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Existence and Stability of Standing and Traveling Wave Solutions Arising from Synaptically Coupled Neuronal Networks

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Existence and Stability of Standing and Traveling
Wave Solutions Arising from Synaptically
Coupled Neuronal Networks

by

Melissa A. Stoner

A Dissertation

Presented to the Graduate and Research Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

Lehigh University

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Melissa Stoner

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Melissa A Stoner

Existence and Stability of Standing and Traveling Wave Solutions Arising from Synaptically Coupled Neuronal Networks

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Abstract

We consider three variations of neuronal network models and apply mathematical analysis to investigate standing and traveling wave solutions to the models. We consider solutions for both the scalar case and the system of equations. We establish existence and uniqueness of the solutions and determine the stability/instability of the solutions to the integral differential model equations. In addition we investigate the influence of sodium currents on the solutions. We perform a speed analysis to determine the effect of various biological parameters on the wave speed and wave structure.

Introduction

Scientists have observed waves of neuron activity traveling across the surface of the brain in migraine patients. Similar waves have been reproduced pharmacologically in laboratory settings. These waves are propagated via action potentials which are all-or-nothing phenomena that arise when the membrane potential reaches a specified threshold value. This change in membrane potential occurs due to movement of the potassium and sodium ions through passive channels and pumps in the cell membrane. The action potential maintains a constant profile as it travels along the length of the neuron's axon to synaptic terminals which convert the electric signal to a chemical neurotransmitter which is then sent to surrounding neurons [47]. There have been many equations developed to model neurons, each focusing on different aspects of the signal transmission. The first neuron model was proposed in 1952 by Hodgkin and Huxley [30] to describe the propagation of the action potential along the squid giant axon. This is an empirical model used to model a single neuron via curve fitting, hence it has no closed solution. More recently, focus has shifted to consider models of integral-differential equations to model the response of a neuron based on the network of neurons to which it is coupled. The interaction between neurons in a synaptically coupled neuronal network is responsible for the nonlocal term in the model. The first model of this type was proposed by Amari [3] in 1977 and is given by:

$$u_t + u = \alpha \int_{\mathbb{R}} K(x - y)H(u(y, t) - \theta)dy. \quad (1)$$

The variable u represents the membrane potential of a neuron at a position x and time t . The parameter α is the synaptic rate constant, θ is the threshold constant for excitation, H is the Heaviside step function, and K is a function which represents the synaptic coupling between neurons. As stated above, the integral is derived from the interaction between neurons.

David Terman [53] expanded upon this model in 1998 by incorporating a cubic term to represent the sodium ion flow as opposed to the linear term used in Amari's model given by:

$$u_t + u(1 - u)(u - a) = \alpha \int_{\mathbb{R}} K(x - y)H(u(y, t) - \theta)dy. \quad (2)$$

More recently, Pinto and Ermentrout[46] proposed a model which reverted to the linear representation of the sodium ion flow and incorporated a second equation representing the leaking current, which provides the negative feedback responsible for limiting the excitation of the network. The model is given by

$$u_t + u + w = \alpha \int_{\mathbb{R}} K(x - y)H(u(y, t) - \theta)dy \quad (3)$$

$$w_t = \varepsilon(u - \gamma w). \quad (4)$$

All variables are consistent with the previous models, in addition w denotes the leaking current, γ denotes the decay rate, ε controls the fast/slow activation of chemical ion channels and $0 < \varepsilon \ll 1$. By considering the traveling wave solution of the system, we make the following assumptions on the parameters $0 < 2\theta < \alpha$, $0 < \alpha\gamma < (1 + \gamma)\theta$ and $0 < \varepsilon \ll 1$ are constants.

Note that the model equation (1) is a special case of the system (3)-(4) where $\varepsilon = 0$ and $w = 0$. Other adaptations of this model have been considered including variations for mechanisms such as temporal delay and the double threshold case. More recently

Zhang and Hutt [65] and [66] have studied a more complex model designed to accommodate both a pulse transmission delay and a feedback delay as follows:

$$\begin{aligned}
& \frac{\partial u}{\partial t} + f(u) \\
= & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) dy \right] dc \\
+ & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \tag{5}
\end{aligned}$$

where $u = u(x, t)$ represents the membrane potential of a neuron at position x and time t in a synaptically coupled neuronal network, $w = w(x, t)$ represents the leaking current. The functions $\xi \geq 0$ and $\eta \geq 0$ are defined on $(0, \infty)$. The kernel functions K and W are defined on \mathbb{R} . They represent synaptic couplings between neurons. The function ξ represents a statistical distribution of action potential speeds, $\xi \geq 0$ on $(0, \infty)$ and $\int_0^\infty \xi(c)dc = 1$. Additionally, ξ may have compact support (c_1, c_2) , where c_1 and c_2 are positive numbers, denoting the lower and upper bounds of biologically possible speeds, respectively. $a \geq 0, b \geq 0, \alpha \geq 0, \beta \geq 0, \varepsilon > 0, \theta > 0$ and $\Theta > 0$ are constants, representing biological mechanisms. In this model system, for simplicity, we choose the gain function to be the Heaviside step function: $H(u - \theta) = 0$ for all $u < \theta$, $H(\theta) = \frac{1}{2}$, and $H(u - \theta) = 1$ for all $u > \theta$. See [6], [7], [13], [32], [44], [46], [60], [61] for the same or very similar equations.

Work in this area of mathematical neuroscience has developed dramatically since Hodgkin and Huxley developed the first model in the 1950's [30]. Recent work has proved the existence, uniqueness and stability of various types of wave forms for varying kernel functions including traveling waves and pulses [10] [13] [12] [39] [46] [17] [52] [59] [60] [61] [62] [63] [64], standing waves [29] [46] spiral waves [40], etc. Work continues to be done in these areas as model equations incorporate additional features of the neuronal network as well as accounting for differences in neuronal networks based

on its location and function in the brain or nervous system.

We will consider traveling wave solutions to three different models and standing wave solutions to the two delay model. In general, one strives to establish the existence and uniqueness of a wave solution that solves the model equations and satisfies initial conditions. Beyond that we look at the exponential stability of the solution. In a more practical sense, we push to establish the behavior of the various biological mechanisms in the network on the wave and wave speed.

One remaining variation in the models is the kernel function which summarizes the synaptic coupling of neurons in the network. Suppose that the synaptic coupling K is at least piecewise continuous, satisfying the following conditions

$$\int_{\mathbb{R}} K(x)dx = 1, \quad \int_{-\infty}^0 K(x)dx = \int_0^{\infty} K(x)dx = \frac{1}{2}, \quad (6)$$

$$\int_{-\infty}^0 |x|K(x)dx \geq 0, \quad (7)$$

$$|K(x)| \leq C \exp(-\rho|x|) \quad \text{on} \quad \mathbb{R}, \quad (8)$$

for some constants $C > 0$ and $\rho > 0$. We are concerned with the following three classes of synaptic couplings.

(A) Pure excitations between neurons (represented by nonnegative kernel functions).

For examples, $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$ and $K_2(x) = \sqrt{\frac{\rho}{\pi}} \exp(-\rho|x|^2)$ may represent pure excitations, where $\rho > 0$ is a constant. Here, ρ has a biological meaning. It indicates how the excitation of a synaptic coupling is distributed. Roughly speaking, if ρ is large, then a neuron is strongly coupled with neurons in a relatively small region; if ρ is small, then a neuron is strongly coupled with all neurons in a relatively large region.

(B) Lateral inhibitions (represented by Mexican hat kernel functions, that is, each coupling satisfies $K \geq 0$ on $(-M, M)$ and $K \leq 0$ on $(-\infty, -M) \cup (M, \infty)$ for a positive constant M . This implies that neurons close to one another have excitatory con-

nections and neurons far away have inhibitory connections.) For example, $K_3(x) = A \exp(-a|x|^2) - B \exp(-b|x|^2)$ may represent a lateral inhibition, where $A > B > 0$ and $a > b > 0$ are positive constants, such that

$$\frac{A}{a} \geq \frac{B}{b}, \quad A\sqrt{\frac{\pi}{a}} - B\sqrt{\frac{\pi}{b}} = 1, \quad M = \sqrt{\frac{1}{a-b} \ln \frac{A}{B}}.$$

(C) Lateral excitations (represented by upside down Mexican hat kernel functions, that is, each coupling satisfies $K \leq 0$ on $(-M, M)$ and $K \geq 0$ on $(-\infty, -M) \cup (M, \infty)$, for some positive constant M). For example, $K_4(x) = A \exp(-a|x|) - B \exp(-b|x|)$ may represent a lateral excitation, where $0 < A < B$ and $0 < a < b$ are positive constants, such that

$$\frac{A}{a} - \frac{B}{b} = \frac{1}{2}, \quad \frac{A}{a^2} \geq \frac{B}{b^2}, \quad M = \frac{1}{b-a} \ln \frac{B}{A}.$$

See Amari [3], Bressloff and Folias [9], Bressloff, Folias, Prat and Li [45], Coombes, Lord and Owen [13], Coombes and Owen [14], Ermentrout [19], Ermentrout and Terman [21], Laing [40], Laing and Troy [41], Pinto and Ermentrout [46], Terman, Ermentrout and Yew [54], and [60]-[61]-[62]-[63] for more information on various synaptic couplings represented by the kernel functions.

Chapter 1

Standing Waves

1.1 Introduction

To begin our study of standing wave solutions, we will use rigorous mathematical analysis to establish the existence and stability/instability of the standing wave solutions to the nonlinear singularly perturbed system of integral differential equations with $\varepsilon > 0$. First of all, we will obtain explicit standing wave solutions for the system. Then, by constructing and making use of some complex analytic functions, called Evans functions, we will accomplish the stability/instability of the standing wave solutions.

1.1.1 The Model Equations

Consider the following nonlinear singular perturbed system of integral differential equations arising from synaptically coupled neuronal networks

$$\begin{aligned} & \frac{\partial u}{\partial t} + f(u) + w \\ &= (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c} |x-y| \right) - \theta \right) dy \right] dc \\ &+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau, \end{aligned} \quad (1.1)$$

$$\frac{\partial w}{\partial t} = \varepsilon [g(u) - w], \quad (1.2)$$

where the parameters are consistent with the model described on page 4. In addition, either $f(u) + g(u)$ is a cubic polynomial function or $f(u) + g(u)$ is a linear function, say $f(u) + g(u) = m(u - n) + k(u - l)$, where $k > 0$ and $m > 0$ are positive constants, l and n are real constants.

If $\varepsilon = 0$ and $w = 0$ in system (1.1)-(1.2), then we have a scalar integral differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + f(u) \\ = & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) dy \right] dc \\ + & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \end{aligned} \quad (1.3)$$

We will study the existence and stability of standing wave solutions, that is, solutions of the form $(u(x, t), w(x, t)) = (\phi(x), \psi(x))$ for system (1.1)-(1.2) and solutions of the form $u(x, t) = \phi(x)$ for equation (1.3).

First of all, let us find the constant solutions of the system.

If $f(u) + g(u) = m(u - n) + k(u - l)$, $u_0 < \theta$ and $u_0 < \Theta$, then $m(u_0 - n) + k(u_0 - l) = 0$.

Thus

$$u_0 = \frac{kl + mn}{k + m} < \theta.$$

If $f(u) + g(u) = m(u - n) + k(u - l)$, $u_1 > \theta$ and $u_1 > \Theta$, then $m(u_1 - n) + k(u_1 - l) = \alpha + \beta - au_1 - bu_1$. Hence

$$u_1 = \frac{\alpha + \beta + kl + mn}{a + b + k + m} > \Theta.$$

Previously, Amari [3], Guo and Chow [29] and Pinto and Ermentrout [46] have studied the existence and stability of standing wave solutions of some integral differential equations arising from synaptically coupled neuronal networks. However, the existence

and stability of standing wave solutions of system (1.1)-(1.2) has been an open problem for a long time. An interesting feature is that the eigenvalue problem derived from linearization of the nonlinear system is nonlinear in λ (this is the eigenvalue parameter). This difficulty arises because the system involves two kinds of delays and any of the two delays may cause such a difficulty. We are able to overcome the difficulty to find the eigenvalues of the eigenvalue problem by constructing the Evans function and studying its properties.

1.1.2 The Mathematical Assumptions

For the duration of this chapter we assume that

$$\begin{aligned}
K(-x) &= K(x), & \text{for all } x \in \mathbb{R}, & & \text{and } \int_{\mathbb{R}} K(x) dx &= 1, \\
W(-x) &= W(x), & \text{for all } x \in \mathbb{R}, & & \text{and } \int_{\mathbb{R}} W(x) dx &= 1, \\
\int_0^{\infty} \xi(c) dc &= 1, & \int_0^{\infty} \eta(\tau) d\tau &= 1, \\
\int_0^{\infty} \frac{1}{c} \xi(c) dc &< \infty, & \int_0^{\infty} \tau \eta(\tau) d\tau &< \infty, \\
f'(\theta) + g'(\theta) + \frac{a}{2} + b \int_{-\infty}^{-Z_0} W(\zeta) d\zeta &> 0, \\
f'(\Theta) + g'(\Theta) + a \int_{-\infty}^{Z_0} K(\zeta) d\zeta + \frac{b}{2} &> 0, \\
\alpha \neq a\theta, & \quad \beta \neq b\theta, \\
\frac{kl + mn}{k + m} < \theta \leq \Theta < \frac{\alpha + \beta + kl + mn}{a + b + k + m}, \\
k + m + a \int_{-\infty}^x K(\zeta) d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta) d\zeta &> 0, & \text{for all } x \in \mathbb{R}, \\
(\alpha - a\theta)K(x) + (\beta - b\theta)W(x - Z_0) &\geq 0, & \text{for all } x \in \mathbb{R}, \\
(\alpha - a\Theta)K(x) + (\beta - b\Theta)W(x - Z_0) &\geq 0, & \text{for all } x \in \mathbb{R}, \\
(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0) &> 0, \\
(\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0) &> 0, \\
\int_{-\infty}^{-Z_0} W(\zeta) d\zeta &= \frac{(2k + 2m + a)\theta - (2kl + 2mn + \alpha)}{2(\beta - b\theta)}, \\
\int_{-\infty}^{Z_0} K(\zeta) d\zeta &= \frac{(2k + 2m + b)\Theta - (2kl + 2mn + \beta)}{2(\alpha - a\Theta)},
\end{aligned}$$

for some real constant $Z_0 \geq 0$.

Remark 1.1.1. *By an intermediate value theorem, it is easy to show that there exists a real number Z_0' , such that*

$$\int_{-\infty}^{-Z_0'} W(\zeta) d\zeta = \frac{(2k + 2m + a)\theta - (2kl + 2mn + \alpha)}{2(\beta - b\theta)}.$$

Similarly, there exists another real number Z_0'' , such that

$$\int_{-\infty}^{Z_0''} K(\zeta)d\zeta = \frac{(2k + 2m + b)\Theta - (2kl + 2mn + \beta)}{2(\alpha - a\Theta)}.$$

For simplicity, we assume that the model parameters and the kernel functions are chosen in such a way that $Z_0' = Z_0'' = Z_0$. If $\theta = \Theta$ and $(a + b + 2k + 2m)\theta = \alpha + \beta + 2kl + 2mn$, then $Z_0 = 0$.

1.2 Existence of the Standing Wave Solutions

1.2.1 Linear Functions

First of all, we establish the existence of the standing wave solutions to the nonlinear singularly perturbed system of integral differential equations with $\varepsilon > 0$ and $f(u) + g(u) = m(u - n) + k(u - l)$, where $k > 0$ and $m > 0$ are positive constants, l and n are real constants.

Theorem 1.2.1. *Suppose that $a \geq 0$, $b \geq 0$, $k > 0$, $l, m > 0$, $n, \alpha \geq 0$, $\beta \geq 0$, $\varepsilon > 0$, $\theta > 0$ and $\Theta > 0$ are real constants. Let $f(u) + g(u) = m(u - n) + k(u - l)$. Then there exist two standing wave solutions*

$$\phi_1(x) = \frac{kl + mn + \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta}{k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta},$$

$$\phi_2(x) = \frac{kl + mn + \alpha \int_x^{\infty} K(\zeta)d\zeta + \beta \int_{x+Z_0}^{\infty} W(\zeta)d\zeta}{k + m + a \int_x^{\infty} K(\zeta)d\zeta + b \int_{x+Z_0}^{\infty} W(\zeta)d\zeta},$$

to the system of integral differential equations

$$\begin{aligned}
& \frac{\partial u}{\partial t} + f(u) + w \\
&= (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H \left(U \left(y, t - \frac{1}{c} |x-y| \right) - \theta \right) dy \right] dc \\
&+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau, \\
\frac{\partial w}{\partial t} &= \varepsilon [g(u) - w].
\end{aligned}$$

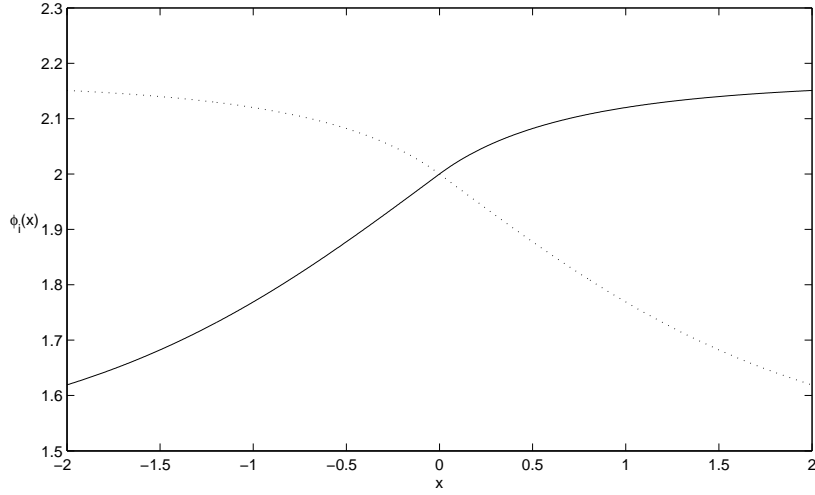


Figure 1.1: Graph of the standing wave solutions $\phi_1(x)$ (solid line) and $\phi_2(x)$ (dash-dotted line) to the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2). In this graph, $a = 2, b = 2, \alpha = 5, \beta = 5, \theta = 2, \Theta = 2, f(u) = u - 1, g(u) = u - 2, K(x) = \frac{1}{2} \exp(-|x|)$ and $W(x) = \frac{1}{2} \exp(-|x|)$.

