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A Study of Graphical Permutations

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A STUDY OF GRAPHICAL PERMUTATIONS

by

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ABSTRACT

A STUDY OF GRAPHICAL PERMUTATIONS

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A permutation π on a set of positive integers $\{a_1, a_2, ..., a_n\}$ is said to be graphical if there exists a graph containing exactly a_i vertices of degree $\pi(a_i)$ for each i $(1 \leq i \leq n)$. It has been shown that for positive integers with $a_1 < a_2 < ... < a_n$, if $\pi(a_n) = a_n$ then the permutation π is graphical if and only if the sum $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and $a_n \leq \sum_{i=1}^{n-1} a_i\pi(a_i)$. This known result has been proved using a criterion of Fulkerson, Hoffman, and McAndrew which requires the verification of $\frac{1}{2}n(n+1)$ inequalities. In this paper, we use a criterion of Tripathi and Vijay to give a shorter proof of this result, requiring only the verification of n inequalities. We also use this criterion to provide a similar result for permutations π such that $\pi(a_{n-1}) = a_n$. We prove that such a permutation is graphical if and only if the sum $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and $a_na_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i\pi(a_i)$. We also consider permutations such that $\pi(a_n) = a_{n-1}$, and then, more generally, those such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$.

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INTRODUCTION

We begin with several basic concepts and definitions from graph theory, as can be found in [1]. A graph G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of V called edges. If u and v are vertices of G that are joined by an edge $\{u, v\}$, often denoted by uv or vu for simplicity, then we say that u and v are adjacent, and that vertex u and edge uv are incident with each other. Similarly, vertex v and edge uv are also incident with each other. To emphasize the vertex set and edge set refer to a specific graph G, we often write $V(G)$ and $E(G)$, respectively. The number of vertices in a graph G is called the order of G and the number of edges is called the size of G. If a graph G has order n, then the size of G is at most $\binom{n}{2}$.

The degree of a vertex v in a graph G is the number of edges that are incident with v. Suppose a graph G has vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $deg(v_i) = d_i$ for each i $(1 \leq i \leq n)$. Then the sequence $d_1, d_2, \ldots, d_n$ is called the degree sequence of G. We note that it is standard in many of the theorems we will consider to list the degree sequence for a given graph in nonincreasing order, so that $d_1 \geq d_2 \geq \cdots \geq d_n$.

A finite sequence s of nonnegative integers is said to be a graphical sequence, or just graphical, if s is a degree sequence for some graph. Note that every graph must have a degree sequence, but not every sequence s of nonnegative integers is graphical. Two necessary conditions are known for a given sequence s to be graphical. First, since each edge must be adjacent to exactly two vertices, each edge contributes to the degree of two different vertices. Thus if we add the values in the degree sequence, each edge is counted twice. The necessary condition, then, is that the sum $\sum_{i=1}^{n} d_i$ is even. Next, if a graph G has order n, then each vertex has at most $n-1$ other vertices that it may be joined to by an edge. Therefore, the second necessary condition is that $d_i \leq n-1$ for each i $(1 \leq i \leq n)$.

It is important to note that while these are necessary conditions, it is possible to have a sequence s of nonnegative integers satisfying both of these conditions which, in fact, is not a graphical sequence. Thus we do not yet have sufficient conditions for when a sequence s is guaranteed to be graphical.

There are, however, known conditions which are both necessary and sufficient for a finite sequence of nonnegative integers to be graphical. One well-known result is due to Havel [6] and Hakimi [4], and is often referred to as the Havel-Hakimi Theorem, although they each gave independent proofs of this result.

Theorem 1 (Havel-Hakimi Theorem). *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$, where $n \geq 2$ and $d_1 \geq 1$, is graphical if and only if the sequence*

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

This theorem provides a method for determining if a given finite sequence of nonnegative integers is graphical. By repeating the algorithm outlined in the theorem, if a sequence is eventually obtained which is known to be graphical, then the original sequence must also be graphical. In particular, if we obtain a sequence in which every term is zero, this is clearly graphical as it would represent the graph with only vertices and no edges.

For example, consider the following sequence.

$$s : 7, 7, 7, 6, 6, 5, 5, 5, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2.$$

We will use the Havel-Hakimi Theorem to illustrate that s is graphical. Let s_n denote the sequence after n applications of the process described in the theorem. In many cases we must reorder the sequence s_n, so that the terms are once again nonincreasing, in order to be able to correctly continue the process outlined in the theorem. Let s'_n denote the sequence s_n after it has been reordered to be nonincreasing. Then by the Havel-Hakimi Theorem, we have the following:

$$s : 7, 7, 7, 6, 6, 5, 5, 5, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2$$

$$s_1 : 6, 6, 5, 5, 4, 4, 4, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2$$

$$s'_1 : 6, 6, 5, 5, 5, 5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 2, 2$$

$$s_2 : 5, 4, 4, 4, 4, 3, 4, 4, 4, 4, 3, 3, 3, 3, 2, 2$$

$$s'_2 : 5, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2$$

$$s_3 : 3, 3, 3, 3, 3, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2$$

$$s'_3 : 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2$$

$$s_4 : 3, 3, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2$$

$$s'_4 : 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2$$

We continue the process until we reach the sequence

$$s_{14} : 0, 0, 0, 0$$

and conclude that the original sequence s is in fact graphical. An example of a graph with degree sequence s is shown in Figure 1.1.

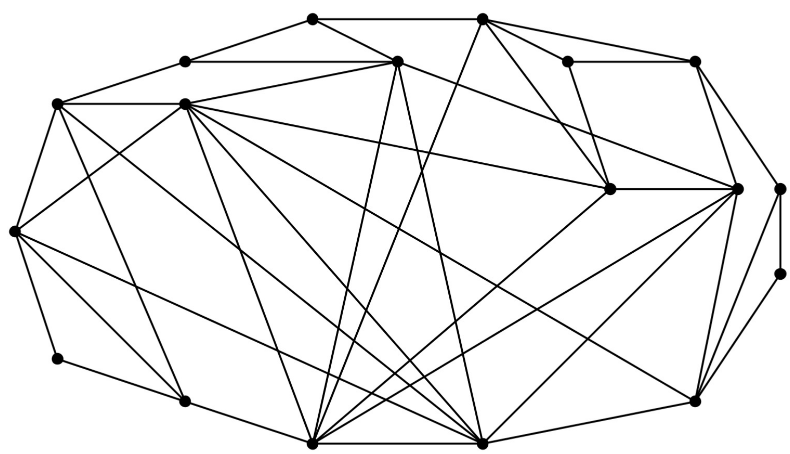

Figure 1.1: Graph with degree sequence $s : 7, 7, 7, 6, 6, 5, 5, 5, 5, 5, 4, 4, 3, 3, 3, 3, 3, 2, 2$.

It is not necessary, however, to reach a sequence consisting of only zeros in order to conclude that the sequence is graphical. It is only necessary to obtain a sequence that is known to be graphical. For example, the graph shown in Figure 1.2 shows that the sequence $s_{10} : 1, 1, 1, 1, 1, 1, 1, 1$ is graphical. Therefore, in the previous example, we could have concluded at this step that the original sequence s is graphical.

