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Average Cayley Genus for Groups with Two Generators of Order Greater Than Two

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AVERAGE CAYLEY GENUS FOR GROUPS WITH TWO GENERATORS

OF ORDER GREATER THAN TWO

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ABSTRACT

Determining the orientable surfaces on which a particular graph may be imbedded is a basic problem in the area of topological graph theory. We look at groups modeled by Cayley graphs. Imbedding Cayley graphs with symmetry is done using Cayley maps. It is of interest to find the average Cayley genus for a particular group and generating set for the group. We consider the group known as the generalized quaternions with generating set Δ, where Δ contains two generators with order greater than two. We find a formula for the average Cayley genus of the generalized quaternions. Moreover, we determine a formula for the average Cayley genus of any finite group that can be generated by two generators with order greater than two. Finally, we find the average Cayley genus of a finite group with generating set consisting of three elements, two with order greater than two and one with order two.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

We start with some fundamental definitions and theorems, which may be found in Chartrand, Lesniak, and Zhang [1]. A **graph** G is a finite nonempty set V of objects called **vertices**, represented by points or nodes in a diagram, together with a possibly empty set E of 2-element subsets of V called **edges**. The **vertex set of a graph** G, denoted by $V(G)$, is the set of all vertices of a graph. The **edge set of a graph** G, denoted by $E(G)$, is the set of all edges of a graph. For the graph G, shown in Figure 1.1, we see $V(G) = \{u, v, w, x\}$ and $E(G) = \{uv, vx, xu, uw, wx\}$.

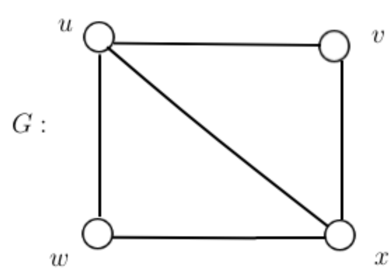


Figure 1.1: The graph G.

Since uv and vx are distinct edges in G that share a vertex, uv and vx are **adjacent edges**. The vertices u and v are called **adjacent vertices** if an edge exists between them. The vertex u and the edge uv are said to be **incident** with each other. The **size** of a graph G is the number of edges of G. The **order** of a graph G is the number of vertices of G. The graph G in Figure 1.1 has size 5 and order 4. The **degree of a vertex** v in a graph G is the number of vertices in G that are adjacent to v. For the graph in Figure 1.1, the degree of v is 2, which we write $deg_G(v) = 2$. If the graph is understood, we simply write $deg(v)$. A graph G is **regular** if all vertices of G have the same degree, and is regular of degree r (or r-regular) if this degree is r. The first theorem of graph theory found in Chartrand, Lesniak, and Zhang [1] states that the sum of all the degrees of each vertex in the graph is twice the size of the graph.

Theorem 1. *If G is a graph of size m, then* $\sum_{v \in V(G)} \deg(v) = 2m.$

For two (not necessarily distinct) vertices u and v in a graph G, a $u-v$ **walk** W is a sequence of vertices, beginning with u and ending at v such that consecutive vertices in W are adjacent. A walk in a graph G where no edge is repeated is a **trail** in G. A walk in a graph in which no vertex is repeated is called a **path**. Using the graph G in Figure 1.2, we can find a walk, trail, and path. An example of a walk is $W = (u, v, u, x, y, w, x, u)$, a trail is $T = (u, v, y, w, v)$, and a path is $P = (u, v, w, y)$. Two vertices u and v in a graph G are **connected** if G contains a $u - v$ path. A graph G is **connected** if every two vertices of G are connected.

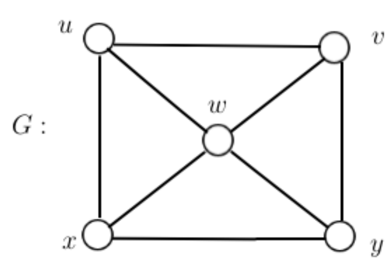


Figure 1.2: Walks in a graph.

A **cycle** C_n is a sequence of n vertices in which consecutive vertices are joined by an edge, and the only vertex repeated is the initial and terminating vertex. An example of a cycle in Figure 1.2 is the set vertices $\{u, v, y, x\}$ with edges $\{uv, vy, yx, xu\}$. This cycle is known as an even cycle, since there are an even number of edges. We also refer to a cycle of length n as an n-cycle. If $n = 3$, we have a 3-cycle, also known as a **triangle**. A triangle is also a cycle of odd length. In Figure 1.3, the set of vertices $\{u, x, w\}$ and the edges $\{ux, xw, wu\}$ form a triangle.

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Shown in Figure 1.3 is a subgraph of the graph G shown in Figure 1.1. A subgraph H of G is a **proper subgraph** of G, if $V(H)$ is a proper subset of $V(G)$ or if $E(H)$ is a proper subset of $E(G)$. A connected subgraph H of a graph G is a **component** of G if H is not a proper subgraph of any other connected subgraph of G. If $V(H) = V(G)$, then H is a **spanning subgraph** of G. The graph H pictured in Figure 1.3 contains a triangle and is a spanning subgraph of G. A graph is **triangle-free** if it has no subgraph that is a triangle. Two triangle-free graphs are shown in Figure 1.4.

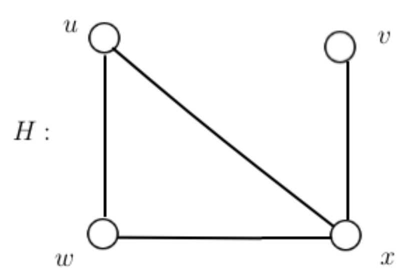


Figure 1.3: A subgraph H of G.

The graphs G and H in Figure 1.4 are also known as bipartite. A graph G is **bipartite** if $V(G) = A \cup B$ (A and B are called partite sets), where $A \cap B = \emptyset$ and every edge of G joins a vertex of A and a vertex of B. The partite sets of the graph G in Figure 1.4 are as follows $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. A graph H is a **complete bipartite graph** if $V(H)$ can be partitioned into two sets A and B so that ab is an edge of H if and only if $a \in A$ and $b \in B$. If $|A| = m$ and and $|B| = n$, then we denote the complete bipartite graph by $K_{m,n}$. The graph

H of Figure 1.4 is $K_{3,3}$.

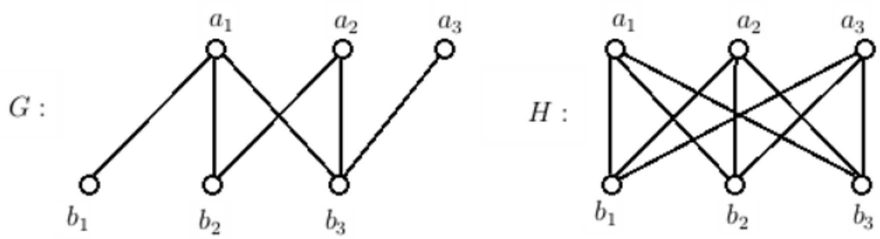


Figure 1.4: A bipartite graph G and a complete bipartite graph H.

A bipartite graph is necessarily triangle-free. This is given by the following theorem found in White [12].

Theorem 2. *A nontrivial graph is bipartite if and only if all its cycles are even.*

A **tree** is a connected acyclic (has no cycles) graph. A **spanning tree** H of a graph G is a spanning subgraph of G that is a tree. In Figure 1.5, H is a spanning tree of G. The following theorem found in Chartrand, Lesniak, and Zhang [1] states that a graph G is a tree if it satisfies any of the two properties in Theorem 3.

Theorem 3. *Let G be a graph of order n and size m. If G satisfies any two of the following three properties, then G is a tree:*

(1) G is connected,

(2) G has no cycles,

(3) $m = n - 1$.

A graph G is called a **planar graph** if G can be drawn in the plane without any two of its edges crossing. Such a drawing of G is called a **planar imbedding**. So the graph G in Figure 1.5 is a planar graph, or we can simply say a plane graph, since G is already drawn as a planar

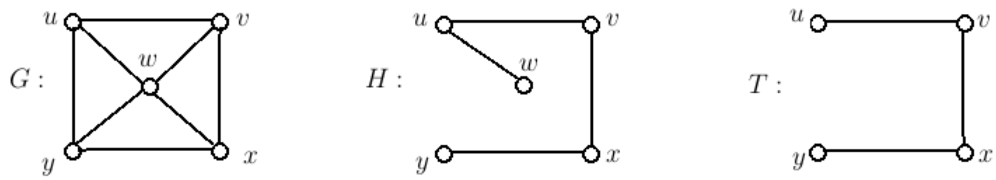
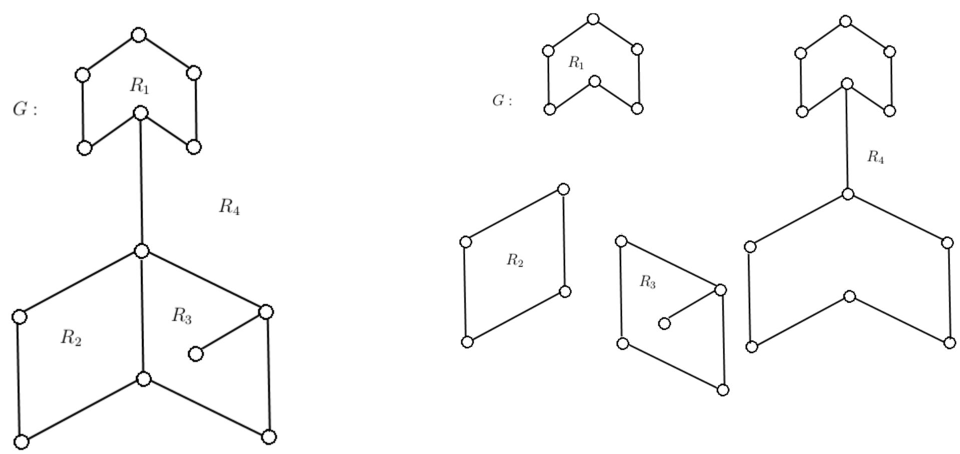


Figure 1.5: A plane graph G, a spanning tree H, and a tree T.

imbedding.

From the planar imbedding we can find the regions of the imbedding. When those points in the plane that correspond to the vertices and edges of a plane graph G are removed from the plane, the resulting connected pieces of the plane are called **regions**. The unbounded region surrounding the imbedded graph is called the **exterior region** of G. Regions are denoted by $R_1, R_2, R_3, ..., R_t$ where t is a natural number. For a region R of a plane graph G, the vertices and edges incident with R form a subgraph of G called the **boundary** of R. In Figure 1.6, we see the graph G and the indicated regions.



(a) A graph G with the regions labled.

(b) The regions of G separated.

Figure 1.6: A graph G.

There is a formula for graphs imbedded on the plane, the theorem stated below is known as the *Euler Identity*. We will be discussing a more general formula shortly.

Theorem 4 (Euler Identity)**.** *For every connected plane graph of order n, size m, and having r regions,*

$$n - m + r = 2.$$

Since not all graphs can be imbedded on a plane, we seek other surfaces on which to imbed them. A torus can be thought of as a doughnut-shape, a tube, or a ring in three dimensions. To create the torus, take the long sides of a rectangle and paste them together to create a cylinder. Then bend the cylinder until both open ends meet. The torus can be represented on a rectangle with opposite sides identified. A **complete graph** is a graph in which each pair of vertices are connected by an edge and denoted by K_n, where n represents the number of vertices of our graph. The following corollaries are found in Chartrand, Lesniak, and Zhang [1].

Corollary 5. *The graph K_5 is nonplanar.*

Corollary 6. *The graph $K_{3,3}$ is nonplanar.*

Consider Corollary 6, it states that the graph of $K_{3,3}$ is nonplanar. Any attempt to draw the graph on the plane will result in crossing edges. To find if $K_{3,3}$ can be imbedded on a torus, we must find an imbedding. Figure 1.7 shows such an imbedding of the graph of $K_{3,3}$ on the torus.

