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Katayoun Bodouhi Kazemi
University of Nevada, Las Vegas, katybod@yahoo.com

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SOLVING DIFFERENTIAL EQUATIONS WITH LEAST SQUARE AND COLLOCATION METHODS

by

Katayoun Bodouhi Kazemi

Master of Science in Engineering Management
George Washington University
2004

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science-Mathematical Science

Department of Mathematical Sciences
College of Sciences
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Master of Science –Mathematical Sciences
Department of Mathematical Sciences

Xin Li, Ph.D.
Examination Committee Chair

Zhonghai Ding, Ph.D.
Examination Committee Member

Rohan Dalpatadu, Ph.D.
Examination Committee Member

David Hatchett, Ph.D.
Graduate College Faculty Representative

Kathryn Hausbeck Korgan, Ph.D.
Graduate College Interim Dean

ABSTRACT

SOLVING DIFFERENTIAL EQUATIONS WITH LEAST SQUARE AND COLLOCATION METHODS

by

Katayoun Bodouhi Kazemi

Dr. Xin Li, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas

In this work, we first discuss solving differential equations by Least Square Methods (LSM). Polynomials are used as basis functions for first-order ODEs and then B-spline basis are introduced and applied for higher-order Boundary Value Problems (BVP) and PDEs. Finally, Kansa's collocation methods by using radial basis functions are presented to solve PDEs numerically. Various numerical examples are given to show the efficiency of the methods.

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ACKNOWLEDGEMENTS

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CHAPTER 1 - INTRODUCTION TO LEAST SQUARE METHODS

Least Square Methods (LSM) have been used to solve differential equations in Finite Element Methods (FEM). For description, we consider the following linear boundary value problem [1]

$$L(y) = f(\mathbf{x}) \quad for \quad \mathbf{x} \in domain \quad \Omega,$$

$$W(y) = g(\mathbf{x}) \quad for \quad \mathbf{x} \in boundary \quad \partial\Omega,$$

where Ω is a domain in R^1 or R^2 or R^3, L is differential operator, and W is the boundary operator. When solving a differential equation by Finite Element Methods (FEM), the solution is given by a sum of weighted basis functions. Using $\phi_i(\mathbf{x})$, $1 \leq i \leq N$, for basis functions, an approximate solution is expressed as

$$\tilde{y} = \sum_{i=1}^{N} q_i \phi_i(\mathbf{x}), \tag{1.1}$$

where q_i's are coefficients (weights) and they can be determined by least square methods, explained below. To be precise, define the residual $R_L(\mathbf{x})$, $R_W(\mathbf{x})$ as follows

$$R_L(\mathbf{x}, \tilde{y}) = L(\tilde{y}) - f(\mathbf{x}) \quad for \quad \mathbf{x} \in domain \quad \Omega,$$

$$R_W(\mathbf{x}, \tilde{y}) = W(\tilde{y}) - g(\mathbf{x}) \quad for \quad \mathbf{x} \in boundary \quad \partial\Omega.$$

Use y_{exact} for exact solution of the boundary value problem. Then, it is obvious that $R_L(\mathbf{x}, y_{exact}) = 0$ and $R_W(\mathbf{x}, y_{exact}) = 0$.

In using LSM, the goal is to find the coefficients q_i's by minimizing the error function in L^2 norm, defined by

$$E = \int_\Omega R_L^2(\mathbf{x}, \tilde{y}) d\mathbf{x} + \int_{\partial\Omega} R_W^2(\mathbf{x}, \tilde{y}) d\mathbf{x}.$$

The best approximate solution is determined by finding the minimal value of E, or

$$\frac{\partial E}{\partial q_i} = 0, \quad for \quad i = 1, .., N,$$

which yields

$$\int_\Omega R_L(\mathbf{x}, \tilde{y}) \frac{\partial R_L}{\partial q_i} d\mathbf{x} + \int_{\partial\Omega} R_W(\mathbf{x}, \tilde{y}) \frac{\partial R_W}{\partial q_i} d\mathbf{x} = 0, \quad i = 1, .., N.$$

The above formulation is in continuous format and the squared residuals are integrated over the domain. This will give N linear equations and by some algebraic manipulation can be written as:

$$Da = b,$$

where D is an $N \times N$ matrix, $a = [q_1, q_2, ..., q_N]^T$, and some column vector b.

The following format is the discrete formulation and the squared residuals are summed at finite points $\mathbf{x}_i$, $1 \leq i \leq k$, in domain, and $\mathbf{x}_i$, $k+1 \leq i \leq m$, on the boundary points. Define

$$E = \sum_{i=1}^{k} R_L^2(\mathbf{x}_i, \tilde{y}) + \sum_{i=k+1}^{m} R_W^2(\mathbf{x}_i, \tilde{y}).$$

Write

$$r = \begin{pmatrix} R_L(a, \mathbf{x}_1) \\ . \\ . \\ R_L(a, \mathbf{x}_k) \\ R_W(a, \mathbf{x}_{k+1}) \\ . \\ . \\ R_W(a, \mathbf{x}_m) \end{pmatrix} = \begin{pmatrix} L\tilde{y}(a, \mathbf{x}_1) - f(\mathbf{x}_1) \\ . \\ . \\ L\tilde{y}(a, \mathbf{x}_k) - f(\mathbf{x}_k) \\ L\tilde{y}(a, \mathbf{x}_{k+1}) - f(\mathbf{x}_{k+1}) \\ . \\ . \\ L\tilde{y}(a, \mathbf{x}_m) - f(\mathbf{x}_m) \end{pmatrix}.$$

The discrete least square solution, which minimizes $E = r^T r$, is then determined by

$$\frac{\partial E}{\partial q_i} = 0, \quad for \quad i = 1, .., N.$$

Choosing the right basis function ϕ_i is very important in LSM. In linear ODE we usually use polynomials. In the next section, polynomials with different degrees are used and compared to solve some first order ODE and second order ODEs in both continuous and discrete formats.

1.1. Example of using LSM to solve a first-order ODE

1.1.1. Continuous Least Square Method

First, we want to solve an example of a first order ordinary differential equation as an illustration to present least squares methods. Assume that we have the following initial value problem of the first order ODE:

$$\frac{dy}{dx} - y = 0, \qquad y(0) = 1,$$

where $0 \leq x \leq 1$.

Let

$$L(x, y) = \frac{dy}{dx} - y$$

Step 1: Choose basis functions. Here we use polynomials. So,

$$\tilde{y} = \sum_{i=1}^{N} q_i x^i + y_0. \tag{1.2}$$

Step 2: For $\tilde{y}$ to satisfy the boundary condition, clearly we must have $y_0 = 1$.

Step 3: From the residual

$$R(x) = \frac{d\tilde{y}}{dx} - \tilde{y}. \tag{1.3}$$

By replacing $\tilde{y}(x)$ from (1.2) into (1.3), we will get:

$$R(x) = \frac{d(\sum_{i=1}^{N} q_i x^i + 1)}{dx} - (\sum_{i=1}^{N} q_i x^i + 1).$$

Step 4: To minimize the square error, we need to set up

$$E = \int_0^1 R^2(x)dx.$$

The best approximate solution is determined by finding the minimal value of E, or

$$\frac{\partial E}{\partial q_i} = 0, \quad for \quad i = 1, .., N$$

$$\int_0^1 R(x)\frac{\partial R}{\partial q_i}dx = 0, \quad i = 1, .., N$$

or

$$\left(R(x), \frac{\partial R(x)}{\partial q_i}\right) = 0 \qquad for \qquad i = 1, 2, 3, ..., N$$

These lead to a linear system which can be solved for q_i's.

The above equations were solved by a Matlab program. The following results were obtained by running the code for different N's.

When N=3, we get the following matrices

$$D = \begin{pmatrix} 0.33 & 0.25 & 0.2 \\ 0.25 & 0.533 & 0.66 \\ 0.2 & 0.66 & 0.94 \end{pmatrix}, \quad b = \begin{pmatrix} -0.5 \\ -0.66 \\ -0.75 \end{pmatrix}, \quad a = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

and the approximate solution is

$$\tilde{y} = 0.2797x^3 + 0.4255x^2 + 1.0131x + 1.$$

Results are shown in the figure 1.1.

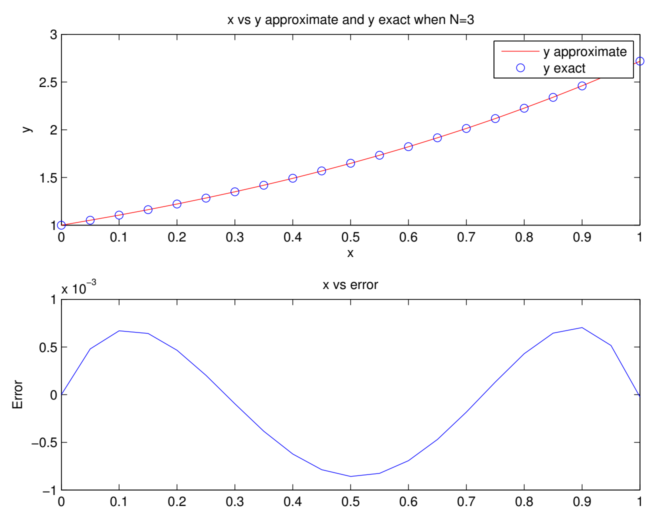

Figure 1.1: Exact solution, approximate solution, and the error for first order ODE example when N=3

When N=5, We get the following matrices

$$D = \begin{pmatrix} 0.33 & 0.25 & 0.2 & 0.16 & 0.14 \\ 0.25 & 0.533 & 0.66 & 0.74 & 0.79 \\ 0.2 & 0.66 & 0.94 & 1.125 & 1.25 \\ 0.16 & 0.74 & 0.125 & 1.125 & 1.6 \\ 0.14 & 0.79 & 1.125 & 1.6 & 1.86 \end{pmatrix}, \quad b = \begin{pmatrix} -0.5 \\ -0.66 \\ -0.75 \\ -0.8 \\ -0.83 \end{pmatrix}, \quad a = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}.$$

The approximate solution is

$$\tilde{y} = 0.0139x^5 + 0.0349x^4 + 0.1702x^3 + 0.4992x^2 + 1.0001x + 1.$$

We know that the exact solution is

$$y_{exact} = e^x.$$

The following figures show the graph of approximate and exact solutions and the errors. The error is defined as

$$error = y_{exact} - \tilde{y}.$$

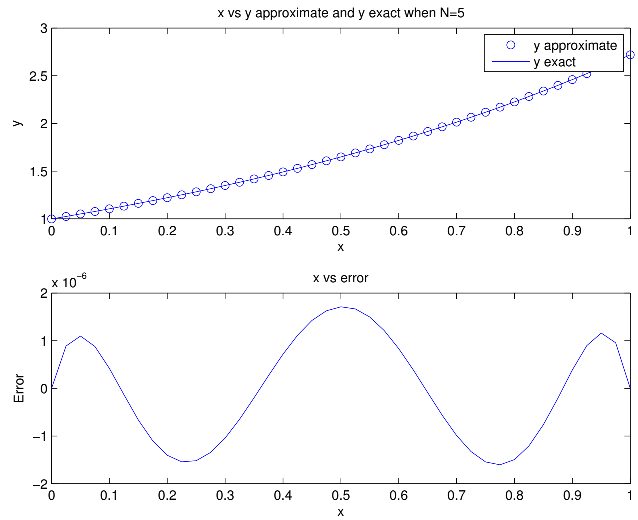

Figure 1.2: Exact solution, approximate solution, and the error for first order ODE example when N=5

1.1.2. Discrete Least Square Methods

For the above example we can solve the first order ODE by least square method in a discrete form. To approach the problem in a discrete method, we approximate the solution in discrete points within the domain and on the boundary points. When $N = 3$, three discrete points $x_1 = 0$, $x_2 = 0.5$ and $x_3 = 1$ were chosen. Matrices r and E were similarly made. Then set

$$\frac{\partial E}{\partial q_i} = 0 \qquad for \qquad i = 1, 2, 3.$$

q_i's are calculated. The approximate solution is

$$\tilde{y} = \frac{2}{3}x^3 + \frac{3}{7}x^2 + x + 1.$$

Figure 1.3 shows the graph of the approximate solution and exact solution as well as the error.

