5-1-2015

Exact Statistical Inferences for Functions of Parameters of the Log-Gamma Distribution

Joseph F. Mcdonald
University of Nevada, Las Vegas, joe@joemath.com

EXACT STATISTICAL INFERENCES FOR FUNCTIONS OF PARAMETERS OF THE LOG-GAMMA DISTRIBUTION

by

Joseph F. McDonald

Bachelor of Science in Secondary Education
University of Nevada, Las Vegas
1991

Master of Science in Mathematics
University of Nevada, Las Vegas
1993

A dissertation submitted in partial fulfillment
of the requirements for the

Doctor of Philosophy - Mathematical Sciences

Department of Mathematical Sciences
College of Sciences
The Graduate College

University of Nevada, Las Vegas
May 2015

UNLV | GRADUATE COLLEGE

We recommend the dissertation prepared under our supervision by

Joseph F. McDonald

entitled

Exact Statistical Inferences for Functions of Parameters of the Log-Gamma Distribution

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy - Mathematical Sciences

Department of Mathematical Sciences

Malwane Ananda, Ph.D., Committee Chair

Amie Amei, Ph.D., Committee Member

Hokwon Cho, Ph.D., Committee Member

Daniel Allen, Ph.D., Graduate College Representative

Kathryn Hausbeck Korgan, Ph.D., Interim Dean of the Graduate College

May 2015

ABSTRACT

Exact Statistical Inferences for Functions of Parameters of the Log-Gamma Distribution

by

Joseph McDonald

Malwane Ananda, Examination Committee Chair
Professor of Mathematical Sciences
University of Nevada, Las Vegas

The log-gamma model has been used extensively for flood frequency analysis and is an important distribution in reliability, medical and other areas of lifetime testing. Conventional methods fails to provide exact solutions for the log-gamma model while asymptotic methods provide approximate solutions that often have poor performance for typical sample sizes. The two parameter log-gamma distribution is examined using the generalized p-value approach. The methods are exact in the sense that the tests and the confidence intervals are based on exact probability statements rather than on asymptotic approximations. Exact tests and exact confidence intervals for the parameter of interest based on a generalized test statistic will be used to compute generalized p-values which can be viewed as extensions to classical p-values. The generalized approach is compared to the classical approach using simulations and published studies. The Type I error and confidence intervals of these exact tests are often better than the performance of more complicated approximate tests obtained by standard methods reported in literature. Statistical inference for the mean, variance and coefficient of variance of the log-gamma distribution are given, and the performances of these procedures over the methods reported in the literature are compared using Monte Carlo simulations.

ACKNOWLEDGEMENTS

I would like to thank my advisor for believing in me and never letting me give up through one of the most difficult periods in my life. I would especially like to thank my wife Kathy and three children, Megan, William and Chloe, for their love and patience. Last but not least, I thank all of my professors whom put their time and their heart into their craft. Without all of you, this achievement would not be possible. Thank all of you from the bottom of my heart.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Introduction

The Log-Gamma distribution, sometimes called the Log-Pearson type 3 distribution, is extensively used in hydrology. It is recommended specifically for flood-frequency analysis by the Water Resources Council. The Log-Gamma distribution and the Negative Log-Gamma distribution are used in life-testing and reliability analysis. Suppose we are interested in predicting the magnitude of the most severe flood in the next 10,000 years. Or perhaps we are concerned if a 10,000-year flood will occur in the next 50 years. Flood-frequency analysis was an empirical process before 1914 using graphical methods with records under 20 years and incomplete records. Warren E. Fuller(1914) [?] related average flood magnitude to recurrence interval which is also called T-year flood intervals. H. Alden Foster (1924) [?] proposed using the Pearson type III, often called the gamma distribution, to analyze floods using a simple function of the mean, standard deviation and skew. Distributions with extreme values that can be used to access risk were established by Leonard Tippett (1902-1985). With the help of R.A. Fisher, Tippet found three asymptotic limits for extreme order statistics each named after their authors; the Gumbel distribution, the Frechet distribution, and the Weibull distributions. Allen Hazen took the logarithms of the flood data in 1924 and introduced using a regional skew coefficient in 1930. National flood insurance programs were developed in the 1960's resulting in a need for uniform flood-frequency analysis techniques. The Log-Pearson III distribution with regional skew information using the method moments of the logarithms of the observed data for estimated parameters was adopted by the Water Resource Council (W.R.C.) in 1967 as the recommended method for flood frequency analysis for the Unite States. This is still the official method for predicting T-year flood intervals as of the writing of this paper. (Bulletin 15 [?]) Manuel A. Benson (1968), chairman of the Work Group on Flow-Frequency Methods Hydrology Committee for the W.R.C. investigated six different methods for flood frequency predictions. Two-parameter Gamma, Gumbel, Log Gumbel, Log Normal, Hazen and the Log Pearson Type III (LP3) were fitted by the programs of more than one agency for the six methods resulting in 14 separate computations. The Work Group recommended the LP3 and was ultimately adopted by the W.R.C. in 1967. Computational ease of finding the method of moments for parameter estimation was one of the major advantages of the LP3. Bobee (1975,1986,1987 and 1989) explored different methods for finding the first three moments including the generalized method of moments and mixed method of moments. [?]

Bernard Bobee (1975) [?] purposed that the method of moments be applied to the original data instead of their logarithms yielding similar results. Condie (1977) [?] proposed using the maximum likelihood method based on Canadian flood data sets concluding his method was superior to the method of moments. Bobee and others have also used a mixed method of moments methods using both original and logarithms of the data. Based on the standard error of the T-year flood, Nozdryn-Poltinicki and Watt (1979) [?] found in their Monte Carlo simulation study of the above methods that the MLE and the MOM were almost comparable. In general, an unusually high bias in all of the parameter estimates were found when testing 1000 random samples of size of 10, 20, 30, 50 and 75. The standardized bias (BIAS), standard error (SE) and the mean root square error (RMSE) were computed. They suggested the use the method of moments recommended by the W.R.C. because of the computational ease.

The Log-Gamma distribution (LG) and the Log-Pearson III (LP3) do not enjoy a clearly well-defined naming convention throughout statistical journals and literature. There is no agreement in research on the names of these distribution within modern articles and literature. Proper attention is needed to identify which distribution is being used when these distributions are referenced. Both distributions are derived from the gamma distribution but they are parameterized differently resulting in different shapes, domains and models. The Log-Gamma distribution will be defined in this paper using the form most often used in hydrology. In this paper the Log-Pearson Type III (LP3) distribution will refer to the following 3 parameter probability density distribution:

$$f_x(x; a, b, c) = \frac{1}{x\,|b|\,\Gamma(a)} \left[\frac{\log x - c}{b}\right]^{a-1} \exp\left[-\frac{\log x - c}{b}\right]. \tag{1.1}$$

where the parameter space is: $a > 0,\ b \neq 0,\ -\infty < c < \infty$

and the domain is:

$$0 < x \leq e^c \qquad b < 0$$

$$e^c \leq x < \infty \qquad b > 0.$$

a, b, and c are the *logscale*, shape and location parameters, respectively, and $\log x$ is the natural logarithm, $\ln x$. The *logscale* is not a true scale parameter but it is a scale parameter for the gamma distribution which can be a useful property. The two parameter distribution is the parametrization that is used most often in this paper. If the location parameter is zero, $c = 0$, we will called this distribution the Log-Gamma (LG) distribution where a and b are the logscale and shape parameters, respectively. The c parameter is a location parameter and is sometimes called a threshold parameter. Furthermore, we will restrict the logscale parameter, b, to positive values only. For the purpose of this paper, consider the following two-parameter Log-Gamma distributions (LG):

$$
\begin{aligned}
f(x;a,b) =& \frac{1}{x\,|b|\,\Gamma(a)}\left[\frac{\log x}{b}\right]^{a-1}\exp\left[-\frac{\log x}{b}\right] \\
=& \frac{(\log x)^{a-1}}{b^a\Gamma(a)}x^{-1/b-1}, \;\; a>0,\; b>0, \;\text{ and }\; x>1.
\end{aligned} \tag{1.2}
$$

The following parameterized is often used where $\beta = \frac{1}{b}$:

$$
f(t;\alpha,\beta) = \frac{\beta^{\alpha}(\log t)^{\alpha-1}}{\Gamma(\alpha)}t^{-\beta-1}, \;\; \alpha>0,\; \beta>0, \;\text{ and }\; t>1. \tag{1.3}
$$

These forms are versatile in the sense they allow us to rewrite the distributions in several forms that allow both computational and mathematical advantages.

Most of the early work on the Log-Gamma distribution was in the form of the Extreme Value, EV, distribution. Prentice and Lawless examined this EV distribution which is an extension of the generalized gamma distribution. The gamma distribution is not used as often as the log-normal, log-logistic and the Weibull for the modeling of lifetime data. The log-normal and the log-logistic are derived from the normal and logistic distributions respectively. We would say that T is log-normally distributed if $Y = \log T$ is normally distributed. The extreme value distribution comes from the Weibull distribution in a similar fashion. If T has a Weibull distribution, then $\log T$ has an extreme value distribution which is also referred to as the Gumbel distribution. The gamma distribution does fit some lifetime data as well as models for insurance, rainfalls, gene expressions and many other uses. One of the extensions from the gamma distribution is the log-gamma distribution or generalized log-gamma distribution which includes the Weibull and the log-normal as special cases. According to Lawless [?], the log-gamma model was originally introduced by specifying that $(T/\alpha)^{\beta}$ has a one-parameter gamma distribution with index parameter $k > 0$. Equivalently, $W = (Y - u_1)/b_1$, where $Y = \log T, u_1 = \log\alpha$ and $b_1 = \beta^{-1}$, has a log-gamma distribution. The motivation for Lawless and Prentice to transform the log-gamma variate $Z = k^{1/2}(W - \log K)$ is the mean and the variance for the gamma distribution both equal k, and as k increases, such that the gamma and log-gamma distributions do not have limits. The mean and variance for W are $E(W) = \psi(k)$, the digamma function and $Var(W) = \psi'(k)$, the trigamma function. For large k, the digamma function and the trigamma function behave like $\log k$ and $1/k$, respectively.

Using definitions from Lawless,(1982) [?] define the one parameter gamma distribution pdf $Y \sim G(k)$ is

$$g(y) = \frac{y^{k-1}e^{-y}}{\Gamma(k)} \text{ where } y > 0, k > 0.$$

The generalized log-gamma model is then the three-parameter family of distributions for which $Z = (Y - u)/b$ has p.d.f.

$$f(z;k) = \frac{k^{k-1/2}}{\Gamma(k)} \exp\left(k^{1/2}z - ke^{zk-1/2}\right) \quad \text{where } z = \frac{y-u}{b}, \ -\infty < z < \infty.$$

u is a location parameter and b is a scale parameter. It is useful to note that as $k \to \infty$, this distribution converges to to the standard normal pdf. We can also use this distribution as a two-parameter family by setting $u = 0$.

Johshson, Kotz and Balakrishnan (1995) [?] calls this pdf

$$f(w) = \frac{1}{\Gamma(k)} \exp(kw - e^{w}) \quad -\infty < w < \infty, \;\; k > 0 \tag{1.4}$$

as the Log-Gamma distribution. Prentice (1974) [?] re-parameterized the generalized gamma density (Stacey, 1962) extending the distribution of the logarithm of a generalized gamma variate. This form is also included in the family of extreme value distributions. This new distribution is clearly a separate distribution than the distribution used in this paper.

Consul and Jain (1974)[?] and other authors use a form of the distribution more closely related to the traditional Log Pearson III or Log-Gamma distribution. This a transformation of the gamma distribution which is useful especially when the values of the variable are very small or very large.

Mathematical Properties of the LP3 Distribution

As noted before, naming conventions for the log-Pearson type III distribution, LP3, and the Log-Gamma distribution, LG, were not consistent. The LP3 and the LG both can be written with 1, 2 or 3 parameters with one location and two shape parameters. Some text will refer to one the of the shape parameters as a logscale parameter because it can be derived for the scale parameter of the Gamma distribution. The random variable x has a Log-Pearson distribution if $y = \log_a x$ has Pearson distribution. The LP3 distribution was often written with base a as $\log_a x$ in early works. The name for the model for this paper will be the two parameter Log Pearson type III (LP3) or the log-gamma distribution (LG).

Hydraulic engineer Bernard Bobee authored several hundred scientific publications in statistical hydrology and played a key role in the establishment of the Review Water Science in France. The popularity of the Log-Gamma distribution was increased resulting in many authors examining the best estimators for the parameters and functions of parameters of the distribution. Lack of computing power and expense of computer time were obstacles that influenced the early choices for best methods of parameter estimation. Bobee [?] (1975) and Bobee and Ashkar [?].

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{1}{x\beta^{\alpha}\Gamma(\alpha)} \left(\log(x)\right)^{\alpha-1} \exp\left(-\log(x)/\beta\right) \\ &= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{-1/\beta-1} \left(\log(x)\right)^{\alpha-1} \sim LG(\alpha,\beta) \end{aligned} \quad (1.5)$$

where $x > 1$, $\alpha > 0$, $\beta > 0$. α and β are both continuous shape parameters.The $\log(x)$ is the natural $\ln(x)$.

As above in equation **??** and equation **??**, sometimes it is more convenient computationally to let $b = 1/\beta$.

$$f(x;a,b) = \frac{b^a}{\Gamma(a)}\, x^{-b-1}\, (\log(x))^{a-1} \sim LG(a,b)$$

Negative Log-Gamma Distribution

The Log-Gamma distribution is an important model used in the analysis more recently in reliability analysis using a Bayesian like approach. Allella (2001) [**?**]

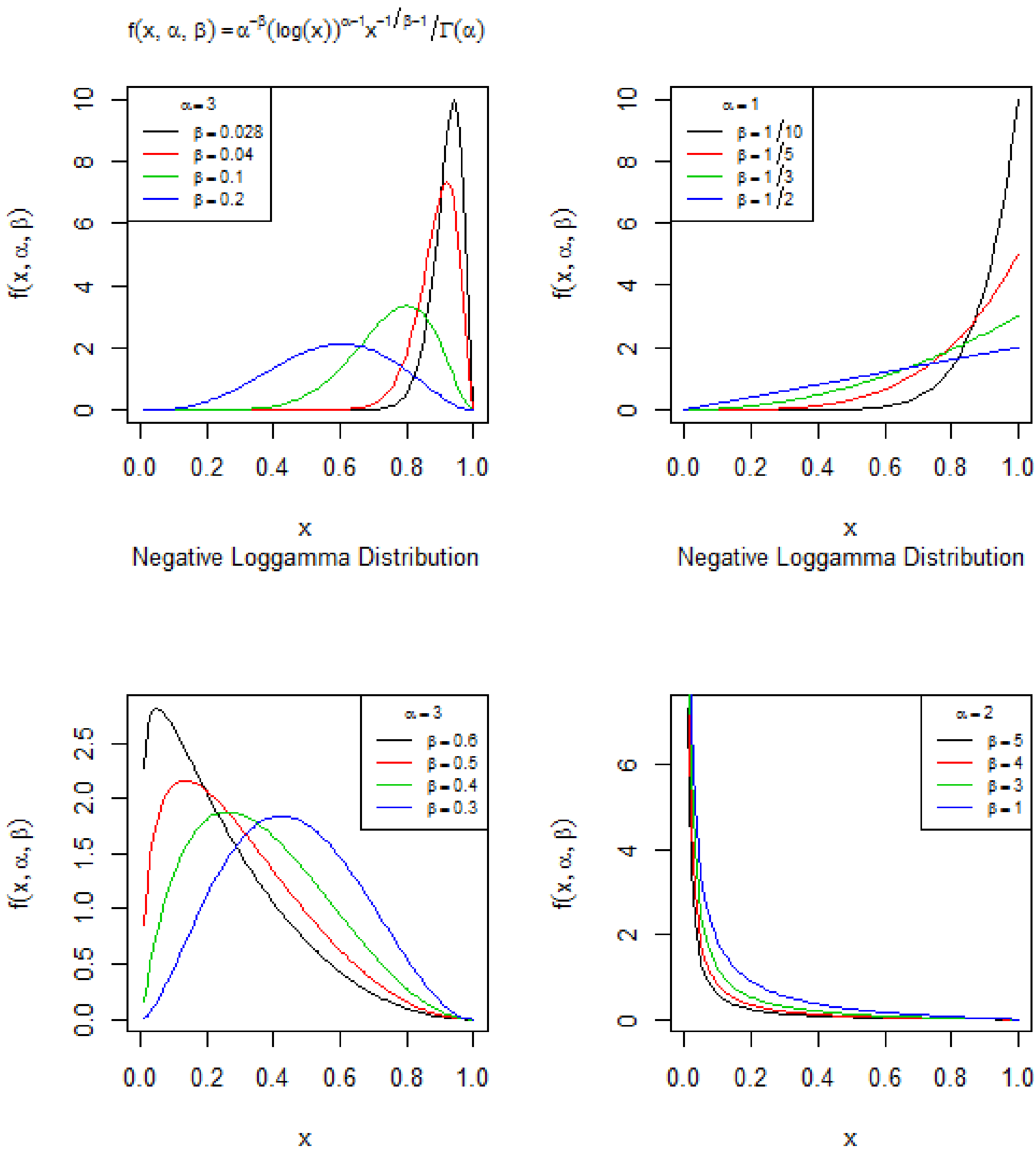
$$f(t) = \frac{b^a}{\Gamma(a)} t^{-b-1}\, (\log(t))^{a-1} \ \text{ for } 1 < t < \infty\,, \quad a > 1,\ b > 0.$$

Let $T = \log R$ where log is the natural logarithm and $0 < R \leq 1$.

$$f(r;a,b) = \frac{1}{b^a\Gamma(a)}\, r^{b-1}\, [-\log r]^{a-1}\,; \quad a, b > 0,\ \ 0 < r \leq 1. \tag{1.6}$$

Allella [**?**] et.all (2001) classifies this form as the Negative Log-Gamma distribution. It is particularly useful for data uncertainty modeling in reliability analysis because the domain consists of values in the interval $[0, 1]$, as required for a reliability variable value. The Negative Log-Gamma (NLG) can be used as a ”conjugate a priori” pdf for components' reliability in the exponential reliability model. Allella showed the Negative Log-Gamma (NLG) distribution using the parametrization $X = -R$ can approximate the reliability pdf of complex ”series-parallel” systems. Martz and Waller (1982) [**?**] have shown that the NLG can be used *a prior* pdf in a Bayesian treatment in reliability assessment and as a posterior system reliability pdf. The

NLG distribution is particular useful in reliability assessment because the domain is $[0, 1]$ and the product of independent NLG random variables is still a NLG random variable. The Negative Log-Gamma distribution is used in uncertainty modeling for reliability analysis of complex systems of many components. [?] (1992) A Bayesian assessment is computed for a system reliability for a r-out-of-k system consisting of k independent and identical components.



Johnson (1994) [?] documented the distribution of $\log X$ where X has the standard gamma distribution based on an thorough investigation by Olshen (1937) [?]. The standard probability density function for the two-parameter gamma distribution is

$$p_x(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad x \geq 0 \text{ where } \beta = 1. \tag{1.7}$$

The moment-generating function of $\log X$ is

$$E\left[e^{t \log X}\right] = E\left[X^t\right] = \frac{\Gamma(\alpha+t)}{\Gamma(\alpha)} \tag{1.8}$$

and the cumulant generating function is $\log \Gamma(\alpha+t) - \log \Gamma(\alpha)$. Consul and Jain (1971) [?] used the model when $Y = -\log X$ has a gamma(α, β) distribution. The pdf is

$$p_Y(y; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \times \frac{(-\log y)^{\alpha-1}}{y^{1+1/\beta}}, \quad 0 < y < 1$$

which is the same as the model I will call negative log-gamma distribution (NLG). The rth moment (about zero) is

$$E\left[Y^r\right] = E\left[e^{tX}\right] = \left(\frac{\beta}{\beta+r}\right)^\alpha = \left(1+\frac{r}{\beta}\right)^{-\alpha} \tag{1.9}$$

Consul and Jain (1971) [?] examined properties of this distribution and also obtained the distributions of the product and the quotient of two independent log-gamma variants.

Historical Remarks

Balakrishnan and Chan (1994) [?] have studied order statistics from the log-gamma distribution and determined their means, variances and covariances. The best linear unbiased estimators (BLUEs) of the location and the scale parameters were determined based on complete samples as well as Type-II censored samples using the mean, variance and the covariances. Lawless [?] applied both type I and Type II censored samples for the log-gama distribution as well as exact methods for uncensured samples using pivotal quantities. The distribution of the pivotal quantities are analytically intractable but can be calculated by simulation to a desired accuracy. These pivotal quantities are available when the data are Type 2 censured.

$$Z_1 = \frac{\widehat{u} - u}{\widehat{b}}, \; Z_2 = \frac{\widehat{b}}{b}, \text{ and } Z_P = \frac{\widehat{u} - y_p}{\widehat{b}}$$

where $u = \log \alpha$ is the location parameter, $b = \beta^{-1}$ is a scale parameter, $Y = \log T$ and $T \sim \text{Gamma}(\alpha, \beta)$. y_p is is the pth value of the ordered sample y.