A torus can also be thought of as a sphere with a handle. A sphere with k handles, $k \geq 0$, is called a **surface of genus** k and is denoted by S_k. The smallest nonnegative integer k such that a graph G can be imbedded on S_k is called the **genus** of G and is denoted $\gamma(G)$. Similar to planar graphs, the graphs imbedded on a surface of genus k also have regions. A region is a **2-cell** if every closed curve in that region can be continuously deformed to a single point. An imbedding of a graph G on some surface is a **2-cell imbedding** if every region in the imbedding

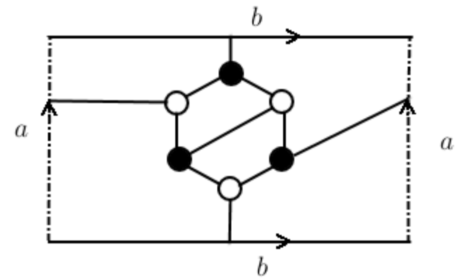


Figure 1.7: Imbedding of $K_{3,3}$ on the torus.

is a 2-cell. The following theorem is found in Lhuilier and Gergonne [8]. Notice that the case $k = 0$ is the *Euler Identity.*

Theorem 7 (Generalized Euler Identity). *If G is a connected graph of order n and size m that is 2-cell imbedded on a surface of genus $k \geq 0$, resulting in r regions, then*

$$n - m + r = 2 - 2k.$$

Finding the genus of a graph is of particular interest. The following theorem provides a lower bound for the genus and can be found in Chartrand, Lesniak, and Zhang [1]. Notice that Theorem 8 can be applied to bipartite graphs, and this will be referred to in Chapter 2.

Theorem 8. *If G is a connected, triangle-free graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{4} - \frac{n}{2} + 1.$$

An upper bound for the genus of a graph is known. The **maximum genus** $\gamma_M(G)$ of G is the maximum integer k for which G can be 2-cell imbedded on S_k. The **Betti number** $\beta(G)$ of a graph G of order n and size m having k components is defined as

$$\beta(G) = m - n + k.$$

(If the graph is connected, then $k = 1$.) The following theorem is found in Nordhaus, Ringeisen, Stewart, and White [9].

Theorem 9. *If G is a connected graph, then*

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor.$$

Furthermore, equality holds if and only if there exists a 2-cell imbedding of G on the surface of genus $\gamma_M(G)$ with exactly one or exactly two regions according to whether $\beta(G)$ is even or odd, respectively.

As stated in the following theorem found in Duke [2], knowing the genus and maximum genus of a graph G implies there is a 2-cell imbedding of G on every surface in between.

Theorem 10 (Duke's Theorem)**.** *If there exist 2-cell imbeddings of a connected graph G on the surfaces S_p and S_q, where $p \leq q$, and k is any integer such that $p \leq k \leq q$, then there exists a 2-cell imbedding of G on the surface S_k.*

A connected graph G is called **upper imbeddable** if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$. As a consequence of Theorem 9, we have the following theorem found in Chartrand, Lesniak, and Zhang [1].

Theorem 11. *Let G be a connected graph with an even (odd) Betti number. Then G is upper imbeddable on a surface S if and only if there exists a 2-cell imbedding of G on S with one region (two regions).*

Another characterization for upper imbeddable graphs is provided next, and this will be used in Chapter 2. A spanning tree T of a connected graph G is a **splitting tree** of G if at most one component of $G - E(T)$ has odd size. The following theorem is found in Jungerman [7] and Xuong [13].

Theorem 12. *A graph G is upper imbeddable if and only if G has a splitting tree.*

Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, ..., v_n\}$. Let $V(i) = \{j \mid v_j \in N(v_i)\}$, where $N(v_i)$ is the set of vertices adjacent to v_i. For each $1 \leq i \leq n$, let $\rho_i : V(i) \to V(i)$ be a cyclic permutation (or a rotation) of $V(i)$. Thus, each ρ_i can be represented by a (permutation) cycle of length $|V(i)| = |N(v_i)| = \deg(v_i)$. The following theorem is found in Heffter [6], Dyck [4], Youngs [14], and Edmonds [5].

Theorem 13 (The Rotational Imbedding Scheme)**.** *Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, ..., v_n\}$. For each 2-cell imbedding of G on a surface, there exists a unique n-tuple $P = (\rho_1, \rho_2, ..., \rho_n)$, where for $i = 1, 2, ...n$, $\rho_i : V(i) \to V(i)$ is a cyclic permutation that describes the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i. Conversely, for each such n-tuple $(\rho_1, \rho_2, ..., \rho_n)$, there exists a 2-cell imbedding of G on some surface such that for $i = 1, 2, ...n$, the subscripts of the vertices adjacent to v_i and in counterclockwise order about v_i are given by ρ_i.*

Graphs can model groups that are generated by elements, this leads to group theory definitions; the following can be found in Dummit and Foote [3]. A nontrivial group Γ is said to be **generated** by nonidentity elements $h_1, h_2, ..., h_k$ called **generators** of Γ if every element of Γ can be expressed as a (finite) product of generators. An equation in a group Γ that the generators satisfy is called a **relation** in Γ. If some group Γ is generated by a generating set $\Delta = \{h_1, h_2, ...h_k\}$ and there is some collection of relations, say $J_1, J_2, ..., J_m$ (each J_i is an equation in elements from $\Delta \cup \{1\}$)

such that any relation among the elements of Δ can be deduced from these, we shall call these generators and relations a **presentation** of Γ and write $\Gamma = \langle \Delta \mid J_1, J_2, ..., J_m \rangle$.

For every group presentation there is an associated **Cayley color graph of Γ with respect to** Δ and is denoted by $D_\Delta(\Gamma)$, where Γ is a finite nontrivial group with generating set $\Delta = \{h_1, h_2, ...h_k\}$. The vertices of $D_\Delta(\Gamma)$ are the elements of Γ. Each generator of the group is associated with a distinct color. Then $D_\Delta(\Gamma)$ is a labeled, directed graph with a color assigned to each edge White [12]. The underlying graph of a Cayley color graph is called a **Cayley graph** and is denoted by $G_\Delta(\Gamma)$, where Δ is the generating set and Γ is the group. Now, $V(G_\Delta(\Gamma)) = \Gamma$ and $E(G_\Delta(\Gamma)) = \{\{g,\ g\delta\} \mid g \in \Gamma,\ \delta \in \Delta \cup \Delta^{-1}\}$, where $\Delta^{-1} = \{\delta^{-1} \mid \delta \in \Delta\}$.

Our interest is to imbed Cayley graphs by use of voltage graphs. The following definitions and theorems can be found in White [12]. A **2-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk. A 2-manifold M is said to be **orientable** if, for every simple closed curved C on M, a clockwise sense of rotation is preserved by traveling once around C. So the surface S_k is a closed orientable 2-manifold. A **multigraph** is a nonempty set of vertices, every two of which are joined by a finite number of edges. Two or more edges that join the same pair of distinct vertices are called **parallel edges**. An edge joining a vertex to itself is called a **loop**. A graph that allows parallel edges and loops (even multiple loops) is a **pseudograph**. A **bouquet** is a pseudograph that has one vertex and m loops.

Let K represent a pseudograph. Each edge in K will be represented as two edges, $uv \in E(K)$, has $e = (u, v)$ and $e^{-1} = (v, u)$. Let $K^* = \{(u, v) \mid uv \in E(K)\}$. A **voltage graph** is a triple (K, Γ, ϕ), where K is a connected pseudograph, Γ is a group, and $\phi : K^* \to \Gamma$ is a function that satisfies $\phi(e^{-1}) = (\phi(e))^{-1}$ for all $e \in K^*$. Each value $\phi(e)$ is called a **voltage**. Let R be a region of the imbedding of the pseudograph K on the surface S and let $|R|_\phi$ be the order of the element $\phi(w)$ in the group Γ, where w is a closed walk at $v \in V(K)$ that bounds R. The **covering graph** $K \times_\phi \Gamma$ for (K, Γ, ϕ) has vertex set $V(K) \times \Gamma$ and each edge $e = (u, v)$ of K determines the edges $(u, g)(v, g\phi(e))$ of $K \times_\phi \Gamma$, for all $g \in \Gamma$.

A continuous function $\rho : \tilde{X} \to X$ from one path connected topological space to another is called a **covering projection** if every point $x \in X$ has a neighborhood U_x which is **evenly covered**. So ρ maps each component of $\rho^{-1}(U_x)$ homeomorphically onto U_x. If $Y \subseteq X$ and $\tilde{Y} \subseteq \tilde{X}$ such that ρ maps $\tilde{Y}$ homeomorphically onto Y, then Y **lifts** to $\tilde{Y}$. We call $\tilde{X}$ a **covering space** for X. A continuous function $\rho : \tilde{X} \to X$ from one path connected topological space to another is called a **branched covering projection** (and $\tilde{X}$ is a **branched covering space** of X) if there exists a finite set $B \subseteq X$ such that the restricted function $\rho : \tilde{X} - \rho^{-1}(B) \to X - B$ is a covering projection. The points of B are called **branch points**. For $b \in B$ and U_b a sufficiently small neighborhood of b, the restricted function $\rho : \tilde{U_b} \to U_b - \{b\}$ is n-fold, for some natural number n, called the **multiplicity of branching** at b, where $\tilde{U_b}$ is a component of $\rho^{-1}(U_b - \{b\})$ in $\tilde{X}$. (If $n = 1$, then there is no branching.) The following theorem stated in White [12] will be referred to in Chapter 2.

Theorem 14. *Let (K, Γ, ϕ) be a voltage graph with rotation scheme P and $\tilde{P}$ the lift of P to $K \times_\phi \Gamma$. Let P and $\tilde{P}$ determine 2-cell imbeddings of K and $K \times_\phi \Gamma$ on orientable surfaces S and $\tilde{S}$, respectively, where $\tilde{S}$ is possibly disconnected. Then there exists a (possibly branched) covering projection $\rho : \tilde{S} \to S$ such that:*

(i) $\rho^{-1}(K) = K \times_\phi \Gamma$;

(ii) if R is a region of the imbedding of K which is a k-gon, then $\rho^{-1}(R)$ has $\dfrac{|\Gamma|}{|R|_\phi}$ components, each of which is a $k|R|_\phi$-gon region of the covering imbedding of $K \times_\phi \Gamma$;

(iii) if $|R|_\phi = n > 1$, then R contains a branch point of multiplicity n. If $n = 1$, then R contains no branch point.