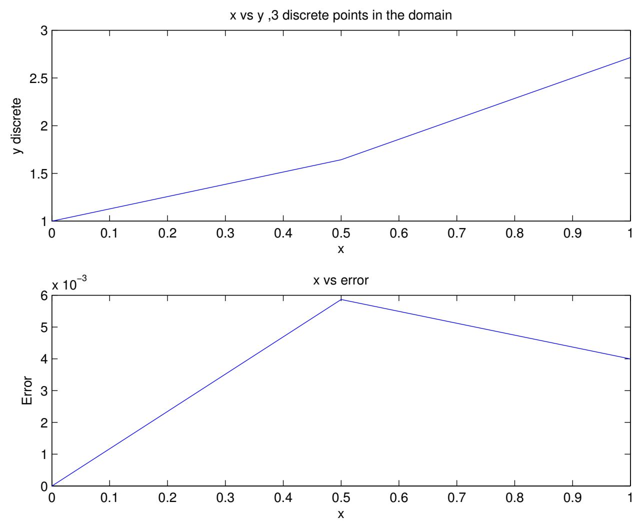

Figure 1.3: Solution of an example of first order ODE by discrete Least Square Mathod with N=3 and 3 points in the domain and on the boundary

When $N = 5$, 5 discrete points $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$ and $x_5 = 1$ were chosen. By following the exact steps as above, we have

$$\tilde{y} = 0.2817x^3 + 0.4288x^2 + 1.006x + 1.$$

Figure 1.4 shows the graph of the approximate solution and exact solution as well as the error.

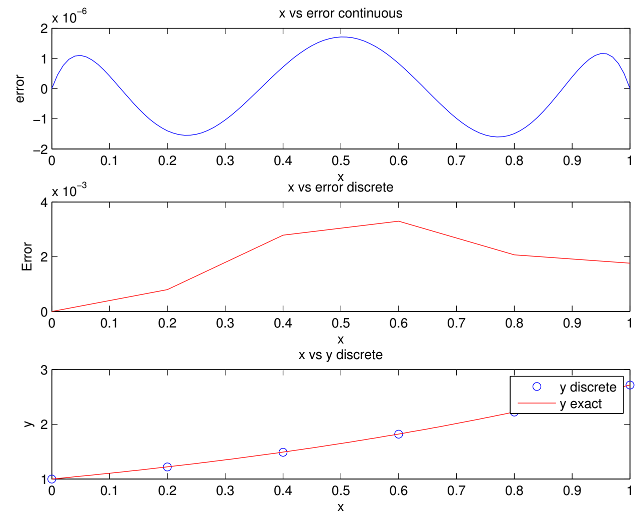

Figure 1.4: Solution of an example of first order ODE by discrete Least Square Mathod with N=3 and 5 points in the domain and boundary

1.2. Example of using LSM to solve a second-order ODE

Now consider the following second order ODE:

$$\frac{dy^2}{dx^2} + \frac{dy}{dx} + 4y = 4x^2 + 10x + 2, \tag{1.4}$$

with Boundary conditions

$$y(0) = 1 \quad and \quad y'(0) = 2.$$

The exact solution is

$$y(x) = 2\exp(-x) - \exp(-4x) + x^2.$$

Now,we use polynomials as basis functions, so the approximate solution will be in the form of

$$\tilde{y} = \sum_{i=1}^{N} q_i \phi_i + y_0. \tag{1.5}$$

To satisfy the boundary conditions, we need to have $y_0 = 1$ and $q_1 = 2$. By following LSM and solving the systems of equations in Matlab, we will get the solution and error below (Figure 1.5). As we saw, polynomials give us acceptable errors as basis

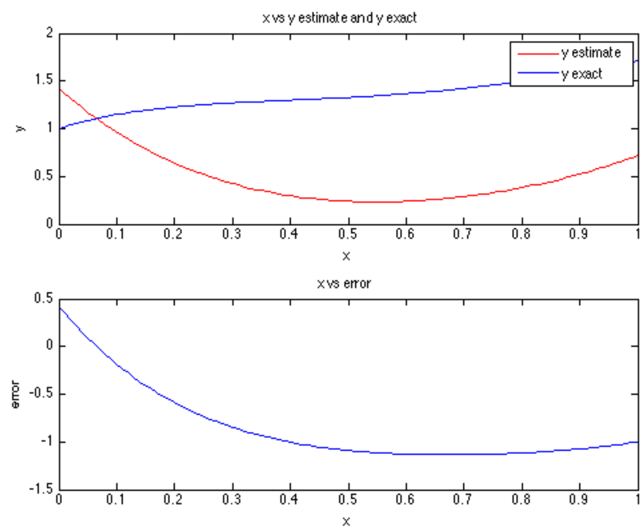

Figure 1.5: Solution and error for example of second order ODE by LSM

functions when we use them in least square method for first order ODEs in both discrete and continuous cases. However, they do not give errors in acceptable range when solving second order ODEs and PDEs. In the next chapter, we introduce B-spline basis functions which satisfy our need.

CHAPTER 2 - LEAST SQUARE METHODS BY B-SPLINES

In this chapter, we will introduce and use B-spline basis functions for solving ODE's and PDE's by LSM.

2.1. B-spline Basis Functions

To introduce B-splines, let U be a set of $n+1$ non-decreasing numbers, $u_0 \leq u_1 \leq ... \leq u_n$, which are called knots, and the half-open interval $[u_i, u_{i+1})$ is the ith knot span for $0 \leq i \leq n-1$. The knots can be spaced uniformly or non-uniformly. They also can appear multiple times and in that case they are called multiple knots.

For $u_i < u_{i+1}$, the ith B-spline of degree 0 is defined by

$$N_{i,0}(u) = \begin{cases} 1, & u_i \leq u < u_{i+1}, \\ 0, & otherwise, \end{cases}$$

and for $k \geq 1$,

$$N_{i,k}(u) = \frac{u - u_i}{u_{i+k} - u_i} N_{i,k-1}(u) + \frac{u_{i+k+1} - u}{u_{i+k+1} - u_{i+1}} N_{i+1,k-1}(u).$$

The above formula is usually referred to as Cox-de Boor recursion formula.

It is obvious that for $k = 0$ the basis functions are all step functions. This is because the basis function $N_{i,0}(u)$ is 1 if u is in the ith knot span $[u_i, u_{i+1})$. For example, for knots $u_0 = 0$, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$ the knot spans are $[0, 1)$, $[1, 2)$, $[2, 3)$. The basis function of degree 0 are $N_{0,0}(u) = 1$ on $[0, 1)$ and 0 elsewhere, $N_{1,0}(u) = 1$ on $[1, 2)$ and 0 elsewhere, $N_{2,0} = 1$ on $[2, 3)$ and 0 elsewhere (Figure 2.1).

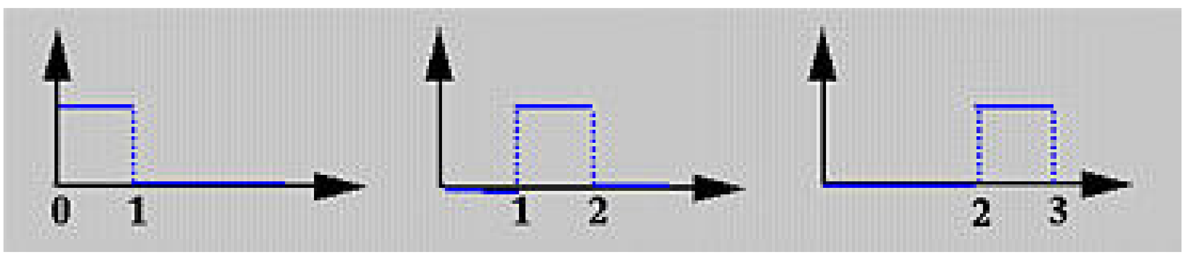

Figure 2.1: Graph of $N_{i,0}(u)$ for knots $0, 1, 2, 3$

Figure 2.2 shows how we can compute $N_{i,k}(u)$ using the triangular computation scheme .

2.2. Computing B-spline basis functions

As an example let's compute the basis functions for k=1. Choose the following knots [2]

$$u_0 = 0, u_1 = 1, u_2 = 2, u_3 = 3.$$

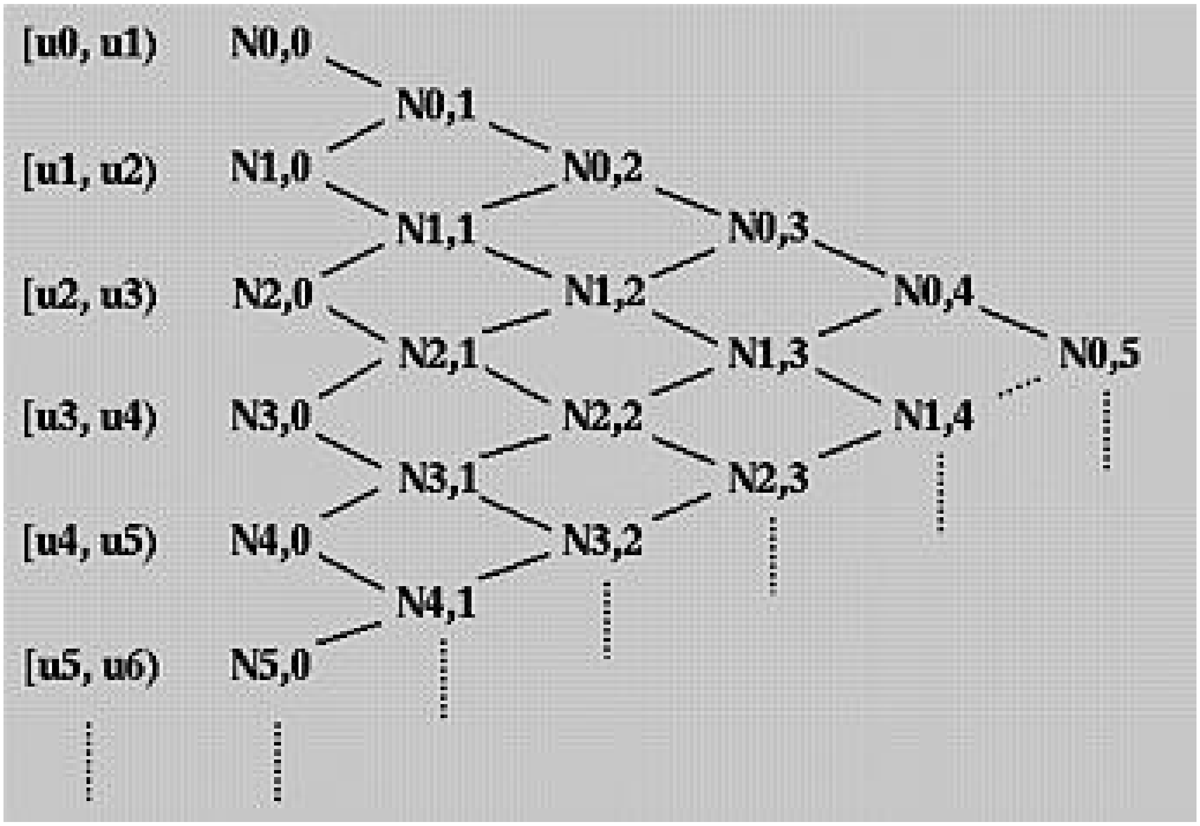

Figure 2.2: Triangular computation scheme for $N_{i,k}(u)$

Then, by definition above,

$$N_{0,1}(u) = \frac{u - u_0}{u_1 - u_0} N_{0,0}(u) + \frac{u_2 - x}{u_2 - u_1} N_{1,0}(u),$$