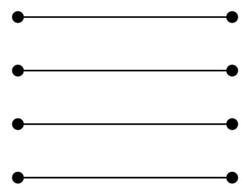

Figure 1.2: Graph with degree sequence $s_{10} : 1, 1, 1, 1, 1, 1, 1, 1$.

An added benefit of the Havel-Hakimi Theorem is that by following the process outlined in the theorem in reverse, it provides a method for constructing a graph with a desired degree sequence. For example, suppose we know that the sequence s_4 is graphical, and we have a graph with s_4 as its degree sequence. We consider the previous step in the process which led to this sequence. In particular, we want to know which value was removed, and which values were reduced as a result. Note that in this case, a 4 was removed from the sequence, and thus the next four values in the sequence were reduced by 1, as shown below.

$$s_3' : \cancel{4},\ \underline{4,\ 4,\ 3,\ 3,}\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 2,\ 2$$

$$s_4 : \underline{3,\ 3,\ 2,\ 2,}\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 2,\ 2$$

This tells us that to construct a graph with degree sequence s_3' from a graph with degree sequence s_4, we will add one new vertex which will have degree 4, and this new vertex will be adjacent to two vertices that previously had degree 3, which will now have degree 4, and two vertices that previously had degree 2, which will now have degree 3. This results in a graph with degree sequence s_3'. By continuing this process back to the original sequence s, we can construct a graph with degree sequence s by adding one new vertex at each step.

Another result that gives necessary and sufficient conditions for a sequence to be graphical is that of Erdös and Gallai [2].

Theorem 2 (Erdös-Gallai Theorem). *A sequence $s : d_1, d_2, \ldots, d_p$ $(p \geq 2)$ of non-negative integers with $d_1 \geq d_2 \geq \cdots \geq d_p$ is graphical if and only if $\sum_{k=1}^{p} d_k$ is even and for each integer n with $1 \leq n \leq p-1$,*

$$\sum_{k=1}^{n} d_k \leq n(n-1) + \sum_{k=n+1}^{p} \min\{n, d_k\}.$$

We will demonstrate the use of the Erdös-Gallai Theorem on the same sequence as in the previous example,

$$s : 7, 7, 7, 6, 6, 5, 5, 5, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2.$$

First note that the sum of the degrees is 82 which is even, so the parity condition is satisfied. Then in order to use the Erdös-Gallai Theorem to conclude that the sequence is graphical, we must verify that the inequality stated in the theorem holds for each value n such that $1 \leq n \leq 17$.

For $n = 1$, we have $7 \leq 17 = \sum_{k=2}^{18} \min\{1, d_k\}$.

For $n = 2$, we have $14 \leq 34 = 2(1) + \sum_{k=3}^{18} \min\{2, d_k\}$.

For $n = 3$, we have $21 \leq 49 = 3(2) + \sum_{k=4}^{18} \min\{3, d_k\}$.

$$\text{For } n = 4, \text{ we have } 27 \leq 60 = 4(3) + \sum_{k=5}^{18} \min\{4, d_k\}.$$

$$\text{For } n = 5, \text{ we have } 33 \leq 69 = 5(4) + \sum_{k=6}^{18} \min\{5, d_k\}.$$

$$\text{For } n = 6, \text{ we have } 38 \leq 74 = 6(5) + \sum_{k=7}^{18} \min\{6, d_k\}.$$

$$\text{For } n = 7, \text{ we have } 43 \leq 81 = 7(6) + \sum_{k=8}^{18} \min\{7, d_k\}.$$

$$\text{For } n = 8, \text{ we have } 48 \leq 90 = 8(7) + \sum_{k=9}^{18} \min\{8, d_k\}.$$

$$\text{For } n = 9, \text{ we have } 53 \leq 101 = 9(8) + \sum_{k=10}^{18} \min\{9, d_k\}.$$

$$\text{For } n = 10, \text{ we have } 58 \leq 114 = 10(9) + \sum_{k=11}^{18} \min\{10, d_k\}.$$

$$\text{For } n = 11, \text{ we have } 62 \leq 130 = 11(10) + \sum_{k=12}^{18} \min\{11, d_k\}.$$

$$\text{For } n = 12, \text{ we have } 66 \leq 148 = 12(11) + \sum_{k=13}^{18} \min\{12, d_k\}.$$

$$\text{For } n = 13, \text{ we have } 69 \leq 169 = 13(12) + \sum_{k=14}^{18} \min\{13, d_k\}.$$

$$\text{For } n = 14, \text{ we have } 72 \leq 192 = 14(13) + \sum_{k=15}^{18} \min\{14, d_k\}.$$

$$\text{For } n = 15, \text{ we have } 75 \leq 217 = 15(14) + \sum_{k=16}^{18} \min\{15, d_k\}.$$

$$\text{For } n = 16, \text{ we have } 78 \leq 244 = 16(15) + \sum_{k=17}^{18} \min\{16, d_k\}.$$

$$\text{For } n = 17, \text{ we have } 80 \leq 274 = 17(16) + \sum_{k=18}^{18} \min\{17, d_k\}.$$

Since all of these inequalities are satisfied, we can conclude by the Erdös-Gallai Theorem that the sequence s is graphical. In general, for a sequence of length p, we must verify $p-1$ inequalities in order to use the Erdös-Gallai Theorem.

In [8], Tripathi and Vijay give a new result, which states that the number of values for which the inequality in the Erdös-Gallai Theorem must be verified in order to conclude that a sequence is graphical can be reduced. In fact, in the case that the degree sequence contains any values that are repeated multiple times, we must only check the inequality in the Erdös-Gallai Theorem at the end of each segment of repeated values. This result is stated formally below. We shall use the notation $(d)_m$ to mean m occurrences of the value d.

Theorem 3 (EG Shortcut). *Let $s : (d_1)_{m_1}, (d_2)_{m_2}, \ldots, (d_\ell)_{m_\ell}$ be a sequence where $d_1 > d_2 > \cdots > d_\ell$, and $m_k \geq 1$ for each k $(1 \leq k \leq \ell)$, with $m_1 + m_2 + \cdots + m_\ell = p$. For each $k = 1, 2, \ldots, \ell$, let $\sigma_k = \sum_{i=1}^{k} m_i$, and let $S_{r,t} = \sum_{i=r}^{t} d_i m_i$. Then the sequence s is graphical if and only if $S_{1,\ell}$ is even, and for each $k = 1, 2, \ldots, \ell$,*

$$S_{1,k} = \sum_{i=1}^{k} d_i m_i \leq \sigma_k(\sigma_k - 1) + \sum_{i=k+1}^{\ell} m_i \cdot \min\{\sigma_k, d_i\}.$$

This theorem is especially useful in the cases where the sequence contains multiple repeated values, or values that are repeated multiple times, because it greatly reduces the number of inequalities from the Erdös-Gallai Theorem that must be verified. Because of this fact, we refer to the theorem as the EG Shortcut theorem. For a sequence of length p, instead of verifying $p - 1$ inequalities, we must now only verify ℓ inequalities, where ℓ is the number of distinct values in the sequence. For instance, in the previous example illustrating the use of the Erdös-Gallai Theorem, we verified 17 different inequalities. Using this updated "shortcut" version of the theorem, we would only need to check six inequalities in order to conclude that the same sequence is graphical.