Proof. Standing wave solutions satisfy $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial w}{\partial t} = 0$. Substituting a solution of the form $(u(x, t), w(x, t)) = (\phi(x), \psi(x))$ into the system (1.1)-(1.2), we get

$$\begin{aligned}
f(\phi(x)) + \psi(x) &= [\alpha - a\phi(x)] \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H(\phi(y) - \theta) dy \right] dc \\
&+ [\beta - b\phi(x)] \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(\phi(y) - \Theta) dy \right] d\tau, \\
0 &= \varepsilon [g(\phi(x)) - \psi(x)].
\end{aligned}$$

Suppose that the standing wave solution satisfies the conditions $\phi < \theta$ on $(-\infty, 0)$, $\phi(0) = \theta$ and $\phi > \theta$ on $(0, \infty)$; $\phi < \Theta$ on $(-\infty, Z_0)$, $\phi(Z_0) = \Theta$ and $\phi > \Theta$ on (Z_0, ∞) , for some real constant $Z_0 \geq 0$. Then the right hand side becomes

$$\begin{aligned}
& [\alpha - a\phi(x)] \int_{\mathbb{R}} K(x-y)H(\phi(y) - \theta)dy \\
& + [\beta - b\phi(x)] \int_{\mathbb{R}} W(x-y)H(\phi(y) - \Theta)dy \\
& = [\alpha - a\phi(x)] \int_0^{\infty} K(x-y)dy + [\beta - b\phi(x)] \int_{Z_0}^{\infty} W(x-y)dy \\
& = [\alpha - a\phi(x)] \int_{-\infty}^x K(\zeta)d\zeta + [\beta - b\phi(x)] \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \\
& = \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \\
& - \left[a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right] \phi(x).
\end{aligned}$$

Hence the first standing wave solution to the integral differential equations (1.1)-(1.2) is given by

$$\phi(x) = \frac{kl + mn + \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta}{k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta}.$$

The derivative is given by

$$\phi'(x) = \frac{[\alpha - a\phi(x)]K(x) + [\beta - b\phi(x)]W(x - Z_0)}{k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta}.$$

In particular

$$\phi'(0) = \frac{(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0)}{k + m + \frac{a}{2} + b \int_{-\infty}^{-Z_0} W(\zeta)d\zeta} > 0,$$

$$\phi'(Z_0) = \frac{(\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0)}{k + m + a \int_{-\infty}^{Z_0} K(\zeta)d\zeta + \frac{b}{2}} > 0.$$

However, this is only a formal solution. We have to show that it is compatible, namely, it satisfies the prescribed conditions. Let $\phi(0) = \theta$ and $\phi(Z_0) = \Theta$, respectively, we have

$$\frac{2kl + 2mn + \alpha + 2\beta \int_{-\infty}^{-Z_0} W(\zeta)d\zeta}{2k + 2m + a + 2b \int_{-\infty}^{-Z_0} W(\zeta)d\zeta} = \theta,$$

$$\frac{2kl + 2mn + 2\alpha \int_{-\infty}^{Z_0} K(\zeta)d\zeta + \beta}{2k + 2m + 2a \int_{-\infty}^{Z_0} K(\zeta)d\zeta + b} = \Theta.$$

Solving the system, we find

$$\int_{-\infty}^{-Z_0} W(\zeta)d\zeta = \frac{(2k + 2m + a)\theta - (2kl + 2mn + \alpha)}{2(\beta - b\theta)},$$

$$\int_{-\infty}^{Z_0} K(\zeta)d\zeta = \frac{(2k + 2m + b)\Theta - (2kl + 2mn + \beta)}{2(\alpha - a\Theta)}.$$

The solution really satisfies the conditions $\phi(0) = \theta$ and $\phi(Z_0) = \Theta$, because

$$\phi(0) = \frac{kl + mn + \alpha \int_{-\infty}^0 K(\zeta)d\zeta + \beta \int_{-\infty}^{-Z_0} W(\zeta)d\zeta}{k + m + a \int_{-\infty}^0 K(\zeta)d\zeta + b \int_{-\infty}^{-Z_0} W(\zeta)d\zeta} = \theta,$$

$$\phi(Z_0) = \frac{kl + mn + \alpha \int_{-\infty}^{Z_0} K(\zeta)d\zeta + \beta \int_{-\infty}^0 W(\zeta)d\zeta}{k + m + a \int_{-\infty}^{Z_0} K(\zeta)d\zeta + b \int_{-\infty}^0 W(\zeta)d\zeta} = \Theta.$$

We also have to verify that the standing wave solution is below and above the threshold θ on $(-\infty, 0)$ and $(0, \infty)$, respectively; and it is below and above the threshold Θ on $(-\infty, Z_0)$ and (Z_0, ∞) , respectively. The following inequalities are equivalent to one another (below, the symbol “ \leq ” means that either we always take “ $<$ ” or we always

take “=” or we always take “>”):

$$\frac{kl + mn + \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta}{k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta} \Leftrightarrow \theta;$$

$$kl + mn + \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \Leftrightarrow \left[k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right] \theta;$$

$$k(l - \theta) + m(n - \theta) + (\alpha - a\theta) \int_{-\infty}^x K(\zeta)d\zeta + (\beta - b\theta) \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \Leftrightarrow 0;$$

$$(\alpha - a\theta) \int_{-\infty}^x K(\zeta)d\zeta + (\beta - b\theta) \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \Leftrightarrow k(\theta - l) + m(\theta - n).$$

Moreover, the following inequalities are equivalent to each other:

$$\frac{kl + mn + \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta}{k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta} \Leftrightarrow \Theta;$$

$$kl + mn + \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \Leftrightarrow \left[k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right] \Theta;$$

$$k(l - \Theta) + m(n - \Theta) + (\alpha - a\Theta) \int_{-\infty}^x K(\zeta)d\zeta + (\beta - b\Theta) \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \Leftrightarrow 0;$$

$$(\alpha - a\Theta) \int_{-\infty}^x K(\zeta)d\zeta + (\beta - b\Theta) \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \Leftrightarrow k(\Theta - l) + m(\Theta - n).$$

Definition 1.2.2. Define the following auxiliary functions

$$\begin{aligned} A(x) &= (\alpha - a\theta) \int_{-\infty}^x K(\zeta) d\zeta + (\beta - b\theta) \int_{-\infty}^{x-Z_0} W(\zeta) d\zeta, \\ B(x) &= (\alpha - a\Theta) \int_{-\infty}^x K(\zeta) d\zeta + (\beta - b\Theta) \int_{-\infty}^{x-Z_0} W(\zeta) d\zeta. \end{aligned}$$

Then

$$\begin{aligned} A(0) &= \frac{1}{2}(\alpha - a\theta) + (\beta - b\theta) \int_{-\infty}^{-Z_0} W(\zeta) d\zeta = k(\theta - l) + m(\theta - n), \\ B(Z_0) &= (\alpha - a\Theta) \int_{-\infty}^{Z_0} K(\zeta) d\zeta + \frac{1}{2}(\beta - b\Theta) = k(\Theta - l) + m(\Theta - n), \end{aligned}$$

and by the assumptions made in Subsection 2.1.2, we have

$$\begin{aligned} A'(x) &= (\alpha - a\theta)K(x) + (\beta - b\theta)W(x - Z_0) \geq 0, & \text{for all } x \in \mathbb{R}, \\ B'(x) &= (\alpha - a\Theta)K(x) + (\beta - b\Theta)W(x - Z_0) \geq 0, & \text{for all } x \in \mathbb{R}, \\ A'(0) &= (\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0) > 0, \\ B'(Z_0) &= (\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0) > 0. \end{aligned}$$

Hence, both $A(x)$ and $B(x)$ are increasing functions on \mathbb{R} . Therefore, we find that $\phi < \theta$ on $(-\infty, 0)$, $\phi(0) = \theta$ and $\phi > \theta$ on $(0, \infty)$. Similarly, $\phi < \Theta$ on $(-\infty, Z_0)$, $\phi(Z_0) = \Theta$ and $\phi > \Theta$ on (Z_0, ∞) .

The existence of the second standing wave solution can be proved very similarly.

Indeed, the equation becomes

$$\begin{aligned} & m[\phi(x) - n] + k[\phi(x) - l] \\ &= [\alpha - a\phi(x)] \int_x^\infty K(\zeta) d\zeta + [\beta - b\phi(x)] \int_{x+Z_0}^\infty W(\zeta) d\zeta. \end{aligned}$$

Therefore, we get the second standing wave solution

$$\phi_2(x) = \frac{kl + mn + \alpha \int_x^\infty K(\zeta) d\zeta + \beta \int_{x+Z_0}^\infty W(\zeta) d\zeta}{k + m + a \int_x^\infty K(\zeta) d\zeta + b \int_{x+Z_0}^\infty W(\zeta) d\zeta}.$$

The derivative is given by

$$\phi_2'(x) = -\frac{[\alpha - a\phi(x)]K(x) + [\beta - b\phi(x)]W(x + Z_0)}{k + m + a \int_x^\infty K(\zeta)d\zeta + b \int_{x+Z_0}^\infty W(\zeta)d\zeta}.$$

In particular

$$\phi_2'(-Z_0) = -\frac{(\alpha - a\Theta)K(-Z_0) + (\beta - b\Theta)W(0)}{k + m + a \int_{-Z_0}^\infty K(\zeta)d\zeta + \frac{b}{2}} < 0,$$

$$\phi_2'(0) = -\frac{(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0)}{k + m + \frac{a}{2} + b \int_{Z_0}^\infty W(\zeta)d\zeta} < 0.$$

It is easy to check that this standing wave solution also satisfies the prescribed conditions. The proof of Theorem 1.2.1 is finished. ■

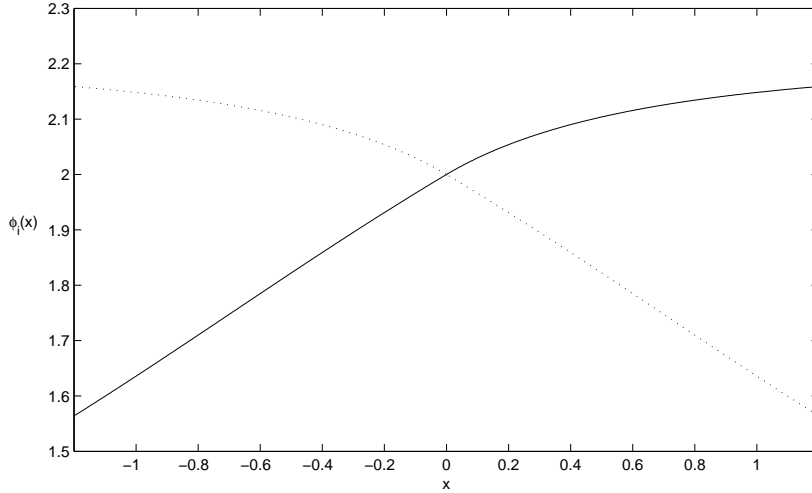


Figure 1.2: Graph of the standing wave solutions $\phi_3(x)$ (solid line) and $\phi_4(x)$ (dash-dotted line) to the scalar integral differential equation (1.3). In this graph, $a = 2$, $b = 2$, $\alpha = 5$, $\beta = 5$, $\theta = 2$, $\Theta = 2$, $f(u) = u - 1$, $K(x) = \frac{1}{2} \exp(-|x|)$ and $W(x) = \frac{1}{2} \exp(-|x|)$.

Corollary 1.2.3. *Suppose that $a \geq 0, b \geq 0, m > 0, n, \alpha \geq 0, \beta \geq 0, \theta > 0$ and $\Theta > 0$ are constants. Let $f(u) = m(u - n)$. Suppose that*

$$\begin{aligned} \theta = \Theta, \alpha + \beta + 2mn &= (a + b + 2m)\theta, & n < \theta < \frac{\alpha + \beta + mn}{a + b + m}, \\ m + a \int_{-\infty}^x K(\zeta) d\zeta + b \int_{-\infty}^x W(\zeta) d\zeta &> 0, & \text{for all } x \in \mathbb{R}, \\ (\alpha - a\theta)K(x) + (\beta - b\theta)W(x) &\geq 0, & (\alpha - a\theta)K(0) + (\beta - b\theta)W(0) > 0. \end{aligned}$$

Then there exist two standing wave solutions

$$\phi_3(x) = \frac{mn + \alpha \int_{-\infty}^x K(\zeta) d\zeta + \beta \int_{-\infty}^x W(\zeta) d\zeta}{m + a \int_{-\infty}^x K(\zeta) d\zeta + b \int_{-\infty}^x W(\zeta) d\zeta},$$

$$\phi_4(x) = \frac{mn + \alpha \int_x^{\infty} K(\zeta) d\zeta + \beta \int_x^{\infty} W(\zeta) d\zeta}{m + a \int_x^{\infty} K(\zeta) d\zeta + b \int_x^{\infty} W(\zeta) d\zeta},$$

to the scalar integral differential equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + m(u - n) \\ &= (\alpha - au) \int_0^{\infty} \xi(c) \left[\int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c} |x - y| \right) - \theta \right) dy \right] dc \\ &+ (\beta - bu) \int_0^{\infty} \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \end{aligned}$$

Proof. The proof follows from Theorem 1.2.1 and it is omitted. ■

Remark 1.2.4. *It is not difficulty to show that $\phi_1(x) = \phi_2(-x)$ for system (1.1)-(1.2) and $\phi_3(x) = \phi_4(-x)$ for equation (1.3), for all $x \in \mathbb{R}$.*

1.2.2 Nonlinear Functions

Now we consider the system (1.1)-(1.2) with $f(u) + g(u) = u(u - 1)(Du - 1)$ and establish the existence of standing wave solutions.

Theorem 1.2.5. *Suppose that $a \geq 0$, $b \geq 0$, $D > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\varepsilon > 0$, $\theta > 0$ and $\Theta > 0$ are real constants. Let $f(u) + g(u) = u(u - 1)(Du - 1)$. Then there exist two standing wave solutions to the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2).*

Proof. It is easy to see that the standing wave solutions to the system (1.1)-(1.2) with $f(u) + g(u) = u(u - 1)(Du - 1)$ satisfy the equation

$$\begin{aligned} \phi(\phi - 1)(D\phi - 1) &= \alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \\ &\quad - \left[a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right] \phi. \end{aligned}$$

Hence we have

$$\begin{aligned} D\phi^3 - (1 + D)\phi^2 &+ \left[1 + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right] \phi \\ &- \left[\alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right] \\ &= 0. \end{aligned}$$

Let

$$\begin{aligned} p &= -\frac{1+D}{D}, \quad q = \frac{1}{D} \left[1 + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right], \\ r &= -\frac{1}{D} \left[\alpha \int_{-\infty}^x K(\zeta)d\zeta + \beta \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta \right]. \end{aligned}$$

Then we get

$$\phi^3 + p\phi^2 + q\phi + r = 0.$$

One real root of the cubic polynomial $u^3 + pu^2 + qu + r$ is given by [61]

$$\begin{aligned} u &= - \left\{ \frac{r}{2} - \frac{pq}{6} + \frac{p^3}{27} + \sqrt{\left(\frac{r}{2} - \frac{pq}{6} + \frac{p^3}{27} \right)^2 + \left(\frac{q}{3} - \frac{p^2}{9} \right)^3} \right\}^{1/3} \\ &\quad - \left\{ \frac{r}{2} - \frac{pq}{6} + \frac{p^3}{27} - \sqrt{\left(\frac{r}{2} - \frac{pq}{6} + \frac{p^3}{27} \right)^2 + \left(\frac{q}{3} - \frac{p^2}{9} \right)^3} \right\}^{1/3} - \frac{p}{3}. \end{aligned}$$

By using this formula, we obtain the first standing wave solution. Very similarly, we obtain the second standing wave solution. The proof of Theorem 1.2.5 is finished. ■

Corollary 1.2.6. *Suppose that $a \geq 0$, $b \geq 0$, $D > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\theta > 0$ and $\Theta > 0$ are real constants. Let $f(u) = u(u - 1)(Du - 1)$. Then there exist two standing wave solutions to the scalar integral differential equation (1.3).*

Proof. The proof follows from Theorem 1.2.5 and it is omitted. ■

1.3 Stability of the Standing Wave Solutions

In this section, we will derive an eigenvalue problem, construct a complex analytic function (namely, Evans function) corresponding to the eigenvalue problem, study properties of the Evans function and then establish the stability or instability of the standing wave solutions.

1.3.1 Derivation of the Eigenvalue Problem

Subtracting the following system of integral differential equations

$$\begin{aligned} & \frac{\partial \phi}{\partial t} + f(\phi) + \psi \\ &= [\alpha - a\phi(x)] \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y)H(\phi(y) - \theta)dy \right] dc \\ &+ [\beta - b\phi(x)] \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y)H(\phi(y) - \Theta)dy \right] d\tau, \\ \frac{\partial \psi}{\partial t} &= \varepsilon[g(\phi) - \psi], \end{aligned}$$

from the following system of integral differential equations

$$\begin{aligned}
& \frac{\partial u}{\partial t} + f(u) + w \\
&= (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c} |x-y| \right) - \theta \right) dy \right] dc \\
&+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau, \\
\frac{\partial w}{\partial t} &= \varepsilon [g(u) - w],
\end{aligned}$$

keeping linear terms and neglecting higher order terms, we obtain the new system of integral differential equations

$$\begin{aligned}
& \frac{\partial v}{\partial t} + f'(\phi(x))v + w = [\alpha - a\phi(x)] \frac{K(x)}{\phi'(0)} \int_0^\infty \xi(c)v \left(0, t - \frac{1}{c} |x| \right) dc \\
&+ [\beta - b\phi(x)] \frac{W(x - Z_0)}{\phi'(Z_0)} \int_0^\infty \eta(\tau)v(Z_0, t - \tau) d\tau \\
&- av \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H(\phi(y) - \theta) dy \right] dc \\
&- bv \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(\phi(y) - \Theta) dy \right] d\tau, \\
\frac{\partial w}{\partial t} &= \varepsilon [g'(\phi(x))v - w].
\end{aligned}$$

Suppose that $(v(x, t), w(x, t)) = (\psi_1(x), \psi_2(x)) \exp(\lambda t)$ is a solution of this system, where λ is a complex number and $\psi_1(x)$ and $\psi_2(x)$ are complex functions. Then

$$\begin{aligned}
& \lambda \psi_1(x) \exp(\lambda t) + f'(\phi(x))\psi_1(x) \exp(\lambda t) + \psi_2(x) \exp(\lambda t) \\
&= [\alpha - a\phi(x)] \frac{K(x)}{\phi'(0)} \int_0^\infty \xi(c)\psi_1(0) \exp \left(\lambda t - \frac{\lambda}{c} |x| \right) dc \\
&+ [\beta - b\phi(x)] \frac{W(x - Z_0)}{\phi'(Z_0)} \int_0^\infty \eta(\tau)\psi_1(Z_0) \exp(\lambda t - \lambda\tau) d\tau \\
&- a\psi_1(x) \exp(\lambda t) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H(\phi(y) - \theta) dy \right] dc \\
&- b\psi_1(x) \exp(\lambda t) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(\phi(y) - \Theta) dy \right] d\tau, \\
&\lambda \psi_2(x) \exp(\lambda t) = \varepsilon [g'(\phi(x))\psi_1(x) \exp(\lambda t) - \psi_2(x) \exp(\lambda t)].
\end{aligned}$$

By canceling out the exponential function $\exp(\lambda t)$, we obtain the eigenvalue problem

$$\begin{aligned}
& \lambda\psi_1(x) + f'(\phi(x))\psi_1(x) + \psi_2(x) \\
= & [\alpha - a\phi(x)] \frac{K(x)}{\phi'(0)} \left[\int_0^\infty \xi(c) \exp\left(-\frac{\lambda}{c}|x|\right) dc \right] \psi_1(0) \\
+ & [\beta - b\phi(x)] \frac{W(x - Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \psi_1(Z_0) \\
- & a \left\{ \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x - y) H(\phi(y) - \theta) dy \right] dc \right\} \psi_1(x) \\
- & b \left\{ \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(\phi(y) - \Theta) dy \right] d\tau \right\} \psi_1(x), \\
& \lambda\psi_2(x) = \varepsilon [g'(\phi(x))\psi_1(x) - \psi_2(x)].
\end{aligned}$$

The eigenvalue problem may be written as

$$\mathcal{L}(\lambda)\psi = \lambda\psi, \quad \text{where} \quad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in [L^\infty(\mathbb{R})]^2,$$

where $\mathcal{L}(\lambda) : [L^\infty(\mathbb{R})]^2 \rightarrow [L^\infty(\mathbb{R})]^2$ is a linear operator.