The Log Gamma Model

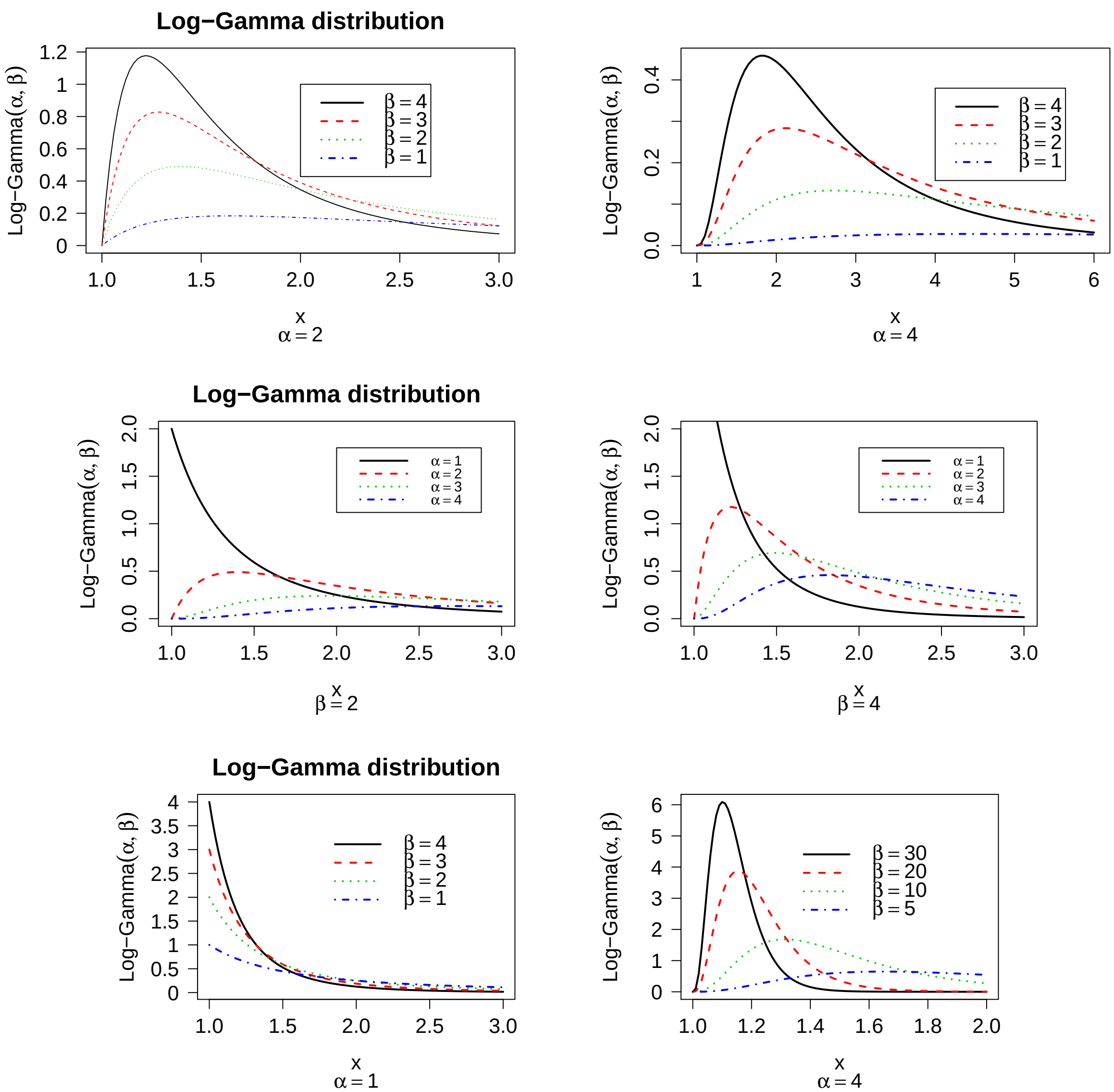
Define the model for this paper is the two parameter Log-Pearson type III or log-gamma distribution.

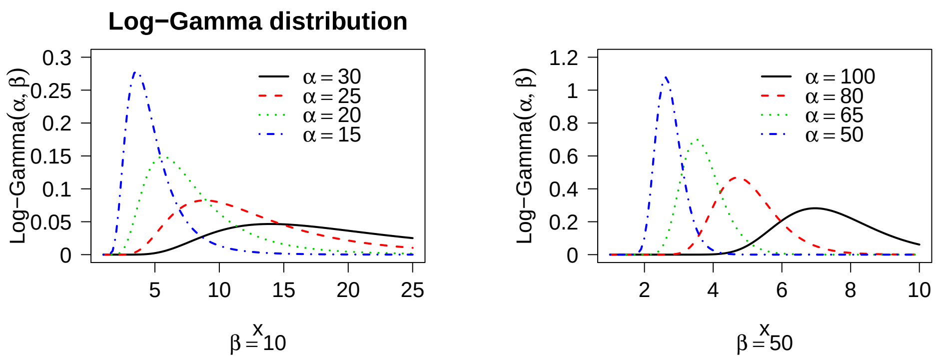
$$\begin{aligned} f(x; \alpha, \beta^*) &= \frac{1}{x\beta^*\alpha\Gamma(\alpha)} \left(\log(x)\right)^{\alpha-1} \exp\left(-\log(x)/\beta^*\right) \\ &= \frac{1}{\Gamma(\alpha)\beta^*\alpha} x^{-1/\beta-1} \left(\log(x)\right)^{\alpha-1} \sim LG(\alpha, \beta^*) \end{aligned} \tag{1.10}$$

It is sometimes written with the logscale parameter as $\beta = 1/\beta^*$

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{\beta^\alpha}{x\Gamma(\alpha)} \left(\log(x)\right)^{\alpha-1} \exp\left(-\beta \log(x)\right) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\beta-1} \left(\log(x)\right)^{\alpha-1} \sim LG(\alpha,\beta) \end{aligned} \tag{1.11}$$

where $x \geq 1$, $\alpha > 0$, $\beta > 0$. α and β are both continuous shape parameters. The $\log(x)$ is the natural logarithm, $\ln(x)$.





Sometimes it is more convenient to let $b = 1/\beta$.

$$f(x; a, b) = \frac{b^a}{\Gamma(a)}\, x^{-b-1} \left(\log(x)\right)^{a-1} \sim LG(a, b)$$

The MLE for Log-Gamma Distribution

We will look at the moments and the MLE for the log-gamma distribution. R. Condie (1977) [**?**] examined the maximum likelihood estimators for the three parameters of a log Pearson Type 3 distribution derived from the logarithmic likelihood function. Condie concluded that the maximum likelihood analysis was superior in terms of the estimate of standard error to the method of moments that is usual technique for flood data. We will use this form as it is written in Condies's paper with renaming the parameters,$a > 0$, $b \neq 0$, and $c > 0$ which are the scale, shape and location parameters respectfully.

$$
\begin{aligned}
f_X(x|a,b,c) &= \frac{1}{x\,|b|\,\Gamma(a)}\left(\frac{\ln x - c}{b}\right)^{a-1}\exp\left(-\frac{\ln x - c}{b}\right); && (1.12)\\
\text{If } b &< 0 \text{ then } 0 < x \leq e^c \;\text{ and if } b > 0,\;\; e^c \leq x < \infty.\\
f_X(x|a,b,c) &= \frac{(\ln x - c)^{a-1}}{b^a\Gamma(b)}x^{-1/b-1}e^{c/b} && (1.13)\\
\text{If } c &= 0, \text{ then } f_X(x|a,b) = \frac{(\ln x)^{a-1}}{b^a\Gamma(a)}x^{-1/b-1} && (1.14)
\end{aligned}
$$

Using the likelihood function and the log likelihood function:

$$
L(\underline{X};a,b,c) = \prod_{i=1}^{n}\left[(\ln X_i - c)/a\right]^{b-1}\exp\left[-(\ln X_i - c)/a\right]/\left[|a|\,\Gamma(b)X_i\right] \quad (1.15)
$$

$$
\begin{aligned}
\ln L(\underline{X};a,b,c) = \ln L(\underline{X};a,b,c) = (b-1)\sum \ln\left[(\ln X_i - c)/a\right] - \\
\frac{1}{a}\sum(\ln X_i - c) \quad - \sum \ln X_i - n\ln|a| - n\ln\Gamma(b) \quad (1.16)
\end{aligned}
$$

This gives us three equations to solve:

$$
\begin{aligned}
\frac{\partial \ln L}{\partial a} &= -n\psi(a) + \sum \ln\left[(\ln X_i - c)/b\right] = 0 && (1.17)\\
\frac{\partial \ln L}{\partial b} &= \frac{1}{b^2}\sum(\ln X_i - c) - \frac{na}{b} = 0 && (1.18)\\
\frac{\partial \ln L}{\partial c} &= -\frac{n}{a} - (b-1)\sum\left[\ln X_i - c\right]^{-1} = 0 && (1.19)
\end{aligned}
$$

Alternate Forms

The Log-Gamma distribution can also be written in terms of the Lower Incomplete Gamma Function. [?] The probability, $P(X < u)$ can be written in terms of the incomplete gamma function where X is form the log-gamma distribution with $shapelog = \alpha$ and the $ratelog = \beta$.

$$\begin{aligned} F(u) = P\left(X < u\right) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{1}^{u} \left(\log(t)\right)^{\alpha-1} t^{-\beta-1}\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_{0}^{\beta \log u} z^{\alpha-1} e^{-z}\, dz \\ &= \frac{\Gamma\left(\alpha; \beta \cdot \log u\right)}{\Gamma(\alpha)} \end{aligned} \tag{1.20}$$

where $\Gamma(\alpha; \beta \cdot \log u)$ is the Lower Complete Gamma Function,

$$\Gamma\left(\alpha; x\right) = \int_{0}^{x} z^{\alpha-1} e^{-z}\, dz.$$

The Log-Gamma distribution is a special case of the generalized extreme value distribution. It is often used in insurance and finance extreme events and is maximin stable, an useful and rare property in this class.

CHAPTER 2

THE METHODOLOGY

Exact Statics

Exact statistical methods is a branch of statistics that was developed to obtain more accurate results for hypothesis testing, confidence intervals, and point estimation. This is accomplished by eliminating procedures based on asymptotic and approximate statistical methods which often require large sample sizes or inconvenient assumptions to yield accurate results. Conventional methods often yield poor results for simple problems when nuisance parameters are introduced or when dealing with small sample sizes. Weerahandi (1995) [?] defined exact statistical methods as being exact in the sense of intervals and tests that are based on exact probability statements instead of being based on asymptotic approximations. According to Weerahandi (1995) [?] "With respect to a specific probability measure, a sample space, and the parameter of interest fixed at the value specified by the null hypothesis, p-value is the exact probability of a well-defined extreme region." The extreme region is a well-defined subset of the sample space with the observed value on its boundary. Inferences are valid for any sample size since statistical tests and confidence intervals are based on exact probability statements. Furthermore, generalized p-values are constructed such that the test variables produce unbiased significance tests. Exactness and unbiasedness are necessary for Fisher's treatment of significance testing. Exact methods do not make distributional assumptions such having equal variances in ANOVA and regression. There are computer programs available such as Stata and XPro that can compute exact methods.

Exact Statistics can have different methodologies such as generalized pivotal quantities for finding exact p-values for ANOVA problems and regression under unequal variances. Tsiu and Weerahandi (1989) [?] and Weerahandi [?] extended Generalized P-values and Generalized Confidence Intervals respectively. Weerahandi (2012) [?] discusses some of the problems of the classical treatment of point estimation in simple problems where Least Squares Estimates(LSE) or Maximum Likelihood Estimation (MLE) give negative results for positive parameters. The MLE based methods are the only systematic method available to tackle any parameter such as a function of variance components. The classical approach to point estimation does not provide a systematic method to incorporate the knowledge that one may have about the parameter space without resorting to Bayesian methods or to ad hoc methods without sound theory supporting it. Weerahandi used these generalized methods by estimating fixed effects of variance components of Linear Mixed Models and predictors of random effects of mixed models. Hanning, Iyer and Patterson (2006) [?] showed these exact methods based on exact probability statements are often asymptotically exact in the classical sense. Gamage et al (2004) [?] applied these extended definitions called Generalized Estimation (GE) to the famous Multi-Variate Behrens-

Fisher problem. Lee and Lin (2004) [?] tackled intervals for the ratio of two normal means. Roy and Mathew (2005) [?] constructed a generalized confidence limit for the reliability function of a two-parameter Exponential distribution. Krishnamoorthy et al (2006) [?], used generalized P-values and Generalized Confidence Intervals (CGI) to model data that has a lognormal distribution. The calculations are easy to compute and the results are applicable to small sample sizes when comparing tests for the ratio or difference of two lognormal means. Chen and Zhou (2006) [?] used Generalized Confidence Intervals for the ratio of two means and the difference of two means for lognormal populations with zeros values, and by Bebu et al (2009) with confidence intervals for limited moments and truncated moments for normal and lognormal models. Other authors are currently applying and extending these methods to other distributions.

Many classical tests for point estimation rely on assumptions such as populations have equal variances or statistical independence. Classical F-tests and t-tests on linear models can fail to detect significant differences in treatment even when the given data gives sufficient evidence otherwise. Weerahandi (2010) [?] Weerahandi(1987) used generalized p-values for comparing parameters of two linear regression models with unequal variances and showed that the generalize p-value is an exact probability of a well-defined unbiased extreme region. Common statistical techniques such as Variance Components and ANOVA under unequal variances do not have classical exact test. Extensions of classical p-values called generalized p-values are defined such that tests are performed based on exact probability statements that are valid for any sample size. Consider a normal population with mean μ and variance σ^2 where $\bar{X}$ and S^2 are the sample mean and the sample variance. We know that:

$$Z = \sqrt{n}(\bar{X} - \mu)/\sigma \sim N(0, 1) \text{ and } U = nS^2/\sigma^2 \sim \chi^2_{n-1}.$$

A problem arises testing the parameter of interest $\rho = \mu/\sigma$, the coefficient of variation, because of the nuisance parameter.

Using the Generalized p-values approach, Weerahandi (1995) [?] investigated ANOVA with equal error variances by extending the classical F-tests to include the unequal variances. Weerahandi showed the classical F-test fails to reject the null hypothesis even when the data provides strong evidence against it. Krutchkoff (1988) [?] reported that the failure of the assumption of equal variances can have catastrophic results in his extensive study of the power of the F-test. Rice and Gaines (1989) extended the p-value given in Barnard (1984)[?] to the one-way ANOVA case. Although Weerahandi's one-way ANOVA case using the generalized F-test is numerically equivalent to the test of Rice and Gaines (1989), the generalized F-test is computationally more efficient and is closely related to the classical F-test. The p-value is also the exact probability of an unbiased and well defined subset of the sample space. Weerahandi showed in several examples that the classical F-test under the assumption of equal variances failed to reject a false null hypothesis that the means were equal whereas the generalized F-test correctly rejected the false null hypothesis.

We are concentrating on exact methods for two-parameter distributions with both parameters unknown. Generalized p-values are extensions of the classical p-values. Most conventional statistical models do not provide exact solutions except for a limited number of problems. Weerahandi (2014) [?] showed that inferences on the most basic distributions such as the two-parameter continuous Uniform Distribution, UNIF(θ_1, θ_2), do not have exact tests. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Uniform distribution with the density

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \text{ where } \theta_1 < x < \theta_2. \tag{2.1}$$

The sufficient statistics for θ_1 and θ_2 are $S = X_{(1)} = \text{Min}\{X_1, X_2, \ldots, X_n\}$ and $T = X_{(n)} = \text{Max}\{X_1, X_2, \ldots, X_n\}$. S and T are minimal sufficient statistics for the parameters and they are also MLE's for θ_1 and θ_2, respectively, as well. Weerahandi (2012) [?] argued that constructing test statistics or pivotal quantities on the parameters or functions of the parameters based on MLE's will often result in with approximate results with inferior performance. The Generalized Likelihood Ratio (GLR) principle may get exact or asymptotic approximations depending on the function of the parameters of interest. The Generalized Test Statistic and the Generalized Likelihood Ratio test statics are equivalent in this case and exact because the are both free of nuisance parameters. But the Generalized Likelihood Ratio approach will not always be free of the nuisance parameter if we are using the coefficient of variation or the second moment. The inferences in a simple case like the uniform distribution for functions of parameters can be difficult or yield no solution at all. Weeranhandi and Gamage (2014) [?] uses the generalized approach as systematic method for finding regular quantities when they exists and for finding generalized pivotal quantities when regular pivotal quantities fail to exist.

GE: Generalized Estimation

This paper examines a method for making inferences about the parameters or the functions of parameters for two-parameter continuous distributions based on exact probability distribution. There is not one approach that will work to find extended p-values by construction for all distributions. The famous Behrens-Fisher problem is a good example to discuss; the interval estimation and hypothesis testing of the difference of the means of two independent normally distributed samples when the variances of the two populations are not assumed to be equal. Exact fixed-level tests based on complete sufficient statistics do not exist according to Linnik (1968), [?] but approximate solutions based on complete sufficient statistics do exist as well as exact conventional tests based on statistics other than complete sufficient statistics. Scheffe (1943) investigated a class of exact solutions to the problem which were inefficient since his methods did not use all the information in the data about the true value of the parameter. Weerahandi (1995) [?] points out that the confidence intervals were longer than those given by approximate solutions. Weerahandi found exact solutions using complete sufficient statistics to the Behrens-Fisher problem using generalized estimation which was formally introduced by Tsui and Weerahandi (1989) [?].

Generalized P-Values

Generalized p-values are extended p-values obtained by extending test variables called Generalized Test Variables, GTV. Generalized p-values are the same as classical p-values except in the way that the extreme region is defined. These generalized p-values are exact probabilities of well-defined extreme regions of the underlying sample space and do not depend on any nuisance parameters.

Generalized P-Value for one sided hypothesis testing:

$$H_0 : \theta \leq \theta_0 \text{ vs } H_1 : \theta > \theta_0$$

Definition 1. If $C_{\mathbf{x}}$ is a generalized extreme region, then p is its generalized p-value for testing H_0.

$$p = \underset{\theta \leq \theta_0}{Sup} \Pr(\mathbf{X} \in C_{\mathbf{x}}(\zeta)|\theta) \tag{2.2}$$

where $\zeta = \{\theta, \delta\}$ where θ is the parameter of interest and δ is a vector of nuisance parameters.

Define: $T = T(\mathbf{X}; \mathbf{x}, \zeta)$ is the generalized test variable.

Define: $T_{obs} = T(\mathbf{X}; \mathbf{x}, \boldsymbol{\zeta_0})$ is the observed test value.

Definition 2. Generalized Test Variable, GTV: Let $T = T(\mathbf{X}; \mathbf{x}, \zeta)$ be a function of $\mathbf{X}$ and θ only. The random quantity T is a *generalized test variable* if it has the following three properties:

Property 1: The observed value, $t_{obs} = t(\mathbf{x};\mathbf{x},\zeta)$ does not depend on unknown parameters.

Property 2: When θ is specified, T has a probability distribution free of nuisance parameters.

Property 3: For fixed $\mathbf{x}$ and δ, $\Pr(T \leq t;\theta)$ is a monotonic function of θ for any given t.

If the generalized test variable is stochastically increasing or decreasing in the parameter of interest, say θ, the generalized p-values can be computed as

$$p = \Pr(T \geq T_{obs}|\theta = \theta_0) \qquad \text{if } T \text{ is increasing.} \tag{2.3}$$

$$p = \Pr(T \leq T_{obs}|\theta = \theta_0) \qquad \text{if } T \text{ is decreasing.} \tag{2.4}$$

These generalized p-values are the same in the following sense; Given a specified probability measure, a sample space, and the parameter of interest fixed at the value specified by the null hypothesis; the p-value is always the exact probability of an unbiased extreme region with the observed sample on its boundary. It measures how well the data supports or contradicts the null hypothesis. P-values smaller than the significance level suggests that the observed data is inconsistent with the null hypothesis whereas larger p-values fail to reject the null hypothesis. Although we can on occasion construct extreme regions using pivotal quantities that are the same as the ones we constructed using the extended approach, the procedure for constructing the extended p-values for each distribution is usually no simple task with a one size fits all approach. Each case has its own difficulties.

Definition 3. Generalized Pivotal Quantity, GPQ: Let $R = r(\mathbf{X}; \mathbf{x}, \theta)$ be a function of $\mathbf{X}$ and possibly $\mathbf{x}$ and θ. The random quantity R is a *generalized pivotal quantity* if it has the following two properties:

Property 1 The distribution of R is free of unknown parameters.

Property 2 The observed value $R, r_{obs} = r(\mathbf{x}; \mathbf{x}, \theta)$, does not depend on nuisance parameters.

Let $\{X_1, X_2, \ldots, X_n\}$ be a random sample from a continuous distribution with the density function, $f(x; \alpha, \beta)$, having two unknown parameters. Let S and T be minimal sufficient statistics for the parameters α and β. Although not a requirement, it is computationally convenient if the sufficient statistics are transformed such that the transformed variables are independent. The approach eventually leads to two independent sufficient statistics but the construction is easier if we start with independent statistics. Weerahandi and Gamage (2014) [**?**]

WLOG, start with making inferences about the parameter α. Let $F_S(s)$ be the Cumulative Distribution function, CDF, of a random sample S. Since $F_S(s)$ is a CDF, then by definition it has a uniform distribution over the unit interval [0,1]. Define the random variable U as

$$U = U(S; \alpha, \beta) = F_S(S) \sim UNIF(0, 1) \tag{2.5}$$

Define the observed values of our sufficient statistics as (S, T) as (s, t) and the observed value of $U(S)$ as $U(s)$. We need to get rid of the nuisance parameter which is β in this case since we are making inferences about α. Therefore, treat $U(s; \alpha, \beta)$ as a function of β for fixed values of α and s. Let the inverse function, u^{-1}, be the equation such that $u^{-1}(u(\beta)) = \beta$. This quantity $R_\beta(S; \alpha, \beta, s)$ will be used to replace the nuisance parameter β in the construction of the generalized pivotal quantity.

$$R_\beta(S; \alpha, \beta, s) = u^{-1}(U(\beta)) = \widehat{\beta}(U) \tag{2.6}$$

The random variable, R_β, must be constructed to satisfy the properties that the observed value of S of s, $R_\beta(s; \alpha, \beta, s) = \beta$ and the distribution of R_β is free of the nuisance parameter β. Define $F_{T|S=s}(t)$ as the conditional cumulative distribution function of T given $S = s$. Again, this distribution is Uniform over the interval $[0, 1]$ since it is a CDF:

$$V(T; s) = F_{T|S=s}(t) \sim \text{UNIF}(0, 1). \tag{2.7}$$

The unconditional distribution of $V(S,T;\alpha,\beta)$ is also Uniform(0,1) and since $V(T;s)$ does not depend on s, it is distributed independently of S. Construct a potential GPQ for α.

$$
\begin{aligned}
R &= \frac{V(S,T;\alpha,\beta)}{v(s,t;\alpha,R_\beta(S;\alpha,\beta,s))} &\quad (2.8)\\
&= \frac{V(S,T;\alpha,\beta)}{v(t,s;\alpha,\hat{\beta}(U))} &\quad (2.9)
\end{aligned}
$$

where

$$V(S,T;\alpha,\beta)\,\text{and}\,U(S;\alpha,\beta)\sim UNIF(0,1)$$

After constructing the Generalized Pivotal Quantity, we want to verify, R, satisfies the following two properties:

1. R becomes 1 at the observed values s of $= S$ and t of T

2. the distribution of R is free of the nuisance parameter β.

We can now make inferences about our parameter of interest say α because by construction, the Generalized Pivotal Quantity depends only on α. Consider making a 90% generalized confidence interval using a Monte Carlo simulation by generating uniform random numbers for U and V. The generalized confidence interval would be $[\min\{A,B\},\max\{A,B\}]$ where A is the value for α such that the 95th percentile of the distribution is equal to 1 and B is the value for α such that the 5th percentile of the distribution is equal to 1. We could switch the roles of α and β to make inferences about β.

Weerahandi (1995)[?] in his book Exact Statistical Methods for Data Analysis demonstrated how to find generalized inferences using a variety of examples each with their own difficulties. Weerahandi and Tsui(1989) [?], Weerahandi (1995) and Weerahandi and Gamage (2104) [?] developed the generalized estimation approach in linear regression with unequal variances, the differences in means of two independent exponential distribution and the differences in means of two independent normal distributions with unequal variances. Mixed Models in general point estimation, the uniform distribution with function of parameters and the gamma distribution were also examined. Each distribution is different with no universal method for developing generalized pivotal quantities (GPQ) and generalized test variables (GTV).

