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A COMPARISON OF THE PRODUCT TOPOLOGY ON TWO TREES WITH THE TREE TOPOLOGY ON THE CONCATENATION OF TWO TREES

by

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Bachelor of Science in Mathematics
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2012

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science - Mathematical Sciences

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ABSTRACT

A game tree is a nonempty set of sequences, closed under subsequences (i.e., if $p \in T$ and p extends q, then $q \in T$). If T is a game tree, then there is a natural topology on [T], the set of paths through T. In this study we consider two types of topological spaces, both constructed from game trees. The first is constructed by taking the Cartesian product of two game trees, T and S: $[T] \times [S]$. The second is constructed by the concatenation of two game trees, T and S: $[T * S]$. The goal of our study is to determine what conditions we must require of the trees T and S so that these two topologies are homeomorphic.

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—Katlyn K. Cox

CHAPTER 1

INTRODUCTION

1.1 Motivation

Determinacy has been studied extensively since the 1950's. The study of determinacy has led to several important results which have impacted areas of modern set theory, such as the study of large cardinals and descriptive set theory. Determinacy with more complicated, (but definable), "pay-offs" is stronger (in consistency strength). One way to get more complicated pay-offs on games of length ω is to play longer open games (length$>\omega$).

In the study of "longer" games—games in which plays have length longer than ω—we see that the long tree has the tree topology, but we may also "split" the tree at ω and view it as the product of two trees. This is used in Ikeda's dissertation [1] and Yosts's thesis [7] .

A specific example of "splitting" trees is given by Ikeda [1]. She explains: "We shall identify the body of the tree $[X^{\leq\omega+n}] = X^{\omega+n}$ with the product $X^\omega \times X^n$. Let

$x = \langle x_0, x_1, \ldots \rangle \in X^\omega$ and $g = \langle g_0, g_1, \ldots, g_{n-1} \rangle \in X^n$. Then

$x^\frown g = \langle x_0, x_1, \ldots, g_0, g_1, \ldots, g_{n-1} \rangle \in X^{\omega+n}$."

However, when a tree has paths which vary in length, it is not clear that splitting is possible. This leads us to natural questions: When can this be done? Is there always a homeomorphism between the product topology of two trees and the concatenated topology of the long tree? If not, for which types of trees does this homeomorphism hold?

Initially, we constructed an example for which the two topologies are homeomorphic, but we were also able to construct a counterexample in which the two topologies are not. With this in mind, we began our studies to answer these questions.

1.2 Introduction to this Thesis

The goal of this thesis is to study the relationship of two common topologies on game trees which arise when studying determinacy. They are the product topology and the tree topology. In this thesis, we provide some basic results as to how the tree topology behaves for "longer" trees. We start by showing that for trees $T = S = \omega^{<\omega}$, the two topologies are homeomorphic. This example drives us to show that there is always a natural bijection (referred to throughout this thesis as the **canonical function**) between the product topology of two trees and the tree topology of the "long" concatenated tree. We are able to introduce lemmas for the canonical function; the first states sufficient conditions to show that the canonical function is continuous, and the second states necessary and sufficient conditions to show that the canonical function is an open map. Using these lemmas, we prove a result that generalizes to more trees for which the canonical function is a homeomorphism between the two topologies. At the end of Chapter 2 and in Chapter 3 we give several examples of interesting trees for which the canonical function produces a homeomorphism between the two topologies. However, in Chapter 4 we show that the canonical function does not always produce a homeomorphism. We give two counterexamples in which we show that the canonical function is not continuous and is not open.

For material in this thesis, the following books and publications are standard references:

Jech's *Set Theory* [2]

Martin's *Borel and Projective Games* [3]

Moschovakis' *Descriptive Set Theory* [4]

Munkres' *Topology* [5]

These and additional references are listed in the Bibliography.

1.3 Definitions from Topology

The following definitions can be found in Munkres' *Topology* [5].

Definition 1.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.

2. The union of elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Definition 1.2. A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**.

Definition 1.3. If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Definition 1.4. If X is any set, the collection of all subsets of X is a topology on X. It is called the **discrete topology**.

Definition 1.5. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that:

1. For each $x \in X$, there is at least one basis element B such that $x \in B$.

2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Remark 1.6. If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as: A subset U of X is open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Definition 1.7. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Definition 1.8. A subset A of a topological space X is said to be **closed** if the set $X \setminus A$ is open.

Definition 1.9. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V) = \{x \in X |\ f(x) \in V\}$ is an open subset of X.

Definition 1.10. A function $f : X \to Y$ is said to be **open** if for every open set U of X, the set $f(U) = \{f(x) |\ x \in U\}$ is open in Y.

Definition 1.11. Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**. Alternatively, a **homeomorphism** is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open if and only if U is open.

1.4 Definitions and Notation for this Thesis

Definition 1.12. Define a **sequence** as any function defined on an ordinal. So, for $f : \alpha \to X$, where $\alpha \in ord$, then

$$f := \langle f(i) |\ i \in \alpha \rangle = \{(i, b) |\ i \in \alpha,\ b \in X \ \text{ and } (i, b) \in f\}$$

Definition 1.13. Let A and B be any sets. $A^B = \{f |\ f : B \to A\}$.

Definition 1.14. Let f be a sequence. Then, we define the **length of f** as $lth(f) = dom(f)$.

Notation 1.15. Suppose f is a sequence of length $\alpha \in ord$. Then, for any $\beta \leq \alpha$, $f \restriction \beta$ is the sequence of length β, such that for all $x \in \beta$, $(f \restriction \beta)(x) = f(x)$.

Definition 1.16. T is a **game tree** on X if and only if T is a non-empty set of sequences, and for all $f \in T$ there exists $\gamma \in ord$, where $f : \gamma \to X$, and $\forall\ \gamma' < \gamma,\ f \restriction \gamma' \in T$.

Remark 1.17. We will refer to a game tree as a tree.

Definition 1.18. A tree T is **non-trivial** if and only if there exists $f \in T$ such that $dom(f) > 0$. A tree T is **trivial** if $T = \{\emptyset\}$.

Definition 1.19. Let T be a tree. The **body of T**, denoted as $[T]$, is defined as $f \in [T]$ if and only if no proper extension of f is in T and:

1. If $dom(f) = \gamma,$ for γ a limit ordinal, then $\forall \gamma' < \gamma,\ f \restriction \gamma' \in T$.

2. If $dom(f) = \gamma,$ for γ a successor ordinal, then $f \in T$.

We say f is a **path through T** if $f \in [T]$.

Definition 1.20. A tree T is **well-founded** if for all $f \in [T],\ dom(f) < \omega$.

Definition 1.21. If A is any set, $\mathcal{P}_{fin}(A) = \{D \subseteq A \mid D \text{ is finite}\}$.

Definition 1.22. $fin(A^B) = \{\tau | \ \exists\ D \in \mathcal{P}_{fin}(B) \text{ and } \tau : D \to A\}$.

Remark 1.23. For any $f \in T$ if $f^\star \in \mathcal{P}_{fin}(f)$, then $dom(f^\star) \in \mathcal{P}_{fin}\big(dom(f)\big)$ and for all $x \in dom(f^\star),\ f^\star(x) = f(x)$.

Definition 1.24. If d is any function, $dom(d) \subseteq ord,\ ran(d) \subseteq X$, and $\alpha \in ord$, then:

1. Shift right by α is defined as $s_R(d, \alpha) = \{(\alpha + \rho, x) \mid (\rho, x) \in d\}$.

2. Shift left by α is defined as $s_L(d, \alpha) = \{(\rho, x)\ |(\alpha + \rho, x) \in d\}$, where $\alpha \leq \gamma$ such that $\gamma = min\{\lambda | \lambda \in dom(d)\}$.

Definition 1.25. Let T be a tree. Then the **Tree Topology** on $[T]$ is the topology generated by the following basis: If $d : A \to X$ and $A \subseteq ord$ where $|A| < \omega$ (i.e. $d \in fin(X^A)$), then $\mathfrak{B}_d^{[T]} = \{f \in [T] |\ f \supseteq d\}$.

Remark 1.26. This is a basis for a topology.

Fact 1.27. Let $A \subseteq [T]$. A is **open in the Tree Topology** if and only if
$\forall f \in A\ \exists f^\star \in \mathcal{P}_{fin}(f)\ \forall g \in [T]\ (g \supseteq f^\star \implies g \in A)$.

Definition 1.28. Let X_α be a set with topology $\mathcal{T}_\alpha$ and $\alpha \in \gamma$, where γ is any ordinal. The **Product Topology** on $\prod_{\alpha\in\gamma} X_\alpha$ is the topology generated by the basis

$$\left\{ \mathfrak{B} = \prod_{\alpha\in\gamma} Y_\alpha, \quad \text{where each } Y_\alpha \text{ is open in } \mathcal{T}_\alpha \text{ and finitely many } Y_\alpha \neq X_\alpha \right\}$$

Remark 1.29. Let $\mathcal{T}_{\otimes}$ denote the product topology on $[T] \times [S]$. A basic open set in $\mathcal{T}_{\otimes}$ is $\mathfrak{B}_{t^{\star}}^{[T]} \times \mathfrak{B}_{s^{\star}}^{[S]} = \{t \in [T] | \ t \supseteq t^{\star}\} \times \{s \in [S] | \ s \supseteq s^{\star}\}$ (where $t^{\star} \in fin(X^A)$ and $s^{\star} \in fin(Y^B)$).

Notation 1.30. Let $(a_1, a_2) \in [T] \times [S]$. To simplify notation, we write $a_1 \times a_2 \in [T] \times [S]$.

Definition 1.31. Let $t \in [T]$ and $s \in [S]$, where $dom(t) = \alpha$ and $dom(s) = \beta$. Define the **concatenation** $x = t^\frown s$ as:

If $i \in \alpha + \beta$, then $x(i) = \begin{cases} t(i), \text{ if } i \in \alpha \\ s(j), \text{ if } i = \alpha + j, \ j \in \beta \end{cases}$

Note that $dom(t^\frown s) = \alpha + \beta$.

Remark 1.32. If $i \in \alpha + \beta$, then $i \in \alpha$ or there exists unique $j \in \beta$ such that $i = \alpha + j$.

Definition 1.33.

$$T * S = \left\{ f \;\middle|\; \begin{array}{l} f \in T, \text{ or} \\ \exists \alpha \text{ such that } f \restriction \alpha \in [T] \text{ and } s_L(f \restriction [\alpha, \ dom(f)), \alpha) \in S \end{array} \right\}$$

Remark 1.34. If we allowed for empty trees, then $T * S = T$, where $S = \emptyset$.

Remark 1.35. If S is trivial ($S = \{\emptyset\}$), then $T * S = T \cup [T]$ and $[T] = [T * S]$. Further, $[T]$ and $[T * S]$ have the same topology.

Remark 1.36. Let $\mathcal{T}_{\circledast}$ denote the tree topology on $[T * S]$. We can express a basic open set in $\mathcal{T}_{\circledast}$ as $\mathcal{B}_d^{[T*S]} = \{f \in [T * S] | \ f \supseteq d\}$.

1.5 Preliminaries

We begin by proving the following lemmas. The first lemma describes that we can split a path through a tree into two unique sequences, such that the concatenation of the two sequences is the same as the original path. The second lemma shows that any path through a concatenated tree can be split into two paths, one of which is a path through the first tree and the other a path through the second tree. It is important to note that each of these paths are also unique by Lemma 1.37.

Lemma 1.37. *If $f : \gamma \to X$ and $\alpha \leq \gamma$, then there exist unique sequences f_1 and f_2 such that $f = f_1^\frown f_2$ where $dom(f_1) = \alpha$.*

Proof. Let $f : \gamma \to X$ and $\alpha \leq \gamma$.

If $\alpha = \gamma$, then let $f_1 = f$ and $f_2 = \emptyset$. Note that $dom(f_1) = \alpha$. Then, by definition of concatenation $f = f_1^\frown f_2$. Also, note that f_1 and f_2 are unique, since f and $\emptyset$ are both unique.

Assume $\alpha < \gamma$. Let $f_1 = f \restriction \alpha$. Note that $dom(f_1) = \alpha$. Let $f_2 = s_L\big(f \restriction [\alpha, \gamma), \alpha\big)$. Then, by definition of concatenation $f = f_1^\frown f_2$.

To show that f_1 and f_2 are unique, consider a second concatenation, $f = g_1^\frown g_2$, where $dom(g_1) = \alpha$. So, $g_1 = f \restriction \alpha$. Since f is a function, then $f_1(\lambda) = f(\lambda) = g_1(\lambda)$ for all $\lambda < \alpha$. Therefore, $f_1 = g_1$. Since $\alpha < \gamma$, there exists a unique β such that $\alpha + \beta = \gamma$. By definition of concatenation, if $dom(f_1) = \alpha$, then $dom(f_2) = \beta$. Similarly, $dom(g_1) = \alpha$. So, $dom(g_2) = \beta$. Hence, $f_2(\mu) = f(\alpha + \mu) = g_2(\mu)$ for all μ such that $\mu < \beta$. Thus, $f_2 = g_2$. We conclude that f_1 and f_2 are unique. □

Lemma 1.38. *Assume T and S are non-trivial trees. Then $f \in [T * S]$ if and only if there exists a unique $\alpha < dom(f)$ such that $f \restriction \alpha \in [T]$, and so $s_L(f \restriction [\alpha, dom(f)), \alpha) \in [S]$.*

Remark 1.39. By Lemma 1.37 $f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, dom(f)), \alpha) \in [S]$ are unique.

