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Structures on a K3 surface

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STRUCTURES ON A K3 SURFACE

by

Nathan P. Rowe

Bachelor of Science
University of Nevada, Las Vegas
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A dissertation submitted in partial fulfullment of
the requirements for the

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Nathan P. Rowe

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Structures on a K3 Surface

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ABSTRACT

Structures on a K3 Surface

by

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In the first part of this paper, we examine properties of K3 surfaces of the form

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) + Axyz - 2 = 0.$$

We show the surface has Picard number $q \geq 12$ by finding 12 curves whose equivalence
classes are linearly independent. These curves have self intersection -2. We find the
lattice representations of the single-coordinate swapping automorphisms in x, y, and
z. We show that we have enough of the Lattice to make accurate predictions of poly-
nomial degree growth under the automorphisms. We describe these automorphisms
in terms of operations on elliptic curves.

In the second part of this paper, we look at curves whose shape is sketched by
the orbit of a point under the composed automorphisms mentioned above. These
curves were studied by Fields Medalist Kurt McMullen. One can prove these curves
are non-algebraic through the use of intersection theory. We offer a simple counting
argument that one such curve is not algebraic. We do this by counting points in $\mathbb{F}_p$
and comparing this to the Hasse-Weil upper bound for such curves.

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CHAPTER 1

INTRODUCTORY MATERIAL

1.1 Preface to the Reader

In this chapter we give some of the background material necessary to understand

the paper. Further background is given as it is needed within the later chapters. It is

assumed that the reader has a basic understanding of fields, and matrix operations.

1.2 Affine vs Projective Space

Here we give a review on some of the concepts of affine and projective space.

Common examples of affine geometries or affine spaces include $\mathbb{R}^n$ and $\mathbb{C}^n$. We can

similarly define an affine space with any field, F.

The One Dimensional Projective Space, $\mathbb{P}(F)$

We will define $\mathbb{P}(F)$, the projective extension of a field F. $\mathbb{P}(F)$ is usually defined

to be the set of lines passing through the origin in $F \times F$. Such a line has the equation

$$\{(x, y) \quad | \quad ax + by = 0\}.$$

Where a and b come from F and a, b are not both zero. Hence we can think of

$\mathbb{P}(F)$ as the set $\{(a, b) \neq (0, 0) \quad | \quad a, b \in F\}$ with the equivalence relation defined by

$(a, b) \equiv (c, d) \Leftrightarrow \{(x, y) \quad | \quad ax + by = 0\} = \{(x, y) \quad | \quad cx + dy = 0\}$.

This equivalence relation simplifies down to $(a, b) \equiv (c, d)$ if and only if there

is some $\lambda \in F$ so that $(a, b) = (\lambda c, \lambda d)$. This allows us to normalize each pair of

coordinates to think of them as being truly 1-dimensional. For this reason we often

write $\mathbb{P}(F) = \{(x_1, x_2) \quad | \quad x_2 = 1 \text{ or } (x_2 = 0 \text{ and } x_1 = 1)\}$. We associate the point

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(x_1, x_2) with the value $\dfrac{x_1}{x_2}$ if $x_2 \neq 0$. The point $(1, 0) \equiv (x_1, 0)$ we think of as a point at ∞. We call the points with $x_2 \neq 0$ the affine part of $\mathbb{P}(F)$ as it is in correspondence with the affine space, F.

The n-Dimensional Space, $\mathbb{P}^n(F)$

There are two natural ways to extend this into multi-dimensional spaces, the most common of which is written $\mathbb{P}^n(F)$. $\mathbb{P}^1(F)$ is the space we just defined by looking at lines through the origin in F^2. We similarly define $\mathbb{P}^n(F)$ to be F^{n+1} modulo lines through the origin.

The projective space, $\mathbb{P}^2(F)$, is the set of 3-tuples, (x, y, z), equipped with the equivalence relation $(x_1, y_1, z_1) \equiv (x_2, y_2, z_2)$ if and only if there is some $\lambda \in F$ such that $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$.. This allows us to normalize the point (x, y, z) to divide away z when possible. If $z \neq 0$, we identify the point $(x, y, z) \in \mathbb{P}^2(F)$ with the point $\left(\dfrac{x}{z}, \dfrac{y}{z} \right) \in F^2$. The collection of points with $z = 0$ we think of as forming a line at infinity. This line is isomorphic to any other line in $\mathbb{P}^2(F)$. That is, it is isomorphic to $\mathbb{P}^1(F)$. In fact, there is nothing special about the line with $z = 0$. removing any one line in $\mathbb{P}^2(F)$ yields a space isomorphic to F^2.

Similarly we have that $\mathbb{P}^n(F)$ is a set of $n + 1$ tuples that are not all zero. We divide away the last component of this $n + 1$ tuple when it is different than zero. Otherwise we think of the point as being located at infinity. $\mathbb{P}^n(F)$ has infinite part isomorphic to $\mathbb{P}^{n-1}(F)$.

We explain $\mathbb{P}^2$ in terms of projections in order to explain the nomenclature. If we pick (x_0, y_0, z_0) to be any point in $\mathbb{Q}^3$ different from the origin, this point defines

a unique line through the origin. If our point has z_0 different than zero, the line it defines will intersect the plane $z = 1$ in exactly one location, namely the point $(x_0/z_0, y_0/z_0, 1)$. If $z_0 = 0$ then we think of our line as intersecting the plane $z = 1$ at infinity. The two free variables left when $z = 0$ form a line at infinity isomorphic to $\mathbb{P}^1$. This idea is called a projection of the space, $\mathbb{Q}^3$ onto the plane $z = 1$. We recognize that this projection is the same idea as $\mathbb{Q}^3$ modulo lines, and hence it produced the space $\mathbb{P}^2$. A similar projection idea can be used to describe $\mathbb{P}^n$.

We will mostly be dealing with $\mathbb{P}^n(\mathbb{C})$, and hence we will write $\mathbb{P}^n$ to mean $\mathbb{P}^n(\mathbb{C})$ for simplicity of notation.

The n-Dimensional Space, $\mathbb{P}^1 \times \mathbb{P}^1 \times ... \times \mathbb{P}^1$

Another natural, though far less common, way to extend this idea of projective space into multiple dimensions is to take the direct product of single dimensional projective lines. For example, The space $\mathbb{P}^1 \times \mathbb{P}^1$ is a two dimensional. In other words, it is surface. We denote its points with $((x_1, x_2), (y_1, y_2))$. Here each of the two components is thought of as a point in $\mathbb{P}^1$ and hence can be normalized to be thought of as 1-dimensional. $\mathbb{P}^1 \times \mathbb{P}^1$ has two projective lines at infinity, $((x_1, x_2), (1, 0))$ and $((1, 0), (y_1, y_2))$. We will often write these lines in affine coordinates as (x, ∞) and (∞, y). These two lines share a point of intersection at $((1, 0), (1, 0))$ which we will write as (∞, ∞).

Most of our work in this paper will be done in the space, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This space is three dimensional with points in the space written $((x_1, x_2), (y_1, y_2), (z_1, z_2))$. The space has affine part isomorphic to $\mathbb{C}^3$ and at infinity it has 3 planes isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Each pair of planes share a line isomorphic to $\mathbb{P}^1$, and all three of these

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lines intersect at the point $((1,0),(1,0),(1,0))$ or (∞,∞,∞). If we attempt to simply account for all of the points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and ignore its structure, we can say that

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{C}^3 \cup \begin{pmatrix} 3 \\ 1 \end{pmatrix} \mathbb{C}^2 \cup \begin{pmatrix} 3 \\ 2 \end{pmatrix} \mathbb{C} \cup (\infty,\infty,\infty).$$

Extending this to n dimensions, we have $\mathbb{P}^1 \times \mathbb{P}^1 \times ... \times \mathbb{P}^1$. Its points are written as a an n-tuple of 2-tuples. Counting points again, we see that

$$\mathbb{P}^1 \times \mathbb{P}^1 \times ... \times \mathbb{P}^1 = \bigcup_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \mathbb{C}^i$$

In the above equation, we think of $\mathbb{C}^0$ as the single point, $(\infty,\infty, \quad ... \quad ,\infty)$.

Equations of Curves in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let $\mathcal{C}_{aff}$ be an algebraic irreducible affine curve in $\mathbb{C}^3$. Since we think of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as a projective extension of $\mathbb{C}^3$, we expect that we should be able to extend this affine curve to a projective curve. We can certainly imbed any affine curve in projective space by simply including all the same points. The problem with this method is that it does not necessarily preserve the continuity of the curve. A curve that is simply imbedded into projected space will be undefined and hence discontinuous at infinity.

A question with a less obvious answer is "can we continuously extend $\mathcal{C}_{aff}$ into a projective curve?" The answer to this question is yes, and furthermore, such an extension is unique. Thus we denote $\mathcal{C}_{proj}$ to be the unique continuous projective extension of the affine curve, $\mathcal{C}_{aff}$. We generate $\mathcal{C}_{proj}$ as follows:

Let $F_{aff}(x,y,z) = 0, G_{aff}(x,y,z) = 0$ be the polynomial equations defining $\mathcal{C}_{aff}$. Let $n_x = \deg_x(F_A(x,y,z))$ be the largest degree of x in $F_{aff}(x,y,z)$. Similarly let $n_y = \deg_y(F_{aff}(x,y,z))$ and $n_z = \deg_z(F_{aff}(x,y,z))$. Also let m_x, m_y, and m_z be the

degrees in x, y, and z for $G_{aff}(x, y, z)$. Then let us define

$$F_{proj}((x_1, x_2), (y_1, y_2), (z_1, z_2)) = F_A(x_1/x_2, y_1/y_2, z_1/z_2)x_2^{n_x}y_2^{n_y}z_2^{n_z}.$$

and

$$G_{proj}((x_1, x_2), (y_1, y_2), (z_1, z_2)) = G_A(x_1/x_2, y_1/y_2, z_1/z_2)x_2^{m_x}y_2^{m_y}z_2^{m_z}.$$

We note that F_{proj} and G_{proj} are polynomials in x_1, x_2, y_1, y_2, z_1, and z_2 and have no variables remaining in the denominator. This explains our choice of each n_i and m_j. We should also note that F_{proj} and G_{proj} are homogeneous in all 6 components. Then the polynomial equations,

$$F_{proj}((x_1, x_2), (y_1, y_2), (z_1, z_2)) = 0, \quad G_{proj}((x_1, x_2), (y_1, y_2), (z_1, z_2)) = 0$$

define our projective curve, $\mathcal{C}_{proj}$. The curve $\mathcal{C}_{proj}$ has affine part isomorphic to $\mathcal{C}_{aff}$ and continuously extends into its infinite part.

We close by stating that this same method can be used to uniquely extend affine surfaces into projective surfaces. We also mention that since the extension from affine coordinates is unique we will often (for simplicity) define a projective entity with its affine equation.