As a special case of this idea of graphical sequences, suppose for a set of nonnegative integers $\{a_1, a_2, \ldots, a_n\}$, there exists a graph G with exactly a_i vertices of degree

a_i for each $i = 1, 2, \ldots, n$. Such a set is called equi-graphical. The following necessary and sufficient condition for when a set of nonnegative integers is equi-graphical is provided in [5].

Theorem 4. *(Hansen and Schultz) Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 2$, be a set of positive integers such that $a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} a_i$ is even. Then S is equi-graphical if and only if*

$$a_n \leq a_1^2 + a_2^2 + \cdots + a_{n-1}^2.$$

For example, consider the set $S = \{2, 3, 7\}$. We can use the above theorem to determine if the set S is equi-graphical. That is, to determine if there exists a graph with the degree sequence $s : 7, 7, 7, 7, 7, 7, 7, 3, 3, 3, 2, 2$. We first note that the sum of the values is even. Next we must check the inequality stated in the theorem, and we have that

$$7 \leq 13 = 2^2 + 3^2.$$

Since the conditions of the theorem hold, we conclude that the sequence s is graphical, and thus the set $S = \{2, 3, 7\}$ is equi-graphical. An example of such a graph is shown in Figure 1.3.

Observe that it is much easier to verify that the sequence above is graphical using this theorem by Hansen and Schultz than by using the criteria found in the Havel-Hakimi Theorem or the Erdös-Gallai Theorem, because it requires only verifying a single inequality. We note, however, that this is a very specialized case and can only be used when we are considering exactly a_i vertices of degree a_i.

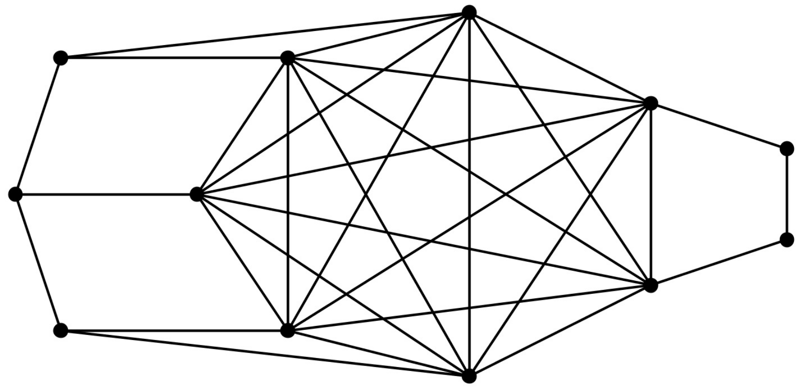

Figure 1.3: Graph representing the equi-graphical set $S = \{2, 3, 7\}$.

We next consider a generalization of this idea. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of distinct positive integers such that $1 \le a_1 < a_2 < \cdots < a_n$, and let π be a permutation of the elements of the set S. We say that the permutation π is a graphical permutation, or that π is graphical, if there exists a graph G containing exactly a_i vertices of degree $\pi(a_i)$ for each i. If a permutation π on a set S as described is graphical, the resulting degree sequence is

$$(a_n)_{\pi^{-1}(a_n)}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

Note that in order to keep our degree sequence listed in nonincreasing order, we need to consider π^{-1}, since each degree a_i will appear $\pi^{-1}(a_i)$ times in the sequence.

As an example of a graphical permutation, consider the set $S = \{2, 4, 5\}$ and the permutation $\pi = (2\ 5\ 4)$. This permutation is graphical if there exists a graph containing exactly 2 vertices of degree 5, 5 vertices of degree 4, and 4 vertices of degree 2. In other words, the permutation π is graphical if the following sequence s is graphical:

$$s : 5, 5, 4, 4, 4, 4, 4, 2, 2, 2, 2.$$

The graph in Figure 1.4 shows that this permutation is graphical.

It should be noted that an equi-graphical set is in fact a graphical permutation of the set in which the permutation being considered is the identity permutation.

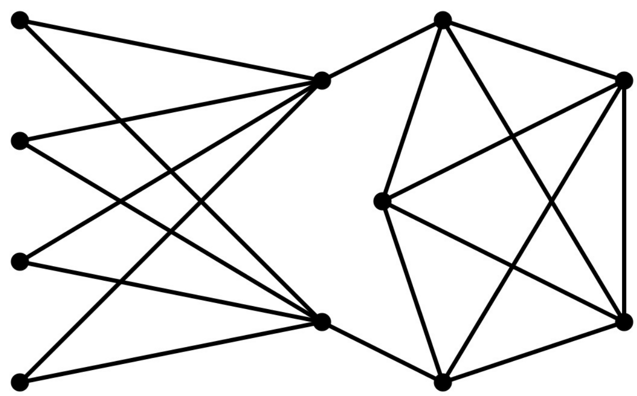

Figure 1.4: Graph representing the permutation $\pi = (2\ 5\ 4)$.

We are particularly interested in the idea of graphical permutations because necessary and sufficient conditions are not yet known for when a permutation in general is graphical. The EG Shortcut theorem proves to be very useful for this problem, because we are always considering sequences that have multiple repeated elements, particularly sequences that have the value $\pi(a_i)$ occurring a_i times, for each i. We will first consider sets of small cardinality and look at specific permutations of these sets. Later, we will look at some permutations more generally.

In [7] certain permutations are studied, namely those permutations π such that

$\pi(a_n) = a_n$. That is, permutations that fix the largest element of the given set. The following characterization of such graphical permutations is provided.

Theorem 5. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_n$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even, and*

$$a_n \leq \sum_{i=1}^{n-1} a_i\pi(a_i).$$

This result is proved in [7] using a theorem of Fulkerson, Hoffman, and McAndrew [3] which gives necessary and sufficient conditions for a sequence to be graphical. We state this theorem below.

Theorem 6. *(Fulkerson, Hoffman, McAndrew) Let $s : d_1, d_2, \ldots, d_n$ be a sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$, where $n \geq 2$. Then s is graphical if and only if for each $k = 1, 2, \ldots, n$, and m with $k + m \leq n$,*

$$\sum_{i=1}^{k} d_i \leq k(n - m - 1) + \sum_{i=n-m+1}^{n} d_i.$$

We would like to point out that this theorem requires the verification of $\frac{1}{2}n(n+1)$ different inequalities for a degree sequence of length n. The benefit of using this result is that it is not necessary to repeatedly determine the minimum of two values, as in the Erdös-Gallai Theorem. Later we give a new proof of Theorem 5 using the EG Shortcut theorem, which requires the verification of significantly fewer inequalities.

