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Generalized Markoff Equations, Euclid Trees, and Chebyshev Polynomials

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GENERALIZED MARKOFF EQUATIONS, EUCLID TREES, AND CHEBYSHEV POLYNOMIALS

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Abstract

The Markoff equation is $x^2+y^2+z^2 = 3xyz$, and all of the positive integer solutions of this equation occur on one tree generated from $(1, 1, 1)$, which is called the Markoff tree. In this paper, we consider trees of solutions to equations of the form $x^2 + y^2 + z^2 = xyz + A$. We say a tree of solutions satisfies the unicity condition if the maximum element of an ordered triple in the tree uniquely determines the other two. The unicity conjecture says that the Markoff tree satisifies the unicity condition. In this paper, we show that there exists a sequence of real numbers $\{c_n\}$ such that the tree generated from $(1, c_n, c_n)$ satisfies the unicity condition for all n, and that these trees converge to the Markoff tree. We accomplish this by first recasting polynomial solutions as linear combinations of Chebyshev polynomials, and showing that these polynomials are distinct. Then we evaluate these polynomials at certain values and use a countability argument. We also obtain upper and lower bounds for these polynomial expressions.

Acknowledgements

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CHAPTER 1

HISTORICAL BACKGROUND

1.1 Markoff Equation and the Unicity Conjecture

This dissertation is about the Markoff equation and the unicity conjecture, which can be traced back to Andrei Andreyevich Markoff and Ferdinand Georg Frobenius, respectively. Markoff worked on approximating irrational numbers by rational numbers, which led him to study the Lagrange spectrum. For those not familiar with the Lagrange spectrum, we provide the following definition.

Definition 1. Pick any real number r. We define the *Lagrange number* of r as $L(r) = \sup L$, where the supremum is taken over all L such that the following holds for infinitely many rational numbers $\frac{p}{q}$:

$$|r - \frac{p}{q}| < \frac{1}{Lq^2}.$$

We define the *Lagrange spectrum* as $\mathcal{L} = \{L(r) : r \in \mathbb{R} \setminus \mathbb{Q}\}$, and the *Lagrange spectrum below* 3 as $\mathcal{L}_{<3} = \{L(r) : r \in \mathbb{R} \setminus \mathbb{Q}, L(r) < 3\}$.

In 1879 and 1880 (see [16] and [17]), Markoff used continued fractions and indefinite quadratic forms to demonstrate that there is a 1-1 correspondence between elements of $\mathcal{L}_{<3}$ and the positive integer solutions of the following equation, which is known as the Markoff equation:

$$x^2 + y^2 + z^2 = 3xyz.$$

Specifically, he showed that

$$\mathcal{L}_{<3} = \left\{ \frac{\sqrt{9m^2 - 4}}{m} : m \in M \right\},$$

where M is the set of all positive integer solutions of the Markoff equation.

We call $(x, y, z) \in \mathbb{R}^3$ an *ordered triple* if $x \le y \le z$, and we call (x, y, z) a *Markoff triple* if it is an ordered triple solution to the Markoff equation with x, y, and z all positive integers. It is easy to show that if (x, y, z) is a Markoff triple then so are $(x, z, 3xz - y)$ and $(y, z, 3yz - x)$. Thus, any ordered triple solution creates additional solutions, forming a tree of solutions for the Markoff equation. In particular, we can generate a tree of solutions from $(1, 1, 1)$, which we refer to as the Markoff tree $\mathfrak{M}$, as shown in Figure 1.1. We say $(1, 1, 1)$ is the *root* of the Markoff tree, and more generally, if (x, y, z) is the triple that generates a tree of solutions $\mathfrak{T}$, then we say (x, y, z) is the *root* of $\mathfrak{T}$. Markoff used a method of descent to show that every Markoff triple descends all way down to $(1, 1, 1)$ in a finite number of steps. Thus, all of the Markoff triples appear in $\mathfrak{M}$. Notice that the Markoff tree behaves like a binary tree starting at the triple $(1, 2, 5)$. The only triples with repeated coordinates are $(1, 1, 1)$ and $(1, 1, 2)$, which J. W. S. Cassels calls singular solutions [6].

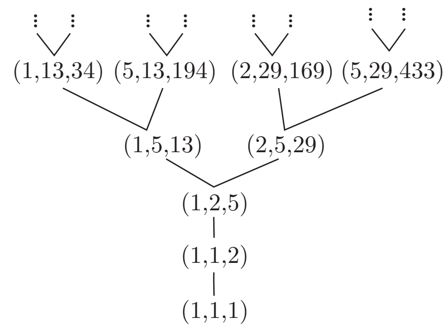

Figure 1.1: The Markoff tree $\mathfrak{M}$.

In 1913, Frobenius conjectured that the largest entry z of a Markoff triple uniquely determines the other two [11]. Another way to say this is that the maximal element is *unique*. This is now known as the *unicity conjecture*. Despite several attempts over the last century, this conjecture remains unsolved. Richard K. Guy has even called the unicity conjecture too difficult for anyone to try to solve [12]. Some partial results of the unicity conjecture have been settled. The following results are due to Arthur Baragar, J. O. Button, Feng-Juan Chen, and Yang-Gao Chen. It is known that if z or $3z \pm 2$ is a prime, twice a prime, or four times a prime then it is unique [3]; or if z is a prime power then it is unique [5]. Currently, it is known that z is unique if $z = k \cdot p^\beta$, with $k \leq 10^{35}, p$ prime and k relatively prime to p [5]; and if $3z \pm 2 = k \cdot p^\beta$, with $k \leq 10^{10}, p$ prime, and k relatively prime to p [7]. The upper bounds for k in these last two results are based on the empirical result that z is unique if $z < 10^{140}$ [3]. In 2007, Ying Zhang provided an elementary proof that if z is a prime power or twice a prime power then z is unique [30]. In 2009, Anitha Srinivasan provided an elementary proof for further results, including the case that if the greatest odd divisor of $3z \pm 2$ is a prime power then z is unique [27].

Even though Frobenius was the first person to state the unicity conjecture, it was Cassel who brought it to the attention of most mathematicians. In his 1957 book *An Introduction to Diophantine Equations* (see [6]), he describes the results of Markoff and Frobenius, and states on page 33 of chapter 2 (titled "The Markoff Chain") that:

"There is a slight ambiguity in the notation F_m since no one has shown that there cannot be two distinct solutions $(m, m_1, m_2), (m, m_1^*, m_2^*)$ occurring in different portions of the tree. No case of this is known and it seems improbable."

1.2 Generalizations of the Markoff Equation

The Markoff equation has been generalized in different ways. In his 1907 paper (see [14]), Adolf Hurwitz introduced the Hurwitz-Markoff equations, which are of the form

$$M_{z,n}: \qquad x_1^2 + x_2^2 + ... + x_n^2 = zx_1x_2...x_n.$$

He showed that if $z > n$ then $M_{z,n}$ has no positive integer solutions, and if $z = n$ then $M_{z,n}$ has exactly one tree of positive integer solutions generated by the root $(1, 1, ..., 1)$. Hurwitz knew that if $z < n$ then there were possibly multiple trees of positive integer solutions for $M_{z,n}$, but he was not able to settle this case. Hurwitz does provide a table of all roots that generate a tree of integer solutions for $M_{z,n}$ for $1 \leq z \leq n \leq 10$ [14]. The case when $z < n$ was completely solved by Norman P. Herzberg. In 1974, he provided a 7 step algorithm that gives every root (that generate a tree of positive integer solutions) for $M_{z,n}$ in Theorem 2 of [13]. Baragar showed that for every $r > 0$, there exists a pair (z_r, n_r) such that M_{z_r,n_r} has at least r roots that generate a tree of positive integer solutions (see Theorems 2.1 and 2.2 in [2]).

Another generalization of the Markoff equation was studied by L. J. Mordell. Mordell looked at generalized Markoff equations of the form

$$x^2 + y^2 + z^2 = axyz + b.$$

In particular, he studied $x^2+y^2+z^2+2xyz = n$ in his 1953 paper (see [19]), which he compared to cubic equations of the form

$$x^3 + y^3 + z^3 + w^3 = m.$$

He proves many results, including that when $n = 2^r$, there are infinitely many solutions when r is odd, but only the solution $x = y = z = 0$ when r is even [19].

In 1980, Gerhard Rosenberger studied equations of the form

$$ax^2 + by^2 + cz^2 = dxyz,$$

with a, b, and c all dividing d, which are called the Markoff-Rosenberger equations [23]. Rosenberger showed that only a finite number of these equations have infinitely many solutions of positive integers. A recent development involves a 2013 paper by Enrique Gonzalez Jimenez and Jose M. Tornero. They show that there can be only a finite number of solutions of Markoff-Rosenberg numbers in arithmetic progression [15].

Many authors have looked at Markoff's equation in the following form:

$$x^2 + y^2 + z^2 = xyz.$$

Later in this work, we study Markoff equations of the form

$$M_A : \qquad x^2 + y^2 + z^2 = xyz + A.$$

Hence, we denote $x^2 + y^2 + z^2 = xyz$ as M_0. It is easy to see that (x, y, z) is a solution of $x^2 + y^2 + z^2 = axyz + b$ if and only if (ax, ay, az) is a solution of M_{a^2b}. In particular, (x, y, z) is a solution to the original Markoff equation if and only if $(3x, 3y, 3z)$ is a solution to M_0. Hence, M_A is the one parameter equivalent of the generalized Markoff equations in two parameters that Mordell studied.

1.3 More Results and Approaches

Over the last century, several methods have been developed to make connections between the Markoff equation and other branches of mathematics, as well as produce partial solutions to the unicity conjecture. There is a connection between integer solutions of the Markoff equation and with algebraic number theory, combinatorics, diophantine approximation, and hyperbolic geometry. For those interested in more details of these connections than this section provides, an excellent source is the book "Markov's Theorem and 100 Years of the Uniqueness Conjecture" by Martin Aigner [1]. Details of the connection between integer solutions of M_0 and exceptional representative sheaves on the complex projective plane $\mathbb{P}^2$ can be found in A. N. Rudakov's paper [24]. Note that we have already mentioned some partial results of the unicity conjecture earlier in the previous sections.

In his 1913 paper, Frobenius provided some evidence to the unicity conjecture by ordering some of the Markoff triples using Farey fractions [11]. In addition, Frobenius established some of the first results about the Markoff numbers and the unicity conjecture, such as if m is an odd Markoff number and a prime number p divides m, then $p \equiv 1 \bmod 4$, and if m is an even Markoff number then $m \equiv 2 \bmod 8$.

In 1955, Harvey Cohn used 2×2 matrices to analyze the Markoff equation [8]. He was the first to observe the connection between M_0 and the following equation, which is known as Fricke's identity:

$$tr(A)^2 + tr(B)^2 + tr(AB)^2 = tr(A)tr(B)tr(AB) + tr(ABA^{-1}B^{-1}) + 2,$$

where A and B are 2 X 2 matrices with integer entries and $tr(A)$ is the trace of the matrix A. It is worth noting that matrices representing Markoff numbers first appear in [11], even though Frobenius never mentions Fricke's identity. It is easy to verify that the triple $(tr(A), tr(B), tr(AB))$ is a Markoff triple if and only if $tr(ABA^{-1}B^{-1}) = -2$.

