2\left(\|v_{n+1}-v_n\|^2_{L^\infty(0,T;Y_1)}+\|\bar{w}_n-\bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)}\right).$$

Thus it follows that

$$\|v_{n+1}-v_n\|^2_{L^\infty(0,T;Y_1)} \leq \frac{2K^2T^2}{1-2K^2T^2}\|\bar{w}_n-\bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)}. \tag{4.45}$$

By (4.45), (4.43) and (4.44) become

$$\|v^0_{n+1}-v^0_n\|^2_{Y_3} \leq \frac{2K^2T^2}{1-2K^2T^2}\|\bar{w}_n-\bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)}, \tag{4.46}$$

$$\|v^1_{n+1}-v^1_n\|^2_{Y_1} \leq \frac{2K^2T^2}{1-2K^2T^2}\|\bar{w}_n-\bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)}. \tag{4.47}$$

Now for system (4.25)-(4.27), if we let $\bar{w}_n(x,t)=\sum_{m=1}^{\infty}a_{n,m}(t)\sin(mx)$, then we obtain

$$\begin{aligned}\bar{w}_n(x,t)=&\sum_{m=1}^{\infty}\sqrt{\frac{2}{\pi}}\int_0^\pi(w_0-v^0_n)(y)\sin(my)\sin(mx)\cos(m^2t)dy\\&+\sum_{m=1}^{\infty}\sqrt{\frac{2}{\pi}}\int_0^\pi\frac{1}{m^2}(w_1-v^1_n)(y)\sin(my)\sin(mx)\sin(m^2t)dy\\&+\sum_{m=1}^{\infty}\frac{\sqrt{2}(\cos(ma)-\cos(mb))}{m\sqrt{\pi}}\int_0^t u_n(s)\sin(mx)\sin(m^2(t-s))ds.\end{aligned}$$

Therefor we may have the following estimation

$$\begin{aligned}&\|\bar{w}_n-\bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)}\\&\leq 3\max_{t\in[0,T]}\left|\sum_{m=1}^{\infty}m^2\left(\sqrt{\frac{2}{\pi}}\int_0^\pi(v^0_n-v^0_{n-1})(y)\sin(my)dy\right)^2\cos^2(m^2t)\right|\\&\quad+3\max_{t\in[0,T]}\left|\sum_{m=1}^{\infty}m^2\left(\sqrt{\frac{2}{\pi}}\int_0^\pi\frac{1}{m^2}(v^1_n-v^1_{n-1})(y)\sin(my)dy\right)^2\sin^2(m^2t)\right|\\&\quad+3\left\|\sum_{m=1}^{\infty}\frac{\sqrt{2}(\cos(ma)-\cos(mb))}{m\sqrt{\pi}}\int_0^t(u_n(s)-u_{n-1}(s))\right.\\&\qquad\qquad\left.\cdot\sin(m^2(t-s))ds\sin(mx)\right\|^2_{L^\infty(0,T;Y_1)}\end{aligned}$$

$$
\begin{aligned}
&\leq 3\sum_{m=1}^{\infty} m^2 \left(\sqrt{\frac{2}{\pi}}\int_0^{\pi}(v_n^0 - v_{n-1}^0)(y)\sin(my)dy\right)^2 \\
&\quad +3\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\sqrt{\frac{2}{\pi}}\int_0^{\pi}(v_n^1 - v_{n-1}^1)(y)\sin(my)dy\right)^2 \\
&\quad +3\Bigg\| \sum_{m=1}^{\infty}\frac{\sqrt{2}(\cos(ma)-\cos(mb))}{m\sqrt{\pi}}\int_0^t (u_n(s)-u_n(s)) \\
&\qquad\qquad\qquad \cdot \sin(m^2(t-s))ds\sin(mx)\Bigg\|^2_{L^\infty(0,T;Y_1)} \\
&\leq 3\|(v_n^0 - v_{n-1}^0)\|^2_{Y_1} + 3\|(v_n^1 - v_{n-1}^1)\|^2_{Y_{-1}} \\
&\quad +3\Bigg\| \sum_{m=1}^{\infty}\frac{\sqrt{2}(\cos(ma)-\cos(mb))}{n\sqrt{\pi}}\int_0^t (u_n(s)-u_{n-1}(s)) \\
&\qquad\qquad\qquad \cdot \sin(m^2(t-s))ds\sin(mx)\Bigg\|^2_{L^\infty(0,T;Y_1)}.
\end{aligned}
$$