Let the labeled orientable 2-cell imbeddings of G, whose vertex set is $\{v_1, v_2, ..., v_n\}$, be denoted as $R(G)$. Observe that $|R(G)| = \prod_{i=1}^{n}(n_i - 1)!$, where $n_i = deg(v_i)$, $1 \leq i \leq n$. The probability model for imbeddings of G, in the area of random topological graph theory, was introduced in White [12], and has sample space Ω consisting of the labled 2-cell imbeddings of G with uniform distribution. So the probability of a given 2-cell imbedding of G is $\frac{1}{|R(G)|}$. Let $G_\Delta(\Gamma)$ be a Cayley graph for a group Γ, as generated by $\Delta \subseteq \Gamma$, where $1 \notin \Delta$, and if $\delta \in \Delta \cap \Delta^{-1}$, then $\delta^2 = 1$. Let $\Delta^* = \Delta \cup \Delta^{-1}$, and let $\rho : \Delta^* \to \Delta^*$ be a cyclic permutation. Recall, $N(x)$ is the set of vertices adjacent to $x \in \Gamma$. As found in White [12], a **Cayley map** (Γ, Δ, ρ) is the imbedding of $G_\Delta(\Gamma)$ with rotational imbedding scheme $\wp = \{\rho_x \mid x \in \Gamma\}$, where each vertex rotation ρ_x is given by $\rho_x(h) = x\rho(x^{-1}h)$ for $x \in \Gamma$ and $h \in N(x)$. The probability model where the sample space Ω consists of all Cayley maps for a fixed finite group Γ and generating set Δ with uniform distribution given by $P(\Gamma, \Delta, \rho) = \frac{1}{(|\Delta^*| - 1)!}$, is introduced in Schultz [11] (see also Schultz [10]), and referred to as random Cayley maps. The **genus random variable** $g : \Omega \to \mathbb{N} \cup \{0\}$ gives the genus of an arbitrary sample point; that is, if the Cayley map (Γ, Δ, ρ) imbeds the Cayley graph $G_\Delta(\Gamma)$ on S_k, then $g(\Gamma, \Delta, \rho) = k$. The **Cayley genus** is the minimum integer k for which $G_\Delta(\Gamma)$ can be 2-cell imbedded as a Cayley map on S_k, we denote this by $\gamma(\Gamma, \Delta) = \min_\rho g(\Gamma, \Delta, \rho)$. The **maximum Cayley genus** is the maximum integer k for which $G_\Delta(\Gamma)$ can be 2-cell imbedded as a Cayley map on S_k, we denote this by $\gamma_M(\Gamma, \Delta) = \max_\rho g(\Gamma, \Delta, \rho)$. The expected value of the genus random variable, called the **average Cayley genus** and denoted by $\overline{\gamma}(\Gamma, \Delta)$, is defined in White [12] and Schultz [11] as $\overline{\gamma}(\Gamma, \Delta) = \frac{1}{(|\Delta^*| - 1)!} \sum_{\Omega} g(\Gamma, \Delta, \rho)$.

CHAPTER 2

GENERALIZED QUATERNIONS AND CORRESPONDING CAYLEY GRAPH

In the previous chapter we discussed Cayley graphs and presentations of groups. In this chapter we will be studying the Cayley graph for a particular group, the generalized quaternions. The generalized quaternions Q_{2^m} (with $m \geq 3$) is the group with presentation:

$$\mathrm{Q}_{2^m} = \langle x,\ y \mid x^{2^{m-1}} = 1,\ y^2 = x^{2^{m-2}},\ y^{-1}xy = x^{-1} \rangle. \tag{2.1}$$

In this presentation, x and y are the generators of Q_{2^m}, where x has order 2^{m-1} and y has order 4. Listing elements, we have

$$\mathrm{Q}_{2^m} = \{1, x, x^2, ..., x^{2^{m-1}-1}, y, xy, x^2y, x^3y, ..., x^{2^{m-1}-1}y\}.$$

We begin by taking a closer look at the relation, $y^{-1}xy = x^{-1}$ in (2.1). This will help to easily identify any given element to one described in the group. The use of the following lemma will appear in Theorem 16.

Lemma 15. *With the presentation given by* (2.1) *for Q_{2^m}, and k an integer, it follows that $yx^k = x^{-k}y$.*

Proof. Clearly if $k = 0$, then the result holds.

First, consider when k is a positive integer. We will show by induction that $yx^k = x^{-k}y$ for any $k \in \mathbb{Z}$.

For the case where $k = 1$, consider the relation $y^{-1}xy = x^{-1}$ given in (2.1). We find an expression for yx using algebra methods. Multiply on the left by y, then on the right by x, and finally on the left by x^{-1}, achieving $yx = x^{-1}y$.

Now, suppose that $yx^k = x^{-k}y$ is true for some $k \in \mathbb{Z}^+$. Then by the above we have that,

$$\begin{aligned} yx^{k+1} &= yx^k x \\ &= x^{-k}yx \\ &= x^{-k}x^{-1}y \\ &= x^{-(k+1)}y \end{aligned}$$

Therefore, we see that $yx^k = x^{-k}y$ for any $k \in \mathbb{Z}^+$.

Note that if $k \in \mathbb{Z}^-$, then we have the expression $yx^{-k} = x^k y$. We manipulate the expression as before to find yx^k. Multiply on the left by x^{-k}, then on the right by x^k, producing $yx^k = x^{-k}y$.

Thus $yx^k = x^{-k}y$ for any integer k. □

We previously mentioned that x has order 2^{m-1}. We next find the order of $x^k y$, where k is an integer with $1 \leq k < 2^{m-1}$. (Note that if $k = 2^{m-1}$, then we have the element $x^{2^{m-1}}y = y$, which has order 4.)

Theorem 16. *For any integer k with $1 \leq k < 2^{m-1}$, the element $x^k y$ in Q_{2^m} has order 4.*

Proof. To find the order of an element a in a group Γ, we find the least positive integer r such that $a^r = 1$. Recall the order of y is 4. Consider the element $x^k y$.

By Lemma 15, observe that

$$
\begin{aligned}
(x^k y)^1 &= x^k y \neq 1 \\
(x^k y)^2 &= x^k y x^k y = x^k x^{-k} y y = y^2 = x^{2^{m-2}} \neq 1 \\
(x^k y)^3 &= (x^k y)^2 x^k y = y^2 x^k y = x^{2^{m-2}} x^k y = x^{2^{m-2}+k} y \neq 1 \\
(x^k y)^4 &= (x^k y)^2 (x^k y)^2 = y^2 y^2 = 1
\end{aligned}
$$

Therefore, the order of $x^k y$ is 4 for any integer k. $\square$

The Cayley graph of Q_{2^m} with $\Delta = \{x, y\}$ is denoted by $G_\Delta(Q_{2^m})$. Note in the presentation of the group, both generators have order greater than two. One of the elements in the generating set of the group is y, which has order 4 and produces 2^{m-2} 4-cycles in our graph. The other element x, has order 2^{m-1} and produces two 2^{m-1}-cycles. We begin our study of $G_\Delta(Q_{2^m})$ with the following theorem stating the graph is bipartite. The graph $G_\Delta(Q_{2^m})$ is 4-regular with vertex set

$V(G_\Delta(Q_{2^m})) = Q_{2^m} = \{1, x, x^2, ..., x^{2^{m-1}-1}, y, xy, x^2y, ..., x^{2^{m-1}-1}y\}$ and edge set

$E(G_\Delta(Q_{2^m})) = \{\{g,\ g\delta\} \mid g \in Q_{2^m},\ \delta \in \Delta \cup \Delta^{-1}\}$, where $\Delta \cup \Delta^{-1} = \{x,\ y,\ x^{-1},\ y^{-1}\}$.

Theorem 17. *For the generalized quaternions Q_{2^m}, with generating set $\Delta = \{x, y\}$, the Cayley graph $G_\Delta(Q_{2^m})$ is bipartite.*

Proof. Let A and B be two sets containing elements of Γ which will represent the vertices of the graph, such that

$$A = \{1, x^2, x^4, ...x^{2^{m-1}-2}, xy, x^3y, ..., x^{2^{m-1}-1}y\}$$

and

$$B = \{x, x^3, ...x^{2^{m-1}-1}, x^2y, x^4y, ..., x^{2^{m-1}-2}y\}.$$

It is obvious that $A \cap B = \emptyset$ and that $V(G_\Delta(Q_{2^m})) = A \cup B$. To show that $G_\Delta(Q_{2^m})$ is bipartite we now show that no two vertices in A are adjacent and no two vertices in B are adjacent, in other words each edge of $G_\Delta(Q_{2^m})$ joins a vertex of A to a vertex of B. Thus we proceed by cases.

Case 1

Let $a \in A$. We consider ax.

Subcase 1.1 First, let $a = x^i$, where i is even. Then $ax = x^i x = x^{i+1}$, and note that $x^{i+1} \in B$ since $i+1$ is odd.

Subcase 1.2 Next, let $a = x^i y$, where i is odd. Then, by Lemma 15,

$$\begin{aligned} ax &= x^i y x \\ &= x^{i-1} y \end{aligned}$$

However, $x^{i-1}y \in B$ since $i-1$ is even.

Therefore, $ax \in B$ for all $a \in A$.

Case 2

Let $b \in B$. We consider bx.

Subcase 2.1 First, let $b = x^i$, where i is odd. Then $bx = x^i x = x^{i+1}$, and note $x^{i+1} \in A$ since $i+1$ is even.

Subcase 2.2 Next, let $b = x^i y$, where i is even. Then, by Lemma 15,

$$\begin{aligned} bx &= x^i y x \\ &= x^{i-1} y \end{aligned}$$

However, $x^{i-1}y \in A$ since $i-1$ is odd.

Therefore, $bx \in A$ for all $b \in B$.

Case 3

Let $a \in A$. We consider ay.

Subcase 3.1 First, let $a = x^i$, where i is even. Then $ay = x^i y$, and $x^i y \in B$ since i is even.

Subcase 3.2 Next, let $a = x^i y$, where i is odd. By the presentation (2.1), we know that $y^2 = x^{2^{m-2}}$. Then,

$$\begin{aligned} ay &= x^i yy \\ &= x^i y^2 \\ &= x^i x^{2^{m-2}} \\ &= x^{i+2^{m-2}} \end{aligned}$$

However, $x^{i+2^{m-2}} \in B$ since $i + 2^{m-2}$ is odd.

Therefore, $ay \in B$ for all $a \in A$.

Case 4

Let $b \in B$. We consider by.

Subcase 4.1 First, let $b = x^i$, where i is odd. Then $by = x^i y$, and $x^i y \in A$ since i is odd.

Subcase 4.2 Next, let $b = x^i y$, where i is even. By the presentation (2.1), we know that $y^2 = x^{2^{m-2}}$. Then,

$$\begin{aligned} by &= x^i yy \\ &= x^i y^2 \\ &= x^i x^{2^{m-2}} \\ &= x^{i+2^{m-2}} \end{aligned}$$

However, $x^{i+2^{m-2}} \in A$ since $i + 2^{m-2}$ is even.

Therefore, $by \in A$ for all $b \in B$.

Hence, A and B are partite sets for $G_\Delta(Q_{2^m})$. Therefore, $G_\Delta(Q_{2^m})$ is bipartite. $\square$

We know the order of the graph to be the number of elements of the group, the order of $G_\Delta(Q_{2^m})$ is 2^m. By Theorem 1, the size is 2^{m+1}. Next, we find the genus of $G_\Delta(Q_{2^m})$ denoted as $\gamma(G_\Delta(Q_{2^m}))$. In order to show the genus of $G_\Delta(Q_{2^m})$ is 1, we first use Theorem 17 and Theorem 8 to show the genus is greater than or equal to 1, and then we find an imbedding on the torus.

Theorem 18. *The genus of $G_\Delta(Q_{2^m})$ is 1.*

Proof. By Theorem 17, $G_\Delta(Q_{2^m})$ is bipartite. Then, $G_\Delta(Q_{2^m})$ is a connected, triangle-free graph with order 2^m and size 2^{m+1}. By Theorem 8,

$$\gamma(G_\Delta(Q_{2^m})) \geq \frac{2^{m+1}}{4} - \frac{2^m}{2} + 1 = 1.$$

So, $\gamma(G_\Delta(Q_{2^m})) \geq 1$.

We now show there exists an imbedding of $G_\Delta(Q_{2^m})$ on the torus. Let C denote the 2^{m-1}-cycle containing the identity element, and let C' denote the 2^{m-1}-cycle not containing the identity element. Now on the sphere, place C inside of C' so that the vertices 1 and y are near each other and the cycles clockwise are $C : 1, x, x^2, ..., x^{2^{m-1}-1}, 1$ and $C' : y, xy, x^2y, ..., x^{2^{m-1}-1}y, y$. The edges $\{x^n, x^ny\}$, where $1 \leq n \leq 2^{m-1}$, are drawn between the vertices of C and C' as shown in Figure 2.1. The remaining edges $\{x^ny, x^{n+2^{m-2}}\}$, where $1 \leq n \leq 2^{m-1}$, are finally drawn over a handle as follows. To represent a handle being added, we draw two dashed circles. One circle is drawn inside C, with the notation of an arrow showing the handle's orientation as counterclockwise. The other circle is drawn outside C', with the notation of the handle's orientation in the clockwise direction. The edges that are near the tail end of the arrow on the inside circle match with the edges near the tail end of the circle on the outside. (As seen in Figure 2.1.)