by replacing the values of the knots we will get

$$N_{0,1}(u) = uN_{0,0}(x) + (2 - u)N_{1,0}(u).$$

If u is in $[0, 1)$, then $N_{0,0} = 1$ and $N_{1,0} = 0$. Therefore, $N_{0,1}(U) = u$. If u is in $[1, 2)$, then $N_{0,0} = 0$ and $N_{1,0} = 1$. Therefore, $N_{0,1}(U) = 2 - u$.

Similarly, $N_{1,1}(u) = u - 1$, if u is in $[1, 2)$ and $N_{1,1}(u) = 3 - u$, if u is in $[2, 3)$. Figure 2.3 shows the graph of $N_{i,1}(x)$.

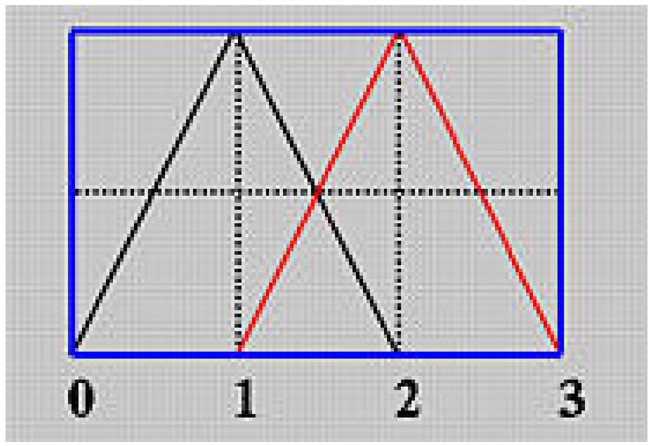

Figure 2.3: Graph of $N_{i,1}(u)$ for knots $0, 1, 2, 3$

Now,

$$N_{0,2}(u) = \frac{u - u_0}{u_2 - u_0} N_{0,1}(u) + \frac{u_3 - u}{u_3 - u_1} N_{1,1}(u)$$

If $0 \leq u < 1$, then, only $N_{0,1}(u) = u$ contributes to $N_{0,2}(u)$, and

$$N_{0,2}(u) = 0.5u^2.$$

For $1 \leq u < 2$ both $N_{0,1}(u) = 2 - u$ and $N_{1,1}(u) = u - 1$ contribute to $N_{0,2}(u)$. In this case,

$$N_{0,2}(u) = (0.5u)(2 - u) + 0.5(3 - u)(3 - u) = 0.5(-3 + 6u - 2u^2).$$

Finally, if $2 \leq x < 3$, only $N_{1,1}(u) = 3 - u$ contributes to $N_{0,2}(u)$,

$$N_{0,2}(u) = (0.5)(3 - u)(3 - u) = 0.5(3 - u)^2$$

Figure 2.4 shows the graph of $N_{i,2}(u)$. It is clear that the curve segments join at the knots and here the graph is smooth. In general, if there are multiple knots, the graph is not smooth.

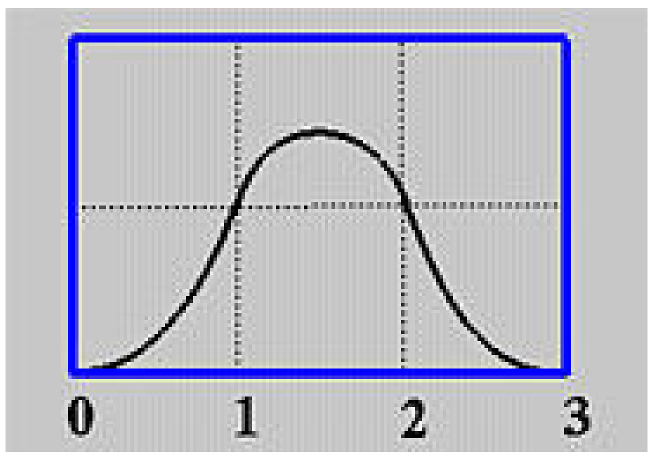

Figure 2.4: Graph of $N_{i,2}(u)$ for knots $0, 1, 2, 3$

Following triangular scheme in Figure 2.2, we are able to compute all $N_{i,k}(u)$s in different knot span. Figures 2.5 and 2.6 show the graphs of $N_{i,3}(u)$ and $N_{i,4}(u)$ in different knot spans.

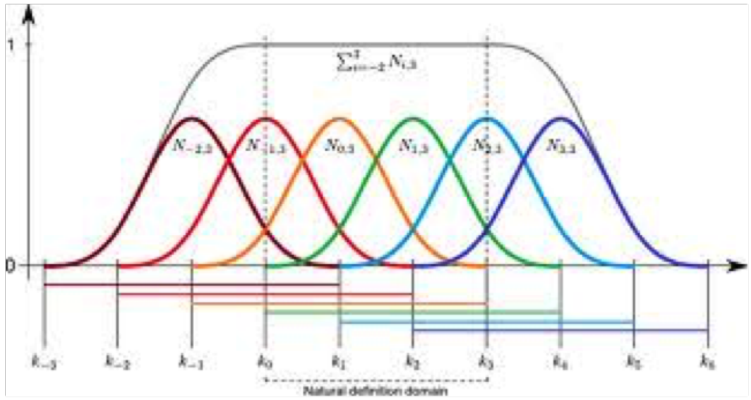

Figure 2.5: Graph of $N_{i,3}(u)$ for knots $0, 1, 2, 3$

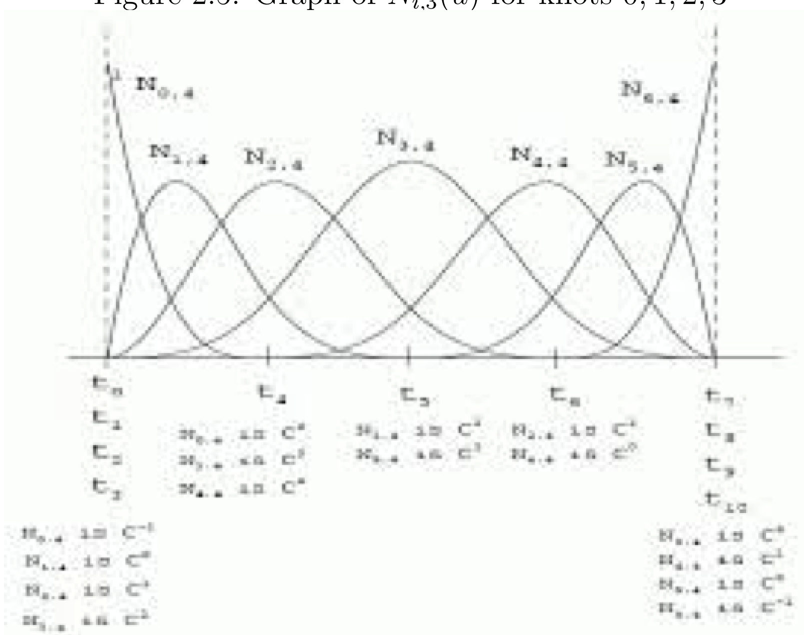

Figure 2.6: Graph of $N_{i,4}(u)$ for knots $0, 1, 2, 3$

By calculating the recursive formula we can observe the following important properties of B-spline basis:

1. Basis function $N_{i,k}(u)$ is non-zero on $[u_i, u_{i+k+1})$

2. On any knot span $[u_i, u_{i+1})$, at most $k+1$ degree k basis functions are non-zero.

3. $N_{i,k}(x)$ is a piece-wise degree k polynomial in each $[u_i, u_{i+1})$, $[u_{i+1}, u_{i+2})$, ...,

$[u_{i+k}, u_{i+k+1})$.

4. For all i, k and u, $N_{i,k}(u)$ is non-negative.

5. $N_{i,k}(u)$ is locally supported. Namely, $N_{i,k}(u)$ is a non-zero piece-wise polynomial on $[u_i, u_{i+k+1})$.

6. The sum of all non-zero degree k basis functions on span $[u_i, u_{i+1})$ is 1.

7. If u_i is a knot of multiplicity m, then $N_{i,k}(u)$ is C^{k-m} continuous at u_i.

2.3. Solving a 2nd-order ODE by B-splines

In the previous chapter, we saw that polynomials are not suitable to be used as basis functions in LSM for solving 2nd order ODEs. Here, we want to show that B-splines give us an acceptable approximate solution by LSM. As we saw, if the knots u_i are distinct or of multiplicity 1, B-spline basis functions of degree k are C^{k-1} continuous. This property is helpful when we use B-spline functions as the basis in Least Square method.

To get a set of basis functions in LSM, as in the paper by Loghmani [3], we use translations and dialations of the standard B-spline functions. To be precise, for an interval $[a, b]$, we consider equal partitions

$$a < a + h < a + 2h < ... < a + 3 \cdot 2^k h = b,$$

where $h = \frac{b-a}{3 \cdot 2^k}$.

Define

$$B_{ki}(t) = B\left(\frac{3 \cdot 2^k}{b-a}(t-a) - i\right), (i = -3, -2, -1, ..., 3 \cdot 2^k - 1)$$

where B is a B-spline and B_{ki} with $k \in N, (i = -3, -2, ..., 3 \cdot 2^k - 1)$ is translation and dilation of B.