Definition 1.3.1. *If there exists a complex number λ_0 and there exists a nontrivial bounded continuous solution ψ_0 on \mathbb{R} , such that $\mathcal{L}(\lambda_0)\psi_0 = \lambda_0\psi_0$, then λ_0 is called an eigenvalue and ψ_0 is called an eigenfunction of the eigenvalue problem.*

If $\psi_1(0) = \psi_1(Z_0) = 0$, then

$$\begin{aligned}
& \left[\lambda + f'(\phi(x)) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\phi(x)) \right. \\
& \left. + a \int_{-\infty}^x K(\zeta) d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta) d\zeta \right] \psi_1(x) \\
= & 0.
\end{aligned}$$

Hence $\psi \equiv 0$ on \mathbb{R} . Therefore, for a function ψ to be a nontrivial eigenfunction of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, it must satisfy the conditions $\psi_1(0) \neq 0$ and $\psi_1(Z_0) \neq 0$.

1.3.2 Derivation of the Evans Function

Note that for the first standing wave solution

$$\begin{aligned}
& \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H(\phi(y) - \theta) dy \right] dc \\
&= \int_{\mathbb{R}} K(x-y) H(H(\phi(y) - \theta)) dy = \int_{-\infty}^x K(\zeta) d\zeta, \\
& \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(\phi(y) - \Theta) dy \right] d\tau \\
&= \int_{\mathbb{R}} W(x-y) H(H(\phi(y) - \Theta)) dy = \int_{-\infty}^{x-Z_0} W(\zeta) d\zeta.
\end{aligned}$$

Letting $x = 0$ and $x = Z_0$ in the eigenvalue problem, respectively, we have

$$\begin{aligned}
& \lambda\psi_1(0) + f'(\theta)\psi_1(0) + \psi_2(0) \\
&= (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \psi_1(0) - \frac{a}{2} \psi_1(0) - b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] \psi_1(0) \\
&+ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \psi_1(Z_0), \\
& \lambda\psi_2(0) = \varepsilon [g'(\theta)\psi_1(0) - \psi_2(0)],
\end{aligned}$$

and

$$\begin{aligned}
& \lambda\psi_1(Z_0) + f'(\Theta)\psi_1(Z_0) + \psi_2(Z_0) \\
&= (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \psi_1(0) \\
&- a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] \psi_1(Z_0) - \frac{b}{2} \psi_1(Z_0) \\
&+ (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \psi_1(Z_0), \\
& \lambda\psi_2(Z_0) = \varepsilon [g'(\Theta)\psi_1(Z_0) - \psi_2(Z_0)].
\end{aligned}$$

In each of the above systems, from the second equation, it is easy to see that

$$\psi_2(0) = \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) \psi_1(0), \quad \psi_2(Z_0) = \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) \psi_1(Z_0).$$

In these two systems, if we plug ψ_2 into the first equation, then we get the following equations

$$\begin{aligned} & \left\{ \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \psi_1(0) \\ &= \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \psi_1(Z_0), \end{aligned}$$

$$\begin{aligned} & \left\{ \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] \right. \\ & \left. + \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \psi_1(Z_0) \\ &= \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\} \psi_1(0). \end{aligned}$$

It is not difficult to find that $\psi_1(0) = 0$ if and only if $\psi_1(Z_0) = 0$. If $\psi_1(0) = 0$ or if $\psi_1(Z_0) = 0$, then

$$\begin{aligned} & \left[\lambda + f'(\phi(x)) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\phi(x)) \right. \\ & \left. + a \int_{-\infty}^x K(\zeta) d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta) d\zeta \right] \psi_1(x) \\ &= 0. \end{aligned}$$

Hence $\psi_1(x) = 0$, for all $x \in \mathbb{R}$. Therefore, if λ_0 is an eigenvalue and $\psi_0 = \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix}$ is an eigenfunction of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, then $\psi_{01}(0) \neq 0$ and $\psi_{01}(Z_0) \neq 0$.

If we multiply these two equations together, then we find that

$$\begin{aligned}
& \left\{ \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\
& \cdot \left\{ \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] \right. \\
& + \left. \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \psi_1(0) \psi_1(Z_0) \\
& = \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& \cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\} \psi_1(0) \psi_1(Z_0).
\end{aligned}$$

Definition 1.3.2. Define the open unbounded simply connected domain $\Omega = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\varepsilon\}$. Also define the domain $\Omega_0 = \{\lambda \in \mathbb{C} : \lambda \text{ satisfies the following conditions}\}$,

$$\begin{aligned}
\operatorname{Re} \lambda &> -f'(\theta) - g'(\theta) - \frac{a}{2} - b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right], \\
\operatorname{Re} \lambda &> -f'(\Theta) - g'(\Theta) - a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] - \frac{b}{2}.
\end{aligned}$$

We define the Evans function $\mathcal{E} = \mathcal{E}(\lambda, \varepsilon)$ for the first standing wave solution of the nonlinear singularly perturbed system (1.1)-(1.2) by

$$\begin{aligned}
\mathcal{E}(\lambda, \varepsilon) &= \left\{ \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\
& \cdot \left\{ \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] \right. \\
& + \left. \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& - \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& \cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\},
\end{aligned}$$

for all $\lambda \in \Omega$. We also define the Evans function $\mathcal{E} = \mathcal{E}(\lambda)$ for the first standing wave

solution of the scalar integral differential equation (1.3) by

$$\begin{aligned} \mathcal{E}(\lambda) = & \left\{ \lambda + f'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\ & \cdot \left\{ \lambda + f'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\ & - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \left. \right\} \\ & - \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\ & \cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\}, \end{aligned}$$

for all $\lambda \in \Omega_0$.

Remark 1.3.3. *The Evans function for the second standing wave solution of (1.1)-(1.2) can be defined very similarly. The Evans function for the second standing wave solution of the scalar equation (1.3) can also be defined similarly. We omit the details.*

Theorem 1.3.4. (I) *The Evans function $\mathcal{E} = \mathcal{E}(\lambda, \varepsilon)$ is a complex analytic function in Ω and it is real valued if the eigenvalue parameter λ is real.*

(II) *The complex number λ_0 is an eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ if and only if $\mathcal{E}(\lambda_0, \varepsilon) = 0$.*

(III) *The algebraic multiplicity of any eigenvalue λ_0 of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ is equal to the order of λ_0 as a zero of the Evans function $\mathcal{E}(\lambda, \varepsilon)$.*

(IV) *The Evans function enjoys the following limit*

$$\lim_{|\lambda| \rightarrow \infty} \frac{\mathcal{E}(\lambda, \varepsilon)}{\lambda^2} = 1.$$

Proof.

(I) Obviously, the assertion is true.

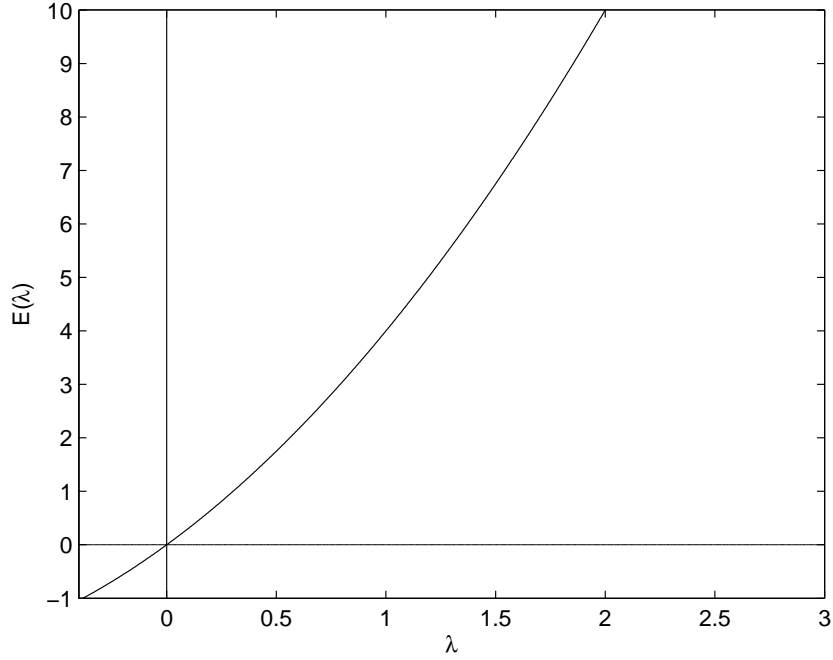


Figure 1.3: Graph of the Evans function $\mathcal{E}(\lambda, \varepsilon)$ for the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) with large ε . In this graph, $a = 2$, $b = 2$, $\alpha = 5$, $\beta = 5$, $\varepsilon = 10$, $\theta = 2$, $\Theta = 2$, $f(u) = u - 1$, $g(u) = u - 2$, $K(x) = \frac{1}{2} \exp(-|x|)$, $W(x) = \frac{1}{2}|x| \exp(-|x|)$ and $\eta(\tau) = \exp(-\tau)$.

(II) If $\lambda_0 \in \Omega$ is an eigenvalue, then there exists a nontrivial bounded continuous solution $\psi_0 = \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix}$ to the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, such that $\psi_{01}(0)\psi_{01}(Z_0) \neq 0$. Therefore, $\mathcal{E}(\lambda_0, \varepsilon) = 0$. On the other hand, if $\mathcal{E}(\lambda_0, \varepsilon) = 0$, then there exists a nontrivial $(\psi_{01}(0), \psi_{01}(Z_0))$, and there exists a nontrivial bounded continuous function $\psi_0 = \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix}$, Therefore, λ_0 is an eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$.

(III) It is very complicated and it is omitted.

(IV) Clearly, the conclusion is correct.

The proof of Theorem 1.3.4 is finished. ■

Definition 1.3.5. Define the positive number ε_0 for the standing wave solutions of the system (1.1)-(1.2) by using the equation

$$\frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon_0) = 0.$$

Theorem 1.3.6. (I) $\lambda = 0$ is a simple eigenvalue of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, that is

$$\begin{aligned} \mathcal{E}(0, \varepsilon) &= 0, & \frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon) &> 0, & \text{for all } \varepsilon \in (\varepsilon_0, \infty), \\ \mathcal{E}(0, \varepsilon) &= 0, & \frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon) &< 0, & \text{for all } \varepsilon \in (0, \varepsilon_0). \end{aligned}$$

(II) For any $\varepsilon \in (0, \varepsilon_0)$, there exist two positive numbers $\lambda_1(\varepsilon) > \lambda_0(\varepsilon) > 0$, such that

$$\frac{\partial \mathcal{E}}{\partial \lambda}(\lambda_0(\varepsilon), \varepsilon) = 0, \quad \mathcal{E}(\lambda_1(\varepsilon), \varepsilon) = 0.$$

(III) In the unbounded domain Ω , the Evans function

$$\mathcal{E}(\lambda, \varepsilon) \neq 0, \quad \text{for all } \varepsilon \in (\varepsilon_0, \infty), \quad \text{for all } \lambda \neq 0, \text{ with } \operatorname{Re} \lambda \geq 0.$$

Proof. Differentiating the traveling wave equations with respect to x , we get

$$\begin{aligned} f'(\phi(x))\phi'(x) + \psi'(x) &= [\alpha - a\phi(x)]K(x) + [\beta - b\phi(x)]W(x - Z_0) \\ &\quad - a\phi'(z) \int_{-\infty}^x K(\zeta)d\zeta - b\phi'(x) \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta, \\ 0 &= [g'(\phi(x))\phi'(x) - \psi'(x)]. \end{aligned}$$

From this system, we see that $\lambda = 0$ is an eigenvalue and $(\phi'(x), \psi'(x))$ is an eigenfunction of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$.

Let us find the derivative of the Evans function. By using the definition, we find that

$$\begin{aligned}
& \frac{\partial \mathcal{E}}{\partial \lambda}(\lambda, \varepsilon) \\
= & \left\{ 1 - \frac{\varepsilon}{(\lambda + \varepsilon)^2} g'(\theta) \right\} \\
& \cdot \left\{ \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\
& - \left. (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& + \left\{ \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\
& \cdot \left\{ 1 - \frac{\varepsilon}{(\lambda + \varepsilon)^2} g'(\Theta) + (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \tau \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& + \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \tau \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& \cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\} \\
& + \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
& \cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \frac{Z_0}{c} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\mathcal{E}'(\lambda) &= \left\{ \lambda + f'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\
&\quad \left. - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
&+ \left\{ \lambda + f'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\
&\cdot \left\{ 1 + (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \tau \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
&+ \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \tau \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
&\cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\} \\
&+ \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
&\cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \frac{Z_0}{c} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right\}.
\end{aligned}$$

In particular, as $\lambda = 0$, we have

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon) &= \left\{ 1 - \frac{1}{\varepsilon} g'(\theta) \right\} \\
&\cdot \left\{ f'(\Theta) + g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) d\tau \right] \right\} \\
&+ \left\{ f'(\theta) + g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\
&\cdot \left\{ 1 - \frac{1}{\varepsilon} g'(\Theta) + (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \tau \eta(\tau) d\tau \right] \right\} \\
&+ \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \tau \eta(\tau) d\tau \right] \right\} \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \xi(c) dc \right] \right\} \\
&+ \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) d\tau \right] \right\} \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^\infty \frac{Z_0}{c} \xi(c) dc \right] \right\} \\
&< 0,
\end{aligned}$$

if $\varepsilon \in (0, \varepsilon_0)$, and $\frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon) > 0$ if $\varepsilon \in (\varepsilon_0, \infty)$.

Moreover

$$\begin{aligned}
\mathcal{E}'(0) &= \left\{ f'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) d\tau \right] \right\} \\
&+ \left\{ f'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\} \\
&\cdot \left\{ 1 + (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \tau \eta(\tau) d\tau \right] \right\} \\
&+ \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \tau \eta(\tau) d\tau \right] \right\} \\
&\cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) dc \right] \right\} \\
&+ \left\{ (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) d\tau \right] \right\} \\
&\cdot \left\{ (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \frac{Z_0}{c} \xi(c) dc \right] \right\} \\
&> 0.
\end{aligned}$$

Let us also find the partial derivative of the Evans function with respect to ε :

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial \varepsilon}(\lambda, \varepsilon) &= \frac{\lambda}{(\lambda + \varepsilon)^2} g'(\theta) \\
&\cdot \left\{ \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\
&- \left. (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right\} \\
&+ \frac{\lambda}{(\lambda + \varepsilon)^2} g'(\Theta) \left\{ \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} \right. \\
&+ \left. b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right\},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 \mathcal{E}}{\partial \varepsilon^2}(\lambda, \varepsilon) &= \frac{2\lambda^2}{(\lambda + \varepsilon)^4} g'(\theta) g'(\Theta) - \frac{2\lambda}{(\lambda + \varepsilon)^3} g'(\theta) \\
&\cdot \left\{ \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\
&- (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^\infty \eta(\tau) \exp(-\lambda\tau) d\tau \right] \left. \right\} \\
&- \frac{2\lambda}{(\lambda + \varepsilon)^3} g'(\Theta) \left\{ \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} \right. \\
&+ b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \left. \right\}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\frac{\partial^2 \mathcal{E}}{\partial \lambda \partial \varepsilon}(\lambda, \varepsilon) &= \frac{\lambda - \varepsilon}{(\lambda + \varepsilon)^3} g'(\theta) \\
&+ \frac{\lambda - \varepsilon}{(\lambda + \varepsilon)^3} g'(\Theta).
\end{aligned}$$

Therefore, we find that

$$\varepsilon_0 = O(1).$$

Fix $\varepsilon \in (0, \varepsilon_0)$. Then $\frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon) < 0$. For all positive, sufficiently large $\lambda > 0$, we find that

$$\frac{\partial \mathcal{E}}{\partial \lambda}(\lambda, \varepsilon) > 0.$$

By using intermediate value theorem, we know that there exists a positive number $\lambda_0(\varepsilon) > 0$, such that

$$\frac{\partial \mathcal{E}}{\partial \lambda}(\lambda_0(\varepsilon), \varepsilon) = 0.$$

Clearly, we know that $\mathcal{E}(0, \varepsilon) = 0$ and that

$$\frac{\partial \mathcal{E}}{\partial \lambda}(\lambda, \varepsilon) < 0,$$

for all real number $\lambda \in (0, \lambda_0(\varepsilon))$. Therefore, $\mathcal{E}(\lambda_0(\varepsilon), \varepsilon) < 0$. Very similarly to before, for all positive, sufficiently large $\lambda > 0$, we find that

$$\mathcal{E}(\lambda, \varepsilon) > 0.$$

By using intermediate value theorem again, we know that there exists a positive number $\lambda_1(\varepsilon) > \lambda_0(\varepsilon) > 0$, such that

$$\mathcal{E}(\lambda_1(\varepsilon), \varepsilon) = 0.$$

Additionally, we also find that

$$\lambda_0 = O(\varepsilon), \quad \lambda_1 = O(1).$$

Lemma 1.3.7. *Suppose that the nonnegative function $\omega \geq 0$ is defined on $(0, \infty)$ and suppose that $0 < \int_0^\infty \omega(x)dx < \infty$. For any complex number $\lambda \neq 0$, if $\operatorname{Re}\lambda \geq 0$, then*

$$\left| \int_0^\infty \exp(\lambda x)\omega(x)dx \right| < \int_0^\infty \omega(x)dx.$$

Proof. See [60]. ■

For all $\varepsilon \in (\varepsilon_0, \infty)$ and for all $\lambda \in \Omega$, if $\operatorname{Re}\lambda \geq 0$, then by using Lemma 1.3.7, we

have the following estimates

$$\begin{aligned}
& \left| \lambda + f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right| \\
> & \left| f'(\theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right|, \\
& \left| \lambda + f'(\Theta) + \frac{\varepsilon}{\lambda + \varepsilon} g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\
& \left. - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right| \\
> & \left| f'(\Theta) + g'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) d\tau \right] \right|, \\
& \left| (\beta - b\Theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right| \\
< & \left| (\beta - b\Theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) d\tau \right] \right|, \\
& \left| (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right| \\
< & \left| (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right|.
\end{aligned}$$

Therefore, we obtain

$$|\mathcal{E}(\lambda, \varepsilon)| > |\mathcal{E}(0, \varepsilon)| = 0.$$

The proof of Theorem 1.3.6 is finished. ■

Corollary 1.3.8. *For the standing wave solutions of the integral differential equation (1.3), there hold the following results*

$$\mathcal{E}(0) = 0, \quad \mathcal{E}'(0) > 0.$$

Moreover, for all complex number $\lambda \neq 0$ with $\operatorname{Re}\lambda \geq 0$, we have

$$\mathcal{E}(\lambda) \neq 0.$$

Proof. The first half is easy to prove. Let us establish the second half. We have the following estimates

$$\begin{aligned}
& \left| \lambda + f'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right| \\
> & \left| f'(\theta) + \frac{a}{2} + b \left[\int_{-\infty}^{-Z_0} W(\zeta) d\zeta \right] - (\alpha - a\theta) \frac{K(0)}{\phi'(0)} \right|, \\
& \left| \lambda + f'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} \right. \\
& \left. - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right| \\
> & \left| f'(\Theta) + a \left[\int_{-\infty}^{Z_0} K(\zeta) d\zeta \right] + \frac{b}{2} - (\beta - b\Theta) \frac{W(0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) d\tau \right] \right|, \\
& \left| (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) \exp(-\lambda\tau) d\tau \right] \right| \\
< & \left| (\beta - b\theta) \frac{W(-Z_0)}{\phi'(Z_0)} \left[\int_0^{\infty} \eta(\tau) d\tau \right] \right|, \\
& \left| (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) \exp\left(-\frac{\lambda}{c} Z_0\right) dc \right] \right| \\
< & \left| (\alpha - a\Theta) \frac{K(Z_0)}{\phi'(0)} \left[\int_0^{\infty} \xi(c) dc \right] \right|.
\end{aligned}$$

Therefore, we obtain

$$|\mathcal{E}(\lambda)| > |\mathcal{E}(0)| = 0.$$

The proof of Corollary 1.3.8 is finished. ■

Remark 1.3.9. *The Evans function for the second standing wave solution of system (1.1)-(1.2) and the Evans function for the second standing wave solution of equation (1.3) also enjoy the properties mentioned in Theorem 1.3.4, Theorem 1.3.6 and Corollary 1.3.8.*

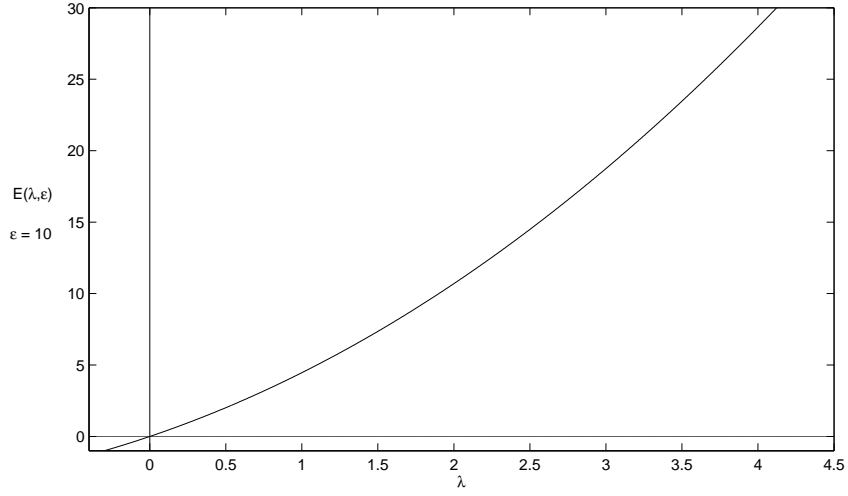


Figure 1.4: Graph of the Evans function $\mathcal{E}(\lambda, \varepsilon)$ for the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) with small ε . In this graph, $a = 2$, $b = 2$, $\alpha = 5$, $\beta = 5$, $\varepsilon = 10$, $\theta = 2$, $\Theta = 2$, $f(u) = u - 1$, $g(u) = u - 2$, $K(x) = \frac{1}{2} \exp(-|x|)$, $W(x) = \frac{1}{2}|x| \exp(-|x|)$ and $\eta(\tau) = \exp(-\tau)$.

