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Periodic Solutions and Positive Solutions of First and Second Order Logistic Type ODEs with Harvesting

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PERIODIC SOLUTIONS AND POSITIVE SOLUTIONS OF FIRST AND

SECOND ORDER LOGISTIC TYPE ODES WITH HARVESTING

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ABSTRACT

It was recently shown that the nonlinear logistic type ODE with periodic harvesting has a bifurcation on the periodic solutions with respect to the parameter ϵ:

$$u' = f(u) - \epsilon h(t).$$

Namely, there exists an ϵ_0 such that for $0 < \epsilon < \epsilon_0$ there are two periodic solutions, for $\epsilon = \epsilon_0$ there is one periodic solution, and for $\epsilon > \epsilon_0$ there are no periodic solutions, provided that $\int_0^T h(t)dt > 0$. In this paper we look at some numerical evidence regarding the behavior of this threshold for various types of harvesting terms, in particular we find evidence in the negative for a conjecture regarding the behavior of this threshold value. Additionally, we look at analagous steady states for the reaction-diffusion IBVP with logistic growth and positive harvesting:

$$-u'' = f(u) - \epsilon h(u), \quad u(0) = u(L) = 0.$$

Using phase plane arguments we show that there is a threshold value of ϵ such that this BVP has no positive solutions.

ACKNOWLEDGEMENTS

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Chapter 1

Introduction

It was Verhulst, in 1838, who first introduced the idea of a differential equation in which the growth rate was bounded by some carrying capacity i.e.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

This equation, along with its solutions is well known in population dynamics. However, we wish to generalize this model as much as possible, and we want it to apply to as many situations as possible. This motivates adding in a harvesting term $h(t)$, and a more general definition of the growth rate function:

$$\frac{du(t)}{dt} = f(u) - h(t).$$

We assume that f has the following properties, for some $M > 0$: $f \in C^2(\mathbb{R}), f(0) = f(M) = 0, f(u) > 0$ for $u \in (0, M), f'(0) > 0, f'(M) < 0$. We call this a logistic type growth function. The growth rate in Verhulst's equation is a very simple example of such a logistic type function. Now, when we are modeling any population our concern is sustainability: in particular we want the solutions to be positive. In this case, it is also natural to add the condition of periodicity to the harvesting. This makes sense, since typically populations that are being harvested from, say fisheries or game populations, undergo a periodic harvesting scheme, where there is a time where animals are no removed, but instead they are stocked. Let's say that $h(t)$ has period T. Obviously, since we are considering a population we are harvesting from the ideal solution to the ODE is going to be periodic. Now, constant harvesting represents the most trivial form of periodic harvesting, and constant solutions represent the most trival form of periodic solutions. It is easily seen that, depending on f, we have a bifurcating value of the constant harvesting term h, which we call h_0 such that for $h < h_0$ we have two constant solutions, for $h = h_0$ we have one constant solution, and for $h > h_0$ we have no constant terms. The question that then arose was whether we could have some similar result for

nonconstant, periodic harvesting and periodic solutions. The answer, as shown in [9], is yes. A scaling parameter is attached to the harvesting term and get the following ODE:

$$\frac{du(t)}{dt} = f(u) - \epsilon h(t). \tag{1.1}$$

It is shown in [9] that, provided that $\int_0^T h(t)dt > 0$ then there is an ϵ_0 such that for $\epsilon > \epsilon_0$ we have exactly two T-periodic solutions, for $\epsilon = \epsilon_0$ we have exactly one periodic solution, and for $\epsilon > \epsilon_0$ we have no periodic solutions. Our goal in this thesis is two-fold. First,we will be doing a review of the proof of this theorem with full details. Additionally we will be looking at evidence on the behavior of this bifurcating value as the harvesting term changes.

Second, we will be considering steady states of the following reaction-diffusion IBVP

$$u_t = u_{xx} + f(u) - \epsilon h(u), u(0,t) = u(L,t) = 0, u(x,0) = u_0(x)$$

where f is logistic and $h > 0$. Our motivation for looking at the steady states is again sustainability: After a long time can the population reach apoint where it ceases to change over time? So we set $u_t = 0$, and since we are considering a time independant solution, we can remove the initial condition as well. This leads to the BVP

$$-u'' = f(u) - \epsilon h(u), \quad u(0) = u(L) = 0$$

Using phase plane arguments we will consider the existence of positive solutions with respect to ϵ and L.

Chapter 2

Preliminaries

In this section we will be going over some basic results that are foundational to my work. We will be doing an extensive review of some basic theorems and how these are applied, ultimately leading up to the main result of [9].

Crandall-Rabinowitz Saddle Node Bifurcation Theorem

Before we introduce the theorem, we will present some motivation. Consider $F(\lambda, x) = x^2 - \lambda$. It is easy to see that for $\lambda < 0$ we have no solutions, for $\lambda = 0$ we have one, and for $\lambda > 0$ we have two solutions. So, $\lambda = 0$ is a bifurcation value for the parameter in the sense that there is a drastic change in the solution set of the equation. We can also see that $F(0,0) = 0$. We can ask, therefore, how do the solutions to $F = 0$ behave in a neighborhood of $(0,0)$? If we visualize this in three dimensions, it is easy to see that they form a parabola with its vertex at the origin. Note that $F_x(0,0) = 0$, and so if we consider this derivative as a linear operator, then we have have that the null space $N(F_u(0,0)) = \mathbb{R}$, and this has dimension 1. Also, the range of this linear operator is just 0, and thus its codimension is 1 as well, and since $F_\lambda(0,0) = 1$, then we see that $F_\lambda(0,0) \notin R(F_u(0,0))$.

This becomes a little less trivial if we consider an example in $\mathbb{R}^2$. Suppose that now

$$F(\lambda, x, y) = (F_1(\lambda, x, y), F_2(\lambda, x, y)) = (x^2 - \lambda, y).$$

We see that $F(0,0,0) = (0,0)$. Now, we consider the Jacobian $\frac{\partial(F_1, F_2)}{\partial(x,y)}(\lambda, 0, 0)$ at $\lambda = 0$:

$$\frac{\partial(F_1(0,0,0), F_2(0,0,0))}{\partial(x,y)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We can see that $N(F_{(x,y)}(0,0,0)) = \{(x,0) : x \in \mathbb{R}\}$ has dimension 1. Also, $R(F_{(x,y)}(0,0,0)) = \{(0,y) : y \in \mathbb{R}\}$ has a dimension of 1, and thus its codimension is 1 as well. Similarly, we have $F_\lambda(0,0,0) = (-1,0) \notin R(F_{(x,y)}(0,0,0))$. Simple algebra indicates that the solutions to $F(\lambda, x, y) = 0$ in a neighborhood of the origin form a parabolic cylinder.

So we notice some patterns in the derivatives of functions that have this bifurcating behavior. We can also talk about the stability of the solutions to $F(\lambda, x, y) = 0$. When we have them, the two solutions are $(\pm\sqrt{\lambda}, 0)$. We can determine stability by looking at the eigenvalues of the Jacobian within a neighborhood of $(0, 0, 0)$, and these are found by looking at the roots of the characteristic equation

$$(2x - \mu)(1 - \mu) = 0.$$

So our two solutions are $\mu = \frac{x}{2}$ and $\mu = 1$. when x is positive, then we have two positive eigenvalues which gives us a repelling node, while if x is negative then we have two eigenvalues of different signs, which gives us a saddle. So among our solutions, one of them is a saddle, and the other is a node. This motivates the name for the following theorem, which generalizes to Banach Spaces the observations we have made.

We shall start by giving some preliminary information that will be useful for the proof of the Saddle Node Bifurcation Theorem. First we give a common result regarding linear transformations. Let X and Y be Banach spaces and recall that the kernel of a linear transformation $\varphi : X \to Y$ is given as

$$\ker(\varphi) = \{x \in X : \varphi(x) = 0\}.$$

Lemma 2.1. *φ is injective if and only if $\ker(\varphi) = \{0\}$.*

The proof is elementary and is usually done in any undergraduate linear algebra course. Also, for the remainder of this paper we will use $N(\varphi)$ and $\ker(\varphi)$ interchangeably.

Proof. Assume φ is injective. Let $\varphi(a) = 0$ and $\varphi(b) = 0$. Then we have that $\varphi(a) = \varphi(b)$ and the injectivity of φ gives that $a = b$. So then the kernel of φ only has a single element, and this element must be zero. Now assume that $\ker(\varphi) = \{0\}$. If $\varphi(a) = \varphi(b)$ we then have that $\varphi(a) - \varphi(b) = 0$, hence $\varphi(a - b) = 0$, and since the kernel is trivial we have that $a - b = 0$ and thus $a = b$ which gives injectivity. $\qquad\square$

We shall be using the following version of the Implicit Function Theorem, which is given in the appendix of [4] without proof, though they note that Theorem 10.2.1 of [6] can be used as a proof .