In 1976, Rosenberger claimed to have a proof of the unicity conjecture, using 2×2 matrices and Fricke's identity [22]. It was shown to have a nontrivial flaw by reviewer Richard T. Bumby. He mentioned that some of Rosenberger's ideas were brilliant but the overall proof was not correct. Specifically, in Lemma 3 on page 363 of [22], Rosenberger's statement of the following inequality is false:

$$tr(AB^2)^n > tr(AB^{2n-1}).$$

The most recent development using matrices is attributed to Norbert Riedel. Riedel claimed to have a solution of the unicity conjecture using nilpotent 3×3 matrices (see [20]), but reviewer S. Perinne has found a nontrivial flaw in each of his preprints. However, Riedel reveals an interesting result on page 9 of his preprint, which seems to suggest that there is something special about the two equations M_0 and M_4. Riedel also appears to be the first to state the connection between Chebyshev polynomials of the first kind (T_n's) and M_4 [20].

Next, we look at methods involving Euclid trees (we note here that we formally define a Euclid tree in Chapter 2). The first person to use this approach was Cohn

[9]. He created a coordinate pair (a, b) for each entry in the Markoff tree and said that $(a, b) = (a', b') + (a'', b'')$ is a *euclidean partition* if and only if $a'b'' - b'a'' = 1$. Cohn used the coordinate pairs to analyze the connection between the Markoff equation and a free semigroup of symbols.

In 1982, Don Zagier established an upper bound for the n^{th} Markoff number, and found the asymptotic behavior of $M(N)$, which is the number of Markoff triples with $x \leq y \leq z \leq N$. Zagier proved the following result:

$$M(N) = C(\ln N)^2 + O(\ln N(\ln \ln N)^2),$$

where C is approximately equal to 0.1807 [29]. He conjectures that the n^{th} Markoff number is approximately $\frac{1}{3}A^{\sqrt{n}}$, where $A = e^{\frac{1}{\sqrt{C}}}$, which is approximately 10.51015. He obtained these results using the $\ln(3x)$ function, which approximately maps the Markoff tree to the Euclid tree [29].

Later, we use Chebyshev polynomials and Euclid trees to represent Markoff numbers in M_A as distinct polynomials, which is one of the main results of this work. These polynomials are labelled $p_{n,j}$ and $q_{n,j}$, and in chapter 2, we show that the pairs (n, j) from these labelings satisfy a similar relationship to the coordinate pairs in a Euclidean partition of [9] (a similar relationship also appears in [11]).

Next, we look at a ring theory approach. Baragar was the first person to show certain types of Markoff numbers are unique using quadratic integer rings (as Baragar points out in [3], Zagier and Cohn knew about this approach, so actually, Baragar was the first person to publish a result showing this approach). He showed the connection between the unicity conjecture and elements of a ring with exactly one pair of principal ideals (see Theorem 1.1 of [3]). Button also used ring theory to show that prime Markoff numbers are unique [4]. Other mathematicians have studied the Markoff equations using abstract algebra, including Srinivasan [27].

Next, we briefly mention the last approach involving hyperbolic geometry. Cohn was the first person to find a connection between the Markoff equation and geodesics in the hyperbolic plane [10]. Cohn considers the geodesics transferred to the perforated torus, and the geodesic of a Markoff form becomes closed under translations of the lattice periods. It is known that the unicity conjecture is equivalent to showing that two simple closed geodesics of the same length are equivalent (on this new surface). Other mathematicians who studied the Markoff equation using hyperbolic geometry are Mark Sheingorn and Caroline Series (see [26] and [25]).

Most of the papers in the literature deal with tree(s) of positive integer solutions to the original Markoff equation or one of its generalizations. Greg McShane and Hugo Parlier wrote one of the very few papers that considers trees of real and not necessarily integral solutions [18]. We will say more about this paper throughout the next few chapters since it contains many important results that are similar to results in this work.

CHAPTER 2

POLYNOMIAL AND EUCLID TREES

2.1 Polynomial Trees

Throughout the rest of this work, we only consider generalized Markoff equations of the form

$$M_A : \qquad x^2 + y^2 + z^2 = xyz + A,$$

where A is any real number. Recall that in Chapter 1, we observed that there is a one-to-one correspondence between the equations $x^2+y^2+z^2 = axyz+b$ and M_{a^2b}. Specifically, (x, y, z) is a solution to $x^2+y^2+z^2 = axyz+b$ if and only if (ax, ay, az) is a solution to M_{a^2b}. In particular, there is a one-to-one correspondence between the Markoff equation and M_0. That is, the triple (x, y, z) is a solution to the Markoff equation if and only if $(3x, 3y, 3z)$ is a solution to M_0.

We define maps τ and σ that generate a tree of solutions corresponding to M_A as follows:

$$\tau(x, y, z) = (x, z, xz - y)$$

$$\sigma(x, y, z) = (y, z, yz - x).$$

Given a triple $\vec{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$, we define $\mathfrak{T}(\vec{r})$ to be the tree rooted at $\vec{r}$ and generated by τ and σ. Later, when we consider $\vec{r}$ as a triple of polynomials instead of numbers, we refer to $\mathfrak{T}(\vec{r})$ as a *polynomial tree*. For ease of notation, we use $\mathfrak{T}(\vec{r})$ and $\mathfrak{T}(r_1, r_2, r_3)$ interchangably (i.e., we use $\mathfrak{T}(r_1, r_2, r_3)$ instead of $\mathfrak{T}((r_1, r_2, r_3))$). We call $\vec{r}$ the *root* of $\mathfrak{T}(\vec{r})$ associated with M_A, and since the root satisfies M_A, we have

$$A = r_1^2 + r_2^2 + r_3^2 - r_1 r_2 r_3.$$

We emphasize here that we consider $\vec{r}$ and A to be real, not necessarily integral. For example, we consider the tree $\mathfrak{T}(3, \pi, \pi)$ associated with the equation $M_{9-\pi^2}$ in section 4.3 after Theorem 11. G. McShane and H. Parlier also looked at trees of solutions with real numbers associated with M_0 [18]. Theorem 11 of this paper is similar to their Theorem 1.3, but our result holds for arbitrary $A \neq 0$, and we use different methods [18].

In cases where $\tau(\vec{x}) = \vec{x}, \sigma(\vec{x}) = \vec{x}$, or $\tau(\vec{x}) = \sigma(\vec{x})$ for any $\vec{x}$ in a tree of solutions with root $\vec{r}$, we use the notation $\mathfrak{T}(\vec{r})$ for the tree that collapses the branches that contain repeated triples, and we use the notation $\mathfrak{T}'(\vec{r})$ for the tree that does not (see Figure 2.1 for an example).

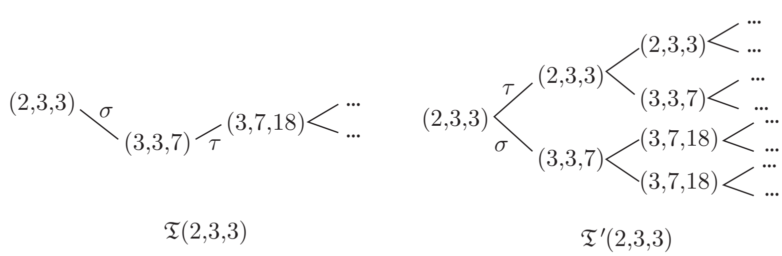

Figure 2.1: The trees $\mathfrak{T}(2, 3, 3)$ and $\mathfrak{T}'(2, 3, 3)$.

In the next section, we show that if $\vec{r}$ is an ordered triple with $2 \leq r_1$ then $\tau(\vec{x})$ and $\sigma(\vec{x})$ are ordered triples whenever $\vec{x}$ is an ordered triple, for every $\vec{x}$ in $\mathfrak{T}(\vec{r})$ (or in $\mathfrak{T}'(\vec{r})$). Hence, we assume from now on that $2 \leq r_1 \leq r_2 \leq r_3$. Of particular interest are polynomial trees rooted at the ordered triple (a, x, x) (see Figure 2.2).

Definition 2. We say that $\mathfrak{T}(\vec{r})$ satisfies the *unicity condition* if the maximum element of any triple in $\mathfrak{T}(\vec{r})$ uniquely determines the other two.

Remark 1. The unicity conjecture states that the maximum element of a Markoff triple uniquely determines the other two. Since all Markoff triples appear in the Markoff tree, this is equivalent to saying that $\mathfrak{T}(3, 3, 3)$, satisfies the unicity condition. We emphasize here that it is possible that for some A, the maximal component of an integer solution may not uniquely determine the other two, yet trees of integer solutions for M_A could still satisfy the unicity condition.

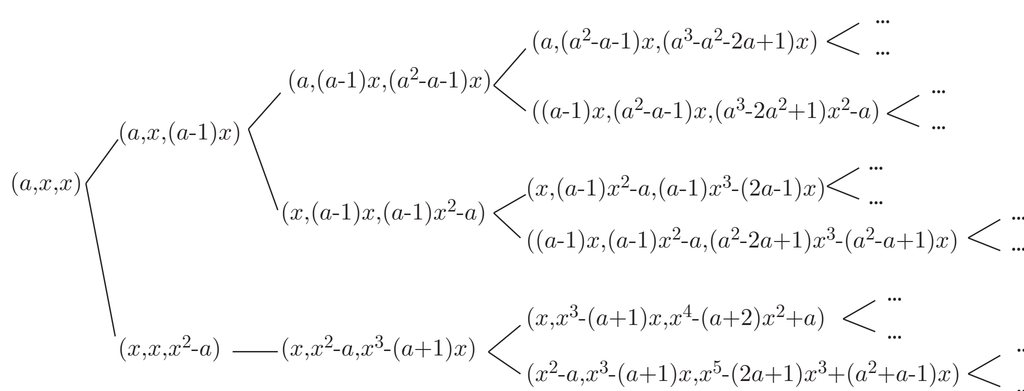

Figure 2.2: The polynomial tree rooted at (a, x, x).

One of our main results is the following theorem (especially when $a = c = 3$), which we prove in section 4.3:

Theorem 12. *For any pair of rational numbers* (a, c) *with* $2 < a \leq c$*, there exists a sequence of real numbers* $\{c_n\}$ *such that the sequence of trees* $\mathfrak{T}(a, c_n, c_n)$ *converges to* $\mathfrak{T}(a, c, c)$*, and* $\mathfrak{T}(a, c_n, c_n)$ *satisfies the unicity condition for every* n*.*

2.2 Proper Ordering of Trees

Throughout this section, we assume that $2 \leq a \leq x$, implying (a, x, x) is an ordered triple. Since (x, y, z) branches to $\tau(x, y, z)$ and $\sigma(x, y, z)$, if (a, x, x) is the root of a tree, then (a, x, x) branches to the two triples $(a, x, (a-1)x)$ and $(x, x, x^2 - a)$. It is straightforward to show that if $2 \leq a \leq x$ then $(a, x, (a-1)x)$ and $(x, x, x^2 - a)$ are both ordered triples. Next, we show that all triples generated from (a, x, x) are ordered triples.