1.3.3 Stability on the interval (ε_0, ∞)

Theorem 1.3.10. *For all $\varepsilon \in (\varepsilon_0, \infty)$, the standing wave solutions ϕ_1 and ϕ_2 of the singularly perturbed system of integral differential equations (1.1)-(1.2) with general functions $f(u)$ and $g(u)$ are exponentially stable.*

Proof. For the standing wave solutions ϕ_1 and ϕ_2 , there exists no nonzero solution to the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ in the region $\text{Re}\lambda \geq -\varepsilon$. Moreover, the neutral eigenvalue $\lambda = 0$ is simple. By using the linearized stability criterion, we find that the standing wave solutions of the system (1.1)-(1.2) are exponentially stable. The proof of Theorem 1.3.10 is finished. ■

Corollary 1.3.11. *The standing wave solutions ϕ_3 and ϕ_4 of the scalar integral differential equation (1.3) are exponentially stable.*

Proof. The proof of Corollary 1.3.11 is very similar to that of Theorem 1.3.10 and

it is omitted. ■

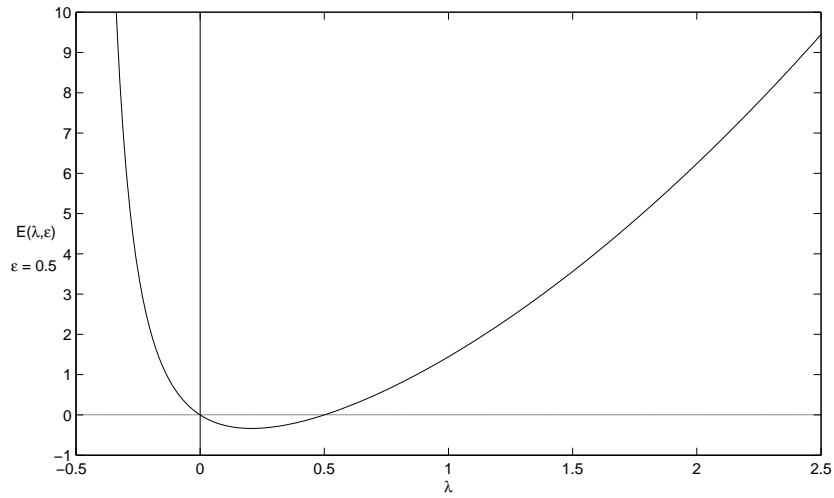


Figure 1.5: Graph of the Evans function $\mathcal{E}(\lambda)$ for the nonlinear integral differential equation (1.3). In this graph, $a = 2, b = 2, \alpha = 5, \beta = 5, \theta = 2, \Theta = 2, f(u) = u - 1, K(x) = \frac{1}{2} \exp(-|x|), W(x) = \frac{1}{2}|x| \exp(-|x|)$ and $\eta(\tau) = \exp(-\tau)$.

1.3.4 Instability on the interval $(0, \varepsilon_0)$

Theorem 1.3.12. *For all $\varepsilon \in (0, \varepsilon_0)$, the standing wave solutions ϕ_1 and ϕ_2 of the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) with general functions $f(u)$ and $g(u)$ are exponentially unstable.*

Proof. For the standing wave solutions ϕ_1 and ϕ_2 , there exists a positive eigenvalue $\lambda_1(\varepsilon) > 0$ to the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$ in the region $\text{Re}\lambda \geq -\varepsilon$. By the linearized stability criterion, we see the standing wave solutions of the system (1.1)-(1.2) are exponentially unstable. The proof of Theorem 1.3.12 is finished. ■

1.3.5 Examples

Choose the model parameters and functions in the following way:

$$\alpha = \beta = 5,$$

$$a = b = 2,$$

$$\theta = \Theta = 2,$$

$$k = m = 1,$$

$$n = 1, \quad l = 2,$$

$$f(u) = u - 1, \quad g(u) = u - 2,$$

$$K(x) = \frac{1}{2} \exp(-|x|), \quad W(x) = \frac{1}{2} |x|^\rho \exp(-|x|),$$

$$\eta(\tau) = \exp(-\tau),$$

$$\varepsilon \in (0, \infty), \quad \rho = 0, 1.$$

Then $Z_0 = 0$ and

$$\begin{aligned} \int_{-\infty}^x K(\zeta) d\zeta &= \frac{1}{2} \exp(x), & \text{for all } x < 0, \\ &= 1 - \frac{1}{2} \exp(-x), & \text{for all } x \geq 0, \\ \int_x^\infty K(\zeta) d\zeta &= \frac{1}{2} \exp(-x), & \text{for all } x \geq 0, \\ &= 1 - \frac{1}{2} \exp(x), & \text{for all } x < 0. \end{aligned}$$

Example 1. Let $\rho = 0$, that is $W(x) = \frac{1}{2} \exp(-|x|)$. Then

$$\begin{aligned} \phi_1(x) &= \frac{3 + 5 \exp(x)}{2 + 2 \exp(x)}, & \text{for all } x < 0, \\ \phi_1(x) &= \frac{13 - 5 \exp(-x)}{6 - 2 \exp(-x)}, & \text{for all } x \geq 0, \\ \phi_2(x) &= \frac{3 + 5 \exp(-x)}{2 + 2 \exp(-x)}, & \text{for all } x \geq 0, \\ \phi_2(x) &= \frac{13 - 5 \exp(x)}{6 - 2 \exp(x)}, & \text{for all } x < 0, \\ \phi_3(x) &= \frac{1 + 5 \exp(x)}{1 + 2 \exp(x)}, & \text{for all } x < 0, \\ \phi_3(x) &= \frac{11 - 5 \exp(-x)}{5 - 2 \exp(-x)}, & \text{for all } x \geq 0, \\ \phi_4(x) &= \frac{1 + 5 \exp(-x)}{1 + 2 \exp(-x)}, & \text{for all } x \geq 0, \\ \phi_4(x) &= \frac{11 - 5 \exp(x)}{5 - 2 \exp(x)}, & \text{for all } x < 0. \end{aligned}$$

Example 2. Let $\rho = 1$, that is $W(x) = \frac{1}{2}|x| \exp(-|x|)$. Then

$$\begin{aligned} \mathcal{E}(\lambda) &= \lambda(\lambda + 3), \\ \mathcal{E}'(\lambda) &= 2\lambda + 3, \\ \mathcal{E}(\lambda, \varepsilon) &= \left[\lambda - 1 + \frac{\varepsilon}{\lambda + \varepsilon} \right] \left[\lambda + 3 + \frac{\varepsilon}{\lambda + \varepsilon} \right], \\ \frac{\partial \mathcal{E}}{\partial \lambda}(\lambda, \varepsilon) &= 2 \left[\lambda + \frac{\varepsilon}{\lambda + \varepsilon} \right] \left[1 - \frac{\varepsilon}{(\lambda + \varepsilon)^2} \right] + 2 \left[1 - \frac{\varepsilon}{(\lambda + \varepsilon)^2} \right]. \end{aligned}$$

Example 3. Let $\rho = 1$, that is $W(x) = \frac{1}{2}|x| \exp(-|x|)$. Then

$$\begin{aligned} \varepsilon_0 &= 1, \\ \lambda_0(\varepsilon) &= \sqrt{\varepsilon} - \varepsilon, \\ \lambda_1(\varepsilon) &= 1 - \varepsilon. \end{aligned}$$

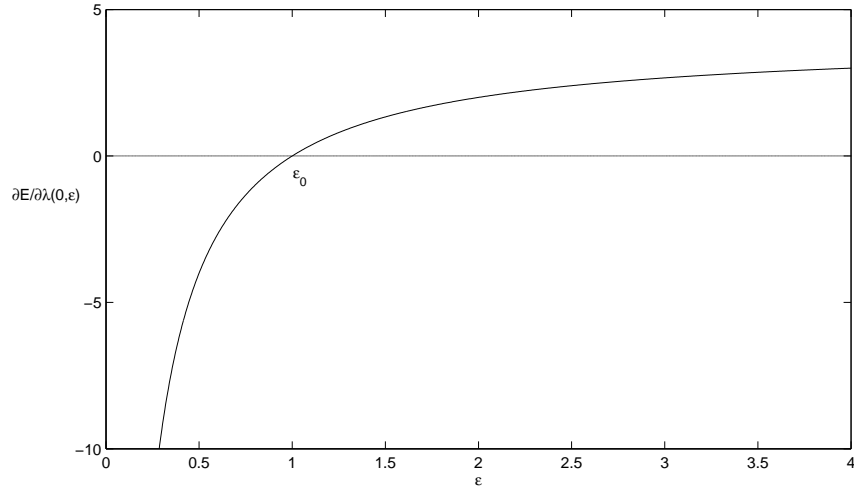


Figure 1.6: Graph of the derivative of the Evans function $\frac{\partial \mathcal{E}}{\partial \lambda}(0, \varepsilon)$ for the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2). In this graph, $a = b = 2$, $\alpha = \beta = 5$, $\varepsilon \in (0, \infty)$, $\theta = \Theta = 2$, $f(u) = u - 1$, $g(u) = u - 2$, $K(x) = \frac{1}{2} \exp(-|x|)$, $W(x) = \frac{1}{2}|x| \exp(-|x|)$ and $\eta(\tau) = \exp(-\tau)$.

1.4 Concluding Remarks

1.4.1 Summary

For the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2), suppose that the functions $\xi \geq 0$ and $\eta \geq 0$ on $(0, \infty)$, suppose that $f(u) + g(u) =$

$m(u - n) + k(u - l)$ and suppose also that there exists a real number $Z_0 \geq 0$, such that

$$\begin{aligned}
K(-x) &= K(x), & \text{for all } x \in \mathbb{R}, & \quad \text{and } \int_{\mathbb{R}} K(x)dx = 1, \\
W(-x) &= W(x), & \text{for all } x \in \mathbb{R}, & \quad \text{and } \int_{\mathbb{R}} W(x)dx = 1, \\
\int_0^\infty \xi(c)dc &= 1, & \int_0^\infty \eta(\tau)d\tau &= 1, \\
\int_0^\infty \frac{1}{c}\xi(c)dc &< \infty, & \int_0^\infty \tau\eta(\tau)d\tau &< \infty, \\
k + m + a \int_{-\infty}^x K(\zeta)d\zeta + b \int_{-\infty}^{x-Z_0} W(\zeta)d\zeta &> 0, & \text{for all } x \in \mathbb{R}, \\
f'(\theta) + g'(\theta) + \frac{a}{2} + b \int_{-\infty}^{-Z_0} W(\zeta)d\zeta &> 0, \\
f'(\Theta) + g'(\Theta) + a \int_{-\infty}^{Z_0} K(\zeta)d\zeta + \frac{b}{2} &> 0, \\
\alpha \neq a\Theta, & \quad \beta \neq b\theta, \\
\frac{kl + mn}{k + m} < \theta \leq \Theta < \frac{\alpha + \beta + kl + mn}{a + b + k + m}, \\
(\alpha - a\theta)K(x) + (\beta - b\theta)W(x - Z_0) &\geq 0, & \text{for all } x \in \mathbb{R}, \\
(\alpha - a\Theta)K(x) + (\beta - b\Theta)W(x - Z_0) &\geq 0, & \text{for all } x \in \mathbb{R}, \\
(\alpha - a\theta)K(0) + (\beta - b\theta)W(-Z_0) &> 0, \\
(\alpha - a\Theta)K(Z_0) + (\beta - b\Theta)W(0) &> 0, \\
\int_{-\infty}^{Z_0} K(\zeta)d\zeta &= \frac{2kl + 2mn + \beta - (2k + 2m + b)\Theta}{2(a\Theta - \alpha)}, \\
\int_{-\infty}^{-Z_0} W(\zeta)d\zeta &= \frac{2kl + 2mn + \alpha - (2k + 2m + a)\theta}{2(b\theta - \beta)},
\end{aligned}$$

then there exist two standing wave solutions $\phi_1 = \phi_1(x)$ and $\phi_2 = \phi_2(x)$, such that $\phi_1 < \theta$ on $(-\infty, 0)$, $\phi_1(0) = \theta$ and $\phi_1 > \theta$ on $(0, \infty)$; $\phi_1 < \Theta$ on $(-\infty, Z_0)$, $\phi_1(Z_0) = \Theta$ and $\phi_1 > \Theta$ on (Z_0, ∞) . Similarly, $\phi_2 > \theta$ on $(-\infty, 0)$, $\phi_2(0) = \theta$ and $\phi_2 < \theta$ on $(0, \infty)$; $\phi_2 > \Theta$ on $(-\infty, -Z_0)$, $\phi_2(-Z_0) = \Theta$ and $\phi_2 < \Theta$ on $(-Z_0, \infty)$.

If $f(u) + g(u) = u(u - 1)(Du - 1)$ is a cubic polynomial function, then these results

are also correct.

Additionally, if $0 < \varepsilon < \varepsilon_0$, then the standing wave solutions are unstable. However, if $\varepsilon > \varepsilon_0$, then the standing wave solutions are stable. The results for the system are surprisingly interesting in mathematical neuroscience.

For the scalar integral differential equation, if $f(u) = m(u - n)$ and

$$\alpha + \beta + 2mn = (a + b + 2m)\theta, \quad n < \theta = \Theta < \frac{\alpha + \beta + kl + mn}{a + b + k + m},$$

$$(\alpha - a\theta)K(x) + (\beta - b\theta)W(x) \geq 0, \text{ for all } x \in \mathbb{R},$$

$$(\alpha - a\theta)K(0) + (\beta - b\theta)W(0) > 0,$$

then there exist two standing waves solutions. If f is a cubic function, then similar results are also true. Additionally, if $(\beta - b\theta)W(0) \geq 0$, then the standing wave solutions are stable.

It is worth mentioning that for the scalar integral differential equation (1.3), the standing wave solutions are always stable. While for the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2), even though the parameter ε plays no role in the existence analysis, it does play a very important role in the stability analysis. In particular, as ε crosses ε_0 , the instability and stability of the standing wave solutions are interchanged.

1.4.2 Open Problems

Under different conditions on the model parameters and functions, there may exist standing pulse solutions rather than standing front solutions. But these problems have not been investigated rigorously.

Chapter 2

Influence of Sodium Currents

2.1 Introduction

Wave propagation in synaptically coupled neuronal networks is of great research interest in computational neuroscience and in applied mathematics. Many mathematical models have been proposed to describe the propagation of nerve impulses. Mathematically, traveling waves share the same properties as nerve impulses: they propagate with constant shapes and constant velocities. Therefore, it is reasonable to use traveling waves to model the propagation of nerve impulses. Traveling waves may also represent the stimuli in the turtle visual cortex [48] and cat visual cortex [28], as well as cortical epilepsy [11] and migraine [43]. Let us briefly review how a nerve impulse is generated. Once a cell membrane potential reaches a threshold, active sodium ion conductance gates are opened and an inward flow of sodium ions results, causing further depolarization. This depolarization increases sodium conductance, consequently inducing more sodium current. This iterative cycle continues driving the membrane potential to sodium reversal potential, and concludes with the closure of the sodium gates. As we can easily see, the sodium current is the first cause of the nerve impulse. Roughly speaking, in a three-dimensional phase space, a traveling pulse solution (that is, a nerve impulse) consists of four pieces: the traveling wave front, the right, the traveling wave back and the left

(recovery period). The traveling wave front is essentially due to the fast movement of sodium ions from the exterior to the interior of the cell membrane and the traveling wave back is essentially due to the movement of the potassium ions from the interior to the exterior of the membrane. Without the sodium current, there would be no traveling wave front, thus no pulse would exist.

The focus of this chapter is to investigate the influence of sodium currents (through voltage gated channels, modeled with nonlinear functions, see [34], [21] and [30]) on wave speeds of traveling wave fronts. There are many biological mechanisms to influence the speeds of traveling wave fronts. Mathematically, we want to apply the model equations derived from neuroscience to investigate how the mechanisms influence the wave speeds. On the other hand, to keep things simple, we may assume that some mechanisms are fixed while others change. Motivated by this idea, we will investigate the influence of sodium currents on wave speeds.

2.1.1 Model equations and biological backgrounds

Consider the following nonlinear singularly perturbed systems of integral differential equations

$$\frac{\partial u}{\partial t} + f(u) + w = \alpha \int_{\mathbb{R}} K(x-y)H(u(y,t) - \theta)dy, \quad (2.1)$$

$$\frac{\partial w}{\partial t} = \varepsilon(g(u) - \gamma w), \quad (2.2)$$

and

$$\frac{\partial u}{\partial t} + f(u) + w = \alpha \int_{\mathbb{R}} K(x-y)H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right)dy, \quad (2.3)$$

$$\frac{\partial w}{\partial t} = \varepsilon(g(u) - \gamma w), \quad (2.4)$$

respectively, where $0 < c < \infty$ in (2.3)-(2.4). See Amari [3], Bressloff [8], Bressloff and Folias [9], Bressloff, Folias, Prat and Li [45], Coombes [12], Coombes, Lord and

Owen [13], Coombes and Owen [14], Ermentrout [19], Ermentrout and McLeod [20], Ermentrout and Terman [21], Folias and Bressloff [26]-[27], Pinto and Ermentrout [46], Pinto, Jackson and Wayne [17], Terman, Ermentrout and Yew [54] and Zhang [60]-[59]. The parameters are consistent with the models previously described. In addition we note that $c > 0$ represents the finite propagation speed of an action potential along an axon, and $\frac{1}{c}|x - y|$ denotes the spatial temporal delay. The function $f(u)$ usually represents sodium currents across the cell membrane. The most popular sodium current function is $f_c(u) = u(u - 1)(Du - 1)$, where $D > 0$ is a constant. Another popular function is $f_0(u) = u$. Biologically speaking, the cubic representation is much better than the linear representation to model sodium current because sodium channels are voltage gated channels (in another word, sodium conductance should be a function of voltage). On the other hand, mathematically, a linear function is much better than a cubic function because a linear function is easy to handle. Note that if $\varepsilon = 0$ and $w = 0$, then (2.1)-(2.2) and (2.3)-(2.4) reduce to the following scalar integral differential equations

$$\frac{\partial u}{\partial t} + f(u) = \alpha \int_{\mathbb{R}} K(x - y)H(u(y, t) - \theta)dy, \quad (2.5)$$

$$(2.6)$$

$$\frac{\partial u}{\partial t} + f(u) = \alpha \int_{\mathbb{R}} K(x - y)H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right) dy, \quad (2.7)$$

respectively, where $0 < c < \infty$ in (2.7). Additionally, if $f(u) = m(u - n)$, where $m > 0$ and $n \in \mathbb{R}$ are real constants, then (2.5) and (2.7) become the particular equations

$$\frac{\partial u}{\partial t} + m(u - n) = \alpha \int_{\mathbb{R}} K(x - y)H(u(y, t) - \theta)dy, \quad (2.8)$$

and

$$\frac{\partial u}{\partial t} + m(u - n) = \alpha \int_{\mathbb{R}} K(x - y)H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right) dy. \quad (2.9)$$

We may interpret the constant m as the sodium conductance and the constant n as the sodium reversal potential. To investigate the speeds of the fast traveling pulse solutions

of (2.1)-(2.2) and (2.3)-(2.4), we will study the speeds of the traveling wave fronts of (2.5) and (2.7), respectively.