Implicit Function Theorem. *Let E, F and G be three Banach Spaces, f a continuous mapping on an open subset A of $E \times F$ into G. Let the map $y \mapsto f(x, y)$ of*

$$A_x = \{y \in F : (x, y) \in A\}$$

into G be differentiable in A_x for each $x \in E$ such that $A_x \neq \emptyset$, and assume the derivative of this map (denoted by f_y) is continuous on A. Let $(x_0, y_0) \in A$ be such that $f(x_0, y_0) = 0$, and $f_y(x_0, y_0)$ is a linear homeomorphism of F onto G. Then there are neighborhoods U of x_0 and V of y_0 in F such that:

(i) $U \times V \subset A$

(ii) There is exactly one function $u : U \to V$ satisfying $f(x, u(x)) = 0$ for $x \in U$.

(iii) The mapping of U of (ii) is continuous.

If, moreover, the mapping f is k-times differentiable on A, then (iii) above may be replaced by

(iv) u is k-times continuously differentiable.

The following two lemmas are given in [11]. The first is a corollary to the Open Mapping Theorem, and the second is a version of the Hahn-Banach Theorem.

Lemma 2.2. *Let X and Y be Banach Spaces, and let T be a bounded linear transformation between them. Then if T is bijective then T^{-1} is continuous.*

Lemma 2.3. *Let B be a closed linear subspace of a Banach space X, and let $\eta \in X \setminus B$, then there exists a linear transformation from X to $\mathbb{R}$ that vanishes on B but not on η.*

We can now state and prove the Saddle Node Bifurcation Theorem of Crandall and Rabinowitz [5]:

Theorem 2.1. *Suppose that X and Y are Banach spaces. Let $(\lambda_0, u_0) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y. Suppose that:*

(i) $\dim N(F_u(\lambda_0, u_0)) = \dim(Y \setminus R(F_u(\lambda_0, u_0))) = 1$ *and* $N(F_u(\lambda_0, u_0)) = span\{w_0\}$.

(ii) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$

Then if Z is a complement of $span\{w_0\}$ in X then the solutions of $F(\lambda, u) = F(\lambda_0, u_0)$ near (λ_0, u_0) form a curve $(\lambda(s), u(s)) = (\lambda(s), u_0 + s w_0 + z(s))$ where $s \mapsto (\lambda(s), z(s))$ is a continuously differentiable function near $s = 0$ and $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$ and $z(0) = z'(0) = 0$.

If F is C^2 then we have

$$\lambda''(0) = -\frac{\langle \ell, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle \ell, F_\lambda(\lambda_0, u_0) \rangle}.$$

The proof given in [5] is only a few lines and leaves out many details. For the sake of completeness we flesh out these details in the following proof, similar to what is done in [12]. Also, one can note the contrast between this theorem and the Implicit Function Theorem, which gives us a unique solution in a neighborhood when we solve for one of the coordinates, while the Saddle Node Bifurcation theorem gives us a curve of solutions for the coordinates in a neighborhood.

Proof. Define the following function $G : \mathbb{R} \times \mathbb{R} \times Z \to Y$ by

$$G(s, \lambda, z) = F(\lambda, u_0 + s w_0 + z) - F(\lambda_0, u_0).$$

Note now that G inherits the same continuity from F, and also that $G(0, \lambda_0, 0) = F(\lambda_0, u_0) - F(\lambda_0, u_0) = 0$. Now we also claim that $G_{(\lambda, z)}(0, \lambda_0, 0)$ is a linear homeomorphism. Linearity follows from the linearity of the derivatives of F. Now, we can show that $G_{(\lambda, z)}(0, \lambda_0, 0)$ is injective using Lemma 2.1. Suppose we have a $(\tau, \psi) \in \ker(G_{(\lambda, z)}(0, \lambda, 0))$, this gives that

$$G_{(\lambda, z)}(0, \lambda_0, 0)[(\tau, \psi)] = \tau F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[\psi] = 0.$$

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Now if $\tau \neq 0$, then we have that $F_\lambda(\lambda_0, u_0) = F_u(\lambda_0, u_0)[\tau^{-1}\psi]$ (since $F_u(\lambda_0, u_0)$ is linear), which contradicts that $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$, and hence we must have that $\tau = 0$, which gives

$$F_u(\lambda_0, u_0)[\psi] = 0.$$

Now this implies that $\psi \in N(F_u(\lambda_0, u_0)) = \text{span}(w_0)$, but since $\psi \in Z$, and Z is a complement of $\text{span}(w_0)$ we must have that $\psi = 0$ because $\text{span}(w_0) \cap Z = \{0\}$. So this gives that

$$\ker(G_{(\lambda,z)}(0, \lambda, 0)) = \{(0, 0)\}.$$

Thus, by Lemma 2.1, this is an injective function. Now, as a consequence of Lemma 2.3, there is a linear functional $\ell : Y \to \mathbb{R}$ such that $\ker(\ell) = R(F_u(\lambda_0, u_0))$ and is nonzero outside of $R(F_u(\lambda_0, u_0))$. Now, we shall use this to show that $G_{(\lambda,z)}(0, \lambda_0, 0)$ is surjective. Let $\theta \in Y$, and consider

$$G_{(\lambda_0,z)}(0, \lambda, 0)[(\tau, \psi)] = \tau F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[\psi] = \theta.$$

Applying ℓ we get

$$\tau\ell(F_\lambda(\lambda_0, u_0)) + \ell(F_u(\lambda_0, u_0))[\psi]) = \ell(\theta).$$

Since $\ker(\ell) = R(F_u(\lambda_0, u_0))$ we have that $\ell(F_u(\lambda_0, u_0)[\psi] = 0$ which gives $\tau\ell(F_\lambda(\lambda_0, u_0)) = \ell(\theta)$. Since $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$ we have $\ell(F_\lambda(\lambda_0, u_0)) \neq 0$ which gives that

$$\tau = \frac{\ell(\theta)}{\ell(F_\lambda(\lambda_0, u_0))},$$

that is to say that τ is uniquely determined by θ. It will be useful for us to note the following: Since Z is a complement of $\text{span}(w_0)$, we have that $Z + \text{span}(w_0) = X$, and so for any element $x \in X$ we have a $z_0 \in Z$ and a $a \in \mathbb{R}$ such that $x = z_0 + aw_0$. This gives that

$$F_u(\lambda_0, u_0)[x] = F_u(\lambda_0, u_0)[z_0 + aw_0] = F_u(\lambda_0, u_0)[z_0] + aF_u(\lambda_0, u_0)[w_0] = F_u(\lambda_0, u_0)[z_0].$$

What this tells us is that $R(F_u(\lambda_0, u_0)|_z) = R(F_u(\lambda_0, u_0))$. Also, since Z is a complement we have that $Z \cap N(F_u(\lambda_0, u_0)) = \{0\}$, and so, as before, we have $F_u(\lambda_0, u_0)|_Z$ is

injective. This gives a well defined inverse of $F_u(\lambda_0, u_0)|_Z$ that has a domain of $R(F_u(\lambda_0, u_0))$, we shall call this inverse K. Now since for any $\psi \in Z$ we have $F_u(\lambda_0, u_0)[\psi] = \theta - \tau F_\lambda(\lambda_0, u_0)$, so then $\theta - \tau F_\lambda(\lambda_0, u_0)$ is in the domain of K, and thus we have

$$\psi = K(\theta - \tau F_\lambda(\lambda_0, u_0))$$

and since τ is completly determined by θ, we see that ψ is also completely determined by θ. This gives us surjectivity of $G_{(\lambda,z)}(0, \lambda_0, 0)$. So since $G_{(\lambda,z)}(0, \lambda_0, 0)$ is continuous and bijective, then as a consequence Lemma 2.2, we get that $(G_{(\lambda_0,z)}(0, \lambda, 0))^{-1}$ is also continuous, and thus is a linear homeomorphism. So we have shown that G satisfies the condition of the Implicit Function Theorem, where our three Banach spaces are $\mathbb{R}$, $\mathbb{R} \times Z$, and Y. So then we have functions $\lambda(s)$ and $u(s)$ where, in a neighborhood of $s = 0$ we have that $G(s, \lambda(s), u(s)) = 0$ and $(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s))$, and the mapping $s \mapsto (\lambda(s), z(s))$ is continuously differentiable, since F is continuously differentiable. Now since $G(0, \lambda(0), z(0)) = 0 = G(0, \lambda_0, 0)$ we then have that $\lambda(0) = \lambda_0$ and $z(0) = 0$. Additionally, note that

$$G_s(0, \lambda_0, 0) = \frac{\partial}{\partial s}(F(\lambda, u_0 + sw_0 + z) - F(\lambda_0, u_0))\Big|_{(0,\lambda_0,0)} = F_u(\lambda_0, u_0)[w_0] = 0.$$