Using this newer theorem requires verification of one inequality for each of the different values in the degree sequence. For a degree sequence of length n this will be at most n inequalities, and if the degree sequence contains repeated values, as is the case for these permutations we are interested in, this can be much fewer than n inequalities.

We also provide new results that consider permutations such that $\pi(a_{n-1}) = a_n$ and also those such that $\pi(a_n) = a_{n-1}$. More generally, we consider which element of the set the largest element a_n is sent to by a permutation, and consider those permutations such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$.

CHAPTER 2

GRAPHICAL PERMUTATIONS ON SETS OF SMALL CARDINALITY

In [5], general results are given for permutations on sets of cardinality two and three. These results are stated as the next two theorems.

Theorem 7. *Let $S = \{a, b\}$ be a set of positive integers such that $1 \leq a < b$. Then the permutation:*

(1) $\pi_1 = (a)(b)$ is graphical if and only if a and b have the same parity and $b \leq a^2$.

(2) $\pi_2 = (a\ b)$ is graphical.

Theorem 8. *Let $S = \{a, b, c\}$ be a set of positive integers such that $1 \leq a < b < c$. Then the permutation:*

(1) $\pi_1 = (a)(b)(c)$ is graphical if and only if $a + b + c$ is even and $c \leq a^2 + b^2$.

(2) $\pi_2 = (a\ b)(c)$ is graphical if and only if c is even and $c \leq 2ab$.

(3) $\pi_3 = (a\ c)(b)$ is graphical if and only if b is even.

(4) $\pi_4 = (b\ c)(a)$ is graphical if and only if a is even.

(5) $\pi_5 = (a\ b\ c)$ is graphical if and only if at most one of a, b, and c is odd, and $bc \leq b(b-1) + ab + ac$.

(6) $\pi_6 = (a\ c\ b)$ is graphical if and only if at most one of a, b, and c is odd.

We note that the conditions given for π_5 as stated above are the corrected conditions stated in [9]. The original result in [5] states only the condition that at most one of a, b, and c is odd. But consider the set $S = \{2, 4, 11\}$. This satisfies the condition that at most one of a, b, and c is odd. However, using the permutation π_5 on this set S would result in the sequence

$$s : 11, 11, 11, 11, 4, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2.$$

Using the algorithm of the Havel-Hakimi Theorem, it is easy to see that the sequence s is not graphical. By applying the algorithm three times, we obtain the sequence

$$s_3 : 8, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0.$$

Recall that a necessary condition for a sequence to be graphical is that the sum of the terms must be even. In this sequence s_3, the sum of the terms is odd, and thus s_3 is not graphical. By the Havel-Hakimi Theorem, the original sequence s is also not graphical, and therefore the permutation π_5 on the set S is not graphical. Note that the set $S = \{2, 4, 11\}$ does not satisfy the added condition that $bc \leq b(b-1)+ab+ac$.

The characterization for graphical permutations on a set of cardinality four is given in [7]. Only the proof in the case of permutation π_{24} is given in [7], while the proofs for the remaining cases appear in [9]. The characterization is as follows.

Theorem 9. *Let $S = \{a, b, c, d\}$ be a set of positive integers such that $1 \leq a < b < c < d$. Then*

(1) $\pi_1 = (a)$ is graphical if and only if $d \leq a^2 + b^2 + c^2$ and $a + b + c + d$ is even.

(2) $\pi_2 = (c\ d)$ is graphical if and only if a and b are of the same parity.

(3) $\pi_3 = (b\ d)$ is graphical if and only if a and c are of the same parity.

(4) $\pi_4 = (a\ d)$ *is graphical if and only if* b *and* c *are of the same parity.*

(5) $\pi_5 = (b\ c)$ *is graphical if and only if* a *and* d *are of the same parity.*

(6) $\pi_6 = (a\ c)$ *is graphical if and only if* b *and* d *are of the same parity.*

(7) $\pi_7 = (a\ b)$ *is graphical if and only if* c *and* d *are of the same parity.*

(8) $\pi_8 = (b\ c\ d)$ *is graphical if and only if* $a^2 + bc + cd + bd$ *is even and* $cd \leq a^2 + bc + bd + c(c-1)$.

(9) $\pi_9 = (a\ c\ d)$ *is graphical if and only if* $ac + b^2 + cd + ad$ *is even and* $cd \leq ac + b^2 + ad + c(c-1)$.

(10) $\pi_{10} = (b\ d\ c)$ *is graphical if and only if* $a^2 + bd + bc + cd$ *is even.*

(11) $\pi_{11} = (a\ d\ c)$ *is graphical if and only if* $ad + b^2 + ac + cd$ *is even.*

(12) $\pi_{12} = (a\ d\ b)$ *is graphical if and only if* $ad + ab + c^2 + bd$ *is even.*

(13) $\pi_{13} = (a\ c\ b)$ *is graphical if and only if* $ac+ab+bc+d^2$ *is even and* $d \leq ab+bc+ac$.

(14) $\pi_{14} = (a\ b\ d)$ *is graphical if and only if* $ab + bd + c^2 + ad$ *is even and* $bd \leq ab + bc + ad + b(b-1)$.

(15) $\pi_{15} = (a\ b\ c)$ *is graphical if and only if* $ab+bc+ac+d^2$ *is even and* $d \leq ab+bc+ac$.

(16) $\pi_{16} = (a\ b)(c\ d)$ *is graphical.*

(17) $\pi_{17} = (a\ c)(b\ d)$ *is graphical.*

(18) $\pi_{18} = (a\ d)(b\ c)$ *is graphical.*

(19) $\pi_{19} = (a\ b\ d\ c)$ *is graphical if and only if* a *and* d *or* b *and* c *have the same parity.*

(20) $\pi_{20} = (a\ d\ b\ c)$ *is graphical if and only if* a *and* b *or* c *and* d *have the same parity.*

(21) $\pi_{21} = (a\ d\ c\ b)$ *is graphical if and only if* a *and* c *or* b *and* d *have the same parity.*

(22) $\pi_{22} = (a\ b\ c\ d)$ *is graphical if and only if* a *and* c *or* b *and* d *have the same parity and* $cd \leq ab + bc + ad + c(c-1)$.

(23) $\pi_{23} = (a\ c\ d\ b)$ *is graphical if and only if* a *and* d *or* b *and* c *have the same parity and* $cd \leq ac + ab + bd + c(c-1)$.

(24) $\pi_{24} = (a\ c\ b\ d)$ *is graphical if and only if* a *and* b *or* c *and* d *have the same parity and either (1) if* $c \leq a+b-1$, *then* $bd \leq ab+bc+ad+b(b-1)$, *or (2) if* $c \geq a+b-1$, *then* $ac + bd \leq bc + ad + (a+b)(a+b-1)$.

We would like to give a correction to the case of the permutation π_{24}. We do not need to consider the two cases as stated in [7]. Instead, regardless of how the size of c compares to the size of $(a + b - 1)$, both of the stated inequalities must always hold in order for the permutation to be graphical. We will prove this new result. We also note that the proof given in [7] uses graph constructions and requires the consideration of six different possible cases. The proof stated here does not require the consideration of graph constructions, but instead uses the EG Shortcut theorem.