1.3 Intersection Theory

We start with the familiar. Recall that in $\mathbb{C}^2$, we have the Fundamental Theorem of Algebra. This states that if $f(x)$ is a polynomial of degree n then the equation, $f(x) = 0$ has exactly n many solutions (counting multiplicity) over the complex numbers.

One step more complicated we may take two algebraic curves X and Y in $\mathbb{P}^2(\mathbb{C})$ with degree n and m respectively. The classical result, Bezout's Theorem, says that if X and Y have no components in common, then the number of intersections between X and Y is exactly equal to mn counting multiplicity over the complex numbers.

This already suggests a method of calculating intersections by passing off to equivalence classes. In this example, the equivalence relation would be defined by $X \equiv Y$ if $\deg(X) = \deg(Y)$. We will see later that it is natural to think of the intersection of two curves as the product of their equivalence classes. Thus

$$\#\{X \cap Y\} = [X] \cdot [Y] = \deg(X)\deg(Y).$$

The Intersection Product

A slightly more complicated example presents itself with the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Here we take two projective curves on the surface that have affine equations $f(x, y) = 0$ and $g(x, y) = 0$. To count the intersections between these two curves we use the formula

$$\#\{f(x, y) \cap g(x, y)\} = \deg_x(f)\deg_y(g) + \deg_y(f)\deg_x(g).$$

This idea further suggests passing off to equivalence classes for the curves f and g. We think of the equivalence classes as belonging to a two dimensional vector space with basis elements D_1 and D_2. Here D_1 represents a curve, C, with $\deg_x(C) = 1$, $\deg_y(C) = 0$; and D_2 represents a curve with $\deg_x(C) = 0$, $\deg_y(C) = 1$.

If we let f have $\deg_x(f) = n_1$ and $\deg_y(f) = n_2$ then $[f] = n_1 D_1 + n_2 D_2$ which we can write in vector notation as $\begin{bmatrix} n_1 & n_2 \end{bmatrix}$. Now let $[g] = \begin{bmatrix} m_1 & m_2 \end{bmatrix}$. We define a product so that $[f] \cdot [g] = \#\{f \cap g\}$. Note that $[f] \cdot [g] = (n_1 D_1 + n_2 D_2) \cdot (m_1 D_1 + m_2 D_2)$.

Hence we see by distribution that

$$[f] \cdot [g] = n_1 m_1 D_1 \cdot D_1 + n_1 m_2 D_1 \cdot D_2 + n_2 m_1 D_1 \cdot D_2 + n_2 m_2 D_2 \cdot D_2$$

It was originally stated that $\#\{f \cap g\} = n_1 m_2 + n_2 m_1$. To satisfy this formula, we must have $D_1 \cdot D_1 = D_2 \cdot D_2 = 0$ and $D_1 \cdot D_2 = 1$.

Let us verify this:

Let $f_1(x, y) = x = 0$, and $f_2(x, y) = y = 0$ so that $D_1 = [f_1]$, and $D_2 = [f_2]$. Since $f_1 = f_2 = 0$ implies that $x = y = 0$, we see that

$$D_1 \cdot D_2 = [f_1] \cdot [f_2] = \#\{f_1(x, y) \cap f_2(x, y)\} = 1.$$

We now have the problem of calculating $D_1 \cdot D_1$ and $D_2 \cdot D_2$. Both of these intersection are self intersections of equivalence classes of curves. Luckily, since D_1 and D_2 both contain multiple curves in their equivalence classes, we need not be able to directly calculate the self intersection of a single curve. We instead pick two different curves in D_1 and find their intersection. Let $f_1(x, y) = x = 0$ as above and $g_1(x, y) = x - 1 = 0$ Both curves belong to D_1 and their intersection is empty since $f_1 = g_1$ only if $x = 0$ and $x = 1$. Hence we see that

$$D_1 \cdot D_1 = [f_1] \cdot [g_1] = \#\{f_1(x, y) \cap g_1(x, y)\} = 0.$$

Similarly we have that

$$D_2 \cdot D_2 = 0,$$

which completes our verification.

We may also write out $[f] \cdot [g]$ as a matrix product. Recall above that $[f] =$

$[n_1, \quad n_2]$, and $[g] = [m_1, \quad m_2]$. Then

$$[f] \cdot [g] = \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} D_1 \cdot D_1 & D_1 \cdot D_2 \\ D_1 \cdot D_2 & D_2 \cdot D_2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}.$$

Since we have calculated that $D_1 \cdot D_1 = D_2 \cdot D_2 = 0$ and $D_1 \cdot D_2 = 1$, we see that

$$[f] \cdot [g] = \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = n_1 m_2 + n_2 m_1.$$

The matrix, $J = [D_i \cdot D_j] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ used above is called the intersection matrix on the surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that $f \equiv g$ if and only if $\begin{bmatrix} \deg_x(f), & \deg_y(f) \end{bmatrix} = \begin{bmatrix} \deg_x(g), & \deg_y(g) \end{bmatrix}$. We also see that $[fg] = [f] + [g]$. Thus, by letting $D_1 = [x]$, and $D_2 = [y]$, we see that $[f] = \deg_x(f)D_1 + \deg_y(f)D_2$.

We extend this to a lattice by considering all linear combination of D_1 and D_2 with integer coefficients. The objects in this lattice are called divisors, and the group of divisors is called the Picard group or Picard lattice. Not all objects in the Picard Lattice represent curves. For example, the zero vector has no curves associated with it. Divisors which do represent curves are called effective divisors, and may be thought of as equivalence classes of curves. The dimension of the Picard group is called the Picard number, q, which we found to be 2. The matrix, $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, found above is called the intersection matrix.

Going back to the simpler example, curves in $\mathbb{P}^2$, the picard group is only 1-dimensional, and the intersection matrix is simply the identity matrix, [1].

The Picard group exists for any smooth surface, and we can always define the intersection of two curves as a product. We provided two very simple examples where

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the intersections number between two curves could be written down purely in terms of the degrees of each curve. This is not always the case. There is often some connection to degree, but it is not always quite as straight forward.

On $K3$ surfaces over $mathbbC$, which we define in a later section, the Picard number, q, varies in $[1, 20]$. Generically a $K3$ surface has Picard number 1. See [1] for a large class of $K3$ surfaces with $q = 3$, and for two classes of $K3$ surfaces with $q = 4$. All $K3$ surfaces over $\mathbb{Q}$ with $q = 20$ have been computed in [7]. In this paper, we present a class of surfaces with $q \geq 12$.

Self Intersection and the Adjunction Formula

Recall that to find J on the surface $\mathbb{P}^1 \times \mathbb{P}^1$, we needed to know

$$D_i \cdot D_j = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

In particular, we needed to know the value of $D_i \cdot D_i$. Above we managed to calculate this value by taking two different curves in the equivalence class D_i and manually computing their intersections. A problem arises when an equivalence class D has only one curve in it. Our original trick fails to be applicable.

To handle such situations, we must invoke the Adjunction Formula as applied to surfaces [4]. This says that for any X, a smooth surface, and any D, a smooth divisor of curves on X,

$$K_D = (K_X + D) \cdot D.$$

K_X above is the canonical divisor of the surface X. Also, since D is a divisor of curves, $K_D = 2g - 2$ where g is the genus of a curve in D. Solving for $D \cdot D$ we get

$$D \cdot D = 2g - 2 - K_X \cdot D$$

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This tells us that if we know the canonical divisor of X and the genus of a curve, C, we can always calculate the self intersection the divisor $[C]$. This is true even for divisors with only one element.

1.4 Surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

For any complete smooth algebraic surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, there are 3 natural divisors to consider.

Let $\mathcal{V}$ be a surface with defining equation $F(x, y, z) = 0$ written in affine form. We let $E_{x=0}$ be the curve on the $\mathcal{V}$ defined by the equation $F(0, y, z) = 0$. The divisor class $[E_{x=0}]$ is exactly those curves which intersect in the same way as $E_{x=0}$. That is to say, a curve, $\mathcal{E}$, is an element of $[E_{x=0}]$ if and only if for any algebraic curve, $\mathcal{C}$, on $\mathcal{V}$, $\#\{\mathcal{C} \cap \mathcal{E}\} = \#\{\mathcal{C} \cap E_{x=0}\}$.

We claim that for any constant value, c, the curve, $E_{x=c}$, is an element of $[E_{x=0}]$.

Proof. Let $\mathcal{C}$ be an arbitrary curve on $\mathcal{V}$ with defining equations, $F(x, y, z) = 0$, and $f(x, y, z) = 0$. We need to show that $\#\{\mathcal{C} \cap E_{x=0}\} = \#\{\mathcal{C} \cap E_{x=c}\}$ to verify the claim.

We first note that

$$\{\mathcal{C} \cap E_{x=0}\} = \{(f(x, y, z) = 0 \cap F(x, y, z) = 0) \cap (x = 0 \cap F(x, y, z) = 0)\},$$

and so

$$\{\mathcal{C} \cap E_{x=0}\} = \{f(0, y, z) = 0 \cap F(0, y, z) = 0\}.$$

Now, the curves defined by $f(0, y, z) = 0$ and $F(0, y, z) = 0$ can be thought of a curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and hence intersect according to our formula in section 1.3. Thus

we see that

$$\#\{(f(0, y, z) = 0) \cap (F(0, y, z) = 0)\} = \deg_y(f) \deg_z(F) + \deg_z(f) \deg_y(F).$$

We similarly find that $\{\mathcal{C} \cap E_{x=c}\} = \{f(c, y, z) = 0 \cap F(c, y, z) = 0\}$. Therefore $\#\{\mathcal{C} \cap E_{x=c}\} = \deg_y(f) \deg_z(F) + \deg_z(f) \deg_y(F)$. Hence we have that

$$\#\{\mathcal{C} \cap E_{x=c}\} = \#\{\mathcal{C} \cap E_{x=0}\}.$$

This proves our claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Since $[E_{x=0}] = [E_{x=c}]$, we may simply refer to this divisor as E_x. We may similarly define the devisors E_y and E_z. These three divisors are present in all such surfaces, $\mathcal{V}$.

$(2, 2, 2)$ Forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

A K3 surface over the complex numbers is defined to be a complete smooth simply connected surface with a trivial canonical divisor. This definition is very technical and offers little insight into what makes $K3$ surfaces interesting. One of the most important things that we can grab from the definition of a $K3$ surface is that for any $\mathcal{V}$ a $K3$ surface, $K_{\mathcal{V}}$, the canonical divisor of $\mathcal{V}$, is just the zero vector. Thus, recalling the adjunction formula from Section 1.3, for any smooth curve C on $\mathcal{V}$ we see that

$$[C] \cdot [C] = 2g - 2 - K_X \cdot [C]$$

simplifies to

$$[C] \cdot [C] = 2g - 2.$$

Note that any smooth curve of genus zero on $\mathcal{V}$ has self intersection -2. For this negative self intersection to make sense, the curve's divisor class must be otherwise

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empty. -2 curves do exist on some $K3$ surfaces and play an important role in the study of $K3$ surfaces. We will find many such curves in the family of surfaces studied in this paper.