On the other hand, since λ and z can depend on s we also have

$$G_s(0, \lambda_0, 0) = G_{(\lambda,z)}(0, \lambda_0, 0)[\lambda'(0), z'(0)].$$

By the injectivity of $G_{(\lambda,z)}(0, \lambda, 0)$ we have that

$$\lambda'(0) = z'(0) = 0.$$

Now, taking the derivative with respect to s of $F(\lambda(s), u(s))$ gives

$$\frac{\partial}{\partial s}F(\lambda(s), u(s)) = F_\lambda(\lambda(s), u(s))\lambda'(s) + F_u(\lambda(s), u(s))u'(s) = 0.$$

Thus

$$\frac{\partial^2}{\partial s^2}F(\lambda(s), u(s)) = \lambda''F_\lambda + (\lambda'(s))^2 + 2\lambda'(s)F_{\lambda u}[u'(s)] + F_{uu}[u'(s), u'(s)] + F_u[u''(s)] = 0$$

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Let $s = 0$ to get that

$$\lambda''(0)F_\lambda(\lambda(0), u(0)) + F_u[u''(0)] + F_{uu}(\lambda_0, u_0)([w_0, w_0] = 0,$$

since $\lambda'(0) = u'(0) = 0$ and $u' = w_0$, since $u(s) = u_0 + sw_0 + z(s)$ and $z'(0) = 0$. Now we apply the linear functional ℓ given by the Hahn Banach theorem to both sides to get

$$\lambda''(0)\langle \ell, F_\lambda(\lambda_0, u_0)\rangle + \langle \ell, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle = 0.$$

Solving gives that

$$\lambda''(0) = -\frac{\langle \ell, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{\langle \ell, F_\lambda(\lambda_0, u_0)\rangle}.$$

This completes the proof. $\qquad\square$

The following theorem will also be used, which is stated without proof, which can be found in [8].

Theorem 2.2. *Suppose F is C^2, and $F(\lambda_0, u_0) = 0$ and F satisfies (i) from above, and $F_\lambda(\lambda_0, u_0) = 0$. Suppose*

$$H \equiv \begin{pmatrix} \langle \ell, F_{\lambda\lambda}(\lambda_0, u_0)\rangle & \langle \ell, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \\ \langle \ell, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle & \langle \ell, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \end{pmatrix}$$

and that $\det(H) > 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is $\{(\lambda_0, u_0)\}$.

Basic Periodic Solutions Results

The following results will be used as well. These are fairly elementary results on the existence and multiplicity of periodic solutions to ODEs. The first one is a well known result that can be found in [1].

Lemma 2.4. *Consider*

$$x' = f(\epsilon, t, x) \tag{2.1}$$

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where $f \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $x \in \mathbb{R}^n$. We suppose that $f(\epsilon, t + T, x) = f(\epsilon, t, x)$ for all $(\epsilon, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, and for $\epsilon = 0$ (2.1) has a T-periodic solution $y = y(t)$. Let $z(\epsilon, t, \xi)$ be the solution of the initial value problem:

$$z' = f(\epsilon, t, \xi), \qquad t > 0, \qquad z(0) = \xi$$

and let $A(\epsilon, t, \xi) = \frac{\partial z(\epsilon, t, \xi)}{\partial \xi}$. Suppose that $\lambda = 1$ is not an eigenvalue of $A(0, T, y(0))$. Then there exists $\delta > 0$ such that for $|\epsilon| < \delta$, there exists a C^1 function $\xi(\epsilon)$ such that $\xi(0) = y(0)$, and (2.1) has a unique T-periodic solution $y_\epsilon(t)$ with $y_\epsilon(0) = \xi(\epsilon)$.

Proof. We note that z must be T-periodic, and thus $z(\epsilon, T, \xi) = z(0) = \xi$. So we define the map

$$F(\epsilon, \xi) = z(\epsilon, T, \xi) - \xi.$$

Since z is continuously differentiable then F must be continuously differentiable as well. Also we see that $F(0, y(0)) = z(0, T, y(0)) - \xi = y(T) - y(0) = 0$, since y is T-periodic. Now we examine the Jacobian of F:

$$F_\xi = \begin{pmatrix} \frac{\partial F_1}{\partial \xi_1} & \cdots & \frac{\partial F_1}{\partial \xi_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial \xi_1} & \cdots & \frac{\partial F_n}{\partial \xi_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial(z_1 - \xi_1)}{\partial \xi_1} & \cdots & \frac{\partial(z_1 - \xi_1)}{\partial \xi_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial(z_n - \xi_n)}{\partial \xi_n} & \cdots & \frac{\partial(z_n - \xi_n)}{\partial \xi_n} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial z_1}{\partial \xi_1} - 1 & \cdots & \frac{\partial z_1}{\partial \xi_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_n}{\partial \xi_n} & \cdots & \frac{\partial z_n}{\partial \xi_n} - 1 \end{pmatrix} = A - I.$$

Now since $\lambda = 1$ is not an eigenvalue of $A(0, T, y(0))$, then we have that $\det(A(0, T, y(0)) - I) \neq 0$, and thus the matrix $F(0, y(0))_\xi = A(0, T, y(0)) - I$ is invertible. So then we have that F satisfies the hypotheses of the Implicit Function Theorem. So there is a C^1 function $\xi(\epsilon)$ and a $\delta > 0$ such that $F(\epsilon, \xi(\epsilon)) = 0$ for $|\epsilon| < \delta$ and $\xi(0) = y(0)$. So now we let $y_\epsilon(t) = z(\epsilon, t, \xi(\epsilon))$, noting that this clearly satisfies the conditions above. $\square$

The following is due to Mawhin [10].

Lemma 2.5. *Assume that f is continuous, $f(x, \cdot)$ is convex on $\mathbb{R}$ for each $x \in I$ and strictly convex on $\mathbb{R}$ for $x \in E \subset I$, where E has positive measure. Then the problem*

$$u'(x) + f(x, u(x)) - s = 0$$

$$u(0) = u(2\pi), \qquad f(x, u) \to +\infty \ as \ |u| \to \infty$$

has at most two solutions.

Proof. We know that the Cauchy Problem associated with this system is uniquely solvable. As a result, any two solutions this system cannot intersect, else the Cauchy problem fails to be uniquely solvable, where the point of intersection is the known value. This means for any pair of solutions of the system u, v, we can say, without loss of generality that $u(x) < v(x)$. Now we argue by contradiction. Assume that the system above has three periodic solutions u, v, and w. As before, we can say that

$$u(x) < v(x) < w(x) \quad \forall x \in I$$

Now consider the expression

$$\frac{(v(x) - u(x))'}{v(x) - u(x)} + \frac{f(x, v(x)) - f(x, u(x))}{v(x) - u(x)} = \frac{v(x)' + f(x, v(x)) - (u(x)' + f(x, u(x)))}{v(x) - u(x)}$$

$$= \frac{s - s}{v(x) - u(x)} = 0$$

Similarly,

$$\frac{(w(x) - u(x))'}{w(x) - u(x)} + \frac{f(x, w(x)) - f(x, u(x))}{w(x) - u(x)} = 0.$$

As a result we see that

$$\int_I \left[\frac{f(x, w(x)) - f(x, u(x))}{w(x) - u(x)} - \frac{f(x, v(x)) - f(x, u(x))}{v(x) - u(x)} \right] dx =$$

$$\int_I \left[-\frac{(w(x) - u(x))'}{w(x) - u(x)} + \frac{(v(x) - u(x))'}{v(x) - u(x)} \right] dx$$

Straightforward substitutions yield

$$\int_I \left[-\frac{(w(x) - u(x))'}{w(x) - u(x)} + \frac{(v(x) - u(x))'}{v(x) - u(x)} \right] dx = \ln(v - u) - \ln(w - u)|_I = 0$$

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since $\ln(v-u)$ and $\ln(w-u)$ are periodic (because u, v, and w are periodic). However, we assumed that f was strictly convex over a set E of positive measure, which is equivalent to saying that for $x \in E$

$$\frac{f(x,w(x))-f(x,u(x))}{w(x)-u(x)} - \frac{f(x,v(x))-f(x,u(x))}{v(x)-u(x)} > 0$$

Since E has positive measure, then we must have that

$$\int_I \left[\frac{f(x,w(x))-f(x,u(x))}{w(x)-u(x)} - \frac{f(x,v(x))-f(x,u(x))}{v(x)-u(x)} \right] dx > 0$$

Which is a contradiction. Thus the system above can have at most two solutions. $\square$

We have the following as a corollary:

Corollary 2.1. *Let*

$$x' = f(x,t), \tag{2.2}$$

Where $f(t+T,x) = f(t,x)$ and $f_{xx}(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, then (2.2) has at most two solutions.