Proof. ($\Rightarrow$) Assume $f \in [T * S]$.

Case 1: Let $dom(f) = \gamma$ for some limit ordinal γ. So, for all $\lambda < \gamma$, $f \restriction \lambda \in T * S$. Then, $f \restriction \lambda \in T$ or there exists $\alpha < \lambda$ such that $f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, \lambda), \alpha) \in S$.

Suppose $f \restriction \lambda \in T$ for all $\lambda < \gamma$. By definition, $f \in [T]$, because γ is a limit ordinal. Now, $f \in [T]$ and $f \in [T * S]$. By Lemma 1.37, $f = f_1^\frown f_2$ where $dom(f_1) = \gamma$. Since $f_1 = f$, then $f_1 \in [T]$. So, $f_2 = \emptyset$ and $dom(f_2) = 0$. Next, because $f = f_1^\frown f_2 \in [T * S]$ and $f_1 \in [T]$, $f_2 \in S$. So, for all $h \in S$, $dom(h) = 0$ which implies S is a trivial tree. By assumption, S is a non-trivial tree. Hence, there exists $\alpha < \gamma$ such that for all $\widehat{\lambda}$ where $\alpha \leq \widehat{\lambda} < \gamma$, $f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, \widehat{\lambda}), \alpha) \in S$.

Say $f \restriction \alpha = f_1$. By Lemma 1.37, $f = f_1^\frown f_2$, where $dom(f_1) = \alpha$ and $f_2 = s_L(f \restriction [\alpha, \gamma), \alpha)$. Also, by Lemma 1.37, f_1 and f_2 are unique. Last, since $\gamma - \alpha$ is a limit ordinal, $s_L(f \restriction [\alpha, \gamma), \alpha) \in [S]$, by definition 1.19, as long as no proper extension of f_2 is in S. Suppose there is some proper extension of f_2 in S. Let $g_2 \supset f_2$ in S. So, there exists $g = f_1^\frown g_2$. Since $f = f_1^\frown f_2$ is unique and $g_2 \supset f_2$, $g \supset f$. But, $f \in [T * S]$. Thus, no proper extension of f exists. Hence, there is no proper extension of $f_2 \in S$.

Case 2: Let $dom(f) = \gamma$ for some successor ordinal γ. By definition 1.19, $f \in T * S$. Then, $f \in T$ or there exists $\alpha < \gamma$ such that $f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, \gamma), \alpha) \in S$. If $f \in T$, then $f \in [T]$, because γ is a successor ordinal. Additionally, $f \in [T * S]$. However, this would imply that S is a trivial tree, as proven in **Case 1**. Hence, there exists $\alpha < \gamma$ such that

$f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, \gamma), \alpha) \in S$.

Say $f \restriction \alpha = f_1$. By Lemma 1.37, $f = f_1^\frown f_2$, where $dom(f_1) = \alpha$ and $f_2 = s_L(f \restriction [\alpha, \gamma), \alpha)$. Also, by Lemma 1.37, f_1 and f_2 are unique. Last, since $\gamma - \alpha$ is a successor ordinal, then $s_L(f \restriction [\alpha, \gamma), \alpha) \in [S]$, by definition 1.19, as long as no proper extension of f_2 is in S. Suppose there is some proper extension of f_2 in S. Let $g_2 \supset f_2$ in S. So, there exists $g = f_1^\frown g_2$. Since $f = f_1^\frown f_2$ is unique and $g_2 \supset f_2$, $g \supset f$. But, $f \in [T * S]$. Thus, no proper extension of f exists. Hence, there is no proper extension of $f_2 \in S$.

($\Leftarrow$) Let f be a sequence such that $f : \gamma \to X$. Suppose there exists $\alpha < dom(f) = \gamma$ such that $f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, \gamma), \alpha) \in [S]$.

First, we wish to show that f has no proper extension in $T * S$. Suppose not. Then, there exists a sequence g, where $g \supset f$. But, $s_L(f \restriction [\alpha, \gamma), \alpha) \in [S]$. So, $s_L(g \restriction [\alpha, dom(g)), \alpha) \notin S$, because no proper extension of $s_L(f \restriction [\alpha, \gamma), \alpha)$ is in S. Thus, $g \notin T * S$.

Suppose $\lambda < \alpha$. Consider $f \restriction \lambda$. Since $f \restriction \alpha \in [T]$, then $f \restriction \lambda \in T$. Thus, $f \restriction \lambda \in T * S$. Suppose $\alpha \leq \lambda < \gamma$. Consider $f \restriction \lambda$. Since $\alpha \leq \lambda$, by Lemma 1.37, $f \restriction \lambda = f_1^\frown f_2$ where $dom(f_1) = \alpha$. Then, by assumption $f_1 \in [T]$, since $f_1 = f \restriction \alpha$. Next, $f_2 = s_L(f \restriction [\alpha, \lambda), \alpha)$. If $\lambda = \alpha$, then $f_2 = \emptyset$. In either case, because $\lambda < \gamma$, then $f_2 \in S$. Thus, $f \restriction \lambda \in T * S$.

If γ is a limit ordinal, then $f \in [T * S]$ and we are done.

Otherwise, let γ be a successor ordinal. Consider f. Since $\alpha < \gamma$, by Lemma 1.37, $f = f_1^\frown f_2$ where $dom(f_1) = \alpha$. By assumption, we have $f_1 \in [T]$, because $f_1 = f \restriction \alpha$. Next, $f_2 = s_L(f \restriction [\alpha, \gamma), \alpha) \in [S]$. Now, since $\lambda - \alpha$ is successor and $f_2 \in [S]$, then $f_2 \in S$. Thus, $f \in T * S$. So, $f \in [T * S]$.

Last, we wish to show that α is unique. Suppose not. Then, there exist an $f \in [T * S]$ and $\alpha < dom(f)$ such that $f \restriction \alpha \in [T]$ and $s_L(f \restriction [\alpha, dom(f)), \alpha) \in [S]$. Additionally, there exists $\beta < dom(f)$ such that $f \upharpoonright \beta \in [T]$ and $s_L(f \upharpoonright [\beta, dom(f)), \beta) \in [S]$, where $\alpha \neq \beta$. Since α and β are ordinals, then either $\alpha < \beta$ or $\beta < \alpha$.

Without loss of generality, assume that $\alpha < \beta$. Say $f_1 = f \upharpoonright \alpha$ and $f_2 = f \upharpoonright \beta$. Since $\alpha < \beta$, $f_2 \supset f_1$. Thus, $f_1 = f_2 \upharpoonright \alpha$.

Case 1: Let β be a successor ordinal. Since $f \upharpoonright \beta \in [T]$, $f \upharpoonright \beta \in T$. Thus, f_1 has a proper extension in T, but $f_1 \in [T]$. Contradiction.

Case 2: Let β be a limit ordinal. So, $\alpha + 1 < \beta$. Since $f \upharpoonright \beta \in [T]$, $f_2 \upharpoonright (\alpha + 1) \in T$. Thus, $f_2 \upharpoonright (\alpha + 1)$ is a proper extension of f_1 in T, but $f_1 \in [T]$. Contradiction.

In both cases, we get a contradiction. Therefore $\alpha = \beta$. So, α is unique. □

CHAPTER 2

THE CANONICAL FUNCTION

We begin this chapter with an example which shows that the product topology on the trees $T = S = \omega^{<\omega}$ is homeomorphic to the tree topology on the concatenated tree $T * S$. This result shows that there are homeomorphisms that exist between the two topologies for certain trees.

2.1 A Basic Homeomorphism

Theorem 2.1. *Let $T = S = \omega^{<\omega}$. Then $[T] \times [S] \cong [T * S]$.*

Proof. Let $T = S = \omega^{<\omega}$.

Define $f : [T] \times [S] \to [T * S]$. Let $a \in [T] \times [S]$ where $a = a_1 \times a_2$, such that $a_1 \in [T]$ and $a_2 \in [S]$, then $f(a) = a_1^\frown a_2$. We wish to show that f is a bijection.

In order to show that f is one-to-one, let $a = a_1 \times a_2, b = b_1 \times b_2 \in [T] \times [S]$. Assume that $f(a) = f(b)$. Then, by the definition of f, we have $a_1^\frown a_2 = b_1^\frown b_2$, so that $(a_1^\frown a_2) \restriction \omega = (b_1^\frown b_2) \restriction \omega$. Since $a_1 \in [T]$, a_1 has length ω, so that $(a_1^\frown a_2) \restriction \omega = a_1$. Similarly, $(b_1^\frown b_2) \restriction \omega = b_1$. Since $(a_1^\frown a_2) \restriction \omega = (b_1^\frown b_2) \restriction \omega$, $a_1 = b_1$. Also, because $a_1^\frown a_2 = b_1^\frown b_2$, $(a_1^\frown a_2) \restriction [\omega, \omega + \omega) = (b_1^\frown b_2) \restriction [\omega, \omega + \omega)$. Since $a_2 \in [S]$, a_2 has length ω, so that $(a_1^\frown a_2) \restriction [\omega, \omega + \omega) = s_R(a_2, \omega)$. Similarly, $(b_1^\frown b_2) \restriction [\omega, \omega + \omega) = s_R(b_2, \omega)$. So, $(a_1^\frown a_2) \restriction [\omega, \omega + \omega) = (b_1^\frown b_2) \restriction [\omega, \omega + \omega)$ which implies $s_R(a_2, \omega) = s_R(b_2, \omega)$. Further,

$s_R(a_2,\omega) = s_R(b_2,\omega)$ implies that $a_2 = b_2$. So, $a_1 = b_1$ and $a_2 = b_2$. Thus, $a_1 \times a_2 = b_1 \times b_2$. Consequently, $a = b$. Therefore, f is one-to-one.

Next, we show that f is onto. Let $y \in [T * S]$. So, $y = y_1^\frown y_2$ for some $y_1 \in [T]$ and $y_2 \in [S]$ by Lemma 1.38. Next, $x = y_1 \times y_2 \in [T] \times [S]$, so $f(x) = y_1^\frown y_2 = y$. Hence, for all $y \in [T * S]$ there exists $x \in [T] \times [S]$ such that $f(x) = y$. So, f is onto.

We conclude that f is a bijection.

Next, we show that f is continuous. Since $f^{-1}\Big(\bigcup\limits_{i\in I} A_i\Big) = \bigcup\limits_{i\in I} f^{-1}(A_i)$, it is enough to show the pre-image of a basic open set is open. Let $\mathcal{B}_d^{[T*S]}$ be a basic open set in $\mathcal{T}_{\circledast}$. So, $\mathcal{B}_d^{[T*S]} = \{t^\frown s \mid t^\frown s \supseteq d\}$ where $d = t^\star \cup s_R(s^\star,\omega)$. Consider $f^{-1}(\mathcal{B}_d^{[T*S]})$. Suppose $x \in f^{-1}(\mathcal{B}_d^{[T*S]})$. Then, $f(x) \in \mathcal{B}_d^{[T*S]}$. So, $f(x) \supseteq d$. Note that $x = \tilde{t} \times \tilde{s}$ for some $\tilde{t} \in [T]$ and $\tilde{s} \in [S]$. So, $lth(\tilde{t}) = \omega$. Since $f(x) = \tilde{t}^\frown\tilde{s} \supseteq d$, then $f(x) \restriction \omega = \tilde{t}$ and $f(x) \restriction [\omega,\omega+\omega) = s_R(\tilde{s},\omega)$. Further, $d \restriction \omega = t^\star$ and $d \restriction [\omega,\omega+\omega) = s_R(s^\star,\omega)$. Thus, $\tilde{t} \supseteq t^\star$ and $\tilde{s} \supseteq s^\star$.

Next, we find an open neighborhood in $\mathcal{T}_\otimes$ containing x. Let

$\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]} = \{t \in [T] \mid t \supseteq t^\star\} \times \{s \in [S] \mid s \supseteq s^\star\}$. Since $x = \tilde{t} \times \tilde{s}$ such that $\tilde{t} \supseteq t^\star$ and $\tilde{s} \supseteq s^\star$, $x \in \mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]}$. Now, choose an arbitrary $z \in \mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]}$. So, $z = \hat{t} \times \hat{s}$ for some $\hat{t} \supseteq t^\star$ and $\hat{s} \supseteq s^\star$. Thus, $f(z) = \hat{t}^\frown\hat{s}$. Since $\hat{t} \supseteq t^\star$, $\hat{s} \supseteq s^\star$, and $lth(\hat{t}) = \omega$, $\hat{t}^\frown\hat{s} \supseteq t^\star \cup s_R(s^\star,\omega)$. So, $f(z) \in \mathcal{B}_d^{[T*S]}$. Hence, $z \in f^{-1}(\mathcal{B}_d^{[T*S]})$ so that $\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]} \subseteq f^{-1}(\mathcal{B}_d^{[T*S]})$. Therefore, f is continuous.