Generically a $K3$ surface has no -2 curves. It also has Picard number 1 and has no automorphisms. However, there are many infinite families of $K3$ surfaces with very interesting automorphisms and larger Picard number. One interesting example of $K3$ surfaces is the family of $(2,2,2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We say a surface, $\mathcal{V}$, is a $(2,2,2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ if its defining affine equation, $F(x, y, z) = 0$ has degree 2 in each of x, y, and z. If we let $\mathcal{V}$ be any smooth, irreducible $(2,2,2)$ form, then $\mathcal{V}$ is a $K3$ surface [12]. A generic $K3$ surface of this form has Picard number $q = 3$. A natural basis for the Picard lattice are the three divisors, E_x, E_y, and E_z which are clearly linearly independent. Such a surface also possesses obvious automorphisms. We call these automorphisms the quadratic swap and describe them in section 2.1.

One of the most important things that we can grab from the definition of a $K3$ surface is that for any $\mathcal{V}$ a $K3$ surface, $K_\mathcal{V}$, the canonical divisor of $\mathcal{V}$, is just the zero vector. Thus, recalling the adjunction formula from Section 1.3, for any smooth curve C on $\mathcal{V}$ we see that

$$[C] \cdot [C] = 2g - 2 - K_X \cdot [C]$$

simplifies to

$$[C] \cdot [C] = 2g - 2.$$

Note that any smooth curve of genus zero on $\mathcal{V}$ has self intersection -2. For this

negative self intersection to make sense, the curve's divisor class must be otherwise empty. We will find many such curves in the family of surfaces studied here.

CHAPTER 2

DESCRIBING AUTOMORPHISMS

2.1 Isometries and the Quadratic Swap

The foci of this paper are properties of a class of K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In affine space, it has the form $\mathcal{V} : F(x, y, z) = 0$ where

$$F(x, y, z) = (x^2 + 1)(y^2 + 1)(z^2 + 1) + Axyz - 2.$$

The obvious basic isometries present on $\mathcal{V}$ are (A) the 3 isometries that swap any 2 coordinates:

$$\phi_{x,y} : (x, y, z) \rightarrow (y, x, z),$$

$$\phi_{x,z} : (x, y, z) \rightarrow (z, y, x),$$

$$\phi_{y,z} : (x, y, z) \rightarrow (x, z, y);$$

(B) the 3 isometries that send any two coordinates to their negative:

$$N_{x,y} : (x, y, z) \rightarrow (-x, -y, z),$$

$$N_{x,z} : (x, y, z) \rightarrow (-x, y, -z),$$

$$N_{y,z} : (x, y, z) \rightarrow (x, -y, -z);$$

and their compositions. The closure of these elements gives us a group of basic isometries of order 24. It is isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The other natural automorphisms to look at are the quadratic swapping maps that fix two coordinates and send the 3rd coordinate to the other quadratic root of

the surface. That is to say, if we let x, and x' be the two roots of $F(X, y_0, z_0) = 0$, then let us define

$$\sigma_x : (x, y_0, z_0) \to (x', y_0, z_0).$$

Similarly we define

$$\sigma_y : (x_0, y, z_0) \to (x_0, y', z_0),$$

and

$$\sigma_z : (x_0, y_0, z) \to (x_0, y_0, z').$$

These 3 automorphisms generate an infinite group. Its presentation is as follows:

$$\left\langle \sigma_x, \sigma_y, \sigma_z : \quad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = e \right\rangle.$$

We can more precisely define the action of the σ maps on a given point. Let (x_1, y_0, z_0), and (x_2, y_0, z_0) be two points on $\mathcal{V}$ with y_0, and z_0 arbitrary. Then we can solve for x_1 and x_2 by solving the quadratic equation

$$F(X, y_0, z_0) = ((1 + y_0^2)(1 + z_0^2))X^2 + (Ay_0z_0)X + ((1 + y_0^2)(1 + z_0^2) - 2) = 0.$$

We obtain two solutions for X from the quadratic equation, namely $x_1 = \dfrac{-b}{2a} + \dfrac{\sqrt{b^2 - 4ac}}{2a}$, and $x_2 = \dfrac{-b}{2a} - \dfrac{\sqrt{b^2 - 4ac}}{2a}$. Notice that $x_1 + x_2 = \dfrac{-b}{a}$ so that $x_2 = -x_1 - \dfrac{b}{a}$.

We see that our map $\sigma_x : (x, y_0, z_0) \to (x', y_0, z_0)$ can be better described by

$$\sigma_x((x, y_0, z_0)) = \left(-x - \frac{Ay_0z_0}{(1 + y_0^2)(1 + z_0^2)}, y_0, z_0 \right).$$

Similarly we obtain that

$$\sigma_y((x_0, y, z_0)) = \left(x_0, -y - \frac{Ax_0z_0}{(1 + x_0^2)(1 + z_0^2)}, z_0 \right),$$

and

$$\sigma_z((x_0, y_0, z)) = \left(x_0, y_0, -z - \frac{Ax_0y_0}{(1 + x_0^2)(1 + y_0^2)} \right).$$

Hence we see that σ_x, σ_y, and σ_z are rational maps.

2.2 The Intersection Matrix

We will soon see that our surface, $\mathcal{V}$, has rational curves present at infinity. We know by the adjunction formula that any smooth rational curve (a smooth curve of genus zero) will have a self intersection of -2. We will find 12 such linearly independent -2 curves. These 12 curves generate a sublattice of the Picard lattice. We will describe the intersection matrix for this sublattice. First we will need to re-write the equation $F(x, y, z)$ as a polynomial in projective space.

We must extend our affine surface equation, $F(x, y, z) = 0$ into the projective equation, $F((x_1, x_2), (y_1, y_2), (z_1, z_2)) = 0$. We do this by plugging $x = \dfrac{x_1}{x_2}$, $y = \dfrac{y_1}{y_2}$, and $z = \dfrac{z_1}{z_2}$ into F. Multiplying by $(x_2)^2(y_2)^2(z_2)^2$, we obtain the equation,

$$F((x_1, x_2), (y_1, y_2), (z_1, z_2)) = (x_1^2 + x_2^2)(y_1^2 + y_2^2)(z_1^2 + z_2^2) + Ax_1x_2y_1y_2z_1z_2 - 2x_2^2y_2^2z_2^2.$$

This is the projective polynomial describing $\mathcal{V}$.

Notice that in the equation above if we let $x = (i, 1) = i$, $y = (1, 0) = \infty$, then $z = (z_1, z_2)$ is free to be anything. This curve is a smooth rational curve. That is to say, it is isomorphic to $\mathbb{P}^1$. This is what tells us that it has genus 0 and is hence a -2 curve. Similarly, each of the twelve -2 curves is free in exactly one variable when one is made equal to $\pm i$ and another equal to infinity. We let $D_1 = ((1, 0), (i, 1), (z_1, z_2))$

which in affine coordinates we will write as (∞, i, z). D_1 through D_{12}, in order, are

D_1 through D_6: $(\infty, i, z), (\infty, -i, z), (\infty, y, i), (\infty, y, -i), (i, \infty, z), (-i, \infty, z),$

D_7 through D_{12}: $(x, \infty, i), (x, \infty, -i), (i, y, \infty), (-i, y, \infty), (x, i, \infty), (x, -i, \infty).$

Knowing any one of these -2 curves and applying all of the isometries found in the previous section will yield the other 11. That is, for any j, D_j is in the orbit of D_1. For example, we obtain D_2 from D_1 by applying the isometry $N_{x,y}$ to D_1 since $-\infty = -(1, 0) = (-1, 0) = (1, 0) = \infty$. We similarly see that $D_8 = \phi_{x,y}(\phi_{y,z}(N_{x,y}(D_1)))$.

We saw in the introduction that for a negative self-intersection to be consistent, the divisor class of the curve must be otherwise empty. Since D_j is a -2 curve, the divisor class, $[D_j]$, has only one curve in it. Thus, we abuse notation and use D_j for both the curve and its divisor class.

Let us find a basis for the span of these 12 divisors. We will first check to see if these curves are linearly independent. To test this, we will construct the matrix of intersections, $J = [D_j \cdot D_k]$. We now compute each of these intersections below.

First we note that each curve has a self-intersection of -2 so that $D_i \cdot D_i = -2$. Now each D_i is fixed in two variables and free in one. If two such curves have the same fixed coordinate with differing values then the two curves cannot intersect. Thus we see that a majority of pairs of curves have an intersection of zero. For example, this is the case with D_1 and D_2 as well as with D_1 and D_7. Every other pair of curves must share exactly one fixed coordinate with the same value. This point of intersection clearly has multiplicity 1 since the curves are perpendicular. With this information

we are able to construct the matrix of intersections,

$$J = [D_j \cdot D_k] = \begin{pmatrix} -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \end{pmatrix}.$$

The matrix J has determinant -135 and hence we see that the 12 curves are linearly independent. This establishes our lower bound of 12 for the Picard number of our surfaces. This also gives us a 12 dimensional sublattice, Λ, of the full Picard Lattice. By letting D_1 through D_{12} be the basis elements of Λ, we may use J as our intersection matrix to define the intersection product between some curves.

The way we do this is as follows: Let C_1, and C_2 be two curves on our surface. Suppose that their divisors, $[C_1]$, and $[C_2]$ are represented in our lattice, Λ. Then we have that

$$\#\{C_1 \cap C_2\} = [C_1] \cdot [C_2] = [C_1]^T J [C_2].$$

The Picard number may be larger than 12 for our surfaces. The discovery of a curves on our surface not already represented in Λ would push the dimension of the Picard Lattice up. Such a curve may or may not exist, so we can only say that the picard number is ≥ 12.

2.3 The Action of σ on Λ

Next we shall look at the action of the quadratic swapping maps σ_x, σ_y, and σ_z on our sublattice, Λ. Every automorphism σ of $\mathcal{V}$ induces an action σ^* on the picard lattice of $\mathcal{V}$. This action is defined by $\sigma^*[C] = [\sigma C]$ for any curve C on $\mathcal{V}$. It turns our that σ^* acts linearly on the picard group, so to solve for σ^*, we need only find its action on a basis. We do not know that we have a basis of the picard group of $\mathcal{V}$, but we can use our basis of Λ to solve for σ_x^*, σ_y^*, and σ_z^* restricted to Λ.