Example: We now consider the following 2nd-order ODE

$$\frac{dy^2}{dx^2} + \frac{dy}{dx} + 4y = 4x^2 + 10x + 2 \tag{2.1}$$

with the boundary condition

$$y(0) = 1 \qquad and \qquad y'(0) = 2.$$

The exact solution is

$$y(x) = 2\exp(-x) - \exp(-4x) + x^2.$$

Since we have a second order ODE, we use the quadratic B-spline basis function.

$$B(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ -2t^2 + 6t - 3, & 1 \le t \le 2, \\ (3-t)^2, & 2 \le t \le 3, \\ 0, & otherwise. \end{cases}$$

Choose $k=1$, so $h=\frac{1}{6}$ and the partitions will be

$$0 \leq \frac{1}{6} \leq \frac{1}{3} \leq \frac{1}{2} \leq \frac{2}{3} \leq \frac{5}{6} \leq 1.$$

Define the approximate solution as

$$\tilde{y} = \sum_{i=-2}^{5} c_i B_{1i}(x).$$

Set the residual as

$$r = \frac{d^2\tilde{y}}{dx^2} + \frac{d\tilde{y}}{dx} + 4\tilde{y} - 4x^2 - 10x - 2,$$

or

$$r = \frac{d^2(\sum_{i=-2}^{5} c_i B_{1i}(x))}{dx^2} + \frac{d(\sum_{i=-2}^{5} c_i B_{1i}(x))}{dx} + 4(\sum_{i=-2}^{5} c_i B_{1i}(x)) - 4x^2 - 10x - 2.$$

To satisfy the boundary conditions we need to have $c_{-1} = \frac{7}{12}$ and $c_{-2} = \frac{5}{12}$.

We use LSM at discrete points $x_0 = 0$, $x_1 = \frac{1}{12}$, $x_2 = \frac{1}{4}$, $x_3 = \frac{5}{12}$, $x_4 = \frac{7}{12}$, $x_5 = \frac{3}{4}$, $x_6 = \frac{11}{12}$, $x_7 = 1$. Then we solve the system of equations in Matlab to find c_i's. The results are given in Figure 2.7 and Table 2.1.

As it can be seen, the error is within acceptable range and B-spline basis is a much better choice than polynomials.

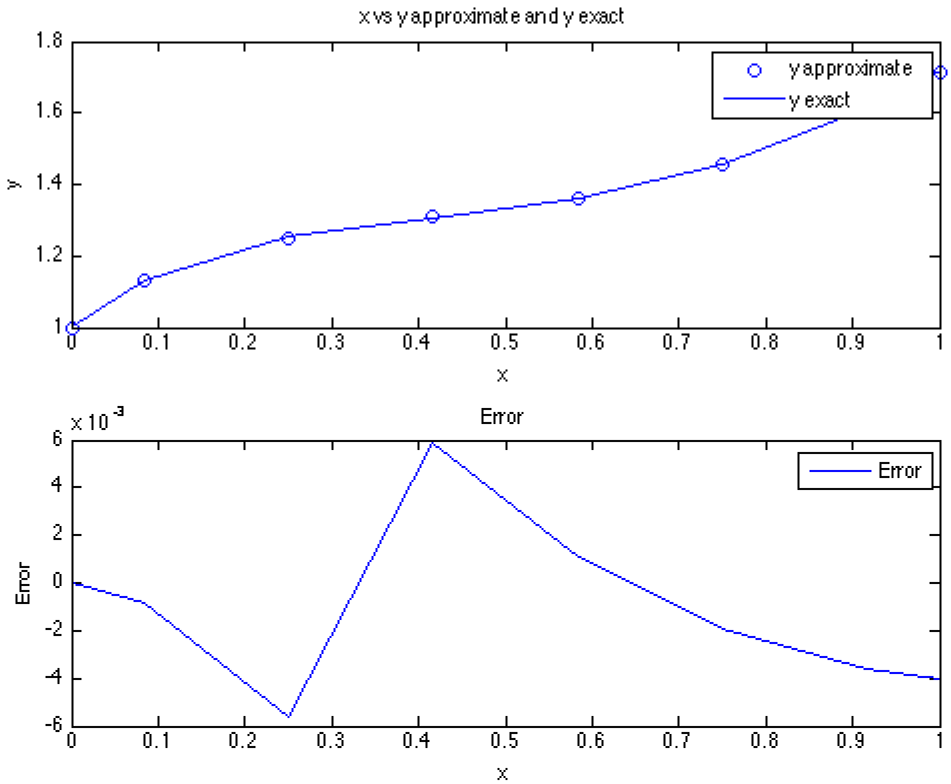

Figure 2.7: Solving a second order ODE by LSM using B-spline basis when k=1

x	y exact	y approximate
0	1	1
0.08	1.1305	1.1297
0.25	1.2522	1.2466
0.42	1.3032	1.3091
0.58	1.3594	1.3605
0.75	1.4574	1.4555
0.92	1.6144	1.6108
1	1.7174	1.7134

Table 2.1: Data for the example of second order ODE when k=1

2.4. More examples of solving higher order ODEs

Example 1: Now consider the following BVP

$$y'''(x) = y(x) - 3e^x, \qquad 0 < x < 1,$$

$$y'(0) = 0, \qquad y(1) = 0, \qquad y(0) = 1.$$

The exact solution is $y(x) = (1-x)e^x$ [5]. For this problem we want to use the following cubic B-spline as the basis function

$$B(t) = \begin{cases} t^3, & 0 \leq t \leq 1, \\ -3t^3 + 12t^2 - 12t + 4, & 1 \leq t \leq 2, \\ 3t^3 - 24t^2 + 60t - 44, & 2 \leq t \leq 3, \\ (4-t)^3, & 3 \leq t \leq 4, \\ 0, & otherwise. \end{cases}$$

Choose $k = 0$, then $h = \frac{1}{3}$ and consider the following partitions

$$0 \leq \frac{1}{3} \leq \frac{2}{3} \leq 1.$$

Define the approximate solution as

$$\tilde{y} = \sum_{i=-3}^{2} c_i B_{0i}(x).$$

Define the residual as

$$r = \frac{d^3\tilde{y}}{dx^3} - \tilde{y} + 3e^x,$$

or

$$r = \frac{d^3(\sum_{i=-3}^{2} c_i B_{0i}(x))}{dx^3} - \sum_{i=-3}^{2} c_i B_{0i}(x)) + 3e^x.$$

To satisfy the boundary condition we need to have

$$c_{-3} = c_{-1}, \qquad c_{-2} = -0.5c_{-1} + \frac{1}{4}, \qquad c_0 = -4c_1 - c_2.$$

We follow least square method calculated at the following discrete points,

$$x_1 = 0,\ x_2 = \frac{1}{9},\ x_3 = \frac{2}{9},\ x_4 = \frac{1}{3},\ x_5 = \frac{4}{9},\ x_6 = \frac{5}{9},\ x_7 = \frac{2}{3},\ x_8 = \frac{7}{9},\ x_9 = \frac{8}{9},\ x_{10} = 1.$$

By solving the systems of equations in Matlab we will get the following results (Table 2.2 and Figure 2.8).

x	y approximate	y exact
0	1	1
0.1111	0.993	0.9933
0.2222	0.9701	0.9713
0.3333	0.9281	0.9304
0.4444	0.8631	0.8665
0.5555	0.7705	0.7745
0.6666	0.6449	0.6492
0.7777	0.4803	0.4837
0.8888	0.2682	0.2702
1	0	0

Table 2.2: Results for example 1, when k=0

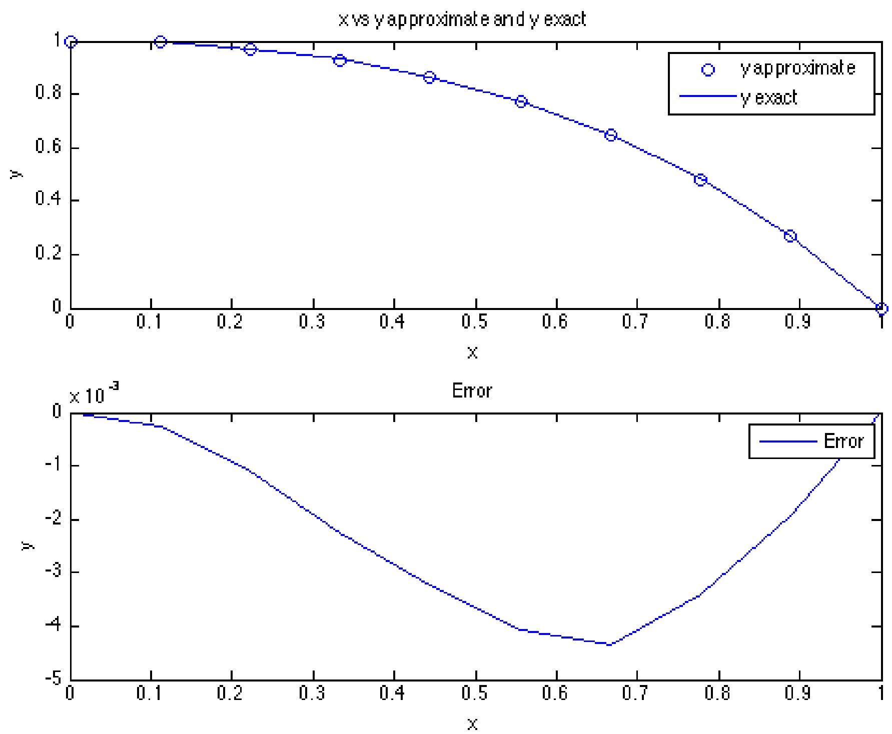

Figure 2.8: Solving a third order ODE by discerete LSM using B-spline basis when k=0

Now, if $k = 1$, then $h = \frac{1}{6}$ and consider the following partitions:

$$0 \leq \frac{1}{6} \leq \frac{1}{3} \leq \frac{1}{2} \leq \frac{2}{3} \leq \frac{5}{6} \leq 1$$

We follow discrete LSM at the following discrete points:

$$x_1 = 0,\ x_2 = \frac{1}{12},\ x_3 = \frac{1}{4},\ x_4 = \frac{5}{12},\ x_5 = \frac{4}{9},\ x_6 = \frac{5}{9},\ x_7 = \frac{7}{12},\ x_8 = \frac{3}{4},\ x_9 = \frac{11}{12},\ x_{10} = 1.$$

. After solving the system of equations in Matlab for c_i's, we will get the following results (Figure 2.9 and Table 2.3).