Proof. Suppose that it has three solutions u_1, u_2, u_3. Then each $u_i \left(\frac{2\pi t}{T} \right)$ satisfies the problem in Lemma (2.5), a contradiction. $\square$

Both of these results will be used repeatedly in the proof that follows.

Chapter 3

Existence and Multiplicity of Periodic Solutions

In this section we will establish the following:

Theorem 3.1. *Let f be a logistic type function, and let $h(t)$ be a continuous, T-periodic function such that $\int_0^T h(t)dt > 0$. Then there exists an ϵ_0 such that the ODE*

$$u' = f(u) - \epsilon h(t)$$

has exactly two T- periodic solutions for $\epsilon < \epsilon_0$, has exactly one T-periodic solution for $\epsilon = \epsilon_0$, andthe ODE has no T periodic solutions for $\epsilon > \epsilon_0$.

Proof. For $\epsilon = 0$ our ODE has two periodic solutions $u_0 = 0$ and $v_0 = M$. Suppose that z satisfies

$$z' = f(z) - \epsilon h(t) \qquad z(0) = \xi$$

Then if $A = \frac{\partial z}{\partial \xi}$ then

$$A' = \frac{\partial A}{\partial t} = f'(z)A,$$

and

$$A(0, 0, \xi) = \frac{\partial z(0, 0, \xi)}{\partial \xi} = \frac{\partial \xi}{\partial \xi} = 1.$$

If $z = u_0$ then we have that $A' = f'(0)A$ with $A(0) = 1$. This gives $A(t) = e^{f'(0)t}$, and thus $A(T) = e^{f'(0)T} \neq 1$ since $f'(0) > 0$. Similarly, when $z = v_0$ we get $A(T) = e^{f'(M)T} \neq 0$ since $f'(M) < 0$. As a result of Lemma 2.4 we have unique periodic solutions for each ϵ in some neighborhood of 0, and we can repeat this process from each solution, provided the solution of

$$A' = f'(z)A, \qquad A(0, 0, \xi) = 1$$

satisfies $A(T) \neq 1$. But the curves (ϵ, u_ϵ) and (ϵ, v_ϵ) cannot be continued indefinitely. Let u_1 be such that $\max_{u \in \mathbb{R}} = f(u_1)$, and $\overline{h} = T^{-1} \int_0^T h(t)dt > 0$. If u is a T-periodic solution:

$$0 = u(T) - u(0) = \int_0^T u'(t)dt = \int_0^T \left[f(u) - \epsilon h(t) \right] dt$$

$$= \int_0^T f(u) - f(u_1)dt + \int_0^T f(u_1) - \epsilon h(t)dt$$

$$= \int_0^T f(u) - f(u_1)dt + Tf(u_1) - \epsilon\overline{h}$$

If $\epsilon > \frac{f(u_1)}{\overline{h}}$ then $Tf(u_1) - \epsilon\overline{h} < 0$, and by definition $\int_0^T f(u) - f(u_1)dt < 0$. So then, for $\epsilon > \frac{f(u_1)}{\overline{h}}$,

$$\int_0^T f(u) - f(u_1)dt + Tf(u_1) - \epsilon\overline{h} < 0,$$

a contradiction. So we must have a degenerate solution along the curve, and let (ϵ_*, u_*) be the first degenerate solution along this curve. At this point we must have $A(T) = 1$. Define $F(\epsilon, u) = z(\epsilon, T, u) - u$, we see that

$$F_u(\epsilon_*, u_*) = A(T) - 1 = 0.$$

So as a linear transformation, $F_u(\epsilon_*, u_*)[\tau] = 0$ for all $\tau \in \mathbb{R}$, so the null space of $F_u(\epsilon_*, u_*) = \mathbb{R}$, and thus has dimension 1, and the codimension of its range is 1. To use the Saddle Node Bifurcation theorem, we need that $F_\epsilon(\epsilon_*, u_*) \neq 0$. So we assume that it is equal to 0 and derive a contradiction. If $F_\epsilon(\epsilon_*, u_*) = 0$ then all the conditions of Theorem 2.2 are satisfied, except the one about the matrix H. Note, since $F_u(\epsilon_*, u_*) = 0$ then the ℓ is the identity mapping.

Now, if $B(t) = F_\epsilon(\epsilon_*, u_*) = \frac{\partial z(\epsilon_*, T, u_*(0))}{\partial \epsilon}$, then B satisfies

$$B' = f'(u_*(t))B - h(t), \quad t > 0, \quad B(0) = 0.$$

Solving this ODE gives that $B(t) = A \int_0^t A(s)^{-1}h(s)ds$. Differentiation confirms. Let $C(t) = F_{\xi\xi}(\epsilon_*, u_*)$, and this can be evaluated by solving

$$C' = f'(u_*(t))c + f''(u_*(t))A^2, \quad C(0) = 0.$$

The solution to this ODE is given by $C(t) = f''(u_*(t))A(t) \int_0^t A(s)ds$, where

$$A(t) = \exp\left(\int_0^t f'(u_*(s))ds\right).$$

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Similarly if $D(t) = F_{\epsilon\xi}(\epsilon_*, u_*)$, then we can use the above equation to get that

$$D' = f'(u_*(t))D + f''(u_*(t))AB, \quad D(0) = 0.$$

Solving this ODE gives $D(t) = f''(u_*(t))A(t)\int_0^t B(s)ds$.

For $E(t) = F_{\epsilon\epsilon}(\epsilon_*, u_*)$ then, as before, we have that:

$$E' = f'(u_*(t))E + f''(u_*(t))B^2, \quad E(0) = 0$$

and the solution to this ODE is given by $E(t) = f''(u_*(t))A(t)\int_0^t A(s)^{-2}b(s)^2 ds$. Note that this also gives that $E(T) < 0$.

So we have that:

$$F_\epsilon(\epsilon_*, u_*) = B(T), \ F_{\xi\xi}(\epsilon_*, u_*) = C(T),$$

$$F_{\epsilon\xi}(\epsilon_*, u_*) = D(T), \ F_{\epsilon\epsilon}(\epsilon_*, u_*) = E(T).$$

So the matrix H in the above theorem takes the form

$$H = H(\epsilon_*, u_*) = \begin{pmatrix} E(T) & D(T) \\ D(T) & C(T) \end{pmatrix}$$

$$\det(H) = E(T)C(T) - D^2(T) =$$

$$(f''(u_*(T))A(T))^2 \left[\int_0^T A(s)ds \int_0^T A(s)^{-1}B(s)^2 ds \right.$$

$$\left. - \left(\int_0^T B(s)ds \right)^2 \right] > 0$$

by the Cauchy-Schwarz Inequality. By Theorem 2.2 this implies that the solution set of $F(\epsilon, u) = 0$ near (ϵ_*, u_*) is the singleton set $\{(\epsilon_*, u_*)\}$. This contradicts that (ϵ_*, u_*) is limit point of a curve of solutions of our ODE. Therefore $F_\epsilon(\epsilon_*, u_*) \neq 0$ and the Saddle Node bifurcation theorem applies.

So then we have a branch of solution extending from $\epsilon = 0$ to $\epsilon = \epsilon_*$. At $\epsilon_a st$ the saddle node bifurcation theorem applies, and thus we have a curve of solutions $(\epsilon(s), u(s))$, such that $\epsilon(0) = \epsilon_*$ and $\epsilon'(0) = 0$. Moreover we have that

$$\epsilon''(0) = -\frac{F_{\xi\xi}(\epsilon_*, u_*)}{F_\epsilon(\epsilon_*, u_*)}$$

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From above we had that $F_{\xi\xi}(\epsilon_*, u_*) = C(T) < 0$. Now if $F_\epsilon(\epsilon_*, u_*) = B(T) > 0$, then we have that $\epsilon''(0) > 0$ and we see that $\epsilon(s)$ is a parabola like curve that open to the right. But the point (ϵ_*, u_*) is a limit point to the curve from the left, and thus if the bifurcation curve opens to the right, we have a problem. So we have that $B(T) < 0$ and thus $\epsilon''(0) < 0$, and hence we have a parabola-like curve that open to the left. The lower branch of this parabola must match up with the curve leading up to (ϵ_*, u_*), else we have a neighborhood of ϵ that has three periodic solutions, which is not possible. Now we can continue the upper branch with decreasing ϵ from (ϵ_*, u_*). If we reach another degenerate solution along this curve, then we can apply the same argument as we did at our last degenerate solution. As a result, we could setup another bifurcating curve, which would give us a neighborhood of ϵ with three periodic solutions, a contradiction. Since we have no more degenerate solutions, we can extend this curve all the way until $\epsilon = 0$ to the constant solution $u = M$. So we have at least two periodic solutions in $(0, \epsilon_*)$, and using Corollary 2.1, we have exactly two periodic solutions.