We will do this explicitly with σ_x on the basis, $\{D_1, ..., D_{12}\}$ of Λ. First we try to find $\sigma_x(D_1) = \sigma_x((\infty, i, z))$. We cannot directly apply our formula from [2.1] since it was written for affine coordinates. Instead we must manually search for a new solution of $F_P(X, i, z)$. We expect that a new solution will be affine, so we work with equation

$$F_A(X, i, z) = AXiz - 2 = 0.$$

Solving for X yields $X = -\dfrac{2i}{Az}$ works so that

$$\sigma_x(D_1) = (\frac{-2i}{Az}, i, z).$$

Similarly we see

$$\sigma_x(D_2) = (\frac{2i}{Az}, -i, z),$$

$$\sigma_x(D_3) = (\frac{-2i}{Ay}, y, i), \text{ and}$$

$$\sigma_x(D_2) = (\frac{2i}{Ay}, y, -i).$$

Next we see that σ_x swaps D_5 and D_6, and swaps D_9 to D_{10}. The map sends D_7, D_8, D_{11}, and D_{12} to themselves.

We have already determined the intersections of each -2 curve with any other, so the only new curves to find intersections with are $\sigma_x(D_1)$ through $\sigma_x(D_4)$. We will make example of just one case, $\sigma_x(D_1)$, as all curves are similar. We quickly see that $\sigma_x(D_1)$ is different than D_2, D_5 through D_8, and D_{12} in the y-component so that $\sigma_x(D_1)$ has intersection zero with with those curves. We also see that if we make $x = 0$ then we get that $z = \infty$. This tells us that $\sigma_x(D_1)$ has intersection zero with D_3 and D_4. If we make $z = \infty$, we see that $x = 0$ so that $\sigma_x(D_1)$ has zero intersection with D_9 and D_{10}. We once again apply our intersection formula on the surface $\mathbb{P}^1 \times \mathbb{P}^1$ to see that $[\sigma_x(D_1)] \cdot D_1 = 1$. This intersection occurs at the point $(\infty, i, 0)$. Similarly $[\sigma_x(D_1)] \cdot D_{11} = 1$, and this intersection occurs at the point $(0, i, \infty)$.

Doing this for all of $\sigma_x(D_1)$ through $\sigma_x(D_4)$, we obtain a 12×12 matrix of intersections, $M = [\sigma_x D_j \cdot D_k] = \sigma_x^* J$.

$$
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2
\end{pmatrix}.
$$

If we let T_x be the matrix defining the action of σ_x^* on Λ then the above matrix, M

is equal to $T_x J$. Solving for T_x we obtain

$$T_x = \begin{pmatrix}
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

We may repeat the above process with σ_y and σ_z to obtain

$$T_y = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

and

$$T_z = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}.$$

We note that, as we would expect, $T_x^2 = I$, $T_y^2 = I$, and $T_z^2 = I$.

The way we use these matrices is as follows: If a curve, C, on our surface has

divisor class $[C]$ in Λ, then

$$[\sigma_x(C)] = T_x^t[C], \quad [\sigma_y(C)] = T_y^t[C], \text{ and } [\sigma_z(C)] = T_z^t[C].$$

2.4 Curves that Fix One Coordinate are in Λ.

In this section, we show that the three natural divisors discussed in section 1.4,

E_x, E_y and E_z, do not extend our intersection matrix J. That is to say, our Picard

sublattice already has within it all of the curves generated by fixing one component.

We define $E_{x=x_0}$ to be the curve formed by fixing $x = x_0$ on $\mathcal{V}$. Then its divisor

class, E_x, contains all such curves with a fixed x component by our work in section

1.3. We will see that E_x is already contained within our lattice. To show this, we

extend the matrix J by adding a 13th row and column representing E_x. We will see

that this extended matrix has determinant zero and is thus linearly dependent.

We must first compute the intersection of E_x with each of the D_i's. This comes

down to checking the degree of each D_i in the x component. Thus we see that E_x has

intersection 1 with D_7, D_8, D_{11}, and D_{12}, and intersection 0 with all other D_i's. We

calculate the self intersection $E_x \cdot E_x$ by choosing two curves in the divisor class E_x.

The curves $E_{x=0}$ and $E_{x=1}$ do not intersect since $0 \neq 1$. Thus we see that $E_x \cdot E_x = 0$.

This gives us all of the necessary information to form our extension of J by

including E_x as the 13th row and column. We call this matrix, $\hat{J}$. Then

$$\hat{J} = \begin{pmatrix} -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The determinant of $\hat{J}$ is zero, and hence E_x is in the span of $\{D_1, ..., D_{12}\}$. We can solve explicitly for E_x. Since E_x has intersection 1 with D_7, D_8, D_{11}, and D_{12} and intersection zero with the other basis elements, we have that

$$JE_x = (0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1).$$

Solving for E_x we get

$$E_x = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0).$$

In other words,

$$E_x = D_1 + D_2 + D_3 + D_4.$$

Similarly we see that

$$E_y = D_5 + D_6 + D_7 + D_8 = (0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0),$$

and

$$E_z = D_9 + D_{10} + D_{11} + D_{12} = (0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1).$$

There is an easy way to have noticed that $E_x = D_1 + D_2 + D_3 + D_4$. If we fix $(x_1, x_2) = (1, 0)$ (i.e. $x = \infty$), then the curve, $E_{x=\infty}$ has the equation

$$F_P((1,0),(y_1,y_2),(z_1,z_2)) = (y_1^2 + y_2^2)(z_1^2 + z_2^2) = 0.$$

This curve is made up of exactly the 4 components, D_1 through D_4. In other words, since the curve $E_{x=\infty}$ decomposes into D_1 through D_4, the divisor class $[E_{x=\infty}] = E_x$ will be exactly $D_1 + D_2 + D_3 + D_4$.

We will need these lattice representations, E_x, E_y, and E_z, in the next section. They will be necessary to compute a curve's degree in x, y, and z using only its representation in our Picard sub-lattice, Λ.

2.5 Example: Predicting Polynomial Degree Growth

Now that we have come up with matrix representations of σ_x^*, σ_y^*, and σ_z^*, as well as the lattice representation of E_x, E_y, and E_z, we can check to see that these matrices accurately predict the intersections of images of curves under the σ maps. If we do this with a curve rationalized in x, y, and z as a function of one variable, we will see that these intersections will correspond to the curve's degrees in each component.

In this section, we work out one example. Let us first find a singular elliptic curve on the surface. A singular elliptic curve will have a parameterization in one variable which helps to serve two purposes. One, it will greatly simplify the computation of the curve's images under the σ maps. Two, given a rationalized projective curve $C = (X(t), Y(t), Z(t))$ with X, Y, and Z all rational expressions written in affine for simplicity, we can calculate $\#\{C \cap E_{x=x_0}\}$ as follows. Let $X(t) = N_X(t)/D_X(t)$ for

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some polynomials, N, D with $\gcd(N, D) = 1$. We want to count the solutions to the equation,

$$\frac{N_X(t)}{D_X(t)} = x_0.$$

Case 1: for $x_0 \neq 0$ or ∞, this is equivalent to counting solutions to $N_X(t) - x_0 D_X(t) = 0$. The fundamental theorem of algebra tells us here that

$$\#\{C \cap E_{x=x_0}\} = \deg(N_X(t) - x_0 D_X(t)) = \max\{\deg(N_X(t)), \deg(D_X(t))\}.$$

Case 2: when $x_0 = 0$ there will be $\deg(N_X(t))$ many solutions varying t through $\mathbb{C}$, but there will also be a solution of order $\max(0, \deg(D_X) - \deg(N_X))$ when $t = 0$. Summing these together, we see that we have the same value for $\#\{C \cap E_{x=x_0}\}$.

Case 3: when $x_0 = \infty$, we again count two kinds of solutions. There will be $\deg(D_X(t))$ solutions which correspond to division by zero. There will also be one solution of multiplicity $\max(\deg(N_X) - \deg(D_X), 0)$ when $t = \infty$. Once again we obtain the same value for $\#\{C \cap E_{x=x_0}\}$.

Since in all cases we had

$$\#\{C \cap E_{x=x_0}\} = \deg(N_X(t) - x_0 D_X(t)) = \max\{\deg(N_X(t)), \deg(D_X(t))\},$$

we define $\deg(X(t)) = \max\{\deg(N_X(t)), \deg(D_X(t))\}$. With this definition, we see that

$$\deg(X(t)) = [C] \cdot E_x = [C]^t J E_x,$$

$$\deg(Y(t)) = [C] \cdot E_y = [C]^t J E_y,$$

and

$$\deg(Z(t)) = [C] \cdot E_z = [C]^t J E_z.$$

This allows us to easily verify that our lattice is making accurate predictions of degree growth.

Let $E_{x=1}$ be the elliptic curve defined by the intersection of $\mathcal{V}$ with the plane, $x = 1$. This curve has affine equation $F(1, y, z) = 0$. I.e.:

$$E_{x=1} : 2(1 + y^2)(1 + z^2) + Ayz - 2 = 0, \ x = 1.$$

We want to rationalize this curve in terms of a single parameter. The reason this will be possible is that there is a singular point on $E_{x=1}$ where $y = 0$, and $z = 0$.

Let us take an arbitrary (1,1) form, $\mathcal{C}$, in y, z. It has the equation,

$$yz + ay + bz + c = 0.$$

We will make $\mathcal{C}$ intersect $E_{x=1}$ at $(y, z) = (0, 0)$ and at one other rational point in $\mathbb{Q}[i]$. $E_{x=1}$ is a (2,2) form so that we expect $\#\{E_{x=1} \cap \mathcal{C}\} = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4$. There will be an automatic double intersection at the point $(0, 0)$ and a single intersection at our third point leaving one degree of freedom to define our fourth point of intersection. This will give us a rationalization of $\mathbb{E}$.

For C to go through the point $(y, z) = (0, 0)$ we see that $c = 0$. Now we need a rational point other than $(0, 0)$ on our curve, $E_{x=1}$. One such point is $(y, z) = (i, \dfrac{-2i}{A})$. We plug $(i, \dfrac{-2i}{A})$ in for y and z into the equation for $\mathcal{C}$ to solve for a in terms of b.

$$2 + ai + b\dfrac{-2i}{A} = 0,$$

thus

$$a = -\dfrac{2i(-1 + ib)}{A}.$$

In the end we have that $\mathcal{C}$ has the equation,

$$yz - \frac{2i(-1+ib)}{A}y + bz = 0.$$

Now we calculate with the help of mathematical software that $E_{x=1}$, and $\mathcal{C}$ have the following solution set:

$$(0,0)^2 \qquad \left(i, \frac{-2i}{A}\right) \qquad \left(-\frac{4ib^2 - 8b - 4i + A^2b}{4b^2 + 8ib - 4 + A^2}, \frac{4ib^2 - 8b - 4i + A^2b}{2(b^2+1)A}\right).$$

This fourth solution is a rationalization of $E_{x=1}$ with free variable, b. We make the substitution $b = t$ to obtain that

$$E_{x=1} = \left(1, -\frac{4it^2 - 8t - 4i + A^2t}{4t^2 + 8it - 4 + A^2}, \frac{4it^2 - 8t - 4i + A^2t}{2(t^2+1)A}\right).$$

We note that

$$[E_{x=1}] = E_x = (1,1,1,1,0,0,0,0,0,0,0,0).$$

Also recall that

$$E_y = (0,0,0,0,1,1,1,1,0,0,0,0),$$

and

$$E_z = (0,0,0,0,0,0,0,0,1,1,1,1).$$

We are now ready to test that our basis properly predicts the degrees of components of $E_{x=1}$ as it is transformed by σ_x, σ_y, and σ_z. Notice that $\deg_x(E_{x=1}) = 0$, $\deg_y(E_{x=1}) = 2$, and $\deg_z(E_{x=1}) = 2$ using the definition of the degree of a rational expression given above. Working in the lattice we also compute that $[E_{x=1}] \cdot E_x = 0$, $[E_{x=1}] \cdot E_y = 2$, and $[E_{x=1}] \cdot E_z = 2$ just as we expect.