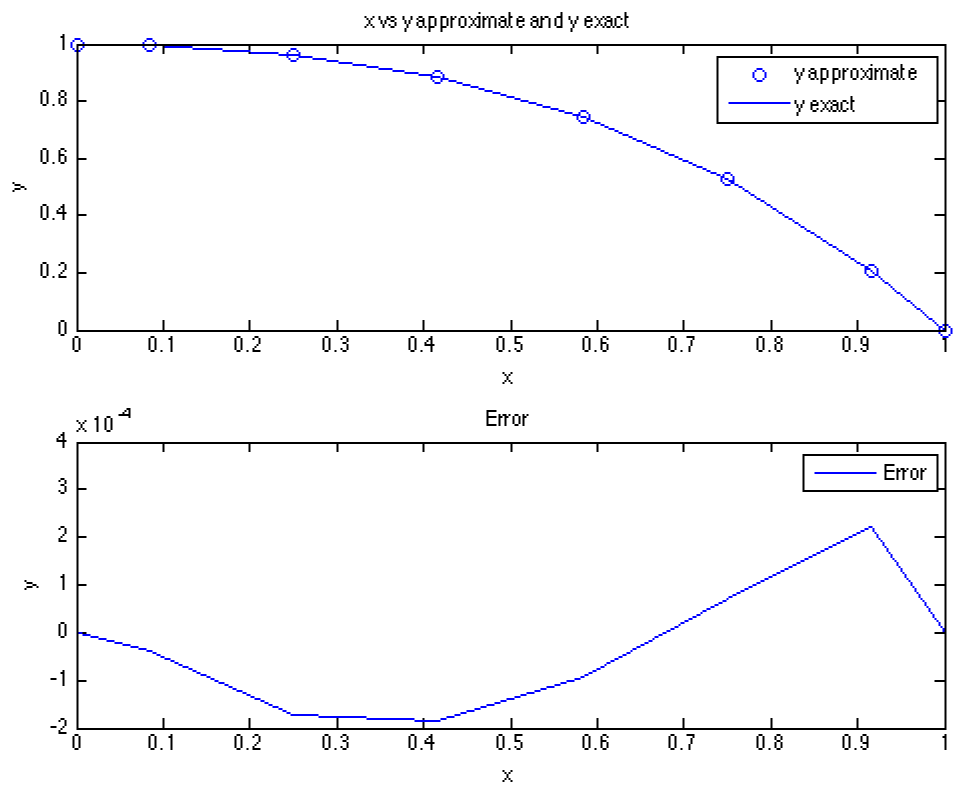

Figure 2.9: Solving third order ODE by discerete LSM using B-spline basis when k=1

The results show that even for $k = 0$, the error is within acceptable range. As we increase k the error becomes smaller.

x	y approximate	y exact	error
0	1	1	5.551E-17
8.33E-2	0.9962	0.9962	-3.762E-5
0.25	0.9628	0.9629	-1.7359E-4
0.4166	0.8846	0.8849	-1.8709E-4
0.5833	0.746	0.7467	-9.377E-5
0.75	0.529	0.5292	7.1409E-5
0.9166	0.2086	0.2084	2.2269E-4
1	6.9388E-18	0	6.9389E-18

Table 2.3: Results for example 1, when k=1

Example 2: Consider a sixth degree linear boundary value problem [6]

$$y^{(6)} + xy = -(24 + 11x + x^3)e^3, \qquad 0 \le x \le 1$$

subject to

$$y(0) = 0, \qquad y(1) = 0,$$

$$y'(0) = 1, \qquad y'(1) = -e,$$

$$y''(0) = 0, \qquad y''(1) = -4e.$$

The exact solution of the above problem is

$$y(x) = x(1-x)e^x.$$

Now, we want to solve this problem with LSM using B-spline basis function. Loghmani [7] suggests using sixth degree spline function,

$$B(t) = \begin{cases} t^6, & 0 \leq t \leq 1, \\ -6t^6 + 42t^5 - 105t^4 + 140t^3 - 105t^2 + 42t - 7, & 1 < t \leq 2, \\ 15t^6 - 210t^5 + 1155t^4 - 3220t^3 + 4935t^2 - 3990t + 1337, & 2 < t \leq 3, \\ -20t^6 + 420t^5 - 3570t^4 + 15680t^3 - 37590t^2 + 47040t - 24178, & 3 < t \leq 4, \\ 15t^6 - 420t^5 + 4830t^4 - 29120t^3 + 96810t^2 - 168000t + 119182, & 4 < t \leq 5, \\ t^6 - 42t^5 + 735t^4 - 6860t^3 + 36015t^2 - 100842t + 117649, & 6 < t \leq 7, \\ 0, \qquad \qquad otherwise. \end{cases}$$

For a fix $k \in N$, equal partitions in $[a, b]$ by the knots.

$$a < a + h < a + 2h < ... < a + 3 \cdot 2^k h = b,$$

where $h = \frac{b-a}{3 \cdot 2^k}$. Define,

$$B_{ki}(t) = B\left(\frac{3 \cdot 2^k}{b - a}(t - a) - i\right), (i = -6, -5, -2, ..., 3 \cdot 2^k - 1).$$

Here we choose $k = 1$ so, $h = \frac{1}{6}$. Form an approximate solution

$$\tilde{y} = \sum_{i=-6}^{5} c_i B_{1i}(x).$$

This solution needs to satisfy the boundary conditions which will give us 6 equations. The rest of equations are found by defining the residual as

$$r = \tilde{y}(6) + x\tilde{y} = -(24 + 11x + x^3)e^3$$

and evaluating them at the discrete points in the domain and on the boundary $x = \{0, \frac{1}{12}, \frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, , \frac{11}{12}, 1\}$. Then. Define $E = rr^T$ and set $\frac{\partial E}{\partial c_i} = 0$. After solving the systems of equations for c_i's, the results are presented below (Figure 2.10).

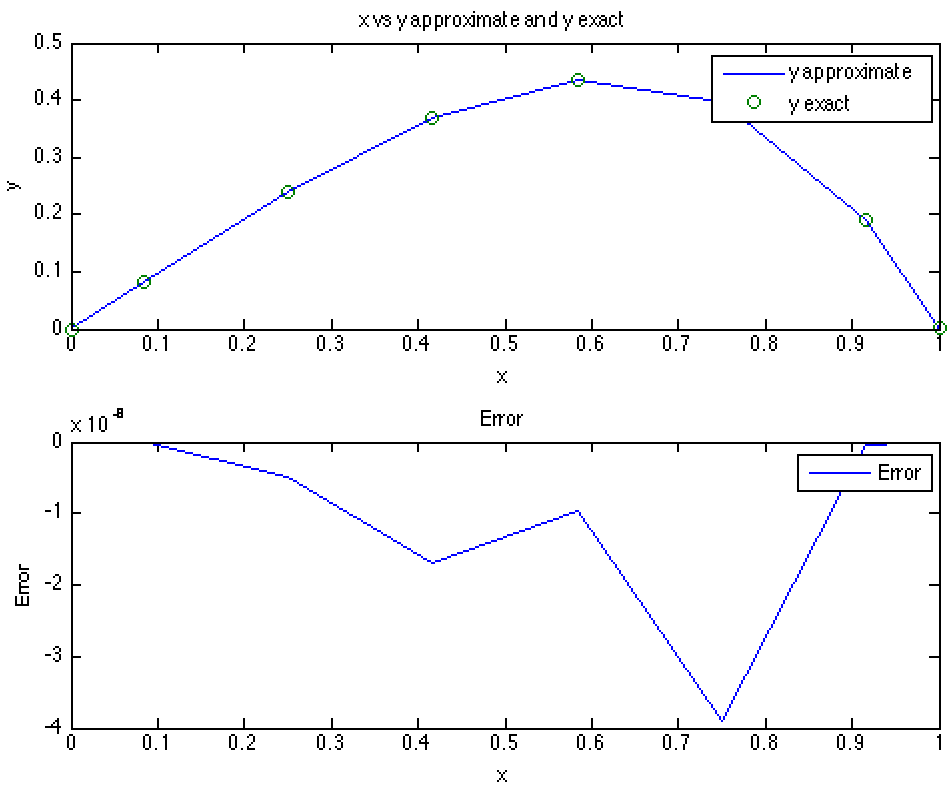

Figure 2.10: Solving a six-order ODE by LSM using B-spline basis when k=1

2.5. Solving a PDE by B-spline basis

Consider the following Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6xy(1-y) - 2x^3$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Suppose that $u(x, y)$ satisfies mixed boundary conditions

$$u(0, y) = 0,$$

$$u(1, y) = y(1-y),$$

$$u(x, 0) = u(x, 1) = 0,$$

Choose $k = 0$, then $h = \frac{1}{3}$ and consider the following partitions $0 \leq \frac{1}{3} \leq \frac{2}{3} \leq 1$ on both x and y axes. Now define the following approximate solution by LSM

$$\tilde{u}(x, y) = \sum_{i=-3}^{2} (\sum_{j=-3}^{2} c_{ij} B_{0i}(x) B_{0j}(y)),$$

where B_{0i} and B_{0j} are dilations and translations of the cubic B-spline.

$c_{ij}, i = -3, -2, ...1, 2$ and $j = -3, -2, ...1, 2$ are the coefficients to be found in LSM. $\tilde{u}(x, y)$ needs to satisfy the boundary points, which gives us 12 equations for 12 discrete points on the boundary.

Now, define the residual as

$$r(x,y) = \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} - 6xy(1-y) - 2x^3.$$

Follow the discrete LSM explained in the earlier chapter to construct $E = \sum r^2$. The sum is over 49 uniform discrete points in the domain.

By LSM we need to have

$$\frac{\partial E}{\partial c_{i,j}} = 0, \quad for \quad i = -3, -2, ...1, 2 \quad and \quad j = -3, -2, ...1, 2$$

The 36 unknown c_{ij} will be found by the above equations and equations from boundary conditions.

The table below shows the exact and approximate solution for discrete points (Table 2.4).

x	y	u exact	u approximate	error
0	0	0	-2.05E-8	2.05E-8
0.2	0	0	-3.2E-3	3.2E-3
0.5	0	0	-3.8E-3	3.8E-3
0.8	0	0	-8.099E-3	8.099E-3
1	0	0	2.7E-3	-2.7E-3
0	0.2	0	1.199E-3	-1.199E-3
0.2	0.2	1.28E-3	-4.7E-3	5.98-3
0.5	0.2	2E-2	1.59E-2	4.1E-3
0.8	0.2	8.192E-2	7.739E-2	4.52E-3
1	0.2	0.160	0.16239	-2.399E-3
0	0.5	0	1.1E-3	-1.11E-3
0.2	0.5	2E-3	-6.999E-4	2.7E-3
0.5	0.5	3.125E-2	2.41E-2	7.15E-3
0.8	0.5	0.128	0.1198	8.2E-3
1	0.5	0.25	0.2465	3.5E-3
0	1	0	0	0
0.2	1	0	1.9E-3	-1.9E-3
0.5	1	0	-9.1E-3	9.14E-3
0.8	1	0	1.2999E-3	-1.299E-3
1	1	0	2.2999E-9	-2.299E-9

Table 2.4: Results for solving the example of PDE by B-spline functions

CHAPTER 3 - DIFFERENTIAL MODELING IN APPLICATIONS

3.1. Mass spring system

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger[8]. The plunger moves vertically up-n-down by a flywheel spinning at constant set speed (Figure 3.1). The constant rotational speed of the flywheel makes the impact force on the sheet metal, and therefore the supporting base, intermittent and cyclic. The heavy base on which the metal sheet is situated has a mass M = 2000 kg. The force acting on the base follows a function: F(t) = 2000 sin(10t), in which t is time in seconds. The base is supported by an elastic pad with an equivalent spring constant $k = 2 \times 10^5$ N/m. If the base is initially depressed down by an amount 0.1 m, what is the resonant vibration situation with the applied load ?

The above problem can be modeled by an ODE as follows

$$2000\frac{d^2x(t)}{dt} + 2 \times 10^5 x(t) = 2000 sin(10t),$$

with initial conditions

$$x(0) = 0.1m, \qquad and \qquad x'(0) = 0,$$

We follow the method discussed in Chapter 2 for solving second order ODE. Here,

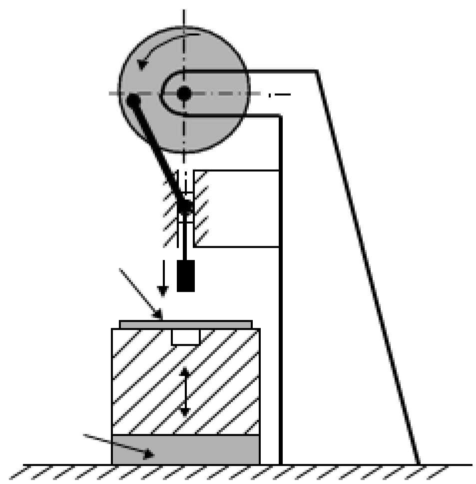

Figure 3.1: Figure for the system

we choose $k = 1$ and use quadratic B-spline basis function. The problem is solved in discrete points $x = \{0, \frac{1}{12}, \frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, 1\}$. By using the Matlab code we get the following approximate solution and error (Figure 3.2).