CHAPTER 3

TESTING THE LOG-SCALE, β, OF THE LG DISTRIBUTION

Introduction

A test statistic for the inference of the shape parameter β will be constructed. We have two minimal sufficient statistics for parameters α and β for the Log-Gamma distribution (LG).

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\beta-1} (\log x)^{\alpha-1} \text{ where } x \geq 1,\ \alpha > 0,\ \beta > 0. \tag{3.1}$$

The minimal sufficient statistics are

$$P = \prod_{i=1}^{n} \log X_i \tag{3.2}$$

$$T = \prod_{i=1}^{n} X_i \sim LG(n\alpha, \beta) \tag{3.3}$$

Although there is no requirement that these statistics be independent, it is computationally advantageous if the two standard sufficient statistics are transformed so that the transformed variables are independent. The approach actually leads to two independent sufficient statistics. The method requires a cumulative distribution function that will be used to handle the nuisance parameter that we will treat as function of our parameter of interest. We will also need a conditional cumulative distribution function of T given $S = s$ which will be uniform over the unit interval. (P, T), the minimal sufficient statistics for (α, β) respectively are complete since the Log-Gamma distribution is in the exponential family. While having complete sufficient statistics is not a requirement for the procedure, this fact will make our computations easier. The likelihood function for the Log-Gamma distribution is:

$$L(\alpha, \beta|\mathbf{X}) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} X_i^{-\beta-1} (\log X_i)^{\alpha-1} = \left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right]^{n} \left(\prod_{i=1}^{n} X_i\right)^{-\beta-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1} \tag{3.4}$$

It is computational advantageous to use $T^* = T^{1/n}$ which keeps the values of the product manageable especially when the number of samples are increased or when the data values are significantly large.

$$T^{1/n} = \left(\prod_{i=1}^{n} X_i\right)^{\frac{1}{n}} \sim LG(n\alpha, n\beta). \tag{3.5}$$

I proposed S to be statistically independent to T.

$$S = \frac{P^{\frac{1}{n}}}{\log T} = \frac{\left(\prod_{i=1}^{n} \log X_i\right)^{\frac{1}{n}}}{\log \prod_{i=1}^{n} X_i} = \frac{\left(\prod_{i=1}^{n} \log X_i\right)^{\frac{1}{n}}}{\sum_{i=1}^{n} \log X_i} \tag{3.6}$$

Finding an Independent Statistic

Basu's Theorem, [**?**], is used to prove T and S are independent. S was found using a technique developed by Godambe, an Indian statistician from his work on estimating functions. Godambe (1980) [**?**] examined ancillary and sufficiency in the presence of a nuisance parameter using definitions proposed from his work, Godambe (1976a) [**?**]. He showed under suitable conditions the conditional likelihood equation provides an estimating function independent of conditioning. In our first case, we need a conditional cumulative distribution $F_{T|S=s}(T)$ where α is our nuisance parameter. One major drawback of Godambe's (1976) [**?**] result is that it requires the existence of a complete sufficient statistic for the parameter of interest while treating the nuisance parameter as fixed (this will be replaced by the values of x). The Log-Gamma and the Gamma distributions are both members of the regular exponential family of distribution fulfilling the the requirement of existence of complete sufficient statistics. There are others procedures available if our distribution was not in this family of distributions. Find statistic T such that it has the following properties given by Godambe (1976). [**?**]

Theorem 4. *The abstract sample space is denoted by* $\mathrm{X} = \{x\}$*, the abstract parameter space is* $\Omega = \{\theta\} = (\alpha, \beta)$ *where* $\alpha \in \Omega_1$ *and* $\beta \in \Omega_2$*, the density function with respect to some measure* μ *on* X *is* $p(x, \theta)$*. Let* (S, T) *is minimal sufficient statistic for* (α, β)*. Let* $p(x, \theta) = f_t(x, \alpha)h(t, \theta)$*, where* h *is the marginal density of* t*.*

- *(i) The conditional density* f_t *of* x *given* t *depends on* θ *only through* α*.*

- *(ii) The class of distributions of* t *corresponding to* $\beta \in \Omega_2$ *is complete for each fixed* $\alpha \in \Omega_1$.

Any statistic t *satisfying conditions (i) and (ii) above is said to be an ancillary statistic with respect to* α. *The marginal distribution of* t *is said to contain no information about* α, *ignoring* β.

Since T is complete and $S = \prod \log(X)^{1/n} / \sum \log(X)$ is ancillary, T and S are independent by Basu's Theorem. Furthermore, S is an ancillary statistic with respect to β so that we can make inferences about β as desired.

Expanding the likelihood function, $L(\alpha, \beta|\mathbf{X})$, and using Godambe's Theorem **??**, we get:

$$f(\mathbf{X}|\alpha, \beta) = f(\mathbf{X}|T, \alpha) f(T|\alpha, \beta) \tag{3.7}$$

$$f(T|\alpha, \beta) = \frac{(\beta)^{n\alpha}}{\Gamma(n\alpha)} T^{-\beta-1} (\log T)^{n\alpha-1} \tag{3.8}$$

And rearranging equation **??** we get...

$$\begin{aligned} f(\mathbf{X}|T, \alpha) &= \frac{f(\mathbf{X}|\alpha, \beta)}{f(T|\alpha, \beta)} = \frac{\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \left(\prod_{i=1}^n X_i\right)^{-\beta-1} \left(\prod_{i=1}^n \log X_i\right)^{\alpha-1}}{\frac{\beta^{n\alpha}}{\Gamma(n\alpha)} T^{-\beta-1} (\log T)^{n\alpha-1}} \\ &\propto \frac{\left(\prod_{i=1}^n \log X_i\right)^{\alpha-1}}{(\log T)^{n\alpha-1}} = \frac{\left(\prod_{i=1}^n \log X_i\right)}{\left(\sum_{i=1}^n \log X_i\right)^n} \propto \frac{\left(\prod_{i=1}^n \log X_i\right)^{1/n}}{\left(\sum_{i=1}^n \log X_i\right)} \end{aligned} \tag{3.9}$$

This result is the same to the proposed statistic S in equation **??**. Therefore, S and T are independent.

Inference about the β Parameter

Let's consider making inferences about the β parameter treating α as the nuisance parameter.

$$H_0 : \beta \leq \beta_0 \text{ vs. } H_1 : \beta > \beta_0 \tag{3.10}$$

Let $F_{S|T=s}(s)$ be the conditional cumulative distribution function of S given $T = t$. We know that the conditional CDF of $W(S;\alpha) = F_{S|T=t}(S)$ given $T = t$ is uniform over the interval $[0, 1]$. Furthermore, by construction let $W = W(S;\alpha) = F_S(S;\alpha)$. The distribution W is uniform over the interval $[0, 1]$ and it is distributed independent of U in our original construction.

Generalized Pivotal Quantity

The CDF of $T \sim LG_{n\alpha}(t;\beta_0)$ is $Pr(T \leq t)$.

$$\begin{aligned} F_T(t;\alpha,\beta_0) &= \frac{\beta_0^{n\alpha}}{\Gamma(n\alpha)} \int_1^t \frac{(\log y)^{n\alpha-1}}{y^{\beta_0+1}}\, dy = LG_{n\alpha}(t;\beta_0) && (3.11) \\ &= \text{plgamma}(t, \text{shapelog} = n\alpha, \text{ratelog} = \beta_0) \\ &= \frac{\beta_0^{n\alpha}}{\Gamma(n\alpha)} \int_0^{\beta_0 \log t} y^{n\alpha-1} e^{-\beta_0}\, dy = G_{n\alpha}(\beta_0 \log t) && (3.12) \\ &= \text{pgamma}(\beta_0 \log t, \text{shape} = n\alpha) \end{aligned}$$

where the latter equation is the lower incomplete gamma function with the shape parameter of $n\alpha$ evaluated at t. Many statistical software packages may not have the Log-Gamma distribution as a built-in function so we can use the gamma function instead. The statistical program R has a package Actuar [**?**] written by Vincent Goulet and Mathieu Pigeon.

β_0 is the hypothesized parameter of interest. Now we are ready to take care of the nuisance parameter, α, by defining a random variable U as an Uniform distribution over the unit interval $[0, 1]$ which will act as our CDF.

$$U(T) = F_T(T; \alpha, \beta) = LG_{n\alpha}\,(t; \beta_0) \tag{3.13}$$

Solve the the equation $u = LG_{n\alpha}\,(t; \beta_0)$ for α called $\widehat{\alpha}\,(u; t)$ by taking its inverse. Since we know that values of t, n and β_0, the inverse function becomes a function of α only thus eliminating the nuisance parameter. This can be done by using a function such as `uniroot` in R by using a random number U, the hypothesized value of β_0 and the observed value of t in $\text{plgamma}(t, \text{shapelog} = n\widehat{\alpha}, \text{ratelog} = \beta_0)$ to find $\widehat{\alpha}$, the value of the nuisance parameter.

Weerahandi and Gamage (2014)[**?**] have shown in the two parameter Uniform distribution with parameters $\text{UNIF}(\alpha, \beta)$ that this step will work with either a closed form distribution or a distribution that is not closed as in our Log-Gamma distribution. Calculation times were significantly reduced when the distribution is simple closed form.

The random quantity $R_a = R_a(T; \alpha, \beta, t) = u^{-1}(U(T))$ becomes α denoted as $\widehat{\alpha}(U)$. By design, the random variable R_a satisfies the following two properties.

1. at the observed values of t of T, $R_a = R_a(t; \alpha, \beta, t) = \alpha$.

2. the distribution of R_a is free of α.

We need to verify our proposed generalized pivotal quantity, GPQ, R is free of unknown parameters at the observed values of s of S and t of T. The random variable R satisfies these two properties:

$$
\begin{aligned}
R &= \frac{W(S, T; \alpha, \beta)}{w(s, t; R_a(T; \alpha, \beta, t), \beta)} &\quad (3.14)\\
&= \frac{W}{w(s, t; \widehat{\alpha}(U), \beta)} &\quad (3.15)
\end{aligned}
$$

where $U = U(T; \alpha, \beta) \sim Unif(0, 1)$ and $W = W(S, T; \alpha, \beta) \sim Unif(0, 1)$ are independent uniform random variables. $\widehat{\alpha}(U) = u^{-1}(U)$ and $u^{-1}()$ is a function of t and β only.

1. at the observed values of s of S and t of T, the value of R becomes 1.

2. the distribution of R is free of the nuisance parameter α.

The distribution of R is therefore a generalized pivotal quantity, GPQ, as defined in the chapter 2 and can be used to make inferences on β such as point estimates and confidence intervals. Functions of α and β such as the mean or coefficient of variance can be theoretically tested as well. Hypothesis testing can be performed using the generalized test variable, GTV, based on the constructed GPQ.

$$R = \frac{W}{w(t, \hat{\alpha}(U; s, \beta_0))} \tag{3.16}$$

$$H_0 : \beta \leq \beta_0 \text{ vs. } H_1 : \beta > \beta_0 \tag{3.17}$$

$$R_0 = \frac{W}{w(s, t; \widehat{\alpha}(U), \beta_0)}$$

The generalized p-value for testing the null hypothesis H_0 against H_1 is

$$p = Pr(R_0 \leq 1) = Pr(W \leq w(s, t; \widehat{\alpha}(U), \beta_0) \tag{3.18}$$

$$= E(w(s, t; \widehat{\alpha}(U), \beta_0)) \tag{3.19}$$

because R stochastically increasing in β and this is an exact probability statement.

Therefore $R = 1$ when the observed values of s and t are evaluated and the distribution or R is free of nuisance parameters α in this example.

Generalized Confidence Intervals Based on p-Values

The GPQ R can be used to find the confidence intervals since it depends only on β. The value for β for which the 2.5th percentile of the distribution is equal to 1. Weerahandi, 1995 [**?**] defines GCI as $\Pr(R \in C_\gamma) = \gamma$ where C_γ is a subset of the sample space depending only on the observed values, $\underline{\mathbf{x}} = \{x_1, x_2, x_3, \ldots, x_n\}$. The main idea of this method is that we are making probability statements based of exact probability statements which does not rely on asymptotic statistical methods that required a large sample size. Tsui and Weerahandi (1989) [?] extended the classical definition to derive exact solutions to such problems as the Behrens-Fisher problem and testing variance components. The confidence intervals for the Log-Gamma distribution will be obtained Monte Carlo simulation since the observed value and the distribution of R are free of nuisance parameters. Confidence intervals for α and β are calculated and discussed in a latter chapter.

Numerical Results of the Simulation

Using a small size of $n = 10$, this performance study is based on 1,000 simulated random samples of size 10 from $X \sim LG(\alpha, \beta)$. Setting the α parameters at values at $\alpha = \{1, 2, 3, 5, 10, 15, 20, 25, 30, 40, 50, 100\}$. Smaller values of β were testing as well such as $0.1, 0.2, 0.5, 0.8$ with a little bias as we get closer to zero for the β value.

The table below summarizes the rate of rejection of the null hypothesis when the intended Type I error is 0.05 when the α parameter is varied from 1 - 100.

$$H_0 : \beta \leq \beta_0 \text{ vs. } H_1 : \beta > \beta_0 \tag{3.20}$$

I also ran 2000 and 3000 iterations for selected values of α. The results are similar

Table 3.1: Case $H_0 : \beta \leq \beta_0,\ \beta_0 = \{1, 2, 5\},\ N = 1000$

α	$\beta_0 = 1$			$\beta_0 = 2$			$\beta_0 = 5$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.012	0.038	0.082	0.006	0.039	0.079	0.009	0.059	0.108
2	0.011	0.045	0.089	0.010	0.047	0.090	0.010	0.052	0.100
3	0.010	0.049	0.097	0.007	0.042	0.088	0.007	0.045	0.103
5	0.010	0.045	0.095	0.009	0.041	0.090	0.011	0.054	0.108
10	0.011	0.057	0.107	0.009	0.045	0.084	0.012	0.059	0.096
15	0.011	0.049	0.102	0.009	0.044	0.086	0.006	0.053	0.104
20	0.009	0.054	0.101	0.010	0.047	0.094	0.012	0.055	0.107
25	0.009	0.045	0.093	0.008	0.047	0.096	0.007	0.052	0.110
30	0.011	0.050	0.107	0.013	0.048	0.100	0.005	0.042	0.103
40	0.010	0.057	0.108	0.017	0.065	0.119	0.015	0.051	0.098
50	0.010	0.057	0.108	0.010	0.055	0.112	0.014	0.061	0.119
100	0.012	0.052	0.102	0.010	0.048	0.100	0.010	0.059	0.130

to the results of 1000 iterations. I tested $\beta_0 = \{10, 15, 20\}$ for selected values of α from 1 to 100.

Table 3.2: Case $H_0 : \beta \leq \beta_0,\ \beta_0 = \{10, 15, 20\},\ N = 1000$

α	$\beta_0 = 10$			$\beta_0 = 15$			$\beta_0 = 20$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.009	0.049	0.106	0.013	0.055	0.112	0.011	0.046	0.091
2	0.020	0.071	0.126	0.012	0.043	0.096	0.008	0.043	0.093
3	0.005	0.052	0.100	0.017	0.059	0.111	0.010	0.037	0.088
5	0.018	0.061	0.112	0.014	0.055	0.097	0.009	0.049	0.098
10	0.008	0.042	0.088	0.011	0.046	0.094	0.006	0.039	0.085
15	0.007	0.050	0.095	0.009	0.048	0.110	0.014	0.047	0.092
20	0.011	0.057	0.110	0.007	0.043	0.093	0.016	0.058	0.101
25	0.011	0.046	0.087	0.009	0.058	0.113	0.010	0.050	0.099
30	0.015	0.051	0.109	0.011	0.059	0.120	0.011	0.040	0.089
40	0.005	0.037	0.089	0.013	0.054	0.103	0.010	0.057	0.101
50	0.009	0.039	0.085	0.003	0.039	0.092	0.010	0.046	0.092
100	0.013	0.058	0.101	0.007	0.059	0.107	0.010	0.045	0.091

Table 3.3: Case $H_0 : \beta \leq \beta_0,\ \beta_0 = \{30, 50, 100\},\ N = 1000$

α	$\beta_0 = 30$			$\beta_0 = 50$			$\beta_0 = 100$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.008	0.041	0.080	0.009	0.046	0.090	0.005	0.044	0.088
2	0.009	0.050	0.092	0.013	0.048	0.089	0.009	0.045	0.097
3	0.009	0.045	0.096	0.008	0.052	0.099	0.019	0.057	0.111
5	0.008	0.046	0.085	0.005	0.054	0.096	0.011	0.056	0.109
10	0.013	0.049	0.094	0.009	0.054	0.112	0.013	0.038	0.082
15	0.008	0.046	0.088	0.012	0.054	0.100	0.011	0.051	0.106
20	0.006	0.039	0.082	0.008	0.050	0.096	0.015	0.054	0.105
25	0.016	0.053	0.116	0.009	0.053	0.100	0.007	0.052	0.103
30	0.012	0.057	0.114	0.013	0.057	0.104	0.011	0.043	0.100
40	0.008	0.052	0.102	0.007	0.051	0.103	0.005	0.047	0.099
50	0.005	0.043	0.095	0.007	0.044	0.090	0.012	0.057	0.112
100	0.009	0.057	0.111	0.016	0.053	0.103	0.008	0.041	0.088

Table 3.4: Case $H_0 : \beta \leq 10$ vs. $H_0 : \beta > 10, N = 1000$ and 2000

(a) N = 1000 iterations

α	0.01	0.05	0.10
1	0.013	0.055	0.112
2	0.012	0.043	0.096
3	0.017	0.059	0.111
5	0.014	0.055	0.097
10	0.011	0.046	0.094
15	0.009	0.048	0.110
20	0.007	0.043	0.093
25	0.009	0.058	0.113
30	0.011	0.059	0.120
40	0.013	0.054	0.103
50	0.003	0.039	0.092
100	0.007	0.059	0.107

(b) N = 2000 iterations

α	0.01	0.05	0.10
1	0.009	0.058	0.112
2	0.009	0.036	0.079
3	0.009	0.048	0.099
5	0.014	0.050	0.100
10	0.008	0.044	0.096
15	0.011	0.057	0.100
20	0.012	0.049	0.094
25	0.011	0.056	0.101
30	0.013	0.052	0.089
40	0.014	0.045	0.105
50	0.012	0.051	0.103
100	0.012	0.048	0.098

Table 3.5: Case $H_0 : \beta \leq 15$ vs. $H_0 : \beta > 15, N = 1000$ and 2000

(a) N = 1000 iterations

α	0.01	0.05	0.10
1	0.013	0.055	0.112
2	0.012	0.043	0.096
3	0.017	0.059	0.111
5	0.014	0.055	0.097
10	0.011	0.046	0.094
15	0.009	0.048	0.110
20	0.007	0.043	0.093
25	0.009	0.058	0.113
30	0.011	0.059	0.120
40	0.013	0.054	0.103
50	0.003	0.039	0.092
100	0.007	0.059	0.107

(b) N = 2000 iterations

α	0.01	0.05	0.10
1	0.009	0.047	0.093
2	0.013	0.046	0.100
3	0.011	0.045	0.093
5	0.009	0.049	0.107
10	0.009	0.040	0.088
15	0.013	0.048	0.102
20	0.011	0.058	0.117
25	0.009	0.047	0.100
30	0.009	0.050	0.095
40	0.009	0.051	0.095
50	0.015	0.045	0.089
100	0.007	0.044	0.098

Table 3.6: Case $H_0 : \beta \leq 20$ vs $H_1 : \beta > 20,\ N = 3000$

α	0.01	0.05	0.10
1	0.007	0.048	0.099
3	0.012	0.045	0.101
5	0.008	0.039	0.087
10	0.009	0.048	0.092
20	0.008	0.052	0.098
30	0.007	0.051	0.108

CHAPTER 4

TESTING THE SHAPE, α, OF THE LG DISTRIBUTION

Let's consider making inferences about the parameter α by looking at the likelihood function $f_X(\mathbf{X}|\alpha, \beta)$. Consider the following the hypothesis:

$$H_0 : \alpha \leq \alpha_0 \text{ vs. } H_1 : \alpha > \alpha_0$$

Find an ancillary statistic for the conditional distribution needed to make inference about α using the procedure given by Godambe [**?**] and Basu (1955) [**?**]. Our new distribution is found using the same method since the

$$f(\mathbf{X}|\alpha, \beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} X_i^{-\beta-1} (\log X_i)^{\alpha-1} = \left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right]^n \left(\prod_{i=1}^{n} X_i\right)^{-\beta-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1} \tag{4.1}$$

where

$$T = \prod_{i=1}^{n} X_i \sim LG(n\alpha, \beta).$$

Using the the same process from Godambe 1976, $f(\mathbf{X}|\alpha, \beta) = f(\mathbf{X}|T, \beta) f(T|\alpha, \beta)$ I used T' for actual calculations. Both T and T' are complete sufficient statistics for β and a fixed α.

$$T' = \left(\prod_{i=1}^{n} X_i\right)^{1/n} \sim LG(n\alpha, n\beta) \tag{4.2}$$

$$\begin{aligned}
f(\mathbf{X}|\alpha,\beta) &= f(\mathbf{X}|\alpha,\beta) = f(\mathbf{X}|T,\beta)f(T|\alpha,\beta) && (4.3)\\
f(T'|\alpha,\beta) &= \frac{(n\beta)^{n\alpha}}{\Gamma(n\alpha)} T^{-n\beta-1}(T)^{n\alpha} && (4.4)\\
f(T|\alpha,\beta) &= \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} T^{-\beta-1}(T)^{n\alpha} && (4.5)
\end{aligned}$$

$$f(\mathbf{X}|\alpha,\beta) = f(\mathbf{X}|T,\beta)f(T|\alpha,\beta) \tag{4.6}$$

Rearrange the terms to find the ancillary statistic.

$$\begin{aligned}
f(\mathbf{X}|T,\beta) &= \frac{f(\mathbf{X}|\alpha,\beta)}{f(T|\alpha,\beta)} = \frac{\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \left(\prod_{i=1}^{n} X_i\right)^{-\beta-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1}}{\frac{(n\beta)^{n\alpha}}{\Gamma(n\alpha)} T^{-n\beta-1} (\log T)^{n\alpha-1}}\\
&= \frac{\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \left(\prod_{i=1}^{n} X_i\right)^{-\beta-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1}}{\frac{(n\beta)^{n\alpha}}{\Gamma(n\alpha)} \left(\prod_{i=1}^{n} X_i^{1/n}\right)^{-n\beta-1} (\log T)^{n\alpha-1}}\\
&= \frac{\Gamma(n\alpha)}{n^{n\alpha}\Gamma(\alpha)^n} \frac{\left(\prod_{i=1}^{n} X_i\right)^{-\beta-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1}}{\left(\prod_{i=1}^{n} X_i\right)^{-\beta-1/n} (\log T)^{n\alpha-1}} && (4.7)
\end{aligned}$$

This is proportional to the following.

$$\propto \quad \frac{\left(\prod_{i=1}^{n} X_i\right)^{1/n-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1}}{\left(\log \prod_{i=1}^{n} X_i^{1/n}\right)^{n\alpha-1}}$$

$$\propto \quad \frac{\left(\prod_{i=1}^{n} X_i\right)^{1} \left(\prod_{i=1}^{n} \log X_i\right)}{\left(\frac{1}{n}\sum_{i=1}^{n} \log X_i\right)^{n}} \tag{4.8}$$

Inference about the α Parameter

Let's consider making inferences about the parameter α by looking at the likelihood function $f_X(\mathbf{X}|\alpha, \beta)$. WLOG, reversing the role of W and U in the previous construction.