Theorem 1. *If* $2 \leq a \leq x$*, then every triple in* $\mathfrak{T}(a, x, x)$ *and* $\mathfrak{T}'(a, x, x)$ *is an ordered triple.*

Proof. First, assume $2 < a$. Suppose that we have an ordered triple (b, c, d) in $\mathfrak{T}(a, x, x)$ with $b > 2$. If $bd - c \leq d$ then $(b-1)d \leq c$, which implies

$$\begin{aligned} d &< (b-1)d && \text{(since } b > 2) \\ &\leq c && \text{(since } bd - c \leq d) \\ &\leq d && \text{(since } b \leq c \leq d), \end{aligned}$$

a contradiction. Thus, $bd - c > d$. Similarly, $cd - b > d$. Hence, all nodes in the tree $\mathfrak{T}(a, x, x)$ are ordered triples.

Now assume $a = 2$. Then $\mathfrak{T}(2, 2, 2)$ only consists of the triple $(2, 2, 2)$ since $(2)(2) - (2) = 2$, and $\mathfrak{T}(2, 2, 2)$ collapses the branches with repeated triples. When $2 = a < x$, we do not branch in the τ direction since $(2)(x) - (x) = x$, and $(2, x, x)$ branches in σ direction to the triple $(x, x, x^2 - 2)$, where $x^2 - 2 > x$ since $x > 2$. Thereafter, our situation is as before and all nodes in the tree $\mathfrak{T}(2, x, x)$ are ordered triples. Therefore, all nodes in the tree $\mathfrak{T}(a, x, x)$ with $2 \leq a \leq x$ are ordered triples, and it follows that all nodes in the tree $\mathfrak{T}'(a, x, x)$ with $2 \leq a \leq x$ are ordered triples as well. $\square$

Theorem 1 shows that polynomials are properly ordered by the maps τ and σ when we evaluate x at values satisfying $2 \leq a \leq x$ (we emphasize here that x is a variable and a is a constant for these polynomials). It is mentioned after Lemmas 2, 3 that this is the same as ordering polynomials by their degrees.

2.3 Euclid Trees

The *Euclid tree* $\mathfrak{E}(\vec{r}_*)$ is the tree rooted at $\vec{r}_* = (r_{*1}, r_{*2}, r_{*3})$ with each r_{*i} a nonnegative integer satisfying $r_{*1} + r_{*2} = r_{*3}$, and defined by the following branching operations:

$$\tau_*(x_*, y_*, z_*) = (x_*, z_*, x_* + z_*)$$

$$\sigma_*(x_*, y_*, z_*) = (y_*, z_*, y_* + z_*).$$

As before, we define $\mathfrak{E}'(\vec{r}_*)$ to be the Euclid tree that does not collapse the branches when $\sigma_*(\vec{x}_*) = \vec{x}_*, \tau_*(\vec{x}_*) = \vec{x}_*$, or $\tau_*(\vec{x}_*) = \sigma_*(\vec{x}_*)$ (see Figure 2.3 for an example). The Euclid trees $\mathfrak{E}(1, 2, 3)$ and $\mathfrak{E}'(0, 1, 1)$ are used in definition 3. Note that $\mathfrak{E}(1, 2, 3)$ is a subtree of $\mathfrak{E}'(0, 1, 1)$ (see Figure 2.3). It follows from the maps σ_* and τ_* that $x_* + y_* = z_*$ for all $(x_*, y_*, z_*) \in \mathfrak{E}(\vec{r}_*)$. The tree $\mathfrak{E}(\vec{r}_*)$ is called a Euclid tree since descending the tree is equivalent to performing the Euclidean algorithm. Note that the descent in these trees is unique, since the Euclidean algorithm is unique. Of particular interest are Euclid trees with $\gcd(r_{*1}, r_{*2}) = 1$, which guarantees that the Euclidean algorithm only involves relatively prime numbers.

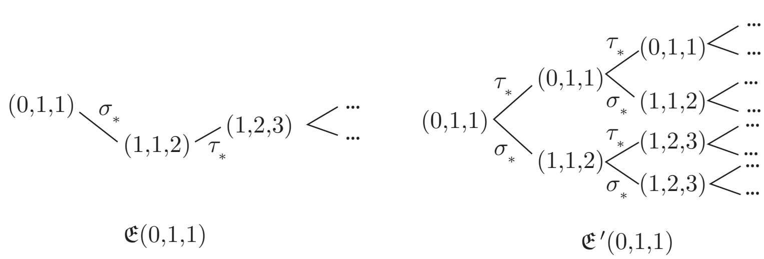

Figure 2.3: The Euclid trees $\mathfrak{E}(0, 1, 1)$ and $\mathfrak{E}'(0, 1, 1)$.

For any composite map $\mu = \sigma^{e_1} \circ \tau^{e_2} \circ ... \circ \sigma^{e_k}$ defined on $\mathfrak{T}(\vec{r})$, we define μ_* on $\mathfrak{E}'(\vec{r}_*)$ by $\sigma_*^{e_1} \circ \tau_*^{e_2} \circ ... \circ \sigma_*^{e_k}$. Then, for any $\vec{r}$ and $\vec{r}_*$, we define:

$$\begin{aligned} \Psi : \qquad & \mathfrak{T}(\vec{r}) \to \mathfrak{E}'(\vec{r}_*) \\ & \mu(\vec{r}) \mapsto \mu_*(\vec{r}_*). \end{aligned}$$

This is the map defined by Cohn and Zagier, under appropriate choices for $\vec{r}$ and $\vec{r}_*$ (see [9] and [29]). Note that Ψ depends on $\vec{r}$ and $\vec{r}_*$. Let us consider $\vec{r}$ and $\vec{r}_*$ with $r_{*1} < r_{*2} < r_{*3}$ and $\deg(r_k) = r_{*k}$ for $k = 1, 2, 3$. Under these conditions, it is easy to see that $\mathfrak{E}'(\vec{r}_*) = \mathfrak{E}(\vec{r}_*)$ and Ψ is invertible.

Before proving the next two lemmas, we mention their significance. Lemma 2 shows that the triples in $\mathfrak{E}(\vec{r}_*)$ uniquely determine the polynomial triples in $\mathfrak{T}(\vec{r})$. Lemma 3 shows that identities (2.1) and (2.2) (which appear on page 21) rely on *unique* m and k in their recursions, a fact that we use in Theorem 7 in Section 3.

Lemma 2. *Let* $r_{*1} < r_{*2} < r_{*3}$ *and* $\deg(r_k) = r_{*k}$ *for* $k = 1, 2, 3$. *Then for every* $\vec{x}$ *in* $\mathfrak{T}(\vec{r})$, *we have* $\Psi(\vec{x}) = (\deg(x_1), \deg(x_2), \deg(x_3))$.

Proof. The result clearly holds for $\vec{x} = \vec{r}$. If we pick any arbitrary nonzero polynomials P_1, P_2, and P_3 such that $\deg(P_1) < \deg(P_2) < \deg(P_3)$, then it is clear that $\deg(P_1P_3 - P_2) = \deg(P_1) + \deg(P_3)$ and $\deg(P_2P_3 - P_1) = \deg(P_2) + \deg(P_3)$. Therefore, it follows inductively from the maps σ, σ_*, τ, and τ_* that $\Psi(\vec{x}) = (\deg(x_1), \deg(x_2), \deg(x_3))$ holds for all $\vec{x}$ in $\mathfrak{T}(\vec{r})$. $\square$

Therefore, Ψ maps ordered triples of polynomials to their degrees, and since Ψ

is invertible, the degree triples uniquely determine the polynomial triples.

Lemma 3. *For any relatively prime integers n and j with $0 < j < \frac{n}{2}$, there exists exactly two solution pairs of integers (n_1, j_1) and $(n_2, j_2) = (n - n_1, j - j_1)$ with each $n_k < n$ such that $|nj_k - n_kj| = 1$ and $0 < j_k < \frac{n_k}{2}$ holds for $k = 1, 2$.*

Proof. It follows from the Euclidean algorithm and the fact that j has a unique inverse in $\mathbb{Z}_n^\times$ that there exists only two positive integer solutions (n_1, j_1) and (n_2, j_2) satisfying $|nj_k - n_kj| = 1$, one for $nj_k - n_kj = 1$ and one for $nj_k - n_kj = -1$. Then $n_2 = n - n_1$ and $j_2 = j - j_1$ holds because

$$\begin{aligned}|n(j - j_1) - j(n - n_1)| &= |nj - nj_1 - nj + jn_1| \\ &= |nj_1 - jn_1| \\ &= 1.\end{aligned}$$

For each k, if $nj_k - b_kj = \pm 1$ then $nj_k = b_kj \pm 1 < \frac{b_kn}{2} \pm 1$. Thus, we can conclude that $j_k < \frac{b_k}{2}$ for each k. □

Let us fix $\vec{r} = (x, x^2 - a, x^3 - (a + 1)x)$ and $\vec{r}_* = (1, 2, 3)$. From before, we know that the tree $\mathfrak{E}(1, 2, 3)$ (which appears in [9] and [29]) consists of all triples $\vec{x_*} = (x_{*1}, x_{*2}, x_{*3})$ with $\gcd(x_{*1}, x_{*2}) = 1, x_{*1} + x_{*2} = x_{*3}$, and $x_{*1} \leq x_{*2} \leq x_{*3}$. Each $\vec{x_*}$ descends (uniquely) to $(1, 2, 3)$. Thus, for any integer $n > 2$, there are $\frac{1}{2}\varphi(n)$ triples $\vec{x_*}$ in $\mathfrak{E}(1, 2, 3)$ so that $x_{*3} = n$, where φ is the Euler totient function (see [29] for details).

We have a bijection Ψ from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ to $\mathfrak{E}(1, 2, 3)$ (see Figure 2.4), from which it follows that each $\vec{x}$ in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ with $\deg(x_1) \leq \deg(x_2) \leq \deg(x_3)$ appears exactly once in the tree. Our main goal is to prove Theorem 7, which shows that each polynomial appears exactly once in the tree as a maximum element.

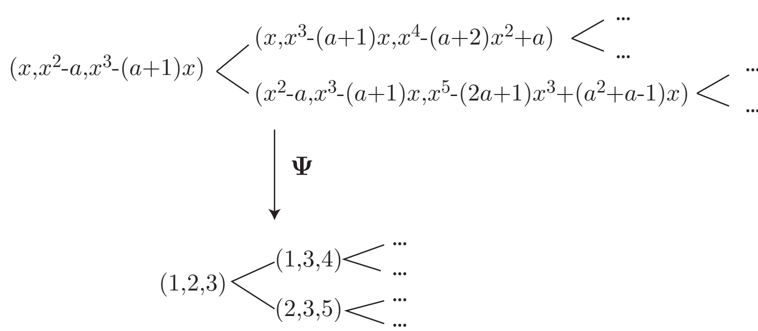

Figure 2.4: The map $\Psi : \mathfrak{T}(x, x^2 - a, x^3 - (a+1)x) \rightarrow \mathfrak{E}(1, 2, 3)$.

Before we continue, we should make a remark about the ordering of polynomial triples. It is natural to assume that polynomials are ordered by their degrees, but by "ordered triple" we mean ordered by magnitude when we plug in a value for x satisfying $2 \leq a \leq x$. By Theorem 1 and the previous two lemmas, these orderings are the same.