Therefore, the genus of $G_\Delta(Q_{2^m})$ is 1 for any $m \geq 3$. $\square$

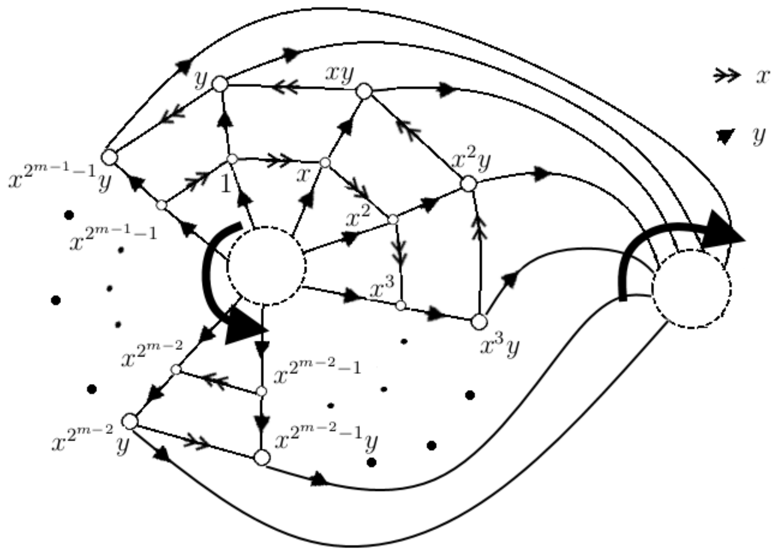


Figure 2.1: $G_\Delta(Q_{2^m})$ on the torus.

The following lemma will help find a splitting tree of $G_\Delta(Q_{2^m})$. A splitting tree will allow us to find the maximum genus of $G_\Delta(Q_{2^m})$.

Lemma 19. *Let T be the subgraph of $G_\Delta(Q_{2^m})$, where the edge set of T is $E(T) = \{\{x^{k+1}y,\ x^k y\}, \{x^n,\ x^n y\} \mid 1 \leq k \leq 2^{m-1}-1,\ 1 \leq n \leq 2^{m-1}\}$. Then T is a spanning tree of $G_\Delta(Q_{2^m})$.*

Proof. Let T be the subgraph of $G_\Delta(Q_{2^m})$ defined by the vertex set $V(T) = \{1, x, x^2, ..., x^{2^{m-1}-1}, y, xy, x^2y, x^3y, ..., x^{2^{m-1}-1}y\}$ and the edge set $E(T) = \{\{x^{k+1}y,\ x^k y\}, \{x^n,\ x^n y\} \mid 1 \leq k \leq 2^{m-1}-1,\ 1 \leq n \leq 2^{m-1}\}$. Observe that T consists of all the edges of C' except the edge $\{xy, y\}$, together with the edges of the form $\{x^n, x^n y\}$, where $1 \leq n \leq 2^{m-1}$ (see Figure 2.2). Note that $V(T) = V(G_\Delta(Q_{2^m}))$. So T is a spanning subgraph of $G_\Delta(Q_{2^m})$. By construction, T is connected and has $2^m - 1$ edges with 2^m vertices.

Therefore, T is a spanning tree of $G_\Delta(Q_{2^m})$. $\square$

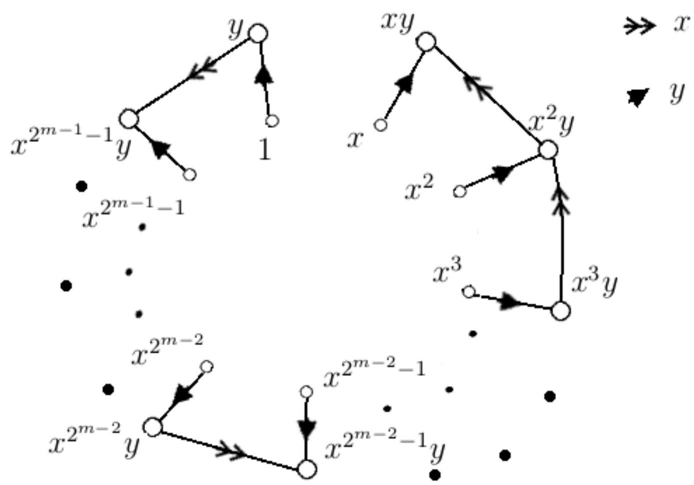


Figure 2.2: The spanning tree T of $G_\Delta(Q_{2^m})$.

The next theorem shows that the spanning tree described in Lemma 19 is in fact a splitting tree of $G_\Delta(Q_{2^m})$.

Theorem 20. *If T is the spanning tree as described in Lemma 19, then T is a splitting tree of $G_\Delta(Q_{2^m})$.*

Proof. We know that $G_\Delta(Q_{2^m})$ is a connected graph. By Lemma 19, T is a spanning tree of $G_\Delta(Q_{2^m})$ with odd size. By construction we know that T is connected. Also, the graph $G_\Delta(Q_{2^m}) - E(T)$ is connected (see Figure 2.3) and has size $2^m + 1$, which is odd. Thus, T is a splitting tree of $G_\Delta(Q_{2^m})$. $\square$

Now we find the maximum genus of $G_\Delta(Q_{2^m})$. This is done by using Theorems 12, 20, and the definitions of Betti number and upper imbeddable.

Theorem 21. *The maximum genus of $G_\Delta(Q_{2^m})$ is 2^{m-1}.*

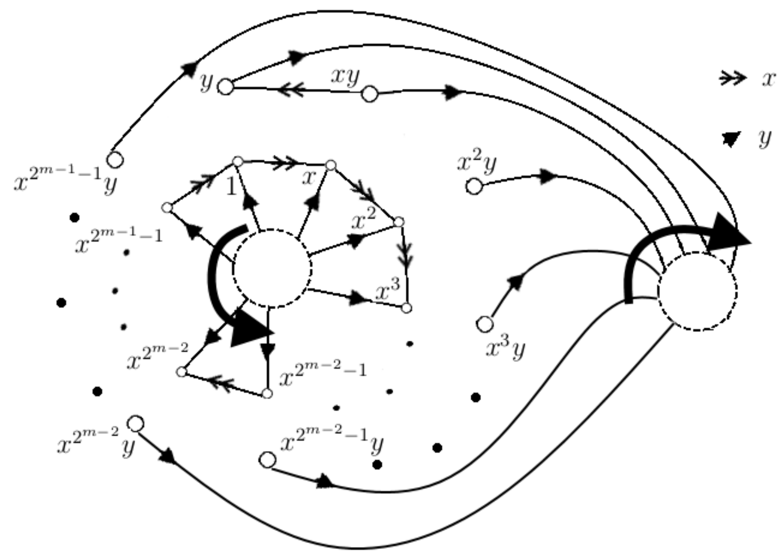


Figure 2.3: $G_\Delta(Q_{2^m}) - E(T)$ has one component of odd size.

Proof. By Theorem 20, we know $G_\Delta(Q_{2^m})$ has a splitting tree T. By Theorem 12, the graph is upper imbeddable. First we find the Betti number, $\beta(G_\Delta(Q_{2^m}))$, of $G_\Delta(Q_{2^m})$. Since our graph is connected we have $k = 1$. Then,

$$\begin{aligned} \beta(G_\Delta(Q_{2^m})) &= 2^{m+1} - 2^m + 1 \\ &= 2^m + 1. \end{aligned}$$

By the definition of upper imbeddable, we know the maximum genus of the graph to be

$$\begin{aligned} \gamma_M(G_\Delta(Q_{2^m})) &= \left\lfloor \frac{\beta(G_\Delta(Q_{2^m}))}{2} \right\rfloor \\ &= \left\lfloor \frac{2^m + 1}{2} \right\rfloor \\ &= \left\lfloor 2^{m-1} + \frac{1}{2} \right\rfloor \\ &= 2^{m-1}. \end{aligned}$$

Therefore, the maximum genus of $G_\Delta(Q_{2^m})$ is 2^{m-1}. $\square$

By Theorem 10, we know that there is an imbedding of $G_\Delta(Q_{2^m})$ on S_k for every k with $1 \leq k \leq 2^{m-1}$. Now we will find the average Cayley genus of $G_\Delta(Q_{2^m})$. We will be looking at six cases. The six cases come from the different permutations of the set $\Delta^* = \{x, y, x^{-1}, y^{-1}\}$. Of the corresponding voltage graph imbeddings, two imbed on the torus. These cases correspond to the cyclic permutations (x, y, x^{-1}, y^{-1}) and (x, y^{-1}, x^{-1}, y). The other four cases, imbed on the sphere and correspond to the cyclic permutations (x, x^{-1}, y, y^{-1}), (x, y, y^{-1}, x^{-1}), (x, y^{-1}, y, x^{-1}), and (x, x^{-1}, y^{-1}, y). Figure 2.4 depicts all six voltage graph imbeddings that will be referenced in the proof of Theorem 22. In Figure 2.4 let v denote the vertex of each voltage graph.

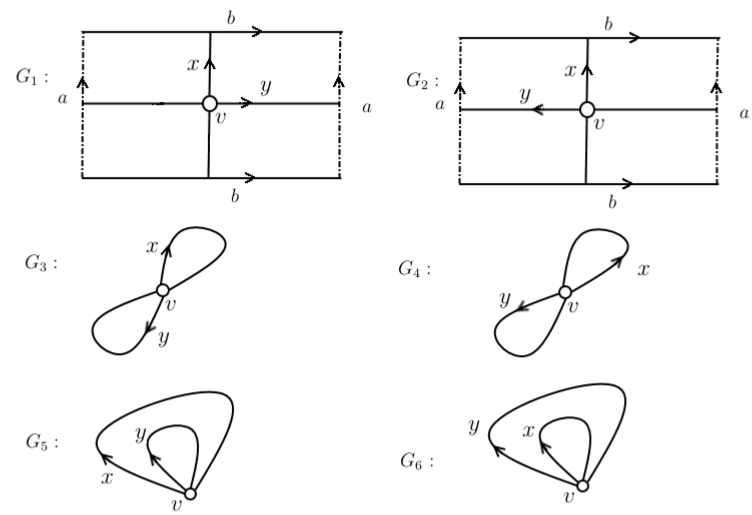


Figure 2.4: Six voltage graph imbeddings for Q_{2^m} and Δ

Theorem 22. *The average Cayley genus for Q_{2^m} and Δ is* $\frac{1}{3}(2^m - 1)$.

Proof. First, note that there are six voltage graph imbeddings associated with $G_\Delta(Q_{2^m})$ as shown in Figure 2.4, one for each cyclic permutation of $\Delta^* = \{x, y, x^{-1}, y^{-1}\}$. Two of the voltage graphs imbed on the torus. The remaining four, imbed on the sphere. We trace each of the regions of the voltage graph imbeddings to find the boundary element. Using Theorem 14 we will calculate the number of regions the voltage graph imbedding lifts to. Then we find the genus of the lifted Cayley map using the Generalized Euler Identity (Theorem 7). We will use t instead of m to denote the size in the Generalized Euler Identity. Then we can find the average Cayley genus of the generalized quaternions with this generating set. Note, the two voltage graph imbeddings that are depicted on the torus are mirror images of each other; and thus, it is enough to find the genus of one of them and multiply by two. We approach the other imbeddings in a similar fashion except we will multiply by four.