We are testing $H_0 : \alpha \leq \alpha_0$ vs. $H_1 : \alpha > \alpha_0$.

$$f(\mathbf{X}|\alpha, \beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} X_i^{-\beta-1} (\log X_i)^{\alpha-1} = \left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right]^n \left(\prod_{i=1}^{n} X_i\right)^{-\beta-1} \left(\prod_{i=1}^{n} \log X_i\right)^{\alpha-1}$$

Let $F_{T|S=s}(t)$ be the conditional cumulative distribution function of T given $S = s$. We know that the conditional CDF of $W(T; s) = F_{T|S=s}(T)$ given $S = s$ is uniform over the interval $[0, 1]$. Furthermore, the unconditional distribution of $W(T, s; \alpha, \beta)$ is UNIF$(0, 1)$ and independent of S since the the distribution does not depend on s by construction.

Again by construction let $W = W(T;\beta) = F_T(T;\beta)$. The distribution W is uniform over the interval $[0,1]$ and it is distributed independent of U as defined below.

Generalized Pivotal Quantity

The CDF of $T \sim LG_{n\beta}(t;\alpha_0)$ is $Pr(T \leq t)$.

$$\begin{aligned} F_T(t;\alpha_0,\beta) &= \frac{\beta^{n\alpha_0}}{\Gamma(n\alpha_0)} \int_1^t \frac{(\log y)^{n\alpha_0-1}}{y^{\beta+1}}\,dy = LG_{n\alpha_0}(t;\beta) \quad (4.9) \\ &= \text{plgamma}(t, \text{shapelog} = n\alpha_0, \text{ratelog} = \beta) \\ &= \frac{\beta^{n\alpha_0}}{\Gamma(n\alpha_0)} \int_0^{\beta \log t} y^{n\alpha_0-1} e^{-\beta y}\,dy = G_{n\alpha_0}(\beta \log t) \quad (4.10) \\ &= \text{pgamma}(\beta \log t, \text{shape} = n\alpha_0) \end{aligned}$$

α_0 is the parameter under the hypothesized testing and β is the nuisance parameter. Let U be a uniform random variable distributed over the unit interval (0,1) which will act as our CDF.

$$U(T) = F_T(S;\alpha,\beta) = LG_{n\beta}(t;\alpha_0) \quad (4.11)$$

Solve the equation $u = LG_{n\beta}(t;\alpha_0)$ or β called $\widehat{\beta}(u;t)$ by taking its inverse. Since we know that values of t, n and α_0, the inverse function becomes a function of β only thus eliminating the nuisance parameter. R^* is free of unknown parameters at the observed values of s of S and t of T.

$$\begin{aligned} R^* &= \frac{W(S,T;\alpha,\beta)}{w(s,t;R_a(T;\alpha,\beta,t),\beta)} \\ &= \frac{W}{w(t,\widehat{\beta}(U;s,\alpha_0))} \end{aligned} \tag{4.12}$$

1. the value of R^* becomes 1 at the observed values s and t of (S,T).

2. the distribution of R^* is free of the nuisance parameter β.

The random variable $R_a = R_a(T;\alpha,\beta,t) = u^{-1}(U(T))$ becomes β as $\widehat{\beta}(U)$. Therefore $R = 1$ when the observed values of s and t are evaluated and the distribution or R is free of nuisance parameters β in this example where $U \sim Unif(0,1)$ and $W \sim Unif(0,1)$ are independent uniform random variables. The distribution of R^* is therefore a generalized pivotal quantity, GPQ, as defined in the chapter 2 and can be used to make inferences on α such as point estimates and confidence intervals. Functions of α and β such as the mean or coefficient of variance can be theoretically testing as well. Hypothesis testing can be performed based of the generalized test variable, GTV, based on the constructed GPQ.

$$H_0 : \alpha \leq \alpha_0 \text{ vs. } H_1 : \alpha > \alpha_0 \tag{4.13}$$

$$R^* = \frac{W}{w(s,t;\widehat{\beta}(U),\alpha_0)}$$

The generalized p-value for testing the null hypothesis H_0 against H_1 is

$$p = Pr(R^* \leq 1) = Pr(W \leq w(s,t;\widehat{\beta}(U),\alpha_0) \tag{4.14}$$

$$= E(w(s,t;\widehat{\beta}(U),\alpha_0)) \tag{4.15}$$

because R^* stochastically increasing in α and this is an exact probability statement.

Table 4.1: Case $H_0 : \alpha \leq \alpha_0,\ \alpha_0 = \{1,2,3\},\ N = 1000$

β	$\alpha_0 = 1$			$\alpha_0 = 2$			$\alpha_0 = 3$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.008	0.043	0.092	0.008	0.040	0.086	0.004	0.037	0.076
2	0.009	0.055	0.095	0.009	0.042	0.091	0.006	0.034	0.078
3	0.011	0.045	0.103	0.011	0.052	0.110	0.012	0.054	0.114
5	0.009	0.051	0.102	0.008	0.053	0.103	0.010	0.045	0.097
10	0.011	0.057	0.107	0.011	0.049	0.099	0.010	0.048	0.092
15	0.009	0.034	0.090	0.009	0.044	0.086	0.009	0.046	0.097
20	0.012	0.049	0.092	0.009	0.042	0.096	0.006	0.042	0.085
25	0.009	0.045	0.093	0.014	0.042	0.084	0.010	0.045	0.105
30	0.010	0.047	0.105	0.015	0.055	0.102	0.010	0.040	0.093
40	0.016	0.060	0.114	0.012	0.054	0.100	0.012	0.059	0.110
50	0.014	0.052	0.099	0.012	0.053	0.100	0.009	0.052	0.091
100	0.016	0.047	0.101	0.010	0.052	0.107	0.011	0.051	0.105

Table 4.2: Case $H_0 : \alpha \leq \alpha_0,\ \alpha_0 = \{5, 10, 15\},\ N = 1000$

β	$\alpha_0 = 5$			$\alpha_0 = 10$			$\alpha_0 = 15$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.000	0.004	0.008	0.005	0.028	0.074	0.000	0.000	0.000
2	0.005	0.029	0.056	0.004	0.046	0.096	0.000	0.000	0.000
3	0.006	0.035	0.069	0.011	0.043	0.092	0.000	0.000	0.000
5	0.008	0.045	0.099	0.008	0.042	0.096	0.000	0.001	0.004
10	0.013	0.052	0.109	0.009	0.048	0.092	0.002	0.022	0.044
15	0.008	0.046	0.103	0.007	0.038	0.084	0.006	0.032	0.066
20	0.010	0.047	0.102	0.012	0.051	0.105	0.009	0.039	0.075
25	0.006	0.039	0.087	0.007	0.035	0.094	0.013	0.040	0.087
30	0.009	0.046	0.095	0.014	0.061	0.102	0.004	0.039	0.085
40	0.010	0.049	0.097	0.011	0.052	0.101	0.009	0.042	0.089
50	0.010	0.059	0.109	0.006	0.032	0.083	0.009	0.046	0.107
100	0.014	0.065	0.116	0.017	0.055	0.112	0.011	0.049	0.102

CHAPTER 5

METHOD OF MOMENTS AND THE MLE OF THE LG DISTRIBUTION

The Method of Moments for the Log-Gamma

One of the popular methods for testing the parameters for the LG is the the Method of Moments (MOM). Different approaches for this technique have been discussed including in chapter 2. The MOM for the Log-Gamma, LG, distribution is obtain using the traditional derivation. Let $x_1, x_2, \ldots, x_n$ be a random sample from a LG distribution with parameters $X \sim LG(\alpha, \beta)$. Define each moment about the origin of the sample by

$$M_r = \frac{1}{n} \sum_{i=1}^{n} x_i^r \tag{5.1}$$

and the theoretical moments define as

$$\mu_r = \int_D x^r f(x)\, dx = \int_1^\infty x^r \frac{\beta^\alpha x^{-\beta-1}}{\Gamma(\alpha)} (\ln x)^{\alpha-1}\, dx = \left(1 - \frac{r}{\beta}\right)^{-\alpha} \tag{5.2}$$

;

when $r < \beta$.

For the two parameter Log-Gamma distribution, we have ...

$$\mu_1 = \left(1 - \frac{1}{\beta}\right)^{-\alpha} \quad \text{and} \quad \mu_2 = \left(1 - \frac{2}{\beta}\right)^{-\alpha} \tag{5.3}$$

and

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \;\; \text{and} \;\; M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2. \tag{5.4}$$

Setting $\mu_1 = M_1$ and $\mu_2 = M_2$, taking the logarithm of both sides and then solving each equation for α we get...

$$\frac{\log M_1}{\log M_2} = \frac{\log\left(1 - \frac{1}{\beta}\right)}{\log\left(1 - \frac{2}{\beta}\right)} \tag{5.5}$$

It is easy to find $\tilde{\beta}$, the method of moments estimate for β given the data using a root function such as `uniroot` in R.

Using $\tilde{\beta}$, find $\tilde{\alpha}$ using

$$M_1 = \left(1 - \frac{1}{\tilde{\beta}}\right)^{-\tilde{\alpha}} \tag{5.6}$$

$$\tilde{\alpha} = -\frac{\log M_1}{\log\left(1 - \frac{1}{\tilde{\beta}}\right)} \tag{5.7}$$

Testing selected values for α and β and varying sample sizes N, each Method of Moments estimator for $\tilde{\alpha}$ and $\tilde{\beta}$ will be compared using the Mean Squared Error, MSE, with respect to the unknown parameter. Selected common values for the parameters and sample sizes are presented in the following tables.

$$MSE\left(\hat{\theta}\right) = E\left[(\hat{\theta} - \theta)^2\right] = Var\left(\hat{\theta}\right) + \left(Bias(\hat{\theta}, \theta)\right)^2 = Var\left(\hat{\theta}\right) + \left[E\left(\hat{\theta}\right) - \theta\right]^2$$

The MOM for the Log-Gamma distribution are poor estimators especially for small sample sizes or large values of β relative to α. The MOM estimates get better for larger values of N. Selected values for β and α were tested using the same method resulting in with similar results. The table shows typical results with $\beta = 5$ and $\alpha = \{1, 2, 5, 10, 20\}$.

Table 5.1: MSE: Method of Moments: $N = 10 - 1000$ Samples

N	Shape	Rate	$\tilde{\alpha}$	$\tilde{\beta}$	MSE $\tilde{\alpha}$	MSE $\tilde{\beta}$
10	1	5	2.784	10.100	4.109	66.609
10	2	5	5.572	9.300	14.757	52.163
10	5	5	7.824	9.168	9.798	47.145
10	10	5	20.512	9.513	116.999	40.619
10	20	5	67.684	10.904	2312.542	60.269
N	Shape	Rate	$\tilde{\alpha}$	$\tilde{\beta}$	MSE $\tilde{\alpha}$	MSE $\tilde{\beta}$
20	1	5	1.087	7.537	0.087	16.817
20	2	5	2.519	7.214	0.472	13.263
20	5	5	4.645	7.216	0.422	11.663
20	10	5	13.018	7.834	10.426	16.854
20	20	5	42.124	8.633	497.622	22.524
N	Shape	Rate	$\tilde{\alpha}$	$\tilde{\beta}$	MSE $\tilde{\alpha}$	MSE $\tilde{\beta}$
100	1	5	1.357	5.726	0.152	2.391
100	2	5	1.878	5.674	0.036	1.928
100	5	5	5.309	5.725	0.180	2.027
100	10	5	10.942	5.938	1.083	2.391
100	20	5	37.974	6.418	324.959	3.921
N	Shape	Rate	$\tilde{\alpha}$	$\tilde{\beta}$	MSE $\tilde{\alpha}$	MSE $\tilde{\beta}$
1000	1	5	1.114	5.127	0.015	0.296
1000	2	5	2.228	5.119	0.056	0.255
1000	5	5	5.001	5.168	0.007	0.292
1000	10	5	10.624	5.255	0.409	0.407
1000	20	5	17.983	5.439	4.111	0.707

Brief History of the MLE and the MOM

R Condie (1977) [?] examined the maximum likelihood estimators for the three parameters of a Log Pearson Type III (LP3) distribution derived from the logarithmic likelihood function. Condie concluded that the maximum likelihood analysis was superior in terms of the estimate of standard error to the method of moments that is the usual technique for flood data. We will use this form as it is written in Condies's paper with renaming the parameters, $a > 0$, $b \neq 0$, and $c > 0$ which are the scale, shape and location parameters respectfully with the exception of letting the location parameter be equal to zero.

Arora and Singh (1988) [?] examined most of the available papers for the MLE of the LP3 because of the popularity of the LP3 used as the based method by the Water Resource Council. Several researchers investigated fitting the distribution for the original data as well as the log-transformed data.

Bobee (1975)[?] suggested parameter estimation based on the moments of the real data. Condie (1977) [?], Phien and Hira (1983)[?] and others, used the method of maximum error of the MLE quantile estimator. Rao (1986)[?] proposed the method of mixed moments (MIX), which conserves the mean and standard deviation of real data, and the mean of log-transformed data, and Ashkar and Bobee (1988)[?] proposed a generalized method of moments. Computationally, the MLE is very difficult compared to the estimation methods available for LP3 such as moments or mixed moments. The W.R.C. choose the Log-Gamma distribution for folld data partly because of the ease computation of the moments estimators. Matalas and Wallis (1973)[?] found such severe computational difficulties associated with solving the MLE equations to the extent of recommending another distribution altogether. Condie concluded on the basis of the asymptotic standard error of the quantile estimator that the MLE estimators are generally superior those fitted by moments.

The MLE for Log-Gamma Distribution

$$
\begin{aligned}
f_X(x|a,b,c) &= \frac{1}{x\,|b|\,\Gamma(a)}\left(\frac{\ln x - c}{b}\right)^{a-1}\exp\left(-\frac{\ln x - c}{b}\right); &(5.8)\\
\text{If } b &< 0 \text{ then } 0 < x \le e^c \text{ and if } b > 0,\ \ e^c \le x < \infty. \\
f_X(x|a,b,c) &= \frac{(\ln x - c)^{a-1}}{b^a\Gamma(b)}x^{-1/b-1}e^{c/b} &(5.9)\\
\text{If } c &= 0, \text{ then } f_X(x|a,b) = \frac{(\ln x)^{a-1}}{b^a\Gamma(a)}x^{-1/b-1} &(5.10)
\end{aligned}
$$

Using the likelihood function:

$$L(\underline{X};a,b,c) = \prod_{i=1}^{n}\left[(\ln X_i - c)/a\right]^{b-1}\exp\left[-(\ln X_i - c)/a\right]/\left[|a|\,\Gamma(b)X_i\right]$$

The log likelihood function:

$$\ln L(\underline{X}; a, b, c) = (b-1)\sum \ln\left[(\ln X_i - c)/a\right] - \frac{1}{a}\sum(\ln X_i - c) - \sum \ln X_i - n\ln|a| - n\ln\Gamma(b) \quad (5.11)$$

This gives us three equations to solve:

$$\frac{\partial \ln L}{\partial a} = -n\psi(a) + \sum \ln\left[(\ln X_i - c)/b\right] = 0 \quad (5.12)$$

$$\frac{\partial \ln L}{\partial b} = \frac{1}{b^2}\sum(\ln X_i - c) - \frac{na}{b} = 0 \quad (5.13)$$

$$\frac{\partial \ln L}{\partial c} = -\frac{n}{a} - (b-1)\sum\left[\ln X_i - c\right]^{-1} = 0 \quad (5.14)$$

which yields the following solutions:

$$\hat{a} = \frac{B}{B-n^2} where\ B = \sum_{i=1}^{n}(\ln X_i - c)\sum_{i=1}^{n}(\ln X_i - c)^{-1} \quad (5.15)$$

$$\hat{b} = \frac{1}{na}\sum_{i=1}^{n}(\ln X_i - c) \quad (5.16)$$

$$\hat{c} = \frac{1}{n}\sum_{i=1}^{n}\ln X_i - \hat{a}\hat{b} \quad (5.17)$$

We will make the location parameter be equal to zero for our comparison giving us these two relations. For our two-parameter Log-Gamma distribution let $\widehat{\alpha} = a$ and

$\widehat{\beta} = 1/b$.

$$\widehat{\alpha} = \frac{B}{B - n^2} \quad \text{where } B = \sum(\ln X_i) \sum(\ln X_i)^{-1} \tag{5.18}$$

$$\widehat{\beta} = \left[\frac{1}{n\widehat{\alpha}} \sum_{i=1}^{n} \ln X_i\right]^{-1} \tag{5.19}$$

The same setup as for the Method of Moments estimator was utilized for $\widehat{\alpha}$ and $\widehat{\beta}$ using the Mean Squared Error, MSE, with respect to the unknown parameter.

Table 5.2: MLE: Maximum Likelihood Estimate: $N = 10 - 1000$ Samples

N	Shape	Rate	$\widehat{\alpha}$	$\widehat{\beta}$	MSE $\widehat{\alpha}$	MSE $\widehat{\beta}$
10	1	5	1.657	9.096	0.997	26.119
10	2	5	2.914	6.850	3.443	6.035
10	5	5	7.033	4.101	20.739	1.138
10	10	5	14.298	4.526	92.037	0.430
10	20	5	29.537	6.214	415.148	1.680
20	1	5	1.429	5.957	0.310	2.818
20	2	5	2.470	4.510	0.881	0.745
20	5	5	6.014	5.693	5.396	0.814
20	10	5	11.924	5.277	24.774	0.217
20	20	5	23.714	7.443	83.497	6.104
100	1	5	1.222	6.119	0.066	1.640
100	2	5	2.110	5.652	0.111	0.593
100	5	5	5.142	6.026	0.568	1.127
100	10	5	10.282	4.713	2.439	0.102
100	20	5	20.697	5.571	9.639	0.341
1000	1	5	1.140	6.222	0.023	1.535
1000	2	5	2.022	4.938	0.015	0.017
1000	5	5	5.013	4.891	0.055	0.017
1000	10	5	9.992	5.014	0.202	0.003
1000	20	5	20.078	4.898	0.722	0.011

The General Estimation, GE, method is compared to the Maximum Likelihood Estimates, MLE, using simulation for testing the β parameter. Type I error at 1%, 5%, 10% Intended Levels are compared with selected values for α from 1 to 100 and the number of samples, $n = 10$.

$$H_0 : \beta \leq \beta_0 \ vs, \ H_1 : \beta \leq \beta_0$$

Table 5.3: Case $H_0 : \beta \leq 10$, $N = 1000$ iterations

α	GE-0.01	MLE-0.01	GE-0.05	MLE-0.05	GE-0.10	MLE-0.10
1	0.008	0.340	0.039	0.491	0.087	0.575
3	0.011	0.388	0.047	0.456	0.091	0.507
5	0.013	0.431	0.052	0.491	0.108	0.528
10	0.012	0.498	0.047	0.545	0.094	0.568
25	0.012	0.547	0.053	0.576	0.105	0.593
50	0.008	0.588	0.056	0.608	0.104	0.615

Table 5.4: Case $H_0 : \beta \leq 5$, $N = 1000$ iterations

α	GE-0.01	MLE-0.01	GE-0.05	MLE-0.05	GE-0.10	MLE-0.10
1	0.007	0.337	0.040	0.489	0.091	0.568
2	0.008	0.352	0.044	0.444	0.091	0.499
3	0.006	0.375	0.049	0.451	0.095	0.508
5	0.008	0.432	0.045	0.502	0.098	0.539
10	0.010	0.482	0.048	0.535	0.099	0.565
15	0.009	0.498	0.051	0.543	0.100	0.567
20	0.010	0.515	0.050	0.542	0.105	0.567
25	0.007	0.546	0.046	0.583	0.097	0.601
30	0.010	0.546	0.058	0.570	0.101	0.585
40	0.008	0.553	0.055	0.588	0.107	0.600
50	0.009	0.578	0.048	0.599	0.096	0.613
100	0.008	0.581	0.045	0.599	0.089	0.615

Table 5.5: Case $H_0 : \beta \leq 1$, $N = 1000$ iterations

α	GE-0.01	MLE-0.01	GE-0.05	MLE-0.05	GE-0.10	MLE-0.10
1	0.012	0.336	0.041	0.481	0.078	0.566
2	0.015	0.373	0.057	0.480	0.100	0.529
3	0.001	0.373	0.043	0.450	0.099	0.502
5	0.015	0.435	0.056	0.493	0.098	0.539
10	0.014	0.500	0.049	0.552	0.106	0.576
15	0.008	0.524	0.052	0.559	0.094	0.589
20	0.005	0.535	0.050	0.571	0.103	0.591
25	0.009	0.548	0.033	0.572	0.091	0.591
30	0.008	0.534	0.045	0.564	0.088	0.579
40	0.012	0.523	0.053	0.558	0.108	0.568
50	0.010	0.601	0.050	0.620	0.094	0.632

The p-values using the MLE in the simulations with sample size 10 are not close to the intended values. Simulations with larger samples perform closer to their intended levels and we achieve parity with samples sizes over 50 for most of the values of α and β. If the number of samples were increased to 100, then the simulations performed to their intended levels for all values of the parameters tested. Clearly the Generalized Estimation process out performs the MLE and the MOM given smaller sample sizes when testing the α and β parameters. The MOM surprisingly performed as well as the MLE for sample sizes of 100 or more. This supports the WRC decision in 1967 (used as early as 1924) to use the MOM since the MOM is easier to compute than the MLE.

The Generalized Estimate method works well for very small values for β as low as 0.1, we still get reasonable results. Here, $\alpha = 2$, $\beta_0 = 0.5$ and $n = 10$, the computed GE and MLE are respectively are 0.048 and 0.419. At $\alpha = 2$, $\beta_0 = 0.1$ and $n = 10$, the computed GE and MLE are respectively are 0.051 and 0.433. As expected, if we increased the sample size to over 40, the results for the MLE tend to be better. If $\alpha = 2$, $\beta_0 = 2$ and $n = 50$, we have the results:

Table 5.6: Comparison between GE and MLE:$H_0 :\leq 2$, $n = 50$

0.01		0.05		0.10	
GE	MLE	GE	MLE	GE	MLE
0.006	0.014	0.049	0.055	0.110	0.121

Numerical Results of the Simulation

A major feature the using exact distributions is very good estimates for both parameters when the sample size is as small as 10. The Method of Moments technique was good for samples as small as 20 when the shape parameter, α, was under 10. The MOM estimates for α was good for values over 20 if the sample sizes were 100 or larger. The logscale parameter, β only performed well when sample sized was 100 or more. Overall the MOM can be used as a limited estimator for the both parameters.