In this chapter, we will investigate the general scalar integral differential equation

$$\frac{\partial u}{\partial t} + f(u) = \alpha \int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) dy, \quad (2.10)$$

where $0 < c \leq \infty$, $\alpha > 0$ and $\theta > 0$ are constants. This model includes (2.5) if $c = \infty$ and includes (2.7) if $0 < c < \infty$. The model equation involves both sodium current function (in particular, $f(u)$ could represent the cubic polynomial function) and the spatial temporal delay. That is why it is more realistic than any previous *scalar* integral differential equations. See [6], [7], [15], [18], [32], [33], [39], [40], [41] and [42] for more information of the model equations (1)-(2), (2.3)-(2.4), (2.5), (2.7), (2.8), (2.9) and (2.10). See [2], [4]-[5], [10], [22], [23], [24], [25], [29], [31], [38], [49], [50], [51], [52], [56]-[57] for related model equations.

As well known, the cubic polynomial function $f_c(u) = u(u-1)(Du-1)$ is the most popular function used to describe sodium currents in neuronal networks, see Borisjuk, Ermentrout, Friedman and Terman [1], Hodgkin and Huxley [30], and Zhang [61]. It is not surprising to see that the linear function $f_0(u) = u$ is the simplest function for sodium current, it may be viewed as a linear approximation of the cubic polynomial function, see [9]-[45], [12]-[15], [18], [46], [17], [52] and [60]. The sodium current may be perturbed by many factors. If $f_0(u) = u$ is the correct function for sodium current, then we may think that $f_+(u) = \frac{1}{D} \sinh(Du)$ and $f_-(u) = \frac{1}{D} \tanh(Du)$ are perturbations of $f_0(u) = u$ on \mathbb{R} , where $D > 0$ is a constant. Note that

$$\lim_{D \rightarrow 0} \frac{\sinh(Du)}{D} = u, \quad \lim_{D \rightarrow 0} \frac{\tanh(Du)}{D} = u.$$

If $f_+(u) = \frac{1}{D} \sinh(Du)$ or $f_-(u) = \frac{1}{D} \tanh(Du)$ describes sodium currents in neuronal networks, then $\varphi_+(u) = \frac{1}{Du} \sinh(Du)$ or $\varphi_-(u) = \frac{1}{Du} \tanh(Du)$ may be viewed

as sodium conductance in these networks (assuming that sodium reversal potential is zero). Here we may interpret D as a biological perturbation parameter. It is worth mentioning that there are other nonlinear functions responsible for the sodium currents. For example, the sodium current functions in the Morris-Lecar equations [21], in the ring model [19] and in the Pelinovsky-Yakhno equations [19], are different from those mentioned above. These sodium current functions are derived also based on Ohm's law, but with different formulations of sodium conductance.

2.1.2 The mathematical assumptions

Mathematically, the function $f(u)$ satisfies $f(n) = 0$, $f'(n) > 0$, and $f'(u) > 0$ for all sufficiently large $u > 0$, where n is a constant, representing sodium reversal potential. Suppose that $0 < 2f(\theta) < \alpha$.

Notice that if the synaptic coupling K is in classes (A) or (C), then the traveling wave front is nonnegative, for all $z \in \mathbb{R}$, see [62], [63], [64]. However, if the synaptic coupling K is in class (B), then the traveling wave front is negative on some interval $(-\infty, -N)$, where $N > 0$ is some constant, depending on K , see [64]. The (n, ∞) indicates that while assumptions are made on the interval (n, ∞) if the synaptic couplings are in classes (A) or (C), the same assumptions should be made on $\mathbb{R} = (-\infty, \infty)$ if the synaptic couplings are in class (B). More assumptions will be made in Section 2 for the functions and parameters of the model equations.

2.1.3 The speed index functions

Motivation of a speed index function. Consider the scalar integral differential equation

$$\frac{\partial u}{\partial t} + u = \alpha \int_{\mathbb{R}} K(x - y) H(u(y, t) - \theta) dy,$$

where α and θ are constants such that $0 < 2\theta < \alpha$. Suppose that the front satisfies

$U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$. Then the traveling wave equation becomes a non homogeneous, first order, linear ordinary differential equation

$$\nu U' + U = \alpha \int_{\mathbb{R}} K(z - y)H(U(y) - \theta)dy = \alpha \int_{-\infty}^z K(x)dx.$$

There exists a unique traveling wave front $u(x, t) = U(x + \nu_0 t)$ to this equation, explicitly

$$\begin{aligned} U(z) &= \alpha \int_{-\infty}^z K(x)dx - \alpha \int_{-\infty}^z \exp\left(\frac{x-z}{\nu_0}\right) K(x)dx, \\ U'(z) &= \frac{\alpha}{\nu_0} \int_{-\infty}^z \exp\left(\frac{x-z}{\nu_0}\right) K(x)dx, \end{aligned}$$

where ν_0 is the wave speed and $z = x + \nu_0 t$ is the moving coordinate. The wave speed ν_0 is the unique positive solution of the equation

$$\alpha \int_{-\infty}^0 \exp\left(\frac{x}{\nu}\right) K(x)dx = \frac{\alpha}{2} - \theta.$$

This last equation is equivalent to $U(0) = \theta$.

The function $\phi(\nu) \equiv \alpha \int_{-\infty}^0 \exp\left(\frac{x}{\nu}\right) K(x)dx$ is called the speed index function for that integral differential equation, see [62]-[64].

Speeds play a very important role in the study of traveling waves of nonlinear integral differential equations. Indeed, once the speed is found, the traveling wave solution is easy to solve by using techniques in differential equations. Moreover, the speeds are closely related to the stability of traveling waves. Intuitively, stable waves are the most important solutions. We have developed a method to construct the speed index functions for equations (2.8) and (2.9), see [64]. The speed index functions are very interesting and important. There exists a solution to the equation $\phi(\nu) = \frac{\alpha}{2} - \theta$, which involves the speed index function and the intrinsic parameters, this unique solution is precisely the speed of the front. Through this, we are able to investigate how the speed depends on various parameters such as α , θ and c , as well as the synaptic coupling K . Many

estimates and asymptotic behaviors of the speed as the parameters approach certain numbers can be investigated very clearly. More appropriately, the speed index function should be called biological mechanism index function because it involves so many parameters (again α , θ and c) representing biological mechanisms. By using properties of the speed index functions, we may be able to prove a simple but elegant identity, which connects the speed of the front of the model (2.9) where there is a spatial temporal delay to the speed of the front of the model (2.8) where there is no delay. For the special case $m = 1$ and $n = 0$, see [64].

It is very difficult to formulate a speed index function for (2.10) with a general nonlinear function $f(u)$. Even if we are able to find the speed index function when $f(u)$ is nonlinear, it may be very complicated, and turns out to be almost useless. Therefore, we will use the speed index functions for (2.8) and (2.9) with the particular linear function $f_l(u) = m(u - n)$ to derive estimates on wave speeds of (2.10) with nonlinear function $f(u)$. Roughly speaking, we are going to treat the problem (2.10) with nonlinear function $f(u)$ as a perturbation of the problems (2.8) and (2.9) with the special linear function $f_l(u) = m(u - n)$.

2.1.4 Known results and open problems

Given any nonnegative kernel function K , the existence, uniqueness and stability of the fast traveling pulse solution of the nonlinear singularly perturbed system of integral differential equations (2.1)-(2.2) and (2.3)-(2.4), respectively, and the existence, uniqueness and stability of the traveling wave front of the scalar integral differential equation (2.5) and (2.7), respectively, have been rigorously established before. See [10], [13], [14], [39], [46], [17], [52], [60]-[59]. Moreover, the existence, uniqueness and/or stability of standing waves [29] and [46], spiral waves [40], and lurching waves [54] of similar

model equations have also been numerically or analytically established (partially). Let us provide some concrete results about the traveling waves.

Theorem 2.1.1. *Let $m > 0$, $n \in \mathbb{R}$, α and θ be four appropriate real constants and let $f(u)$ be a smooth function defined on \mathbb{R} , such that $f(n) = 0$, $m = f'(n) > 0$ and $0 < 2f(\theta) < \alpha$, and the equation $f(U) = \alpha$ has a unique solution $U = \beta > \theta$, such that $f'(\beta) > 0$. Then there exists a unique traveling wave front $U = U_{\text{front}}(\cdot)$ to equation (2.10), such that $U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$, where $z = x + \mu_0 t$ and μ_0 represents the unique positive speed. The traveling wave front satisfies the differential equations*

$$\begin{aligned} \mu_0 U' + f(U) &= \alpha \int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y|\right) - \theta\right) dy \\ &= \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx, \end{aligned}$$

where $s(x) = -1$ for $x < 0$, $s(0) = 0$ and $s(x) = 1$ for $x > 0$. The traveling wave front enjoys the decay estimates

$$\begin{aligned} |U(z)| &\leq C \exp(-\rho|z|), & \text{on } & (-\infty, 0), \\ |U(z) - \beta| &\leq C \exp(-\rho|z|), & \text{on } & (0, \infty), \\ |U'(z)| &\leq C \exp(-\rho|z|), & \text{on } & \mathbb{R}, \end{aligned}$$

where $C > 0$ and $\rho > 0$ are positive constants.

See [13], [61] and [64].

Theorem 2.1.2. *(Pinto and Ermentrout [46]) Consider the following nonlinear singularly perturbed system of integral differential equations*

$$\frac{\partial u}{\partial t} + u + w = \alpha \int_{\mathbb{R}} K(x-y) H(u(y, t) - \theta) dy, \quad (2.11)$$

$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \quad (2.12)$$

Suppose that the parameters α , γ and θ satisfy the conditions $0 < 2\theta < \alpha$ and $0 < \alpha\gamma < (1 + \gamma)\theta$. Then there exists a number ε_0 , $0 < \varepsilon_0 \ll 1$. For each fixed singular perturbation parameter $\varepsilon \in (0, \varepsilon_0)$, there are exactly two positive wave speeds $\nu_{\text{slow}}(\varepsilon)$ and $\nu_{\text{fast}}(\varepsilon)$, with $0 < \nu_{\text{slow}}(\varepsilon) < \nu_{\text{fast}}(\varepsilon)$. There exists a slow traveling pulse solution $(U, W) = (U_{\text{slow}}(\varepsilon, \cdot), W_{\text{slow}}(\varepsilon, \cdot))$ corresponding to the slow wave speed ν_{slow} , and there exists a fast traveling pulse solution $(U, W) = (U_{\text{fast}}(\varepsilon, \cdot), W_{\text{fast}}(\varepsilon, \cdot))$ corresponding to the fast wave speed ν_{fast} . Both the fast traveling pulse solution and the slow traveling pulse solution satisfy the traveling wave equations

$$\begin{aligned}\nu(\varepsilon)U' + U + W &= \alpha \int_{\mathbb{R}} K(z - y)H(U(y) - \theta)dy, \\ \nu(\varepsilon)W' &= \varepsilon(U - \gamma W),\end{aligned}$$

and the homogeneous boundary conditions

$$\lim_{z \rightarrow \pm\infty} (U(\varepsilon, z), W(\varepsilon, z)) = \lim_{z \rightarrow \pm\infty} (U_z(\varepsilon, z), W_z(\varepsilon, z)) = (0, 0).$$

The same results are also correct for the systems (2.1)-(2.2) and (2.3)-(2.4) under appropriate conditions on f and g .

The slow pulse is unstable because there exists a positive eigenvalue to an associated linear operator and biologically the slow pulse is not very interesting. The fast traveling pulse solution is of great importance and that is why Zhang [60] studied its stability.

By the construction and application of Evans functions and by coupling ideas in differential equations and functional analysis, Zhang analyzed the spectrum (in particular, eigenvalues) of an associated linear differential operator $\mathcal{L}(\varepsilon)$, which is obtained by linearizing the nonlinear system about the fast traveling pulse solution. By using linearized stability criterion, he demonstrated the exponential stability of the fast traveling pulse solution in [60]. This is the first mathematically rigorous stability result in the area of nonlinear nonlocal neuronal networks.

Theorem 2.1.3. *Suppose that the positive parameters $\alpha > 0$, $\gamma > 0$, $\varepsilon > 0$ and $\theta > 0$ satisfy the conditions $0 < 2\theta < \alpha$, $0 < \alpha\gamma < (1 + \gamma)\theta$, and $\varepsilon \in (0, \varepsilon_0)$ with $0 < \varepsilon_0 \ll 1$. Then the unique fast traveling pulse solution $(U, W) = (U_{\text{fast}}(\varepsilon, \cdot), W_{\text{fast}}(\varepsilon, \cdot))$ is exponentially stable in the sense of $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ -norm, as $t \rightarrow \infty$.*

See [60].

Some preliminary results on the speeds of the traveling wave fronts of (2.8) and (2.9) have been obtained in [64], where $m = 1$ and $n = 0$. Zhang has investigated the influence of synaptic rate constant α , threshold θ and action potential speed c on the wave speed.

Moreover, there have been some other nice results on similar model equations. It seems that lateral inhibition supports and stabilizes standing waves (see Guo and Chow [29], Pinto and Ermentrout [46]) and lateral excitation leads to Turing-Hopf bifurcation in delayed neuronal networks (see Atay and Hutt [6]).

The existence results in Theorem 2.1.1 and Theorem 2.1.2 can be proved by using fixed point theorem or other techniques in dynamical systems, see Coombes, Lord and Owen [13], Coombes and Owen [14], Ermentrout and McLeod [20], Pinto and Ermentrout [46], Pinto, Jackson and Wayne [17], Sandstede [52] and Zhang [60]-[61].

Suppose that $f \in C^\infty(\mathbb{R})$. If $c = \infty$, then the traveling wave front of (2.10) is as smooth as the kernel function K . If $0 < c < \infty$ and $K(0) \neq 0$, then the front is at most C^1 smooth.

If f is a nonlinear function such that the equation $f(u) = \alpha$ has no solution which is larger than the threshold, then there exists no traveling wave front to equation (2.10). If neither the equation $f(u) = 0$ nor the equation $f(u) = \alpha$ has a solution, then the sodium current function f causes wave propagation failure. For example, if D is sufficiently large (i.e too much biological noises are present), then the equation $\frac{1}{D} \tanh(Du) = \alpha$

has no solution at all. If $f(u) = \frac{1}{2}(1 + \tanh u)$ is used to describe a sodium current, then it causes wave propagation failure because neither the equation $\frac{1}{2}(1 + \tanh u) = 0$ nor the equation $\frac{1}{2}(1 + \tanh u) = \alpha$ has a solution, where $\alpha > 1$.

For the integral differential equations (2.8) and (2.9), we have constructed the speed index function and the stability index function by

$$\phi(\mu) = \alpha \int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu}x\right) K(x)dx, \quad \mathcal{E}(\lambda) = 1 - \frac{1}{\phi(\mu_0)}\phi\left(\frac{\mu_0}{\lambda+1}\right),$$

where the wave speed μ_0 is the unique solution of the equation $\phi(\mu) = \frac{\alpha}{2} - \theta$, see [62]-[64]. The stability of the traveling wave front is determined completely by the zeros of the stability index function $\mathcal{E}(\lambda)$. The speed index function and the stability index function may be used to study bifurcation of waves of the model equations, as the parameters α , θ and c vary.

Summary

The mathematical study of the traveling waves are motivated by several important works in biology, see [11], [31], [36], [43], [16], [48], [38], [28], [35] and [55].

By using mathematical analysis as a main technical tool of investigation, we study how wave speed of the traveling wave front of a scalar integral differential equation is influenced by sodium currents.

If the sodium current function f in equation (2.10) is linear and the synaptic coupling K is an exponential function or a delta function, then we can compute the wave speed exactly. If $f_+(u) = \frac{1}{D} \sinh(Du)$ is used to describe the sodium current function, then the speed is a decreasing function of D , where $D > 0$ is a real parameter. If $f_-(u) = \frac{1}{D} \tanh(Du)$ is used to describe the sodium current function, then the speed is an increasing function of D , where $D > 0$ is a real parameter. If $f_c(u) = u(u-1)(Du-1)$ is used to describe the sodium current function, then the speed is an increasing function of

D , where $D > 0$ is a parameter. For any nonlinear sodium current function $f(u)$ and for any synaptic coupling K in the three classes, we can derive upper and lower bounds for the wave speed. We can also compare the wave speeds of (2.10) with different sodium currents. In particular, stronger/weaker sodium current yields faster/slower propagation speed, respectively, see Theorem 2.2.10 and Corollary 2.2.11.

These results are correct for the case $c = \infty$ (without spatial temporal delay) and the case $0 < c < \infty$ (with finite spatial temporal delay).

When investigating the influence of sodium current on wave speed, we must keep in mind that there are many reasonable ways to formulate the sodium current function $f(u)$, see [34], [21] and [30]. That is why we study how the wave speed is influenced by $f(u)$. The sodium current function $f(u)$ in (2.10) is similar to that in the Hodgkin-Huxley equations, which is an empirical model where the sodium current is derived using Ohm's law and also using curve fitting through exponential functions. It is not a physiological model based on physical laws or biological theory. The Fitzhugh-Nagumo equations is a simplified version of the Hodgkin-Huxley equations, see [37].

We hope the speed index functions will be helpful in estimating the speeds of spiral waves and the speeds of other interesting waves and we want to find connections and applications of our results to neuroscience.

2.2 The Mathematical Analysis

In this section, we are going to provide rigorous mathematical investigations/analysis of influence of sodium currents on wave speeds of (2.10), where the sodium current is modeled with the nonlinear function $f(u)$ and the synaptic coupling is modeled with the kernel function K . We will use the speed index functions for (2.8) and (2.9) to establish estimates and comparisons of wave speeds of the general equation (2.10).

Definition 2.2.1. Let $m > 0$, $c > 0$, $\alpha > 0$, $\mu > 0$ and $\nu > 0$ be positive parameters.

Define the speed index functions ϕ and ψ by

$$\phi(\mu) = \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx, \quad (2.13)$$

$$\psi(\nu) = \alpha \int_{-\infty}^0 \exp\left(m \frac{x}{\nu}\right) K(x) dx. \quad (2.14)$$

The definition is motivated by Subsection 3.1.4 and the papers [62]-[64].

Please see Figure 2.2 below for the graphs of three speed index functions.