Theorem 10. *Let* $S = \{a, b, c, d\}$ *be a set of positive integers such that* $1 \leq a < b < c < d$. *Then the permutation* $\pi_{24} = (a\ c\ b\ d)$ *is graphical if and only if* a *and* b *or* c *and* d *have the same parity, and* $bd \leq ab + bc + ad + b(b-1)$ *and* $ac + bd \leq bc + ad + (a+b)(a+b-1)$.

Proof. The sequence that must be considered is as follows:

$$d, \cdots, d, c, \cdots, c, b, \cdots, b, a, \cdots, a,$$

where d is listed b times, c is listed a times, b is listed c times, and a is listed d times. Using the notation that is used in the statement of the EG Shortcut theorem, this would be

$$(d)_b, (c)_a, (b)_c, (a)_d.$$

We first consider the parity condition. The sum of the terms in this sequence is $db + ac + bc + ad$. This sum is even if and only if a and b or c and d have the same parity.

Then by the EG Shortcut theorem, the sequence is graphical if and only if the inequality stated in the Erdös-Gallai Theorem holds for each value of σ_k $(1 \leq k \leq 4)$. Note that the inequality corresponding to $\sigma_1 = b$ is the following:

$$bd \leq ab + bc + ad + b(b-1).$$

This is the first inequality stated in the result. Similarly, the inequality corresponding to $\sigma_2 = a + b$ is the following:

$$ac + bd \leq bc + ad + (a+b)(a+b-1).$$

This is the second inequality stated in the result. We note that the inequality corresponding to $\sigma_3 = a + b + c$ is the following:

$$ac + bd + bc \leq ad + (a+b+c)(a+b+c-1).$$

This inequality is true whenever the inequality corresponding to σ_2 is true. And finally, the resulting inequality corresponding to $\sigma_4 = a + b + c + d$ is the following:

$$ac + bd + bc + ad \leq (a + b + c + d)(a + b + c + d - 1),$$

which is always true. Therefore, by the EG Shortcut theorem, the permutation $\pi_{24} = (a\ c\ b\ d)$ is graphical if and only if $bd \leq ab + bc + ad + b(b-1)$ and $ac + bd \leq bc + ad + (a + b)(a + b - 1)$. $\square$

In [7], it is said that the permutation π_{24} is the most interesting case, likely due to the fact that it is the only permutation on the sets of cardinality four requiring two inequalities. We next consider the corresponding permutation on a set of cardinality six.

Theorem 11. *Let $S = \{a, b, c, d, e, f\}$ be a set of positive integers such that $1 \leq a < b < c < d < e < f$. Let π be the permutation of S as follows: $\pi = (a\ d\ b\ e\ c\ f)$. Then π is graphical if and only if the sum $cf + be + ad + ce + bd + af$ is even and the following inequalities hold:*

$$(1)\ cf \leq c(c-1) + bc + ac + ce + bd + af$$

$$(2)\ cf + be \leq (b + c)(b + c - 1) + ce + bd + af + a \cdot \min\{b + c, d\}$$

$$(3)\ cf + be + ad \leq (a + b + c)(a + b + c - 1) + ce + bd + af$$

Proof. The sequence being considered in this case is as follows:

$$(f)_c, (e)_b, (d)_a, (c)_e, (b)_d, (a)_f.$$

First assume π is graphical, and thus the above sequence is graphical. Then there exists a graph containing exactly c vertices of degree f, b vertices of degree e, a vertices of degree d, e vertices of degree c, d vertices of degree b, and f vertices of degree a. The sum of the degrees in this graph is $cf + be + ad + ce + bd + af$ and so this sum must be even. Since we are assuming π is graphical, by the EG Shortcut, the sequence must satisfy the inequality stated in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq 6)$. In particular, we will look at the inequalities corresponding to σ_1, σ_2, and σ_3. For this degree sequence, $\sigma_1 = c$ and the inequality corresponding to σ_1 is the following:

$$cf \leq c(c-1) + bc + ac + ce + bd + af.$$

So by the EG Shortcut, inequality (1) is true. The inequality corresponding to $\sigma_2 = b + c$ is the following:

$$cf + be \leq (b+c)(b+c-1) + ce + bd + af + a \cdot \min\{b+c, d\},$$

and so again by the EG Shortcut, inequality (2) holds. Similarly, for $\sigma_3 = a + b + c$ the corresponding inequality for this degree sequence which must be true by the EG Shortcut is the following:

$$cf + be + ad \leq (a+b+c)(a+b+c-1) + ce + bd + af,$$

and so inequality (3) must also hold.

For the converse, we assume inequalities (1), (2), and (3) as stated in the theorem hold, and that the sum $cf+be+ad+ce+bd+af$ is even. Then in order for the sequence to be graphical, by the EG Shortcut, the inequality in the Erdös-Gallai Theorem must hold for each value of σ_k $(1 \leq k \leq 6)$. We first note that the inequalities corresponding to σ_1, σ_2, and σ_3 are in fact the same inequalities stated in the theorem as (1), (2), and (3), respectively. These inequalities hold by assumption. We must verify, then, the inequalities corresponding to σ_4, σ_5, and σ_6.

For $\sigma_4 = a+b+c+e$, the inequality that needs to be verified is the following:

$$cf + be + ad + ce \leq (a+b+c+e)(a+b+c+e-1) + bd + af.$$

We know that the inequality corresponding to σ_3 is true. Note that

$$ce \leq e(a+b) + e(a+b+c-1) + e^2$$

is clearly also true. We will add ce to the left hand side of the inequality corresponding to σ_3, and we will add $e(a+b)+e(a+b+c-1)+e^2$ to the right hand side. This gives us precisely the inequality corresponding to σ_4 above.

Next we consider $\sigma_5 = a+b+c+d+e$. The inequality corresponding to σ_5 that must be verified is the following:

$$cf + be + ad + ce + bd \leq (a+b+c+d+e)(a+b+c+d+e-1) + af.$$

We will again use the fact that we already know that the inequality corresponding to σ_3 is true. We now note that

$$ce + bd \leq (a+b+c)(d+e) + (a+b+c-1)(d+e) + (d+e)^2 - ce - bd$$

is also a true inequality. Then we will add $ce+bd$ to the left hand side of the inequality corresponding to σ_3, and we will add $(a+b+c)(d+e)+(a+b+c-1)(d+e)+(d+e)^2-ce-bd$ to the right hand side. This gives us the inequality corresponding to σ_5 as stated above.

The inequality corresponding to $\sigma_6 = a+b+c+d+e+f$ is the following:

$$cf + be + ad + ce + bd + af \leq (a+b+c+d+e+f)(a+b+c+d+e+f-1).$$

For this case, the inequality is always true.