Definition 3. For each polynomial x_3 that appears as a maximum element in $\mathfrak{T}(\vec{r})$ where $\vec{r} = (x, x^2 - a, x^3 - (a+1)x)$, we assign two parameters n and j in the following way. For each $\vec{x}$, there exists a composite map μ (depending on $\vec{x}$)

such that $\vec{x} = \mu(\vec{r})$. We let $\Psi(\vec{x}) = \mu_*(1,2,3) = (x_{*1}, x_{*2}, x_{*3})$ in $\mathfrak{E}(1,2,3)$ and $\mu_*(0,1,1) = (y_{*1}, y_{*2}, y_{*3})$ in $\mathfrak{E}'(0,1,1)$. Then we let $n = x_{*3}$ and $j = y_{*3}$. We now define x_3 as $p_{n,j}$. We also define $p_{1,0} = x$ and $p_{2,1} = x^2 - a$.

The following corollary shows that this definition of $p_{n,j}$ is well-defined. In Section 3, Theorem 7 shows that $p_{n,j}$ is unique.

Corollary 4. *The pair* (n,j) *uniquely determines* $p_{n,j}$.

Proof. First, we want to show that j satisfies the properties in Lemma 3. Clearly, it does for the triple $(x, x^2 - a, x^3 - (a+1)x)$. Suppose we have a triple (x_1, x_2, x_3) in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ where for $k = 1,2,3$, each x_k is associated with parameters (n_k, j_k), and that $n_1 + n_2 = n_3$, $j_1 + j_2 = j_3$, and $n_2 j_1 - n_1 j_2 = \pm 1$. We just need to show that $n_3 j_2 - n_2 j_3 = \pm 1$ holds (with $n_3 j_1 - n_1 j_3 = \pm 1$ being shown in a similar way). So,

$$
\begin{aligned}
n_3 j_2 - n_2 j_3 &= (n_1 + n_2) j_2 - n_2 (j_1 + j_2) \\
&= n_1 j_2 - n_2 j_1 = \mp 1.
\end{aligned}
$$

Also, note that if $j_1 < \frac{n_1}{2}$ and $j_2 < \frac{n_2}{2}$ then $j_3 = j_1 + j_2 < \frac{n_1 + n_2}{2} = \frac{n_3}{2}$.

It now follows from the previous paragraph and from Lemmas 2 and 3 that $p_{n,j}(x)$ is well-defined, and when it is a maximum element (with respect to degrees), it uniquely determines $\vec{x}$. $\square$

Thus, $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x) = \mathfrak{T}(p_{1,0}, p_{2,1}, p_{3,1})$, as in Figure 2.5. As a result of Lemma 3 and Corollary 4, and the definitions of the maps τ and σ, we get the following recursion, which holds throughout all of $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$, where m and k are unique and dependent on n and j:

$$p_{n,j} = (p_{m,k})(p_{n-m,j-k}) - (p_{n-2m,j-2k}). \tag{2.1}$$

Now we take a look at $\mathfrak{T}(a, x, (a-1)x)$. We use $q_{n,j} = q_{n,j}(x)$ to denote the polynomials in this tree. We define the $q_{n,j}$'s in a similar way as the $p_{n,j}$'s. That is to say, if the composite map μ maps $(x, x^2 - a, x^3 - (a+1)x)$ to $(p_{n_1,j_1}, p_{n_2,j_2}, p_{n_3,j_3})$ then μ maps $(a, x, (a-1)x)$ to $(q_{n_1,j_1}, q_{n_2,j_2}, q_{n_3,j_3})$. For example, $q_{1,0} = a, q_{2,1} = x, q_{5,2} = (a-1)x^2 - a$ and $q_{7,3} = (a-1)x^3 - (2a-1)x$. Then each $q_{n,j}$ occurs exactly once in $\mathfrak{T}(a, x, (a-1)x)$, and the following recursion

$$q_{n,j} = (q_{m,k})(q_{n-m,j-k}) - (q_{n-2m,j-2k}), \tag{2.2}$$

holds throughout this tree where m and k are unique and depend on n and j.

Since the degrees of $a, x,$ and $(a-1)x$ are 0, 1, and 1, respectively, $q_{n,j}$ is a

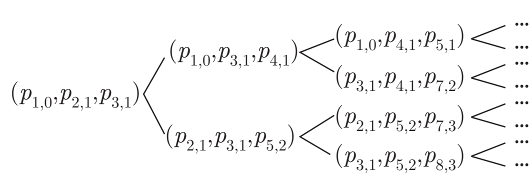

Figure 2.5: The tree $\mathfrak{T}(p_{1,0}, p_{2,1}, p_{3,1}) = \mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$.

polynomial of degree j, not degree n. Also, in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$, there are only a finite number of polynomials of degree n for each n, which are all monic, but in $\mathfrak{T}(a, x, (a-1)x)$, there are infinitely many polynomials of degree n for each n which, besides $q_{2,1} = x$, are all not monic when $a > 2$ (this will be shown later in the proof of Theorem 7). Since $q_{2,1} = x$ does not represent the maximum element of any triple in $\mathfrak{T}(a, x, (a-1)x)$, all of the polynomials that represent the maximum element of a triple in $\mathfrak{T}(a, x, (a-1)x)$ are not monic. We use the fact that the polynomials in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ are all monic and the maximal polynomials in $\mathfrak{T}(a, x, (a-1)x)$ are all not monic in Theorem 7.

Remark 2. There is an alternate way to define the parameters n and j for the Markoff tree $\mathfrak{T}(3, 3, 3)$ using Farey fractions, worth mentioning because Farey fractions were shown to give an ordering of some of the Markoff numbers (see [11] or [1]). For those who are familiar with the Farey fraction indexing of the Markoff numbers, we use the notation that $(1, 2, 5)$ corresponds to $(m_{\frac{0}{1}}, m_{\frac{1}{1}}, m_{\frac{1}{2}})$ [1]. Similarly, in the tree $\mathfrak{T}(3, 3, 3)$, the triple $(3, 6, 15)$ corresponds to $(3m_{\frac{0}{1}}, 3m_{\frac{1}{1}}, 3m_{\frac{1}{2}})$. The polynomials of $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$, when evaluated at $x = a = 3$, satisfy

$$p_{n,j} = 3m_{\frac{j}{n-j}}.$$

This can be easily verified by using induction and identity (2.1).

CHAPTER 3

CHEBYSHEV POLYNOMIALS

3.1 Markoff Numbers as Chebyshev Polynomials

Let us consider triples in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ of the form $\tau^{k-2}(x, x^2 - a, x^3 - (a+1)x) = (p_{1,0}, p_{k,1}, p_{k+1,1})$. The τ map implies that $p_{k+1,1}(x) = x \cdot p_{k,1}(x) - p_{k-1,1}(x)$ holds for all $k \geq 2$. This recursion is similar to the recursion for the Chebyshev polynomials.

Chebyshev polynomials are a sequence of orthogonal polynomials (we discuss orthogonality later) that were first studied by Chebyshev. They have many interesting properties and applications in various branches of mathematics, and Chebyshev polynomials are important in approximation theory because their roots are used as nodes in polynomial interpolation [21]. In the next two paragraphs, we list some important identities and facts about the Chebyshev polynomials that are used to establish identities (3.1) - (3.6) (see [21] for details).

The Chebyshev polynomials of the first and second kind, denoted $T_n = T_n(x)$ and $U_n = U_n(x)$, respectively, are defined recursively as follows:

$$T_{n+1}(x) = (2x)T_n(x) - T_{n-1}(x), \qquad (n \geq 1), \qquad \text{and}$$

$$T_0 = 1, T_1 = x,$$

for the first kind, and

$$U_{n+1}(x) = (2x)U_n(x) - U_{n-1}(x), \qquad (n \geq 1), \qquad \text{and}$$

$$U_0 = 1, U_1 = 2x,$$

for the second kind. The first few such polynomials with nonnegative indices are:

$$\begin{array}{ll} T_0 = 1 & U_0 = 1 \\ T_1 = x & U_1 = 2x \\ T_2 = 2x^2 - 1 & U_2 = 4x^2 - 1 \\ T_3 = 4x^3 - 3x & U_3 = 8x^3 - 4x \\ T_4 = 8x^4 - 8x^2 + 1 & U_4 = 16x^4 - 12x^2 + 1 \\ T_5 = 16x^5 - 20x^3 + 5x & U_5 = 32x^5 - 32x^3 + 6x. \end{array}$$

The Chebyshev polynomials of the first kind are orthogonal on the interval (-1, 1) with respect to the weight $\frac{1}{\sqrt{1-x^2}}$, and the inner product $\int_{-1}^{1} T_n(x)T_m(x)\,\frac{dx}{\sqrt{1-x^2}}$.

The Chebyshev polynomials of the second kind are orthogonal on [-1, 1] with respect to the weight $\sqrt{1-x^2}$, and the inner product $\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,dx$.

Remark 3. The following characterization shows a connection between Chebyshev polynomials of the second kind and twin primes (see [28] for details). The following statements are equivalent:

(i) n and $n+2$ are primes,

(ii) $U_n(\frac{x}{2})+1$ has exactly two irreducible factors, and

(iii) $U_n(\frac{x}{2})-1$ has exactly two irreducible factors.

The Chebyshev polynomials of the first kind satisfy the following for all x:

$$T_n(\cos x) = \cos(nx).$$

The Chebyshev polynomials of the second kind satisfy the following:

$$U_n(\cos x) = \frac{\sin((n+1)x)}{\sin(x)}.$$

Noting that $\cos(-nx) = \cos(nx)$, we are motivated to define $T_{-n} = T_n$. Similarly, we are motivated to define $U_{-n} = -U_{n-2}$. The following identities are derived from the product to sum formulas for cosine and sine:

$$\begin{aligned} T_jT_k &= \frac{1}{2}(T_{j+k} + T_{j-k}) \\ T_jU_k &= \frac{1}{2}(U_{k+j} + U_{k-j}) \\ U_jU_k &= \frac{T_{j+k+2} - T_{j-k}}{2(T_1^2-1)}. \end{aligned}$$

Now, notice that when we fix $a=1$, we get the following relationship between

$p_{n,j}$ and the Chebyshev polynomials of the second kind:

$$p_{1,0}(2x) = 2x = U_1(x), \quad \text{and}$$

$$p_{2,1}(2x) = 4x^2 - 1 = U_2(x).$$

We can prove $p_{k,1}(2x) = U_k(x)$, for all $k \geq 2$ by induction using identity (2.1). Also, notice that when we fix $a = 2$, we get the following relationship between $p_{n,j}$ and the Chebyshev polynomials of the first kind:

$$p_{1,0}(2x) = 2x = 2T_1(x), \quad \text{and}$$

$$p_{2,1}(2x) = 4x^2 - 2 = 2T_2(x).$$

We can prove $p_{k,1}(2x) = 2T_k(x)$, for all $k \geq 2$ by induction using identity (2.1) as well.