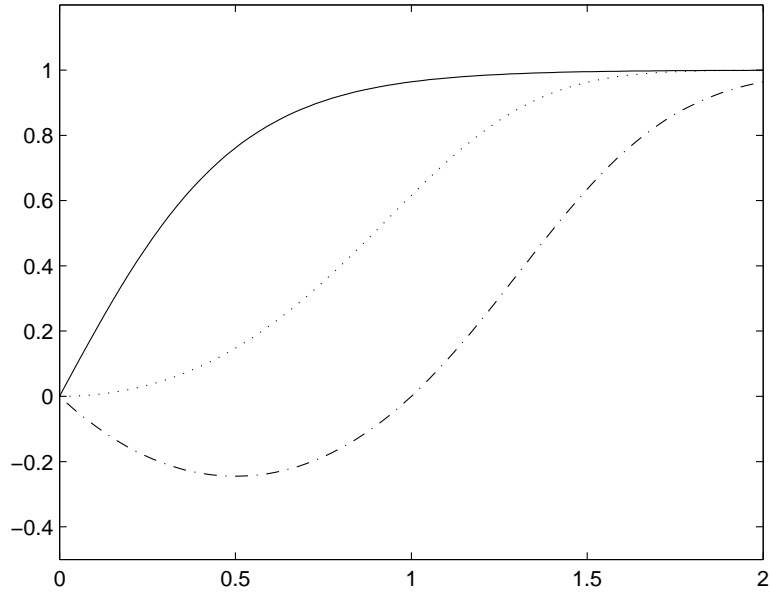


Figure 2.1: Graph of three speed index functions (A) Speed index function for a pure excitation (solid curve). (B) Speed index function for a lateral inhibition (dotted curve). (C) Speed index function for a lateral excitation (dash-dotted curve).

Influence of Sodium Currents: Constant Conductance

In this case, the sodium currents are modeled with linear functions. In particular, we are concerned with $f_0(u) = u$ and $f_l(u) = m(u - n)$, where $m > 0$ and n are real

constants. As before, m represents the sodium conductance and n represents the sodium reversal potential.

2.2.1 Influence of spatial temporal delay on wave speeds

Consider the model equations (2.8) and (2.9), that is

$$\frac{\partial u}{\partial t} + m(u - n) = \alpha \int_{\mathbb{R}} K(x - y) H(u(y, t) - \theta) dy,$$

and

$$\frac{\partial u}{\partial t} + m(u - n) = \alpha \int_{\mathbb{R}} K(x - y) H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right) dy.$$

See Coombes, Lord and Owen [13], Pinto and Ermentrout [46] for such integral differential equations. We want to associate the wave speeds between the case $0 < c < \infty$ and the case $c = \infty$. We will use $\nu_0(f_l)$ and $\mu_0(f_l)$ to represent the wave speeds of (2.8) and (2.9) with the special function $f_l(u) = m(u - n)$, respectively.

Theorem 2.2.2. *If $n < \theta < n + \frac{\alpha}{2m}$, then*

$$\frac{1}{\mu_0(f_l)} = \frac{1}{c} + \frac{1}{\nu_0(f_l)}.$$

There also hold the estimates

$$0 < \mu_0(f_l) < \min\{c, \nu_0(f_l)\}.$$

Proof. By using the uniqueness, we find

$$\frac{1}{\mu_0(f_l)} = \frac{1}{c} + \frac{1}{\nu_0(f_l)}.$$

The proof of Theorem 2.2.2 is finished. ■

Corollary 2.2.3. (I) Let $f_l(u) = m(u - n)$ in equation (2.10), where $m > 0$ and $n \in \mathbb{R}$ are constants, such that $n < \theta < n + \frac{\alpha}{2m}$. Then the wave speed enjoys the limits

$$\lim_{c \rightarrow 0} \mu_0 = 0, \quad \lim_{c \rightarrow \infty} \mu_0 = \nu_0,$$

where ν_0 is the unique solution of

$$\alpha \int_{-\infty}^0 \exp\left(\frac{m}{\nu_0}x\right) K(x) dx = \frac{\alpha}{2} - m\theta + mn.$$

(II) Suppose that $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$, where $\rho > 0$ is a constant. Then the wave speed of (2.10) satisfies

$$\frac{1}{\mu_0} = \frac{1}{c} + \frac{\rho(\theta - n)}{\frac{\alpha}{2} - m\theta + mn}.$$

As a function of the three parameters m , n and c , the speed μ_0 is a decreasing function of m , an increasing function of n and also an increasing function of c .

The speed enjoys the limits

$$\begin{aligned} \lim_{m \rightarrow 0} \mu_0 &= \frac{c\alpha}{\alpha + 2c\rho(\theta - n)}, & \lim_{m \rightarrow \alpha/(2(\theta - n))} \mu_0 &= 0, \\ \lim_{n \rightarrow \theta - \frac{\alpha}{2m}} \mu_0 &= 0, & \lim_{n \rightarrow \theta} \mu_0 &= c, \\ \lim_{c \rightarrow 0} \mu_0 &= 0, & \lim_{c \rightarrow \infty} \mu_0 &= \frac{\alpha - 2m\theta + 2mn}{2\rho(\theta - n)}. \end{aligned}$$

(III) Suppose that the synaptic coupling $K_2(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $\rho > 0$ is a constant, then the wave speed

$$\frac{1}{\mu_0} = \frac{1}{c} + \frac{1}{\nu_0} = \frac{1}{c} + \frac{1}{m\rho} \ln \frac{\alpha}{\alpha - 2m\theta + 2mn}.$$

It enjoys the limits

$$\begin{aligned} \lim_{m \rightarrow 0} \mu_0 &= \frac{c\alpha\rho}{\alpha\rho + 2c(\theta - n)}, & \lim_{m \rightarrow \alpha/(2(\theta-n))} \mu_0 &= 0, \\ \lim_{n \rightarrow \theta - \frac{\alpha}{2m}} \mu_0 &= 0, & \lim_{n \rightarrow \theta} \mu_0 &= c, \\ \lim_{c \rightarrow 0} \mu_0 &= 0, & \lim_{c \rightarrow \infty} \mu_0 &= m\rho \frac{1}{\ln \frac{\alpha}{\alpha - 2m\theta + 2mn}}. \end{aligned}$$

Proof. It is straightforward and omitted. ■

Please see Figure 2.2.1, Figure 2.2.1 and Figure 2.2.1 for the dependence of the wave speed on the parameters (m, n) , (α, θ) and (c, ρ) .

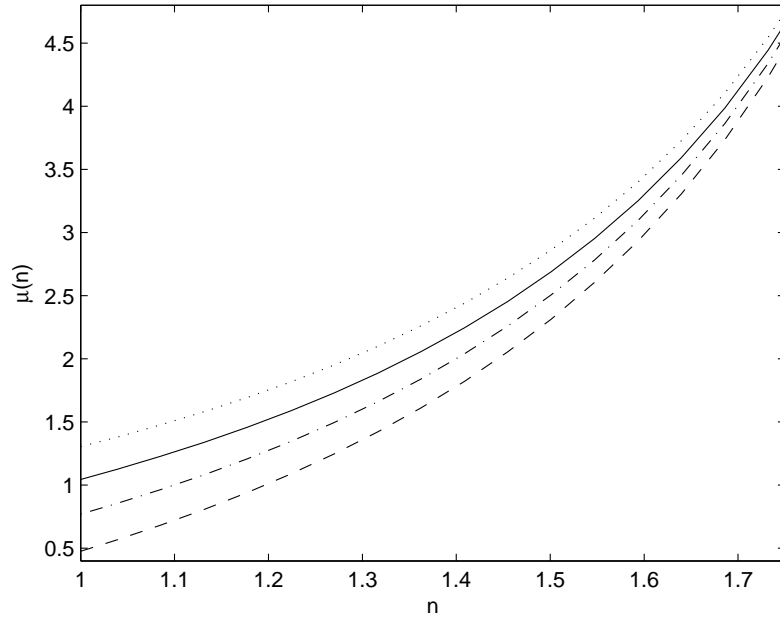


Figure 2.2: In this graph, $c = 10$, $\alpha = 5$, $\theta = 2$, $\rho = 1$, $f(u) = m(u - n)$ and $K(x) = \frac{1}{2} \exp(-|x|)$. Influence of sodium conductance m and sodium reversal potential n on speeds: $\mu = \mu(m, n)$, where $m \in [1, 2]$ and $n \in [1, 1.75]$. The wave speed μ is a decreasing function of m and it is an increasing function of n . For the dotted curve, $m = 1.0$. For the solid curve, $m = 1.333$. For the dash-dotted curve, $m = 1.666$. For the dashed curve, $m = 2.0$.

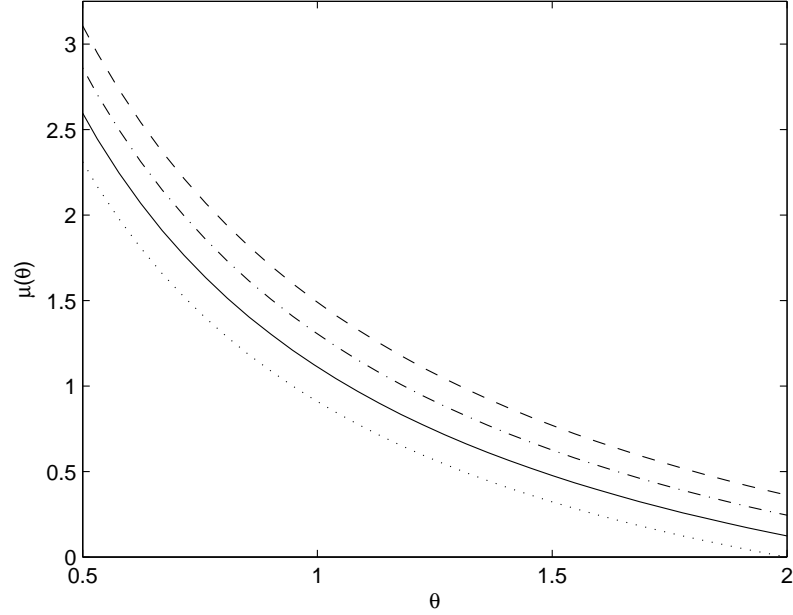


Figure 2.3: In this graph, $m = 1$, $n = 0$, $c = 10$, $\rho = 1$, $f(u) = u$ and $K(x) = \frac{1}{2} \exp(-|x|)$. Influence of synaptic rate constant α and threshold θ on speed: $\mu = \mu(\alpha, \theta)$, where $\alpha \in [4, 6]$ and $\theta \in [0.5, 2]$. The wave speed μ is an increasing function of α and it is a decreasing function of θ . For the dotted curve, $\alpha = 4.0$. For the solid curve, $\alpha = 4.5$. For the dash-dotted curve, $\alpha = 5.0$. For the dashed curve, $\alpha = 5.5$.

2.2.2 Influence of related threshold on wave speeds

We would like to investigate the influence of related thresholds, $m(\theta - n)/\alpha$, on the wave speeds of (2.10).

Theorem 2.2.4. Consider (2.10) with two pairs of parameters $(\alpha_1, \theta_1, n_1)$ and $(\alpha_2, \theta_2, n_2)$ and two functions $f_1(u) = m(u - n_1)$ and $f_2(u) = m(u - n_2)$, such that

$$0 < \frac{m(\theta_1 - n_1)}{\alpha_1} < \frac{m(\theta_2 - n_2)}{\alpha_2}.$$

We have the estimate $\mu_2(\alpha_2, \theta_2, n_2) < \mu_1(\alpha_1, \theta_1, n_1)$. In particular, if $\theta_1 - n_1 < \theta_2 - n_2$ and $\alpha_1 = \alpha_2$, then $\mu_2(\alpha_2, \theta_2, n_2) < \mu_1(\alpha_1, \theta_1, n_1)$. If $\alpha_1 > \alpha_2$ and $\theta_1 - n_1 = \theta_2 - n_2$,

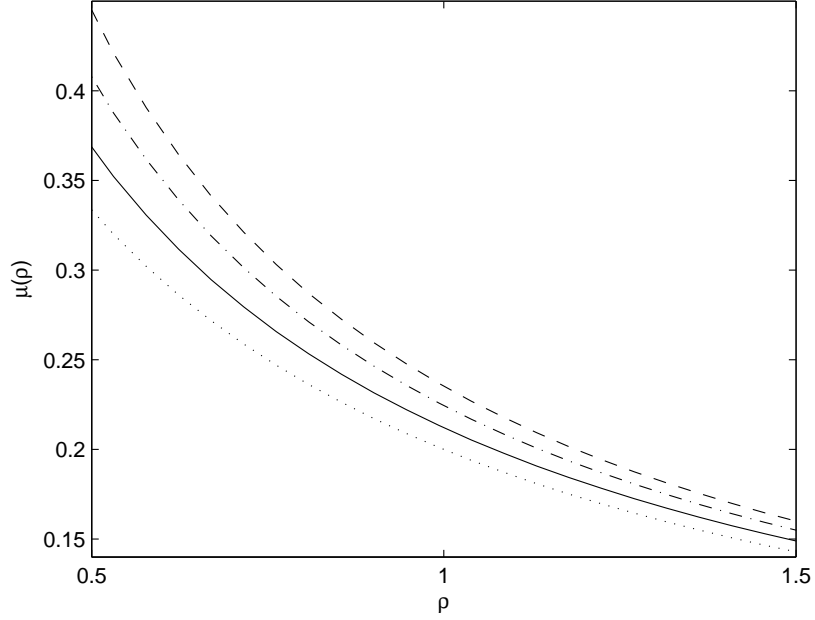


Figure 2.4: In this graph, $m = 1$, $n = 0$, $\alpha = 5$, $\theta = 2$ and $f(u) = u$. Let the synaptic coupling $K(x) = \frac{\rho}{2} \exp(-\rho|x|)$. Influence of speed of action potential c and distribution constant ρ on wave speed: $\mu = \mu(c, \rho)$, where $c \in [1, 4]$ and $\rho \in [0.5, 1.5]$. The wave speed μ is an increasing function of c and it is a decreasing function of ρ . For the dotted curve, $c = 1.0$. For the solid curve, $c = 1.4$. For the dash-dotted curve, $c = 2.2$. For the dashed curve, $c = 4.0$.

then $\mu_2(\alpha_2, \theta_2, n_2) < \mu_1(\alpha_1, \theta_1, n_1)$.

Proof. The wave speeds of (2.10) satisfy the following equations

$$\frac{1}{2} - \int_{-\infty}^0 \exp\left(m \frac{c - \mu_1}{c\mu_1} x\right) K(x) dx = \frac{m\theta_1 - mn_1}{\alpha_1},$$

and

$$\frac{1}{2} - \int_{-\infty}^0 \exp\left(m \frac{c - \mu_2}{c\mu_2} x\right) K(x) dx = \frac{m\theta_2 - mn_2}{\alpha_2}.$$

Since

$$\frac{d}{d\mu} \left[\frac{1}{2} - \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx \right]$$

$$= -\frac{m}{\mu^2} \int_{-\infty}^0 |x| \exp\left(m \frac{c-\mu}{c\mu} x\right) K(x) dx < 0,$$

the function

$$\frac{1}{2} - \int_{-\infty}^0 \exp\left(m \frac{c-\mu}{c\mu} x\right) K(x) dx$$

is a strictly decreasing function of μ . Thus, we find that

$$\mu_2(\alpha_2, \theta_2, n_2) < \mu_1(\alpha_1, \theta_1, n_1), \quad \text{if} \quad \frac{m(\theta_1 - n_1)}{\alpha_1} < \frac{m(\theta_2 - n_2)}{\alpha_2}.$$

The proof of Theorem 2.2.4 is completed. ■

Suppose that a neuron located at position x is coupled with only two neurons which are located at $x \pm \rho$, where $\rho > 0$ is a constant. This coupling may be described with delta functions.

Corollary 2.2.5. *Let $f(u) = u$ and $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $\rho > 0$ is a constant. Consider equation (2.10) with two groups of parameters (α_1, θ_1) and (α_2, θ_2) , where*

$$0 < \frac{\theta_1}{\alpha_1} < \frac{1}{2}, \quad 0 < \frac{\theta_2}{\alpha_2} < \frac{1}{2}.$$

Then the speeds satisfy

$$\frac{1}{\mu(\alpha_1, \theta_1, \rho)} = \frac{1}{c} + \frac{1}{\rho} \ln \frac{\alpha_1}{\alpha_1 - 2\theta_1}, \quad \frac{1}{\mu(\alpha_2, \theta_2, \rho)} = \frac{1}{c} + \frac{1}{\rho} \ln \frac{\alpha_2}{\alpha_2 - 2\theta_2}.$$

If

$$\frac{\theta_1}{\alpha_1} < \frac{\theta_2}{\alpha_2},$$

then there holds the estimate

$$\mu_2(\alpha_2, \theta_2) < \mu_1(\alpha_1, \theta_1).$$

Proof. *It is straightforward and omitted.* ■

Please see Figure 2.2.2 for the dependence of the wave speed on the parameters α and θ .

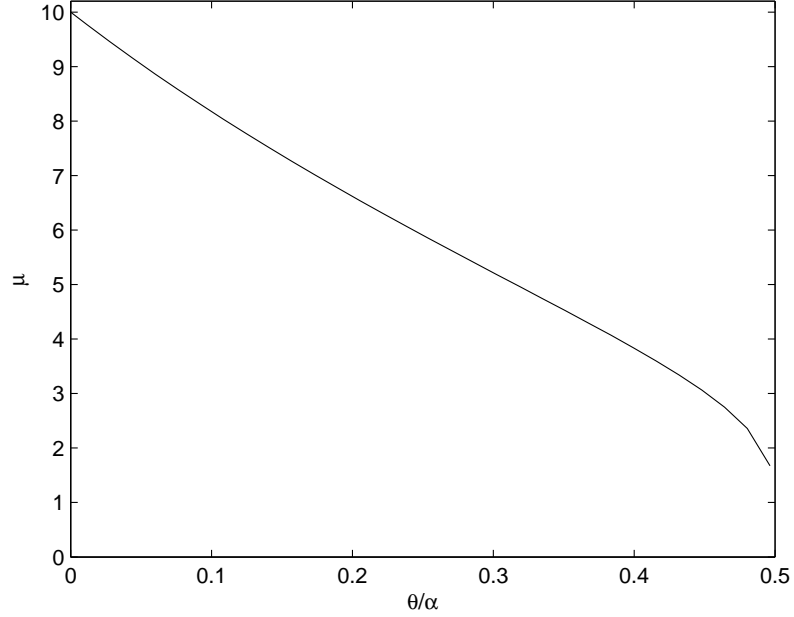


Figure 2.5: In this graph, $c = 10$, $\rho = 10$, $f(u) = u$ and $K(x) = \frac{1}{2}[\delta(x+\rho) + \delta(x-\rho)]$. Graph of $\mu = \mu_0(\theta/\alpha)$, where $\theta/\alpha \in [0, 0.5]$. The wave speed μ is a decreasing function of θ/α .

2.2.3 Influence of on-center and off-center kernel functions

We would like to investigate the influence of an on-center kernel function and an off-center kernel function on the wave speeds of (2.10).