We have shown that the inequality holds for each value of σ_k $(1 \leq k \leq 6)$, and so by the EG Shortcut theorem, the sequence is graphical. Thus, the permutation $\pi = (a\ d\ b\ e\ c\ f)$ is graphical. $\square$

CHAPTER 3

GENERAL RESULTS

We now consider some general results for graphical permutations on sets of n elements. We are particularly interested in the largest elements of the given set on which the permutations will act. Specifically, we prove results for permutations which send large elements to other large elements, and give necessary and sufficient conditions for when such permutations are graphical.

Suppose $S = \{a_1, a_2, \ldots, a_n\}$ is a set of positive integers satisfying $1 \leq a_1 < a_2 < \cdots < a_n$. In [7], permutations π on such a set S such that $\pi(a_n) = a_n$, are considered, and necessary and sufficient conditions are given for permutations of this type to be graphical. This result was stated in Chapter 1 as Theorem 5. We restate it here and provide a different proof than that given in [7], now using the EG Shortcut.

Theorem 12. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_n$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even, and*

$$a_n \leq \sum_{i=1}^{n-1} a_i\pi(a_i).$$

Proof. The sequence to be considered in this case is as follows:

$$(a_n)_{a_n}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

Note that in order to match our current notation, the value of σ_k $(1 \leq k \leq n)$ as stated in the EG Shortcut theorem will be as follows throughout this proof:

$$\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

First assume that π is graphical, that is, assume that there exists a graph G with exactly a_i vertices of degree $\pi(a_i)$ for each i $(1 \leq i \leq n)$. Then the sum of the degrees, $\sum_{i=1}^{n} a_i\pi(a_i)$ must be even. By the EG Shortcut, the degree sequence must satisfy the inequality from the Erdös-Gallai Theorem for each value σ_k $(1 \leq k \leq n)$. In particular, the degree sequence must satisfy the inequality for $\sigma_1 = \pi^{-1}(a_n) = a_n$, which is the following:

$$a_n a_n \leq a_n(a_n - 1) + \sum_{i=1}^{n-1} a_i\pi(a_i).$$

Note that $a_i < a_n$ for each $i < n$, so the minimum in the inequality is always chosen to be the value of a_i, thus resulting in the above inequality. Then by simplifying, we have

$$a_n a_n \leq a_n a_n - a_n + \sum_{i=1}^{n-1} a_i\pi(a_i),$$

which is equivalent to

$$a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i),$$

as desired.

For the converse, assume that $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and that

$$a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i).$$

By the EG Shortcut, we must verify the inequality in the Erdös-Gallai Theorem for each value σ_k $(1 \leq k \leq n)$. In general, for σ_k we need to show that

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq \sigma_k(\sigma_k - 1) + \sum_{i=1}^{n-k} \pi^{-1}(a_i) \cdot \min\{\sigma_k, a_i\}.$$

Using notation corresponding to our specific sequence, and since $\pi^{-1}(a_n) = a_n$ and $a_i < a_n$ for each $i < n$, this can be written as

$$(a_n)^2 + \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i) \leq \left(a_n + \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(a_n - 1 + \left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\right)$$
$$+ \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i). \qquad (\star)$$

Recall that we are assuming $a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i)$, which we can instead write as

$$a_n \leq \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i)$$

by separating the summation into two parts, first with the terms from 1 up to $n-k$ and then the remaining terms from $n-k+1$ up to n. Then note that since $a_n > a_i$ for each $i < n$, we obtain

$$a_n \leq \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + a_n \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i).$$

Also note that

$$\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right) + a_n - 1\right) - \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i)$$

is positive, and so we can add this to the right hand side, to obtain

$$\begin{aligned} a_n \leq &\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right) + a_n - 1\right) \\ &+ \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + a_n \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) - \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i). \end{aligned}$$

Then by adding $(a_n)^2 - a_n$ to both sides, this is equivalent to

$$\begin{aligned} (a_n)^2 \leq a_n(a_n - 1) + &\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right) + a_n - 1\right) \\ &+ a_n \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) - \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i). \end{aligned}$$

And then by factoring the right hand side, and some rearranging, this is equivalent to $(\star)$ as desired. Thus we have shown that the inequality holds for each value of σ_k

$(1 \leq k \leq n)$, and so by the EG Shortcut, the sequence, and therefore the permutation π, is graphical. $\square$

We point out that this proof is shorter than the original proof given in [7] and also does not require the consideration of multiple cases. We mentioned in Chapter 1 that a benefit of using Theorem 6 to prove this result in [7] was that it did not require the determination of the minimum of two values, as in the Erdös-Gallai Theorem. By using the EG Shortcut theorem to prove this result, since $\pi(a_n) = a_n$ and thus $\sigma_1 = a_n$, we also avoided the need to determine the minimum of two values, since a_n is the largest value of the set.

Next we consider permutations π such that $\pi(a_{n-1}) = a_n$. A similar result is given in this case, which gives the necessary and sufficient conditions for these types of permutations to be graphical.

Theorem 13. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_{n-1}) = a_n$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even, and*

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i\pi(a_i).$$

Proof. The sequence to be considered is as follows:

$$(a_n)_{a_{n-1}}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

In order to match our current notation, the value of σ_k as stated in the EG Shortcut will be as follows throughout this proof:

$$\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

Assume that π is graphical, and so the sequence above is graphical. Then there exists a graph G with exactly a_i vertices of degree $\pi(a_i)$ for each i $(1 \leq i \leq n)$. Since such a graph exists, clearly the sum of the degrees, $\sum_{i=1}^{n} a_i\pi(a_i)$ must be even. By the EG Shortcut, the degree sequence must satisfy the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$. In particular, the degree sequence must satisfy the inequality corresponding to $\sigma_1 = a_{n-1}$, which is the following:

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i=1}^{n-1} \pi^{-1}(a_i) \cdot \min\{a_{n-1}, a_i\}.$$

Since our summation is taken up to $n-1$, and $a_i \leq a_{n-1}$ for each $i \leq n-1$, this is equivalent to

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n} a_i \pi^{-1}(a_i).$$

Finally, since $a_n \pi^{-1}(a_n) = a_n a_{n-1} = a_{n-1}\pi(a_{n-1})$, this is equivalent to

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i \pi(a_i),$$

as desired.

For the converse, assume that $\sum_{i=1}^{n} a_i\pi(a_i)$ is even, and

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1}-1) + \sum_{i \neq n-1} a_i \pi(a_i).$$

By the EG Shortcut, we must verify the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$. Thus in general, for σ_k we need to show that

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq \sigma_k(\sigma_k - 1) + \sum_{i=1}^{n-k} \pi^{-1}(a_i) \cdot \min\{\sigma_k, a_i\}.$$

Using our notation for this particular permutation, and since $a_i \leq a_{n-1}$ for each $i \leq n-1$, after some rearranging this is equivalent to

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1}-1) + \left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + 2a_{n-1} - 1\right)$$
$$+ \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) - \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i).$$

Recall that we are assuming that

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1}-1) + \sum_{i \neq n-1} a_i \pi(a_i),$$

and so it suffices to show that

$$\sum_{i\neq n-1} a_i\pi(a_i) \leq \left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + 2a_{n-1} - 1\right) + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i)$$
$$- \sum_{i=n-k+1}^{n-1} a_i\pi^{-1}(a_i). \qquad (\star)$$

Since $a_{n-1}\pi(a_{n-1}) = a_{n-1}a_n = a_n\pi^{-1}(a_n)$, we note that

$$\sum_{i\neq n-1} a_i\pi(a_i) = \sum_{i=1}^{n-1} a_i\pi^{-1}(a_i) = \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i) + \sum_{i=n-k+1}^{n-1} a_i\pi^{-1}(a_i),$$

by first converting from π to π^{-1}, and then separating the summation into two parts.