Next, we show that the $p_{n,j}$'s with an arbitrary value of a can always be written in terms of the $p_{n,j}$'s that use the specific values of $a = 1$ and $a = 2$. To help avoid confusion with the parameter a, when $a = 1$ we define

$$u_1 = p_{1,0}, \quad \text{and}$$

$$u_k = p_{k,1}, \qquad (k \geq 2),$$

and when $a = 2$, we define

$$t_1 = p_{1,0}, \quad \text{and}$$

$$t_k = p_{k,1}, \qquad (k \geq 2).$$

We use the notation t_n and u_n for the polynomials at the fixed values of $a = 2$ and $a = 1$, respectively, because of their direct connection to the Chebyshev polynomials T_n and U_n. Hence, we have the following identities (which hold for all j, k):

$$t_j t_k = t_{j+k} + t_{j-k}, \tag{3.1}$$

$$t_j u_k = u_{k+j} + u_{k-j}, \tag{3.2}$$

$$u_j u_k = \frac{t_{j+k+2} - t_{j-k}}{t_1^2 - 4}, \tag{3.3}$$

$$t_{k+1} = t_1 t_k - t_{k-1}, \tag{3.4}$$

and

$$u_{k+1} = u_1 u_k - u_{k-1}. \tag{3.5}$$

Notice that the first three identities (3.1, 3.2, 3.3) come from the Chebyshev polynomial identities derived from the product to sum formulas of sine and cosine, and the last two identities come from identities (2.1) and (2.2). Let us consider $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ for any real a. We now show that $p_{n,1}$ with the general

a can be represented in terms of $p_{n,1}$ with a fixed as 1 or 2. Observe that

$$\begin{aligned}
p_{1,0} &= x \\
&= t_1 - (a-2)u_{-1} \qquad (\text{ since } u_{-1} = 0), \\
p_{2,1} &= x^2 - a \\
&= t_2 - (a-2)u_0, \\
p_{3,1} &= x^3 - (a+1)x = t_3 - (a-2)u_1.
\end{aligned}$$

We show that $p_{n,1} = t_n - (a-2)u_{n-2}$, for all n. Assume by induction that we have shown that $p_{n,1} = t_n - (a-2)u_{n-2}$ holds for all n up to k. Then

$$\begin{aligned}
p_{k+1,1} &= (p_{k,1})(p_{1,0}) - p_{k-1,1} \qquad (\text{by } (2.1)) \\
&= (t_k - (a-2)u_{k-2})(t_1) - (t_{k-1} - (a-2)u_{k-3}) \\
&\qquad (\text{by induction hypothesis}) \\
&= (t_1 t_k - t_{k-1}) - (a-2)(u_1 u_{k-2} - u_{k-3}) \qquad (\text{since } t_1 = u_1) \\
&= t_{k+1} - (a-2)u_{k-1} \qquad (\text{by } (3.4) \text{ and } (3.5)).
\end{aligned}$$

Therefore, by mathematical induction, we have obtained the following result for $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$:

$$p_{n,1} = t_n - (a-2)u_{n-2}, \qquad n \geq 2. \tag{3.6}$$

Hence, for every real number a, we can represent all the polynomials as linear combinations of Chebyshev polynomials. In a similar way, we can show by induction that the polynomials in $\mathfrak{T}(a, x, (a-1)x)$ satisfy

$$q_{n,1} = (u_{n-2}(a) - u_{n-3}(a))x = (u_{n-2}(a) - u_{n-3}(a))t_1(x).$$

We emphasize here that the t_n's and u_n's are polynomials of the variable x, and that the expression $u_{n-2}(a) - u_{n-3}(a)$ is just the coefficient of $t_1 = x$. For example,

$$\begin{aligned} q_{5,1}(x) &= (a^3 - a^2 - 2a + 1)x \\ &= (u_3(a) - u_2(a))x \\ &= (u_3(a) - u_2(a))t_1(x), \end{aligned}$$

and it is still a polynomial of degree 1. Therefore, we can represent all the polynomials of $\mathfrak{T}(a, x, (a-1)x)$ as linear combinations of Chebyshev polynomials as well.

Representing the polymonials of $\mathfrak{T}(a, x, x)$ as linear combinations of Chebyshev polynomials makes it easier to show that all of these polynomials are distinct and uniquely determine its triple in the tree of solutions (we show this in chapter 4). In the next section, we look at M_4, which Zagier showed fails the unicity condition as part of his analysis of calculating the number of Markoff numbers below a certain bound (M_4 appears as equation 13 of [29]).

3.2 The Equation M_4

In this section, we use the representation of Markoff numbers as Chebyshev polynomials in the case when $a = 2$. Note that for every $x, \mathfrak{T}(2, x, x)$ is a tree of solutions for the Markoff equation M_4. This can be easily seen by the following:

$$4 + x^2 + x^2 = 2 \cdot x \cdot x + 4.$$

We now show that each polynomial tree associated with M_4 of the form $\mathfrak{T}(2, x, x)$ does not satisfy the unicity condition.

Lemma 5. *The equation M_4 fails the unicity condition infinitely many times.*

Proof. By observation, we see that $(2, x, x)$ is a solution for any real x. When $x = 2$, the entire tree collapses to the only solution of $(2, 2, 2)$. For $x > 2$, the root $(2, x, x)$ moves in σ direction to $(x, x, x^2 - 2)$ which moves to $\mathfrak{T}(t_1, t_2, t_3)$. From before when $a = 2$, the polynomials of $\mathfrak{T}(t_1, t_2, t_3)$ satisfy $p_{n,1} = t_n$, for all $n \geq 1$. Then

$$\begin{aligned}
p_{n,2} &= p_{2k+1,2} \\
&= p_{k,1}p_{k+1,1} - p_{1,0} \qquad \text{(by (2.1))} \\
&= t_k t_{k+1} - t_1 \\
&= t_{2k+1} \qquad \text{(by (3.4))}.
\end{aligned}$$

Assume $p_{n,j} = t_n$, holds for all n and for all $j < J$. Then, for any n, there exists unique integer pair (m, j) with $j < J$ such that

$$
\begin{aligned}
p_{n,J} &= p_{m,j}p_{n-m,J-j} - p_{n-2m,J-2j} \qquad \text{(by (2.1))} \\
&= t_m t_{n-m} - t_{n-2m} \qquad \text{(by the induction hypothesis)} \\
&= t_n. \qquad \text{(by (3.4))}
\end{aligned}
$$

Since n was arbitrary, $p_{n,j} = t_n$, for all n and j. Hence, the $\frac{1}{2}\varphi(n)$ polynomials of degree n that appear (in each tree) are the same, and the result follows. □

Remark 4. Since M_4 has (t_j, t_k, t_{j+k}) as a solution for all real x and any relatively prime integers j, k, we obtain the following identity:

$$T_j(x)^2 + T_k(x)^2 + T_{j+k}(x)^2 = 2T_j(x)T_k(x)T_{j+k}(x) + 1.$$

Since each $t_n(2\cos\theta) = 2\cos(n\theta)$, we also obtain the following trigonometric identity:

$$\cos(j\theta)^2 + \cos(k\theta)^2 + \cos((j+k)\theta)^2 = 2\cos(j\theta)\cos(k\theta)\cos((j+k)\theta) + 1,$$

where j, k are relatively prime positive integers. This result seems to be known, and it is easy to verify that this trigonometric identity holds for all real j, k by using the sum addition formula for cosine and the Pythagorean identity. Also, Riedel mentions that M_4 has (t_j, t_k, t_{j+k}) as a solution for all real x on page 9 of [20].

It is known that $U_n = 2(T_n + T_{n-2} + ... + T_1)$ when n is odd, and $U_n = 2(T_n + T_{n-2} + ... + T_2) + 1$ when n is even (see page 9 of [20]). Hence, $u_n = t_n + t_{n-2} + ... + t_1$ when n is odd, and $u_n = t_n + t_{n-2} + ... + 1$ when n is even. Thus, identity (3.6) can be written as

$$p_{n,1} = t_n - (a-2)(t_{n-2} + t_{n-4} + ...). \tag{3.7}$$

Identity (3.7) is used in the proof of Theorem 7, which is the main technical result of this paper.

3.3 Extra Identity of Chebyshev Polynomials

This section is optional for the reader. It contains an identity that the author finds interesting, but is not used in the rest of the text. Let $x = s + s^{-1}$. It is already known from previous work on Chebyshev polynomials that

$$T_n(1) = 1, \text{and } U_n(1) = n + 1,$$

holds for all $n \geq 0$. Recall that this corresponds with $t_n(2) = 2, u_n(2) = n + 1$, for all $n \geq 0$, since $t_n(2x) = 2T_n(x)$ and $u_n(2x) = U_n(x)$. We find expressions for arbitrary values of s, hence, x, using the following lemma.

Lemma 6. *For all $n \geq 0$ and for all $s > 0, t_n(s + s^{-1}) = s^n + s^{-n}$,*
$u_n(s + s^{-1}) = \frac{s^{n+1} - s^{-n-1}}{s - s^{-1}}$.

Proof. Pick any $s > 0$ and let $g_n(s+s^{-1}) = s^n + s^{-n}$ and $h_n(s+s^{-1}) = \frac{s^{n+1}-s^{-n-1}}{s-s^{-1}}$, for all n. Note that

$$\begin{aligned}
g_0(s+s^{-1}) &= s^0 + s^0 = 2, \\
g_1(s+s^{-1}) &= s^1 + s^{-1} = s + s^{-1}, \\
h_0(s+s^{-1}) &= \frac{s^1 - s^{-1}}{s - s^{-1}} = 1, \text{and} \\
h_1(s+s^{-1}) &= \frac{s^2 - s^{-2}}{s - s^{-1}} \\
&= \frac{(s - s^{-1})(s + s^{-1})}{s - s^{-1}} \\
&= s + s^{-1}.
\end{aligned}$$

Thus, $g_0(x) = 2, g_1(x) = x, h_0(x) = 1$, and $h_1(x) = x$ hold for $x = s + s^{-1}$. To prove $g_n = t_n$ and $h_n = u_n$ holds for all $n \geq 0$, it suffices to show that $g_{n+1}(x) = xg_n(x) - g_{n-1}(x)$ and $h_{n+1}(x) = xh_n(x) - h_{n-1}(x)$. Then

$$\begin{aligned}
(s+s^{-1})g_n(s+s^{-1}) - g_{n-1}(s+s^{-1}) &= (s+s^{-1})(s^n + s^{-n}) - (s^{n-1} + s^{-(n-1)}) \\
&= s^{n+1} + s^{-n+1} + s^{n-1} + s^{-n-1} - s^{n-1} - s^{-n+1} \\
&= s^{n+1} + s^{-(n+1)} = g_{n+1}(s+s^{-1}), \qquad \text{and} \\
(s+s^{-1})h_n(s+s^{-1}) - h_{n-1}(s+s^{-1}) &= (s+s^{-1})(\frac{s^{n+1} - s^{-n-1}}{s - s^{-1}}) - (\frac{s^n - s^{-n}}{s - s^{-1}}) \\
&= \frac{s^{n+2} - s^{-n} + s^n - s^{-n-2}}{s - s^{-1}} - \frac{s^n - s^{-n}}{s - s^{-1}} \\
&= \frac{s^{n+2} - s^{-n-2}}{s - s^{-1}} = h_{n+1}(s+s^{-1}).
\end{aligned}$$

Thus, $g_n = t_n$ and $h_n = u_n$, for all $n \geq 0$, as desired. $\square$

Here are a few examples:

$$
\begin{aligned}
t_n(\frac{5}{2}) &= \frac{4^n + 1}{2^n}, \\
u_n(\frac{5}{2}) &= \frac{4^{n+1} - 1}{3 \cdot 2^n}, \\
t_n(\frac{10}{3}) &= \frac{9^n + 1}{3^n}, \\
u_n(\frac{10}{3}) &= \frac{9^{n+1} - 1}{8 \cdot 3^n}, \\
t_n(\frac{\pi^2 + 1}{\pi}) &= \frac{\pi^{2n} + 1}{\pi^n}, \\
u_n(\frac{\pi^2 + 1}{\pi}) &= \frac{\pi^{2n+2} - 1}{\pi^{n+2} - \pi^n}.
\end{aligned}
$$

CHAPTER 4

THE UNICITY CONDITION

4.1 Distinct Polynomials

Using the results of chapters 2 and 3, we now show that all of the polynomials in $\mathfrak{T}(a, x, x)$ are distinct, and hence, uniquely determine the triple they appear in.