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Now we test to see that the lattice continues to make accurate prediction of degrees of $E_{x=1}$ under the actions of σ_x, σ_y, and σ_z. $E_{x=1}$ is fixed by σ_y and σ_z, so we look at $\sigma_x(E_{x=1})$.

Since $E_{x=1}$ is rationalized in the form, $(1, Y(t), Z(t))$, we can apply our function, $\sigma_x((x, y, z)) = \left(-x - \dfrac{Ayz}{(1+y^2)(1+z^2)}, y, z\right)$, to $E_{x=1}$. The y and z components will remain fixed, but x changes to x' where

$$x' = \frac{4A^4t^2 - 16A^2 - 80t^2A^2 + 32it^3A^2 - 32itA^2 + 384t^2 - 384it^3 + 128it - 128t^4}{4A^4t^2 - 48t^2A^2 + 32it^3A^2 - 32itA^2 + 16A^2 + 384t^2 - 128 - 128it^3 + 384it}.$$

Notice the degree of x' with respect to t is 4. It is of utmost importance that we simplify x' as much as possible so that if P is the numerator of x', and Q the denominator, then $\gcd(Q, P) = 1$.

Now we find the lattice representation of $\sigma_x(E_{x=1})$ which is given by $[\sigma_x(E_{x=1})] = T_x^t[E_{x=1}]$. We obtain

$$T_x^t[E_{x=1}] = (-1, -1, -1, -1, 2, 2, 1, 1, 2, 2, 1, 1).$$

We find its intersection with E_x, E_y, and E_z to test that our lattice makes accurate predictions. We get

$$(T_x^t[E_{x=1}]) \cdot E_x = 4,$$

$$(T_x^t[E_{x=1}]) \cdot E_y = 2,$$

$$(T_x^t[E_{x=1}]) \cdot E_z = 2.$$

All of these values match up properly with the degrees of $\sigma_x(E_{x=1})$ in x, y, and z. The author has also tested that the lattice accurately predicts polynomial degrees for further images of $E_{x=1}$ under the action of all the σ maps. Accurate predictions

of degrees were verified up to compositions of length 10. Due to computational difficulties, this was done with $A = 2$ instead of A variable. See the table below for partial results.

Table 2.1. Some tested images of $E_{x=1}$ with $A = 2$

Image	degree in x	degree in y	degree in z
E_x	0	2	2
$\sigma_x E_x$	4	2	2
$\sigma_y \sigma_x E_x$	4	8	2
$\sigma_z \sigma_y \sigma_x E_x$	4	8	12
$\sigma_x \sigma_z \sigma_y \sigma_x E_x$	24	8	12
$\sigma_x \sigma_y \sigma_x E_x$	14	8	2
$\sigma_z \sigma_x \sigma_y \sigma_x E_x$	14	8	22
$\sigma_y \sigma_x \sigma_y \sigma_x E_x$	14	22	2
$\sigma_z \sigma_y \sigma_x \sigma_y \sigma_x E_x$	14	22	38
$\sigma_x \sigma_y \sigma_x \sigma_y \sigma_x E_x$	30	22	2
$\sigma_x \sigma_z \sigma_y \sigma_x \sigma_y \sigma_x E_x$	72	22	38
$\sigma_y \sigma_z \sigma_y \sigma_x \sigma_y \sigma_x E_x$	14	72	38
$\sigma_x \sigma_y \sigma_z \sigma_y \sigma_x \sigma_y \sigma_x E_x$	104	72	38
$\sigma_z \sigma_y \sigma_z \sigma_y \sigma_x \sigma_y \sigma_x E_x$	14	72	122

This is evidence that our basis for Λ is large in the sense that it is closed under the action of the σ automorphisms. This, unfortunately, is not evidence that we have the entire Picard lattice. See [1] for an example of surface with a strict sublattice of the Picard lattice which similarly makes accurate predictions of degrees for curves in the sublattice.

2.6 σ Maps as Elliptic Curve Operations

The curve we get on $\mathcal{V}$ from fixing one of x, y, or z has self intersection zero, so by the adjunction formula these curves have genus 1. Thus each such curve is an elliptic curve. For this reason, we can think of $\mathcal{V}$ as a continuum of elliptic curves glued together along an axis. Such a continuum of curves is called a fibering of our surface. The rich structure and background of Elliptic curves leads us naturally to search for an additional automorphism of $\mathcal{V}$.

The group structure on elliptic curves can be used to generate a automorphism of our surface. To do this, we will need a continuous way of choosing our additive zero on $E_{x=x_0}$ for each x_0 and similarly with $E_{y=y_0}$, and $E_{z=z_0}$. We will make use of the rational curves, D_1 through D_{12}, to give us this continuous way of choosing our zero. In this way, the map sending P to $-P$ on a fixed elliptic curve can be extended to a map on the whole of $\mathcal{V}$.

We demonstrate this for elliptic curves defined by fixing $x = x_0$. The elliptic curve, $E_{x=x_0}$, takes the affine form $F(x_0, y, z)$, and hence is a (2,2) form. A (2,2) form and any (1,1) form will intersect exactly 4 times. This means that if we have 3 known points of intersection, then it will define our fourth point of intersection. Elliptic curve addition takes advantage of this fact.

We first define a trinary operation,

$$*_{x_0} : E_{x=x_0}^3 \to E_{x=x_0}.$$

where $*_{x_0}$ is defined by following: Let C be a $(1,1)$ form made to go through the points P_1, P_2, and P_3. Then $*_{x_0}(P_1, P_2, P_3) = P_4$ where P_4 is the fourth intersection

of $E_{x=x_0}$ and C.

We use this map to establish an additive group on $E_{x=x_0}$. Our addition map will be of the form , $-(P_1+P_2) = *_{x_0}(P_1, P_2, \mathcal{O}')$ where $\mathcal{O}'$ is a fixed point defined in terms of x_0. We will derive $\mathcal{O}'$ from our choice of the point, $\mathcal{O}$, where $\mathcal{O}$ is the additive zero of our group.

Note that if $+$ is to be an additive operation on $E_{x=x_0}$ then $\mathcal{O}+\mathcal{O}$ must equal $\mathcal{O}$. I.e. $*_{x_0}(\mathcal{O},\mathcal{O},\mathcal{O}') = \mathcal{O}$. This gives us a way of defining $\mathcal{O}'$ in terms of $\mathcal{O}$ since we see that $*_{x_0}(\mathcal{O},\mathcal{O},\mathcal{O}) = \mathcal{O}'$.

As mentioned above, we must choose our $\mathcal{O}$ in a special way so that this automorphism defined on individual fibers extends to an automorphism of the entire surface. We choose $\mathcal{O}$ to be positioned on one of D_7, D_8, D_{11}, and D_{12} since these four curves give us one point on each fiber. Our addition will be different depending on which of these curves we use for our choice of $\mathcal{O}$. Here we will work out our addition with the choice that $\mathcal{O}$ is on D_7. That is, we define $\mathcal{O}$ to be the intersection of $E_{x=x_0}$ and D_7 so that $\mathcal{O} = (x_0, \infty, i)$.

With this choice of $\mathcal{O}$, we now derive $\mathcal{O}' = *_{x_0}(\mathcal{O},\mathcal{O},\mathcal{O})$. We must find the equation for C, a (1,1) form in the fixed $x = x_0$ plane that has a triple intersection with $E_{x=x_0}$ at the point (x_0, ∞, i). Let us begin by finding the first and second derivatives of E_{x_0} at $\mathcal{O}$.

E_{x_0} has the affine curve equation $F_A(x_0, y, z) = 0$. We will do our derivative calculation in affine coordinates and then make the substitution $Y = \dfrac{1}{y}$ before plugging in the point $\mathcal{O}$. This will allow for simpler calculation.

Let C have the affine equation $yz + ay + bz + c = 0$. Making our substitution we

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get that $z + a + bzY + cY = 0$. Letting $(Y, z) = (0, i)$ be a solution we obtain that $a = -i$. We next let $\frac{dY}{dz}(C) = \frac{dY}{dz}(E_{x=x_0})$. We compute that

$$\frac{dY}{dz}(C) = -\frac{1 + bY}{bz + c},$$

and

$$\frac{dY}{dz}(E_{x=x_0}) = -\frac{2z + 2zY^2 + 2zx_0^2 + 2zx_0^2Y^2 + Ax_0Y}{-2Y + 2Yz^2 + 2Yx_0^2 + 2Yx_0^2 * z^2 + Ax_0z}.$$

Plugging in $(Y, z) = (0, i)$, we obtain the constraint,

$$\frac{1}{bi + c} = \frac{2(1 + x_0^2)}{Ax_0}.$$

Similarly we let $\frac{d^2Y}{(dz)^2}(C) = \frac{d^2Y}{(dz)^2}(E_{x_0})$ to obtain the constraint,

$$\frac{2b}{-b^2 + 2ibc + c^2} = \frac{2i(-8 - 16x_0^2 - A^2x_0^2 - x_0^4A^2 - 8x_0^4)}{A^3x_0^3}.$$

These two constraints uniquely define b, and c. solving for the two, we obtain

$$b = -\frac{i(A^2x_0^2 + 8x_0^2 + 8)}{4Ax_0(1 + x_0^2)},$$

and

$$c = \frac{A^2x_0^2 - 8x_0^2 - 8}{4Ax_0(1 + x_0^2)}.$$

Putting this all together, we see that C has affine coordinates,

$$yz - iy - \frac{i(A^2x_0^2 + 8x_0^2 + 8)}{4Ax_0(1 + x_0^2)}z + \frac{A^2x_0^2 - 8x_0^2 - 8}{4Ax_0(1 + x_0^2)} = 0.$$

Notice that $(y, z) = (\frac{2i}{Ax_0}, -i)$ is a solution to the above equation and is also a point on $E_{x=x_0}$. Since the other 3 intersections of C and E_{x_0} were forced to be at the point, $\mathcal{O}$, we have that $\mathcal{O}' = (x_0, \frac{2i}{Ax_0}, -i)$.