First, we consider the imbedding on the torus, let G_1 be the imbedding represented by the cyclic permutation $(x,\ y,\ x^{-1},\ y^{-1})$ of Δ^*. The voltage graph imbedding G_1 has one region with boundary element $xyx^{-1}y^{-1} = x^2$, and recall $ord(x^2) = 2^{m-2}$. By Theorem 14, the region lifts to four regions. The genus of the Cayley map is,

$$\begin{aligned} k &= \frac{t}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{2^{m+1}}{2} - \frac{2^m}{2} - \frac{4}{2} + 1 \\ &= 2^m - 2^{m-1} - 2 + 1 \\ &= 2^{m-1} - 1. \end{aligned}$$

The imbedding of $G_\Delta(Q_{2^m})$ lifted from G_1 is on a surface of genus $2^{m-1} - 1$. Similarly, the imbedding of $G_\Delta(Q_{2^m})$ lifted from G_2 is also on a surface of genus $2^{m-1} - 1$. Thus, we have $2(2^{m-1} - 1)$ contributing to the average Cayley genus.

Next, we consider an imbedding on the sphere. Let G_5, as shown in Figure 2.4, be the imbedding represented by the cyclic permutation $(x,\ y,\ y^{-1},\ x^{-1})$ of Δ^*. The voltage graph imbedding of G_5 has three regions with boundary elements x^{-1}, y, and xy^{-1}. Let R_1 be the region bounded by the element x^{-1}. Let R_2 be the region bounded by the element y. Recall that $ord(x^{-1}) = ord(x) = 2^{m-1}$, and $ord(y) = 4$. Let R_3 be the region bounded by the element xy^{-1}. By rewriting y^{-1} as y^3, and then replacing y^2 by $x^{2^{m-2}}$ gives us the element $x^{2^{m-2}+1}y$, and note by Theorem 16 $ord(x^{2^{m-2}+1}y) = 4$. By Theorem 14, each region $R_1,\ R_2$, and R_3 lifts to 2, 2^{m-2}, and 2^{m-2} regions respectively. Thus, the regions of G_5 lift to $(2 + 2^{m-2} + 2^{m-2})$ regions. The genus of the Cayley map is,

$$
\begin{aligned}
k &= \frac{t}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\
&= \frac{2^{m+1}}{2} - \frac{2^m}{2} - \frac{2 + 2^{m-2} + 2^{m-2}}{2} + 1 \\
&= 2^m - 2^{m-1} - \frac{2^{m-1} + 2}{2} + 1 \\
&= 2^{m-1} - 2^{m-2} \\
&= 2^{m-2}.
\end{aligned}
$$

The imbedding of $G_\Delta(Q_{2^m})$ lifted from G_5 is on a surface of genus 2^{m-2}. Note that the boundary elements will be similar for each voltage graph imbedding, and can be written in the form x or x^{-1} for region one, y or y^{-1} for region two, and $x^{-1}y^{-1}$ or $x^{-1}y$ or xy for the third region. Therefore, we know that the remaining three voltage graph imbeddings on the sphere lift to $G_\Delta(Q_{2^m})$ on the surface of genus 2^{m-2}. Thus, we have $4(2^{m-2})$ contributing to the average Cayley genus.

Therefore, the average Cayley genus for the generalized quaternions is,

$$
\begin{aligned}
\bar{\gamma}(Q_{2^m}, \Delta) &= \frac{2(2^{m-1} - 1) + 4(2^{m-2})}{6} \\
&= \frac{2^m - 2 + 2^m}{6} \\
&= \frac{2^{m+1} - 2}{6} \\
&= \frac{1}{3}(2^m - 1).
\end{aligned}
$$

□

It is interesting to notice that the difference between the Cayley genus and the genus of the Cayley graph is $2^{m-2} - 1$. Also, the difference between the maximum genus of the Cayley graph and the maximum Cayley genus is $2^{m-1} - (2^{m-1} - 1) = 1$.

CHAPTER 3

AVERAGE CAYLEY GENUS OF A FINITE GROUP WITH TWO GENERATORS OF ORDER GREATER THAN TWO

Note that a finite group Γ with generating set $\Delta = \{x, y\}$, where the order of x and the order of y are greater than two will create the same six types of voltage graph imbeddings that we saw in the proof of Theorem 22. All such groups will have two voltage graph imbeddings on the torus and four on the sphere. The layout of the following proof is similar to the proof of Theorem 22. The next theorem can be found in Schultz [11], we include it with more details in the proof. In algebra, it is well known that for any element g in Γ that $ord(g^{-1}) = ord(g)$.

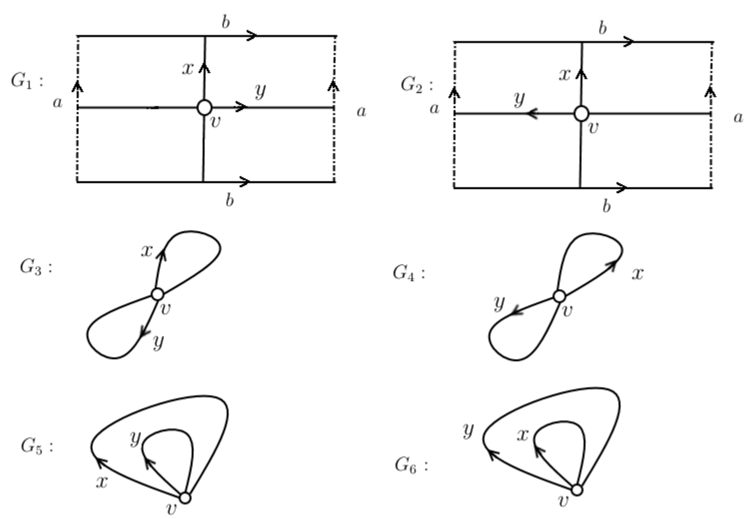


Figure 3.1: Six voltage graph imbeddings for Γ and Δ

Theorem 23. *The average Cayley genus of a finite group Γ with generating set $\Delta = \{x,\ y\}$ where* *ord*(*x*) *and ord*(*y*) *are greater than two, is given by the formula* $\bar{\gamma}(\Gamma, \Delta) = \dfrac{|\Gamma|}{2} - \dfrac{|\Gamma|}{6ord(xyx^{-1}y^{-1})} - \dfrac{|\Gamma|}{3ord(x)} - \dfrac{|\Gamma|}{3ord(y)} - \dfrac{|\Gamma|}{6ord(xy^{-1})} - \dfrac{|\Gamma|}{6ord(x^{-1}y^{-1})} + 1.$

Proof. Let Γ be a finite group with generating set $\Delta = \{x, y\}$, where $ord(x)$ and $ord(y)$ are greater than two. Note that $|V(G_\Delta(\Gamma))| = |\Gamma|$ and $|E(G_\Delta(\Gamma))| = 2|\Gamma|$. There are six voltage graph imbeddings associated with the six cyclic permutations of $\Delta^* = \{x, y, x^{-1}, y^{-1}\}$, as shown in Figure 3.1. We will consider three cases, each case consisting of two imbeddings that are mirror images of each other. The first case is on the torus with the two voltage graph imbeddings represented by G_1 and G_2. The next case is on the sphere consisting of the two voltage graph imbeddings represented by G_3 and G_4. The last case is also on the sphere consisting of the two voltage graph imbeddings represented by G_5 and G_6. We begin by tracing the regions of a voltage graph imbedding to find the boundary element of each region, keeping the boundary on our left. Recall to find the boundary element we trace the region by keeping the boundary on our left. Using Theorem 14 we calculate the number of regions the voltage graph imbedding lifts to. Then we find the genus of the surface of the lifted Cayley map using the Generalized Euler Identity (Theorem 7). Lastly, we calculate the average Cayley genus of a finite group Γ with generating set Δ.

First, we consider an imbedding on the torus, let G_1 be the imbedding represented by the cyclic permutation (x, y, x^{-1}, y^{-1}). See Figure 3.1. The voltage graph imbedding G_1 has one region with boundary element $xyx^{-1}y^{-1}$. By Theorem 14, the region lifts to $\dfrac{|\Gamma|}{ord(xyx^{-1}y^{-1})}$ regions. Using Theorem 7, we find the genus of the surface that the Cayley map represented by G_1 is imbedded on.

$$
\begin{aligned}
k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\
&= \frac{2|\Gamma|}{2} - \frac{|\Gamma|}{2} - \frac{\dfrac{|\Gamma|}{ord(xyx^{-1}y^{-1})}}{2} + 1 \\
&= \frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(xyx^{-1}y^{-1})} + 1.
\end{aligned}
$$

The imbedding of $G_{\Delta}(\Gamma)$ lifted from G_1 is on a surface of genus $\frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(xyx^{-1}y^{-1})} + 1$. Since there are two imbeddings in this case, we have $|\Gamma| - \frac{|\Gamma|}{ord(xyx^{-1}y^{-1})} + 2$ contributing to the average Cayley genus.

Next, we consider an imbedding on the sphere. Let G_3, as shown in Figure 3.1, be the imbedding represented by the cyclic permutation (x, x^{-1}, y, y^{-1}). This voltage graph imbedding G_3 has three regions with boundary elements x, y, and $x^{-1}y^{-1}$. Let R_1 be the region bounded by the element x. Let R_2 be the region bounded by the element y. Let R_3 be the region bounded by the element $x^{-1}y^{-1}$. Let the order of the elements be denoted by $ord(x)$, $ord(y)$, and $ord(x^{-1}y^{-1})$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(x)}$, $\frac{|\Gamma|}{ord(y)}$, and $\frac{|\Gamma|}{ord(x^{-1}y^{-1})}$ regions respectively. Thus, the regions of G_3 lift to $\left(\frac{|\Gamma|}{ord(x)} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(x^{-1}y^{-1})}\right)$ regions. Using Theorem 7, we find the genus of the surface that the Cayley map represented by G_3 is imbedded on.

$$
\begin{aligned}
k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\
&= \frac{2|\Gamma|}{2} - \frac{|\Gamma|}{2} - \frac{\left(\dfrac{|\Gamma|}{ord(x)} + \dfrac{|\Gamma|}{ord(y)} + \dfrac{|\Gamma|}{ord(x^{-1}y^{-1})}\right)}{2} + 1 \\
&= \frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(x^{-1}y^{-1})} + 1.
\end{aligned}
$$

The imbedding of $G_{\Delta}(\Gamma)$ lifted from G_3 is on a surface of genus $\frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(x^{-1}y^{-1})} + 1$. Since there are two voltage graph imbeddings on the sphere that lift to $G_{\Delta}(\Gamma)$ on the surface of genus $\frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(x^{-1}y^{-1})} + 1$, we have $|\Gamma| - \frac{|\Gamma|}{ord(x)} - \frac{|\Gamma|}{ord(y)} - \frac{|\Gamma|}{ord(x^{-1}y^{-1})} + 2$ contributing to the average Cayley genus.

Next, we consider another imbedding on the sphere. Let G_5, as shown in Figure 3.1, be the imbedding represented by the cyclic permutation (x, y, y^{-1}, x^{-1}). This voltage graph imbedding G_5 has three regions with boundary elements x^{-1}, y, and xy^{-1}. Let R_1 be the region bounded by the element x^{-1}. Let R_2 be the region bounded by the element y. Let R_3 be the region bounded by the element xy^{-1}. Let the order of the elements be denoted by $ord(x^{-1})$, $ord(y)$, and $ord(xy^{-1})$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(x^{-1})}$, $\frac{|\Gamma|}{ord(y)}$, and $\frac{|\Gamma|}{ord(xy^{-1})}$ regions respectively. Thus, the regions of G_5 lift to $\left(\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xy^{-1})}\right)$ regions. Using Theorem 7, we find the genus of the surface that the Cayley map represented by G_5 is imbedded on.

$$\begin{aligned}
k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\
&= \frac{2|\Gamma|}{2} - \frac{|\Gamma|}{2} - \frac{\left(\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xy^{-1})}\right)}{2} + 1 \\
&= \frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1.
\end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_5 is on a surface of genus $\frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1$. Since there are two voltage graph imbeddings on the sphere that lift to $G_\Delta(\Gamma)$ on the surface of genus $\frac{|\Gamma|}{2} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1$, we have $|\Gamma| - \frac{|\Gamma|}{ord(x^{-1})} - \frac{|\Gamma|}{ord(y)} - \frac{|\Gamma|}{ord(xy^{-1})} + 2$ contributing to the average Cayley genus.