Now if for some $\epsilon > \epsilon_*$ we have a periodic solution, then using the same arguments as above, we can extend a curve from this solution all the way back to $\epsilon = 0$. But, since we have at most 2 solutions, this curve will match up with the branches of the curve we defined above. As a result this curve cannot extend beyond ϵ_*, since we have a limit point there. So we cannot have a solution beyond ϵ_*. So all our periodic soutions lie on this curve and the result is proved. $\qquad\square$

Chapter 4

The Behavior of the Threshold

In the proof of the Theorem 1.1 in [9] an upper bound is given for the threshold value
which is based on the average harvesting. In particular we had that

$$\epsilon_0 \leq \frac{T \max f(u)}{\int_0^T h(t)dt}$$

So when the total harvesting is the same, then we see that the upper bound for theis
threshold remains unchanged. For example we could consider the set of harvesting
functions $h_a(t) = 1 + a\cos t$. For each h_a we have

$$\epsilon_0 \leq \frac{2\pi \max f(u)}{\int_0^{2\pi} h_a(t)dt} = 2\pi \max f(u)$$

We note that the difference between each h_a is the total variation of the function.
One of the hypotheses was that $\int_0^{2\pi} h(t)dt > 0$, meaning that we are going to be
harvesting more than we are stocking. However, if we increase the variation of the
harvesting we are going to be harvesting more from the population, and it will make it
more difficult for the population to recover, even though we are stocking more. So we
can intuitively conclude that if we increase the variation of the harvesting function, it
should lower the threshold value. This is conjectured at the end of [9]. In this section
we are going to look at classes of function that have the same average harvesting, but
have larger variation. It is easy to setup some of these classes, for example $1 + a\cos t$
or $2 + b\sin t$, but we will also be looking at some more complicated piecewise linear
harvesting functions. In order to determine the threshold value, we will be relying on
Mathematica to numerically solve these ODEs.

First we will consider the above example: the class of functions $h(t) = 1 + a\cos t$.
To evaluate this threshold we made use of Mathematica's NDSolve function. We used
a trial and error approach to deterimine the threshold value, and as figures 4.1 - 4.5
indicate, it behaves as conjectured.
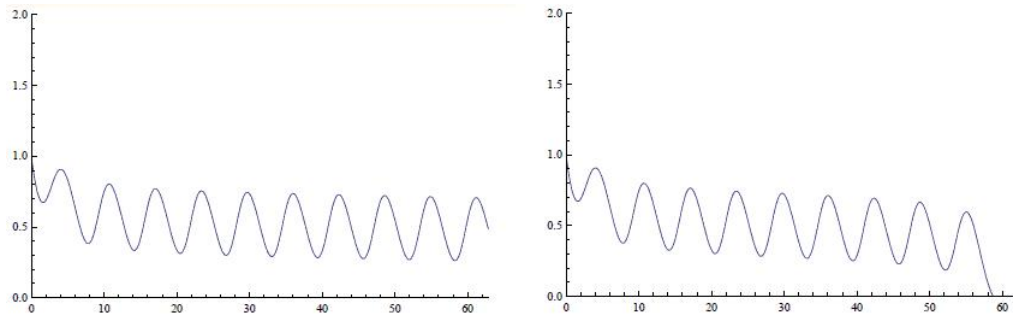
Figure 4.1: Left: $\epsilon = .225$ Right: $\epsilon = .226$
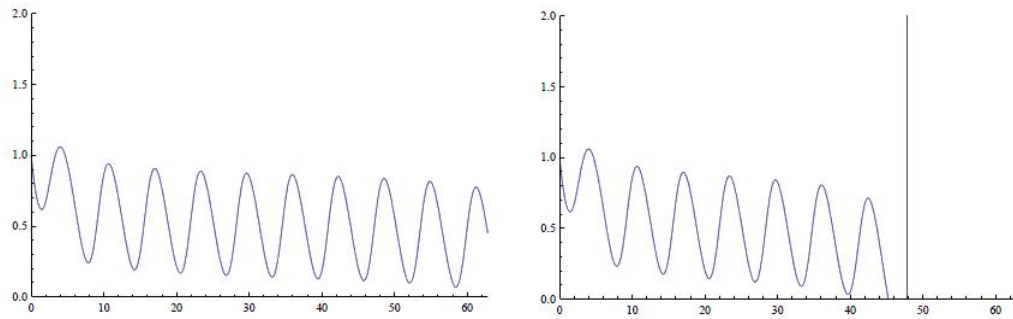

Figure 4.2: Left: $\epsilon = .186$ Right $\epsilon = .187$
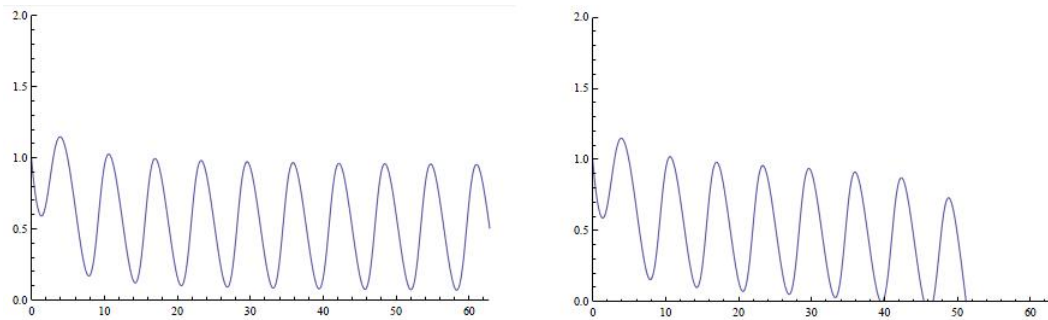

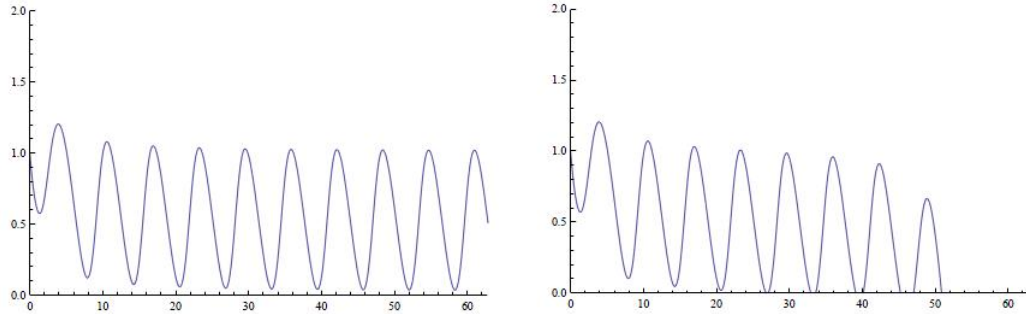Figure 4.3: Left: $\epsilon = .153$ Right $\epsilon = .154$

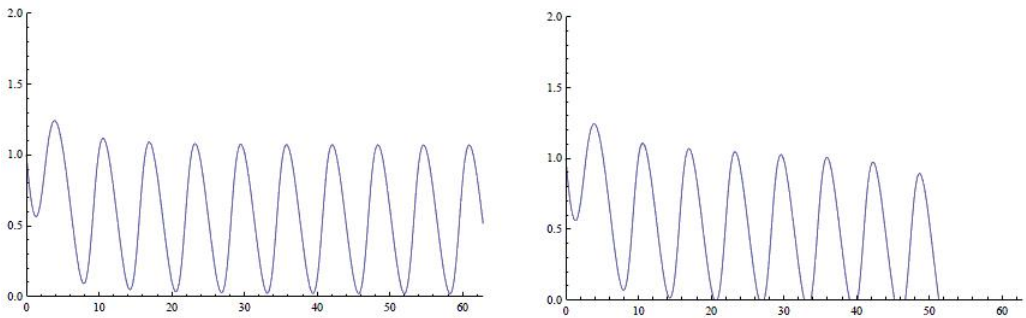Figure 4.4: Left: $\epsilon = .129$ Right $\epsilon = .13$

Figure 4.5: Left: $\epsilon = .111$ Right $\epsilon = .112$

Next we consider $h(t) = 1 + a\sin t$. It turns out that these, while producing different solutions, have the same threshold values as $1 + a\cos t$. This is expected, since the change to sin shifts the harvesting term to the right by $\frac{\pi}{2}$, and so the solutions are shifted as well.