Theorem 7. *All entries in the polynomial tree $\mathfrak{T}(a, x, x)$ are distinct when $a > 2$. More specifically, any two polynomials of degree n from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ differ by a polynomial whose degree is exactly $n-2$; any two polynomials of degree n from $\mathfrak{T}(a, x, (a-1)x)$ differ by a polynomial whose degree is exactly n; and any two polynomials of degree n with one from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ and one from $\mathfrak{T}(a, x, (a-1)x)$ differ by a polynomial whose degree is exactly n.*

Proof. When $a > 2$, we do not get trees associated with $A = 4$, which does not have distinct polynomials by Lemma 5. For simplicity, let $\alpha = a - 2$. It is clear that polynomials of different degrees are different, so we only need to show that polynomials of the same degree are different.

First, we look at $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$. Notice that

$$p_{2k+1,2} = p_{k,1}p_{k+1,1} - p_{1,0} \qquad \text{(by (2.1))}$$

$$= (t_k - \alpha t_{k-2} - \langle * \rangle)(t_{k+1} - \alpha t_{k-1} - \langle * \rangle) - t_1$$

$$\text{(by identity (3.7))}$$

$$= t_k t_{k+1} - \alpha(t_k t_{k-1} + t_{k+1}t_{k-2}) - \langle * \rangle$$

$$= t_{2k+1} - 2\alpha t_{2k-1} - \langle * \rangle,$$

where $\langle * \rangle$ denotes lower degree terms. Hence, $p_{n,j} = t_n - j\alpha t_{n-2} - \langle * \rangle$, holds for all n, and for $j = 1, 2$. Assume $p_{n,j} = t_n - j\alpha t_{n-2} - \langle * \rangle$, holds for all n, and for all $j < J$. Then, for any n, there exists a unique pair of integers (m, j) (uniqeness guaranteed by Lemma 3) with $j < J$ such that

$$p_{n,J} = p_{m,j}p_{n-m,J-j} - p_{n-2m,J-2j} \qquad \text{(by (2.1))}$$

$$= (t_m - j\alpha t_{m-2} - \langle * \rangle)(t_{n-m} - (J-j)\alpha t_{n-m-2} - \langle * \rangle) -$$

$$(t_{n-2m} - (J-2j)\alpha t_{n-2m-2} - \langle * \rangle) \qquad \text{(by the induction hypothesis)}$$

$$= t_m t_{n-m} - \alpha(j t_{m-2} t_{n-m} + (J-j)t_{n-m-2}t_m) - \langle * \rangle$$

$$= t_n - J\alpha t_{n-2} - \langle * \rangle.$$

Since n was arbitrary, we obtain the following result:

$$p_{n,j} = t_n - j\alpha t_{n-2} - \langle * \rangle,$$

which holds for all n and j. Therefore, for any $j_1 \neq j_2$,

$p_{n,j_1} - p_{n,j_2} = (j_1 - j_2)\alpha t_{n-2} - \langle * \rangle$, which is a polynomial of degree $n-2$. In particular, they are not equal.

Next, we look at $\mathfrak{T}(a, x, (a-1)x)$. Recall that $q_{n,j}$ is a polynomial of degree j not n. We know that $q_{n,1} = (u_{n-2}(a) - u_{n-3}(a))x$. Theorem 1 guarantees that $n_1 > n_2$ implies $q_{n_1,1} > q_{n_2,1}$, as long as $a > 2$. Let us refer to the leading coefficient of $q_{n,1}$ as C_n ($C_2 = 1, C_3 = a-1$, etc.). Since $q_{2k+1,2} = q_{k,1}q_{k+1,1} - q_{1,0}$ by identity (2.2), it follows that the leading term of $q_{2k+1,2}$ is $(C_k C_{k+1})x^2$. More generally, it can be established that the leading term of $q_{n,j}$ is $(C_k^{j-r} C_{k+1}^r)x^j$, where $n = kj + r$, with $0 < r < j$.

We already have mentioned that $n_1 > n_2$ implies that the leading coefficient of $q_{n_1,1}$ is bigger than the leading coefficient of $q_{n_2,1}$. Pick any $j > 1$. Let $n_1 > n_2$. We show that the leading coefficient of $q_{n_1,j}$ is bigger than the leading coefficient of $q_{n_2,j}$. By applying Euclidean division, there exists integers k_1, k_2, r_1, and r_2 such that $n_1 = jk_1 + r_1$ and $n_2 = jk_2 + r_2$.

Case 1: $k_1 = k_2, r_1 > r_2$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-r_1} C_{k_1+1}^{r_1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_1}^{j-r_2} C_{k_1+1}^{r_2}$. Since

$$C_{k_1}^{j-r_1} C_{k_1+1}^{r_1} = \left(\frac{C_{k_1+1}}{C_{k_1}} \right)^{r_1 - r_2} C_{k_1}^{j-r_2} C_{k_1+1}^{r_2}$$

and

$$\left(\frac{C_{k_1+1}}{C_{k_1}} \right)^{r_1 - r_2} > 1,$$

the result follows for this case.

Case 2: $k_1 > k_2, r_1 \geq r_2$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-r_1} C_{k_1+1}^{r_1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_2}^{j-r_2} C_{k_2+1}^{r_2}$. Since $k_1 > k_2$ implies $C_{k_1} > C_{k_2}$, we get:

$$\begin{aligned} C_{k_1}^{j-r_1} C_{k_1+1}^{r_1} &> C_{k_2}^{j-r_1} C_{k_2+1}^{r_1} \\ &\geq C_{k_2}^{j-r_2} C_{k_2+1}^{r_2} \qquad \text{(by case 1)} \end{aligned}$$

Thus, the result follows in this case.

Case 3: $k_1 - k_2 = 1, r_1 = 1, r_2 = j - 1$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-1} C_{k_1+1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_1-1} C_{k_1}^{j-1}$. Since $C_{k_1+1} > C_{k_1-1}$, the result follows in this case.

Case 4: $k_1 > k_2, r_1 < r_2$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-r_1} C_{k_1+1}^{r_1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_2}^{j-r_2} C_{k_2+1}^{r_2}$. Then

$$\begin{aligned} C_{k_1}^{j-r_1} C_{k_1+1}^{r_1} &\geq C_{k_2+1}^{j-r_1} C_{k_2+2}^{r_1} \qquad \text{(by case 2, since } k_1 - k_2 \geq 1) \\ &\geq C_{k_2+1}^{j-1} C_{k_2+2} \qquad \text{(by case 1, since } r_1 \geq 1) \\ &> C_{k_2} C_{k_2+1}^{j-1} \qquad \text{(by case 3)} \\ &\geq C_{k_2}^{j-r_2} C_{k_2+1}^{r_2} \qquad \text{(by case 1, since } j - 1 \geq r_2). \end{aligned}$$

Thus, the result follows in this case. These four cases represent all possibilities.

Hence, all polynomials are distinct in $\mathfrak{T}(a, x, (a-1)x)$. Furthermore, when two polynomials have the same degree n, their leading coefficients are different, and therefore, their difference is a polynomial of degree n.

We now know that the maximal polynomials are monic polynomials in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ but not in $\mathfrak{T}(a, x, (a-1)x)$ (as long as $a > 2$), so it is clear that any two polynomials of degree n with one from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ and the other from $\mathfrak{T}(a, x, (a-1)x)$ are distinct and have a difference that is exactly degree n. $\square$

The previous two results show that when $a = 2$, the polynomials of the same degree are all the same, but when $a > 2$, all the polynomials are different (we also know the exact degree when we subtract any two polynomials of the same degree). However, this does not imply that in the latter case, the polynomials are all distinct at a specific value, which is what the unicity condition implies. However, it suggests that there is something special about M_4 and why it should fail the unicity condition more significantly than any other parameters.

4.2 Trees Satisfying the Unicity Condition up to Level N

Suppose for fixed a that there exist two different polynomials in $\mathfrak{T}(a, c, c)$, say p_{n_1,j_1} and p_{n_2,j_2}, such that $p_{n_1,j_1}(c) = p_{n_2,j_2}(c) = m$, for some real numbers m and c with $m \geq c \geq a$. This is equivalent to the tree $\mathfrak{T}(a, c, c)$ failing the unicity condition, because m is no longer unique as a maximum element in $\mathfrak{T}(a, c, c)$ (note that we are using the same a, c, and m as in the previous sentence). More generally, the statement "(a_i, b_i, m) is an ordered triple solution of $\mathfrak{T}(a, c, c)$ for $i = 1, 2$ implies $a_1 = a_2, b_1 = b_2$" is equivalent to the statement "$(p_{n_1,j_1} - p_{n_2,j_2})(c) \neq 0, (p_{n_1,j_1} - q_{n_1,j_1})(c) \neq 0, (p_{n_1,j_1} - q_{n_2,j_2})(c) \neq 0, (p_{n_2,j_2} - q_{n_1,j_1})(c) \neq 0, (p_{n_2,j_2} - q_{n_2,j_2})(c) \neq 0$, and $(q_{n_1,j_1} - q_{n_2,j_2})(c) \neq 0$ in $\mathfrak{T}(a, c, c)$, for all n_1, n_2, j_1, j_2". For the sake of brevity, this last statement can be phrased as "$(\lambda_1 - \lambda_2)(c) \neq 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all n_1, j_1, n_2, j_2. Also, when we say "for all n_1, n_2, j_1, j_2", we mean for all n_1, n_2, j_1, j_2 such that $p_{n_1,j_1}, q_{n_1,j_1}, p_{n_2,j_2}$ and q_{n_2,j_2} represent distinct spots in $\mathfrak{T}(a, c, c)$. Thus, we have the following definition.

Definition 4. When $(\lambda_1 - \lambda_2)(c) \neq 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all n_1, j_1, n_2, j_2, we say $\mathfrak{T}(a, c, c)$ satisfies the unicity condition, and when $(\lambda_1 - \lambda_2)(c) \neq 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all $n_1, j_1, n_2, j_2 \leq N$, we say $\mathfrak{T}(a, c, c)$ satisfies the unicity condition up to level N.

In the following proof, we start with (a, x, x) with a being constant but x being a variable to create the polynomials $p_{n,j}$ and $q_{n,j}$, which are functions of x instead of numbers associated with a specific M_A. Then we find rational numbers q that give us the $\mathfrak{T}(a, q, q)$'s satisfying the unicity condition up to level N, and then we find the parameters associated with $\mathfrak{T}(a, q, q)$; i.e., we want $A = a^2 + 2q^2 - aq^2$. Notice that if a is rational, then the maps τ and σ always produce polynomials with rational coefficients in $\mathfrak{T}(a, x, x)$, because the coefficients are just products and differences of rational numbers. Hence, if a is rational, then $\mathfrak{T}(a, c, c)$ contains polynomials (of c now) with rational coefficients, even if A and c are irrational.