Therefore, the average Cayley genus for a finite group Γ with generating Δ is, $\bar{\gamma}(\Gamma, \Delta) = \frac{|\Gamma|}{2} - \frac{|\Gamma|}{6ord(xyx^{-1}y^{-1})} - \frac{|\Gamma|}{3ord(x)} - \frac{|\Gamma|}{3ord(y)} - \frac{|\Gamma|}{6ord(xy^{-1})} - \frac{|\Gamma|}{6ord(x^{-1}y^{-1})} + 1.$ □

Next, we consider the different types of groups whose Cayley graph can be imbedded using Theorem 23. First, we consider the semi-direct product of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a, b \mid a^3 = b^4 = 1 \text{ and } bab^{-1} = a^{-1}\rangle$. By routine manipulation with the the relation given in the presentation of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ we see that $ord(aba^{-1}b^{-1}) = 6$, $ord(ab^{-1}) = 4$, and $ord(a^{-1}b^{-1}) = 4$.

Corollary 24. *Let* $\Gamma = \mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a, b \mid a^3 = b^4 = 1$ *and* $bab^{-1} = a^{-1}\rangle$ *and let* $\Delta = \{a, b\}$*. Then the average Cayley genus is* $\bar{\gamma}(\Gamma, \Delta) = \dfrac{10}{3}$.

It is well known that the symmetric group on n elements S_n can be generated by a transposition $(i,\ i+1)$ and an n-cycle as found in Dummit and Foote [3]. By using two cycles, neither of which is a transposition, we can generate a transposition to then generate the group S_n. Let c be the n-cycle $(1\ 2\ 3\ ...\ n-1\ n)$, and let d be the $(n-1)$-cycle $(n\ n-1\ ...\ 3\ 2)$. Note that $c^n = d^{n-1} = (1)$. Then performing multiplication in the usual way, we see that $c \cdot d = (1\ 2)$. Thus, $\Delta = \{c, d\}$ is a generating set for S_n. So, to find the order of the element $cdc^{-1}d^{-1}$, we will use the fact that $c \cdot d = (1\ 2)$.

$$\begin{aligned}(cdc^{-1}d^{-1}) &= (1\ 2)(n\ n-1\ ...\ 3\ 2\ 1)(2\ 3\ ...\ n-1\ n) \\ &= (1\ n\ 2)(3)(4)(5)...(n-1)(n).\end{aligned}$$

Now that we know the element $(cdc^{-1}d^{-1})$ reduces to a 3-cycle, this implies that the order of $(cdc^{-1}d^{-1})$ is three. To find $ord(cd^{-1})$, we must consider when n is even or when n is odd. If n is even, we have $cd^{-1} = (1\ 2\ 4\ 6\ 8\ ...\ n-2\ n\ 3\ 5\ 7\ ...\ n-3\ n-1)$. This is an n-cycle, which has order n. Then if n is odd, we have $cd^{-1} = (1\ 2\ 4\ 6\ 8\ ...\ n-3\ n-1)(\ 3\ 5\ 7\ ...\ n-2\ n)$. Now, the cycles that are produced are disjoint. We know that the order of the product of two disjoint cycles in S_n is the least common multiple, LCM, of the orders of each cycle. The two cycles generated are an $\dfrac{n+1}{2}$-cycle and an $\dfrac{n-1}{2}$-cycle. Thus, $ord(cd^{-1}) = \dfrac{n^2-1}{4}$. Now we will find $ord(c^{-1}d^{-1})$. Multiplying in the usual way we find $c^{-1}d^{-1} = (1\ n)$, which is a transposition, and has order 2.

Corollary 25. *The average Cayley genus of the symmetric group* S_n *where* $n \geq 4$ *and is even with generating set* $\Delta = \{c, d\}$*, where* c *and* d *are the cycles described above, is* $\bar{\gamma}(S_n, \Delta) = \dfrac{13n!}{36} - \dfrac{(n-1)!}{2} - \dfrac{n(n-2)!}{3} + 1$.

Corollary 26. *The average Cayley genus of the symmetric group S_n where $n \geq 4$ and is odd with generating set $\Delta = \{c, d\}$, where c and d are the cycles described above, is $\bar{\gamma}(S_n, \Delta) = \frac{13n!}{36} - \frac{(n-1)!}{3} - \frac{n(n-2)!}{3} - \frac{2n(n-2)!}{3(n+1)} + 1$.*

CHAPTER 4

ADDING ONE GENERATOR OF ORDER TWO

Let Γ be a finite group with generating set $\Delta = \{x, y, z\}$, where the orders of x and y are greater than two and the order of z is two. Adding a generator of order two, creates a spoke in one of the regions of the voltage graph imbedding. Now each vertex in $G_\Delta(\Gamma)$ has degree 5; so there are $(5-1)! = 24$ possible voltage graph imbeddings, and also 24 Cayley maps for this group and generating set. We know that the order of $G_\Delta(\Gamma)$ is $|\Gamma|$ and the size of $G_\Delta(\Gamma)$ is $\frac{5|\Gamma|}{2}$ (found using Theorem 1). In this proof we will consider the mirror images of the imbeddings. All mirror images are drawn in pairs in Figures 4.1, 4.2, and 4.3.

Theorem 27. *The average Cayley genus for a finite group Γ with generating set $\Delta = \{x,\ y,\ z\}$, where $ord(x)$ and $ord(y)$ are greater than two and $ord(z)$ is two, is given by the formula*

$$\begin{aligned}\bar{\gamma}(\Gamma,\Delta) = \frac{3|\Gamma|}{4} \quad & -\frac{|\Gamma|}{4ord(x)} - \frac{|\Gamma|}{4ord(y)} - \frac{|\Gamma|}{12ord(xy^{-1})} - \frac{|\Gamma|}{12ord(x^{-1}y)} \\ & -\frac{|\Gamma|}{24ord(xz)} - \frac{|\Gamma|}{24ord(y^{-1}z)} - \frac{|\Gamma|}{24ord(x^{-1}z)} - \frac{|\Gamma|}{24ord(yz)} \\ & -\frac{|\Gamma|}{24ord(x^{-1}yz)} - \frac{|\Gamma|}{24ord(x^{-1}zy)} - \frac{|\Gamma|}{24ord(xzy^{-1})} \\ & -\frac{|\Gamma|}{24ord(xy^{-1}z)} - \frac{|\Gamma|}{24ord(xyx^{-1}y^{-1}z)} - \frac{|\Gamma|}{24ord(xzyx^{-1}y^{-1})} \\ & -\frac{|\Gamma|}{24ord(xyx^{-1}zy^{-1})} - \frac{|\Gamma|}{24ord(xyzx^{-1}y^{-1})} + 1\end{aligned}$$

Proof. Let Γ be a finite group with generating set $\Delta = \{x, y, z\}$, where $ord(x)$ and the $ord(y)$ are greater than two and $ord(z)$ is two. Note that the order of $G_\Delta(\Gamma)$ is $|\Gamma|$ and the size of $G_\Delta(\Gamma)$ is $\frac{5|\Gamma|}{2}$. There are 24 voltage graph imbeddings associated with $G_\Delta(\Gamma)$, as shown in Figures 4.1, 4.2, and 4.3, one for each permutation of $\Delta^* = \{x, y, x^{-1}, y^{-1}, z\}$. Each cyclic permutation corresponds to an imbedding of the voltage graph that consists of one vertex, two loops, and one spoke. For this proof we have twelve different cases to consider. First, we trace the regions of the voltage graph imbedding to find the boundary element of each region. Then, using Theorem 14 we will calculate the number of regions the imbedding lifts to. Next, we find the genus of the surface of the lifted Cayley map using the Generalized Euler Identity (Theorem 7). Lastly, we calculate the average Cayley genus for Γ with generating set Δ. Consider the first

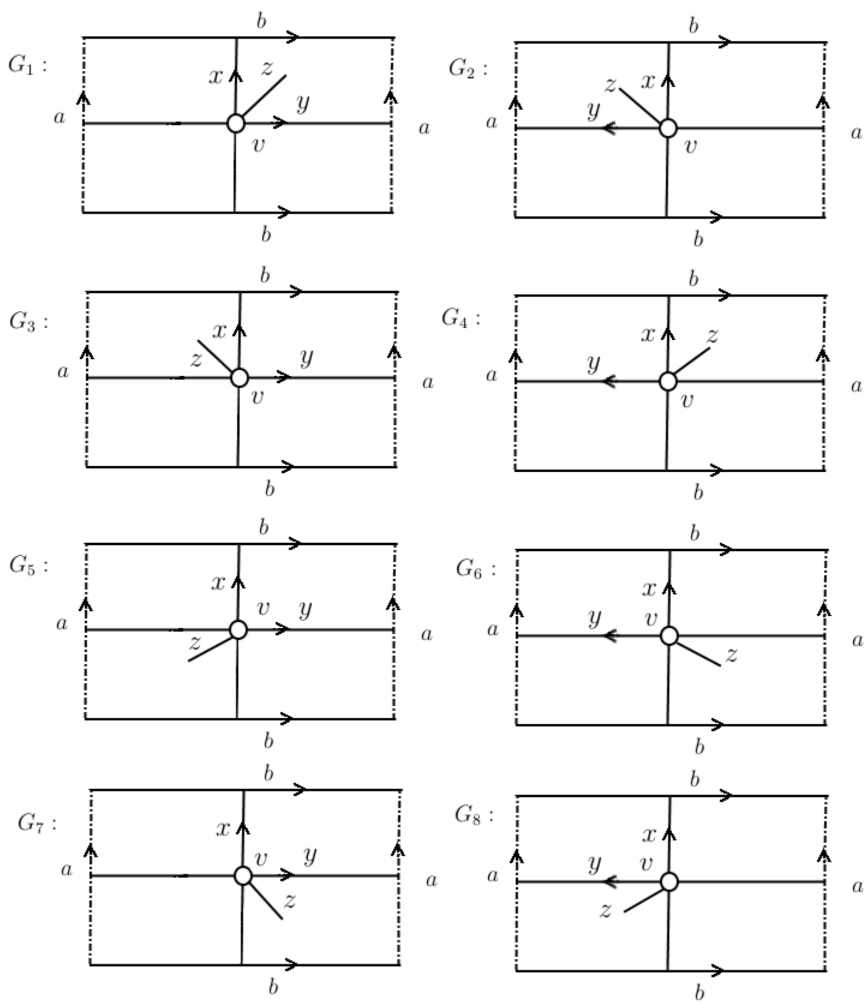


Figure 4.1: Eight voltage graph imbeddings for Γ and Δ on the torus

two voltage graph imbeddings of $G_\Delta(\Gamma)$ on the torus. Let G_1 be the imbedding represented by the cyclic permutation $(x, z, y, x^{-1}, y^{-1})$ of Δ^*. The imbedding has one region that is bounded by the element $xyx^{-1}y^{-1}z$. By Theorem 14, the one region lifts to $\dfrac{|\Gamma|}{ord(xyx^{-1}y^{-1}z)}$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_1 is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \frac{\frac{|\Gamma|}{ord(xyx^{-1}y^{-1}z)}}{2} + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(xyx^{-1}y^{-1}z)} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_1 is on a surface of genus $\dfrac{3|\Gamma|}{4} - \dfrac{|\Gamma|}{2ord(xyx^{-1}y^{-1}z)} + 1$. Since there are two imbeddings in this case, we have $\dfrac{3|\Gamma|}{2} - \dfrac{|\Gamma|}{ord(xyx^{-1}y^{-1}z)} + 2$ contributing to the average Cayley genus.