Last, we show that f is open. Since $f\Big(\bigcup\limits_{i\in I} A_i\Big) = \bigcup\limits_{i\in I} f(A_i)$, it is enough to show the image of a basic open set is open. Let $\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]}$ in $\mathcal{T}_\otimes$ where

$\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]} = \{t \in [T] \mid t \supseteq t^\star\} \times \{s \in [S] \mid s \supseteq s^\star\}$. Now, we consider $f(\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]})$. Pick an arbitrary $y \in f(\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]})$. Then, $y = f(y_1 \times y_2) = y_1^\frown y_2$ for some $y_1 \in \mathfrak{B}_{t^\star}^{[T]}$ and $y_2 \in \mathfrak{B}_{s^\star}^{[S]}$.

So, $y_1 \supseteq t^\star$ and $y_2 \supseteq s^\star$. Since $lth(y_1) = \omega$, $y \supseteq t^\star \cup s_R(s^\star, \omega)$.

Next, we find an open neighborhood in $\mathcal{T}_\circledast$ containing y. Consider $\mathcal{B}_d^{[T*S]} = \{t^\frown s \in [T*S] |\ t^\frown s \supseteq d\}$, where $d = t^\star \cup s_R(s^\star, \omega)$. Hence, $y \in \mathcal{B}_d^{[T*S]}$. Next, pick an arbitrary $q \in \mathcal{B}_d^{[T*S]}$. So, $q \supseteq d$. By Lemma 1.38, there exist unique q_1 and q_2 such that $q = q_1^\frown q_2$ where $q_1 = q \restriction \omega \in [T]$ and $q_2 = s_L(q \restriction [\omega, \omega+\omega), \omega) \in [S]$. Next, $d \restriction \omega = t^\star$ and $s_L(d \restriction [\omega, \omega+\omega), \omega) = s^\star$. So, $q_1 \supseteq t^\star$ and $q_2 \supseteq s^\star$. Let $p \in [T] \times [S]$, where $p = q_1 \times q_2$. Thus, $p \in \mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]}$. Further, $f(p) = q$. So, $f(p) \in f(\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]})$. Therefore, $\mathcal{B}_d^{[T*S]} \subseteq f(\mathfrak{B}_{t^\star}^{[T]} \times \mathfrak{B}_{s^\star}^{[S]})$. Thus, f is an open map.

We have shown that $f : [T] \times [S] \to [T*S]$ is a bijection which is continuous and open. Therefore $[T] \times [S] \cong [T*S]$. □

The function that is used in Theorem 2.1 is a natural choice. Throughout the rest of this thesis, instead of finding any homeomorphism, we devote our studies to the use of this natural function. We will refer to this function as the "**Canonical Function**." Our next step is to show that regardless of the two trees we use, the canonical function will produce a bijection.

2.2 Results for the Canonical Function

Theorem 2.2. *Let T and S be any non-trivial trees. Then there exists a bijection $f : [T] \times [S] \to [T*S]$, defined by $f(a) = a_1^\frown a_2$ for $a = a_1 \times a_2 \in [T] \times [S]$.*

Proof. Let $a \in [T] \times [S]$, where $a = a_1 \times a_2$ such that $a_1 \in [T]$ and $a_2 \in [S]$. Define $f : [T] \times [S] \to [T*S]$, by $f(a) = a_1^\frown a_2$.

We wish to show that f is one-to-one. Let $a, b \in [T] \times [S]$. So, $a = a_1 \times a_2$ and

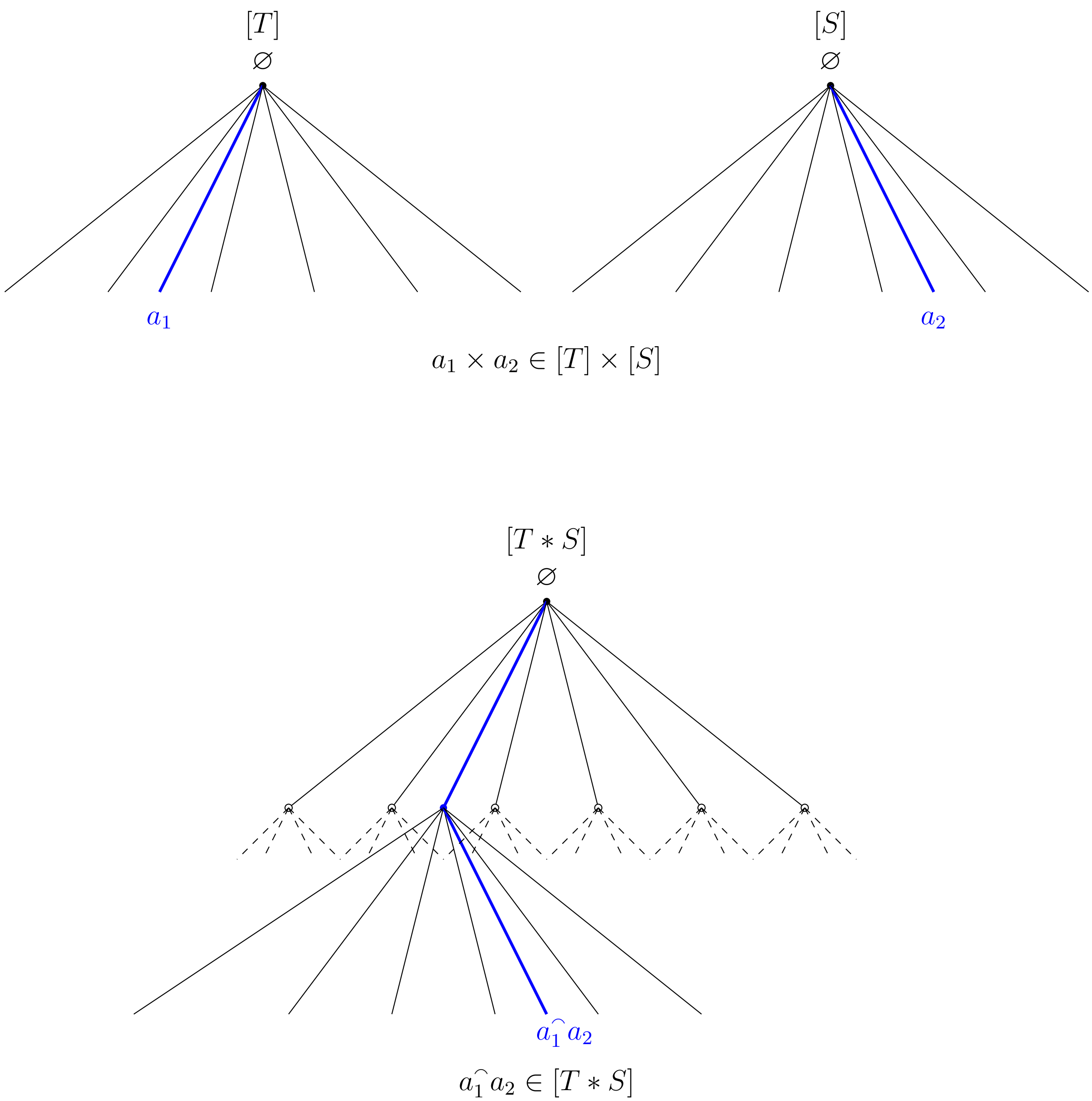

Figure 2.1: The path $a_1 \times a_2 \in [T] \times [S]$ and corresponding path $a_1^\frown a_2 \in [T * S]$.

$b = b_1 \times b_2$ for some $a_1, b_1 \in [T]$ and $a_2, b_2 \in [S]$. Assume $f(a) = f(b)$. By the definition of f, $f(a), f(b) \in [T * S]$. Since $f(a) = a_1^\frown a_2$ and $f(b) = b_1^\frown b_2$, $a_1^\frown a_2 = b_1^\frown b_2$. Suppose that $lth(a_1) = \alpha$, for some ordinal α, and that $lth(b_1) = \beta$, for some ordinal β. Since $a_1, b_1 \in [T]$ and $f(a) \restriction \alpha = a_1$, $f(a) \restriction \alpha \in [T]$. Similarly, $f(b) \restriction \beta = b_1$. So, $f(b) \restriction \beta \in [T]$. By assumption, $f(a) = f(b)$. Thus, $f(a) \restriction \beta \in [T]$. However, by Lemma 1.38, there must be a unique ordinal α such that $f(a) \restriction \alpha \in [T]$. We have $f(a) \restriction \alpha \in [T]$ and $f(a) \restriction \beta \in [T]$, where α is unique, so $\alpha = \beta$. Hence, $f(b) \restriction \alpha = b_1$. Thus, $f(a) = f(b)$. So, $f(a) \restriction \alpha = f(b) \restriction \alpha$, which implies $a_1 = b_1$.

Since $f(a) = f(b)$, then $dom(f(a)) = dom(f(b))$. Also, we have shown that $a_1 = b_1$, where $dom(a_1) = \alpha$ for some ordinal α. So, $f(a) = a_1^\frown a_2$ and $f(a) = a_1^\frown b_2$. However, by Lemma 1.37, $f(a) = a_1^\frown a_2$ is unique for $dom(a_1) = \alpha$. Hence, $a_2 = b_2$.

Since $a_1 = b_1$ and $a_2 = b_2$, $a_1 \times a_2 = b_1 \times b_2$. Thus, $a = b$. Therefore, f is a one-to-one function.

To show that f is onto, let $y \in [T * S]$. By Lemma 1.38, $y = y_1^\frown y_2$ such that $y_1 \in [T]$ and $y_2 \in [S]$. Now, suppose $x = y_1 \times y_2$. Then $x \in [T] \times [S]$, because $y_1 \in [T]$ and $y_2 \in [S]$. Thus, $f(x) = y_1^\frown y_2 = y$. So, f is onto.

We have shown that f is one-to-one and onto. Therefore, f is a bijection. □

In order to show that two topological spaces are homeomorphic, we must show that there exists a bijection between the two topological spaces that is both continuous and open. By Theorem 2.2, we have shown that we always have a bijection between the product topology and the tree topology. So, to prove a homeomorphism exists between $[T] \times [S]$ and $[T * S]$, it is enough to show that for the given trees T and S, the canonical function is continuous and open.

Our next goal is to find conditions that imply the canonical function is continuous and open. We have found sufficient conditions that prove the canonical function is continuous. This result is stated in Lemma 2.3. Additionally, we have found necessary and sufficient conditions that prove the canonical function is open. This result is stated in Lemma 2.9.

Lemma 2.3. *Let T and S be any non-trivial trees. If for all $p \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(p)$ such that for all $q \in \mathfrak{B}^{[T]}_{d^\star}$, $lth(q) = lth(p)$, then the canonical function is continuous.*

Proof. Let T and S be any non-trivial trees. Assume for all $p \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(p)$ such that for all $q \in \mathfrak{B}^{[T]}_{d^\star}$, $lth(q) = lth(p)$. Recall $\mathfrak{B}^{[T]}_{d^\star} = \{t \in [T] |\ t \supseteq d^\star\}$. Consider a basic open set in $\mathcal{T}_{\circledast}$, say $\mathcal{B}^{[T*S]}_d$, where $\mathcal{B}^{[T*S]}_d = \{y \in [T * S] |\ y \supseteq d\}$. Consider $f^{-1}(\mathcal{B}^{[T*S]}_d)$. Suppose $x \in f^{-1}(\mathcal{B}^{[T*S]}_d)$. So, $f(x) \in \mathcal{B}^{[T*S]}_d$.

Next, we find an open neighborhood in $\mathcal{T}_{\otimes}$ containing x. Note that $x = p_1 \times s$ for some $p_1 \in [T]$ and $s \in [S]$. Suppose $lth(p_1) = \alpha$ for some ordinal α. Because $f(x) \in \mathcal{B}^{[T*S]}_d$, $f(x) = p_1^\frown s \supseteq d$. By Lemma 1.38, α is unique, since $f(x) \restriction \alpha = p_1 \in [T]$. Also,
$s = s_L\Big(f(x) \restriction \big[\alpha, dom(f(x))\big), \alpha\Big) \in [S]$. By assumption, there exists $d^\star \in \mathcal{P}_{fin}(p_1)$ such that for all $q_1 \in \mathfrak{B}^{[T]}_{d^\star}$, $lth(q_1) = lth(p_1)$. Define $d_1 = (d \restriction \alpha) \cup d^\star$ and
$d_2 = s_L\Big(d \restriction \big[\alpha, dom(f(x))\big), \alpha\Big)$. So, $d \subseteq d_1 \cup s_R(d_2, \alpha)$. Then, $p_1 \supseteq d_1$ and $s \supseteq d_2$. Thus, $\mathfrak{B}^{[T]}_{d_1} \times \mathfrak{B}^{[S]}_{d_2} = \{u \in [T] |\ u \supseteq d_1\} \times \{v \in [S] |\ v \supseteq d_2\}$ is a neighborhood of x. If $d_1 = \emptyset$, then $\mathfrak{B}^{[T]}_{d_1} \times \mathfrak{B}^{[S]}_{d_2} = [T] \times \mathfrak{B}^{[S]}_{d_2}$. If $d_2 = \emptyset$, then $\mathfrak{B}^{[T]}_{d_1} \times \mathfrak{B}^{[S]}_{d_2} = \mathfrak{B}^{[T]}_{d_1} \times [S]$.