Recall from Section 4 above that $\sigma_y(x_0, \infty, -i) = (x_0, \frac{2i}{Ax_0}, -i)$. It is interesting to note then that $\mathcal{O}' = \sigma_y(N_{y,z}(\mathcal{O}))$.

Now that we have defined $\mathcal{O}$, and found $\mathcal{O}'$, let us define

$$\lambda_7 : E_{x_0} \rightarrow E_{x_0}$$

to be the elliptic map on E_{x_0} sending P to $-P$ with $\mathcal{O}$ and $\mathcal{O}'$ defined as above. Thus, since $-P = -(P + \mathcal{O}) = *_{x_0}(P, \mathcal{O}, \mathcal{O}')$, we see that $\lambda_7(P) = *_{x_0}(P, \mathcal{O}, \mathcal{O}')$.

We wish to investigate this further by evaluating $\lambda_7(P)$ for P arbitrary. Let $P = (x_0, y_0, z_0)$ be an arbitrary point on E_{x_0}. Since $F_A(x_0, y_0, z)$ is quadratic in z, we can also write $P = (x_0, y_0, z_1)$ or $P = (x_0, y_0, z_2)$ where z_1 and z_2 are the two roots of the quadratic. Let us work under the assumption that $z_0 = $

$$z(A, x_0, y_0) = \frac{-Ax_0 y_0 + \sqrt{A^2 x_0^2 y_0^2 + 4 - 8x_0^2 y_0^2 - 4y_0^4 - 8y_0^4 x_0^2 - 4x_0^4 - 8x_0^4 y_0^2 - 4x_0^4 y_0^4}}{2(1 + y_0^2 + x_0^2 + x_0^2 y_0^2)}.$$

Half of the points on $E_{x=x_0}$ should be of this form, while the other half will have its z coordinate be the conjugate root. By encoding our point this way, we have made the coordinates of our point witness that P is an element of $E_{x=x_0}$. The author has found that without writing z this way, some mathematical software packages are unable to solve the equation in the next step.

Let C be a (1,1) form that passes through the 3 points, $\mathcal{O}$, $\mathcal{O}'$, and P. This (1,1) form must then intersect E_{x_0} in exactly one other point. Let our (1,1) form take the affine form,

$$C : yz + ay + bz + c = 0.$$

We will solve for a, b, and c. We first have C pass through $\mathcal{O}$. To do this, we will

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once again make the substitution $Y = \dfrac{1}{y}$ to obtain the new equation,

$$C : z + a + bzY + cY = 0$$

Since C passes through $\mathcal{O}$, $(Y, z) = (0, i)$ is a solution. This gives us that $a = -i$.

We may return once again to using the coordinates y, and z before having C pass through $\mathcal{O}'$. Letting $(y, z) = (\dfrac{2i}{Ax_0}, -i)$ be a solution to equation for C, and letting $a = -i$, we obtain the *, $c = ib - \dfrac{4}{Ax_0}$. This gives us that C has the equation, $yz - iy + bz + ib - \dfrac{4}{Ax_0} = 0$, and hence,

$$b = \frac{-Ax_0yz + IyAx_0 + 4}{Ax_0(z + I)}.$$

We now plug in our point P into the above equation and solve for b in terms of y_0, x_0, and A. The formula for $b = b(A, x_0, y_0)$ is too big to include here. We also get an equally large formula for $c = c(A, x_0, y_0)$ by plugging our new value for b into the *, $c = ib - \dfrac{4}{Ax_0}$.

We have determined all 3 variables of C, so we need only to calculate the 4th intersection of C and $E_{x=x_0}$. Let us think of x_0, y_0, and A fixed so that the equation,

$$yz(A, x_0, y) - iy + b(A, x_0, y_0)z(A, x_0, y) + c(A, x_0, y_0) = F_A(x_0, y, z(A, x_0, y)),$$

has only one variable, y. Note that the point $(x_0, y, z(A, x_0, y))$ is guaranteed to be a solution to $F_A(x_0, y, Z_y) = 0$ so that the above equation simplifies down to

$$yz(A, x_0, y) - iy + b(A, x_0, y_0)z(A, x_0, y) + c(A, x_0, y_0) = 0$$

Turning to algebraic software, we obtain the following solution set for y:

$$y = \infty, \frac{2i}{Ax_0}, y_0, -y_0.$$

The first 3 solutions were those that we programmed into C. The new solution discovered is $y = -y_0$. Thus the point, $(-y_0, z(A, x_0, -y_0))$, is our fourth point of intersection. What we have shown here is that

$$\lambda_7((x_0, y_0, z(A, x_0, y_0)) = (x_0, -y_0, z(A, x_0, -y_0)).$$

where plugging in $A, x_0, -y_0$ yields

$$z(A, x_0, -y_0) = \frac{A x_0 y_0 + \sqrt{A^2 x_0^2 y_0^2 + 4 - 8x_0^2 y_0^2 - 4y_0^4 - 8y_0^4 x_0^2 - 4x_0^4 - 8x_0^4 y_0^2 - 4x_0^4 y_0^4}}{2(1 + y_0^2 + x_0^2 + x_0^2 y_0^2)}.$$

We note that $z(A, x_0, -y_0)$ is the negative conjugate of $z(A, x_0, y_0)$. That is, suppose z_1 and z_2 are the two solutions to the quadratic, $F_A(x_0, y_0, Z)$. Now if we let $z_1 = z(A, x_0, y_0)$ then $z_2 = -Z_{-y_0}$. Therefore $\lambda_7(x_0, y_0, z_1) = (x_0, -y_0, -z_2)$, and thus $\lambda_7(P) = N_{y,z}(\sigma_z(P))$ where $N_{y,z}$ is the isometry sending y to $-y$ and z to $-z$.

We similarly define λ_1 through λ_{12} so that λ_1, λ_2, λ_5, and λ_6 are all elliptic maps from $E_{z_0} \to E_{z_0}$. λ_3, λ_4, λ_9, and λ_{10} are all maps from $E_{y_0} \to E_{y_0}$, and λ_7, λ_8, λ_{11}, and λ_{12} are the maps from $E_{x_0} \to E_{x_0}$.

If we similarly go through the above process with each of λ_1 through λ_{12}, we see that

$$\lambda_1 = N_{x,y} \circ \sigma_y, \qquad \lambda_2 = N_{x,y} \circ \sigma_y,$$

$$\lambda_3 = N_{x,z} \circ \sigma_z, \qquad \lambda_4 = N_{x,z} \circ \sigma_z,$$

$$\lambda_5 = N_{x,y} \circ \sigma_x, \qquad \lambda_6 = N_{x,y} \circ \sigma_x,$$

$$\lambda_7 = N_{y,z} \circ \sigma_z, \qquad \lambda_8 = N_{y,z} \circ \sigma_z,$$

$$\lambda_9 = N_{x,z} \circ \sigma_x, \qquad \lambda_{10} = N_{x,z} \circ \sigma_x,$$

$$\lambda_{11} = N_{y,z} \circ \sigma_y, \qquad \lambda_{12} = N_{y,z} \circ \sigma_y.$$

Thus, each of the σ maps can be thought of as the the composition of a basic isometry with an elliptic curve map sending a point P to $-P$.

This gives us insight into to form of the map $\sigma_i \circ \sigma_j$. Let us compose the two maps λ_1, and λ_5. If we let $P = (x, y, z)$ then we see

$$\sigma_y \circ \sigma_x(P) = \lambda_1 \circ \lambda_5(P) = *_z(*_z(P, \mathcal{O}_5, \mathcal{O}'_5), \mathcal{O}_1, \mathcal{O}'_1).$$

Here $*_z(P, \mathcal{O}_5, \mathcal{O}'_5)$ is just $\mathcal{O}_5 - P$. Thus we see that

$$\sigma_y \circ \sigma_x(P) = *_z(\mathcal{O}_5 - P, \mathcal{O}_1, \mathcal{O}'_1) = -(\mathcal{O} - P) = P - \mathcal{O}$$

Hence we see that $(\sigma_y \circ \sigma_x)^n(P) = P - [n]\mathcal{O}$. Similarly we see that for all $i \neq j$,

$$(\sigma_i \circ \sigma_j)^n(P) = P + [n]K_{i,j}$$

for some point $K_{i,j}$ of infinite order.

In [10] it is shown that such an elliptic map will have the height of $(\sigma_y \circ \sigma_x)^n(P)$ grow quadratically with n. In the next chapter we will investigate some of the properties of the the composed automorphisms, $\sigma_i \circ \sigma_j \circ \sigma_k$. We will show that heights of points grow exponentially under the action of this automorphism. This tells us that it must not be similarly defined as elliptic curve map sending P to $P + K$ for any K.

CHAPTER 3

A COUNTING ARGUMENT

3.1 Elliptic Islands

Curt McMullen studied the surfaces with affine equation,

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) + Axyz - 2 = 0$$

in [5]. The focus of his paper was the presence of what he called "Elliptic Islands" on the surface for various values of A. Note that the term "elliptic" here comes from dynamics and does not have a connection to elliptic curves or curves of genus 1.

If we define σ_x, σ_y, and σ_z as in section 2.1, then these islands are sketched by the orbit of a point on the surface under the combined automorphism, $\sigma_x \circ \sigma_y \circ \sigma_z$. McMullen noticed that for values of A close to 2, the surface is dominated by these islands, but as we let A increase to 8, the surface becomes dominated by ergodic orbits. See figures 3.1 through 3.3 for images of the Elliptic Islands and ergodic parts with various values of A.