The MLE technique yields good estimations when the sample size is larger. This is not the case when the sample size falls below 100 depending on the values of the parameters as well. The Generalized Estimation technique out performed the MOM and MLE techniques based on comparing the MSE of each parameter.

CHAPTER 6

TESTING THE MEAN: μ

The generalized estimation method is used for testing the mean of the Log-Gamma distribution using a generalized pivotal quantity. Exact pivotal quantities can not be constructed so we will use the generalized pivotal quantity method developed in the previous chapters. The mean is a function of both parameters α and β. We will tackle the variance in the next chapter.

Definitions

$$E\left[X\right] = \int_{1}^{\infty} x \cdot \frac{x^{-\beta-1}\left(\log(x)\right)^{\alpha-1}}{\beta^{-\alpha}\Gamma(\alpha)}\,dx = \left(1-\frac{1}{\beta}\right)^{-\alpha} \text{ for } \beta > 1 \tag{6.1}$$

$$E\left[X^2\right] = \int_{1}^{\infty} x^2 \cdot \frac{x^{-\beta-1}\left(\log(x)\right)^{\alpha-1}}{\beta^{-\alpha}\Gamma(\alpha)}\,dx = \left(1-\frac{2}{\beta}\right)^{-\alpha} \text{ for } \beta > 2 \tag{6.2}$$

The mean:

$$\mu = EX = \left(1-\frac{1}{\beta}\right)^{-\alpha} \text{ for } \beta > 1 \tag{6.3}$$

The Variance:

$$\sigma^2 = EX^2 - \mu^2 = \left(1 - \frac{2}{\beta}\right)^{-\alpha} - \left(1 - \frac{1}{\beta}\right)^{-2\alpha} \quad \text{for } \beta > 2 \tag{6.4}$$

Method

The framework for testing the ratelog parameter, β, in chapter 2 is used for testing the mean of the Log-Gamma distribution. The ratelog parameter, β, can be written as $\beta = (1-\mu^{-1/\alpha})^{-1}$ or as $b = 1-\mu^{-1/\alpha}$ whichever computes more efficiently, treating α as the nuisance parameter. We can also treat β as the nuisance parameter by re-parameterizing α.

$$\beta = \left(1 - \mu^{-\frac{1}{\alpha}}\right)^{-1} \qquad \text{or} \quad \alpha = -\frac{\log\mu}{\log\left(1 - \frac{1}{\beta}\right)} = \frac{\log\mu}{\log\left(\frac{\beta}{\beta-1}\right)}\,; \beta > 1. \tag{6.5}$$

Now we can replace β with $(1 - {\mu_0}^{-1/\alpha})^{-1}$ and use the `uniroot` function in R or a Newton-Raphson type method to solve for $\widehat{\alpha}$ in order to eliminate the nuisance parameter. Set $F_T(t; \widehat{\alpha}, \mu_0) = U(T)$ given t and μ_0 to find $\widehat{\alpha}$. U is an Uniform random variable on the interval $[0, 1]$. We find the inverse function numerically treating the equation $u = LG_{n\alpha}(t)$ as a function of α.

$$\text{Let } T = \prod_{i=1}^{n} X_i \sim LG(n\alpha, \beta) \tag{6.6}$$

I use $T^* = \left(\prod_{i=1}^{n} X_i\right)^{\frac{1}{n}} \sim LG(n\alpha, n\beta)$ because it was computationally more efficient when using larger values for n, the sample size. A distribution that does not have a closed form can work although it is computationally advantageous to use a known distribution as we will see in this mean approach. Using a unknown distribution drastically slows down performance time. The CDF of $T^* \sim LG(n\alpha, n\beta)$ is used to eliminate the nuisance parameter α using the previous derivation for testing the logscale, β. Recall, T (or T^*) is a complete sufficient statistic for β.

$$f(T^*|\alpha,\beta) = \frac{(n\beta)^{n\alpha}}{\Gamma(n\alpha)} T^{-n\beta-1} (\log T)^{n\alpha} \tag{6.7}$$

We can compute log-gamma distribution in R's `actuar` package or use the use the gamma distribution which is more widely available in almost all statistical software packages. Refer to the equation: **??**.

We need to construct a conditional cumulative distribution of S given $T = t$, denoted as $F_{S|t=t}(S) = F_S(s)$. Let

$$U = U(S;\alpha) = F_S(S;\alpha) \tag{6.8}$$

Let $U \sim \text{UNIF}(0,1)$ be a random variable such that is does not depend on ratelog parameter, β. For simplicity of notation, we call the inverse for α as $\widehat{\alpha}(s;u)$. Reversing the roles from testing the ratelog β value, finding $\widehat{\alpha}$ using an unknown open-form distribution on S is more complicated and time consuming. $\widehat{\alpha}'$s are computed by using a Monte Carlo method and a root find algorithm such as `uniroot` in R. This time we find candidates for $\widehat{\alpha}$ for each random number, U, that satisfies $u = F_S(s; \alpha)$.

With the roles reversed, let W be the Uniform random variable such that

$$W = LG_{n\alpha,n\beta}(t; \mu) \sim \text{UNIF}(0, 1) \tag{6.9}$$

A Generalized Pivotal Quantity, GPQ, for making inferences about μ can now be constructed for handling the nuisance parameter α by substituting $\widehat{\alpha}(s, U)$ into W. Define the random quantity $R_\mu = R_\mu(S; \alpha, \beta, s) = \widehat{\alpha}(U(S))$.

The Generalized Pivotal Quantity, GPQ, constructed becomes

$$R = \frac{W}{w(s, \hat{\alpha}(t; U, \mu_0))} \tag{6.10}$$

which satisfies the requirement for a GPQ. The distribution R is free of unknown parameters and at the observed values s of S and t of Tbecomes 1. In this case, R is increasing in the mean μ. The hypothesis for the mean is:

$$H_0 : \mu \geq \mu_0 \quad vs. \quad H_1 : \mu < \mu_0 \quad \text{where} \quad \mu = (1 - 1/\beta)^{-\alpha}. \tag{6.11}$$

$$p = \Pr\left(\frac{W}{w(s, \hat{\alpha}(t; U, \mu_0))} \leq 1\right) \tag{6.12}$$

$$= \Pr\left(W \leq w(s, \hat{\alpha}(t; U, \mu_0))\right)$$

$$= E\left(w(s, \hat{\alpha}(t; U, \mu_0))\right) \tag{6.13}$$

Numerical Results of the Simulation

I did some testing with 1000, 2000, 3000 iterations. This method is significantly slower to find the values for the unknown distribution but it still yields very good results for simulation studies. Using a size 10 from $X \sim LG(\alpha, \beta)$ and generating uniform random variables, $U \sim \text{UNIF}(0, 1)$ and $W \sim \text{UNIF}(0, 1)$, the Type I error of the test is unbiased when the shape parameter is much larger than the scale value or vice a versus. As you can see by the tables, the p-values are 0.005 - 0.01 lower than the intended values when this occurs. I ran some samples for $n = 3000$ iterations for $\mu = 3$ given $\alpha = 5$, with $\beta \approx 5.0695$ computed a Type I error at sizes 0.01, 0.05 and 0.10 with results at 0.0117, 0.048, and 0.0963 respectively. These results were typical for most the parameter configurations that were computed. I used 1000 iterations for most of the simulation because of the length of time to run each simulations. Simulation were computed for various values of the mean $\mu_0 = \{2, 3, 5, 10, 20, 30, 50, 100, 200\}$. Some of the simulations are included below.

The log-gamma mean can be very large if the logshape parameter, α, is significantly larger than the the logscale parameter, β. For example, if $\alpha = 50$ and $\beta = 3$, the mean would be 637,621,500. This method work for values as large as 10^8 for the

Table 6.1: Case $H_0 : \mu \geq \mu_0,\ \mu_0 = \{2, 3, 5\},\ N = 1000$

α	$\mu_0 = 2$			$\mu_0 = 3$			$\mu_0 = 5$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.011	0.054	0.098	0.014	0.049	0.091	0.016	0.057	0.099
2	0.010	0.060	0.095	0.013	0.048	0.095	0.012	0.052	0.090
3	0.010	0.053	0.102	0.010	0.053	0.098	0.018	0.049	0.090
5	0.014	0.058	0.094	0.014	0.052	0.084	0.012	0.044	0.096
10	0.014	0.063	0.106	0.009	0.052	0.099	0.004	0.041	0.088
20	0.014	0.049	0.094	0.011	0.050	0.105	0.006	0.044	0.090
30	0.013	0.053	0.108	0.007	0.040	0.092	0.007	0.035	0.089
50	0.015	0.056	0.107	0.013	0.048	0.089	0.010	0.049	0.093

mean. $\mu_0 = 10^8$, $\alpha = 10$ and 10000 iterations computed a Type I error at sizes 0.01, 0.05 and 0.10 with results 0.01, 0.053, and 0.092 respectively.

Table 6.2: Case $H_0 : \mu \geq \mu_0,\ \mu_0 = \{10, 20, 30\},\ N = 1000$

α	$\mu_0 = 10$			$\mu_0 = 20$			$\mu_0 = 30$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.011	0.058	0.106	0.012	0.056	0.107	0.009	0.048	0.090
2	0.013	0.051	0.103	0.015	0.051	0.096	0.007	0.046	0.093
3	0.014	0.052	0.096	0.011	0.047	0.095	0.010	0.044	0.095
5	0.014	0.045	0.085	0.014	0.045	0.085	0.007	0.044	0.090
10	0.009	0.046	0.090	0.009	0.046	0.090	0.007	0.044	0.094
20	0.008	0.050	0.101	0.010	0.041	0.089	0.012	0.070	0.112
30	0.009	0.048	0.085	0.007	0.044	0.091	0.010	0.064	0.106
50	0.013	0.053	0.083	0.012	0.045	0.082	0.012	0.046	0.096

Table 6.3: Case $H_0 : \mu \geq \mu_0$, $\mu_0 = \{50, 100, 200\}$, $N = 1000$

α	$\mu_0 = 50$			$\mu_0 = 100$			$\mu_0 = 200$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.013	0.056	0.108	0.010	0.053	0.104	0.015	0.054	0.117
2	0.015	0.051	0.096	0.009	0.043	0.092	0.017	0.055	0.101
3	0.011	0.047	0.095	0.009	0.054	0.098	0.009	0.053	0.092
5	0.014	0.045	0.085	0.009	0.044	0.096	0.012	0.059	0.114
10	0.009	0.047	0.090	0.014	0.049	0.101	0.010	0.053	0.097
20	0.011	0.053	0.101	0.012	0.048	0.098	0.009	0.042	0.088
30	0.014	0.042	0.092	0.010	0.047	0.095	0.011	0.046	0.088
50	0.015	0.057	0.103	0.009	0.048	0.098	0.011	0.047	0.087

CHAPTER 7

TESTING THE VARIANCE: σ^2

The generalized estimation method is used for testing the variance of the Log-Gamma distribution using a generalized pivotal quantity. Exact pivotal quantities can not be constructed so we will use the generalized pivotal quantity method developed in the previous chapters. The variance is a function of both parameters α and β.

Definitions

The variance is defined as **??**. Construct the hypothesis test:

$$H_0 : \sigma^2 \geq \sigma_0^2 \ vs. \ H_1 : \sigma^2 < \sigma_0^2$$

The Variance:

$$\sigma^2 = \left(1 - \frac{2}{\beta}\right)^{-\alpha} - \left(1 - \frac{1}{\beta}\right)^{-2\alpha} \text{ for } \beta \ > 2\,\alpha > 0. \tag{7.1}$$

The R package `actuar` has built in functions for the moments of the log-gamma distribution. A root finding algorithm can be used to find β treating α as the nuisance parameter. Reversing the process treating β as the nuisance parameter and solving for α is equally difficult computationally.

The R code in `actuar`

```
var <- mlgamma(2,a,b) - mlgamma(1,a,b)^2
```

where mlgamma(n, logshape = a, logscale = b) gives the moments of the log-gamma distribution if $b > 2$.

Let W be the Uniform random variable such that

$$W = LG_{n\alpha,n\beta}(t;\mu) \sim \text{UNIF}(0,1) \tag{7.2}$$

Method

The framework for testing the variance uses the same test statistic as the mean and the β inference testing. There is some computational difficulty using the `uniroot` function to find α and β from the hypothesized variance for the simulation. The technique used for finding the inference for β is combined with an algorithm for separating the variance into parameters of α and β. The hypothesis test for σ is

$$H_0 : \sigma \geq \sigma_0 \ vs. \ H_1 : \sigma < \sigma_0.$$

The generalized p-value for testing the null hypothesis H_0 is given by

$$p = \Pr\left(\frac{W}{w(s, \hat{\alpha}(t; U, \sigma_0^2))} \le 1\right) \tag{7.3}$$

$$= \Pr\left(W \le w(s, \hat{\alpha}(t; U, \sigma_0^2))\right)$$

$$= \mathrm{E}\left(w(s, \hat{\alpha}(t; U, \sigma_0^2))\right) \tag{7.4}$$

Numerical Results of the Simulation

Numerical testing with 1000 and 3000 iterations were performed. Remarkably, finding the variance has the same speed as the β parameter simulation. Using a size 10 from $X \sim LG(\alpha, \beta)$ and generating uniform random variables, $U \sim \mathrm{UNIF}(0,1)$ and $W \sim \mathrm{UNIF}(0,1)$, the Type I error of the test is unbiased for most values of β and α. I ran samples of $n = 3000$ and 5000 iterations for test values $\mu = 3$ given $\alpha = 5$, with $\beta \approx 5.0695$ computed a Type I error at sizes 0.01, 0.05 and 0.10 with results at 0.0117, 0.048, and 0.0963 respectively. These results were typical for most the parameter configurations that were computed. I used 1000 iterations for most of the simulation because of the length of time to run each simulations. There were very little performance difference between 1000 samples and 5000 samples. Simulation were computed for various values of the variance $\sigma_0^2 = \{2, 3, 5, 10, 20, 30, 50, 100, 200\}$. Some of the simulations are included below.

Table 7.1: Case $H_0 : \sigma^2 \geq \sigma_0^2,\ \sigma_0^2 = \{10, 20, 30\},\ N = 1000$

α	$\sigma_0^2 = 10$			$\sigma_0^2 = 20$			$\mu_0 = 30$		
P-Value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1	0.011	0.058	0.106	0.012	0.056	0.107	0.009	0.048	0.090
2	0.013	0.051	0.103	0.015	0.051	0.096	0.007	0.046	0.093
3	0.014	0.052	0.096	0.011	0.047	0.095	0.010	0.044	0.095
5	0.014	0.045	0.085	0.014	0.045	0.085	0.007	0.044	0.090
10	0.009	0.046	0.090	0.009	0.046	0.090	0.007	0.044	0.094
20	0.008	0.050	0.101	0.010	0.041	0.089	0.012	0.070	0.112
30	0.009	0.048	0.085	0.007	0.044	0.091	0.010	0.064	0.106
50	0.013	0.053	0.083	0.012	0.045	0.082	0.012	0.046	0.096

CHAPTER 8

TESTING THE COEFFICIENT OF VARIANCE: $\rho = CV$

The coefficient of variance is the ratio of the standard deviation, σ, and the non-zero mean, μ, of a probability distribution. The coefficient of variation is a standardized measure of dispersion of a probability distribution. The value is independent of the units in the measurement resulting in a dimensionless number which could be used to compare data sets with different units. Numerical problems arise when the mean is close to zero causing the coefficient of variation to approach infinity and is very sensitive to small changes in the mean. An advantage for using the coefficient of variation over the variance is that it is a scale-invariant quantity.

Definitions

Define Coefficient of Variance as $CV = \rho$.

$$\rho = \frac{\sigma}{\mu} = \frac{\sqrt{(1-2/\beta)^{-\alpha} - (1-1/\beta)^{-2\alpha}}}{(1-1/\beta)^{-\alpha}}. \tag{8.1}$$

or we can write this as:

$$\rho = \sqrt{\frac{(1-2/\beta)^{-\alpha}}{(1-1/\beta)^{-2\alpha}} - 1}$$

The hypothesis test for ρ is

$$H_0 : \rho \geq \rho_0 \text{ vs. } H_1 : \rho < \rho_0.$$

The generalized p-value for testing the null hypothesis H_0 is given by

$$\begin{aligned} p &= \Pr\left(\frac{W}{w(s, \hat{\alpha}(t; U, \rho_0))} \leq 1\right) && (8.2) \\ &= \Pr\left(W \leq w(s, \hat{\alpha}(t; U, \rho_0))\right) \\ &= \mathrm{E}\left(w(s, \hat{\alpha}(t; U, \rho_0))\right) && (8.3) \end{aligned}$$

Method

Using the framework outlined chapter 2, use $T^* = \left(\prod_{i=1}^{n} X_i\right)^{\frac{1}{n}} \sim LG(n\alpha, n\beta)$ to eliminate the nuisance parameter α. The hypothesized coefficient of variance value, ρ_0, is used to determine the value of β by re-parameterizing β in terms of ρ_0 allowing the treatment of α as the nuisance parameter as before.

Let $U = U(T) = F_T(T; \alpha, \beta) = LG_{n\alpha,n\beta} \sim \text{UNIF}(0, 1)$. The inverse of α is denoted as $\widehat{\alpha}(u; t)$. We find $\widehat{\alpha}$ by creating a Monte Carlo simulation of distribution of T to find the value of $\widehat{\alpha}$ using `uniroot` in R.

The package `actuar` in R has the built-in function mlgamma(order, shapelog, ratelog) which calculates the moments of the log-gamma distribution where order =

1 is the first moment EX, order = 2 is the second moment EX^2 and order = k is EX^k. Use `uniroot` or some other root finding algorithm to find either $a = \alpha$ or $b = \beta$. cv is the hypothesized coefficient of variation.

```
cv=sqrt(mlgamma(2,a,b)-mlgamma(1,a,b)^2)/(mlgamma(1,a,b))
```

Numerical Results of the Simulation

Table 8.1: Case $H_0 : \rho \geq \rho_0,\ \rho_0 = \{0.25, 0.50, 1\},\ N = 1000$

P-Value		0.01	0.05	0.10		0.01	0.05	0.10		0.01	0.05	0.10
α	β	$\rho_0 = 0.25$			β	$\rho_0 = 0.5$			β	$\rho_0 = 1$		
3	8.07	0.014	0.050	0.095	4.74	0.014	0.059	0.107	3.20	0.013	0.051	0.095
5	10.11	0.015	0.059	0.106	5.79	0.006	0.046	0.093	3.77	0.010	0.055	0.099
7	11.77	0.011	0.046	0.100	6.64	0.013	0.066	0.118	4.25	0.006	0.037	0.080
9	13.20	0.014	0.062	0.116	7.39	0.008	0.045	0.105	4.67	0.011	0.051	0.107
11	14.49	0.008	0.063	0.100	8.01	0.012	0.061	0.112	5.04	0.012	0.062	0.108
13	15.66	0.012	0.062	0.125	8.67	0.017	0.061	0.106	5.38	0.012	0.059	0.113
15	16.74	0.012	0.055	0.108	9.23	0.011	0.042	0.091	5.70	0.009	0.044	0.083
17	17.76	0.010	0.043	0.100	9.76	0.012	0.052	0.102	6.00	0.012	0.050	0.095

Table 8.2: Case $H_0 : \rho \geq \rho_0,\ \rho_0 = \{2, 3, 5\},\ N = 1000$

P-Value		0.01	0.05	0.10		0.01	0.05	0.10		0.01	0.05	0.10
α	β	$\rho_0 = 2$			β	$\rho_0 = 3$			β	$\rho_0 = 5$		
3	2.55	0.010	0.056	0.110	2.37	0.014	0.054	0.101	2.22	0.012	0.039	0.097
5	2.91	0.019	0.062	0.112	2.65	0.017	0.058	0.111	2.44	0.014	0.057	0.102
7	3.21	0.009	0.049	0.098	2.89	0.012	0.045	0.094	2.64	0.015	0.052	0.108
9	3.47	0.020	0.057	0.110	3.10	0.013	0.039	0.094	2.81	0.013	0.056	0.112
11	3.71	0.020	0.058	0.108	3.30	0.012	0.055	0.106	2.98	0.013	0.058	0.101
13	3.93	0.011	0.060	0.107	3.48	0.018	0.054	0.100	3.12	0.014	0.053	0.097
15	4.14	0.018	0.058	0.114	3.65	0.015	0.049	0.106	3.26	0.006	0.052	0.107
17	4.33	0.006	0.048	0.108	3.81	0.015	0.048	0.095	3.39	0.011	0.049	0.111

Table 8.3: Case $H_0 : \rho \geq \rho_0,\ \rho_0 = \{10, 15, 20\},\ N = 1000$

P-Value		0.01	0.05	0.10		0.01	0.05	0.10		0.01	0.05	0.10
α	β	$\rho_0 = 10$			β	$\rho_0 = 15$			β	$\rho_0 = 20$		
10	2.64	0.013	0.057	0.109	2.54	0.015	0.058	0.108	2.49	0.010	0.059	0.111
15	2.94	0.012	0.053	0.100	2.81	0.009	0.048	0.098	2.74	0.010	0.053	0.118
20	3.20	0.011	0.056	0.113	3.05	0.014	0.055	0.107	2.97	0.012	0.051	0.099
25	3.44	0.009	0.051	0.009	3.27	0.012	0.064	0.109	3.17	0.014	0.051	0.105
30	3.64	0.017	0.062	0.106	3.46	0.016	0.046	0.094	3.35	0.012	0.064	0.103
35	3.85	0.009	0.050	0.103	3.64	0.010	0.044	0.086	3.52	0.010	0.041	0.096
40	4.03	0.011	0.049	0.102	3.81	0.011	0.049	0.099	3.68	0.011	0.055	0.099
50	4.37	0.010	0.052	0.101	4.12	0.011	0.051	0.101	3.98	0.006	0.051	0.113