Now, choose an arbitrary $z \in \mathfrak{B}^{[T]}_{d_1} \times \mathfrak{B}^{[S]}_{d_2}$. So, $z = z_1 \times z_2$ such that $z_1 \supseteq d_1$ and $z_2 \supseteq d_2$. By assumption, since $z_1 \supseteq d_1$, then $z_1 \supseteq d^\star$. So, $z_1 \in \mathfrak{B}^{[T]}_{d^\star}$. Hence, $lth(z_1) = lth(p_1) = \alpha$. Further, $f(z) = z_1^\frown z_2$. So, $z_1^\frown z_2 \supseteq d_1 \cup s_R(d_2, \alpha)$. Because $d \subseteq d_1 \cup s_R(d_2, \alpha)$, $f(z) \supseteq d$. So,

$f(z) \in \mathcal{B}_d^{[T*S]}$. Thus, $z \in f^{-1}(\mathcal{B}_d^{[T*S]})$. So, $\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]} \subseteq f^{-1}(\mathcal{B}_d^{[T*S]})$. Since x was chosen arbitrarily, we have shown for all $x \in f^{-1}(\mathcal{B}_d^{[T*S]})$ there exists $\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]} \in \mathcal{T}_\otimes$ such that $x \in \mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]} \subseteq f^{-1}(\mathcal{B}_d^{[T*S]})$. Therefore, f is continuous. □

Remark 2.4. The converse of Lemma 2.3 does not hold. In the following example, we provide a tree, T, that does not satisfy the conditions in Lemma 2.3; however, for a specific tree, S, we see that the canonical function is continuous.

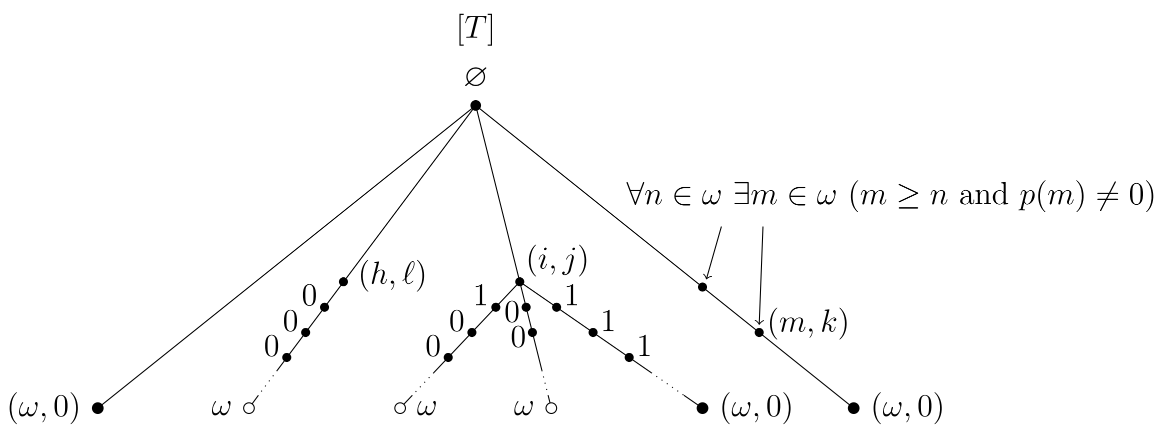

Figure 2.2: $[T]$ defined in Example 2.5.

Example 2.5.

Let $p \in [T]$ *iff* $\begin{cases} p \in \omega^{\omega+1}, & \text{if } \forall n \in \omega\ \exists m \in \omega\ (m \geq n \text{ and } p(m) \neq 0) \text{ and } p(\omega) = 0 \\ p \in \omega^{\omega}, & \text{if } \exists n \in \omega\ \forall m \in \omega\ (m \geq n \implies p(m) = 0) \end{cases}$

Let $[S] = \{0\}^\omega$. *Then, the canonical function is continuous.*

Remark 2.6. Let f be a sequence with $ran(f) \subseteq \omega$, then

$$f \in T \textit{ iff } \left\{ \begin{array}{l} lth(f) < \omega, \text{ or} \\ \\ lth(f) = \omega \text{ and } \forall n \in \omega\ \exists m \in \omega\ (m \geq n \text{ and } f(m) \neq 0), \text{ or} \\ \\ lth(f) = \omega + 1 \text{ and } f(\omega) = 0 \text{ and } \forall n \in \omega\ \exists m \in \omega\ (m \geq n \text{ and } (f \restriction \omega)(m) \neq 0) \end{array} \right\}$$

Proof. Let T and S be defined as above. First, note that for any $t \in [T]$ with $lth(t) = \omega$, for all $d^\star \in \mathcal{P}_{fin}(t)$, there exists $t_2 \in \mathfrak{B}^{[T]}_{d^\star}$ with $lth(t_2) = \omega + 1$. See Figure 2.2. We wish to show that for T and S, $f : [T] \times [S] \to [T * S]$ is continuous.

Let $\mathcal{B}^{[T*S]}_d = \{t^\frown s \in [T * S] |\ t^\frown s \supseteq d\}$ be any basic open neighborhood of $\mathcal{T}_{\circledast}$. Suppose $x_1 \in f^{-1}(\mathcal{B}^{[T*S]}_d)$. So, $f(x_1) \in \mathcal{B}^{[T*S]}_d$. Then, $f(x_1) \supseteq d$. Let $x_1 = p \times s \in [T] \times [S]$.

Case 1: Assume $lth(p) = \omega + 1$. By the definition of $[T]$, for all $n \in \omega$ there exists $m \in \omega$ such that $m \geq n$ and $p(m) \neq 0$ and $p(\omega) = 0$. Now, since $p^\frown s \supseteq d$, define $d_1 = d \restriction \omega + 1$ and $d_2 = s_L\Big(d \restriction [\omega+1, \omega+\omega), \omega+1\Big)$. So, $p \supseteq d_1$ and $s \supseteq d_2$. Note that $s = \vec{0}$, so $ran(d_2) \subseteq \{0\}$. Since $p(\omega) = 0$, $p \supseteq \{(\omega, 0)\}$. Let $d^\star = d_1 \cup \{(\omega, 0)\}$. Note that either $\omega \in dom(d_1)$ and $d_1(\omega) = 0$, in which case $\{(\omega, 0)\} \subseteq d_1$, or $\omega \notin dom(d_1)$ and $d_1 \subseteq d^\star$. So, $\mathfrak{B}^{[T]}_{d^\star} \times \mathfrak{B}^{[S]}_{d_2}$ is an open neighborhood of x_1. Suppose $x_2 \in \mathfrak{B}^{[T]}_{d^\star} \times \mathfrak{B}^{[S]}_{d_2}$ where $x_2 = q \times s$, for some $q \in \mathfrak{B}^{[T]}_{d^\star}$ and $s \in \mathfrak{B}^{[T]}_{d_2}$. Then, $q \supseteq d^\star$. Since $q(\omega) = 0$, $lth(q) = \omega + 1$. So, $lth(q) = lth(p)$. Thus, $f(x_2) = q^\frown s \supseteq d^\star \cup s_R(d_2, \omega+1)$. Further, $d_1 \subseteq d^\star$, so $d = d_1 \cup s_R(d_2, \omega+1) \subseteq d^\star \cup s_R(d_2, \omega+1)$. Hence, $f(x_2) \supseteq d$. Therefore, $f(x_2) \in \mathcal{B}^{[T*S]}_d$. Thus, $\mathfrak{B}^{[T]}_{d^\star} \times \mathfrak{B}^{[S]}_{d_2} \subseteq f^{-1}(\mathcal{B}^{[T*S]}_d)$.

Case 2: Assume $lth(p) = \omega$. So, by the definition of $[T]$, there exists $n \in \omega$ for all $m \in \omega$ such that $m \geq n \implies p(m) = 0$. Now, since $p^\frown s \supseteq d$, define $d_1 = d \restriction \omega$ and $d_2 = s_L\Big(d \restriction [\omega, \omega + \omega), \omega\Big)$. So, $p \supseteq d_1$ and $s \supseteq d_2$. Note that $s = \vec{0}$, so $ran(d_2) \subseteq \{0\}$. So, $\mathfrak{B}^{[T]}_{d_1} \times \mathfrak{B}^{[S]}_{d_2}$ is an open neighborhood of x_1. Suppose $x_2 \in \mathfrak{B}^{[T]}_{d_1} \times \mathfrak{B}^{[S]}_{d_2}$ where $x_2 = q \times s$. Since $q \in \mathfrak{B}^{[T]}_{d_1}$, $q \supseteq d_1$.

If $lth(q) = \omega$, then $f(x_2) = q^\frown s \supseteq d_1 \cup s_R(d_2, \omega)$. Thus, $f(x_2) \supseteq d$. So, $f(x_2) \in \mathcal{B}_d^{[T*S]}$.

Next, if $lth(q) = \omega + 1$, then $q(\omega) = 0$. Since $q(\omega) = 0$,

$f(x_2) \supseteq d_1 \cup \{(\omega, 0)\} \cup s_R(d_2, \omega) = \hat{d}$. If $\omega \in dom(s_R(d_2, \omega))$, then $\{(\omega, 0)\} \subseteq s_R(d_2, \omega)$. So, $d = \hat{d}$. Thus, $d \subseteq \hat{d}$. Otherwise, $\omega \notin dom(s_R(d_2, \omega))$. So, $d \subseteq \hat{d}$. Either way, since $d \subseteq \hat{d}$, $f(x_2) \supseteq d$. So, $f(x_2) \in \mathcal{B}_d^{[T*S]}$. Therefore, $\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]} \subseteq f^{-1}(\mathcal{B}_d^{[T*S]})$.

Using both cases, we have shown that for all $x \in f^{-1}(\mathcal{B}_d^{[T*S]})$ there exists $\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]} \in \mathcal{T}_\otimes$ such that $x \in \mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]} \subseteq f^{-1}(\mathcal{B}_d^{[T*S]})$. Hence, $f^{-1}(\mathcal{B}_d^{[T*S]})$ is open. Therefore, f is continuous. □

Remark 2.7. If we define T, as in Example 2.5, and let $S = \omega^{<\omega}$, then the canonical function is not continuous.

Conjecture 2.8. *Let T be any non-trivial tree. Assume for all non-trivial trees, S, the canonical function is continuous. Then for all $p \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(p)$ such that for all $q \in \mathfrak{B}_{d^\star}^{[T]}$, $lth(q) = lth(p)$.*

Lemma 2.9. *Let T and S be any non-trivial trees. The canonical function is an open map if and only if given any $h_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $h_2 \in \mathfrak{B}_{d_2}^{[S]}$ there exists d such that $h_1^\frown h_2 \in \mathcal{B}_d^{[T*S]}$ and for all $g = g_1^\frown g_2 \in \mathcal{B}_d^{[T*S]}$ (where $g_1 \in [T]$ and $g_2 \in [S]$), $g_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $g_2 \in \mathfrak{B}_{d_2}^{[S]}$.*

Proof. Let T and S be any non-trivial trees.

($\Rightarrow$) Assume that f is an open map. Recall $\mathfrak{B}_{d_1}^{[T]} = \{t \in [T] | \ t \supseteq d_1\}$ and $\mathfrak{B}_{d_2}^{[S]} = \{s \in [S] | \ s \supseteq d_2\}$. Note that $\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]}$ is a basic open set in $\mathcal{T}_\otimes$. Let $h_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $h_2 \in \mathfrak{B}_{d_2}^{[S]}$. So, $x = h_1 \times h_2 \in [T] \times [S]$. Then, $f(x) = h_1^\frown h_2 \in [T * S]$. Note that $f(x) \in f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$. Since f is open, there exists $\mathcal{B}_d^{[T*S]} \in \mathcal{T}_\circledast$ such that $f(x) \in \mathcal{B}_d^{[T*S]} \subseteq f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$, where $\mathcal{B}_d^{[T*S]} = \{t^\frown s \in [T * S] | \ t^\frown s \supseteq d\}$. Thus, there exists

d such that $f(x) = h_1^\frown h_2 \in \mathcal{B}_d^{[T*S]} \subseteq f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$.

Suppose $g \in \mathcal{B}_d^{[T*S]}$. So, $g \in f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$. Then, there exists $z \in \mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]}$ such that $f(z) = g$. Because $z \in \mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]}$, $z = g_1 \times g_2$, for some $g_1 \in \mathfrak{B}_{d_1}^{[T]}$ and for some $g_2 \in \mathfrak{B}_{d_2}^{[S]}$. Note that $f(z) = g_1^\frown g_2 = g$. Last, g was arbitrary, so for all $g = g_1^\frown g_2 \in \mathcal{B}_d^{[T*S]}$ (where $g_1 \in [T]$ and $g_2 \in [S]$), $g_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $g_2 \in \mathfrak{B}_{d_2}^{[S]}$.