Then, by Lemma 4.4, we have

$$
\begin{aligned}
&\|\bar{w}_n - \bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)} \\
&\qquad \leq 3\|v_n^0 - v_{n-1}^0\|^2_{Y_3} + 3\|v_n^1 - v_{n-1}^1\|^2_{Y_1} + 3C_0\|u_n - u_{n-1}\|^2_{L^2(0,T)}. \qquad (4.48)
\end{aligned}
$$

Now assume $\{\phi_n, \psi_n\}$ and $\{\phi_{n-1}, \psi_{n-1}\}$ are two pairs of solutions of systems (4.32)-(4.34) and (4.35)-(4.37), correspond to $\{v_n^0, v_n^1\}$ and $\{v_{n-1}^0, v_{n-1}^1\}$ respectively.

Thus

$$
u_n(t) = \frac{\partial \phi_n}{\partial x}(b,t) - \frac{\partial \phi_n}{\partial x}(a,t), \quad u_{n-1}(t) = \frac{\partial \phi_{n-1}}{\partial x}(b,t) - \frac{\partial \phi_{n-1}}{\partial x}(a,t),
$$

Since

$$
\begin{aligned}
\langle \Lambda\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\rangle &= \left\langle \left\{\frac{\partial \psi}{\partial t}(x,0), -\psi(x,0)\right\}, \{\phi^0,\phi^1\}\right\rangle \\
&= \int_0^{\pi}\left(\frac{\partial \psi}{\partial t}(x,0)\phi^0 - \psi(x,0)\phi^1\right)dx.
\end{aligned}
$$

We have the following important estimate

$$
\begin{aligned}
&\|u_n(t) - u_{n-1}(t)\|^2_{L^2(0,T)} \\
&= \left\langle \left\{ \frac{\partial \psi_n}{\partial t}(x,0) - \frac{\partial \psi_{n-1}}{\partial t}(x,0), -\psi_n(x,0) + \psi_{n-1}(x,0) \right\}, \{\phi_n^0 - \phi_{n-1}^0, \phi_n^1 - \phi_{n-1}^1\} \right\rangle \\
&= \left\langle \left\{ \frac{\partial \psi_n}{\partial t}(x,0) - \frac{\partial \psi_{n-1}}{\partial t}(x,0), -\psi_n(x,0) + \psi_{n-1}(x,0) \right\}, \right. \\
&\qquad\qquad \left. \Lambda^{-1} \left\{ \frac{\partial \psi_n}{\partial t}(x,0) - \frac{\partial \psi_{n-1}}{\partial t}(x,0), -\psi_n(x,0) + \psi_{n-1}(x,0) \right\} \right\rangle \\
&\leq \|\Lambda^{-1}\| \left\langle \left\{ \frac{\partial \psi_n}{\partial t}(x,0) - \frac{\partial \psi_{n-1}}{\partial t}(x,0), -\psi_n(x,0) + \psi_{n-1}(x,0) \right\}, \right. \\
&\qquad\qquad \left. \left\{ \frac{\partial \psi_n}{\partial t}(x,0) - \frac{\partial \psi_{n-1}}{\partial t}(x,0), -\psi_n(x,0) + \psi_{n-1}(x,0) \right\} \right\rangle \\
&= \|\Lambda^{-1}\| \int_0^\pi \left[\left(\frac{\partial \psi_n}{\partial t}(x,0) - \frac{\partial \psi_{n-1}}{\partial t}(x,0) \right)^2 + (-\psi_n(x,0) + \psi_{n-1}(x,0))^2 \right] dx \\
&= \|\Lambda^{-1}\| \left[\int_0^\pi \left(\frac{\partial \bar{w}_n}{\partial t}(x,0) - \frac{\partial \bar{w}_{n-1}}{\partial t}(x,0) \right)^2 dx + \int_0^\pi (\bar{w}_n(x,0) - \bar{w}_{n-1}(x,0))^2 \, dx \right] \\
&= \|\Lambda^{-1}\| \left[\int_0^\pi \left(v_n^1(x) - v_{n-1}^1(x) \right)^2 dx + \int_0^\pi \left(v_n^0(x) - v_{n-1}^1(x) \right)^2 dx \right] \\
&\leq \|\Lambda^{-1}\| \left[\|v_n^1 - v_{n-1}^1\|^2_{Y_0} + \|v_n^0 - v_{n-1}^0\|^2_{Y_0} \right] \\
&\leq \|\Lambda^{-1}\| \left[\|v_n^1 - v_{n-1}^1\|^2_{Y_1} + \|v_n^0 - v_{n-1}^0\|^2_{Y_3} \right]. \qquad (4.49)
\end{aligned}
$$