Table 8.4: Case $H_0 : \rho \geq \rho_0,\ \rho_0 = \{25, 30, 50\},\ N = 1000$

P-Value		0.01	0.05	0.10		0.01	0.05	0.10		0.01	0.05	0.10
α	β	$\rho_0 = 25$			β	$\rho_0 = 30$			β	$\rho_0 = 50$		
10	2.45	0.011	0.059	0.107	2.42	0.015	0.058	0.108	2.35	0.005	0.051	0.103
15	2.69	0.009	0.063	0.106	2.66	0.009	0.048	0.098	2.57	0.009	0.047	0.094
20	2.91	0.012	0.056	0.093	2.86	0.014	0.055	0.107	2.76	0.011	0.046	0.097
25	3.09	0.018	0.055	0.110	3.05	0.012	0.064	0.109	2.93	0.010	0.053	0.108
30	3.28	0.012	0.060	0.113	3.22	0.016	0.046	0.094	3.09	0.012	0.049	0.098
35	3.44	0.014	0.055	0.105	3.38	0.010	0.044	0.086	3.23	0.017	0.063	0.114
40	3.59	0.011	0.052	0.099	3.53	0.011	0.049	0.099	3.37	0.012	0.069	0.129
50	3.87	0.010	0.051	0.100	3.80	0.011	0.051	0.101	3.63	0.007	0.048	0.102

CHAPTER 9

THE CONFIDENCE INTERVALS

Figure 9.1: Log-Gamma Confidence Interval with $\alpha = 10$ and $\beta = 10$

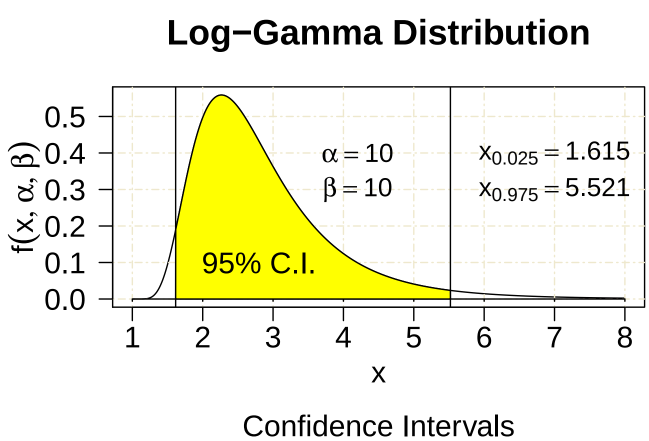
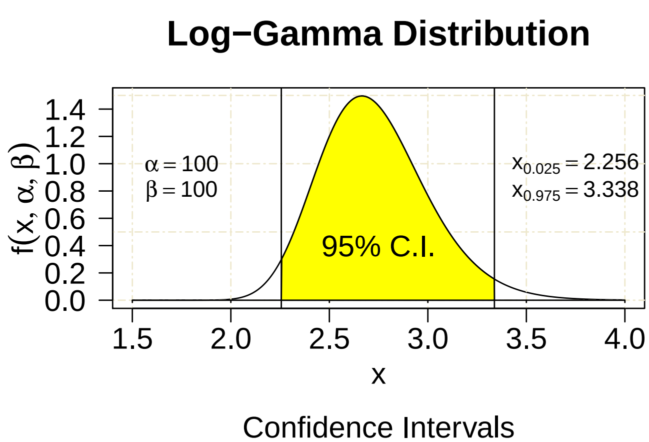


Figure 9.2: Log-Gamma Confidence Interval with $\alpha = 100$ and $\beta = 100$



We want to construct a confidence interval using the generalized estimation approach as defined in Chapter two. Our goal is to construct confidence intervals for continuous distributions when there are no nuisance parameters by using pivotal quantities or generalized pivotal quantities. Let $\mathbf{X}$ be a random sample from a distribution where $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ and $A(\mathbf{X})$ and $B(\mathbf{X})$ be two statistics such that $\Pr[A(\mathbf{X}) \leq \theta \leq \mathbf{B}(\mathbf{X})] = \gamma$, the desired confidence coefficient. There are many situations where exact confidence intervals are unavailable or are too computationally difficult especially when there are a nuisance parameters. In our case, we have $\theta = \{\alpha, \beta\}$.

1. Case 1: Find the confidence interval for α treating β as the nuisance parameter.

2. Case 2: Find the confidence interval for β treating α as the nuisance parameter.

Eliminating the nuisance parameters using the conditional densities defined (equation 3.3) for the Monte Carlo simulations.

C.I. using the Generalized Estimation

One benefit of using the generalized method is that we have eliminated the negative values on the lower bounds as the case when $\alpha = 2$ and $\beta = 3$. The 90% C.I. for M.L.E is $(-1.219, 4.912)$ where the lower bound -1.219 is out of the parameter space for β which is positive values. The length of the interval is 6.131 which is twice as large as the G.E. 90% C.I., $(1.673, 5.329)$ with length 3.656. The M.L.E. is generally good to find the values of each parameter when the sample space is large but even when the the sample space is large, there may be multiple MLE's for the location parameter in the 3 parameter log-gamma distribution, Rao (1986) [?]. One solution for the negative values in the confidence intervals is to take the maximum of the the value and zero or just replace the values of that is negative with zero. Much of the analysis of the log-gamma distribution has been concerning the T year flood events. Condie, (1977) [?] and others, have determine that the the MLE method for estimating parameters or a Mixed method of moments and MLEs were superior to fitting by moments alone. Of course, if using a small sample size those estimates my not be very good. Weerahandi (1995) [?] points out the that Generalized Confidence Intervals may not always have the confidence coefficient γ as defined. This happens when we construct our pivotal quantity on two minimal sufficient statistics which are not complete. Therefore we do not utilize all of the information in the data concerning our parameter.

I used the p-value code derived for General Estimation for the kernel of the Confidence Intervals. From above, the values of $\Pr(R \leq 1) = 1 - \gamma$ where $R = W/w(s, t, \widehat{\alpha}(U), \beta_{\gamma/2})$ by solving for $\beta_{\alpha/2}$ and $\beta_{1-\gamma/2}$ for the inverse process using the bisection method. No appreciable benefits were gained above 200 Monte Carlo simulations. The α, logshape, and the β, logscale parameters were evaluated with $n = 10$ and $n = 20$ sample sizes for parameters: $\alpha = \{1, 2, 3, 5, 10, 20\}$ with $\beta = \{1, 2, 3, 5, 10, 20\}$ for 90% and 95% confidence intervals.

The generalized confidence interval is given by $[min\{A, B\}, max\{A, B\}]$. Let A be a value of α for which the 97.5th percentile of the distribution is equal to 1, and B be the value of β for which the 2.5th percentile of the distribution of is equal to 1 where $\Pr(R \leq 1) = a$ and $\Pr(R \leq 1) = 1 - a$. Here, $a = 0.05/2 = 0.025$. The interval is an exact generalized interval based on exact probability statement. As expected in our case, the GPQ out performed the MLE based method below especially in smaller sample sizes.

Generalized Pivotal Quantity for the C.I. for Beta parameter

The same basic technique was used to find the confidence intervals that was used to find the P-values for the hypothesis testing for the parameters and functions of parameters.

For example, take the 95% confidence interval for $\alpha = 10$ and $\beta = 5$. Hypothesis testing such as $H_0 : \beta \leq 2.719$ against $H_1 : \beta > 2.719$ yields a p-values of 0.057245 which is right on the lower boundary of the 95% confidence interval which we would expect especially because the test for the β parameter is unbiased. Similarly, the upper bound of the confidence interval, 10.2106, yields a p-value of 0.052535 for the test of $H_0 : \beta \geq 10.21$ against $H_1 : \beta < 10.21$.

Comparing the confidence intervals using the GE method with the confidence intervals using asymptotic methods such as Fisher Information theory which produces

Table 9.1: 90% General Estimation C.I. for the Lograte = β

N	α	β	Interval	Length
200	2	**2**	(1.141, 3.582)	2.441
200	3	**2**	(1.155, 3.620)	2.465
200	5	**2**	(1.148, 3.472)	2.324
200	10	**2**	(1.218, 3.655)	2.437
200	2	**3**	(1.798, 5.621)	3.823
200	5	**3**	(1.566, 5.378)	3.812
200	10	**3**	(1.753, 5.269)	3.516
200	2	**5**	(2.868, 9.007)	6.139
200	3	**5**	(2.891, 8.885)	5.994
200	5	**5**	(2.548, 8.811)	6.263
200	10	**5**	(2.956, 8.856)	5.951
200	2	**10**	(5.865, 18.370)	12.5044
200	3	**10**	(5.671, 17.519)	11.848
200	5	**10**	(5.886, 17.818)	11.932
200	10	**10**	(5.956, 17.870)	11.914

Table 9.2: 95% General Estimation C.I. for the Lograte = β

N	α	β	Interval	Length
200	2	**2**	(1.037, 4.026)	2.989
200	3	**2**	(1.026, 3.925)	2.899
200	5	**2**	(1.019, 3.837)	2.818
200	10	**2**	(1.026, 3.925)	2.899
200	2	**3**	(1.541, 6.037)	4.496
1000	5	**3**	(1.514, 5.622)	4.108
200	10	**3**	(1.596, 5.943)	4.346
200	2	**5**	(2.733, 10.690)	7.957
200	3	**5**	(2.590, 9.924)	7.334
1000	5	**5**	(2.803, 9.868)	7.065
300	10	**5**	(2.719, 10.121)	7.402
200	2	**10**	(5.022, 19.742)	14.720
200	3	**10**	(5.118, 19.635)	14.517
200	5	**10**	(4.868, 18.349)	13.481
200	10	**10**	(5.326, 19.809)	14.483

Table 9.3: 90% General Estimation C.I. for the Shapelog = α

N	α	β	Interval	Length
200	**2**	2	(1.238, 3.465)	2.227
200	**2**	3	(1.219, 3.374)	2.155
200	**2**	5	(1.242, 3.426)	2.184
200	**2**	10	(1.297, 3.580)	2.283
200	**5**	2	(2.565, 9.028)	6.464
200	**5**	3	(2.850, 9.063)	6.214
200	**5**	5	(2.985, 8.940)	5.954
200	**5**	10	(3.110, 9.049)	5.939
200	**10**	2	(4.058, 21.596)	17.5381
200	**10**	3	(4.058, 21.596)	17.5381
200	**10**	5	(5.048, 19.202)	14.154
200	**10**	10	(5.699, 17.983)	12.284

good estimates when the sample size is large but not so good for small sample sizes.

Generalized Pivotal Quantity for the C.I. for Alpha parameter

The confidence intervals for α using the general estimation method is significantly better in terms of length and eliminating negative values for the parameters,

Frequentists often use confidence intervals as method of interval estimation using repeated sampling. The confidence interval would contain the true population parameter γ % percent of the time where the probability statement is based on the confidence interval, not the population parameter.

As in the classical approach for finding point estimators, our estimator needs to be an observable random quantity free of all nuisance parameters. Weeranhandi (2014) [**?**] extends the concept of generalized point estimates to generalized confidence

Table 9.4: 95% General Estimation C.I. for the Shapelog $=\alpha$

N	α	β	Interval	Length
200	**2**	2	(1.107, 3.782)	2.675
200	**2**	3	(1.079, 3.633)	2.554
200	**2**	5	(1.110, 3.727)	2.617
200	**2**	10	(1.105, 3.694)	2.589
200	**5**	2	(2.263, 10.311)	8.047
200	**5**	3	(2.408, 9.590)	7.182
200	**5**	5	(2.641, 9.804)	7.163
200	**5**	10	(2.592, 9.257)	6.665
200	**10**	2	(3.131, 29.245*)	26.113
200	**10**	3	(3.308, 25.557)	22.250
200	**10**	5	(4.222, 21.045)	16.823
200	**10**	10	(4.957, 19.553)	14.595

intervals using generalized p-values. Our goal is to construct exact confidence intervals without having to resort to asymptotic confidence intervals that are accurate only when we have a large sample size by finding a pivotal quantity that does not depend on the unknown true parameter or on any nuisance parameter based on observed values of **X**.

Figure 9.3: Log-Gamma Confidence Interval with $\alpha = 20$ and $\beta = 10$

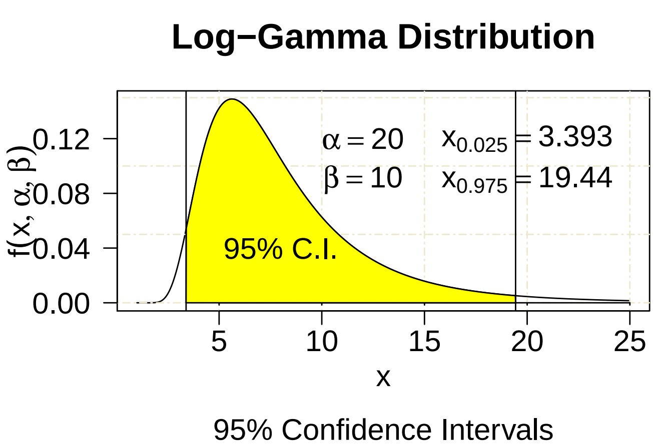
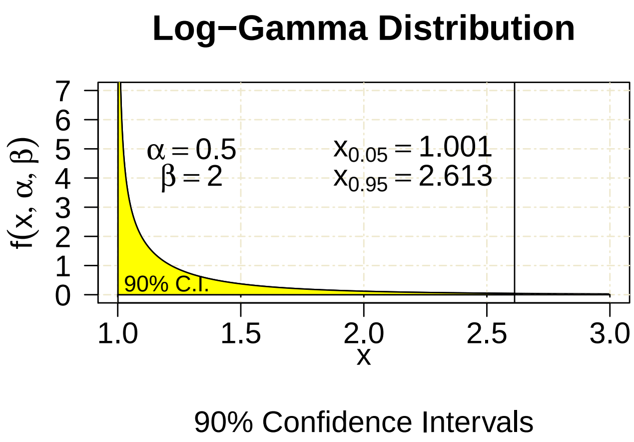


Figure 9.4: Log-Gamma Confidence Interval with $\alpha = 0.5$ and $\beta = 2$



C.I. using the MLE

The Maximum Likelihood Estimates, MLE, are used to compare to the GE and the MOM methods.

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}x^{-\beta-1}}{\Gamma(\alpha)}(\log x)^{\alpha-1}$$

$$f(\mathbf{x};\alpha,\beta) = \prod_{i=1}^{n}\frac{\beta^{\alpha}x_i^{-\beta-1}}{\Gamma(\alpha)}(\log x_i)^{\alpha-1} = \frac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n}\prod_{i=1}^{n}x_i^{-\beta-1}(\log x_i)^{\alpha-1} \tag{9.1}$$

$$\log L(\alpha,\beta) = \log\left[\frac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n}\prod_{i=1}^{n}x_i^{-\beta-1}\prod_{i=1}^{n}(\log x_i)^{\alpha-1}\right] \tag{9.2}$$

$$\begin{aligned} l_n(\alpha,\beta) &= n\alpha\log\beta - n\log\Gamma(\alpha) + (-\beta-1)\log\left(\prod_{i=1}^{n}x_i\right) + (\alpha-1)\log\prod_{i=1}^{n}(\log x_i) \\ &= n\alpha\log\beta - n\log\Gamma(\alpha) - (\beta+1)\sum_{i=1}^{n}\log x_i + (\alpha-1)\sum_{i=1}^{n}\log(\log x_i) \\ &= n\alpha\log\beta - n\log\Gamma(\alpha) - n(\beta+1)\bar{y}_n + n(\alpha-1)\bar{z}_n \end{aligned}$$

$$\text{where } \bar{y}_n = \frac{1}{n}\sum_{i=1}^{n}\log x_i \text{ and } \bar{z}_n = \frac{1}{n}\sum_{i=1}^{n}\log(\log x_i). \tag{9.3}$$

The expected Fisher information matrix is:

$$I_n(\alpha,\beta) = -\begin{bmatrix} \frac{\partial l_n(\alpha,\beta)}{\partial\alpha^2} & \frac{\partial l_n(\alpha,\beta)}{\partial\beta\partial\alpha} \\ \frac{\partial l_n(\alpha,\beta)}{\partial\alpha\partial\beta} & \frac{\partial l_n(\alpha,\beta)}{\partial\beta^2} \end{bmatrix} = \begin{bmatrix} n\,\psi'(\alpha) & -\frac{n}{\beta} \\ -\frac{n}{\beta} & \frac{n\alpha}{\beta^2} \end{bmatrix} \tag{9.4}$$

where the digamma function is $\psi(\alpha) = \dfrac{d}{d\alpha}\log\Gamma(\alpha) = \dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)}$

and the trigamma function is $\psi'(\alpha) = \psi_1(\alpha) = \dfrac{d^2}{d\alpha^2}\log\Gamma(\alpha) = \dfrac{d}{d\alpha}\,\psi(\alpha).$

We will use the MLE's for the α and β parameters as initial values for expected Fisher information.

$$\begin{pmatrix} \hat{\alpha}_n \\ \hat{\beta}_n \end{pmatrix} \approx N\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \mathbf{I}_n(\theta)^{-1}\right)$$

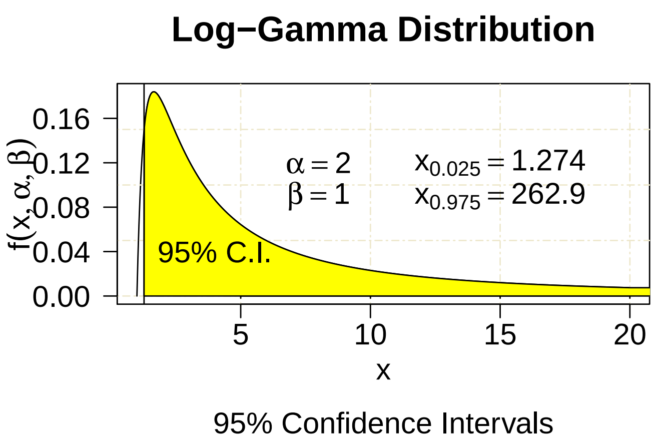
We can also use the observes Fisher information.

$$\begin{pmatrix} \hat{\alpha}_n \\ \hat{\beta}_n \end{pmatrix} \approx N\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \mathbf{J}_n(\theta)^{-1}\right) \text{ where } \mathbf{J}_n(\theta) = -\nabla^2 l_n(\theta).$$

A similar method using a Newton-Raphson algorithm yields slightly better approximations. There are several limitations to this technique including the requirement for close initial estimates which can be troublesome for the log-gamma distributions, large standard errors even with small sample sizes, and negative parameter values causing the algorithm to crash unexpectedly.

Varying sample sizes were evaluated with selected examples with small sample sizes are provided below. The Newton-Raphson algorithm is used to find the two unknown parameters using $\mathbf{J}(\theta)$ the observed information matrix. This method, as well as the MLE method is often unpredictable for small sample sizes.

Figure 9.5: Log-Gamma Confidence Interval with $\alpha = 2$ and $\beta = 1$



$$\theta^{(s+1)} = \theta^{(s+1)} + \mathbf{J}^{-1}\left(\theta^{(s)}\right) s\left(\theta^{(s)}\right)$$

where $s(\theta)$ is the score function and $J(\theta)$ is the observed information matrix. Let $n = 10$, $\alpha = 4$ and $\beta = 5$:

$$\hat{\alpha} = 2.885,\ \hat{\beta} = 3.959$$

$$\mathbf{J}(\widehat{\theta}) = \begin{bmatrix} 8.269 & -5.051 \\ -5.051 & 3.681 \end{bmatrix},\ \mathbf{J}^{-1}(\widehat{\theta}) = \begin{bmatrix} 0.748 & 1.026 \\ 1.026 & 1.698 \end{bmatrix}$$

The standard errors are $\widehat{\alpha}_{\text{se}} = \sqrt{0.748} = 0.865$ and $\widehat{\beta}_{\text{se}} = \sqrt{1.680} = 1.296$. The values of the MLE used as initial estimates were $\alpha = 4.990$ and $\beta = 6.750$. The Newton Ralpson method is a vast improvement over the MLE method where the estimates varied greatly for each approximation using the `uniroot` function in R.

The Confidence Intervals were obtained using this technique. I ran 1000 iterations to show that the MLE is an asymptotically consistent estimator.

N	Shape	Rate	a	b	LB_a	UB_a	LB_b	UB_b
1000	1	5	1.172	5.974	-2.647	4.635	2.003	8.134
1000	2	5	2.035	4.876	-1.669	5.613	1.659	7.791
1000	5	5	5.037	5.016	1.424	8.706	1.978	8.109
1000	10	5	9.791	4.886	6.124	13.406	1.807	7.939
1000	20	5	18.904	4.729	15.26	22.541	1.663	7.794

We compared the estimates for the parameters for the log-gamma distribution using the Generalized Estimation method with a parametric bootstrap method using the MLE and the method of moments for each parameter. We can see by thee tables that the Generalized Estimation method gave more narrow confidence intervals and did not give values that were outside of the parameter space such as negative values.