($\Leftarrow$) Let $\mathfrak{B}_{d_1}^{[T]} = \{t \in [T] |\ t \supseteq d_1\}$ and $\mathfrak{B}_{d_2}^{[S]} = \{s \in [S] |\ s \supseteq d_2\}$. Assume for any $h_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $h_2 \in \mathfrak{B}_{d_2}^{[S]}$, there exists d such that $h_1^\frown h_2 \in \mathcal{B}_d^{[T*S]}$, where $\mathcal{B}_d^{[T*S]} = \{t^\frown s \in [T * S] |\ t^\frown s \supseteq d\}$, and for all $g = g_1^\frown g_2 \in \mathcal{B}_d^{[T*S]}$ (where $g_1 \in [T]$ and $g_2 \in [S]$), $g_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $g_2 \in \mathfrak{B}_{d_2}^{[S]}$. Now, $\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]}$ is a basic open set in $\mathcal{T}_\otimes$. Consider $f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$. Let $y \in f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$. Then, there exists $q = \tilde{t} \times \tilde{s}$, for some $\tilde{t} \in \mathfrak{B}_{d_1}^{[T]}$ and for some $\tilde{s} \in \mathfrak{B}_{d_2}^{[S]}$, such that $f(q) = y$. Then, $f(q) = \tilde{t}^\frown \tilde{s} = y$. Since $\tilde{t} \in \mathfrak{B}_{d_1}^{[T]}$ and $\tilde{s} \in \mathfrak{B}_{d_2}^{[S]}$, by assumption, there exists d such that $y = \tilde{t}^\frown \tilde{s} \in \mathcal{B}_d^{[T*S]}$. Further, $\mathcal{B}_d^{[T*S]}$ is an open neighborhood of y in $\mathcal{T}_\circledast$. Choose an arbitrary $z \in \mathcal{B}_d^{[T*S]}$. By Lemma 1.38, there exist unique z_1 and z_2 such that $z = z_1^\frown z_2$ where $z_1 \in [T]$ and $z_2 \in [S]$. Next, by assumption since $z \in \mathcal{B}_d^{[T*S]}$, $z_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $z_2 \in \mathfrak{B}_{d_2}^{[S]}$. Let $x = z_1 \times z_2$. Note that $x \in [T] \times [S]$. So, $f(x) = z_1^\frown z_2 = z$. Then, $f(x) = z \in f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$. Therefore, $\mathcal{B}_d^{[T*S]} \subseteq f(\mathfrak{B}_{d_1}^{[t]} \times \mathfrak{B}_{d_2}^{[S]})$. Since y was chosen arbitrarily we have shown for all $y \in f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$ there exists $\mathcal{B}_d^{[T*S]} \in \mathcal{T}_\circledast$ such that $y \in \mathcal{B}_d^{[T*S]} \subseteq f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$. Hence, $f(\mathfrak{B}_{d_1}^{[T]} \times \mathfrak{B}_{d_2}^{[S]})$ is open. So, f is an open map. $\square$

Below is a simple example to show the usefulness of Theorem 2.2, Lemma 2.3, and Lemma 2.9. In this example, we use a well-founded tree, that we call "R" throughout the rest of this thesis. See Figure 2.3.

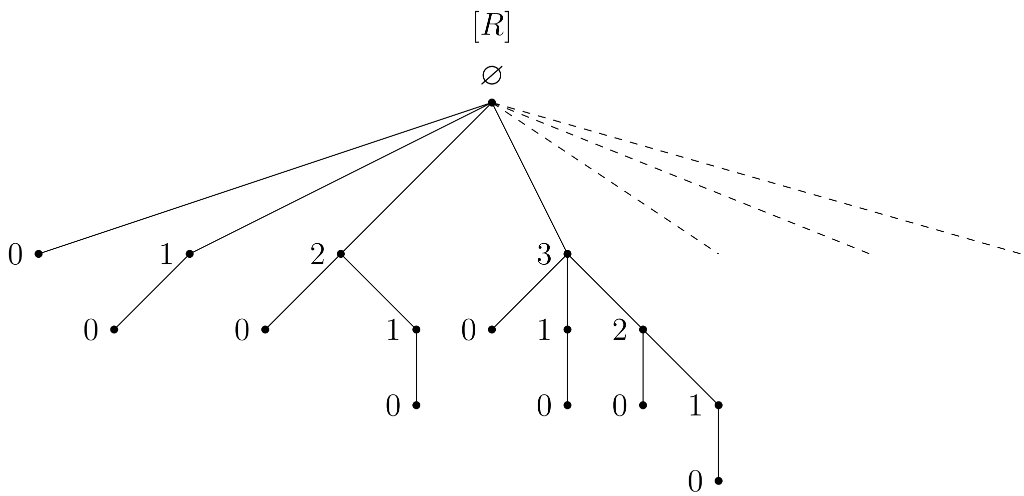

Figure 2.3: Well-founded Tree R.

Example 2.10. *Let* $T = \omega^{<\omega}$ *and* $R = \{a \in \omega^{<\omega} |\ \forall i < j < lth(a),\ a_i > a_j\}$. *Then* $[T] \times [R] \cong [T * R]$.

Proof. Let $T = \omega^{<\omega}$ and $R = \{a \in \omega^{<\omega} |\ \forall i < j < lth(a),\ a_i > a_j\}$. Since T and R are non-trivial trees, by Theorem 2.2, the canonical function is a bijection.

Next, we use Lemma 2.3 to show that f is continuous. Let $p \in [T]$. Then, $lth(p) = \omega$. Suppose $p^\star \in \mathcal{P}_{fin}(p)$. Let $\mathfrak{B}_{p^\star}^{[T]} = \{t \in [T] |\ t \supseteq p^\star\}$. Now, suppose $q \in \mathfrak{B}_{p^\star}^{[T]}$. Then, $q \in [T]$. So, $lth(q) = \omega$. Hence, $lth(q) = lth(p)$. Thus, for all $t \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(t)$ such that for all $q \in \mathfrak{B}_{d^\star}^{[T]}$, $lth(q) = lth(t)$. By Lemma 2.3, f is continuous.

Next, we use Lemma 2.9 to show that f is an open map. Let $h_1 \in \mathcal{B}_{d_1}^{[T]}$ and $h_2 \in \mathcal{B}_{d_2}^{[R]}$. So, $h_1 \supseteq d_1$ and $h_2 \supseteq d_2$. Since $h_1 \in [T]$, $dom(h_1) = \omega$. Let $d = d_1 \cup s_R(d_2, \omega)$. Let $x = h_1 \times h_2 \in [T] \times [R]$. So, $f(x) = h_1^\frown h_2$. Since $h_1 \supseteq d_1$ and $h_2 \supseteq d_2$, $h_1^\frown h_2 \supseteq d_1 \cup s_R(d_2, \omega)$. Therefore, $h_1^\frown h_2 \supseteq d$. Let $\mathcal{B}_d^{[T*R]} = \{y \in [T * R] |\ y \supseteq d\}$. So, $h_1^\frown h_2 \in \mathcal{B}_d^{[T*R]}$.

Now, suppose that $g \in \mathcal{B}_d^{[T*R]}$. So, $g \in [T * R]$ and $g \supseteq d$. Since f is onto there exists an $s \in [T] \times [R]$ such that $f(s) = g$. Say $s = g_1 \times g_2$, where $g_1 \in [T]$ and $g_2 \in [R]$. Thus,

$f(s) = g_1^\frown g_2 = g$. Since $g_1 \in [T]$, $dom(g_1) = \omega$. Thus, $g_1 = g \upharpoonright \omega$. Recall $d = d_1 \cup s_R(d_2, \omega)$. So, $d_1 = d \upharpoonright \omega$. Thus, $g_1 \supseteq d_1$. So, $g_1 \in \mathcal{B}_{d_1}^{[T]}$. Also, $g_2 = s_L\Big(g \upharpoonright \big[\omega, dom(g)\big), \omega\Big)$ and $d_2 = s_L\Big(d \upharpoonright \big[\omega, dom(g)\big), \omega\Big)$. Thus, $g_2 \supseteq d_2$. So, $g_2 \in \mathcal{B}_{d_2}^{[R]}$. By Lemma 2.9, f is open.

We have shown that f is a bijection which is continuous and open. Therefore, $[T] \times [R] \cong [T * R]$. $\square$

CHAPTER 3

EXAMPLES OF HOMEOMORPHISMS

In this chapter, we use the results from Chapter 2 to show that a homeomorphism exists between $\mathcal{T}_{\otimes}$ and $\mathcal{T}_{\circledast}$ for trees that are defined in a more general manner. In fact, as shown in Theorem 3.1, we can show that the existence of a homeomorphism sometimes only depends on the first tree. The second tree can be any non-trivial tree.

3.1 T has uniform length

Theorem 3.1. *Let T be any non-trivial tree such that there exists $\gamma \in ord$, for all $p \in [T],\ dom(p) = \gamma$, and let S be any non-trivial tree. Then $[T] \times [S] \cong [T * S]$.*

Proof. Let T be any non-trivial tree such that $\exists\ \gamma \in ord,\ \forall\ p \in [T],\ dom(p) = \gamma$ and let S be any nontrivial tree. By Theorem 2.2, the canonical function is a bijection.

We wish to show that f is continuous, using Lemma 2.3. Consider $p_1 \in [T]$. Then, $lth(p_1) = \gamma$. Suppose $p_1 \supseteq p^\star$ for some $p^\star \in \mathcal{P}_{fin}(p_1)$. Let $\mathfrak{B}^{[T]}_{p^\star} = \{t \in [T] |\ t \supseteq p^\star\}$. Now, suppose $q_1 \in \mathfrak{B}^{[T]}_{p^\star}$. Then, $q_1 \in [T]$. So, $lth(q_1) = \gamma$. Hence, $lth(q_1) = lth(p_1)$. Thus, for all $p \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(p)$ such that for all $q \in \mathfrak{B}^{[T]}_{d^\star},\ lth(q) = lth(p)$. So, by Lemma 2.3, f is continuous.

Next, we wish to show that f is an open map, using Lemma 2.9. Recall $\mathfrak{B}^{[T]}_{d_1} = \{t \in [T] |\ t \supseteq d_1\}$ and $\mathfrak{B}^{[S]}_{d_2} = \{s \in [S] |\ s \supseteq d_2\}$. Let $h_1 \in \mathfrak{B}^{[T]}_{d_1}$ and $h_2 \in \mathfrak{B}^{[S]}_{d_2}$. Since

$h_1 \in \mathfrak{B}^{[T]}_{d_1}$, $h_1 \in [T]$. So, $dom(h_1) = \gamma = lth(h_1)$. Consider $x = h_1 \times h_2 \in [T] \times [S]$. Then, $f(x) = h_1^\frown h_2 \in [T * S]$. Since $h_1 \supseteq d_1$, $h_2 \supseteq d_2$, and $lth(h_1) = \gamma$, then $h_1^\frown h_2 \supseteq d_1 \cup s_R(d_2, \gamma)$. Define $d = d_1 \cup s_R(d_2, \gamma)$. Let $\mathcal{B}^{[T*S]}_d = \{t^\frown s \in [T * S] | \ t^\frown s \supseteq d\}$, so $f(x) = h_1^\frown h_2 \in \mathcal{B}^{[T*S]}_d$. Since h_1 and h_2 are arbitrary, for any $h_1 \in \mathfrak{B}^{[T]}_{d_1}$ and $h_2 \in \mathfrak{B}^{[S]}_{d_2}$ there exists d such that $f(x) = h_1^\frown h_2 \in \mathcal{B}^{[T*S]}_d$.

Suppose $g \in \mathcal{B}^{[T*S]}_d$. Then, $g \in [T * S]$ and $g \supseteq d$. By Lemma 1.38, there exist unique g_1 and g_2 such that $g = g_1^\frown g_2$, where $g_1 = g \upharpoonright \gamma \in [T]$ and $g_2 = s_L\Big(g \upharpoonright \big[\gamma, dom(g)\big), \gamma\Big) \in [S]$. Next, $d \upharpoonright \gamma = d_1$ and $d_2 = s_L\Big(d \upharpoonright \big[\gamma, dom(g)\big), \gamma\Big)$. So, $g_1 \supseteq d_1$ and $g_2 \supseteq d_2$. Thus, $g_1 \in \mathfrak{B}^{[T]}_{d_1}$ and $g_2 \in \mathfrak{B}^{[S]}_{d_2}$. Last, g was arbitrary, so for all $g = g_1^\frown g_2 \in \mathcal{B}^{[T*S]}_d$ (where $g_1 \in [T]$ and $g_2 \in [S]$), $g_1 \in \mathfrak{B}^{[T]}_{d_1}$ and $g_2 \in \mathfrak{B}^{[S]}_{d_2}$. So, by Lemma 2.9, f is an open map.