Therefore

$$
\|\bar{w}_n - \bar{w}_{n-1}\|^2_{L^\infty(0,T;Y_1)} \leq 3(1 + C_0\|\Lambda^{-1}\|) \left[\|v_n^0 - v_{n-1}^0\|^2_{Y_3} + \|v_n^1 - v_{n-1}^1\|^2_{Y_1} \right]. \quad (4.50)
$$

Thus by (4.46), (4.47) and (4.50), we obtain

$$
\begin{aligned}
&\left[\|v_{n+1}^0 - v_n^0\|^2_{Y_3} + \|v_{n+1}^1 - v_n^1\|^2_{Y_1} \right] \\
&\leq \frac{6K^2T^2}{1 - 2K^2T^2} (1 + C_0\|\Lambda^{-1}\|) \left[\|v_n^0 - v_{n-1}^0\|^2_{Y_3} + \|v_n^1 - v_{n-1}^1\|^2_{Y_1} \right].
\end{aligned}
$$

This is for all $n \geq 1$

$$
\left\| F(\{v_n^0, v_n^1\}) - F(\{v_{n-1}^0, v_{n-1}^1\}) \right\|^2_{Y_3 \times Y_1}
$$

$$\leq \frac{6K^2T^2}{1-2K^2T^2}(1+C_0\|\Lambda^{-1}\|)\|\{v_n^0, v_n^1\} - \{v_{n-1}^0, v_{n-1}^1\}\|^2_{Y_3\times Y_1}.$$

From this we have the conclusion: when T is small enough, say

$$T^2 < \frac{1}{K^2(6\|\Lambda^{-1}\|C_0+8)},$$

we have

$$(1+C_0\|\Lambda^{-1}\|)\frac{6K^2T^2}{1-2K^2T^2} < 1,$$

which implies that F is a contraction mapping. By the Banach Contraction Mapping Principle, there is a unique solution $\{v^0, v^1\} \in Y_3 \times Y_1$ such that

$$\{v^0, v^1\} = F(\{v^0, v^1\}).$$

Let $v(x,0) = v^0(x)$ and $\dfrac{\partial v}{\partial t}(x,0) = v^1(x)$ in system (4.29)-(4.31), we will get a control function $u(t)$ such that the nonlinear system (4.1)-(4.3) is exactly L^2-controllable. Thus we have proved the Theorem 4.2. □

CHAPTER 5

FUTURE WORK

In this dissertation, we have studied the control problems of the 1-D Lazer-McKenna suspension bridge equation. A natural question is: can the results obtained in this dissertation be extended to the high dimensional cases?

- Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\Gamma \triangleq \partial\Omega$, $\omega \subset \Omega$ and $w^0(x)$, $w^1(x)$ are in proper Hilbert spaces. $u(x, y, t)$ to be the control. Consider

$$\begin{cases} w_{tt} + \Delta^2 w + w^+ = u(x,y,t)\chi_\omega, & x \in \Omega,\ t \in (0,T), \\ w(x,t) = 0, \quad \Delta w(x,t) = 0, & x \in \Gamma,\ t \in (0,T), \\ w(x,0) = w^0(x), \qquad w_t(x,0) = w^1(x), & x \in \Omega. \end{cases}$$

  To study the exact controllability of this system, the key step is to establish a Carleman estimate for the high dimensional cases. Similar ideas in Chapter 3 may apply.

The initial suspension bridge model introduced by Lazer and McKenna [Lazer and McKenna (1990)] consists two coupled beam and wave equations. The general suspension bridge model proposed by Ahmed [Ahmed and Harbi (1998)] is a system of four coupled equations. Can we still apply the same ideas developed in this dissertation to show the exact controllability of these coupled systems?

- Let $\Omega = (0, L) \subset \mathbb{R}$, $\omega = (a, b) \subset \Omega$, and $u^0(x)$, $u^1(x)$, $w^0(x)$, $w^1(x)$ be in proper

Hilbert spaces. Let $u(x,t)$ denote the control function. Consider

$$\begin{cases} m_c u_{tt} - Qu_{xx} - K(w-u)^+ = 0, & x \in \Omega,\ 0 < t < T, \\ m_b w_{tt} + EIw_{xxxx} + K(w-u)^+ = u(x,t)\chi_\omega, & x \in \Omega,\ 0 < t < T, \\ u(0,t) = u(L,t) = 0, & 0 < t < T, \\ w(0,t) = w(L,t) = 0, & 0 < t < T, \\ w_{xx}(0,t) = w_{xx}(L,t) = 0, & x \in \Omega, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega, \\ w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), & x \in \Omega. \end{cases}$$

The exact controllability of this system is a very challenging problem to study. A more challenging problem is to study the exact controllability of the general suspension bridge system developed by Ahmed [Ahmed and Harbi (1998)] with propriate controls.

In 2002, P. J. McKenna and K. S. Moore showed numerically in [McKenna and Moore (2002)] that, with w^+ being replaced by a nonlinear smooth function $f(w) = -\frac{1}{a}(e^{aw}-1)$ in the Lazer-McKenna suspension bridge equation, the solutions replicate more accurately the nonlinear phenomena in the collapse of the First Tacoma Narrows Suspension Bridge.

- Let $\Omega = (0,L)$, a, $b \in \Omega$ and u to be the control function. Consider

$$\begin{cases} w_{tt} + w_{xxxx} + f(w) = u(t)\frac{d}{dx}[\delta_b(x) - \delta_a(x)], & x \in \Omega,\ t > 0, \\ w(0,t) = w(L,t) = 0, \quad w_{xx}(0,t) = w_{xx}(L,t) = 0, & t > 0, \\ w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), & x \in \Omega. \end{cases}$$

  Can we show the system is exact controllable? Moreover, it is also important to develop numerical simulations of nonlinear control systems.

APPENDIX

AN EXAMPLE OF THE HILBERT UNIQUENESS METHOD

Here is a quick example for this -so called the Hilbert Uniqueness Method (HUM) from [Lions (1988)]: Let $\Omega \subset \mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$, and $x^0 \in \mathbb{R}^n$ be given. Let $m(x) = \{x_k - x_k^0\}$ and

$$\Gamma(x^0) = \{x | x \in \Gamma,\, m(x)\nu(x) \geq 0\}$$

where $\nu(x)$ is the unit outward normal vector to Γ. Let $\Gamma_*(x^0) = \Gamma \setminus \Gamma(x^0)$. Consider the following wave equation system with boundary control

$$\frac{\partial^2 y}{\partial t^2} - \triangle y = 0, \qquad (x,t) \in \Omega \times (0,T), \tag{0.1}$$

$$y = \begin{cases} u, & \text{on}\, \Gamma(x^0) \times (0,T), \\ 0, & \text{on}\, \Gamma_*(x^0) \times (0,T), \end{cases} \tag{0.2}$$

$$y(0) = y^0, \quad \frac{\partial y}{\partial t}(0) = y^1, \qquad x \in \Omega, \tag{0.3}$$

where u is the control function.