So the conjecture appears to hold for a fairly large class of functions. Now we are going to look at a piecewise linear harvesting function. Recall that the proof of the main result required that $\int_0^T h(t)dt > 0$. So our piecewise linear function will be taking on a sin shape, roughly, but it's integral will be larger than 0. In particular, let's say that

$$g^*(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 1 \le t \le 2 \\ 1 - \frac{t}{2} & 2 \le t \le 3 \\ \frac{t}{2} - 2 & 3 \le t \le 4 \end{cases}$$

We can then extend this periodically by defining $g(t) = g^*(t - \lfloor \frac{t}{4} \rfloor)$. The graph is given
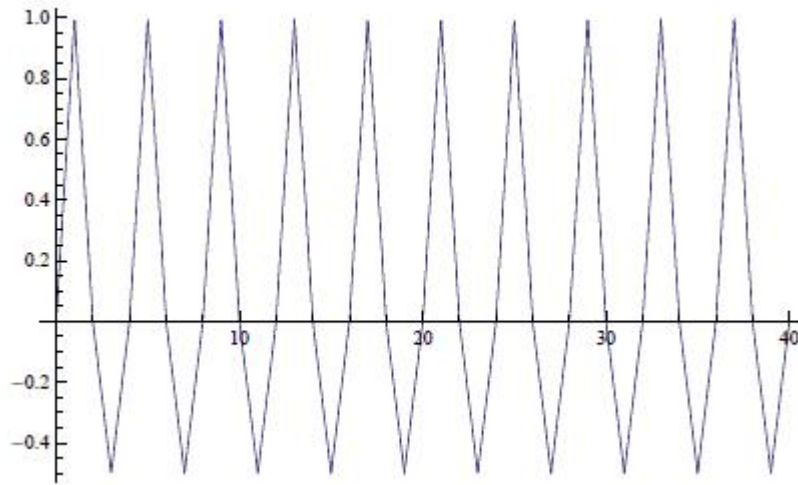
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Figure 4.6: Piecewise linear harvesting function $g(t)$.

in Fig. 4.6. Now, by merely considering areas we can see that

$$\int_0^4 g(t)dt = 1 - \frac{1}{2} = \frac{1}{2}$$

Again, we use Mathematica to determine the threshold values when this is used at the harvesting function.
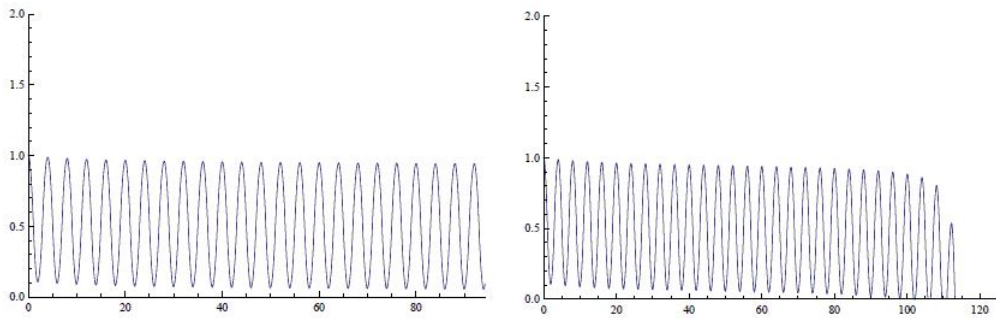

Figure 4.7: Left: $\epsilon = 1.171$.   Right: $\epsilon = 1.172$.

Now, we can actually use this function to define a class of functions that have the same area over the period, but have a larger variation. To do this, we take the first half of the period where g is positive and scale the function by some constant $\alpha > 1$. Then

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in the last half where g is negative, we want to find a $\beta > 1$ such that the function

$$g_\alpha(t) = \begin{cases} \alpha g(t) & 0 \le t \le 2 \\ \beta g(t) & 2 \le t \le 4 \end{cases} \qquad \text{has } \int_0^4 g_\alpha(t)dt = \frac{1}{2}.$$

Finding this β is easy, for if we calculate the integral by areas we have

$$\int_0^4 g_\alpha(t)dt = \alpha - \frac{\beta}{2} = \frac{1}{2}.$$

Solving for β gives

$$\beta = 2\alpha - 1.$$

So then, using these values, we can form a whole class of piecewise linear harvesting terms that all have the same average harvesting. So we take $\alpha = 1.5$, which gives $\beta = 2$. The plot for this function is given in Figure 4.8. Again we can use Mathematica
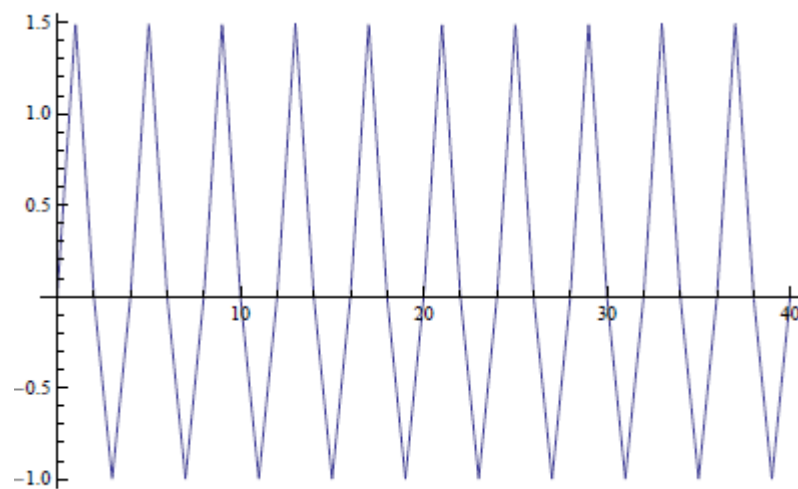

Figure 4.8: $g_{\frac{3}{2}}(t)$

to find the solutions. The threshold is found to be $\epsilon_0 = .840$. So the threshold drops, as we suspected. But now we will look at this piecewise linear harvesting function and compare it to a similar harvesting term that is smooth. So we consider $g(t)$ as given above, and we also introduce $h(t) = .125 + .7\sin\left(\frac{\pi t}{2}\right)$. The plot of these functions is given in Figure 4.11. Note now that the total variation of $g(t)$ is 1.5, whilst the total
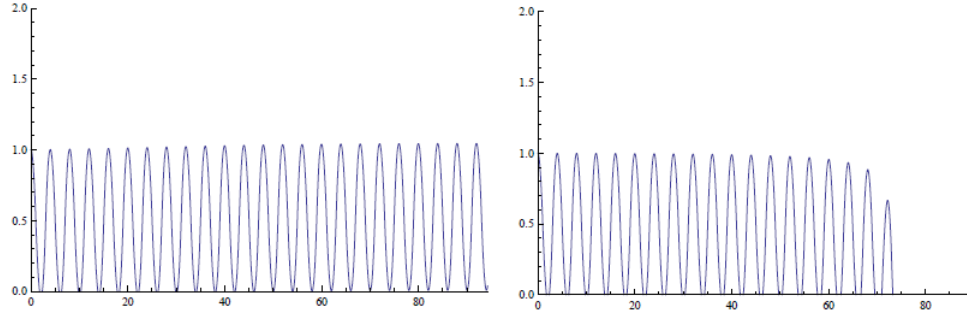
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Figure 4.9: Left: $\epsilon = .84$.   Right: $\epsilon = .841$.