We also note that

$$\sum_{i=1}^{n-k} a_i\pi^{-1}(a_i) + \sum_{i=n-k+1}^{n-1} a_i\pi^{-1}(a_i) \leq \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i) + a_{n-1}\cdot \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i),$$

since $a_i \leq a_{n-1}$ for each $i \leq n-1$. Thus we have shown that

$$\sum_{i\neq n-1} a_i\pi(a_i) \leq \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i) + a_{n-1}\cdot \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i).$$

Then notice that

$$\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + a_{n-1} - 1\right) - \sum_{i=n-k+1}^{n-1} a_i\pi^{-1}(a_i)$$

must be positive, and so we add this expression to the right hand side to obtain

$$\begin{aligned}\sum_{i\neq n-1} a_i\pi(a_i) \leq \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i) + a_{n-1}\cdot \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) \\ + \left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + a_{n-1} - 1\right) \\ - \sum_{i=n-k+1}^{n-1} a_i\pi^{-1}(a_i).\end{aligned}$$

After factoring the right hand side, this is equivalent to ($\star$) as desired. Thus we have shown that the sequence satisfies the inequality stated in the Erdös-Gallai Theorem for each value of σ_k ($1 \leq k \leq n$). Then by the EG Shortcut, the sequence is graphical, and so the permutation π is graphical. □

Now instead of looking at which element the permutation sends to the largest element a_n, as in the previous result, we will consider which element a_n is sent to by the permutation. We will again be particularly interested in the next largest element of the set, a_{n-1}.

Theorem 14. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_{n-1}$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even.*

Proof. The sequence being considered corresponding to this permutation is as follows:

$$(a_n)_{\pi^{-1}(a_n)}, (a_{n-1})_{a_n}, (a_{n-2})_{\pi^{-1}(a_{n-2})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

To match our current notation for the permutation π, the value of σ_k as stated in the

EG Shortcut will be as follows throughout this proof:

$$\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

By the EG Shortcut theorem, this sequence is graphical if and only if the sum $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and the inequality stated in the Erdös-Gallai Theorem is satisfied for each value of σ_k $(1 \leq k \leq n)$.

We will first consider the inequality corresponding to $\sigma_1 = \pi^{-1}(a_n)$, which is

$$a_n\pi^{-1}(a_n) \leq \left(\pi^{-1}(a_n)\right)\left(\pi^{-1}(a_n) - 1\right) + \sum_{i=1}^{n-1} \pi^{-1}(a_i) \cdot \min\{a_i, \pi^{-1}(a_n)\},$$

or equivalently, since $\pi^{-1}(a_{n-1}) = a_n$,

$$\begin{aligned} a_n\pi^{-1}(a_n) \leq \left(\pi^{-1}(a_n)\right)\left(\pi^{-1}(a_n) - 1\right) + a_n \cdot \min\{a_{n-1}, \pi^{-1}(a_n)\} \\ + \sum_{i=1}^{n-2} \pi^{-1}(a_i) \cdot \min\{a_i, \pi^{-1}(a_n)\}. \end{aligned}$$

Observe that $\pi^{-1}(a_n) \leq a_{n-1}$, and therefore $\min\{a_{n-1}, \pi^{-1}(a_n)\} = \pi^{-1}(a_n)$. Thus the inequality we need to verify is equivalent to

$$a_n\pi^{-1}(a_n) \leq \left(\pi^{-1}(a_n)\right)\left(\pi^{-1}(a_n) - 1\right) + a_n\pi^{-1}(a_n) + \sum_{i=1}^{n-2} \pi^{-1}(a_i) \cdot \min\{a_i, \pi^{-1}(a_n)\},$$

which is clearly true. Therefore, the inequality corresponding to σ_1 holds.

For $k \geq 2$, notice that σ_k is a sum containing a_n, and thus $\min\{\sigma_k, a_i\} = a_i$ in each inequality corresponding to σ_k with $k \geq 2$. Knowing this, and using our particular sequence, what we need to show for each σ_k ($k \geq 2$) is

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) \right) \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1 \right) + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i). \tag{$\star$}$$

Since $a_i \leq a_n$ for each $i \leq n$, it is easy to see that

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq a_n \cdot \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

Then since the expression

$$\left(\sum_{\substack{i \geq n-k+1 \\ i \neq n-1}} \pi^{-1}(a_i) \right) \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1 \right) - a_n + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i)$$

must be positive, we can add this to the right hand side to obtain

$$\begin{aligned} \sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq a_n \cdot \sum_{i=n-k+1}^{n} \pi^{-1}(a_i) + \left(\sum_{\substack{i \geq n-k+1 \\ i \neq n-1}} \pi^{-1}(a_i) \right) \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1 \right) \\ - a_n + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i), \end{aligned}$$

or equivalently,

$$\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq a_n \cdot \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) + \left(\sum_{\substack{i\geq n-k+1 \\ i\neq n-1}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i)$$

Then by factoring the right hand side, we obtain

$$\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq \left(a_n + \sum_{\substack{i\geq n-k+1 \\ i\neq n-1}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i),$$

which is equivalent to the inequality $(\star)$. Thus the sequence satisfies the inequality corresponding to σ_k, $k \geq 2$.

We have shown that the sequence satisfies the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$, and so by the EG Shortcut, the sequence is graphical. Therefore the permutaion π is graphical. $\square$

Next, we will generalize this idea. We consider which element of the set the largest element a_n is sent to by the permutation. We will look at permutations such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$. Notice that the previous theorem is the case such that $j = 1$.

Theorem 15. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$. Let σ_k be defined as $\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i)$. Then π is graphical if and* *only if* $\sum_{i=1}^{n} a_i \pi(a_i)$ *is even, and the inequality in the Erdös-Gallai Theorem holds for each value of* σ_k $(1 \leq k \leq j)$.

Proof. The sequence being considered for this permutation is as follows:

$$(a_n)_{\pi^{-1}(a_n)}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots, (a_{n-j+1})_{\pi^{-1}(a_{n-j+1})}, (a_{n-j})_{a_n}, (a_{n-j-1})_{\pi^{-1}(a_{n-j-1})}, \ldots$$
$$\ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

First assume that π is graphical, and thus the above sequence is graphical. Then clearly the sum $\sum_{i=1}^{n} a_i \pi(a_i)$ is even. By the EG Shortcut, the sequence must satisfy the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$. Therefore, the inequalities corresponding to the values $\sigma_1, \sigma_2, \ldots, \sigma_j$ must all be satisfied, as desired.