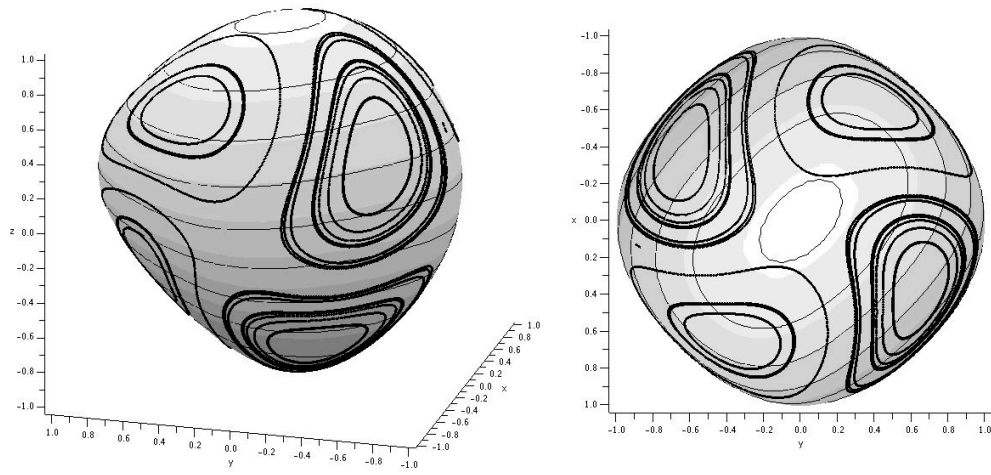**Figure 3.1.** Some orbits of points on the surface with $A = 2$

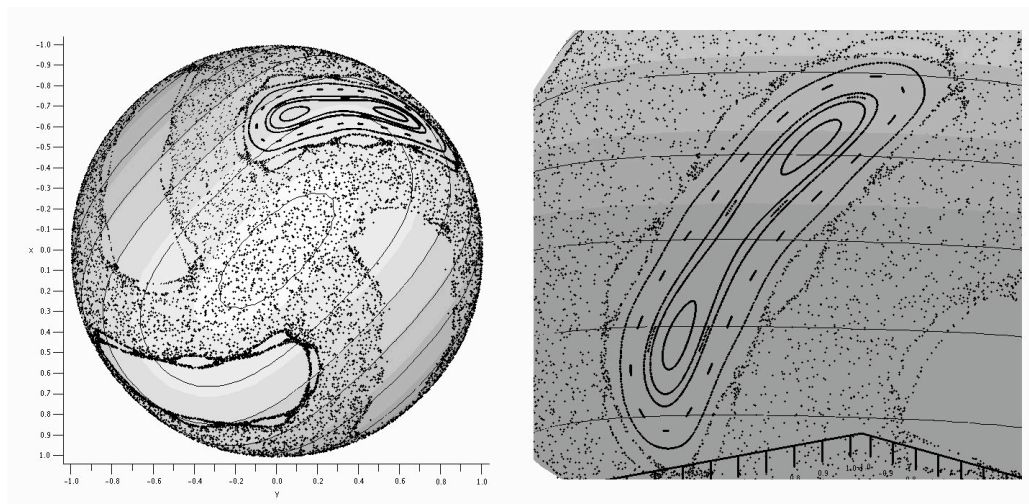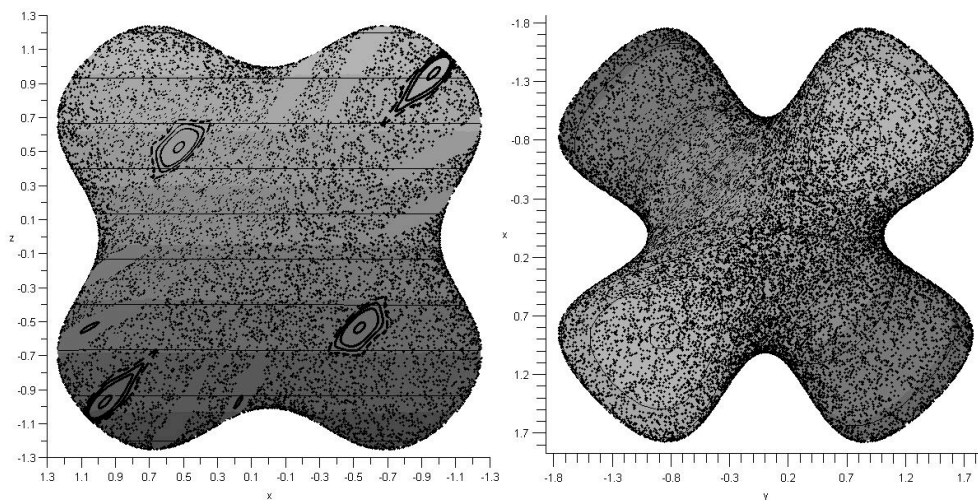**Figure 3.2.** Orbits of points on the surface with $A = 3$

**Figure 3.3.** Left: $A = 6$, Right: $A = 8$

3.2 An Orbit on the Surface with $A = 6$

In this paper we are interested in a particular orbit. Let $P = (1, 1, -1)$, then P is on the surface with $A = 6$. The orbit of P forms elliptic islands. We will use this orbit to argue that these elliptic islands are non-algebraic curves.

Figure 3.4 shows the surface with $A = 6$ and the orbit of P. It will be important to note later that orbit shown here contains more than 18119 many points.

Figure 3.4. The Orbit of point P
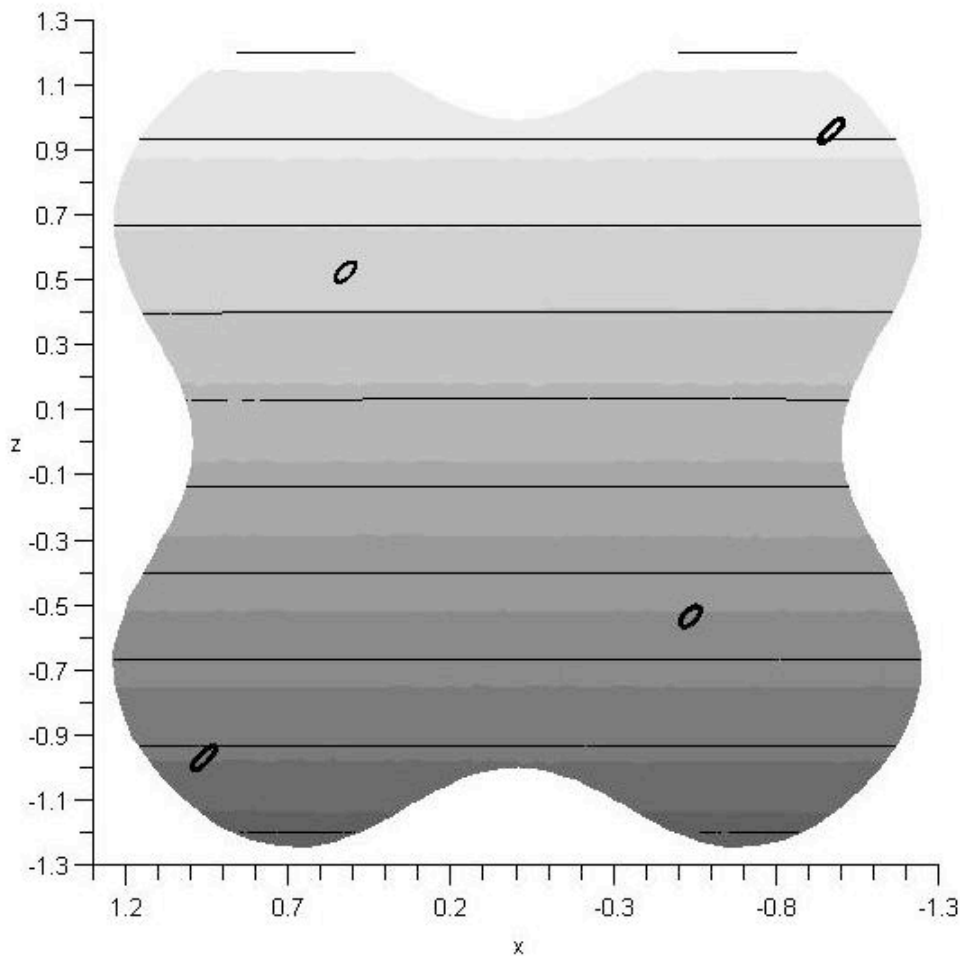
All points in the orbit of P are visible in the above image. Note that the orbit contains exactly 4 real Elliptical Island components.

The author counted the number of points present in the orbit of P in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ for various values of p. See Table 3.1 for a selection of the tested values, and see Figure 3.5 for a plot of p vs the number of points in the orbit of P for all tested values.

Table 3.1. The Orbit of P in $\mathbb{F}_p$ for selected values of p

p	# of points in the orbit of P	# divided by p
3	2	0.67
7	8	1.14
11	4	0.36
19	26	1.37
43	134	3.12
79	152	1.92
83	32	0.39
467	2324	4.98
4519	12	0.0027
10099	84154	8.33
17807	32	0.0018
18119	169330	9.35

Figure 3.5. Graph of p vs the number of points in the orbit of P
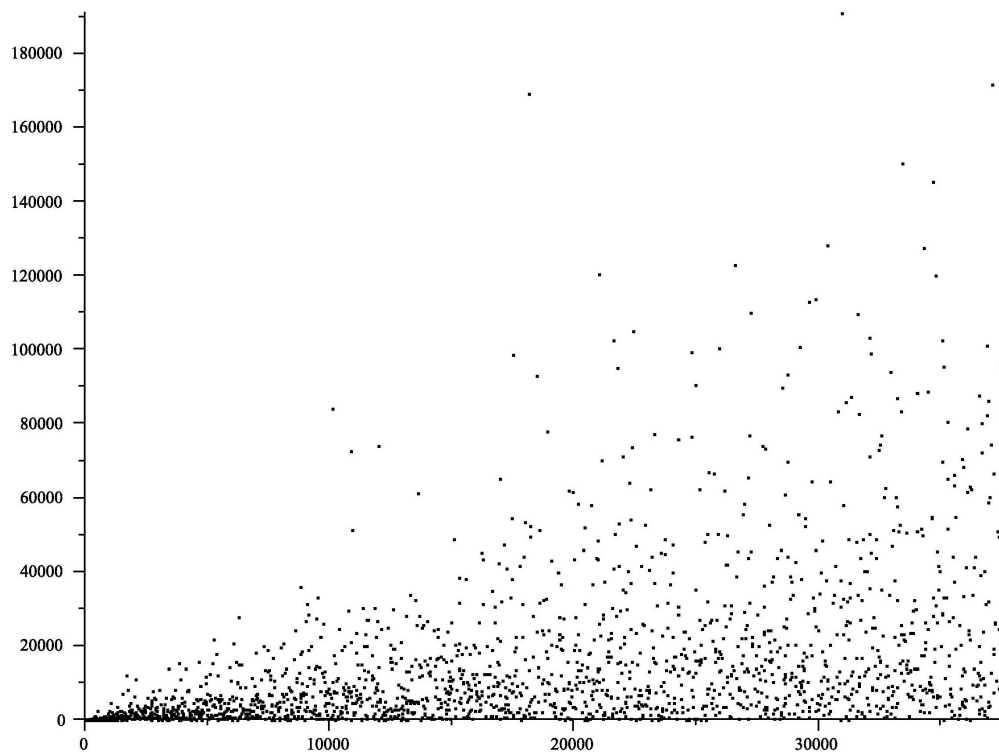
Notice that for $p = 18,119$ we have 169,930 points in the orbit of P.

Evidence that the Orbit of P is Infinite

We can offer only heuristic evidence that the orbit of P contains infinitely many points.