variation of this $h(t)$ is 1.4, but $\int_0^4 h(t)dt = 4(.125) = .5$, the same average harvesting as $g(t)$. So if the conjecture given at the end of [9] is true, the we should expect the threshold for $g(t)$ to be smaller that that of $h(t)$. Recall that the threshold for $g(t)$ was $\epsilon = 1.171$. We numerically solve the ODE to get the threshold for $h(t)$ as 1.09. So our numerical experiments seem to indicate that the conjecture need not be true. We can consider a similar situation with $g_{\frac{3}{2}}(t)$ and $h(t) = .125 + 1.2\sin\left(\frac{\pi t}{2}\right)$. As before, note that the total variation of $g_{\frac{3}{2}}(t)$ is 2.5, whilst the total variation of this $h(t)$ is 2.4 and consider Figures 4.11 and 4.12. Recall that the threshold for $g_{\frac{3}{2}}(t)$ was $\epsilon = .84$. We numerically solve the ODE to get the threshold for $h(t)$ as .746. So again, this seems to point to the fact that the conjecture need not be true. We have observed that the conjecture holds within limited classes of functions, in particular smooth functions.
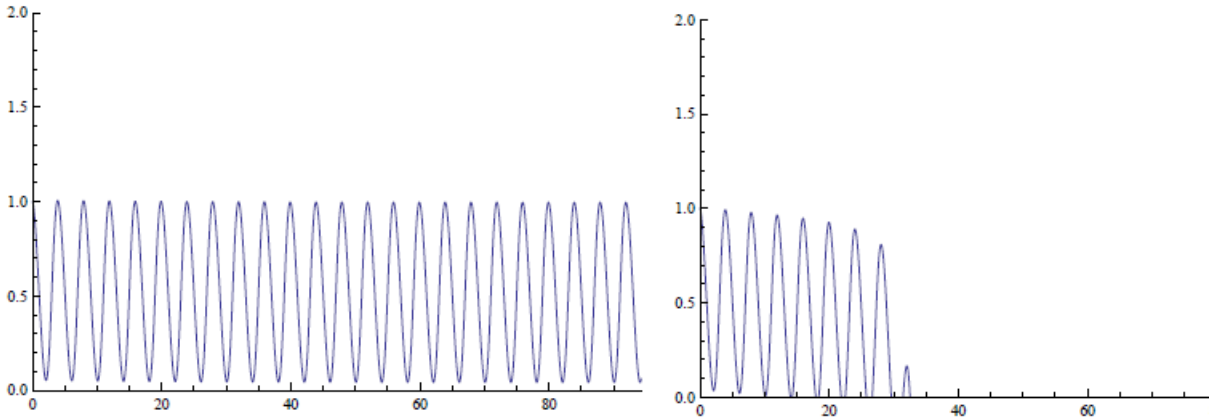

Figure 4.10: Solutions for $h(t) = .125 + .7\sin\left(\frac{\pi t}{2}\right)$. Left: $\epsilon = 1.09$.   Right: $\epsilon = 1.1$.
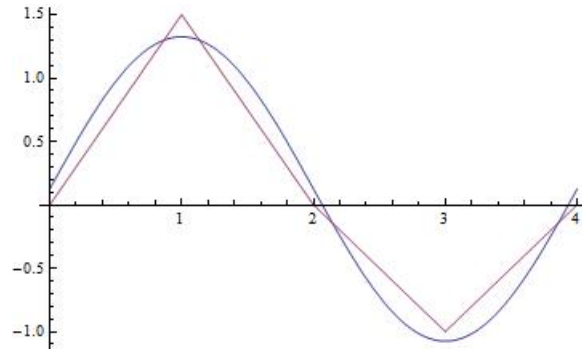
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Figure 4.11: $g_{\frac{3}{2}}(t)$ and $h(t) = .125 + 1.2\sin\left(\frac{\pi t}{2}\right)$.
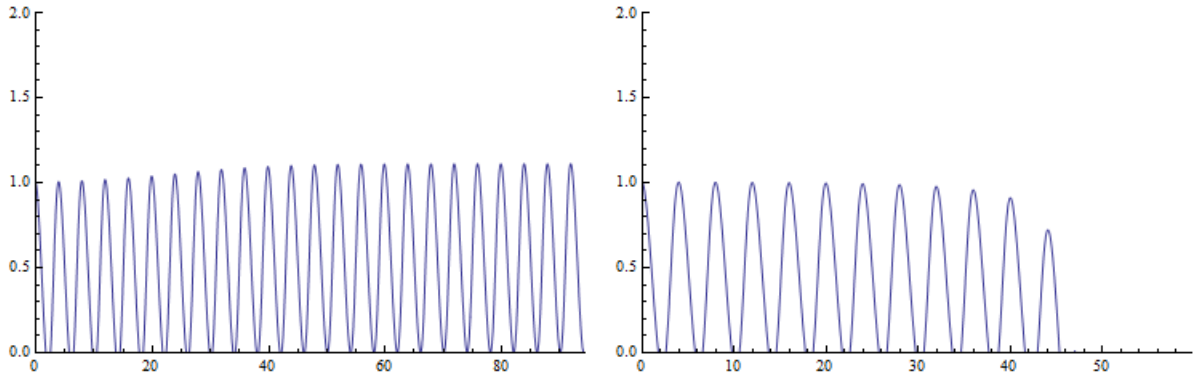
Figure 4.12: Solutions for $h(t) = .125 + 1.2\sin\left(\frac{\pi t}{2}\right)$. Left: $\epsilon = .746$.   Right: $\epsilon = .747$

But when we compare these to function that are not smooth the conjecture does not hold. It is also worth noting that the smoother harvesting function spend a longer time at greater harvesting that the piecewise linear harvesting terms. This sustained time of heavier harvesting could be what contributes to the lower threshold. Further experimentation with the numerics may lead to a firmer idea about what the actual cause of the lower threshold is.

Chapter 5

Positive Solutions of the Second Order Logistic Type ODE

Having now considered the case of harvesting over an entire population, it is natural for us to move into more space conscious considerations. We will be considering the well known reaction-diffusion equation with Dirichlet boundary conditions, a logistic type growth term, and a positive, bounded, continuous harvesting term that depends on the population (instead of time):

$$u_t = u_{xx} + f(u) - \epsilon h(u), \quad u(0,t) = u(L,t) = 0, u(x,0) = u_0(x).$$

Since we are considering this in the context of an actual population we are going to be interested primarily in steady state solutions ($u_t(x,t) = 0 \forall t > 0$), for if we are considering a biological model, then the steady state represents sustainability. Obviously only positive solutions will make sense, as a negative population has no physical meaning. So we have

$$0 = u_{xx} + f(u) - \epsilon h(u), \qquad u(0) = u(L) = 0.$$

Naturally, this can be considered an ODE, and for convenience of notation we will consider the variable as t instead of x:

$$-u'' = f(u) - \epsilon h(u), \qquad u(0) = u(L) = 0. \tag{5.1}$$

We wish to learn about the positive solutions as we change the value of the parameter ϵ.

The Phase Plane

We will be using the phase plane to show our results in this section. Many undergraduate ODE textbooks will contain a discussion on the use and construction of phase planes, for example [7] or [2]. Along these lines, we will be constructing the phase plane for the above ODE. First we can write it as a system of first order ODEs:
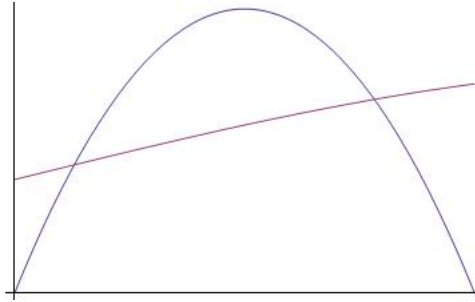
$$u' = v, \qquad -v' = f(u) - \epsilon h(u).$$

Figure 5.1: Intersections of $f(u)$ and $\epsilon h(u)$.

Now, this fits the definition of a conservative system, and thus the total energy in the system remains constant. This gives

$$\frac{1}{2}v^2 - F(u) + \epsilon H(u) = E \tag{5.2}$$

where E is a constant, and F and H are the primitives of f and h repsectively. This equation gives the level curves of the phase plane. Now we need to find the equilibrium points, that is to say, the constant solutions of the ODE, and classify them. This requires that $f(u) - \epsilon h(u) = 0$, which gives $f(u) = \epsilon h(u)$. Now we can see that since $h(u) > 0$, and since $f(u)$ is bounded, then there is a sufficiently large ϵ such that there are no equilibrium points. If there is a an equilibrium point, then the fact that $f(0) = 0$ and $h(u) > 0$ means that that at the first intersection we have that $h(u)$ crosses $f(u)$ from above to below. Its qualitative behavior is similar to that of Figure 5.1. However, due to the general nature of $h(u)$, it need not only intersect $f(u)$ at only one place, but rather we can say that it will intersect at at least two places. This leads to at least two equilibrium points on the phase plane, let us call these points $u = a_0$ and $u = a_1$, noting that $a_0 \leq a_1$. Now we want to classify these equilibrium points. We will be using the method and theory of classification as given in chapter 2 of [7]. The general idea to classify them is to consider the behavior of the phase plane in a small neighborhood of the equilibrium points using a Taylor series expansion near that point. This linearization leads us to consider the eigenvalues of the following matrix at the

equilibirum points

$$\begin{pmatrix} v_u & v_v \\ \frac{\partial}{\partial u}(\epsilon h(u) - f(u)) & \frac{\partial}{\partial v}(\epsilon h(u) - f(u)) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \epsilon h_u(u) - f_u(u)) & 0 \end{pmatrix}$$

So at $u = a_0$, the eigenvalues are found by solving

$$\lambda^2 + f_u(a_0) - \epsilon h_u(a_0)$$

Now, since $f(0) = 0$ and $h(u) > 0$, then at the first intersection $u = a_0$, we have $\epsilon h(u)$ diving below $f(u)$, and thus $\epsilon h_u(a_0) < f_u(a_0$, which gives that $f_u(a_0) - \epsilon h_u(a_0) > 0$, and thus $\lambda^2 + f_u(a_0) - \epsilon h_u(a_0)$ has two complex solutions with no real part, and as a result we can say that this equilibrium point is a center. For $u = a_1$, the situation is reversed, and so $\lambda^2 + f_u(a_1) - \epsilon h_u(a_1)$ will have two distinct, real, nonzero solutions, and thus this point is a saddle.