We have shown that $f : [T] \times [S] \to [T * S]$ is a bijection which is continuous and open. Therefore, $[T] \times [S] \cong [T * S]$. □

In Theorem 3.1 above, we were able to prove a homeomorphism exists between the two topological spaces if the paths through the first tree T have the same length. As stated before, the second tree can be any non-trivial tree S. After proving this result, we wish to find examples of trees which have varying path lengths to use as our first tree. For our next example, we use the same well-founded tree R that was used in Example 2.10.

3.2 Interesting Examples

Example 3.2. *Let* $R = \{a \in \omega^{<\omega} | \ \forall i < j < lth(a), \ a_i > a_j\}$ *and let* S *be any non-trivial tree. Then* $[R] \times [S] \cong [R * S]$.

Proof. Since R and S are non-trivial trees, by Theorem 2.2, the canonical function is a bijection.

Next, we use Lemma 2.3 to show that f is continuous. Let $p \in [R]$. Then, there is a $k \in \omega$ such that $p(k) = 0$. By definition of R, $lth(p) = k+1$. Consider $p^\star = \{(k,0)\}$, where $p^\star \in \mathcal{P}_{fin}(p)$. So, $\mathcal{B}^{[R]}_{p^\star} = \{r \in [R] |\ r \supseteq p^\star\}$. Suppose $q \in \mathcal{B}^{[R]}_{p^\star}$. So, $q \supseteq p^\star$. Then, $q(k) = 0$. Again, by definition of R, $lth(q) = k+1$. Therefore, $lth(p) = lth(q)$. Thus, for all $r \in [R]$ there exists $d^\star \in \mathcal{P}_{fin}(r)$ such that for all $\tilde{r} \in \mathcal{B}^{[R]}_{d^\star}$, $lth(\tilde{r}) = lth(r)$. So, by Lemma 2.3, f is continuous.

Last, we use Lemma 2.9 to show that f is open. Let $h_1 \in \mathcal{B}^{[R]}_{d_1}$ and $h_2 \in \mathcal{B}^{[S]}_{d_2}$. So, $h_1 \supseteq d_1$ and $h_2 \supseteq d_2$. Since $h_1 \in [R]$, $h_1(k) = 0$ for some $k \in \omega$. So, $dom(h_1) = k+1$. Because $dom(h_1)$ is finite, $h_1 \in \mathcal{P}_{fin}(h_1)$. Define $d = h_1 \cup s_R(d_2, k+1)$. Since $h_1 \supseteq d_1$, $d \supseteq d_1$. Let $x = h_1 \times h_2 \in [R] \times [S]$, so $f(x) = h_1^\frown h_2$. Since $h_1 \supseteq h_1$ and $h_2 \supseteq d_2$, $h_1^\frown h_2 \supseteq h_1 \cup s_R(d_2, k+1)$. So, $h_1^\frown h_2 \supseteq d$. Let $\mathcal{B}^{[R*S]}_d = \{y \in [R*S] |\ y \supseteq d\}$. So, $h_1^\frown h_2 \in \mathcal{B}^{[R*S]}_d$.

Now, suppose that $g \in \mathcal{B}^{[R*S]}_d$. So, $g \in [R*S]$ and $g \supseteq d$. Further, $g \supseteq h_1$. Thus, $g(k) = 0$. Recall $h_1 \in [R]$ with $dom(h_1) = k+1$. So, $h_1(n) \neq 0$ for all $n < k$. Hence, $g(n) \neq 0$ for all $n < k$. By Lemma 1.38, there exist unique g_1 and g_2 such that $g = g_1^\frown g_2$, where $g \restriction (k+1) = g_1 \in [R]$ and $s_L\Big(g \restriction \big[k+1, dom(g)\big), k+1\Big) = g_2 \in [S]$. Recall $d = h_1 \cup s_R(d_2, k+1)$. So, $d \restriction (k+1) = h_1$. Thus, $g_1 \supseteq h_1$. So, $g_1 \in \mathcal{B}^{[R]}_{d_1}$. Also, $s_L\Big(d \restriction \big[k+1, dom(g)\big), k+1\Big) = d_2$. Thus, $g_2 \supseteq d_2$. So, $g_2 \in \mathcal{B}^{[S]}_{d_2}$. Therefore, by Lemma 2.9, f is an open map.

We have shown that f is a bijection which is continuous and open. Therefore, $[R] \times [S] \cong [R*S]$. □

In the next example, we define a tree which has varying path lengths. In this case, we want a tree that has paths that are of length ω or greater. To construct this tree, the domain of each path is based on the move in the first position, and defined by $\omega \cdot (t(0)+1)$. This yields paths that have lengths which are multiples of ω. See Figure 3.1.

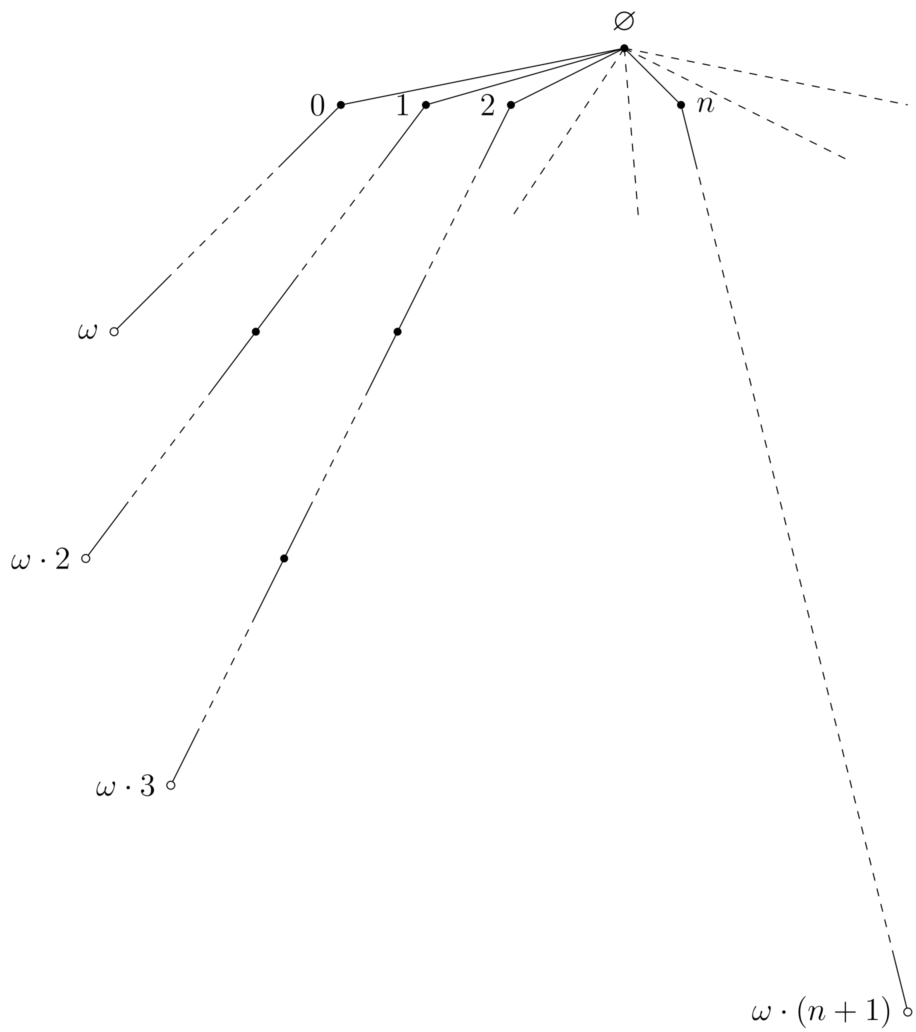

Figure 3.1: $[T] = \{t \mid dom(t) = \omega \cdot k,\ k = t(0)+1,\ ran(t) \subseteq \omega\}$.

Example 3.3. *Let* $[T] = \{t \mid dom(t) = \omega \cdot k,\ k = t(0)+1,\ ran(t) \subseteq \omega\}$. *Let* S *be any non-trivial trivial tree. Then* $[T] \times [S] \cong [T * S]$.

Proof. Since T and S are non-trivial trees, by Theorem 2.2, the canonical function is a bijection.

Next, we use Lemma 2.3 to show that f is continuous. Let $p \in [T]$. Then $p(0) = n$, for some $n \in \omega$. Consider $p^\star = \{(0,n)\}$, where $p^\star \in \mathcal{P}_{fin}(p)$. So, $\mathcal{B}^{[T]}_{p^\star} = \{t \in [T] | \ t \supseteq p^\star\}$. Suppose $q \in \mathcal{B}^{[T]}_{p^\star}$. So, $q \supseteq p^\star$. Then, $q(0) = n$. So, $q(0) + 1 = n + 1 = k$. Hence, $dom(q) = \omega \cdot k = lth(p) = lth(q)$. Therefore, for all $t \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(t)$ such that for all $\tilde{t} \in \mathcal{B}^{[T]}_{d^\star}$, $lth(\tilde{t}) = lth(t)$. So, by Lemma 2.3 f is continuous.

Last, we use Lemma 2.9 to show that f is open. Let $h_1 \in \mathcal{B}^{[T]}_{d_1}$ and $h_2 \in \mathcal{B}^{[S]}_{d_2}$. So, $h_1 \supseteq d_1$ and $h_2 \supseteq d_2$. Also, $h_1(0) = n$ for some $n \in \omega$. Next, $h_1(0) + 1 = n + 1 = k$. Since $h_1 \in [T]$, $dom(h_1) = \omega \cdot k = lth(h_1)$. Let $d = \{(0,n)\} \cup d_1 \cup s_R(d_2, \omega \cdot k)$. Because $h_1 \supseteq \{(0,n)\}$ and $h_1 \supseteq d_1$, either $d_1(0) = n$ or $0 \notin dom(d_1)$. Let $x = h_1 \times h_2 \in [T] \times [S]$. So, $f(x) = h_1^\frown h_2$. Since $h_1 \supseteq \{(0,n)\} \cup d_1$ and $h_2 \supseteq d_2$, $h_1^\frown h_2 \supseteq \{(0,n)\} \cup d_1 \cup s_R(d_2, \omega \cdot k)$. So, $h_1^\frown h_2 \supseteq d$. Let $\mathcal{B}^{[T*S]}_d = \{y \in [T*S] | \ y \supseteq d\}$. So, $h_1^\frown h_2 \in \mathcal{B}^{[T*S]}_d$.

Now, suppose that $g \in \mathcal{B}^{[T*S]}_d$. So, $g \in [T*S]$ and $g \supseteq d$. Since f is onto there exists an $r \in [T] \times [S]$ such that $f(r) = g$. Say $r = g_1 \times g_2$, where $g_1 \in [T]$ and $g_2 \in [S]$. Thus, $f(r) = g_1^\frown g_2 = g$. Next, since $g \supseteq d$, $g(0) = n$. So, $g_1(0) = n$. Thus, $dom(g_1) = \omega \cdot k$. Therefore, $g \restriction (\omega \cdot k) = g_1$. Recall $d = \{(0,n)\} \cup d_1 \cup s_R(d_2, \omega \cdot k)$, so $d \restriction (\omega \cdot k) = \{(0,n)\} \cup d_1$. Thus, $g_1 \supseteq \{(0,n)\} \cup d_1$. So, $g_1 \supseteq d_1$. Therefore, $g_1 \in \mathcal{B}^{[T]}_{d_1}$. Also, $s_L\Big(g \restriction \big[\omega \cdot k, dom(g)\big), \omega \cdot k\Big) = g_2$ and $s_L\Big(d \restriction \big[\omega \cdot k, dom(g)\big), \omega \cdot k\Big) = d_2$. Thus, $g_2 \supseteq d_2$, because $g \supseteq d$. So, $g_2 \in \mathcal{B}^{[S]}_{d_2}$. Therefore, by Lemma 2.9, f is open.

Therefore, $[T] \times [S] \cong [T*S]$. □

Prior to the example below, all of the trees we have used either have paths which are limit length or finite length. In the following example, we define a tree T where paths have two possible lengths: limit length ω or successor length $\omega+1$. In this tree, notice that each path's length is determined by the path restricted to ω being in a clopen set, or by being in the clopen complement.

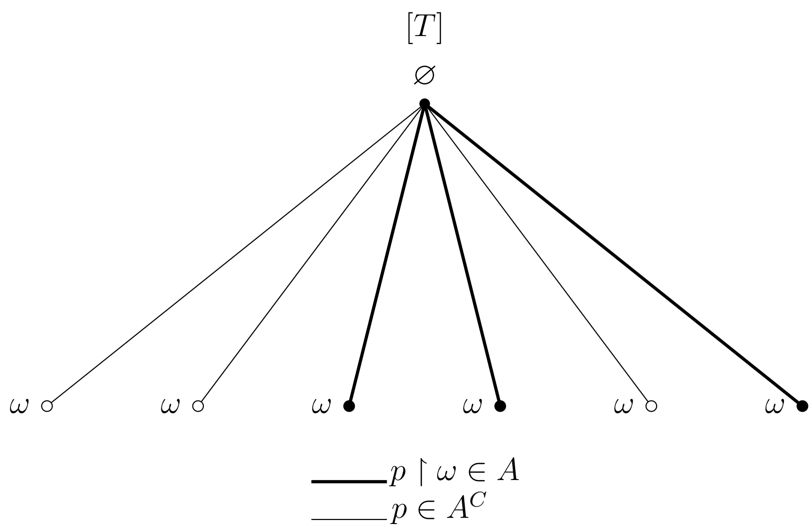

Figure 3.2: $[T]$ defined in Example 3.4.