Table 9.5: 95% Confidence Intervals for α and β: $\alpha = 2$

N	Shape	Rate	$\widehat{\alpha}$	$\widehat{\beta}$	95% C.I. $\widehat{\alpha}$	95% C.I. $\widehat{\beta}$
10	2	1	4.325	2.428	(0.684, 7.966)	(-0.638, 5.494)
	2	2	1.744	1.847	(-1.897, 5.384)	(-1.219, 4.912)
	2	5	1.353	2.548	(-2.288, 4.994)	(-0.518, 5.613)
	2	10	2.234	7.300	(-1.407, 5.874)	(4.235, 10.366)
	2	20	1.371	10.648	(-2.270, 5.012)	(7.582, 13.713)
20	2	1	3.290	1.569	(-0.351, 6.931)	(-1.497, 4.634)
	2	2	3.768	4.026	(0.127, 7.409)	(0.961, 7.092)
	2	5	2.749	5.811	(-0.891, 6.390)	(2.745, 8.876)
	2	10	1.961	7.723	(-1.679, 5.602)	(4.658, 10.789)
	2	20	2.654	24.209	(-0.987, 6.295)	(21.144, 27.275)
30	2	1	1.886	1.029	(-1.754, 5.527)	(-2.037, 4.094)
	2	2	2.699	3.163	(-0.942, 6.340)	(0.097, 6.228)
	2	5	2.349	5.314	(-1.292, 5.990)	(2.248, 8.380)
	2	10	2.986	14.375	(-0.655, 6.627)	(11.309, 17.440)
	2	20	1.977	16.760	(-1.664, 5.618)	(13.694, 19.825)
50	2	1	1.284	0.678	(-2.357, 4.925)	(-2.388, 3.743)
	2	2	2.039	1.811	(-1.602, 5.680)	(-1.254, 4.877)
	2	5	1.979	4.646	(-1.662, 5.620)	(1.581, 7.712)
	2	10	1.966	10.445	(-1.675, 5.606)	(7.379, 13.510)
	2	20	2.343	21.354	(-1.298, 5.984)	(18.288, 24.419)
100	2	1	1.708	1.021	(-1.933, 5.348)	(-2.044, 4.087)
	2	2	2.343	2.233	(-1.297, 5.984)	(-0.833, 5.298)
	2	5	2.776	7.853	(-0.865, 6.417)	(4.787, 10.919)
	2	10	1.948	10.927	(-1.692, 5.589)	(7.862, 13.993)
	2	20	2.377	24.040	(-1.264, 6.018)	(20.975, 27.106)

Table 9.6: 95% Confidence Intervals for α and β: $\alpha = 5$

N	Shape	Rate	$\widehat{\alpha}$	$\widehat{\beta}$	95% C.I. $\widehat{\alpha}$	95% C.I. $\widehat{\beta}$
10	5	1	6.170	1.669	(2.604, 9.886)	(-1.376, 4.756)
	5	2	4.155	1.600	(-0.272, 7.010)	(-1.768, 4.363)
	5	5	6.390	5.972	(2.505, 9.787)	(2.679, 8.810)
	5	10	9.000	19.728	(5.182, 12.464)	(16.275, 22.407)
	5	20	7.135	26.722	(3.108, 10.390)	(22.211, 28.342)
20	5	1	5.587	1.122	(1.800, 9.082)	(-1.973, 4.159)
	5	2	4.278	1.659	(0.889, 8.171)	(-1.309, 4.822)
	5	5	3.727	4.014	(0.645, 7.926)	(1.550, 7.682)
	5	10	6.140	12.273	(1.977, 9.258)	(8.162, 14.294)
	5	20	4.071	17.038	(-0.272, 7.009)	(11.034, 17.165)
30	5	1	4.171	0.958	(0.530, 7.811)	(-2.108, 4.024)
	5	2	4.280	1.643	(0.639, 7.921)	(-1.423, 4.709)
	5	5	5.543	5.802	(1.903, 9.184)	(2.736, 8.867)
	5	10	4.218	9.751	(0.578, 7.859)	(6.686, 12.817)
	5	20	3.637	15.476	(-0.004, 7.278)	(12.411, 18.542)
50	5	1	3.793	0.830	(0.296, 7.578)	(-2.204, 3.927)
	5	2	5.729	2.234	(2.174, 9.456)	(-0.799, 5.333)
	5	5	5.991	6.502	(2.223, 9.505)	(3.298, 9.430)
	5	10	4.470	8.821	(0.759, 8.041)	(5.617, 11.748)
	5	20	5.134	19.685	(1.919, 9.200)	(18.251, 24.382)
100	5	1	6.082	1.146	(2.441, 9.723)	(-1.920, 4.212)
	5	2	4.837	2.027	(1.196, 8.477)	(-1.039, 5.092)
	5	5	6.132	5.999	(2.491, 9.773)	(2.933, 9.065)
	5	10	5.501	10.993	(1.860, 9.142)	(7.927, 14.059)
	5	20	4.350	17.459	(0.709, 7.991)	(14.393, 20.524)

Table 9.7: 95% Confidence Intervals for α and β: $\beta = 2$

N	Shape	Rate	$\widehat{\alpha}$	$\widehat{\beta}$	95% C.I. $\widehat{\alpha}$	95% C.I. $\widehat{\beta}$
10	1	2	1.191	2.387	(-2.450, 4.832)	(-0.678, 5.453)
	2	2	1.663	1.973	(-1.978, 5.304)	(-1.092, 5.039)
	5	2	6.967	3.178	(3.327, 10.608)	(0.112, 6.244)
	10	2	10.436	1.956	(6.796, 14.077)	(-1.110, 5.021)
	20	2	37.972	3.642	(34.331, 41.613)	(0.577, 6.708)
20	1	2	0.612	1.065	(-3.028, 4.253)	(-2.001, 4.131)
	2	2	1.846	1.647	(-1.795, 5.486)	(-1.419, 4.712)
	5	2	2.958	1.196	(-0.682, 6.599)	(-1.870, 4.261)
	10	2	22.515	4.773	(18.875, 26.156)	(1.708, 7.839)
	20	2	21.107	2.229	(17.466, 24.747)	(-0.836, 5.295)
30	1	2	1.653	2.874	(-1.987, 5.294)	(-0.192, 5.940)
	2	2	1.618	1.656	(-2.023, 5.259)	(-1.409, 4.722)
	5	2	6.835	2.539	(3.194, 10.475)	(-0.527, 5.604)
	10	2	13.809	2.616	(10.168, 17.449)	(-0.450, 5.682)
	20	2	19.536	1.896	(15.895, 23.177)	(-1.170, 4.961)
50	1	2	1.311	2.624	(-2.329, 4.952)	(-0.442, 5.689)
	2	2	2.065	1.895	(-1.576, 5.705)	(-1.171, 4.961)
	5	2	5.106	2.009	(1.465, 8.747)	(-1.056, 5.075)
	10	2	7.663	1.473	(4.022, 11.304)	(-1.592, 4.539)
	20	2	18.433	1.836	(14.793, 22.074)	(-1.229, 4.902)
100	1	2	0.944	1.82	(-2.697, 4.585)	(-1.246, 4.886)
	2	2	2.148	2.361	(-1.493, 5.788)	(-0.705, 5.427)
	5	2	5.605	2.182	(1.964, 9.246)	(-0.884, 5.248)
	10	2	8.574	1.673	(4.933, 12.215)	(-1.392, 4.739)
	20	2	20.528	1.994	(16.887, 24.169)	(-1.072, 5.059)

Table 9.8: 95% Confidence Intervals for α and β: $\beta = 5$

N	Shape	Rate	$\widehat{\alpha}$	$\widehat{\beta}$	95% C.I. $\widehat{\alpha}$	95% C.I. $\widehat{\beta}$
10	1	5	1.549	5.322	(-2.092, 5.190)	(2.256, 8.388)
	2	5	7.629	18.841	(3.988, 1.270)	(15.776, 21.907)
	5	5	5.356	5.191	(1.715, 8.997)	(2.126, 8.257)
	10	5	4.724	2.669	(1.083, 8.365)	(-0.397, 5.734)
	20	5	36.517	8.874	(32.876, 40.158)	(5.808, 11.940)
20	1	5	0.963	3.650	(-2.678, 4.604)	(0.585, 6.716)
	2	5	1.321	3.733	(-2.320, 4.961)	(0.668, 6.799)
	5	5	4.849	4.607	(1.208, 8.490)	(1.541, 7.673)
	10	5	9.787	4.461	(6.147, 13.428)	(1.396, 7.527)
	20	5	18.244	4.630	(14.604, 21.885)	(1.564, 7.696)
30	1	5	0.710	3.413	(-2.931, 4.351)	(0.348, 6.479)
	2	5	1.751	4.009	(-1.89, 5.391)	(0.943, 7.075)
	5	5	5.846	5.220	(2.205, 9.487)	(2.155, 8.286)
	10	5	9.375	4.310	(5.734, 13.016)	(1.244, 7.375)
	20	5	19.808	5.550	(16.167, 23.449)	(2.485, 8.616)
50	1	5	0.952	4.581	(-2.689, 4.593)	(1.515, 7.646)
	2	5	2.231	5.133	(-1.410, 5.871)	(2.068, 8.199)
	5	5	6.880	6.507	(3.239, 10.520)	(3.442, 9.573)
	10	5	11.332	6.077	(7.691, 14.972)	(3.011, 9.143)
	20	5	19.976	4.765	(16.335, 23.617)	(1.699, 7.831)
100	1	5	1.020	5.746	(-2.620, 4.661)	(2.680, 8.811)
	2	5	2.344	6.067	(-1.296, 5.985)	(3.001, 9.133)
	5	5	5.210	5.046	(1.569, 8.851)	(1.980, 8.112)
	10	5	8.901	4.505	(5.260, 12.542)	(1.439, 7.570)
	20	5	23.032	5.638	(19.391, 26.673)	(2.572, 8.704)

CHAPTER 10

GOODNESS OF FIT BALL BEARINGS

Ball Bearings

Data Set 1: The first data set is as follows; (see below). The data given here arose in tests on endurance of deep groove ball bearings. Statisticians have used this well-known data set to compare different extreme value distributions such as the Weibull, gamma, log-gamma, censored data, etc...

- Lieblein and Zelen (1956) [?] used log lifetimes and then used two-parameter Weibull distribution
- Lawless (1982)[?] and Balakrishnan and Chan (1995a,b) [?][?] assumed a generalized gamma distribution for the original data and hence a log-gamma distribution for the log-lifetimes.
- Chien-Tai Lin, Sam J. S. Wu and Balakrishnan (2006)[?] Log-Gamma Distribution Based on Progressively Type-II Censored Data

The assumptions in the original study by were given by Lieblein and Zelen (1956) [?]. Some of the pertinent assumptions were:

- Each test group can be regarded as a random sample from a homogeneous population of ball bearings.

- The probability distribution of the number of revolutions to fatigue failure is of the same form for each test group, although its parameters may differ from group to group.

- Differences in the measured life of bearings classed as identical, tested at the same load, reflect only the inherent variability of fatigue life, and are free from systematic errors that may arise from different test conditions, materials, manufacturing methods, etc.

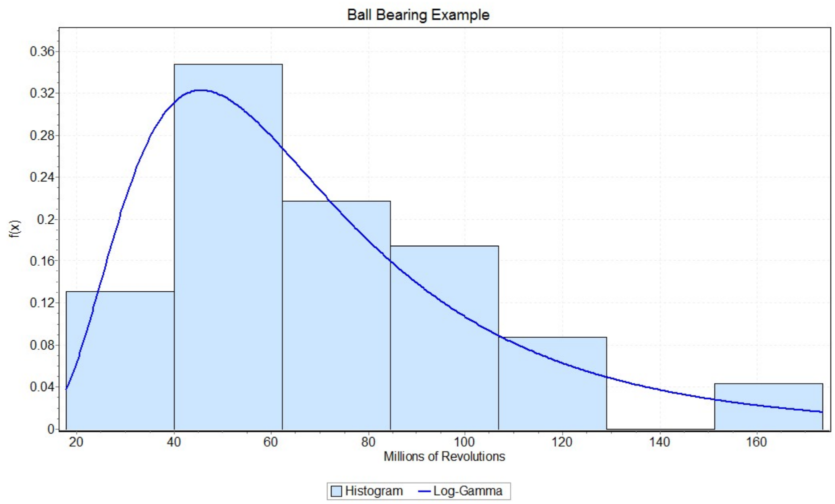
- The Weibull distribution was used in the original study as the fatigue-life distribution although other methods performed as well or better in subsequent studies using the same data sets.

The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

The log-lifetimes are used in the Weibull, Gamma and Log-Gamma studies: 2.884, 3.365, 3.497, 3.726, 3.741, 3.820, 3.888, 3.948, 3.950, 3.991, 4.017, 4.217, 4.229, 4.229, 4.232, 4.432, 4.534, 4.591, 4.655, 4.662, 4.851, 4.852, 5.156.

Using Chi-Squared Goodness of Fit test for these three distributions, Log-Gamma is

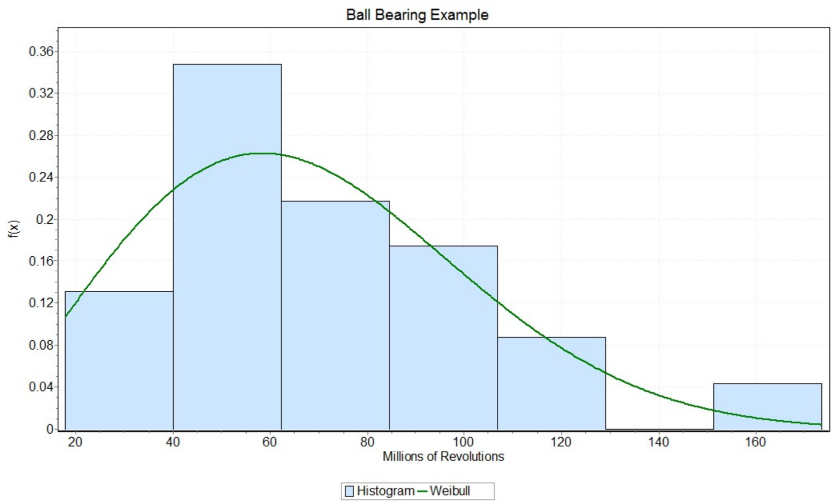
Figure 10.1: Log-Gamma Distribution with $\alpha = 60.6$ and $\beta = 14.6$



the best, followed by Gamma and then Weibull. The three parameter Log Pearson and the two parameter Log-Gamma performed equally well on using the Chi-Squared goodness of fit test. The Log-Gamma distribution was compared to 65 distributions with one to six parameters using Easy Fit [reference] by comparing the Kolmogorov-Smirnov test, Anderson-Darling tests and the Chi-Squared test. The Log-Gamma distribution provided a very good fit especially for a two parameter distribution. The Log-Gamma distribution was a better fit for two out of the tests than the 3 parameter Log-Pearson distribution.

The estimates for the parameters for the Log-Gamma distribution using the Generalized Estimation method with a parametric bootstrap method using the MLE and the method of moments for each parameter.

Figure 10.2: Weibull Distribution with $\alpha = 2.2$ and $\beta = 77.2$



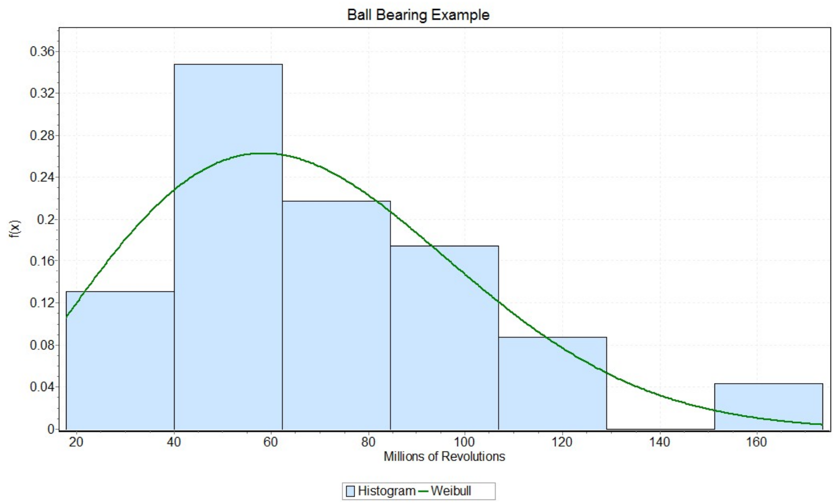
Ball Bearing Example

Notice by talking the log of the data the graph of the distribution is more centered with less skew. Using the MLE's of the shape parameter,$\alpha = 59.17$, and the logscale parameter, $\beta = 14.25$, fitting this data on the log-gamma distribution the 90% confidence for β is $(8.668262, 25.876486)$ with length 17.208.

The true coverage for these parameters is using the simulation with fixed alpha and beta. DO we know the true values of this?

As stated in the introduction sections, Lawless (1982) [**?**], Lawless (1980) [**?**] and Prentice (1974) [**?**] used a "transformed log-gamma distribution" which is a re-parameterizations of the the Log-Pearson / Log-Gamma distribution that is directly derived from the gamma distribution. With this clever parametrization, the log-gamma distribution has the Normal and the Extreme Value as distributions in this new family. Lifetime analysis can be analyzed with the assumption of Weibull

Figure 10.3: Gamma Distribution with $\alpha = 3.7$ and $\beta = 19.4$



or LogNormal with an adjust of the K parameter.

Starting with a one parameter gamma distribution, $T \sim Gamma(k)$, where k is a shape parameter or sometimes called a index parameter in this case.

$$f(t) = \frac{t^{k-1}e^{-t}}{\Gamma(k)} \text{where } t > 0 \text{ and } k > 0. \tag{10.1}$$

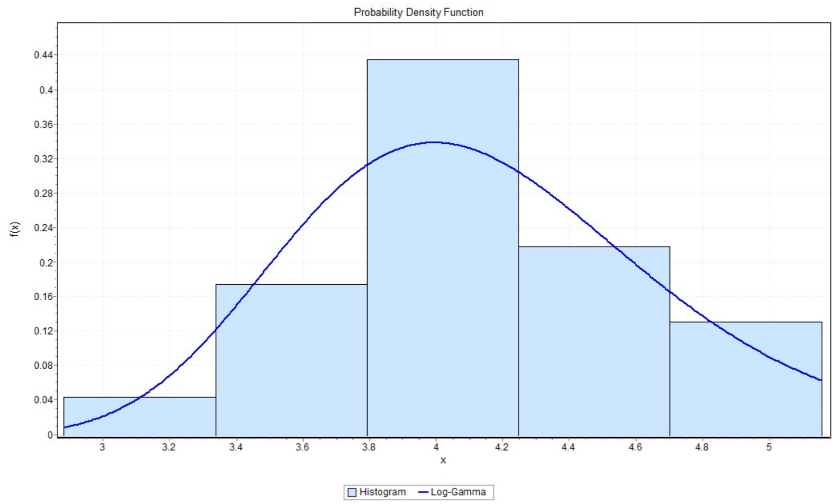
$$\text{Let } W = \log T \text{ or } T = \exp(W). \quad \frac{dT}{dW} = e^W$$

. Substitute into equation 8.1.

$$f_W(w) = \frac{(e^w)^{k-1}e^{-e^w}}{\Gamma(k)} \cdot e^w = \frac{\exp(kw - e^w)}{\Gamma(k)} \tag{10.2}$$

Finally, we want to substitute $W = k^{-1/2}Z + \log K$ and $dW = k^{-1/2}dZ$ because as k approaches infinity, the mean and variance of become infinite. This is the motivation

Figure 10.4: Log-Gamma Distribution using log lifetimes with $\alpha = 112.4$ and $\beta = 79.4$



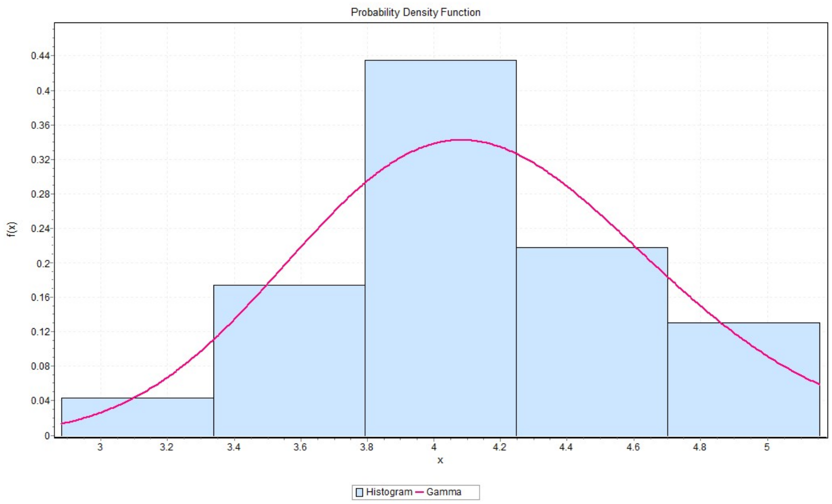
for the new variate in the form of W.

$$f_Z(z;k) = \frac{\exp\left\{k(k^{-\frac{1}{2}}z + \log k) - \exp(k^{-\frac{1}{2}}z + \log k)\right\} k^{-\frac{1}{2}}}{\Gamma(k)}$$

$$= \frac{k^{k-1/2}}{\Gamma(k)} \exp\left[k^{1/2}z - k\exp(k^{-1/2}z)\right], \quad -\infty < z < \infty. \qquad (10.3)$$

When $k = 1$ we get the Extreme Value distribution and when $k = \infty$, we get the Normal distribution corresponding to Weibull and the log-Normal distributions for T. Two important comment by Lawless; the pdf f(z;k) changes very little as as k increases from 1 to ∞ and except for very large sample sizes, k is difficulty to estimate precisely. This is the preferred parametrization for several authors in literature mainly because it is now a little easier to find MLE's confidence intervals as well as hypothesis testing. Obviously the domain contains all real numbers as opposed to $1, \infty$ such as the Log-Gamma distribution. We can now add a location and scale parameter to the

Figure 10.5: Gamma Distribution using log lifetimes with $\alpha = 60.6$ and $\beta = 14.6$



distribution by letting...

$$W_1 = \frac{Y - u}{b} \sim LG(\alpha, \beta)$$

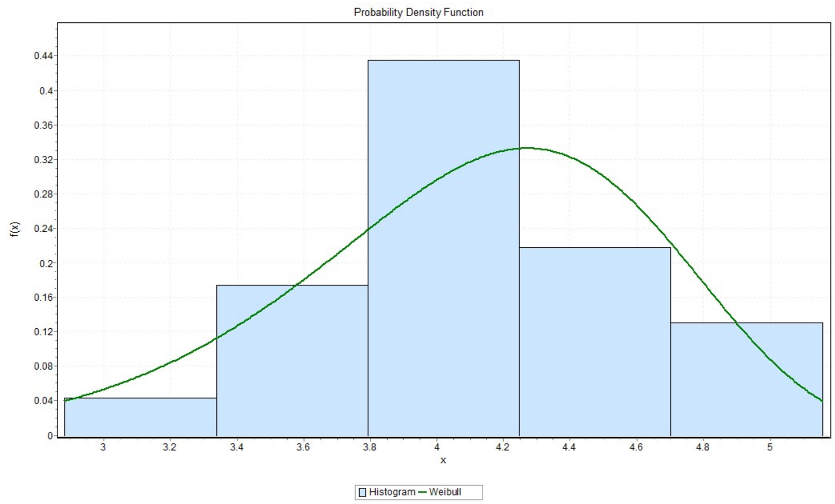
where $u = \log \alpha$ and $b = \beta^{-1}$

If we let $W = \frac{Y-\mu}{\sigma}$, where $\sigma = b/\sqrt(k)$ and $\mu = u + b \log k$.

Some notes on the ball bearing. Using the numbers from Lawless, 1980, we get MLE for $\widehat{\mu} = 4.230$ and $\widehat{\sigma} = 0.510$ using $k = 10.6$. Suing the Log-Gamma distribution I got $\alpha = \log(112) = 4.718$ and $\beta = 0.01259 * \sqrt(10.6) =$

Conditional and unconditional methods produced almost exactly the same results although Lawless has shown theoretical grounds for preferring the conditional approach. The conditional distribution were conditioned on the observed value of the ancillary statistic for the sample in question. $a_i = (y_i - \bar{\mu})/\tilde{\sigma},\ (i = 1, ...n)$. According

Figure 10.6: Weibull Distribution using log lifetimes with $\alpha = 8.6$ and $\beta = 4.3$



to Lawless and Prentice, approximations are not good unless unless the sample sizes are fairly large since we resort to using maximum likelihood-based large sample procedures even after transforming the log-gamma distribution. These result were better but have limitations. Furthermore, approximating k has its own difficulties where good inference procedures are often difficult when k is unknown, assumed based on agreement or a range of plausible values.