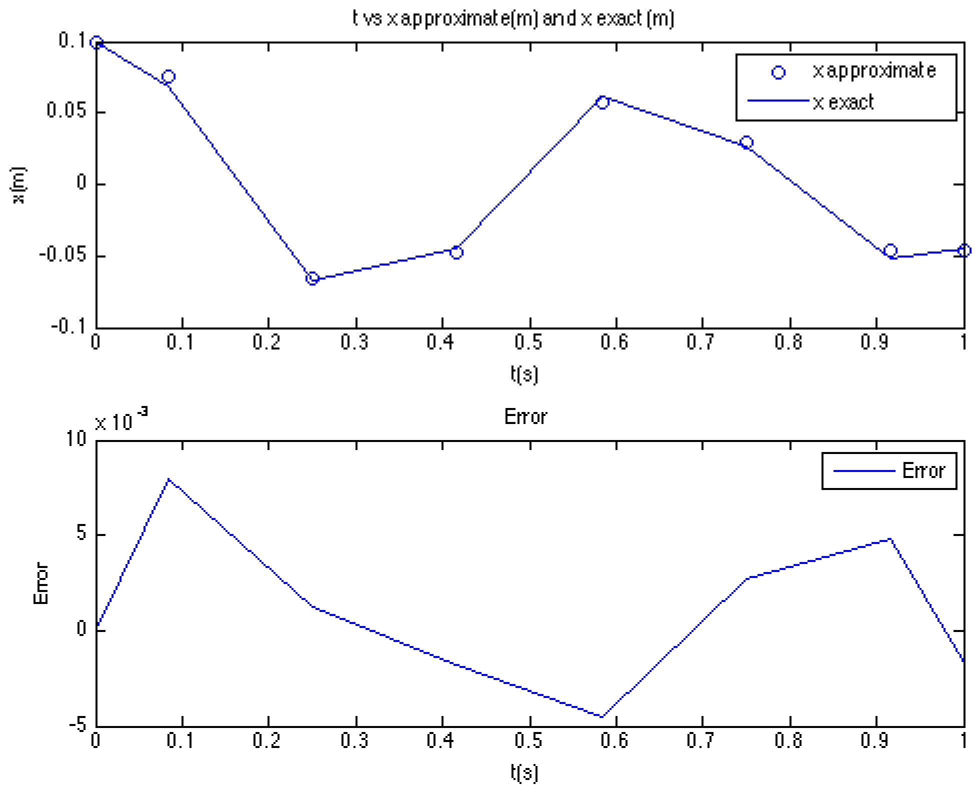

Figure 3.2: x approximate vs t and error for mass spring sysytem

3.2. Wave modeling

Consider one meter power line that is attached to poles at the end. Suddenly a bird lands on the line, 1/3 away from one of the poles and flies away immediately. The initial triangular shape modeled by the function (See Figure 3.3)

$$f(x) = \begin{cases} -\frac{3}{10}x & 0 \leq x \leq \frac{1}{3}, \\ \frac{3(x-1)}{20} & \frac{1}{3} \leq x \leq 1. \end{cases}$$

How does the power line vibrate after the bird releasing the cord until it sets to rest? The problem can be modeled as initial boundary value problem of a wave equation [9].

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0,$$

where $c = \frac{1}{\pi}\frac{m}{s}$, with the boundary conditions

$$u(0,t) = 0 \quad and \quad u(1,t) = 0 \quad for \;\; all \;\; t > 0,$$

and the initial conditions

$$u(x,0) = f(x).$$

The exact solution to the above PDE is

$$u(x,t) = \frac{9}{10\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} \sin n\pi \cos nt.$$

We use quadratic B-spline basis function. Choose $k = 1$, so $h = \frac{1}{6}$ and the partitions on x axis will be $0 \leq \frac{1}{6} \leq \frac{1}{3} \leq \frac{1}{2} \leq \frac{2}{3} \leq \frac{5}{6} \leq 1$. Furthermore, on t axis $h = \frac{10}{6}$ and the partitions are $0 \leq \frac{10}{6} \leq \frac{20}{6} \leq 5 \leq \frac{40}{6} \leq \frac{50}{6} \leq 10$. After setting up the residuals at the following discrete points and minimizing the sum of square errors at these points, we will get the following errors (Table 3.1).

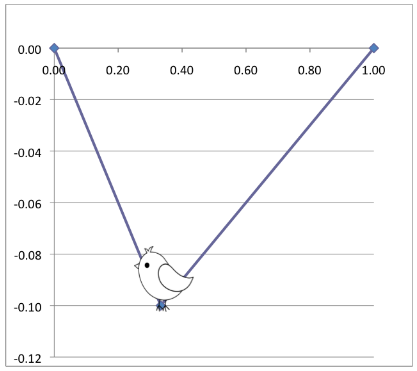

Figure 3.3: Initial position of the bird on the wire

x(m)	t(sec)	u exact	U approximate	error
0	2	0	0	0
0.2	2	0.329	0.3285	-5E-4
0.33329	2	0.478	0.4778	-1E-3
0.5	2	0.33729	0.33739	1E-4
0.8	2	9.21E-2	9.32E-2	1.1E-3
1	2	0	0	0
0	4	0	0	0
0.2	4	0.32869	0.32872	22E-5
0.33329	4	0.54759	0.54759	0
0.5	4	0.5602	0.5603	9.99E-5
0.8	4	0.31419	0.31409	-9.9E-5
1	4	0	0	0
0	8	0	0	0
0.2	8	0.3284	0.32827	-1.3E-4
0.33329	8	0.2536	0.25363	-6.99E-5
0.5	8	0.1147	0.114678	-2.199E-5
0.8	8	0.1295	0.12967	-1.7E-4
1	8	0	0	0

Table 3.1: The solution and the error for wave modeling

3.3. Diffusion equation

A doctor administers an intravenous injection of an allergy fighting medicine to a patient suffering from an allergic reaction. The injection takes a total time T=5sec. The blood in the vein flows with mean velocity u, such that blood over a region of length $L = uT$ contains the injected chemical. The concentration of the chemical in the blood is C_0. What is the distribution of the chemical in the vein when it reaches the heart 80 s later?

The problem can be solved as a diffusion equation [11].

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

with initial distribution:

$$C(x,t_0) = \begin{cases} C_0 & -\frac{L}{2} < x < \frac{L}{2}, \\ 0 & otherwise. \end{cases}$$

The analytical solution is

$$C(x,t) = \frac{c_0}{2}\left(erf(\frac{x+\frac{L}{2}}{\sqrt{4Dt}}) - erf(\frac{x-\frac{L}{2}}{\sqrt{4Dt}})\right),$$

where erf is the error function and is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

The following data was found by talking to a physician

$T = 3$sec, $u = 2$ cm/s, $C_0 = 100$ microgram/cc, $D = 3.24 \times 10^{-3} cm^2/s$.

Therefore, $L = uT = 6$cm. Now we will have:

$$C(x,t) = 50(erf(\frac{x+3}{\sqrt{12.96 \times 10^{-3} t}}) - erf(\frac{x-3}{\sqrt{12.96 \times 10^{-3} t}}))$$

we use quadratic B-spline basis and choose k=1. Now, on the x axis we have $h = 1$ and the partitions are $-3 \le -2 \le -1 \le 0 \le 1 \le 2 \le 3$. Furtheremore, on the t axis $h = \frac{3}{40}$ and we chose the partitions accordingly. We follow the discrete method at $x = \{-3, -2, -1, 0, 1, 2, 3\}$ and $t = \{0, 10, 20, 30, 40, 50, 60, 70, 80\}$. The result for $t = 80sec$ is presented below (Figure 3.4).

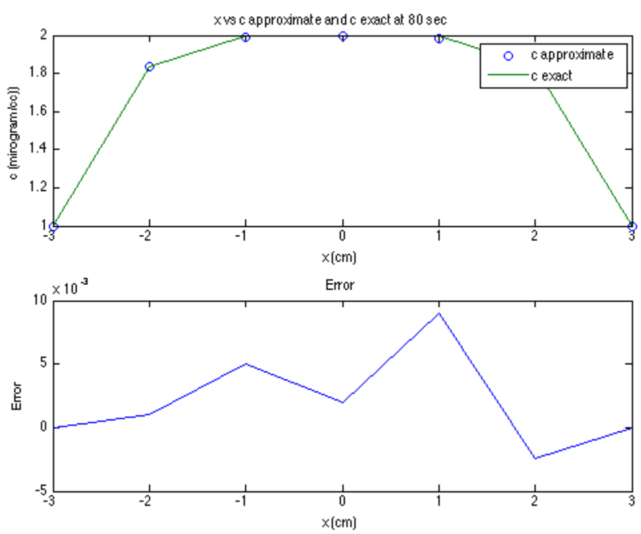

Figure 3.4: Approximate and exact concentration distribution of medicine and error at 80 sec

CHAPTER 4 - COLLOCATION METHODS BY RADIAL BASIS FUNCTIONS

4.1. Radial Basis Functions

Let $\phi : [0, +\infty) \to R$ be a univariate continuous function with $\phi(r) = \phi(\|\mathbf{x}\|)$, where $r = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + ... + x_s^2}$ for $\mathbf{x} = (x_1, x_2, ..., x_s)$ in R^s. Below are a few popular Radial Basis Functions (RBF).

1. Gaussian radial basis:

$$\phi(r) = e^{-c^2 r^2},$$

where c is the constant shape parameter given by

$$c^2 = \frac{1}{2\sigma^2},$$

where σ^2 is the variance of the normal distribution (Figure 4.1).

2. Multiquadric RBF:

$$\phi(r) = \sqrt{r^2 + c^2} \quad for \;\; some \;\; c > 0.$$

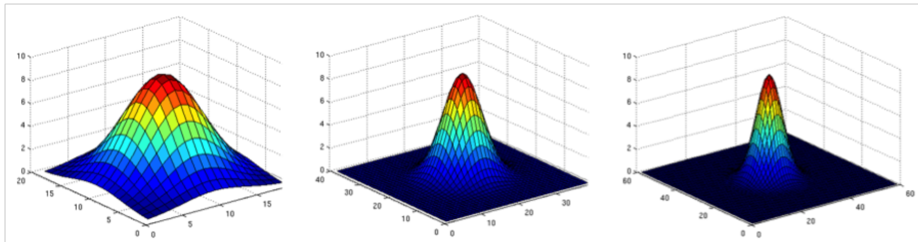

Figure 4.1: Gaussian radial basis: left is centered at (10,10) with c=5, middle is centered at (20,20) with c=3, right is centered at (30,30) with c=1

3. Inverse Multiquadric Function (IMQ)

$$\phi(r) = \frac{1}{\sqrt{r^2 + c^2}} \quad for \;\; some \;\; c > 0.$$

4. Generalized Inverse Multiquadric Functions:

$$\phi(r) = (r^2 + c^2)^{-\alpha}, \quad where \quad c > 0 \quad and \quad \alpha > 0.$$

The Gaussian and Inverse Multiuadric functions have the property

$$\phi(r) \to 0 \qquad as \qquad \|r\| \to \infty,$$

but this is not strictly necessary in solving differential equations below.