Consider the next two voltage graph imbeddings of $G_\Delta(\Gamma)$ on the torus. Let G_3 be the imbedding represented by the cyclic permutation $(x, y, x^{-1}, y^{-1}, z)$ of Δ^*. The imbedding has one region that is bounded by the element $xyx^{-1}zy^{-1}$. By Theorem 14, the one region lifts to $\dfrac{|\Gamma|}{ord(xyx^{-1}zy^{-1})}$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_3 is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \frac{\frac{|\Gamma|}{ord(xyx^{-1}zy^{-1})}}{2} + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(xyx^{-1}zy^{-1})} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_3 is on a surface of genus $\dfrac{3|\Gamma|}{4} - \dfrac{|\Gamma|}{2ord(xyx^{-1}zy^{-1})} + 1$. Since there are two imbeddings in this case, we have $\dfrac{3|\Gamma|}{2} - \dfrac{|\Gamma|}{ord(xyx^{-1}zy^{-1})} + 2$ contributing to the average Cayley genus.

Consider the next two voltage graph imbeddings of $G_\Delta(\Gamma)$ on the torus. Let G_5 be the imbedding represented by the cyclic permutation $(x, y, x^{-1}, z, y^{-1})$ of Δ^*. The imbedding has one region that is bounded by the element $xyzx^{-1}y^{-1}$. By Theorem 14, the one region lifts to $\dfrac{|\Gamma|}{ord(xyzx^{-1}y^{-1})}$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_5 is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \frac{\frac{|\Gamma|}{ord(xyzx^{-1}y^{-1})}}{2} + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(xyzx^{-1}y^{-1})} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_5 is on a surface of genus $\dfrac{3|\Gamma|}{4} - \dfrac{|\Gamma|}{2ord(xyzx^{-1}y^{-1})} + 1$. Since there are two imbeddings in this case, we have $\dfrac{3|\Gamma|}{2} - \dfrac{|\Gamma|}{ord(xyzx^{-1}y^{-1})} + 2$ contributing to the average Cayley genus.

Consider the last two voltage graph imbeddings of $G_\Delta(\Gamma)$ on the torus. Let G_7 be the imbedding represented by the cyclic permutation $(x, y, z, x^{-1}, y^{-1})$ of Δ^*. The imbedding has one region that is bounded by the element $xzyx^{-1}y^{-1}$. By Theorem 14, the one region lifts to $\dfrac{|\Gamma|}{ord(xzyx^{-1}y^{-1})}$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_7 is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \frac{\frac{|\Gamma|}{ord(xzyx^{-1}y^{-1})}}{2} + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(xzyx^{-1}y^{-1})} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_7 is on a surface of genus $\dfrac{3|\Gamma|}{4} - \dfrac{|\Gamma|}{2ord(xzyx^{-1}y^{-1})} + 1$. Since there are two imbeddings in this case, we have $\dfrac{3|\Gamma|}{2} - \dfrac{|\Gamma|}{ord(xzyx^{-1}y^{-1})} + 2$ contributing to the average Cayley genus.

The next imbeddings we consider are depicted in Figure 4.2 on the sphere. Consider the first two voltage graph imbeddings on the sphere. Let G_9 be the imbedding represented by the cyclic permutation $(x, z, x^{-1}, y^{-1}, y)$ of Δ^*. Let R_1 be the region bounded by the element xz. Let R_2

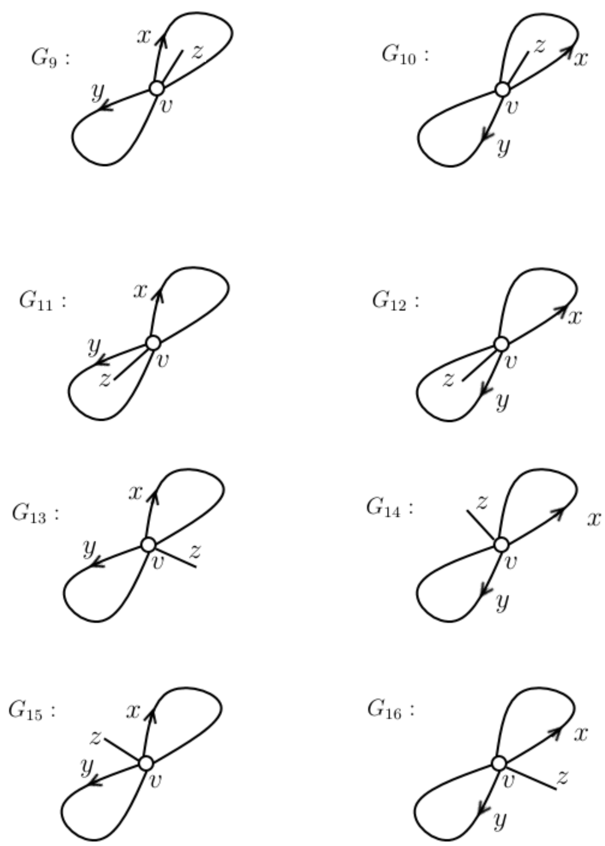


Figure 4.2: Eight voltage graph imbeddings for Γ and Δ on the sphere

be the region bounded by the element y^{-1}. Let R_3 be the region bounded by the element $x^{-1}y$. Let the order of the elements be denoted by $ord(xz)$, $ord(y^{-1})$, and $ord(x^{-1}y)$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(xz)}$, $\frac{|\Gamma|}{ord(y^{-1})}$, and $\frac{|\Gamma|}{ord(x^{-1}y)}$ regions respectively. The regions of G_9 lift to a total of $\left(\frac{|\Gamma|}{ord(xz)} + \frac{|\Gamma|}{ord(y^{-1})} + \frac{|\Gamma|}{ord(x^{-1}y)}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_9 is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(xz)} + \frac{|\Gamma|}{ord(y^{-1})} + \frac{|\Gamma|}{ord(x^{-1}y)}}{2}\right) + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(xz)} - \frac{|\Gamma|}{2ord(y^{-1})} - \frac{|\Gamma|}{2ord(x^{-1}y)} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_9 is on a surface of genus $\frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(xz)} - \frac{|\Gamma|}{2ord(y^{-1})} - \frac{|\Gamma|}{2ord(x^{-1}y)} + 1$. Since there are two imbeddings in this case, we have $\frac{3|\Gamma|}{2} - \frac{|\Gamma|}{ord(xz)} - \frac{|\Gamma|}{ord(y^{-1})} - \frac{|\Gamma|}{ord(x^{-1}y)} + 2$ contributing to the average Cayley genus.

Consider the next two cases of voltage graph imbeddings on the sphere, as seen in Figure 4.2. Let G_{11} be the imbedding represented by the cyclic permutation $(x, x^{-1}, y^{-1}, z, y)$ of Δ^*. Let R_1 be the region bounded by the element x. Let R_2 be the region bounded by the element $y^{-1}z$. Let R_3 be the region bounded by the element $x^{-1}y$. Let the order of the elements be denoted by $ord(x)$, $ord(y^{-1}z)$, and $ord(x^{-1}y)$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(x)}$, $\frac{|\Gamma|}{ord(y^{-1}z)}$, and $\frac{|\Gamma|}{ord(x^{-1}y)}$ regions respectively. The regions of G_{11} lift to a total of

$\left(\frac{|\Gamma|}{ord(x)} + \frac{|\Gamma|}{ord(y^{-1}z)} + \frac{|\Gamma|}{ord(x^{-1}y)}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{11} is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x)} + \frac{|\Gamma|}{ord(y^{-1}z)} + \frac{|\Gamma|}{ord(x^{-1}y)}}{2}\right) + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y^{-1}z)} - \frac{|\Gamma|}{2ord(x^{-1}y)} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{11} is on a surface of genus $\frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y^{-1}z)} - \frac{|\Gamma|}{2ord(x^{-1}y)} + 1$. Since there are two imbeddings in this case, we have $\frac{3|\Gamma|}{2} - \frac{|\Gamma|}{ord(x)} - \frac{|\Gamma|}{ord(y^{-1}z)} - \frac{|\Gamma|}{ord(x^{-1}y)} + 2$ contributing to the average Cayley genus.

Now, consider the next two cases of voltage graph imbeddings on the sphere, as seen in Figure 4.2. Let G_{13} be the imbedding represented by the cyclic permutation $(x, x^{-1}, z, y^{-1}, y)$ of Δ^*. Let R_1 be the region bounded by the element x. Let R_2 be the region bounded by the element y^{-1}. Let R_3 be the region bounded by the element $x^{-1}yz$. Let the order of the elements be denoted by $ord(x)$, $ord(y^{-1})$, and $ord(x^{-1}yz)$ appropriately. By Theorem 14, each region R_1, R_2, and

R_3 lifts to $\dfrac{|\Gamma|}{ord(x)}$, $\dfrac{|\Gamma|}{ord(y^{-1})}$, and $\dfrac{|\Gamma|}{ord(x^{-1}yz)}$ regions respectively. The regions of G_{13} lift to a total of

$\left(\dfrac{|\Gamma|}{ord(x)} + \dfrac{|\Gamma|}{ord(y^{-1})} + \dfrac{|\Gamma|}{ord(x^{-1}yz)}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{13} is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x)} + \frac{|\Gamma|}{ord(y^{-1})} + \frac{|\Gamma|}{ord(x^{-1}yz)}}{2}\right) + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y^{-1})} - \frac{|\Gamma|}{2ord(x^{-1}yz)} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{13} is on a surface of genus $\dfrac{3|\Gamma|}{4} - \dfrac{|\Gamma|}{2ord(x)} - \dfrac{|\Gamma|}{2ord(y^{-1})} - \dfrac{|\Gamma|}{2ord(x^{-1}zy)} + 1$. Since there are two imbeddings in this case, we have $\dfrac{3|\Gamma|}{2} - \dfrac{|\Gamma|}{ord(x)} - \dfrac{|\Gamma|}{ord(y^{-1})} - \dfrac{|\Gamma|}{ord(x^{-1}yz)} + 2$ contributing to the average Cayley genus.

Consider the last two cases shown in Figure 4.2 of voltage graph imbeddings on the sphere. Let G_{15} be the imbedding represented by the cyclic permutation $(x, x^{-1}, y^{-1}, y, z)$ of Δ^*. Let R_1 be the region bounded by the element x. Let R_2 be the region bounded by the element y^{-1}. Let R_3 be the region bounded by the element $x^{-1}zy$. Let the order of the elements be denoted by $ord(x)$, $ord(y^{-1})$, and $ord(x^{-1}zy)$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\dfrac{|\Gamma|}{ord(x)}$, $\dfrac{|\Gamma|}{ord(y^{-1})}$, and $\dfrac{|\Gamma|}{ord(x^{-1}zy)}$ regions respectively. The regions of G_{15} lift to a total of

$\left(\dfrac{|\Gamma|}{ord(x)} + \dfrac{|\Gamma|}{ord(y^{-1})} + \dfrac{|\Gamma|}{ord(x^{-1}zy)}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{15} is imbedded on.

$$
\begin{aligned}
k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\
&= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x)} + \frac{|\Gamma|}{ord(y^{-1})} + \frac{|\Gamma|}{ord(x^{-1}zy)}}{2} \right) + 1 \\
&= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y^{-1})} - \frac{|\Gamma|}{2ord(x^{-1}zy)} + 1.
\end{aligned}
$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{15} is on a surface of genus $\frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x)} - \frac{|\Gamma|}{2ord(y^{-1})} - \frac{|\Gamma|}{2ord(x^{-1}zy)} + 1$. Since there are two imbeddings in this case, we have $\frac{3|\Gamma|}{2} - \frac{|\Gamma|}{ord(x)} - \frac{|\Gamma|}{ord(y^{-1})} - \frac{|\Gamma|}{ord(x^{-1}zy)} + 2$ contributing to the average Cayley genus.