But now what if there is no intersection of $\epsilon h(u)$ and $f(u)$? Let us suppose that we have ϵ large enough such that $\epsilon h(u) > f(u)$. Now, by differentiation of both sides of equation 5.2 with respect to u we have that

$$v v_u = \epsilon h(u) - f(u) \quad \Rightarrow \quad v_u = \frac{\epsilon h(u) - f(u)}{v}.$$

Since $\epsilon h(u) > f(u)$, we have that if $v > 0$, then $v_u > 0$ and $v < 0$ gives that $v_u < 0$, no matter the value of u. So then for postive u, the level curves of the phase plane will always be moving away from the u axis.

So we have two regimes of the phase plane. Moreover, the continuity of our logistic type function and harvesting function allow us to say that there is an ϵ_0 such that for $\epsilon < \epsilon_0$ we have two intersections of $f(u)$ and $\epsilon h(u)$, and thus a saddle and a center on the phase plane. For $\epsilon > \epsilon_0$, we have no equilibrium points, and the slope of the phase plane has the same sign as v. So as our ϵ moves away from zero, we transition between these regimes. See figure 5.2. In fact, due to the convexity of $f(u)$, we must have that the first center and saddle will be getting closer to each other as the ϵ approaches this threshold.
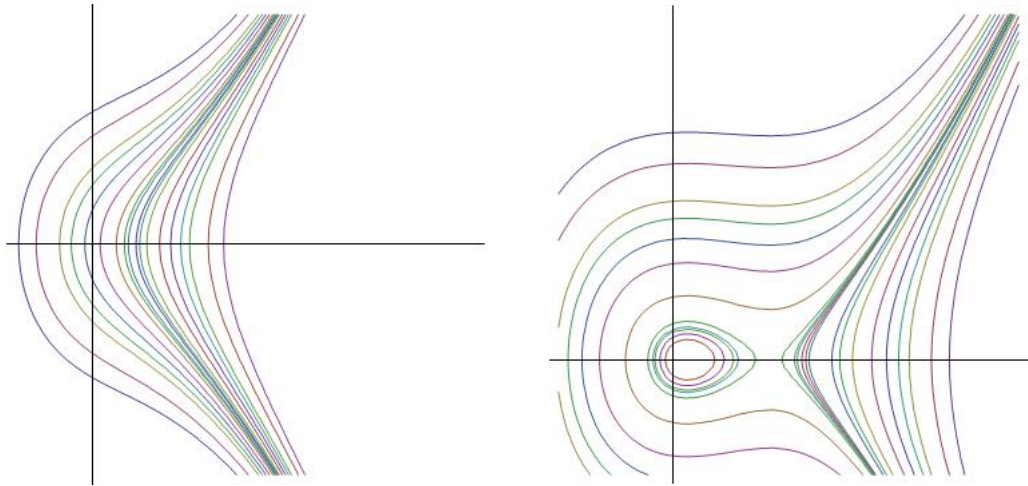
Figure 5.2: Left:For $\epsilon > \epsilon_0$. Right: For $\epsilon < \epsilon_0$

The Boundary Value Problem

So what does all this mean for the BVP (5.1)? Well in order to have a positive solution, we must have that u starts at zero, goes positive, and then after some must return to 0 at L. If $\epsilon > \epsilon_0$ then the level curves never cross the u-axis, and thus our positive solutions cannot return to $u = 0$. As a result, we can say that for $\epsilon > \epsilon_0$ (5.1) has no positive solutions. Since at $u = a_0$ we have a center, then we have that the solutions or orbiting around this point, until they reach the saddle at $u = a_1$. The orbits will produce solutions that will start at 0 and end at 0, and these are our candidates for positive solutions. But not every one of these represents the solution to the BVP (5.1), since it may not return to zero over the interval $(0, L)$. In order to determine what the length of the interval over which the BVP is satisfied we will be using the so called Time Map. Recall that we constructed the phase plane using

$$\frac{1}{2}v^2 + \epsilon H(u) - F(u) = E$$

We can solve for v and recall that $v = \frac{du}{dt}$ and we get

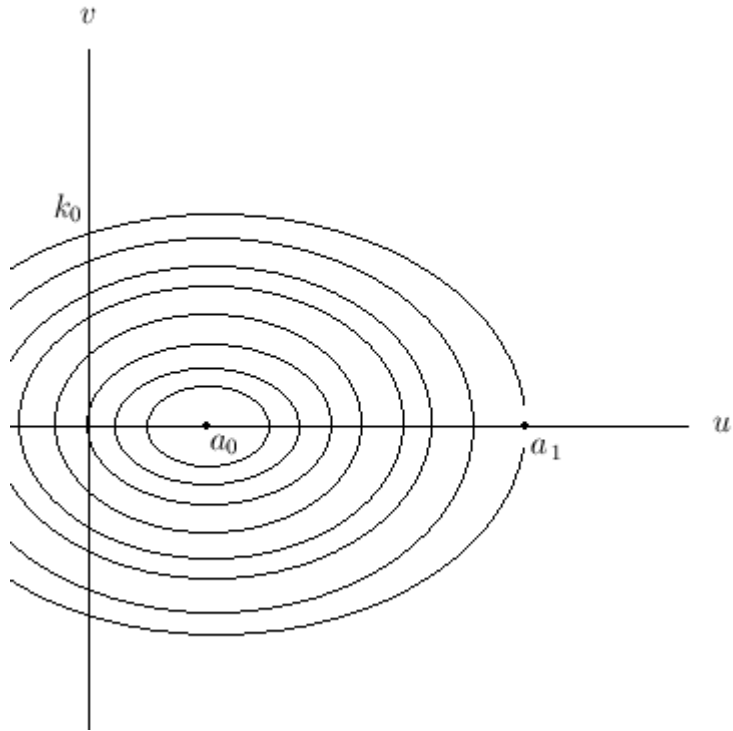$$\frac{du}{dt} = \sqrt{2}\sqrt{E + F(u) - \epsilon H(u)}.$$

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Figure 5.3: The Phase Plane

This can be thought of as a separable differential equation, this gives

$$T(k) = \frac{1}{\sqrt{2}} \int_0^{w(k)} \frac{du}{\sqrt{E + F(u) - \epsilon H(u)}}.$$

We say that we are integrating along one of the level curves on the phase plane that starts at $(0, k)$ and goes to $(w(k), 0)$, and what this integral gives half the length of the interval for which the solution passes over. So we can use this time map to determine the values of L for which we have postive solutions without doing any explicit calculations. Now, one of the orbits around the equilibrium point $(a_0, 0)$ goes from $(0, k_0)$ to $(a_1, 0)$, see Figure 5.2 . But the point $(a_1, 0)$ is an orbit itself, since it is an equilbrium point, and thus the line that proceeds from k_0 does not reach the u-axis in finite time (or in this case "time" refers to the length of the interval for which (5.1) has a positive solution). So as our starting point along the v-axis goes from 0 to k_0, then we have that the time map starts at some finite value $\frac{L_0}{2}$ and goes to infinity. Continuity of the time map indicates that we must hit every value in between. The preceeding discussion

has yielded the following result:

Theorem 5.1. *Given the BVP (5.1) there exists an ϵ_0 such that for $\epsilon > \epsilon_0$ there are no positive solutions and for $\epsilon < \epsilon_0$ there is an L_0 such that for any $L \geq L_0$ the BVP (5.1) has at least one positive solution.*

Now when we consider the intersections of $f(u)$ and $\epsilon h(u)$, we see that they get farther aprt as $\epsilon \to 0$, due to the convexity of f. This causes the time map to get smaller. As a result, if our ϵ is small, the the space required to maintain the positive solutions becomes small, that is, the length of our smallest interval for which there is a positive solution gets smaller. Similarly, as ϵ approaches its threshold we see that the smallest length of the interval is going to become larger, and thus more space is required to have a postive solution, and thus have a steady state. This result is fairly unspecific and only establishes existence. A more in-depth analysis of the time map could yield a result on the multiplicity of the solutions as ϵ changes, see [3] as an example.

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