Example 3.4. *Let* $p \in [T]$ *iff* $\begin{cases} p \in \omega^{\omega+1}, & \text{if } p \restriction \omega \in A \\ p \in \omega^{\omega}, & \text{if } p \in A^C \end{cases}$,
where $A \subseteq \omega^\omega$ *is any (fixed) clopen set. Let* S *be any non-trivial trivial tree. Then* $[T] \times [S] \cong [T * S]$.

Proof. First, note that because A is a clopen set, A^C is a clopen set. Since T and S are non-trivial trees, by Theorem 2.2, the canonical function is a bijection.

Next, we use Lemma 2.3 to show that f is continuous. Let $p \in [T]$.

Case 1: Assume that $p \restriction \omega \in A$. Then, $lth(p) = \omega + 1$. So, for some $k \in \omega$, $p(\omega) = k$. Let

$d^\star = \{(\omega, k)\} \in \mathcal{P}_{fin}(p)$. Suppose that $q \in \mathfrak{B}^{[T]}_{d^\star}$. Then, $q \supseteq d^\star$. Thus, $lth(q) = \omega + 1$. Hence, $lth(q) = lth(p)$.

Case 2: Assume that $p \restriction \omega \in A^C$. Then, $lth(p) = \omega$ and $p = p \restriction \omega$. Since A^C is open, A^C can be written as a countable union of basic open sets. Say $A^C = \bigcup_{i \in \omega} \mathfrak{B}^{[T]}_i$. Since $p \in A^C$, there exists $n \in \omega$ such that $p \in \mathfrak{B}^{[T]}_n$. Recall, by definition of a basic open set in the tree topology, $\mathfrak{B}^{[T]}_n = \{t \in [T] | t \supseteq d^\star_n\}$ where $d^\star_n \in fin(\omega^\omega)$. Thus, $p \supseteq d^\star_n$. Now, suppose $q \in \mathfrak{B}^{[T]}_n$. Then, $q \restriction \omega \in A^C$. So, $lth(q) = \omega$. Hence, $lth(q) = lth(p)$.

In both cases, we have met the conditions of Lemma 2.3. Thus, f is continuous.

Next, we use Lemma 2.9 to show that f is an open map.

Case 1: Let $h_1 \restriction \omega \in A$. Then, for some $n \in \omega$, $h_1 \in \mathfrak{B}^{[T]}_n = \{t \in [T] | t \supseteq d^\star_n\}$ where $d^\star_n \in fin(\omega^{\omega+1})$. So, $h_1 \supseteq d^\star_n$. Let $h_2 \in \mathfrak{B}^{[S]}_{d_2}$. So, $h_2 \supseteq d_2$. Since $h_1 \restriction \omega \in A$, $dom(h_1) = \omega + 1$. Define $d = d^\star_n \cup s_R(d_2, \omega + 1)$. Suppose $x = h_1 \times h_2 \in [T] \times [S]$. Then, $f(x) = h_1^\frown h_2 \in [T * S]$. Since $h_1 \supseteq d^\star_n$, $h_2 \supseteq d_2$, and $lth(h_1) = \omega + 1$, we have $h_1^\frown h_2 \supseteq d$. Recall $\mathcal{B}^{[T*S]}_d = \{t^\frown s \in [T * S] |\ t^\frown s \supseteq d\}$. So, $f(x) = h_1^\frown h_2 \in \mathcal{B}^{[T*S]}_d$. Next, suppose $g \in \mathcal{B}^{[T*S]}_d$. Then, $g \in [T * S]$ and $g \supseteq d$. By Lemma 1.38, there exist unique g_1 and g_2 such that $g = g_1^\frown g_2$, where $g_1 = g \restriction \omega + 1 \in [T]$ and $g_2 = s_L\Big(g \restriction \big[\omega + 1, dom(g)\big), \omega + 1\Big) \in [S]$. Next, $d \restriction \omega + 1 = d^\star_n$ and $d_2 = s_L\Big(d \restriction \big[\omega + 1, dom(g)\big), \omega + 1\Big)$. So, $g_1 \supseteq d^\star_n$ and $g_2 \supseteq d_2$. Thus, $g_1 \in \mathfrak{B}^{[T]}_n$ and $g_2 \in \mathfrak{B}^{[S]}_{d_2}$.

Case 2: Let $h_1 \in A^C$. Then, for some $n \in \omega$, $h_1 \in \mathfrak{B}^{[T]}_n = \{t \in [T] | t \supseteq d^\star_n\}$ where $d^\star_n \in fin(\omega^\omega)$. So, $h_1 \supseteq d^\star_n$. Let $h_2 \in \mathfrak{B}^{[S]}_{d_2}$. So, $h_2 \supseteq d_2$. Since $h_1 \in A^C$, $dom(h_1) = \omega$. Define $d = d^\star_n \cup s_R(d_2, \omega)$. Suppose $x = h_1 \times h_2 \in [T] \times [S]$. Then, $f(x) = h_1^\frown h_2 \in [T * S]$. Since $h_1 \supseteq d^\star_n$, $h_2 \supseteq d_2$, and $lth(h_1) = \omega$, we have $h_1^\frown h_2 \supseteq d$. Let $\mathcal{B}^{[T*S]}_d = \{t^\frown s \in [T * S] |\ t^\frown s \supseteq d\}$. So, $f(x) = h_1^\frown h_2 \in \mathcal{B}^{[T*S]}_d$. Next, suppose $g \in \mathcal{B}^{[T*S]}_d$. Then, $g \in [T * S]$ and $g \supseteq d$. By

Lemma 1.38, there exist unique g_1 and g_2 such that $g = g_1^\frown g_2$, where $g_1 = g \restriction \omega \in [T]$ and $g_2 = s_L\Big(g \restriction \big[\omega, dom(g)\big), \omega\Big) \in [S]$. Next, $d^\star_n = d \restriction \omega$ and $d_2 = s_L\Big(d \restriction \big[\omega, dom(g)\big), \omega\Big)$. So, $g_1 \supseteq d^\star_n$ and $g_2 \supseteq d_2$. Thus, $g_1 \in \mathfrak{B}^{[T]}_n$ and $g_2 \in \mathfrak{B}^{[S]}_{d_2}$.

In both cases, we have met the conditions of Lemma 2.9. Hence, f is an open map.

We have shown that $f : [T] \times [S] \to [T * S]$ is a bijection which is continuous and open. Therefore, $[T] \times [S] \cong [T * S]$. □

CHAPTER 4

COUNTEREXAMPLES

In this chapter, we show that the canonical function does not always produce a homeomorphism. In both of the counterexamples below, the canonical function is not continuous and not open. In both of these counterexamples, we use a construction for the first tree that is similar to the construction of the first tree used in Example 3.4. However, path lengths of the first tree in our counterexamples are defined by sets that are not clopen.

Theorem 4.1. *There exist trees for which the canonical function is not a homeomorphism.*

Proof. See counterexamples below. □

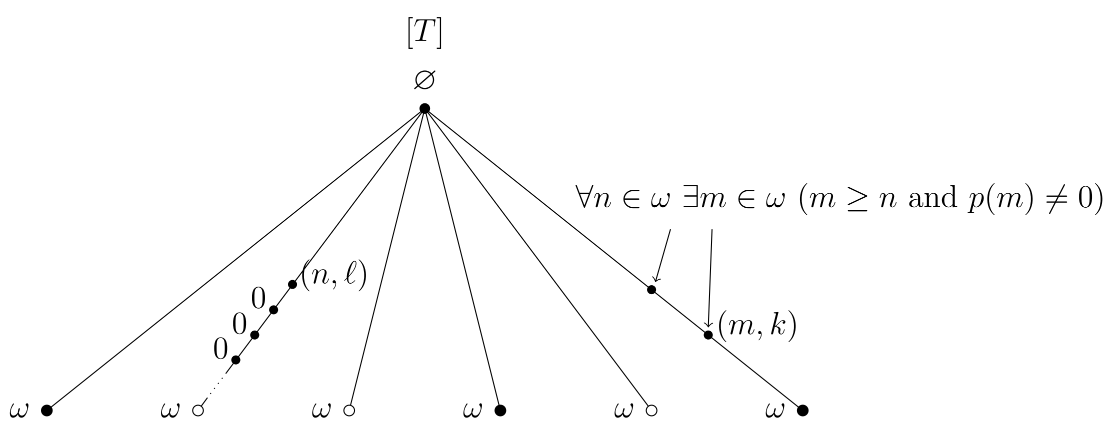

Figure 4.1: $[T]$ defined in Counterexample 4.2.

Counterexample 4.2. *This is an example of a tree, T, that has lengths of paths defined by sets that are both not open and not closed. In this case, the canonical function is not continuous and not open.*

First define the tree T.

$$\text{Let } p \in [T] \text{ iff } \begin{cases} p \in \omega^{\omega+1}, & \text{if } \forall n \in \omega\ \exists m \in \omega\ (m \geq n \text{ and } p(m) \neq 0) \\ \\ p \in \omega^{\omega}, & \text{if } \exists n \in \omega\ \forall m \in \omega\ (m \geq n \implies p(m) = 0) \end{cases}$$

Let $[S] = \omega^\omega$.

Proof. Let T and S be defined as above. Suppose $\mathcal{B}_d^{[T*S]} = \{p^\frown q \in [T*S] |\ p^\frown q \supseteq d\} \in \mathcal{T}_\circledast$, where $d = \{(\omega, 0)\}$. We show $f^{-1}(\mathcal{B}_d^{[T*S]})$ is not an open set. Let $p = \vec{0}$ and $q = \vec{0}$. Then, let $x = p \times q \in [T] \times [S]$. So, $lth(p) = \omega$. Thus, $f(x) = p^\frown q = \vec{0}$. So, $f(x) \supseteq d$. Hence, $f(x) \in \mathcal{B}_d^{[T*S]}$. Thus, $x \in f^{-1}(\mathcal{B}_d^{[T*S]})$. Let $p^\star \in \mathcal{P}_{fin}(p)$ and $q^\star \in \mathcal{P}_{fin}(q)$. Consider $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$, where $\mathfrak{B}_{p^\star}^{[T]} = \{t \in [T] | t \supseteq p^\star\}$ and $\mathfrak{B}_{q^\star}^{[S]} = \{s \in [S] | s \supseteq q^\star\}$. $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$ is an arbitrary basic open neighborhood in $\mathcal{T}_\otimes$ containing x. We show $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]} \nsubseteq f^{-1}(\mathcal{B}_d^{[T*S]})$.

Consider $z \in \mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$. Let $z = \tilde{p} \times \tilde{q}$ such that $\tilde{q} = q$ and $\tilde{p} = p_1^\frown p_2^\frown\{1\}$. Define $p_1 = \vec{0}$, so that $p_1 \supseteq p^\star$ and $p_2 = \overrightarrow{0,1}$. So, $\tilde{p} \in [T]$ and $lth(\tilde{p}) = \omega + 1$. Note $\tilde{p}(\omega) = 1$. So, $f(z) = \tilde{p}^\frown \tilde{q} \not\supseteq d$. Thus, $f(z) \notin \mathcal{B}_d^{[T*S]}$. So, $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]} \nsubseteq f^{-1}(\mathcal{B}_d^{[T*S]})$. Since $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$ is arbitrary, $f^{-1}(\mathcal{B}_d^{[T*S]})$ is not open. Therefore, f is not continuous.

Suppose $[T] \times \mathfrak{B}_{c_2}^{[S]} = [T] \times \{q \in [S] |\ q \supseteq c_2\} \in \mathcal{T}_\otimes$, where $c_2 = \{(0,0)\}$. We show $f([T] \times \mathfrak{B}_{c_2}^{[S]})$ is not an open set. Suppose $p = \vec{0}$ and $q = \{0\}^\frown\vec{1}$. Let $x = p \times q \in [T] \times [S]$. Hence, $lth(p) = \omega$. Also, $q \supseteq c_2$, because $q(0) = 0$. So, $x \in [T] \times \mathfrak{B}_{c_2}^{[S]}$. Then,

$f(x) = p^\frown q \in f([T] \times \mathfrak{B}_{c_2}^{[S]})$. Let $p^\star \in \mathcal{P}_{fin}(p)$. Consider $\mathcal{B}_c^{[T*S]} = \{t^\frown s \in [T*S] |\ t^\frown s \supseteq c\}$, where $c \supseteq p^\star \cup s_R(c_2, \omega)$. Since $p \supseteq p^\star$, $q \supseteq c_2$, and $lth(p) = \omega$, then $f(x) \supseteq c$. Thus, $\mathcal{B}_c^{[T*S]}$ is an arbitrary basic open neighborhood in $\mathcal{T}_\circledast$ containing $f(x)$.