This fact is used after Theorem 11.

Theorem 8. *Given any rational number $a > 2$, any positive integer N, and any subset X of $[a, \infty)$ that contains infinitely many rational numbers, there exists infinitely many rational numbers $q \in X$ such that $\mathfrak{T}(a, q, q)$ satisfies the unicity condition up to level N.*

Proof. For any $a > 2$, the polynomials are all distinct by Theorem 7. Let $F_{n_1,n_2,j_1,j_2} = \{x \in X : (\lambda_1 - \lambda_2)(c) = 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all $n_1, j_1, n_2, j_2\}$. Since p_{n_1,j_1} and p_{n_2,j_2} are distinct polynomials, their difference is a nonzero polynomial of degree at most the maximum of n_1 and n_2. Hence, $p_{n_1,j_1} = p_{n_2,j_2}$ for at most $\max\{n_1, n_2\}$ real numbers. Using a similar argument for the other five polynomial differences, the set F_{n_1,n_2,j_1,j_2} contains only a finite number of points in X. Since $j_1 < \frac{n_1}{2} \leq \frac{N}{2}$ and $j_2 < \frac{n_2}{2} \leq \frac{N}{2}$, the collection of F_{n_1,n_2,j_1,j_2} for all positive integers n_1, n_2, j_1, j_2 with $n_1, n_2 \leq N$ is finite. Therefore, the union of all F_{n_1,n_2,j_1,j_2} must be a finite subset in X. Thus, $\mathfrak{T}(a, q, q)$ satisfies the unicity condition up to level N, for infinitely many rational $q \in X$, as desired. □

Remark 5. Let us fix any pair of rational numbers (a, c) with $2 < a \leq c$. As a consequence of Theorem 8 (using $X = [c, c + \epsilon]$), given any $\epsilon > 0$, we now know that there are infinitely many rational numbers q in the interval $(c, c+\epsilon)$ such that $\mathfrak{T}(a, q, q)$ satisfies the unicity condition up to level N, for any large N. Specifically, we obtain the following corollary:

Corollary 9. *Given any arbitrarily large N and any small $\epsilon > 0$ (each independent* *of each other), there exists infinitely many rational numbers $q \in [3, 3+\epsilon]$ such that $\mathfrak{T}(3, q, q)$ satisfies the unicity condition up to level N.*

We can use a diagonalization argument from this last corollary to obtain the following result:

Corollary 10. *There exists a sequence of rational numbers $\{q_n\}$ with each $q_n \geq 3$ such that for each n, $\mathfrak{T}(3, q_n, q_n)$ satisfies the unicity condition up to level n, and $q_n \to 3$ as $n \to \infty$.*

Proof. Pick any small $\epsilon > 0$. By previous corollary, there exists infinitely many rational numbers in the interval $(3, 3+\epsilon]$, call them $q_{(1,1)}, q_{(1,2)}, q_{(1,3)}$, ..., such that $\mathfrak{T}(3, q_{(1,k)}, q_{(1,k)})$ satisfies the unicity condition up to level 1, for each k. Next, we apply the previous corollary to obtain infinitely many rational numbers in the interval $(3, q_{(1,1)}]$, call them $q_{(2,1)}, q_{(2,2)}, q_{(2,3)}$, ..., such that $\mathfrak{T}(3, q_{(2,k)}, q_{(2,k)})$ satisfies the unicity condition up to level 2, for each k. Inductively, we apply the previous corollary to obtain infinitely many rational numbers in the interval $(3, q_{(n,n)}]$, call them $q_{(n+1,1)}, q_{(n+1,2)}, q_{(n+1,3)}$, ..., such that $\mathfrak{T}(3, q_{(n+1,k)}, q_{(n+1,k)})$ satisfies the unicity condition up to level $n+1$, for each k. Then we take the diagonal elements $q_n = q_{(n,n)}$, and form the desired sequence. $\square$

Remark 6. Since all of the zeros of a finite collection of polynomials are clearly bounded, for every N there exists x_N such that for every rational $a > 2, \mathfrak{T}(a, c, c)$ satisfies the unicity condition up to level N for every rational number $c > x_N$. If for some a, all of the zeros of all the differences of any two polynomials are uniformly bounded, say by M, then $\mathfrak{T}(a,c,c)$ satisfies the unicity condition for all $c > M$.

Remark 7. The argument in Theorem 8 cannot work for $a = 2$ because all of the polynomials of the same degree are the same in M_4. Hence, every real number satisfies $(p_{n,j_1} - p_{n,j_2})(x) = 0$, so $F_{n,n,j_1,j_2} = X$, instead of a finite number of points in X. This once again shows that M_4 is special.

Theorem 8 can be expanded to trees $\mathfrak{T}(a,c,c)$ for any real a and any real c, as we show in the next section.

4.3 Trees Satisfying the Unicity Condition

Theorem 11. *Given any real number $a > 2$, the set of all real numbers $x \geq a$ such that $\mathfrak{T}(a,x,x)$ satisfies the unicity condition is the complement of a countable set. In particular, the set of all real numbers $x \geq a$ such that $\mathfrak{T}(a,x,x)$ satisfies the unicity condition is uncountable and dense in $[a,\infty)$.*

Proof. Just like in Theorem 8, when $a \neq 2$, the polynomials are all distinct. Therefore, the sets F_{n_1,n_2,j_1,j_2} (which were defined in Theorem 8) only contain a finite number of points. When we consider the union of all the F_{n_1,n_2,j_1,j_2}'s, we get a countable union of finite sets. Therefore, the complement of this union, which is the set of all real numbers $x \geq a$ such that $\mathfrak{T}(a,x,x)$ satisfies the unicity condition, is the complement of a countable set, and in particular, is uncountable and dense in $[a,\infty)$, as desired. □

Theorem 11 shows that for the ordered pairs (a, c) with $2 < a \le c$, $\mathfrak{T}(a, c, c)$ satisfies the unicity condition for almost every c, i.e., for all but a set of measure zero (in Theorem 1.3 of [18], they use the phrase "Baire dense"). However, the existence of a tree $\mathfrak{T}(a, c, c)$ with a and c both rational or algebraic is still an open problem. When a is rational, all of the polynomials in $\mathfrak{T}(a, c, c)$ have rational coefficients. Hence, as a consequence of Theorem 11, $\mathfrak{T}(3, \omega, \omega)$ satisfies the unicity condition for every transcendental number ω (note the similarity between this last statement and the last sentence of [18]). For example, in $M_{9-\pi^2}$, $\mathfrak{T}(3, \pi, \pi)$ satisfies the unicity condition since π is transcendental. Since transcendental numbers are dense among the real numbers, we now know that there exists a sequence of real numbers $\{c_n\}$ such that $c_n \to 3$ as $n \to \infty$ and $\mathfrak{T}(3, c_n, c_n)$ satisfies the unicity condition for all n. The concept of $c_n \to 3$ as $n \to \infty$ is related to the concept of the sequence of trees $\mathfrak{T}(3, c_n, c_n)$ converging to $\mathfrak{T}(3, 3, 3)$, which we define in Section 5.3. More generally, we obtain the following theorem with our main result as a consequence:

Theorem 12. *For any pair of rational numbers (a, c) with $2 < a \le c$, there exists a sequence of real numbers $\{c_n\}$ such that the sequence of trees $\mathfrak{T}(a, c_n, c_n)$ converge to $\mathfrak{T}(a, c, c)$, and $\mathfrak{T}(a, c_n, c_n)$ satisfies the unicity condition for every n.*

Note that Theorem 12 does not mention whether $\mathfrak{T}(a, c, c)$ satisfies the unicity condition or not. Hence, Theorem 12 even holds for pairs of rational numbers (a, c) such that $\mathfrak{T}(a, c, c)$ fails the unicity condition.

CHAPTER 5

POLYNOMIAL METHODS

5.1 Rational Root Theorem

In this section, we show new methods involving polynomials for analyzing the unicity condition. For now, let us consider the tree $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$. The next result shows that there are several cases where the polynomials are distinct from each other at a specific value. The proof of the following theorem only uses a simple induction argument and the rational root theorem.

Theorem 13. *Let a be an integer.*

a) Pick any rational number c such that c does not divide $2a$ or c is not an integer. If n and m are neither both odd, both congruent to 0 mod 4, nor both congruent to 2 mod 4, then $p_{n,j_1}(c) \neq p_{m,j_2}(c)$.

b) Let a, c both be odd integers. If exactly one of $\{n + j_1, m + j_2\}$ is divisible by 3, then $p_{n,j_1}(c) \neq p_{m,j_2}(c)$.

c) Let a be even and c be odd integer. If exactly one of $\{n, m\}$ is divisible by 3, then $p_{n,j_1}(c) \neq p_{m,j_2}(c)$.

Proof. a) It is easy to see that $p_{n,1}(0) = 0$ when n is odd, and $p_{n,1}(0) = (-1)^{\frac{n}{2}}a$ when n is even. Inductively, using $p_{n,j} = p_{m,k}p_{n-m,j-k} - p_{n-2m,j-2k}$, we can also show that for all $j, p_{n,j}(0) = 0$ when n is odd, and $p_{n,j}(0) = (-1)^{\frac{n}{2}}a$ when n is even (the reader may notice that the induction fails if m and $n-m$ are both even, but this cannot happen since n, m, and $n-m$ all have to be relatively prime). If n and m are neither both odd, both congruent to 0 mod 4, nor both congruent to 2 mod 4, then, $(p_{n,j_1} - p_{m,j_2})(0) = \pm a$ [or $2a$]. Since $p_{n,j_1} - p_{m,j_2}$ is a monic polynomial with integer coefficients, by the rational root theorem, divisors of $2a$ are the only possible rational numbers that might be zeros. Hence, if c is not an integer or does not divide $2a$ then $p_{n,j_1}(c) \neq p_{m,j_2}(c)$.

The proofs for parts b) and c) are similar, so they will just be outlined here. We first substitute $y = x+2-c$. With this substitution, we go from $a, x, x^2-a, ...$ to $a, y+c-2, y^2-2(c-2)y+((c-2)^2-a), ...$, but we will still be using the notation $p_{n,j}$ to describe the corresponding polynomials of variable y. Note that $x = c$ if and only if $y = 2$. Next, we conclude (using $p_{n,j} = p_{m,k}p_{n-m,j-k} - p_{n-2m,j-2k}$) that if a is odd and c is odd then $p_{n,j}(0)$ is even if and only if $n+j$ is divisible by 3, and if a is even and c is odd then $p_{n,j}(0)$ is even if and only if n is divisible by 3. Hence, by rational root theorem, the result follows. $\square$

This approach can be generalized, but it is limited in the fact that it only shows parts of the trees satisfying the unicity condition instead of the entire tree satisfying the unicity condition. Another approach is demonstrated in the next section.