We wish to find a u (in a suitable Hilbert space) such that

$$y(x,T) = \frac{\partial y}{\partial t}(x,T) = 0. \tag{0.4}$$

We start with the wave equation

$$\begin{array}{ll} \dfrac{\partial^2 \phi}{\partial t^2} - \triangle \phi = 0, & \text{in}\, \Omega \times (0,T), \\ \phi = 0, & \text{on}\, \Gamma \times (0,T), \\ \phi(0) = \phi^0, \quad \dfrac{\partial \phi}{\partial t}(0) = \phi^1, & \text{in}\, \Omega. \end{array} \tag{0.5}$$

Given ϕ^0, ϕ^1 in appropriate Hilbert spaces, (0.5) admits a unique solution. We then solve

$$\begin{aligned} &\frac{\partial^2\psi}{\partial t^2} - \triangle\psi = 0, && \text{in}\,\Omega\times(0,T), \\ &\psi = \begin{cases} \partial\phi/\partial\nu, & \text{on}\,\Gamma(x^0)\times(0,T), \\ 0, & \text{on}\,\Gamma_*(x^0)\times(0,T), \end{cases} \\ &\psi(T) = \frac{\partial\psi}{\partial t}(T) = 0, && \text{in}\,\Omega. \end{aligned} \tag{0.6}$$

This is a non-homogeneous boundary value problem. The space where the solution lies will depend on the properties of ϕ^0, ϕ^1.

Since (0.6) always admits at least a weak solution. Therefore, given ϕ^0 and ϕ^1, we have defined, in a unique fashion,

$$\Lambda\{\phi^0,\phi^1\} = \left\{\frac{\partial\psi}{\partial t}(0), -\psi(0)\right\}.$$

Let us assume for a moment that−in appropriate Hilbert spaces−Λ (which depends on T) becomes invertible (for T large enough).

Then the problem is solved. Indeed, for given y^0, y^1, we solve

$$\Lambda\{\phi^0,\phi^1\} = \{y^1, -y^0\}.$$

We then solve (0.5) and we choose

$$u = \frac{\partial\phi}{\partial\nu} \qquad \text{on}\,\Gamma(x^0)\times(0,T).$$

Thus

$$y(u) = \psi;$$

Hence (0.4) holds and we have constructed a control u driving the state variable of system (0.1)-(0.3) to rest at time T.

To prove Λ is invertible, define the scalar product

$$\langle\Lambda\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\rangle = \int_\Omega \left(\frac{\partial\psi}{\partial t}(0)\phi^0 - \psi(0)\phi^1\right)dx.$$

By multiplying (0.6) by ϕ and using integration by parts, we obtain

$$\langle \Lambda\{\phi^0, \phi^1\}, \{\phi^0, \phi^1\}\rangle = \int_{\Gamma(x^0)\times(0,T)} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt.$$

The key step is to prove, for proper T,

$$\left(\int_{\Gamma(x^0)\times(0,T)} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt\right)^{\frac{1}{2}} \tag{0.7}$$

defines a norm on the set of initial data $\{\phi^0, \phi^1\}$. The norm (0.7) is then equivalent to the usual norm of $H_0^1(\Omega) \times L^2(\Omega)$.

Thus Λ defines an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$, where $H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega)$. Therefore one can conclude that

> *Let T be large enough, say $T > T_0$. For any y^0, y^1 given in $L^2(\Omega) \times H^{-1}(\Omega)$, there exists $u \in L^2(\Gamma(x^0) \times (0,T))$ such that the control u drives the system from $\{y^0,\ y^1\}$ at $t = 0$ to rest at $t = T$.*

The proof of (0.7) is equivalent to the proof of the following observability inequality:

$$\int_{\Gamma(x^0)\times(0,T)} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \geq C(T - T_0)\left[\|\phi^0\|^2_{H_0^1(\Omega)} + \|\phi^1\|^2_{L^2(\Omega)}\right].$$

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