Theorem 2.2.6. *Let $f_1(u) = m(u - n)$ in (2.10). Given a kernel function $K(\cdot)$ in class (A), (B) or (C), let $K(d, \cdot)$ be defined by*

$$K(d, x) = \begin{cases} K(x + d), & \text{for all } x \leq -d, \\ K(x - d), & \text{for all } x \geq +d, \\ 0, & \text{for all } |x| < d, \end{cases}$$

where $d > 0$ is a parameter. Then the wave speed of (2.10) is an increasing function of d , that is, $\mu(d_2) > \mu(d_1) > \mu(0)$, for all $d_2 > d_1 > 0$. Furthermore

$$\lim_{d \rightarrow \infty} \mu(d) = c \quad \text{and} \quad \lim_{d \rightarrow \infty} \{d[c - \mu(d)]\} = \frac{c^2}{m} \ln \frac{\alpha}{\alpha - 2m\theta + 2mn}.$$

Proof. We have defined the speed index function $\phi(\mu)$ by

$$\phi(\mu) = \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx.$$

For $K(d, \cdot)$, we have

$$\begin{aligned} \phi(d, \mu) &= \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(d, x) dx \\ &= \alpha \int_{-\infty}^{-d} \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x + d) dx \\ &= \alpha \exp\left(-m \frac{c - \mu}{c\mu} d\right) \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx \\ &= \exp\left(-m \frac{c - \mu}{c\mu} d\right) \phi(\mu). \end{aligned}$$

Hence for each fixed number $\mu \in (0, c)$, if $\phi(\mu) > 0$, then the function $\phi(d, \mu)$ is monotonically decreasing in d , for all kernel functions in classes (A), (B) and (C). The speed $\mu(d)$ is a solution of the equation $\phi(d, \mu(d)) = \frac{\alpha}{2} - m\theta + mn$. Explicitly

$$\alpha \int_{-\infty}^0 \exp\left[m \frac{c - \mu(d)}{c\mu(d)} (x - d)\right] K(x) dx = \frac{\alpha}{2} - m\theta + mn.$$

Differentiating this equation with respect to d , we obtain

$$m\alpha \int_{-\infty}^0 \left\{ \frac{-\mu(d) - \mu'(d)(x - d)}{[\mu(d)]^2} + \frac{1}{c} \right\} \exp\left[m \frac{c - \mu(d)}{c\mu(d)} (x - d)\right] K(x) dx = 0.$$

That is

$$\begin{aligned} &\mu'(d) \int_{-\infty}^0 (d - x) \exp\left[m \frac{c - \mu(d)}{c\mu(d)} (x - d)\right] K(x) dx \\ &= \left\{ \mu(d) - \frac{[\mu(d)]^2}{c} \right\} \int_{-\infty}^0 \exp\left[m \frac{c - \mu(d)}{c\mu(d)} (x - d)\right] K(x) dx. \end{aligned}$$

Recall that

$$\begin{aligned} &\int_{-\infty}^0 (d - x) \exp\left[m \frac{c - \mu(d)}{c\mu(d)} (x - d)\right] K(x) dx > 0, \\ &\int_{-\infty}^0 \exp\left[m \frac{c - \mu(d)}{c\mu(d)} (x - d)\right] K(x) dx > 0, \\ &0 < \mu(d) < c, \quad \text{for all } d \geq 0. \end{aligned}$$

Thus, $\mu'(d) > 0$. Note that K may be in any of the three classes, i.e. the synaptic coupling could be a pure excitation, lateral inhibition or lateral excitation. Regardless if the spatial temporal delay is present (either $c = \infty$ or $0 < c < \infty$), the wave speed is an increasing function of d . Furthermore, it is straightforward to show that the limit

$$\lim_{d \rightarrow \infty} \mu(d) = c,$$

exists. Additionally, suppose that $d[c - \mu(d)] \rightarrow L$, as $d \rightarrow \infty$, where $L \geq 0$ is a constant. From

$$\exp \left[-m \frac{c - \mu(d)}{c\mu(d)} d \right] \phi(\mu(d)) = \frac{\alpha}{2} - m\theta + mn,$$

we find

$$\exp \left(-\frac{mL}{c^2} \right) \phi(c) = \frac{\alpha}{2} - m\theta + mn.$$

But $\phi(c) = \frac{\alpha}{2}$. Thus

$$L = \frac{c^2}{m} \ln \frac{\alpha}{\alpha - 2m\theta + 2mn}.$$

Therefore

$$\lim_{d \rightarrow \infty} \{d[c - \mu(d)]\} = \frac{c^2}{m} \ln \frac{\alpha}{\alpha - 2m\theta + 2mn}.$$

The proof of Theorem 2.2.6 is completed. ■

The interesting point is that the result is true for all $0 < c < \infty$ and for all $d > 0$.

Corollary 2.2.7. *Let $f_l(u) = m(u - n)$ and $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $m > 0$, $n \in \mathbb{R}$ and $\rho > 0$ are constants. Then*

$$K(d, x) = \frac{1}{2}[\delta(x + d + \rho) + \delta(x - d - \rho)].$$

The wave speeds $\mu(0)$ and $\mu(d)$ satisfy

$$\frac{1}{\mu(0)} = \frac{1}{c} + \frac{1}{m\rho} \ln \frac{\alpha}{\alpha - 2m\theta + 2mn},$$

and

$$\frac{1}{\mu(d)} = \frac{1}{c} + \frac{1}{m(\rho + d)} \ln \frac{\alpha}{\alpha - 2m\theta + 2mn},$$

respectively. The speed $\mu(d)$ is an increasing function of d and $\mu(0) < \mu(d) < c$, for all $d > 0$.

Proof. It is straightforward and omitted. ■

Please see Figure 2.2.3 for the dependence of the wave speed on the parameter d .

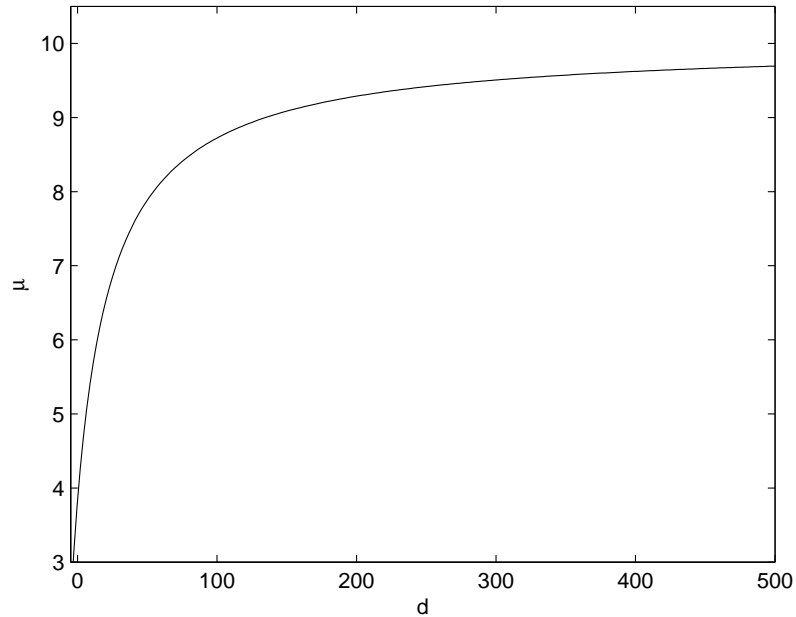


Figure 2.6: In this graph, $c = 10$, $\alpha = 5$, $\theta = 2$, $\rho = 10$ and $f(u) = u$. Let $K(d, x) = \frac{1}{2}[\delta(x + \rho + d) + \delta(x - \rho - d)]$. The graph of the wave speed $\mu = \mu_0(d)$. The wave speed μ is an increasing function of d .

Influence of Sodium Currents: Voltage Related Conductance

In this part, the sodium currents are modeled with nonlinear functions. In particular, we are concerned with $f_+(u) = \frac{1}{D} \sinh(Du)$, $f_-(u) = \frac{1}{D} \tanh(Du)$, where $D > 0$ is a constant, and the cubic polynomial function $f_c(u) = u(u-1)(Du-1)$, where $D > 0$ is also a constant. We will use $\nu_0(f)$ and $\mu_0(f)$ to represent the wave speeds of (2.5) and (2.7) with general nonlinear function $f(u)$, respectively.

2.2.4 Approximations of wave speeds

Given a nonlinear function $f(u)$, we would like to use a reasonable linear function $f_l(u)$ to approximate the nonlinear function. There are many ways to do this. Suppose that $f_l(u) = m(u - n)$. To determine the constants m and n , we use an average idea. The average of f and f_l over $[0, \theta]$ should be the same, and the average of f' and f_l' over $[0, \theta]$ should also be the same:

$$\begin{aligned}\frac{1}{\theta} \int_0^\theta f(u) du &= \frac{1}{\theta} \int_0^\theta m(u - n) du, \\ \frac{1}{\theta} \int_0^\theta f'(u) du &= \frac{1}{\theta} \int_0^\theta m du.\end{aligned}$$

Solving the system, assuming that $f(\theta) > f(0)$, we find

$$m = \frac{f(\theta) - f(0)}{\theta}, \quad n = \frac{\theta}{2} - \frac{1}{f(\theta) - f(0)} \int_0^\theta f(u) du.$$

Therefore, we obtain the linear function

$$f_l(u) = \frac{f(\theta) - f(0)}{\theta} u - \frac{f(\theta) - f(0)}{2} + \frac{1}{\theta} \int_0^\theta f(u) du.$$

This approximation makes sense because if f itself is linear, then $f_l = f$.

Given any nonlinear function $f(u)$, if $f(\theta) > f(0)$, then we may generate the linear function $f_l(u)$ in the above way. Let us consider an approximate equation (2.10*) with

the linear function $f_l(u) = m(u - n)$:

$$\frac{\partial u}{\partial t} + m(u - n) = \alpha \int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c} |x - y| \right) - \theta \right) dy. \quad (2.10^*)$$

The approximate wave speed μ_{appr} satisfies the equation

$$\alpha \int_{-\infty}^0 \exp \left(m \frac{c - \mu}{c\mu} x \right) K(x) dx = \frac{\alpha}{2} - m\theta + mn.$$

Theorem 2.2.8. *The wave speed satisfies the estimate*

$$|\mu_0 - \mu_{\text{appr}}| \leq \ln \left[1 + \max_{u \in [0, \theta]} |f(u) - m(u - n)| \right].$$

Proof. Denote by μ_0 and μ_{appr} the wave speeds of the traveling wave fronts of the equations. Then (see Appendix)

$$\begin{aligned} U_{\text{appr}}(z) &= n + \frac{\alpha}{m} \int_{-\infty}^{cz/(c+s(z)\mu_{\text{appr}})} K(x) dx \\ &\quad - \frac{\alpha}{m} \int_{-\infty}^z \exp \left[\frac{m}{\mu_{\text{appr}}} (x - z) \right] \frac{c}{c + s(x)\mu_{\text{appr}}} K \left(\frac{cx}{c + s(x)\mu_{\text{appr}}} \right) dx, \end{aligned}$$

and

$$\begin{aligned} U(z) &= n + \frac{\alpha}{m} \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \\ &\quad - \frac{\alpha}{m} \int_{-\infty}^z \exp \left[\frac{f'(n)}{\mu_0} (x - z) \right] \frac{c}{c + s(x)\mu_0} K \left(\frac{cx}{c + s(x)\mu_0} \right) dx \\ &\quad + \frac{1}{\mu_0} \int_{-\infty}^z \exp \left[f'(n) \frac{x - z}{\mu_0} \right] [f'(n)(U(x) - n) - f(U(x))] dx. \end{aligned}$$

Letting $z = 0$ in these solution representations, noticing that $U_{\text{appr}}(0) = U(0) = \theta$ and

$\int_{-\infty}^0 K(x) dx = \frac{1}{2}$, we have

$$\theta = n + \frac{\alpha}{2m} - \frac{\alpha}{m} \int_{-\infty}^0 \exp \left(\frac{mx}{\mu_{\text{appr}}} \right) \frac{c}{c - \mu_{\text{appr}}} K \left(\frac{cx}{c - \mu_{\text{appr}}} \right) dx,$$

and

$$\begin{aligned}\theta &= n + \frac{\alpha}{2m} - \frac{\alpha}{m} \int_{-\infty}^0 \exp\left(\frac{mx}{\mu_0}\right) \frac{c}{c - \mu_0} K\left(\frac{cx}{c - \mu_0}\right) dx \\ &+ \frac{1}{\mu_0} \int_{-\infty}^z \exp\left(\frac{mx}{\mu_0}\right) [m(U(x) - n) - f(U(x))] dx.\end{aligned}$$

By using an intermediate value theorem, we may write

$$\begin{aligned}&\frac{1}{\mu_0} \int_{-\infty}^z \exp\left(\frac{mx}{\mu_0}\right) [m(U(x) - n) - f(U(x))] dx \\ &= \pm \int_{-\infty}^0 \exp\left(\kappa x + \frac{c - \mu_{\text{appr}}}{c\mu_{\text{appr}}} x\right) K(x) dx,\end{aligned}$$

for some real number κ . Hence we get

$$\theta = n + \frac{\alpha}{2m} - \frac{\alpha}{m} \int_{-\infty}^0 \exp\left(\frac{mx}{\mu_{\text{appr}}}\right) \frac{c}{c - \mu_{\text{appr}}} K\left(\frac{cx}{c - \mu_{\text{appr}}}\right) dx,$$

and

$$\begin{aligned}\theta &= n + \frac{\alpha}{2m} - \frac{\alpha}{m} \int_{-\infty}^0 \exp\left(\frac{mx}{\mu_0}\right) \frac{c}{c - \mu_0} K\left(\frac{cx}{c - \mu_0}\right) dx \\ &\pm \int_{-\infty}^0 \exp\left(\kappa x + \frac{c - \mu_{\text{appr}}}{c\mu_{\text{appr}}} x\right) K(x) dx.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\alpha}{m} \int_{-\infty}^0 \exp\left(\frac{mx}{\mu_{\text{appr}}}\right) \frac{c}{c - \mu_{\text{appr}}} K\left(\frac{cx}{c - \mu_{\text{appr}}}\right) dx \\ &= \frac{\alpha}{m} \int_{-\infty}^0 \exp\left(\frac{mx}{\mu_0}\right) \frac{c}{c - \mu_0} K\left(\frac{cx}{c - \mu_0}\right) dx \\ &\quad \pm \int_{-\infty}^0 \exp\left(\kappa x + \frac{c - \mu_{\text{appr}}}{c\mu_{\text{appr}}} x\right) K(x) dx.\end{aligned}$$

By uniqueness of the wave speed, the proof of Theorem 2.2.8 is finished. ■

Corollary 2.2.9. *Application 1. Let*

$$\begin{aligned}f_+(u) &= \frac{1}{D} \sinh(Du), \\ K_1(x) &= \frac{\rho}{2} \exp(-\rho|x|), \\ K_2(x) &= \frac{1}{2} [\delta(x + \rho) + \delta(x - \rho)],\end{aligned}$$

where $D > 0$ and $\rho > 0$ are constants. We have

$$\begin{aligned} m &= \frac{1}{D\theta} \sinh(D\theta), & n &= \frac{\theta}{2} - \frac{\cosh(D\theta) - 1}{D \sinh(D\theta)}, \\ mn &= \frac{1}{2D} \sinh(D\theta) - \frac{1}{D^2\theta} [\cosh(D\theta) - 1]. \end{aligned}$$

The approximate speeds μ_{appr} and ν_{appr} satisfy

$$\begin{aligned} \frac{1}{\mu_{\text{appr}}} &= \frac{1}{c} + \frac{1}{\nu_{\text{appr}}}, \\ \nu_{\text{appr}} &= \frac{\alpha - 2m\theta + 2mn}{2\rho(\theta - n)}, \text{ for } K_1(x) = \frac{\rho}{2} \exp(-\rho|x|), \\ \nu_{\text{appr}} &= m\rho \frac{1}{\ln \frac{\alpha}{\alpha - 2m\theta + 2mn}}, \text{ for } K_2(x) = \frac{1}{2} [\delta(x + \rho) + \delta(x - \rho)]. \end{aligned}$$

Application 2. Let

$$\begin{aligned} f_-(u) &= \frac{1}{D} \tanh(Du) \\ K_1(x) &= \frac{\rho}{2} \exp(-\rho|x|), \\ K_2(x) &= \frac{1}{2} [\delta(x + \rho) + \delta(x - \rho)], \end{aligned}$$

where $D > 0$ and $\rho > 0$ are constants. We have

$$\begin{aligned} m &= \frac{1}{D\theta} \tanh(D\theta), & n &= \frac{\theta}{2} - \frac{\ln[\cosh(D\theta)]}{D \tanh(D\theta)}, \\ mn &= \frac{1}{2D} \tanh(D\theta) - \frac{1}{D^2\theta} \ln[\cosh(D\theta)]. \end{aligned}$$

The approximate speeds μ_{appr} and ν_{appr} satisfy

$$\begin{aligned} \frac{1}{\mu_{\text{appr}}} &= \frac{1}{c} + \frac{1}{\nu_{\text{appr}}}, \\ \nu_{\text{appr}} &= \frac{\alpha - 2m\theta + 2mn}{2\rho(\theta - n)}, \text{ for } K_1(x) = \frac{\rho}{2} \exp(-\rho|x|), \\ \nu_{\text{appr}} &= m\rho \frac{1}{\ln \frac{\alpha}{\alpha - 2m\theta + 2mn}}, \text{ for } K_2(x) = \frac{1}{2} [\delta(x + \rho) + \delta(x - \rho)]. \end{aligned}$$

Application 3. Let

$$\begin{aligned} f_c(u) &= u(u-1)(Du-1), \\ K_1(x) &= \frac{\rho}{2} \exp(-\rho|x|) \\ K_2(x) &= \frac{1}{2} [\delta(x+\rho) + \delta(x-\rho)], \end{aligned}$$

where $D > 0$ and $\rho > 0$ are constants. We have

$$\begin{aligned} m &= (\theta - 1)(D\theta - 1), \\ n &= \frac{\theta}{2} - \frac{1}{\theta(\theta - 1)(D\theta - 1)} \left(\frac{D}{4}\theta^4 - \frac{1+D}{3}\theta^3 - \frac{1}{2}\theta^2 \right), \\ mn &= \frac{1}{2}\theta(\theta - 1)(D\theta - 1) - \frac{D}{4}\theta^3 + \frac{1+D}{3}\theta^2 - \frac{1}{2}\theta. \end{aligned}$$

The approximate speeds μ_{appr} and ν_{appr} satisfy

$$\begin{aligned} \frac{1}{\mu_{\text{appr}}} &= \frac{1}{c} + \frac{1}{\nu_{\text{appr}}}, \\ \nu_{\text{appr}} &= \frac{\alpha - 2m\theta + 2mn}{2\rho(\theta - n)}, \text{ for } K_1(x) = \frac{\rho}{2} \exp(-\rho|x|), \\ \nu_{\text{appr}} &= m\rho \frac{1}{\ln \frac{\alpha}{\alpha - 2m\theta + 2mn}}, \text{ for } K_2(x) = \frac{1}{2} [\delta(x+\rho) + \delta(x-\rho)]. \end{aligned}$$

Proof. It is straightforward and omitted. ■

2.2.5 Comparison of wave speeds

The next theorem investigates the influence of sodium currents on the wave speeds of (2.10).