Figure 10.7: Log-Gamma vs.Gamma vs. Weibull

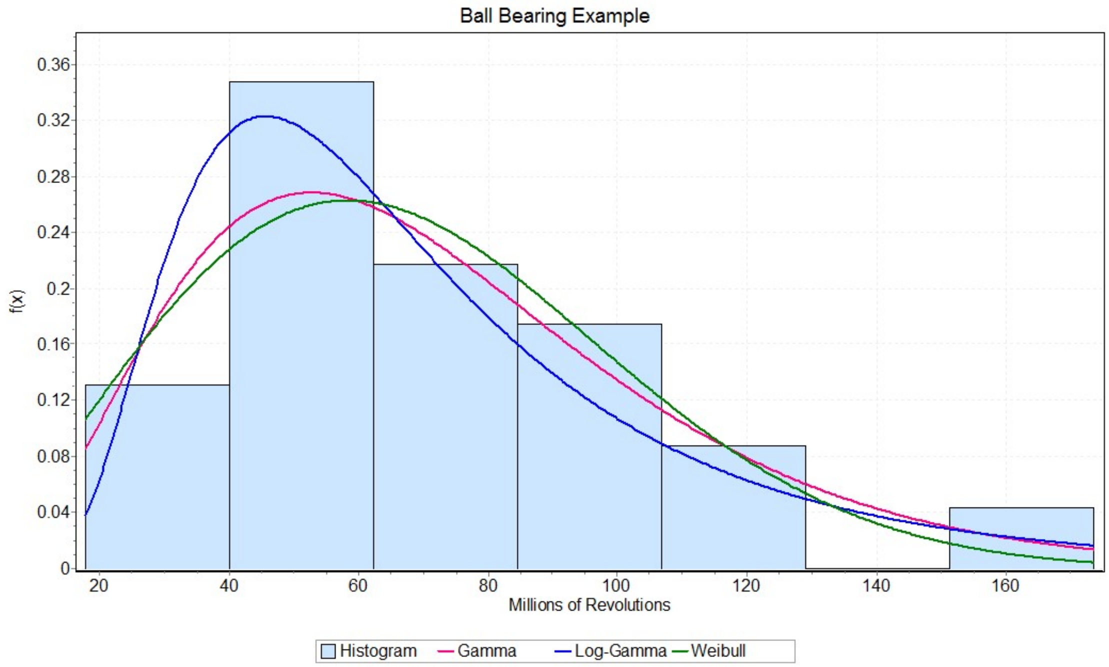


Figure 10.8: Log-Gamma vs.Gamma vs. Weibull using log lifetimes

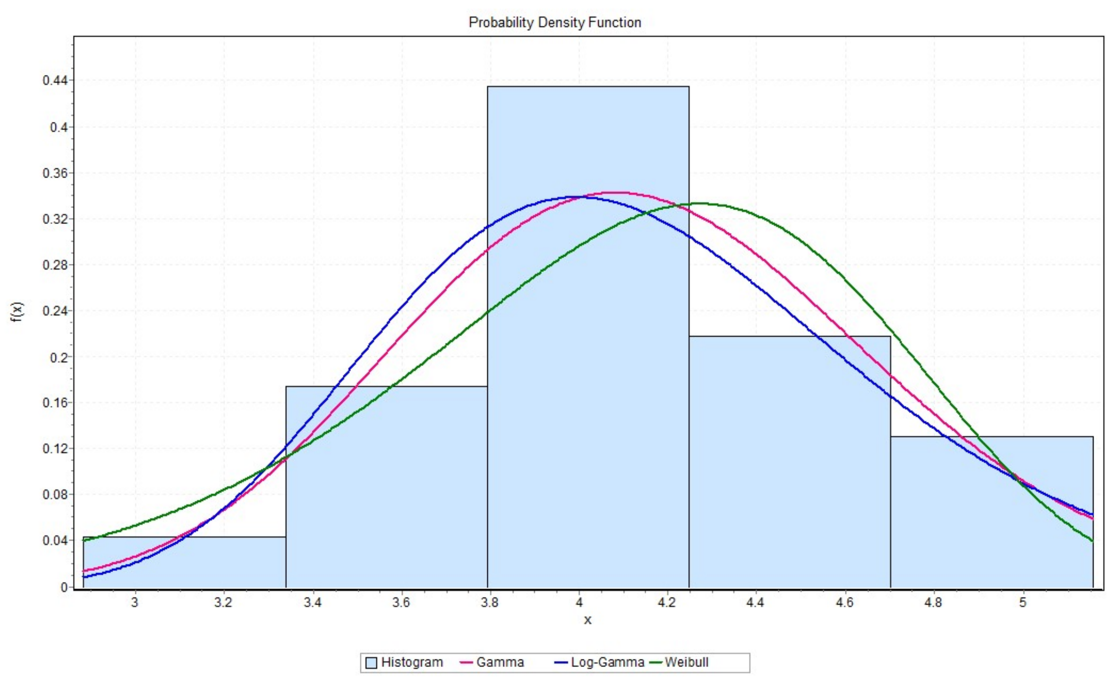


Figure 10.9: Ball Bearing CI with Data and Log Data

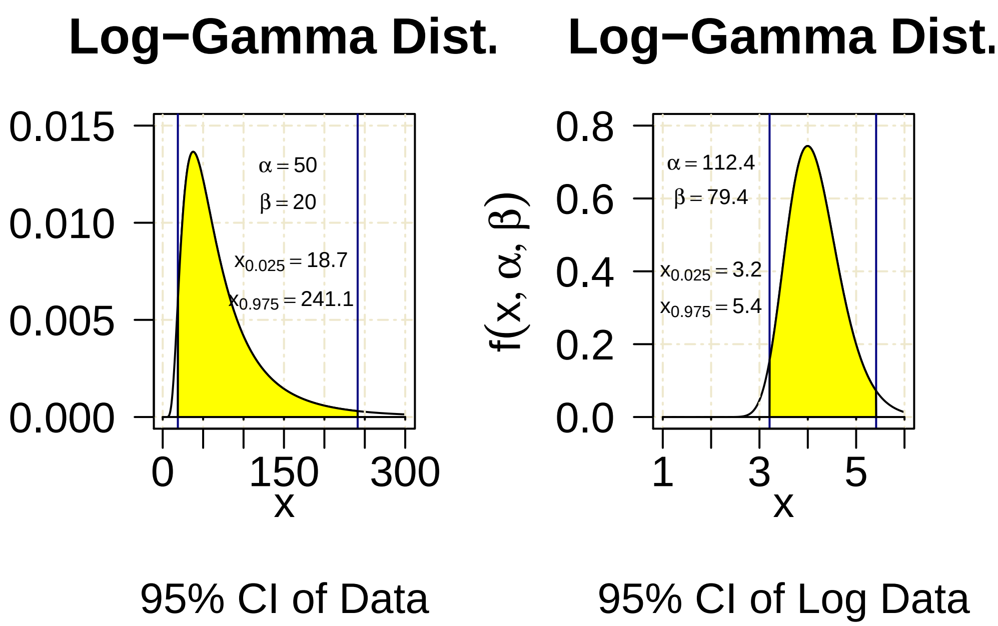


Table 10.1: Fit tests for Log-Gamma Distribution

Log-Gamma					
Kolmogorov-Smirnov					
Sample Size	23				
Statistic	0.08804				
P-Value	0.9871				
Rank	3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.21645	0.24746	0.2749	0.30728	0.32954
Reject?	No	No	No	No	No
Anderson-Darling					
Sample Size	23				
Statistic	0.22898				
P-Value	none				
Rank	3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	No	No	No	No	No
Chi-Squared					
Sample Size	23				
Statistic	0.85779				
P-Value	0.65123				
Rank	18				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	3.2189	4.6052	5.9915	7.824	9.2103
Reject?	No	No	No	No	No

CHAPTER 11

TESTING THE POWER OF THE PARAMETERS

This method can also be used to test real data sets for the mean, variance, and each of the parameters, α and β. For example, the p-value for testing the mean of the log-gamma distribution can be calculated by modifying the code used for the simulation by testing the data set and the hypothesized value for the parameter or functions of parameters such as the coefficient of variance.

The β Parameter

Recall the hypothesis test for the β is:

$$H_0 : \beta \geq \beta_0 \;\; vs. \;\; H_1 : \beta < \beta_0$$

The parameters α and β will be called a and b, respectively, while α and β represent the probability of a type I error and the probability of a type II error, respectively.

Figure 11.1: Power of the generalized test for Log-Scale parameter, $\alpha = 0.05$.

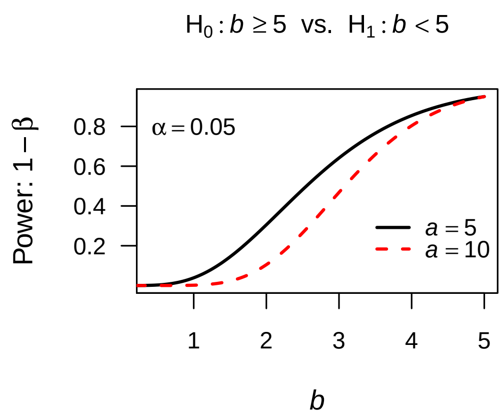
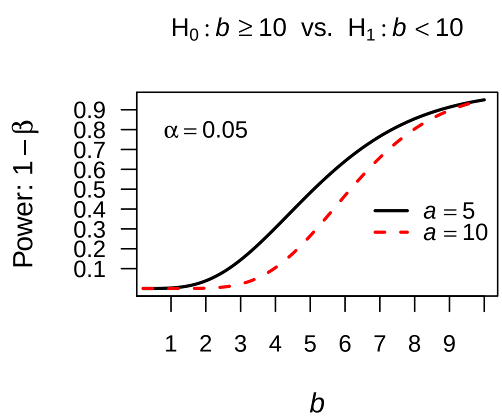


Figure 11.2: Power of the generalized test for Log-Scale parameter, $\alpha = 0.05$.



Testing the μ_0 value for data sets

Recall the hypothesis test for the μ is:

$$H_0 : \mu \geq \mu_0 \;\; vs. \;\; H_1 : \mu < \mu_0$$

where $\mu = (1 - 1/\beta)^{-\alpha}$. Sample sizes as low as $n = 5$ were too small to give accurate results.

Table 11.1: Case $H_0 : \mu \geq \mu_0,\ \alpha = \{2, 5, 10\}$

μ_0	$\alpha = 3, \beta = 5$			μ_0	$\alpha = 5, \beta = 5$			μ_0	$\alpha = 5, \beta = 10$		
Samples	5	10	20		5	10	20		5	10	20
2	0.777	0.998	0.994	2.5	0.927	0.977	0.999	4	0.699	0.784	0.995
3	0.412	0.565	0.466	3.0	0.727	0.728	0.878	6	0.504	0.499	0.950
4	0.312	0.203	0.167	3.5	0.548	0.391	0.430	8	0.351	0.264	0.813
5	0.268	0.071	0.118	4.0	0.369	0.192	0.161	10	0.231	0.121	0.611
6	0.255	0.050	0.066	4.5	0.252	0.094	0.063	12	0.157	0.052	0.392
7	0.200	0.043	0.050	5.0	0.222	0.063	0.025	14	0.103	0.021	0.221
8	0.134	0.047	0.044	5.5	0.219	0.047	0.014	16	0.062	0.008	0.106
9	0.093	0.022	0.036	6.0	0.205	0.032	0.006	18	0.041	0.003	0.051
10	0.055	0.021	0.033	7.0	0.188	0.013	0.002	20	0.025	0.001	0.022

CHAPTER 12

CONCLUSION

The Log-Gamma Distribution is used in hydrology, finance and reliability testing. Accurate testing of the parameters and functions of the parameters is difficult using current methods based on the Maximum Likelihood Estimates and the Method of Moments especially for small sample sizes. Significantly more accurate results using the Generalized Estimation method for typically used values in the parameters spaces for sample sizes as low as 10 were produced for each parameter, the mean, the variance, and the coefficient of variance. Other functions of parameters can be testing using the basic code and modifying the simulation part of the code. The length of computation time was significantly increased for more complicated functions of parameters.

There were computational difficulties testing the shape parameter α when the α value was larger than the β value. The test was not valid for α larger than β value. This computational problem did not occur for functions of the parameters which included the α and β parameters. The Generalized approach introduced by Weerahandi for two parameter distributions is accomplished for each distribution is dependent on the distribution of the sufficient statistics. Transforming the standard sufficient statistics into independent sufficient statistics is not an easy task. More research for an easier, systemic approach for this step is warranted in order to tackle other two parameter distributions such as the Weibull or the Laplace distribution.

BIBLIOGRAPHY

[1] Allella, F., Chiodo, E., Lauria, D. and Pagano, M., Negative Log-Gamma distribution for data uncertainty modelling inreliability analysis of complex systems, *International Journal of Quality and Reliability Management*, **18**, No. 3, 307-232, (2001).

[2] Arora, K. and Singh, V.P., On the method of maximum likelihood estimation for the log-Pearson type 3 distribution, *Stochastic Hydrology and Hydraulics*, Springer-Verlag, **2**, No. 2, 155-160, (1988).

[3] Barnard, G. A., Comparing the Means of Two Independent Samples. *Applied Statistics* **33**, No. 3, 266-271, (1984).

[4] Basu, D., On Statistics Independent of Complete Sufficient Statistic., *Sankhya* , **15**, No. 4, 377-380, (1955).

[5] Balakrishnan, N. and Chan P. S., Log-gamma order statistics and linear estimation of parameters. *Computational Statistics and Data Analysis* (1994).

[6] Balakrishnan, N., Chan, P. S., Maximum likelihood estimation for the loggamma distribution under Type-II censored samples and associated inference. Balakrishnan, N. J., ed. *Advances in Reliability.* Boca Raton, FL, CRC Press, 409-437,(1995a).

[7] Balakrishnan, N., Chan, P. S. Maximum likelihood estimation for the three parameter log-gamma distribution under Type-II censoring. Balakrishnan, N. J., ed. *Advances in Reliability.* Boca Raton, FL: CRC Press, 439-451, (1995b).

[8] Bebu, I. and Mathew, T., Confdence intervals for limited moments and truncated moments in normal and lognormal models. *Statistics and Probability Letters*, **79**, 375-380, (2009).

[9] Benson, M., Uniform Flood-Frequency Estimating Methods for Federal Agencies, *Water Resources Research*, **4**, No. 5, 891-908, (1968).

[10] Bhaumik, D. K., Kapur, K., and Gibbons, R. D., Testing Parameters of a Gamma Distribution for Small Samples. *Technometrics*, **51**, 326-334, (2009).

[11] Bobee, B., The Log Pearson Type 3 Distribution and Its Application in Hydrology. *Water Resources Research*, **11**, No. 5, 681-689, (1975).

[12] Bobee, B. and Ashkar, F., The generalized method of moments as applied to the log Pearson type 3 *J. Hydraul. Eng. Am. Soc. Civ. Eng.*, (1988).

[13] Buntao, N., and Niwitpong, S., Confidence intervals for the difference of coefficients of variation for lognormal distributions and delta-lognormal distributions. *Applied Mathematical Sciences*,**6**, No. 134, 6691-6704, (2012).

[14] Chen, Y., and Zhou, X. Generalized confidence intervals for the ratio or difference of two means for lognormal populations with zeros, *UW Biostatistics Working Paper Series*, Working Paper 296.(2006).

[15] Condie, R., The log Pearson type 3 distribution: The T-year event and its asymptotic standard error by maximum likelihood theory., *Water Resour. Res.*, **13**, No. 6, 987-991, (1977).

[16] Consul, P.C. and Jain, G.C., On the log-gamma distribution and its properties, *Statistische Theorie*, **12**, 100-106, (1971).

[17] C. Dutang, V. Goulet and M. Pigeon actuar: An R Package for Actuarial Science. *Journal of Statistical Software*, **25**, NO. 7, 1-37. (2008). URL http://www.jstatsoft.org/v25/i07

[18] Engelhardt, M., and Bain, L. J., Uniformly Most Powerful Unbiased Tests on the Scale Parameter of a Gamma Distribution With a Nuisance Shape Parameter, *Technometrics*, **19**, 7781, (1977).

[19] Paul Embrechts, Claudia Kliippelberg, and Thomas Mikosch, Modeling Extremal Events, Springer Berlin Heidelberg, 283-370, (1997).

[20] H. Alden Foster, Theoretical Frequency Curves and Their Application to Engineering Problem, *Transactions of the American Society of Civil Engineers*, **88**, No. 1, 142-173, (1924).

[21] Godambe, V. P., On sufficiency and ancillarity in the presence of a nuisance parameter, *Biometrika* **67**, 155-162, (1980).

[22] Godambe, V. P., Conditional likelihood and unconditional optimum estimating equations., *Biometrika* **63**, 277-284, (1976).

[23] Gamage, J., Mathew, T., and Weerahandi, S. (2004). Generalized p-values and generalized confidence regions for the multivariate BehrensFisher problem and MANOVA, *Journal of Multivariate Analysis*, **88**, 177-189. (2004)

[24] Grice, J. V., and Bain, L. J. , Inferences Concerning the Mean of the Gamma Distribution, *Journal of the American Statistical Association*, **75**, No. 372, 929-933, (1980).

[25] Hannig, J., Iyer, H. and Patterson, P., Fiducial Generalized Confidence Intervals, *Journal of the American Statistical Association*, **101**, No. 473, 254-269, (2006).

[26] Johnson, Kotz and Balakrishnan, Continuous Univariate Distribtutions Volume 1, 2nd edition, New York, Wiley and Sons, (1994).

[27] Krishnamoorthy, K., Mathew, T., and Mukherjee, S. , Normal Based Methods for a Gamma Distribution: Prediction and Tolerance Intervals and Stress-Strength Reliability, *Technometrics*, **50**, 69-78, (2008).

[28] Krishnamoorthy, K., Mathew, T., and Ramachandran, G. Generalized P-values and confidence intervals : A novel approach for analyzing lognormally distributed exposure data, *Journal of Occupational and Environmental Hygiene*, **3**, 642-650, (2006).

[29] Krutchkoff, R. G., One-way fixed effects analysis of variance when the error variances may be unequal *Journal of Statistical and Computational Simulations*, **30**, 259-271, (1988).

[30] Chien-Tai Lin , Sam J. S. Wu and N. Balakrishnan, Inference for Log-Gamma Distribution Based on Progressively Type-II Censored Data, *Communications in Statistics - Theory and Methods*, **35:7**, 1271-1292, (2006).

[31] Lawless, J. F., Inference in the Generalized Gamma and Log Gamma Distributions, *Technometrics*, **22**, No. 3, 409-419, (1980).

[32] Lawless, J. F. *Statistical Models and Methods for Lifetime Data* New York, John Wiley, (1982).

[33] Lee, J. C., and Lin, S. H. Generalized confidence intervals for the ratio of means of two normal populations. *Journal of Statistical Planning and Inference*,**123**, 49-60, (2004).

[34] Lieblein, J., Zelen, M., Statistical investigation of the fatigue life of deep groove ball bearings. *Journal of Resesearch at the National Bureau of Standards*, **57**, No. 5, 273-316, (1956).

[35] Linnik, Y. Statistical Problems with Nuisance Parameters, Translation of Mathematical monograph. *New York: American Mathematical Society*, No. 20, (1968).

[36] Harry Franklin Martz, Ray A. Waller, Bayesian Reliability Analysis, New York: Wiley, (1982).

[37] Matalas, N. C., and J. R. Wallis , Eureka! It fits a Pearson type: 3 distribution, *Water Resources Research*, **9**, No. 2, 281289, (1973).

[38] Nozdryn-Plotinicki, M.J. and Watt, M.E., Assessment of fitting techniques for the Log-Pearson type 3 distribution using Monte Carlo simulation, *Water Resources Research*, **15**, No. 3, 714-718, (1979).

[39] Olshen, A. C., Transformations of the Pearson Type III Distribution. *Annals of Mathematical Statistics*, **9**, No. 3, 176-200, (1938).

[40] Phien, H.N. and Hira, M.A., Log Pearson type-3 distribution: parameter estimation, *Journal of Hydrology*, **64**, 25-37, (1983).

[41] Prentice, R. L., A Log Gamma Model and Its Maximum Likelihood Estimation, *Biometrika*, **61**, 539-544, (1974).

[42] Rao, D.V., Fitting log Pearson type 3 distribution by maximum likelihood. Paper presented at the International Symposium on Flood frequency and risk analyses, held at Louisiana State University, Baton Rouge, LA (1986).

[43] Rice, W. R. and Gaines, S. D., One-way analysis of variance with unequal variances, *Proceedings of the National Academy of Sciences*, **86**, 8183-8184, (1989).

[44] Roy, A. and Mathew, T., A Generalized confidence limit for the reliability function of a two-parameter Exponential distribution, *Journal of Statistical Planning and Inference*, **128**, 505-517, (2005).

[45] Shiue, W. K., Bain, L. J. and Engelhardt, M., Test of Equal Gamma-Distribution Means With Unknown and Unequal Shape Parameters, *Technometrics*, **30**, No. 2, 169-174, (1988).

[46] Tian, L., Inferences on the mean of zero-inflated lognormal data: the generalized variable approach, *Statistics in Medicine*, **24**, 3223-3232, (2006).

[47] Tsui, K. and Weerahandi, S., Generalized p-values in significance testing of hypotheses in the presence of nuisance parameters, *Journal of the American Statistical Association*, **84**, 602-607, (1989).

[48] Weerahandi, S., Generalized Point Estimation with Application to Small Response Estimation, *Communications in Statistics - Theory and Methods*, **41**, 4069-4095, (2012).

[49] Weerahandi, S., Testing regression equality with unequal variances, *Econometrica*, **55**, 1211-1215, (1987).

[50] Weerahandi, Samaradasa, ANOVA under Unequal Error Variances, *Biometrics*, **51**, No. 2, 589-599, (1995).

[51] Weerahandi, Samaradasa, Exact Statistical Methods for Data Analysis, Springer-Verlag New York Inc., (1995).

[52] Weerahandi, S. Generalized Confidence Intervals *Journal of the American Statistical Association*, **88**, No. 423, 899-905. (1993).

[53] Weerahandi, S and Gamage, J., A General Method of Inference for Two-Parameter Continuous Distributions. *Pending*, , (2014).

[54] W.R.C. (Water Resources Council), A uniform technique for determining flood flow frequencies. *Water Resour. Res.*, Washington, D.C., Bull. 15.(1967).

[55] W.R.C. (Water Resources Council), Flood Flow Frequencies. *Water Resour. Res.*, Washington, D.C., Bull. 17B. (1981).

VITA

Graduate College
University of Nevada, Las Vegas

Joseph McDonald

Home Address:
2105 Paganini Ave
Las Vegas, Nevada 89052

Degrees:
Bachelor of Science, Secondary Education, 1991
University of Nevada, Las Vegas

Masters of Science, Mathematical Sciences, 1993
University of Nevada, Las Vegas

Doctor of Philosophy, Mathematical Sciences, 2015
University of Nevada, Las Vegas

Dissertation Title: Exact Statical Inferences for Functions of Parameters of the Log-Gamma Distribution

Dissertation Examination Committee:
Chairperson, Dr. Malwane Ananda, Ph.D.
Committee Member, Dr. Amei Amei, Ph.D.
Committee Member, Dr. Hokwon Cho, Ph.D.
Graduate College Representative, Dr. Daniel Allen, Ph.D.