4.2. Kansa collocation methods

When we are given scattered data $\{\mathbf{x}_i, f_i\}$, $i = 1, ..., N, \mathbf{x}_i \in R^s, f_i \in R$, our goal is to find an interpolant of the form [13]

$$P(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \mathbf{x} \in R^S,$$

for some RBF ϕ, such that

$$P(\mathbf{x}_i) = f_i, i = 1, ..., N.$$

Setting up the above equation will lead to

$$\begin{pmatrix} \phi(\|\mathbf{x}_1 - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_1 - \mathbf{x}_2\|) & ... & \phi(\|\mathbf{x}_1 - \mathbf{x}_N\|) \\ \phi(\|\mathbf{x}_2 - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_2 - \mathbf{x}_2\|) & ... & \phi(\|\mathbf{x}_2 - \mathbf{x}_N\|) \\ & . & & \\ & . & & \\ & . & & \\ \phi(\|\mathbf{x}_N - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_N - \mathbf{x}_2\|) & ... & \phi(\|\mathbf{x}_N - \mathbf{x}_N\|) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2. \\ . \\ . \\ . \\ c_N \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ . \\ . \\ . \\ f(\mathbf{x}_N) \end{pmatrix}.$$

This linear system can be written as $Ac = f$, where the entries of A are

$$A_{ij} = \phi(\|\mathbf{x} - \mathbf{x}_j\|).$$

It is known that A is non-singular for some radial functions like Multiquadratics, Gaussians and Inverse Multiquadratic basis functions.

There are different ways to choose collocation points. The following figures present several popular collocation points.

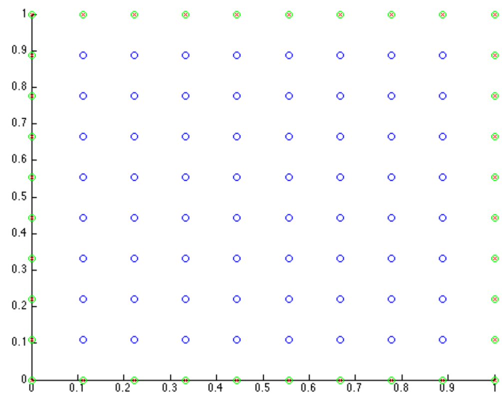

Figure 4.2: 100 uniform points

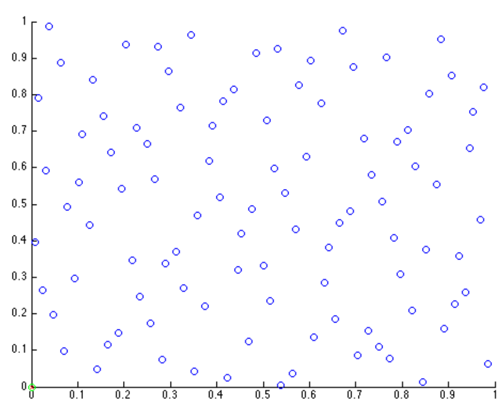

Figure 4.3: 100 Halton points

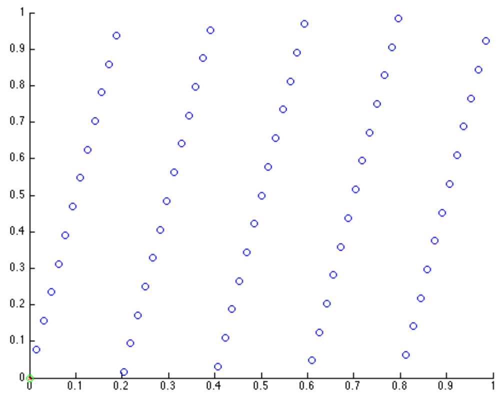

Figure 4.4: 100 Fence points

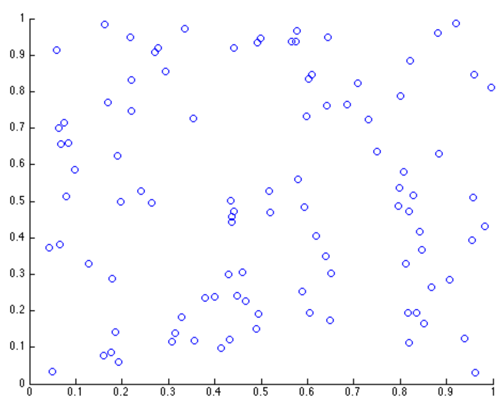

Figure 4.5: 100 Random points

Now consider solving a partial differential equation in the form of

$$Lu(x) = f(x), \quad x \in \Omega,$$

with Dirichlet boundary conditions

$$u(x) = g(x), \quad x \in \partial\Omega.$$

In Kansa's collocation method, we choose the approximate solution in the form [10]

$$\tilde{u}(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x}_i - \mathbf{x}_j^c\|),$$

where $\mathbf{x}_j^c$, $j = 1, .., N$, are the the centers and $\mathbf{x}_i \in \Omega$, $i = 1, .., k$, or $\mathbf{x}_i \in \partial\Omega$, $i = k+1, .., M$, are the collocation points. Most of the times the centers and the collocation points are the same set of points, or $\mathbf{x}_i = \mathbf{x}_i^c$. In this work we set the collocation points and the centers equal, so $M = N$.

Now, we want to match the approximate and exact solution for the differential equations and boundary conditions at the collocation points. So matrix A becomes

$$A = \begin{pmatrix} \tilde{A}_L \\ \\ \tilde{A} \end{pmatrix},$$

where

$$Ac = f = \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ . \\ . \\ f(\mathbf{x}_k) \\ g(\mathbf{x}_{k+1}) \\ . \\ . \\ g(\mathbf{x}_N) \end{pmatrix},$$

$$(\tilde{A}_L)_{ij} = L\phi(\|\mathbf{x}_i - \mathbf{x}_j\|), for \;\; \mathbf{x}_i \in \Omega,$$

$$(\tilde{A})_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|), for \;\; \mathbf{x}_i \in \partial\Omega,$$

where $\mathbf{x}_j$, $j = 1, .., N$, are the centers. In this method RBFs that we can use are the popular RBFs listed in section 4.1, but Kansa specifically used Multiquadratic form. The above method is called Kansa Mesh-free method. The method collocates the RBFs at each node without the need for mesh. The popular numerical solution of PDEs by finite element methods needs creating mesh on the domain. This can be difficult for irregular domain or higher dimensional domains. That is why the above methods attract attention.

4.3. Solving PDEs by RBFs

Example 1: Solving Poisson's equation with Dirichlet boundary condition:

Consider the following Poisson's equation [12]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6xy(1-y) - 2x^3$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, where the Dirichlet boundary conditions are

$$u(0,y) = 0, \qquad u(1,y) = y(1-y),$$

$$u(x,0) = u(x,1) = 0.$$

This problem can be solved analytically, and the exact solution is

$$u(x,y) = y(1-y)x^3.$$

We follow Kansa collocation method with Inverse Multiquadric and Gaussian RBF. 100 uniform and Halton collocation points have been used with in the domain and on the boundary. For Gaussian and IMQ RFB, $c = 3$ was used. The results have been compared (See Figures 4.6-4.7). As it can be seen uniform collocation form gives us a better error in comparison with the Halton collocation points. Also, Gaussian RBF was a better choice in comparison with IMQ.

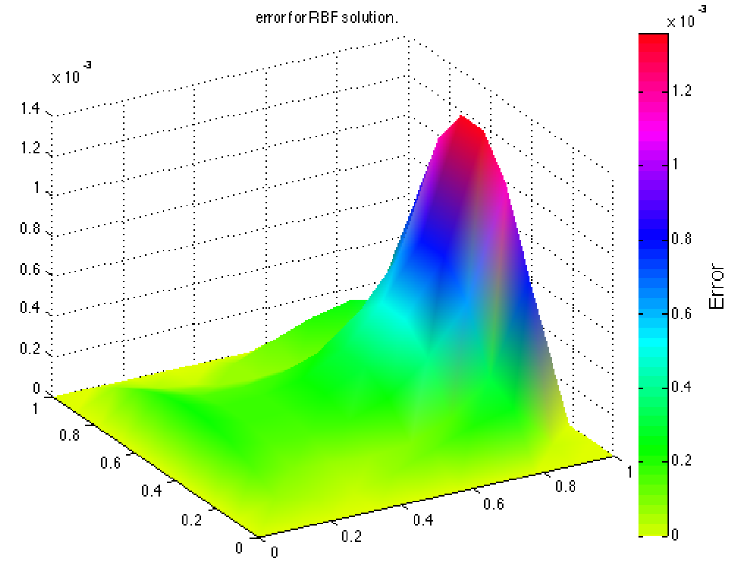

Figure 4.6: Error for example of Poisson's equation with 100 uniform collocation points by Gaussian RBF

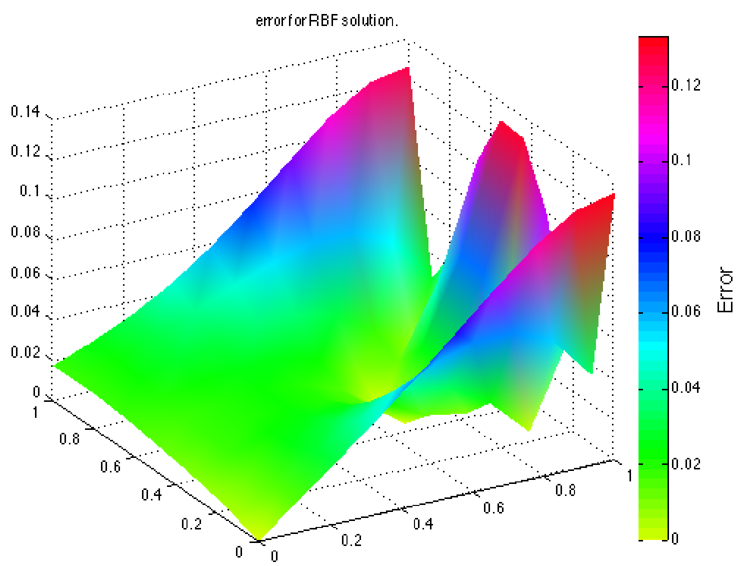

Figure 4.7: Error for example of Poisson's equation with 100 Halton collocation points by Gaussian RBF

Example 2: Poisson's equation with Neumann boundary conditions:

Consider the following Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2(2y^3 - 3y^2 + 1) + 6(1 - x^2)(2y - 1),$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the boundary conditions

$$u(0, y) = 2y^3 - 3y^2 + 1,$$

$$u(1, y) = 0,$$

$$\frac{\partial u(x, 0)}{\partial y} = \frac{\partial u(x, 1)}{\partial y} = 0.$$

The analytical solution of this equation is

$$u(x, y) = (1 - x^2)(2y^3 - 3y^2 + 1).$$

We use Kansa collocation method with 100 Halton collocation points. Here IMQ gives us the best error. We choose $c = 3$ and $c = 10$ in this example. As it shows, $c = 3$ gives slightly better errors (Figures 4.8 and 4.9).