For the converse, we assume the sum $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and that the inequalities corresponding to the values $\sigma_1, \sigma_2, \ldots, \sigma_j$ are all satisfied. We show that the remaining inequalities corresponding to the values for σ_k $(j+1 \leq k \leq n)$ are also satisfied.

We notice that for $k > j$, the value of σ_k is a summation containing a_n, since $\pi^{-1}(a_{n-j}) = a_n$. Thus the inequality corresponding to σ_k, where $j+1 \leq k \leq n$ is

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) \right) \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1 \right) + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i). \quad (\star)$$

Since $a_i \leq a_n$ for each $i \leq n$, we know that

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq a_n \cdot \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

Observe that the expression

$$\left(\sum_{\substack{i\geq n-k+1\\ i\neq n-j}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right)-a_n+\sum_{i=1}^{n-k} a_i\pi^{-1}(a_i)$$

must be positive, and so by adding this to the right hand side, we obtain

$$\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq a_n \cdot \sum_{i=n-k+1}^{n} \pi^{-1}(a_i) + \left(\sum_{\substack{i\geq n-k+1\\ i\neq n-j}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right) - a_n + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i).$$

Then notice that by factoring the right hand side, this is equivalent to

$$\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq \left(a_n + \sum_{\substack{i\geq n-k+1\\ i\neq n-j}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right) + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i).$$

Since $a_n = \pi^{-1}(a_{n-j})$, this inequality is equivalent to the inequality $(\star)$. Thus, we have shown that the sequence satisfies the inequality in the Erdös-Gallai Theorem for each value of σ_k $(j+1 \leq k \leq n)$. Recall that by assumption, the inequality is satisfied for values of σ_k $(1 \leq k \leq j)$. Thus, by the EG Shortcut theorem, the sequence is graphical, and thus the permutation π is graphical. $\square$

CHAPTER 4

CONCLUSIONS AND CONJECTURES

We first would like to point out that the notation which became standard for degree sequences and related theorems is not necessarily the best notation for the case of graphical permutations. We give a restatement of the EG Shortcut theorem, specific to the notation that comes from considering graphical permutations.

Theorem 16. *Let* $S = \{a_1, a_2, \ldots, a_n\}$ *be a set of integers such that* $1 \leq a_1 < a_2 < \cdots < a_n$. *Let* π *be a permutation on the set* S *such that* $\sum_{i=1}^{n} a_i \pi(a_i)$ *is even. Let* $S_k = \sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i)$ *and let* $T_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i)$. *Then the permutation* π *is graphical if and only if the following holds for each value of* k $(1 \leq k \leq n)$:

$$S_k \leq T_k(T_k - 1) + \sum_{\substack{i \leq n-k \\ a_i \leq T_k}} a_i \pi^{-1}(a_i) + \sum_{\substack{i \leq n-k \\ a_i > T_k}} T_k \pi^{-1}(a_i).$$

In Chapter 3, we gave some results for permutations on a set of n elements, which are largely dependent upon the largest element of the given set, a_n. The most general result was Theorem 15, which considers permutations such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$. This result further reduces how many different inequalities must be verified in order to conclude that a given permutation is graphical. If $\pi(a_n) = a_{n-j}$, then we must verify the inequality stated in the Erdös-Gallai Theorem for j different values. Thus it is easier to conclude a permutation π on a set S is graphical when it sends its largest element a_n to another large element of the set, which results in a smaller value of j.

While this general result still does not give us one single inequality to verify in order to conclude a permutation π is graphical, it does allow us to immediately identify the maximum number of inequalities which would need to be verified, based on how the permutation acts on the largest element a_n.

We would like to note that this is not the best possible case, in the sense that it does not give the absolute least amount of inequalities which are necessary to verify. For example, recall Theorem 11, which gave a result for a particular permutation on a set of six elements, $\pi = (a\ d\ b\ e\ c\ f)$. We only needed to verify three inequalities for this permutation, those corresponding to σ_1, σ_2, and σ_3, and remaining inequalities followed. However, since $\pi(f) = a$, or in different notation $\pi(a_6) = a_1$, for this permutation, we have $j = 5$. By Theorem 15, we should need to verify the inequality stated in the Erdös-Gallai Theorem for five different values. This tells us that there is possibly a better result which could give a true minimum amount of inequalities which would be necessary to verify.

Consider a generalization of this permutation on a set of $2n$ elements, $S = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$, such that $1 \leq a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}$. The permutation in question is $\pi = (a_1\ a_{n+1}\ a_2\ a_{n+2} \ldots\ a_{n-1}\ a_{2n-1}\ a_n\ a_{2n})$. This type of permutation can be defined as follows:

$$\pi(a_p) = \begin{cases} a_{p+n} & \text{for} \quad p \leq n \\ a_{p-n+1} & \text{for} \quad n < p < 2n \\ a_1 & \text{for} \quad p = 2n \end{cases}$$

We give the following conjecture for this type of permutation.

Conjecture 1. *Let* $S = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$ *be a set of* $2n$ *positive integers such that* $1 \leq a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}$*. Let* π *be the permutation of the set* S *as defined above, with* $\sum_{i=1}^{2n} a_i\pi(a_i)$ *even. Let* $\sigma_k = \sum_{i=(2n)-k+1}^{2n} \pi^{-1}(a_i)$*. Then* π *is graphical if and only if the inequality in the Erdös-Gallai Theorem holds for each value of* σ_k $(1 \leq k \leq n)$*.*

This conjecture says that for this type of permutation on a set of even cardinality, we need to verify only half of the inequalities required by the EG Shortcut theorem. It would be an interesting problem to prove this conjecture for this specific type of permutation.

Suppose instead of only focusing on the largest element a_n, we instead look at the largest product $a_i\pi(a_i)$ for the given permutation on the set. This has been considered in conjectures in previous papers, including [7] and [9], however the proposed conjectures have since been shown to be false. We conclude with a conjecture which uses a similar idea.

Conjecture 2. *Let* $S = \{a_1, a_2, \ldots, a_n\}$ *be a set of integers with* $1 \leq a_1 < a_2 < \cdots <$

a_n $(n \geq 2)$ and let π be a permutation on S such that $\pi(a_n) \neq a_n$ and $\sum_{i=1}^{n} a_i\pi(a_i)$ is even. Let m be the smallest such that

$$a_m\pi(a_m) = \max\{a_i\pi(a_i) | 1 \leq i \leq n\}.$$

Then if $a_m < \pi(a_m)$, then π is graphical if and only if the inequality in the Erdös-Gallai Theorem holds for values $\sigma_1, \sigma_2, \ldots, \sigma_{n-m}$.

Note that this conjecture only considers the case when $a_m < \pi(a_m)$. Suppose that m is as defined in Conjecture 2, and $a_m \geq \pi(a_m)$. In this case, further investigation is necessary to determine conditions for such a permutation to be graphical.

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