The last set of cases depicted in Figure 4.3, where the voltage graph is imbedded on the sphere. Let G_{17} be the imbedding represented by the cyclic permutation $(x, y, y^{-1}, x^{-1}, z)$ of Δ^*. Let R_1 be the region bounded by the element $x^{-1}z$. Let R_2 be the region bounded by the element

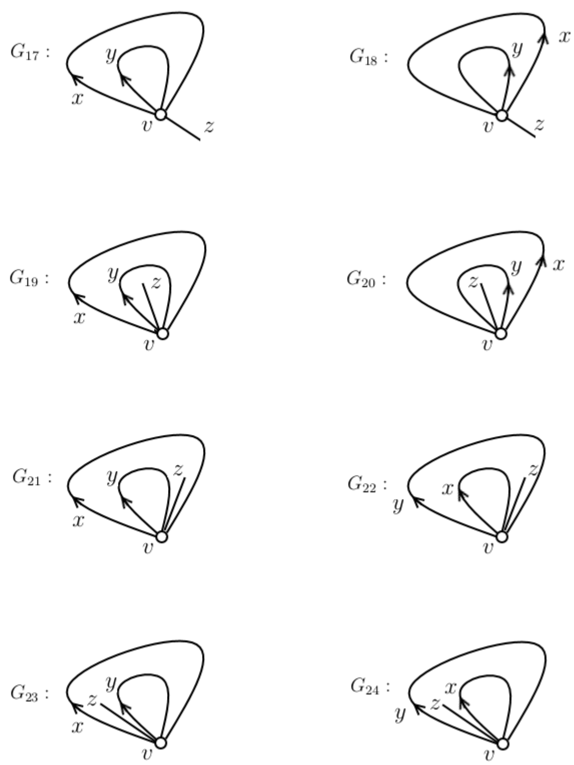


Figure 4.3: Eight voltage graph imbeddings for Γ and Δ on the sphere

y. Let R_3 be the region bounded by the element xy^{-1}. Let the order of the elements be denoted by $ord(x^{-1}z)$, $ord(y)$, and $ord(xy^{-1})$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(x^{-1}z)}$, $\frac{|\Gamma|}{ord(y)}$, and $\frac{|\Gamma|}{ord(xy^{-1})}$ regions respectively. The regions of G_{17} lift to a total of $\left(\frac{|\Gamma|}{ord(x^{-1}z)} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xy^{-1})}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{17} is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x^{-1}z)} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xy^{-1})}}{2}\right) + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1}z)} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{17} is on a surface of genus $\frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1}z)} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1$. Since there are two imbeddings in this case, we have $\frac{3|\Gamma|}{2} - \frac{|\Gamma|}{ord(x^{-1}z)} - \frac{|\Gamma|}{ord(y)} - \frac{|\Gamma|}{ord(xy^{-1})} + 2$ contributing to the average Cayley genus.

Next consider the second case depicted in Figure 4.3, where the voltage graph is imbedded on the sphere. Let G_{19} be the imbedding represented by the cyclic permutation $(x, y, z, y^{-1}, x^{-1})$ of Δ^*. Let R_1 be the region bounded by the element x^{-1}. Let R_2 be the region bounded by the element yz. Let R_3 be the region bounded by the element xy^{-1}. Let the order of the elements be denoted by $ord(x^{-1})$, $ord(yz)$, and $ord(xy^{-1})$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(x^{-1})}$, $\frac{|\Gamma|}{ord(yz)}$, and $\frac{|\Gamma|}{ord(xy^{-1})}$ regions respectively. The regions of G_{19} lift to a total of $\left(\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(yz)} + \frac{|\Gamma|}{ord(xy^{-1})}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{19} is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(yz)} + \frac{|\Gamma|}{ord(xy^{-1})}}{2}\right) + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(yz)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{19} is on a surface of genus $\frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(yz)} - \frac{|\Gamma|}{2ord(xy^{-1})} + 1$. Since there are two imbeddings in this case, we have $\frac{3|\Gamma|}{2} - \frac{|\Gamma|}{ord(x^{-1})} - \frac{|\Gamma|}{ord(yz)} - \frac{|\Gamma|}{ord(xy^{-1})} + 2$ contributing to the average Cayley genus.

Now, consider the next two cases of voltage graph imbeddings on the sphere, as seen in Figure 4.3. Let G_{21} be the imbedding represented by the cyclic permutation $(x, y, y^{-1}, z, x^{-1})$ of Δ^*. Let R_1 be the region bounded by the element x^{-1}. Let R_2 be the region bounded by the element y. Let R_3 be the region bounded by the element xzy^{-1}. Let the order of the elements be denoted by $ord(x^{-1})$, $ord(y)$, and $ord(xzy^{-1})$ appropriately. By Theorem 14, each region R_1, R_2, and R_3 lifts to $\frac{|\Gamma|}{ord(x^{-1})}$, $\frac{|\Gamma|}{ord(y)}$, and $\frac{|\Gamma|}{ord(xzy^{-1})}$ regions respectively. The regions of G_{21} lift to a total of

$\left(\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xzy^{-1})}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{21} is imbedded on.

$$\begin{aligned} k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\ &= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xzy^{-1})}}{2}\right) + 1 \\ &= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xzy^{-1})} + 1. \end{aligned}$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{21} is on a surface of genus $\frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xzy^{-1})} + 1$. Since there are two imbeddings in this case, we have $\frac{3|\Gamma|}{2} - \frac{|\Gamma|}{ord(x^{-1})} - \frac{|\Gamma|}{ord(y)} - \frac{|\Gamma|}{ord(xzy^{-1})} + 2$ contributing to the average Cayley genus.

Consider the last two cases shown in Figure 4.3 of voltage graph imbeddings on the sphere. Let G_{23} be the imbedding represented by the cyclic permutation $(x, z, y, y^{-1}, x^{-1})$ of Δ^*. Let R_1 be the region bounded by the element x^{-1}. Let R_2 be the region bounded by the element y. Let R_3 be the region bounded by the element $xy^{-1}z$. Let the order of the elements be denoted by $ord(x^{-1})$, $ord(y)$, and $ord(xy^{-1}z)$ appropriately. By Theorem 14, each region R_1, R_2, and R_3

lifts to $\dfrac{|\Gamma|}{ord(x^{-1})}$, $\dfrac{|\Gamma|}{ord(y)}$, and $\dfrac{|\Gamma|}{ord(xy^{-1}z)}$ regions respectively. The regions of G_{23} lift to a total of

$\left(\dfrac{|\Gamma|}{ord(x^{-1})} + \dfrac{|\Gamma|}{ord(y)} + \dfrac{|\Gamma|}{ord(xy^{-1}z)}\right)$ regions. By Theorem 7 we find the genus of the surface that the Cayley map represented by G_{23} is imbedded on.

$$
\begin{aligned}
k &= \frac{m}{2} - \frac{n}{2} - \frac{r}{2} + 1 \\
&= \frac{\frac{5|\Gamma|}{2}}{2} - \frac{|\Gamma|}{2} - \left(\frac{\frac{|\Gamma|}{ord(x^{-1})} + \frac{|\Gamma|}{ord(y)} + \frac{|\Gamma|}{ord(xy^{-1}z)}}{2}\right) + 1 \\
&= \frac{3|\Gamma|}{4} - \frac{|\Gamma|}{2ord(x^{-1})} - \frac{|\Gamma|}{2ord(y)} - \frac{|\Gamma|}{2ord(xy^{-1}z)} + 1.
\end{aligned}
$$

The imbedding of $G_\Delta(\Gamma)$ lifted from G_{23} is on a surface of genus $\dfrac{3|\Gamma|}{4} - \dfrac{|\Gamma|}{2ord(x^{-1})} - \dfrac{|\Gamma|}{2ord(y)} - \dfrac{|\Gamma|}{2ord(xy^{-1}z)} + 1$. Since there are two imbeddings in this case, we have $\dfrac{3|\Gamma|}{2} - \dfrac{|\Gamma|}{ord(x^{-1})} - \dfrac{|\Gamma|}{ord(y)} - \dfrac{|\Gamma|}{ord(xy^{-1}z)} + 2$ contributing to the average Cayley genus.

Recall that for any $g \in \Gamma$ $ord(g) = ord(g^{-1})$. Therefore the average Cayley genus of a finite group with two generators of order greater than two and one generator with order two is given by the formula,

$$
\begin{aligned}
\bar{\gamma}(\Gamma, \Delta) = \frac{3|\Gamma|}{4} \quad & - \frac{|\Gamma|}{4ord(x)} - \frac{|\Gamma|}{4ord(y)} - \frac{|\Gamma|}{12ord(xy^{-1})} - \frac{|\Gamma|}{12ord(x^{-1}y)} \\
& - \frac{|\Gamma|}{24ord(xz)} - \frac{|\Gamma|}{24ord(y^{-1}z)} - \frac{|\Gamma|}{24ord(x^{-1}z)} - \frac{|\Gamma|}{24ord(yz)} \\
& - \frac{|\Gamma|}{24ord(x^{-1}yz)} - \frac{|\Gamma|}{24ord(x^{-1}zy)} - \frac{|\Gamma|}{24ord(xzy^{-1})} \\
& - \frac{|\Gamma|}{24ord(xy^{-1}z)} - \frac{|\Gamma|}{24ord(xyx^{-1}y^{-1}z)} - \frac{|\Gamma|}{24ord(xzyx^{-1}y^{-1})} \\
& - \frac{|\Gamma|}{24ord(xyx^{-1}zy^{-1})} - \frac{|\Gamma|}{24ord(xyzx^{-1}y^{-1})} + 1
\end{aligned}
$$

□

BIBLIOGRAPHY

[1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 5th ed., CRC Press, Boca Raton, FL, 2011. MR2766107

[2] R. A. Duke, *The genus, regional number, and Betti number of a graph*, Canad. J. Math. **18** (1966), 817–822. MR0196731

[3] D. S. Dummit and R. M. Foote, *Abstract algebra*, 3rd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR2286236

[4] W. Dyck, *Beiträge zur Analysis situs*, Math. Ann. **32** (1888), no. 4, 457–512, DOI 10.1007/BF01443580 (German). MR1510522

[5] J. Edmonds, *A combinatorial representation for polyhedral surfaces*, Notices Amer. Math. Soc. **7** (1960), 646.

[6] L. Heffter, *Ueber das Problem der Nachbargebiete*, Math. Ann. **38** (1891), no. 4, 477–508, DOI 10.1007/BF01203357 (German). MR1510685

[7] M. Jungerman, *A characterization of upper-embeddable graphs*, Trans. Amer. Math. Soc. **241** (1978), 401–406, DOI 10.2307/1998852. MR492309

[8] Lhuilier and Gergonne, *Géométrie. Mémoire sur la polyédrométrie; contenant une démonstration directe du théorème d'Euler sur les polyèdres, et un examen des diverses exceptions auxquelles ce théorème est assujetti*, Ann. Math. Pures Appl. [Ann. Gergonne] **3** (1812/13), 169–189 (French). MR1555676

[9] E. A. Nordhaus, R. D. Ringeisen, B. M. Stewart, and A. T. White, *A Kuratowski-type theorem for the maximum genus of a graph*, J. Combinatorial Theory Ser. B **12** (1972), 260–267. MR0299523

[10] M. Schultz, *Random Cayley maps*, ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–Western Michigan University. MR2694540

[11] M. Schultz, *Two probability models for Cayley graph imbeddings*, 1999. Y. Alavi, A.J. Schwenk (Eds.), Proceedings of the Eight Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications.

[12] A. T. White, *Graphs of groups on surfaces*, North-Holland Mathematics Studies, vol. 188, North-Holland Publishing Co., Amsterdam, 2001. Interactions and models. MR1852593

[13] N. H. Xuong, *Upper-embeddable graphs and related topics*, J. Combin. Theory Ser. B **26** (1979), no. 2, 226–232, DOI 10.1016/0095-8956(79)90059-5. MR532590

[14] J. W. T. Youngs, *Minimal imbeddings and the genus of a graph*, J. Math. Mech. **12** (1963), 303–315. MR0145512

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