In section 2.2, we saw the matrix representations of σ_x, σ_y, and σ_z. Those were labelled T_x, T_y, and T_z respectively. Let $R = T_x T_y T_z =$

$$
R = \begin{pmatrix}
2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & -1 & -1 & -2 & -1 \\
2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & -1 & -1 & -1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0
\end{pmatrix}.
$$

R has 6 eigenvalues on the complex unit circle and 6 eigenvalues that are the roots of the following irreducible polynomial over $\mathbb{Q}$:

$$
f(x) = x^6 - 5x^5 - 6x^4 - 5x^3 - 6x^2 - 5x + 1.
$$

See figure 3.6 for a plot of $f(x)$.

Figure 3.6. plot of the polynomial $f(x) = x^6 - 5x^5 - 6x^4 - 5x^3 - 6x^2 - 5x + 1$
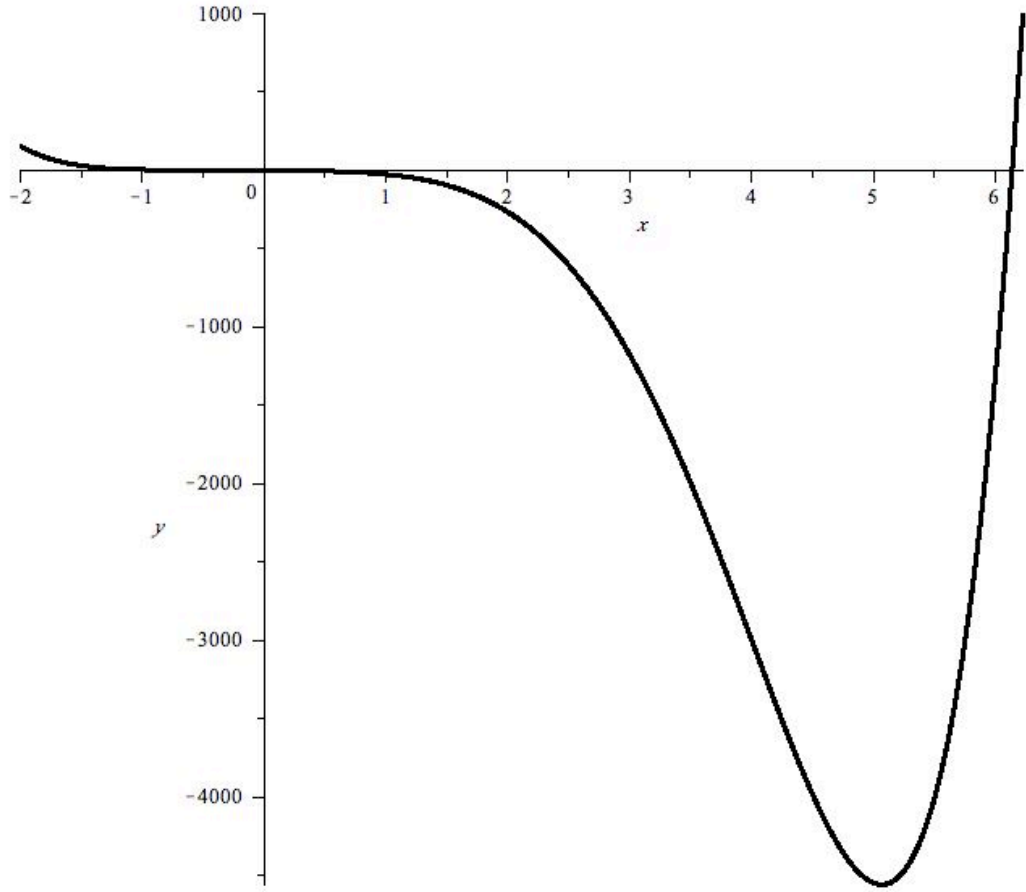
Note that $f(x)$ has one root larger than one, $\alpha \approx 6.139301847$. $f(x)$ is a Salem polynomial and α is a Salem number [2]. This number tells us that for all but finitely many points on $\mathcal{V}$, the height of the point grows exponentially under the automorphism, $\sigma_x \circ \sigma_y \circ \sigma_z$, with common ratio equal to α [11] [9]. Here the height of a point, $Q = (x_0, y_0, z_0)$ is defined as follows:

$$h(Q) = \ln(H(x_0) + H(y_0) + H(z_0)).$$

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Where H is the function defined by

$$H\left(\frac{p}{q}\right) = \max(|p|, |q|).$$

For our point, $P = (1, 1, -1)$, $h(P) = \ln(1+1+1) \approx 1.098$. Let P_0 denote our original point, and let $P_{n+1} = \sigma_x \circ \sigma_y \circ \sigma_z(P_n)$. We calculate that $P_1 = (-7/13, 1/5, -1/2)$ and that $h(P_1) = \ln(13 + 5 + 2) \approx 2.996$.

Table 3.2 shows the heights of the sequence P_n for the first few n. The coordinates of P_9 are rational numbers with numerator and denominator each around a million digits. Calculating P_9 took approximately 7 minutes with a 2.5 GHz Intel Core 2 Duo processor.

Table 3.2. The height of P_n under the action of $\sigma_x \circ \sigma_y \circ \sigma_z$

n	$height(P_n)$	$height(P_n)/height(P_{n-1})$
0	1.098612	N/A
1	2.995732	2.726833027
2	16.68609	5.569954099
3	104.6982	6.274579485
4	643.4880	6.146122960
5	3.953×10^3	6.143520283
6	2.427×10^4	6.139532494
7	1.490×10^5	6.139444004
8	9.148×10^5	6.139303332
9	5.616×10^6	6.139304939

This experimentally verifies that the height of P grows at the predicted rate for points of infinite order. Indeed, we can estimate α accurate to the hundred-thousands place based on the data presented in Table 3.2. Though this does not suffice for proof, it is extremely strong evidence that the orbit of P has infinitely many points.

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The Orbit of P Does Not Sketch an Algebraic Curve

Here we argue that the island sketched by the orbit of P cannot be an algebraic curve. We do this under the assumption that the orbit of P is infinite.

Theorem 3.1. *If the orbit of P is infinite and sketches an algebraic curve, then this curve has at least 10 components.*

Proof. Suppose the orbit of P is infinite and sketches an algebraic curve, C. That is, suppose every point in the orbit of P lies on the curve, C.

Since each point obtained this way is rational, C contains infinitely many rational points. Now Falting's Theorem tells us that any algebraic curve with infinitely many rational points must have genus either zero or one.

By the Fundamental Theorem of Algebra, a curve with genus 0 (a rational curve) has no more than $p + 1$ many points in the finite field, $\mathbb{F}_p$.

A result due to Hasse tells us that the number of points on an elliptic curve over the finite field, $\mathbb{F}_p$, is bounded by $p + 2\sqrt{p} + 1$. [10]

We have counted 169930 points in the orbit of P in $\mathbb{F}_{18119}$ telling us that C has at least this many points in $\mathbb{F}_{18119}$. Now since $\lfloor 9 * (18119 + 2\sqrt{18119} + 1) \rfloor = 165502 \leq$ 169930, C must have at least 10 components. This completes our proof. $\square$

Now since $P = (1, 1, -1)$ is a real-valued point, and $\sigma_x \circ \sigma_y \circ \sigma_z$ is a rational map, each element in the orbit of P will be a real valued point. Thus we can conclude that C must have at least 10 *real* components. Now since we plotted more than 18119 many points when making figure 3.4, all 10 of these components must be visible. We have already seen images of the real part of C. It has only 4 visible components.

Thus we conclude that C must not be an algebraic curve.

3.3 Ideas for Further Research

Here we offer some ideas for further research along the lines of this paper.

Idea #1: Fully Describe $Aut(\mathcal{V})$

Originally when I started working on this topic with my advisor, Dr. Arthur Baragar, our hope was to describe the entire automorphic group $Aut(\mathcal{V})$. We have evidence that the collection of isometries found in this paper is not the entire automorphic group. Much effort was done to try to find even a single additional automorphism of the surface.

For example, originally when I explored the automorphism generating by sending P to $-P$ with $\mathcal{O}$ chosen to be on one of the -2 curves, I was looking for something new. Learning that this map was simply a σ map composed with another isometry was interesting, but not what I had hoped for. It was rather surprising that this map did not yield a new automorphism on the surface. See [1] for an example where a similarly defined map does yield a new automorphism.

One idea left that may yield new isometries of the surface is related to the first idea. It is sometimes the case that on $K3$ surfaces with elliptic curve fibrations that the rank of every elliptic curve in the fibering is larger than one. We have shown in this paper that each elliptic fiber has rank at least 1. This is clear since there are infinately many $\mathbb{Q}[i]$-rational points on every fiber. The composed maps $\sigma_i \circ \sigma_j$ can be thought of as maps sending P to $P + K$ for some point K of infinite order. If we

can find other generators of infinite order in the additive group for each elliptic fiber, this might give us new automorphisms of the surface. These automorphisms will also have the form of sending P to $P + Q$ where Q is some other point of infinite order in the fiber.

See [1] for an example of such a surface with elliptic curve rank always ≥ 2.

Idea #2: Prove the Picard Number, $q = 12$

We have not offered any good evidence in this paper that $q = 12$. We know that our surface belongs to an infinite family with picard number generically equal to 12. This is easy to prove. If we look at the family of surfaces with affine equation

$$(1 + x^2)(1 + y^2)(1 + z^2) + \text{ (lower order terms) } = 0,$$

This family has 8 free variables so that the dimension of the moduli space of these surfaces is 8.

Now we note that all of the lower order terms have no affect on what happens at infinity. Thus the $12 - 2$ curves found in this paper will be present for all surfaces in this moduli space. Hence we see that this 8 dimensional family of $K3$ surfaces has picard number, $q \geq 12$. Now it has been shown that the picard number and dimension of the moduli space for a surface sum to 20 for all but a countable collection of lower dimensional families of surfaces. Thus we see that generically $q = 12$ in our family of surfaces.

This result may fail in countably many lower dimensional subfamilies of $K3$ surfaces, so we do not know that the picard number of our 1 dimensional family of $K3$ surfaces is 12. This does, however, give us reason to believe it might be.

There is a known method to calculate q which works for some $K3$ surfaces. Van Luijk gave a method for finding upper bounds for q. When one of these upper bounds coincides with the lower bound for q ($q \geq 12$ in our case), we know the value of q exactly. This method is computationally expensive and requires advanced algorithms. We must count the number of points on the surface in $\mathbb{F}_{2^i}$ and $\mathbb{F}_{3^j}$ for $i \leq 10$ and $j \leq 9$. Checking point by point is unfeasible since there are $(3^9 + 1)^3 \approx 7.6 \times 10^{12}$ many points to check in $\mathbb{F}_{3^9}$.

Instead of checking point by point, one can use the theory of elliptic curves to quickly count how many point are on each elliptic fiber as we vary x. This method still requires that we count the points on $3^{10} + 1$ many elliptic curves, so we would need a very fast algorithm at computing these numbers. This faster method was also conceived of by Ronald van Luijk and was successfully implemented in [3].

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