5.2 Upper and Lower Bounds

In this section, we obtain upper and lower bounds for the polynomials $p_{n,j}$'s in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$. Once we establish these bounds, we show that $p_{n,j} < p_{n,1}$ when $j > 1$. For simplicity, we let $p_n = p_{n,1}$. Next, we prove the following lemma.

Lemma 14. *For all $y \geq x$, $p_x \cdot p_y - p_{y-x} = p_{y+x} - (a-2)u_{x-2}p_y$.*

Proof. Let $\alpha = a - 2$ for simplicity. Recall that $p_n = t_n - \alpha u_{n-2}$, for all n. Then

$$\begin{aligned}
p_x p_y - p_{y-x} &= (t_x - \alpha u_{x-2})(t_y - \alpha u_{y-2}) - (t_{y-x} - \alpha u_{y-x-2}) \\
&= t_x t_y - \alpha t_x u_{y-2} - \alpha u_{x-2}(t_y - \alpha u_{y-2}) - t_{y-x} + \alpha u_{y-x-2} \\
&= t_{y+x} - \alpha(u_{y+x-2} + u_{y-x-2}) - \alpha u_{x-2} p_y + \alpha u_{y-x-2} \text{ (by (1), (2))} \\
&= (t_{y+x} - \alpha u_{y+x-2}) - \alpha u_{x-2} p_y \\
&= p_{y+x} - \alpha u_{x-2} p_y.
\end{aligned}$$

□

When $a \geq 3$ and $x \geq 2$, then $(a-2)u_{x-2} \geq 1$. Hence, for $2 \leq x \leq y$, we get $p_x \cdot p_y < p_{y+x}$. We now establish upper and lower bounds for the polynomials $p_{n,j}$.

Lemma 15. *For all n and j (where $n = jk + r$),*

$$p_k^{j-r} \cdot p_{k+1}^r - (j-1)p_1^r \cdot p_k^{j-2} \leq p_{jk+r,j} \leq p_k^{j-r} \cdot p_{k+1}^r,$$

with the inequalities being strict for $j \geq 3$.

Proof. Result is clear for $j = 1$ (both inequalities cannot be strict here), and since $p_{2k+1,2} = p_k \cdot p_{k+1} - p_1$, result holds for $j = 2$, with right side inequality being strict. Since

$$\begin{aligned} p_{3k+1,3} &= p_k \cdot p_{2k+1,2} - p_{k+1} \\ &= p_k^2 \cdot p_{k+1} - (p_1 \cdot p_k + p_{k+1}), \end{aligned}$$

and since

$$p_1 \cdot p_k > p_{k+1},$$

the result holds for $n = 3k + 1$. Similarly,

$$p_{3k+2,3} = p_k \cdot p_{k+1}^2 - (p_1 \cdot p_{k+1} + p_k)$$

and

$$p_1 \cdot p_{k+1} < p_1^2 \cdot p_k \text{ and } p_k < p_1^2 \cdot p_k$$

implies that the result holds for $n = 3k + 2$. Therefore, the result holds for $j = 3$.

Assume we have proved the result for all $j \leq I - 1$. Pick any n associated with I. From before, there exists a unique pair (m, J) such that $p_{n,I} = p_{m,J} \cdot p_{n-m,I-J} - p_{n-2m,I-2J}$. Also, there exists a k and r such that $n = Ik + r$, from which it follows that $m = Jk + s$, for some $s < r$. Then,

$$\begin{aligned} p_{n,I} &< (p_k^{J-s} \cdot p_{k+1}^s)(p_k^{I-J-r+s} \cdot p_{k+1}^{r-s}) - p_{n-2m,I-2J} \\ &< p_k^{I-r} \cdot p_{k+1}^r, \quad \text{since } p_{n-2m,I-2J} > 0. \end{aligned}$$

Thus, upper bound holds for $j = I$. Also,

$$\begin{aligned} p_{n,I} &> (p_k^{J-s} \cdot p_{k+1}^s - (J-1)p_1^s \cdot p_k^{J-2})(p_k^{I-J-r+s} \cdot p_{k+1}^{r-s} - (I-J-1)p_1^{r-s} \cdot p_k^{I-J-2}) \\ &\qquad - p_k^{I-2J-r+2s} \cdot p_{k+1}^{r-2s} \\ &> p_k^{I-r} \cdot p_{k+1}^r - (J-1)p_1^s \cdot p_k^{I-r+s-2} \cdot p_{k+1}^{r-s} - (I-J-1)p_1^{r-s} \cdot p_k^{I-s-2} \cdot p_{k+1}^s \\ &\qquad - p_k^{I-2J-r+2s} \cdot p_{k+1}^{r-2s}. \end{aligned}$$

We know that $p_1 \cdot p_k > p_{k+1}$, so $(p_1 \cdot p_k)^m > p_{k+1}^m$, for every positive integer m. Thus,

$$\begin{aligned} p_1^s \cdot p_k^{I-r+s-2} \cdot p_{k+1}^{r-s} &= p_1^r \cdot p_k^{I-2} \cdot (p_1^{s-r} \cdot p_k^{s-r} \cdot p_{k+1}^{r-s}) \\ &< p_1^r \cdot p_k^{I-2}, \end{aligned}$$

because $(p_1 \cdot p_k)^{r-s} > p_{k+1}^{r-s}$, and

$$\begin{aligned} p_1^{r-s} \cdot p_k^{I-s-2} \cdot p_{k+1}^s &= p_1^r \cdot p_k^{I-2} \cdot (p_1^{-s} \cdot p_k^{-s} \cdot p_{k+1}^s) \\ &< p_1^r \cdot p_k^{I-2}, \end{aligned}$$

because $(p_1 \cdot p_k)^s > p_{k+1}^s$. Also,

$$p_k^{I-2J-r+2s} \cdot p_{k+1}^{r-2s} = p_1^r \cdot p_k^{I-2} \cdot (p_1^{-r} \cdot p_k^{-2J-r+2s+2} \cdot p_{k+1}^{r-2s})$$
$$< p_1^r \cdot p_k^{I-2},$$

because $p_k^{2s} \cdot p_{k+1}^r < p_{k+1}^{2s} \cdot p_1^r \cdot p_k^{2J-2+r}$.

Therefore, we get that

$$\begin{aligned} p_{n,I} &> p_k^{I-r} \cdot p_{k+1}^r - [(J-1) + (I-J-1) + 1](p_1^r \cdot p_k^{I-2}) \\ &= p_k^{I-r} \cdot p_{k+1}^r - (I-1)p_1^r \cdot p_k^{I-2}. \end{aligned}$$

Thus, the lower bound holds for $j = I$. Therefore, by induction on j, for every $p_{n,j}$,

$$p_k^{j-r} \cdot p_{k+1}^r - (j-1)p_1^r \cdot p_k^{j-2} \leq p_{jk+r,j} \leq p_k^{j-r} \cdot p_{k+1}^r.$$

□

Pick any $j > 1$. For $p_{n,j}$, there exists an l and an s such that $n = jl + s$. Thus, $p_n > p_l \cdot p_{n-l} > ... > (p_l)^{j-s}(p_{l+1})^s$. Therefore, for all n and for all $j < \frac{n}{2}$ with $\gcd(n, j) = 1$,

$$p_n > p_{n,j}.$$

The previous two sections only dealt with the subtree $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$. When x is evaluated at a, the subtree $\mathfrak{T}(a, x, (a-1)x)$ collapses, and the tree

$\mathfrak{T}(a,x,x)$ only produces the subtree $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$. Hence, when trying to analyze trees of the form $\mathfrak{T}(a,a,a)$ for $a > 2$, it suffices to only consider the subtree $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$.

5.3 Convergence of Trees

In this section, we explain what we mean by convergence of a sequence of trees. Let $\{c_n\}$ be a sequence of real numbers such that $\mathfrak{T}(a, c_n, c_n)$ satisfies the unicity condition (or satisifies the unicity condition up to level N) for all n. Let $c_n \to c$ as $n \to \infty$ in the usual sense, i.e., for all $\epsilon > 0$, there exists $N(\epsilon) = N > 0$ such that if $n > N$ then $|c_n - c| < \epsilon$.

All polynomial functions are continuous, so each $p_{m,j}$ and $q_{m,j}$ that appear in $\mathfrak{T}(a,c,c)$ is continuous. Notice that each polynomial occurs in the same spot of every tree, regardless of parameters. For example, $p_{5,2} = x^5 - (2a+1)x^3 + (a^2 + a - 1)x$ occurs as the maximum element of a triple at $\tau(x, x^2 - a, x^3 - (a+1)x)$, for each $\mathfrak{T}(a, c_n, c_n)$ and for $\mathfrak{T}(a,c,c)$. Hence, we have the following definition.

Definition 5. We say that the sequence of the trees $\mathfrak{T}(a, c_n, c_n)$ converges to the tree $\mathfrak{T}(a,c,c)$, if for each $p_{m,j}$ and $q_{m,j}$, $p_{m,j}(c_n)$ converges to $p_{m,j}(c)$ and $q_{m,j}(c_n)$ converges to $q_{m,j}(c)$. We let $p_{m,j}(c_n)$ converge to $p_{m,j}(c)$ in the usual sense, i.e., for every $\epsilon > 0$, there exists $M > 0$ (where $M = M(p_{m,j})$ depends on the polynomial) such that if $n > M$ then $|p_{m,j}(c_n) - p_{m,j}(c)| < \epsilon$.

Since the M's may vary from polynomial to polynomial, we do not necessarily have uniform convergence.

Since all the trees of generalized Markoff equations consist of ordered triples of polynomials, we say $\mathfrak{T}(a, c_n, c_n)$ converges to $\mathfrak{T}(a, c, c)$, when for each $p_{m,j}$, $p_{m,j}(c_n)$ converges to $p_{m,j}(c)$ (and for each $q_{n,j}$ as well).

CHAPTER 6

CONCLUSION

The Markoff equation and its generalizations have been studied for over a century with many connections to several branches of mathematics. Some partial results of the unicity conjecture have been settled, but it still remains unsolved. In this paper, we recasted the solutions to generalized Markoff equations as polynomials and considered analyzing trees of solutions of these equations satisfying a unicity condition. Finding trees that satisfy the unicity condition is difficult, especially when we only consider trees with rational solutions. However, we were able to find uncountably many parameters that contain a tree that satisfies the unicity condition, and infinitely many rational ones that contain a tree that satisfies the unicity condition up to level N, for arbitrary large N.

Furthermore, for every pair of rational numbers (a, c) with $2 < a \leq c$, we were able to show that there exists a sequence of real numbers $\{c_n\}$ such that $\mathfrak{T}(a, c_n, c_n)$ satisfies the unicity condition for all n, and $c_n \to c$ as $n \to \infty$. All of these results were obtained by rewriting the ordered triple of solutions of M_A as linear combinations of Chebyshev polynomials. In particular, the product to sum formula for Chebyshev polynomials of the first kind (the T_n's) was very crucial to obtaining the key result, which was that all of the polynomials of a tree associated with M_A are all distinct from each other (except when $A = 4$, which fails the unicity condition significantly).

Future works will consist of putting a bound on roots that the polynomials have in common, in order to find trees of rational solutions satisfying the unicity condition; tightening up the upper and lower bounds obtained in Chapter 5 so we can order the numbers in these trees; and analyzing other diophantine equations similar to generalized Markoff equations, and seeing if the techniques used in this work can be applied to other equations.

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