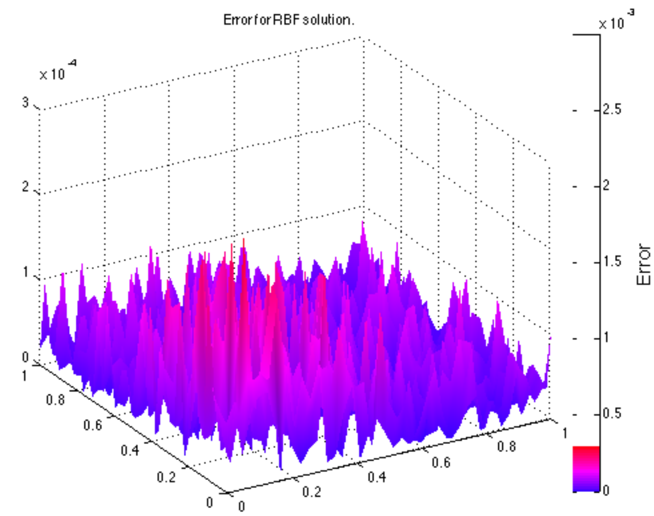

Figure 4.8: Error of an example of Poisson's equation with Neumann boundary condition with IMQ RBF ($c = 3$) (using 100 Halton collocation points)

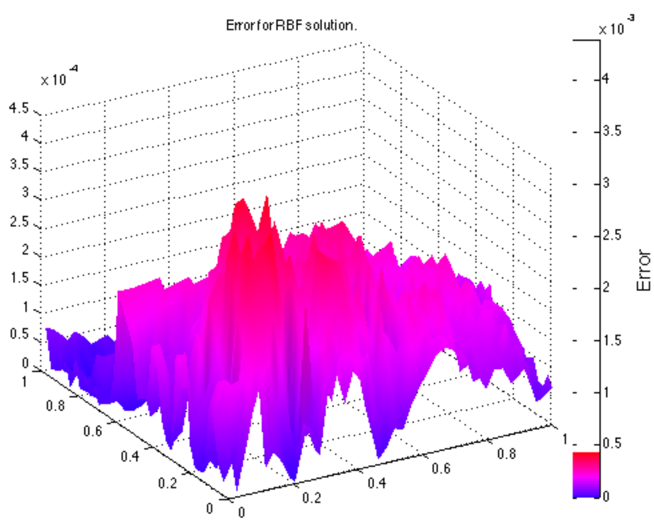

Figure 4.9: Error of an example of Poisson's equation with Neumann boundary condition with IMQ RBF ($c = 10$) (using 100 Halton collocation points)

Example 3: Elliptic equation with variable coefficients:

Consider the following elliptic equation with variable coefficients and homogeneous Dirichlet boundary conditions

$$-\nabla(\alpha(x,y)\nabla u(x,y)) = F(x,y)$$

on the region $\Omega = [0,1] \times [0,1]$, where

$$\alpha(x,y) = 1 + x + 2y^2,$$

$$F(x,y) = x(1-x)(2+2x-4y+12y^2) + y(1-y)(1+4x+4y^2),$$

and boundary condition of

$$u(x,y) = 0, \quad on \;\; \partial\Omega.$$

The analytical solution is [50]

$$u(x,y) = xy(1-x)(1-y).$$

The above example was solved by Kansa collocation method. We choose IMQ RBF and $c = 3$. 100 uniform collocation points were used (Figure 4.10).

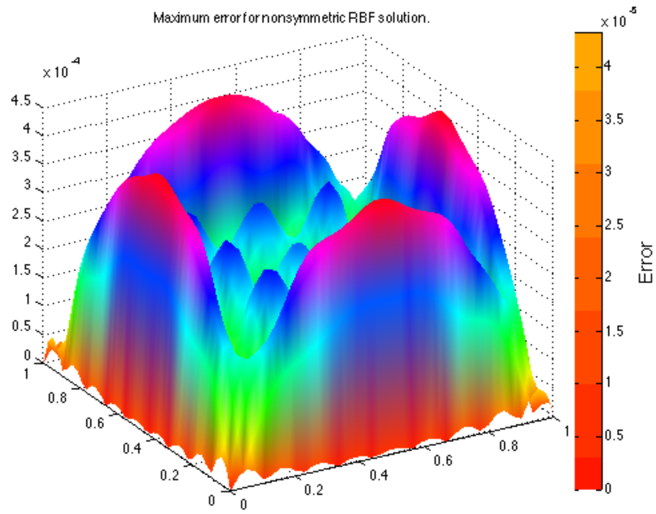

Figure 4.10: Error of an example of Elliptic equation with variable coefficient with IMQ RBF ($c = 3$) (using 100 uniform collocation points)

Example 4: Solving a PDE with mixed boundary condition:

Consider

$$\nabla^2 u + u = (2+3x)e^{x-y}, \quad (x,y) \in [0,1] \times [0,1]$$

with the following mixed boundary conditions

$$u(0,y) = 0, \quad u(x,0) = xe^x,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 2e^{1-y}, \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = -xe^{x-1}.$$

The exact solution of this problem is [14]

$$u(x,y) = xe^{x-y}.$$

The problem was solved by using 100 Uniform collocation points by IMQ RBF ($c = 3$). The result is followed (Figure 4.11).

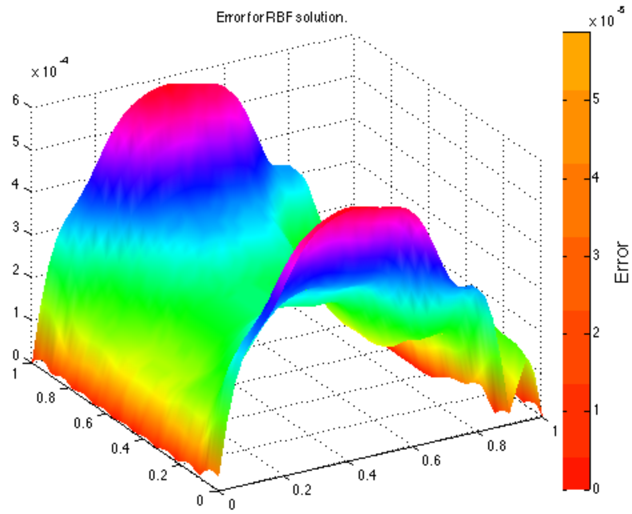

Figure 4.11: Error of an example of PDE with mixed boundary conditions by IMQ RBF ($c = 3$) (using 100 uniform collocation points)

4.4. Conclusion

Least square Method for Boundary Value Problems using B-splines discussed in chapter 2, is an improved form of using B-spline basis functions [3]. Numerical Analysis contains little literature on higher order BVPs. In the papers by Loghmani all the examples are higher order ODEs, but he suggested that the method works for any linear and non-linear PDEs and system of elliptic PDEs. We used the method for Poisson's equation and the error was acceptable. As it can be seen in the examples, the accuracy of the method is efficient even with large partitioning of the domain ($k = 0$ and $k = 1$). The Matlab program can take longer time to run for solving PDEs. As we discussed earlier, meshfree methods introduced by Kansa attract attention. In most popular finite element method, creating mesh was one of the disadvantages. Creating mesh can be time consuming and costly especially for higher dimensional and irregular shaped domains. Kansa Method is rather simple to use. He suggests changing the shape parameter c to improve accuracy. The problem for this method is that for a constant shape parameter c, the matrix A may become singular for certain sets of centers $\mathbf{x}_i$. However, there is an approach that gives us strategies to select a set of centers from possible points that ensure the non-singularity of the collocation matrix [Ling et al. (2006)]. Kansa's method has been extended by several researchers to solve non-linear PDEs, systems of elliptic PDEs and time-dependent parabolic and hyperbolic PDEs. Different methods have been suggested to improve stability and it is an active research area.

REFERENCES

[1] Eason, Ernest D. A review of least square methods for solving partial differential equations. International Journal for Numerical Methods In Engineering, Vol. 10, 1021-1046 (1976)

[2] www.cs.mtu.edu/ shene/COURSES/cs3621/NOTES/spline/B-spline/bspline-ex-1.html

[3] Loghmani, M. AHMADINIA, Numerical solution of third-order boundary value problems. Iranian Journal of Science & Technology, Transaction A, Vol. 30, No. A3 (2006) 291−295

[4] Loghmani G.B, Application of least square method to arbitrary-order problems with separated boundary conditions. Journal of Computational and Applied Mathematics 222 (2008) 500−510

[5] Yahya Qaid, Hasan.The numerical solution of third-order boundary value problems by the modified decomposition method. Advances in Intelligent Transportation Systems (AITS), Vol. 1, No. 3, 2012.

[6] Viswanadham, Kasi and Showri Raju. B-spline Collocation Method For Sixth Order Boundary Value Problems. Global Journal of researches in engineering Numerical, Volume 12 Issue 1 Version 1.0 March 2012 Type: Double Blind Peer Reviewed International Research Journal Publisher: Global Journals Inc. (USA) ISSN: 0975-5861

[7] G.B. Loghmani a,*, M. Ahmadinia. Numerical solution of sixth order boundary value problems. Applied Mathematics and Computation 186 (2007) 992999

[8] Tai-Ran Hsu. Application of Second Order Differential Equations in Mechanical Engineering Analysis.
http://www.engr.sjsu.edu/trhsu/Chapter%204%20Second%20order%20DEs.pdf

[9] Diffusion of point source and biological dispersal.
www.resnet.wm.edu/ jxshix/math490/lecture-chap3.pdf

[10] Fasshauer, Gregory (2007). Meshfree Approximation Methods With Matlab. World science publishing company.

[11] Advective Diffusion Equation. https://ceprofs.civil.tamu.edu/ssocolofsky/cven489/downloads/book/ch2.pdf

[12] An example solution of Poisson's equation in 2-d
$http://farside.ph.utexas.edu/teaching/329/lectures/node71.html$

[13] Meshfree methods. $www.math.iit.edu/\ fass/590/notes/Notes590_Ch1Part3Print.pdf$

[14] Liu and Y.T. Gu.(2005). An introduction to meshfree methods and their programming. Netherland. Springer.

CURRICULUM VITAE

Katayoun (Kathy) Kazemi

1901 Soaring ct, Las Vegas, NV 89134

Tel:(702) 218-4078 E-mail:kazemik3@unlv.nevada.edu

Education

- M.S. in Engineering Management, George Washington University, Washington DC, 2014
- B.S. in Mechanical Engineering, Tehran Polytechnic University, Tehran, 2000. Thesis: Comparing efficiency of cyclones in series versus parallel connection.

Employment

- Graduate Assistant and Mathematics Instructor
  University of Nevada, Las Vegas, NV (1/2012-present)
- Mathematics Instructor Nevada State College, Las Vegas, NV (8/2004-1/2012)
- Mathematics Instructor University of phoenix, Las Vegas, NV (2004-2011)
- Mechanical Engineer Power Plant Projects Management Co. (MAPNA), Tehran, Iran (4/2000-1/2002)