Now, consider $v \in \mathcal{B}_c^{[T*S]}$. Let $v = \tilde{p}^\frown\tilde{q}$. Define $\tilde{q} = \vec{1}$. Further, let $\tilde{p} = p_1^\frown p_2^\frown\{0\}$, where $p_1 = \vec{0}$ so that $p_1 \supseteq p^\star$ and $p_2 = \overrightarrow{0,1}$. So, $\tilde{p} \in [T]$ and $lth(\tilde{p}) = \omega + 1$. Because $\tilde{p} \supseteq p^\star$ and $\tilde{p}(\omega) = 0$, $v \supseteq c$. However, consider $f^{-1}(v) = \tilde{p} \times \tilde{q}$. Since $\tilde{q} = \vec{1}$, $\tilde{q} \not\supseteq c_2$. So, $\tilde{q} \notin \mathfrak{B}_{c_2}$. Thus, $v \notin f([T] \times \mathfrak{B}_{c_2}^{[S]})$. So, $\mathcal{B}_c^{[T*S]} \nsubseteq f([T] \times \mathfrak{B}_{c_2}^{[S]})$. Since $\mathcal{B}_c^{[T*S]}$ is arbitrary, $f([T] \times \mathfrak{B}_{c_2}^{[S]})$ is not open. Therefore, f is not an open map. □

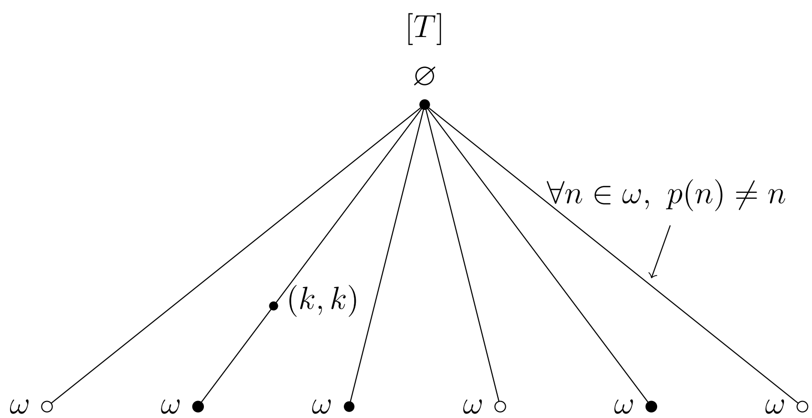

Figure 4.2: $[T]$ defined in Counterexample 4.3.

Counterexample 4.3. *This is an example of a tree, T, that has two possible path lengths. If the sequence has length $\omega + 1$, then the sequence restricted to ω is in the open set A (defined below). If the sequence has length ω, then the sequence is not in A. Instead, the sequence is in the complement of A which is closed. In this case, the canonical function, $f : [T] \times [S] \to [T * S]$ is not continuous and not open.*

Let $A \subseteq \omega^\omega$, where $A = \bigcup_{n\in\omega} \mathcal{O}_n$ such that $\mathcal{O}_n = \{t \in \omega^\omega |\ t(n) = n\}$.

$$\text{Let } p \in [T] \text{ iff } \begin{cases} p \in \omega^{\omega+1}, & \text{if } p \restriction \omega \in A \\ p \in \omega^\omega, & \text{if } p \notin A \end{cases}$$

Let $[S] = \omega^\omega$.

Proof. Let T and S be defined as above. Suppose $\mathcal{B}_d^{[T*S]} = \{p^\frown q \in [T*S] |\ p^\frown q \supseteq d\} \in \mathcal{T}_\circledast$, where $d = \{(\omega, 0)\}$. We wish to find an element of $f^{-1}(\mathcal{B}_d^{[T*S]})$. Let $p = \{1\}^\frown\vec{0}$ and $q = \vec{0}$. So, $p(0) = 1$ and for all $m \in \omega$, $p(m+1) = 0$. Thus, $p \restriction \omega \notin A$. So, $lth(p) = \omega$. Let $x = p \times q$. Note that $f(x) = p^\frown q = \{1\}^\frown\vec{0}$, so $f(x) \supseteq d$. Hence, $f(x) \in \mathcal{B}_d^{[T*S]}$. Thus, $x \in f^{-1}(\mathcal{B}_d^{[T*S]})$. Let $p^\star \in \mathcal{P}_{fin}(p)$ and $q^\star \in \mathcal{P}_{fin}(q)$. Consider $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$, where $\mathfrak{B}_{p^\star}^{[T]} = \{t \in [T] | t \supseteq p^\star\}$ and $\mathfrak{B}_{q^\star}^{[S]} = \{s \in [S] | s \supseteq q^\star\}$. So, $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$ is an arbitrary basic open neighborhood in $\mathcal{T}_\otimes$ containing x.

Since $p^\star$ is a finite set, there exists $k \in \omega,\ k \neq 0$ such that $k \notin dom(p^\star)$. Now, consider $z = \tilde{p} \times q$ where $\tilde{p} = \{1\}^\frown\vec{0}^\frown\{k\}^\frown\vec{0}^\frown\{1\}$. So, $\tilde{p}(k) = k$. Thus, $\tilde{p} \in A$. So, $lth(\tilde{p}) = \omega + 1$. Notice for all $j \in dom(p^\star),\ j > 0,\ \tilde{p}(j) = 0$ and $\tilde{p}(0) = 1 = p(0)$. So, $\tilde{p} \supseteq p^\star$. Hence, $z \in \mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$. However, since $lth(\tilde{p}) = \omega + 1$ and $\tilde{p} = \{1\}^\frown\vec{0}^\frown\{k\}^\frown\vec{0}^\frown\{1\}$, $\tilde{p}(\omega) = 1$. So, $f(z) = \tilde{p}^\frown\tilde{q} \not\supseteq d$. Thus, $f(z) \notin \mathcal{B}_d^{[T*S]}$. So, $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]} \not\subseteq f^{-1}(\mathcal{B}_d^{[T*S]})$. Since $\mathfrak{B}_{p^\star}^{[T]} \times \mathfrak{B}_{q^\star}^{[S]}$ is arbitrary, $f^{-1}(\mathcal{B}_d^{[T*S]})$ is not open. Therefore, f is not continuous.

Suppose $[T] \times \mathfrak{B}_{c_2}^{[S]} = [T] \times \{q \in [S] |\ q \supseteq c_2\} \in \mathcal{T}_\otimes$, where $c_2 = \{(0,0)\}$. We wish to find an element of $f([T] \times \mathfrak{B}_{c_2}^{[S]})$. Suppose $p = \{1\}^\frown\vec{0}$ and $q = \{0\}^\frown\vec{1}$. So, $p \in [T]$ and $q \in [S]$. Let $x = p \times q$. Since $p(0) = 1$ and for all $m \in \omega$, $p(m+1) = 0$, then $p \notin A$. So, $lth(p) = \omega$. Next, $q \supseteq c_2$, because $q(0) = 0$. So, $x \in [T] \times \mathfrak{B}_{c_2}^{[S]}$. Hence, $f(x) = p^\frown q \in f([T] \times \mathfrak{B}_{c_2}^{[S]})$. Next, we find an open neighborhood of $f(x)$. Let $p^\star \in \mathcal{P}_{fin}(p)$. Consider $\mathcal{B}_c^{[T*S]} = \{t^\frown s \in [T*S] |\ t^\frown s \supseteq c\}$, where $c \supseteq p^\star \cup s_R(c_2, \omega)$. Since $p \supseteq p^\star$, $q \supseteq c_2$, and $lth(p) = \omega$, then $f(x) \supseteq c$. Thus, $\mathcal{B}_c^{[T*S]}$ is an arbitrary basic open neighborhood in $\mathcal{T}_\circledast$ containing $f(x)$.

Since $p^\star$ is a finite set, there exists $k \in \omega,\ k \neq 0$ such that $k \notin dom(p^\star)$. Define

$\tilde{p} = \{1\}^\frown\vec{0}^\frown\{k\}^\frown\vec{0}$. So, $\tilde{p}(k) = k$. Thus, $\tilde{p} \in A$. So, $lth(\tilde{p}) = \omega + 1$. Define $\tilde{q} = \vec{1}$. Let $v = \tilde{p}^\frown\tilde{q}$. Notice for all $j \in dom(p^\star)$, $j > 0$, $\tilde{p}(j) = 0$ and $\tilde{p}(0) = 1 = p(0)$. So, $\tilde{p} \supseteq p^\star$. Because $\tilde{p} \supseteq p^\star$ and $\tilde{p}(\omega) = 0$, $v \supseteq c$. Therefore, $v \in \mathcal{B}_c^{[T*S]}$. However, $f^{-1}(v) = \tilde{p} \times \tilde{q}$. Since $\tilde{q} = \vec{1}$, $\tilde{q} \not\supseteq c_2$. So, $\tilde{q} \notin \mathfrak{B}_{c_2}^{[S]}$. Thus, $v \notin f([T] \times \mathfrak{B}_{c_2}^{[S]})$. So, $\mathcal{B}_c^{[T*S]} \not\subseteq f([T] \times \mathfrak{B}_{c_2}^{[S]})$. Since $\mathcal{B}_c^{[T*S]}$ is arbitrary, $f([T] \times \mathfrak{B}_{c_2}^{[S]})$ is not open. Therefore, f is not an open map. $\square$

CHAPTER 5

CONCLUSION

In this thesis we provided some basic results as to how the product topology on two trees behaves in comparison to the tree topology for longer trees.

In Chapter 2, we were able to show that the canonical function is always a bijection between the product topology of two trees and the tree topology of the long concatenated tree. We were also able to state results that more easily allow us to prove a homeomorphism exists between the two topological spaces, the product topology for $[T] \times [S]$ and the tree topology for $[T * S]$, given trees T and S. We explored conditions that we must require of the trees T and S to guarantee that the canonical function produces a homeomorphism. We gave sufficient conditions to show that the canonical function is a continuous map. Additionally, we gave a conjecture that states that if the canonical function is continuous for all non-trivial trees S, then T satisfies the condition that every path in $[T]$ is contained in an open neighborhood such that all paths in that neighborhood have the same length. We were successful in finding necessary and sufficient conditions for T to prove that the canonical function is open.

In Chapter 3, we used our proven lemmas to present several examples of trees T and S for which the canonical function is a homeomorphism. Lastly, in Chapter 4 we gave counterexamples that show the canonical function does not always produce a homeomorphism. In our counterexamples, we show that the canonical function is both not continuous and not open.

The original purpose of this study was to show that for some trees T and S, the two topological spaces are homeomorphic, and to find trees T and S for which the two topological spaces are not homeomorphic. We were able to complete our first goal. However, we were only able to prove that the canonical function is not a homeomorphism for certain trees T and S. This does not prove that there is not a homeomorphism between the two topological spaces for those given trees. Future work may include finding other functions which produce a homeomorphism for these more complicated trees, or it may be possible to show that no such function exists.

RESULTS

Lemma 1.37: If $f : \gamma \to X$ and $\alpha \leq \gamma$, then there exist unique sequences f_1 and f_2 such that $f = f_1^\frown f_2$ where $dom(f_1) = \alpha$.

Lemma 1.38: Assume T and S are non-trivial trees. Then $f \in [T * S]$ if and only if there exists a unique $\alpha < dom(f)$ such that $f \restriction \alpha \in [T]$ and so $s_L(f \restriction [\alpha, dom(f)), \alpha) \in [S]$.

Theorem 2.1: Let $T = S = \omega^{<\omega}$. Then $[T] \times [S] \cong [T * S]$.

Theorem 2.2: Let T and S be any non-trivial trees. Then there exists a bijection $f : [T] \times [S] \to [T * S]$, defined by $f(a) = a_1^\frown a_2$, for $a = a_1 \times a_2 \in [T] \times [S]$.

Lemma 2.3: Let T and S be any non-trivial trees. If for all $p \in [T]$ there exists $d^\star \in \mathcal{P}_{fin}(p)$ such that for all $q \in \mathfrak{B}_{d^\star}^{[T]},\ lth(q) = lth(p)$, then the canonical function is continuous.

Lemma 2.9: Let T and S be any non-trivial trees. The canonical function is an open map if and only if given any $h_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $h_2 \in \mathfrak{B}_{d_2}^{[S]}$ there exists d such that $h_1^\frown h_2 \in \mathcal{B}_d^{[T*S]}$ and for all $g = g_1^\frown g_2 \in \mathcal{B}_d^{[T*S]}$ (where $g_1 \in [T]$ and $g_2 \in [S]$), $g_1 \in \mathfrak{B}_{d_1}^{[T]}$ and $g_2 \in \mathfrak{B}_{d_2}^{[S]}$.

Theorem 3.1: Let T be any non-trivial tree such that there exists $\gamma \in ord$, for all $p \in [T],\ dom(p) = \gamma$, and let S be any non-trivial tree. Then $[T] \times [S] \cong [T * S]$.

Theorem 4.1: There exist trees for which the canonical function is not a homeomorphism.

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