Theorem 2.2.10. *Let $m > 0$ and $n \in \mathbb{R}$ be two real constants, and let f and g be two real functions such that*

$$f(u) \leq m(u - n) \leq g(u), \quad \text{on } (n, \infty).$$

Let $u(x, t) = U_f(x + \mu_f t)$ and $u(x, t) = U_g(x + \mu_g t)$ be the traveling wave fronts of

$$\frac{\partial u}{\partial t} + f(u) = \alpha \int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c} |x - y| \right) - \theta \right) dy,$$

and

$$\frac{\partial u}{\partial t} + g(u) = \alpha \int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c} |x - y| \right) - \theta \right) dy,$$

respectively, such that $U_f(0) = U_g(0) = \theta$, $U_f < \theta$ and $U_g < \theta$ on $(-\infty, 0)$, $U_f > \theta$ and $U_g > \theta$ on $(0, \infty)$, and the limits

$$\lim_{z \rightarrow -\infty} U_f(z) = U_f^-, \quad \lim_{z \rightarrow -\infty} U_g(z) = U_g^-,$$

exist, where U_f^- and U_g^- are real constants. Then $\mu_f \geq \mu_g$. In particular, if additionally $f \neq g$, then

$$\mu_f > \mu_g.$$

Proof. First of all, the fronts satisfy the traveling wave equations

$$\mu_f U_f' + f(U_f) = \alpha \int_{\mathbb{R}} K(z - y) H \left(U_f \left(y - \frac{\mu_f}{c} |z - y| \right) - \theta \right) dy,$$

where $z = x + \mu_f t$ and $' = \frac{d}{dz}$, and

$$\mu_g U_g' + g(U_g) = \alpha \int_{\mathbb{R}} K(z - y) H \left(U_g \left(y - \frac{\mu_g}{c} |z - y| \right) - \theta \right) dy,$$

where $z = x + \mu_g t$. Using the assumptions

$$f(u) \leq m(u - n) \leq g(u), \quad \text{on } (n, n + \theta),$$

we get

$$\mu_f U_f' + m(U_f - n) \geq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_f)} K(x) dx,$$

and

$$\mu_g U_g' + m(U_g - n) \leq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_g)} K(x) dx,$$

on $(-\infty, 0)$. Now, we find

$$\begin{aligned} m[U_f(z) - n] &\geq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_f)} K(x) dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left[\frac{m}{\mu_f}(x-z)\right] \frac{c}{c+s(x)\mu_f} K\left(\frac{cx}{c+s(x)\mu_f}\right) dx, \end{aligned}$$

and

$$\begin{aligned} m[U_g(z) - n] &\leq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_g)} K(x) dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left[\frac{m}{\mu_g}(x-z)\right] \frac{c}{c+s(x)\mu_g} K\left(\frac{cx}{c+s(x)\mu_g}\right) dx, \end{aligned}$$

on $(-\infty, 0)$. Let $z = 0$, recall that $\int_{-\infty}^0 K(x) dx = \frac{1}{2}$, we have

$$m(\theta - n) = m[U_f(0) - n] \geq \frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu_f}{c\mu_f} x\right) K(x) dx.$$

Similarly, we have

$$m(\theta - n) = m[U_g(0) - n] \leq \frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu_g}{c\mu_g} x\right) K(x) dx.$$

Note that

$$\begin{aligned} &\frac{d}{d\mu} \left[\frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx \right] \\ &= -\frac{\alpha m}{\mu^2} \int_{-\infty}^0 |x| \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx < 0. \end{aligned}$$

Therefore, it follows from the above inequalities that $\mu_f \geq \mu_g$. Moreover, if $f \neq g$, then $\mu_f > \mu_g$. Now we have finished the proof of Theorem 2.2.10. \blacksquare

Corollary 2.2.11. *Let $f_0(u) = u$ and $f_c(u) = u(u - 1)(Du - 1)$ on \mathbb{R} , where $D > 0$ is a constant. Suppose that the functions $f_+(u)$ and $f_-(u)$ satisfy $f_-(u) \leq f_0(u) \leq f_+(u)$ on $(0, \infty)$ and $f_c(u) \leq f_-(u)$ on $(0, \theta)$. Let μ_+ , μ_0 , μ_- and μ_c denote the wave speeds of (2.10) with the functions $f_+(u)$, $f_0(u)$, $f_-(u)$ and $f_c(u)$, respectively. Then we have the estimates*

$$\mu_+ < \mu_0 < \mu_- < \mu_c.$$

Proof. Note that

$$f_c(u) \leq f_-(u) \leq f_0(u) \leq f_+(u) \quad \text{on } (0, \theta).$$

Applying the general estimates in Theorem 2.2.10, we complete the proof of Corollary 2.2.11. ■

Please see Figure 2.2.5 for the graphs of four sodium current functions.

2.2.6 Estimates of the wave speeds

The next theorem provides the estimates of the wave speeds of (2.10) where the sodium current function is bounded by a linear function.

Theorem 2.2.12. (I) *Suppose that $f(u) \geq m(u - n)$ on (n, ∞) , such that $n < \theta < n + \frac{\alpha}{2m}$. Then*

$$\alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \leq \frac{\alpha}{2} - m\theta + mn.$$

In particular, let the synaptic coupling $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$, where $\rho > 0$ is a constant, then

$$\frac{1}{c} + \frac{\rho(\theta - n)}{\frac{\alpha}{2} - m\theta + mn} \leq \frac{1}{\mu_0}.$$

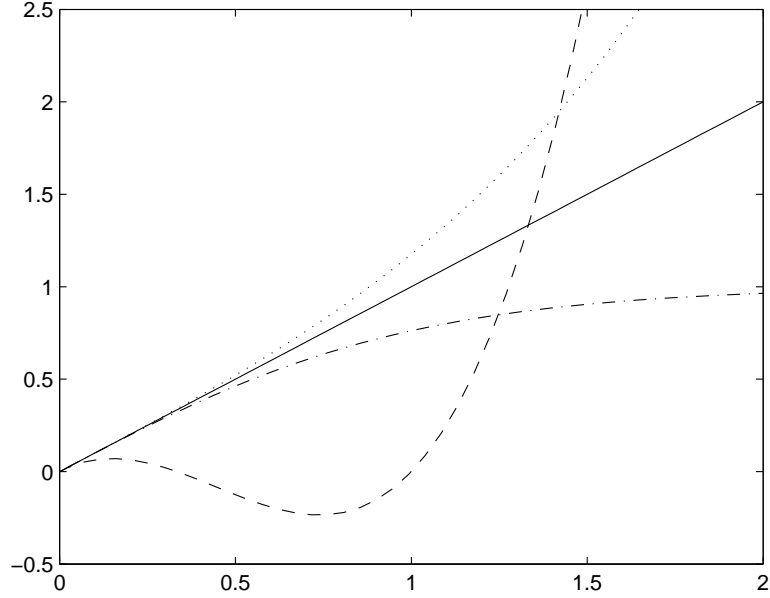


Figure 2.7: Graph of four sodium current functions: $f_+(u) = \sinh u$ (dotted curve), $f_0(u) = u$ (solid curve), $f_-(u) = \tanh u$ (dash-dotted curve) and $f_c(u) = u(u - 1)(3u - 1)$ (dashed curve).

Let the synaptic coupling $K_2(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where δ represents the Dirac delta impulse function and $\rho > 0$ is a constant, then

$$\frac{1}{c} + \frac{1}{m\rho} \ln \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} - m\theta + mn} \leq \frac{1}{\mu_0}.$$

(II) Suppose that $f(u) \leq m(u - n)$ on (n, ∞) , such that $n < \theta < n + \frac{\alpha}{2m}$. Then

$$\frac{\alpha}{2} - m\theta + mn \leq \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx.$$

In particular, let the synaptic coupling $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$, where $\rho > 0$ is a constant, then

$$\frac{1}{\mu_0} \leq \frac{1}{c} + \frac{\rho(\theta - n)}{\frac{\alpha}{2} - m\theta + mn}.$$

Let the synaptic coupling $K_2(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $\rho > 0$ is a constant, then

$$\frac{1}{\mu_0} \leq \frac{1}{c} + \frac{1}{m\rho} \ln \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} - m\theta + mn}.$$

(III) Suppose that there are real constants $m_k > 0$ and $n_k \in \mathbb{R}$, for $k = 1, 2$, such that

$$m_1(u - n_1) \leq f(u) \leq m_2(u - n_2),$$

on (n_1, ∞) . Then

$$\alpha \int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \leq \frac{\alpha}{2} - m_1\theta + m_1n_1,$$

$$\frac{\alpha}{2} - m_2\theta + m_2n_2 \leq \alpha \int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx.$$

In particular, let the synaptic coupling $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$, $\rho > 0$ is a constant, then

$$\frac{1}{c} + \frac{\rho(\theta - n_1)}{\frac{\alpha}{2} - m_1\theta + m_1n_1} \leq \frac{1}{\mu_0} \leq \frac{1}{c} + \frac{\rho(\theta - n_2)}{\frac{\alpha}{2} - m_2\theta + m_2n_2},$$

provided that

$$n_1 < \theta < n_1 + \frac{\alpha}{2m_1}, \quad \text{and} \quad n_2 < \theta < n_2 + \frac{\alpha}{2m_2}.$$

Let the synaptic coupling $K_2(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $\rho > 0$ is a constant, then

$$\frac{1}{c} + \frac{1}{m_1\rho} \ln \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} - m_1\theta + m_1n_1} \leq \frac{1}{\mu_0} \leq \frac{1}{c} + \frac{1}{m_2\rho} \ln \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} - m_2\theta + m_2n_2}.$$

Proof. Let $u(x, t) = U(z)$ be the traveling wave front of (2.10), where $z = x + \mu_0 t$, such that $U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$, and the limit $U \rightarrow U^-$ exists, as $z \rightarrow -\infty$, where U^- is a real constant. Then the integral differential equation (2.10) becomes

$$\mu_0 U' + f(U) = \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx.$$

Moreover

$$\mu_0 U' + m_1(U - n_1) \leq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx,$$

and

$$\mu_0 U' + m_2(U - n_2) \geq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx,$$

on $(-\infty, 0)$. Solving these differential inequalities, we find

$$\begin{aligned} m_1[U(z) - n_1] &\leq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left[\frac{m_1}{\mu_0}(x - z)\right] \frac{c}{c + s(x)\mu_0} K\left(\frac{cx}{c + s(x)\mu_0}\right) dx, \end{aligned}$$

and

$$\begin{aligned} m_2[U(z) - n_2] &\geq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left[\frac{m_2}{\mu_0}(x - z)\right] \frac{c}{c + s(x)\mu_0} K\left(\frac{cx}{c + s(x)\mu_0}\right) dx, \end{aligned}$$

on $(-\infty, 0)$. In particular, letting $z = 0$, we have

$$m_1(\theta - n_1) \leq \frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx,$$

and

$$m_2(\theta - n_2) \geq \frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx.$$

In other words, we have the estimates

$$\alpha \int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \leq \frac{\alpha}{2} - m_1\theta + m_1n_1,$$

and

$$\frac{\alpha}{2} - m_2\theta + m_2n_2 \leq \alpha \int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx.$$

For the synaptic coupling $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$, where $\rho > 0$ is a constant, we have

$$\frac{1}{c} + \frac{\rho(\theta - n_1)}{\frac{\alpha}{2} - m_1\theta + m_1n_1} \leq \frac{1}{\mu_0} \leq \frac{1}{c} + \frac{\rho(\theta - n_2)}{\frac{\alpha}{2} - m_2\theta + m_2n_2}.$$

For the synaptic coupling $K_2(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $\rho > 0$ is a constant, we have

$$\frac{1}{c} + \frac{1}{m_1\rho} \ln \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} - m_1\theta + m_1n_1} \leq \frac{1}{\mu_0} \leq \frac{1}{c} + \frac{1}{m_2\rho} \ln \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} - m_2\theta + m_2n_2}.$$

The proof of Theorem 2.2.12 is completed. ■

Corollary 2.2.13. (I) Let $f(u) = u$. Then

$$\alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_0}{c\mu_0} x\right) K(x) dx = \frac{\alpha}{2} - \theta.$$

(II) Let $f(u)$ satisfy $u \leq f(u) \leq \frac{f(\theta)}{\theta}u$ on $(0, \theta)$. Then there hold the estimates

$$\alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \leq \frac{\alpha}{2} - \theta,$$

$$\frac{\alpha}{2} - f(\theta) \leq \alpha \int_{-\infty}^0 \exp\left(\frac{f(\theta)}{\theta} \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx.$$

(III) Let $f(u)$ satisfy $\frac{f(\theta)}{\theta}u \leq f(u) \leq u$ on $(0, \theta)$. Then there hold the estimates

$$\alpha \int_{-\infty}^0 \exp\left(\frac{f(\theta)c - \mu_0}{\theta c\mu_0}x\right) K(x)dx \leq \frac{\alpha}{2} - f(\theta),$$

$$\frac{\alpha}{2} - \theta \leq \alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_0}{c\mu_0}x\right) K(x)dx.$$

(IV) Let $f(u) = u(u-1)(Du-1)$ satisfy $u + f(\theta) - \theta \leq f(u) \leq u$ on $(0, \theta)$, where $D > 0$ is a constant. Then there hold the estimates

$$\frac{\alpha}{2} - \theta \leq \alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_0}{c\mu_0}x\right) K(x)dx \leq \frac{\alpha}{2} - f(\theta).$$

Proof. The proof of Corollary 2.2.13 follows from Theorem 2.2.12. ■

2.2.7 More delicate estimates

We will provide more delicate estimates on the wave speeds.

Theorem 2.2.14. (I) Suppose that

$$f(u) \geq f_l(u) = m(u - n), \quad \text{on} \quad (n, \infty).$$

Then $\mu_0(f) \leq \mu_0(f_l)$ and

$$\frac{1}{\mu_0(f)} \geq \frac{1}{\mu_0(f_l)} = \frac{1}{c} + \frac{1}{\nu_0(f_l)}.$$

(II) Suppose that

$$f(u) \leq f_l(u) = m(u - n), \quad \text{on} \quad (n, \infty).$$

Then $\mu_0(f) \geq \mu_0(f_l)$ and

$$\frac{1}{\mu_0(f)} \leq \frac{1}{\mu_0(f_l)} = \frac{1}{c} + \frac{1}{\nu_0(f_l)}.$$

(III) Suppose that

$$m_1(u - n_1) = f_{l_1}(u) \leq f(u) \leq f_{l_2}(u) = m_2(u - n_2), \quad \text{on} \quad (n, \infty).$$

Then $\mu_0(f_{l_2}) \leq \mu_0(f) \leq \mu_0(f_{l_1})$ and

$$\frac{1}{c} + \frac{1}{\nu_0(f_{l_1})} = \frac{1}{\mu_0(f_{l_1})} \leq \frac{1}{\mu_0(f)} \leq \frac{1}{\mu_0(f_{l_2})} = \frac{1}{c} + \frac{1}{\nu_0(f_{l_2})}.$$

Proof. By combining the results of Theorem 2.2.2, Theorem 2.2.10 and Theorem 2.2.12, we finish the proof of Theorem 2.2.14 immediately. \blacksquare

Corollary 2.2.15. Suppose that $m_1(u - n_1) = f_{l_1}(u) \leq f(u) \leq f_{l_2}(u) = m_2(u - n_2)$, for two positive constants m_1 and m_2 and two real constants n_1 and n_2

(I) Let $K_1(x) = \frac{\rho}{2} \exp(-\rho|x|)$, where $\rho > 0$ is a constant. Then

$$\frac{1}{c} + \frac{1}{\nu_0(f_{l_1})} \leq \frac{1}{\mu_0(f)} \leq \frac{1}{c} + \frac{1}{\nu_0(f_{l_2})},$$

where

$$\nu_0(f_{l_1}) = \frac{\alpha - 2m_1\theta_1 + 2m_1n_1}{2\rho(\theta - n_1)},$$

and

$$\nu_0(f_{l_2}) = \frac{\alpha - 2m_2\theta_2 + 2m_2n_2}{2\rho(\theta - n_2)},$$

are the solutions of the equations

$$\alpha \int_{-\infty}^0 \exp\left[\frac{m_1}{\nu_0(f_{l_1})}x\right] K_1(x)dx = \frac{\alpha}{2} - m_1\theta + m_1n_1,$$

and

$$\alpha \int_{-\infty}^0 \exp\left[\frac{m_2}{\nu_0(f_{l_2})}x\right] K_2(x)dx = \frac{\alpha}{2} - m_2\theta + m_2n_2,$$

respectively.

(II) Let $K_2(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$, where $\rho > 0$ is a constant. Then

$$\frac{1}{c} + \frac{1}{\nu_0(f_{l_1})} \leq \frac{1}{\mu_0(f)} \leq \frac{1}{c} + \frac{1}{\nu_0(f_{l_2})},$$

where

$$\frac{1}{\nu_0(f_{l_1})} = \frac{1}{m_1\rho} \ln \frac{\alpha}{\alpha - 2m_1\theta + 2m_1n_1},$$

and

$$\frac{1}{\nu_0(f_{l_2})} = \frac{1}{m_2\rho} \ln \frac{\alpha}{\alpha - 2m_2\theta + 2m_2n_2}.$$

Proof. The proof follows from Theorem 2.2.14. ■

2.2.8 Asymptotic behaviors of the wave speeds

The next theorem investigates the asymptotic behaviors of the wave speeds as the parameters tend to zero or infinity.

Theorem 2.2.16. (I) Consider the integral differential equation (2.10) with the sodium current function $f_+(u) = \frac{1}{D} \sinh(Du)$. Given the positive constants $\alpha > 0$ and $\theta > 0$, there exist two positive numbers $D_{\sinh}^* > 0$ and $D_{\sinh}^{**} > 0$, defined by

$$\frac{1}{D_{\sinh}^*} \sinh(D_{\sinh}^* \theta) = \frac{\alpha}{2},$$

and

$$\frac{1}{D_{\sinh}^{**}} \sinh(D_{\sinh}^{**} \theta) = \alpha.$$

(I-1) For any $D \in (0, D_{\sinh}^*)$, there exists a unique traveling wave front $U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$. Additionally,

$$\lim_{z \rightarrow -\infty} U(z) = 0, \quad \lim_{z \rightarrow \infty} U(z) = \beta_{\sinh}, \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0,$$

where $\beta_{\sinh} > \theta$ is a constant such that $\frac{1}{D} \sinh(D\beta_{\sinh}) = \alpha$.

(I-2) For any $D \in (D_{\sinh}^*, D_{\sinh}^{**})$, there exists a unique traveling wave front $U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U > \theta$ on $(-\infty, 0)$ and $U < \theta$ on $(0, \infty)$. Additionally,

$$\lim_{z \rightarrow -\infty} U(z) = \beta_{\sinh}, \quad \lim_{z \rightarrow \infty} U(z) = 0, \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0,$$

where $\beta_{\sinh} > \theta$ is a constant such that $\frac{1}{D} \sinh(D\beta_{\sinh}) = \alpha$. (I-3) The wave speed enjoys the limits

$$\lim_{D \rightarrow 0} \mu_0(D) = \mu_0, \quad \lim_{D \rightarrow D_{\sinh}^*} \mu_0(D) = 0,$$

where $\mu_0 > 0$ is the wave speed of the front of equation (2.7) with $f_0(u) = u$. For any $D \in (D_{\sinh}^{**}, \infty)$, there exists no traveling wave front.

(II) Consider the integral differential equation (2.10) with the sodium current function $f_-(u) = \frac{1}{D} \tanh(Du)$. Let $\alpha > 0$ and $\theta > 0$ be constants, such that $0 < \theta < \alpha$. There are two cases to consider. Case One: If $\frac{\alpha}{2} < \theta$, then there exist two positive numbers $D_{\tanh}^* > 0$ and $D_{\tanh}^{**} > 0$, $D_{\tanh}^{**} < D_{\tanh}^*$, defined by

$$\frac{1}{D_{\tanh}^*} \tanh(D_{\tanh}^* \theta) = \frac{\alpha}{2},$$

and

$$\frac{1}{D_{\tanh}^{**}} \tanh(D_{\tanh}^{**} \theta) = \alpha.$$

(II-1) For any $D \in \left(D_{\tanh}^*, \frac{1}{\alpha}\right)$, there exists a unique traveling wave front

$U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$.

Additionally,

$$\lim_{z \rightarrow -\infty} U(z) = 0, \quad \lim_{z \rightarrow \infty} U(z) = \beta_{\tanh}, \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0,$$

where $\beta_{\tanh} > \theta$ is a constant such that $\frac{1}{D} \tanh(D\beta_{\tanh}) = \alpha$.

(II-2) For any $D \in (D_{\tanh}^{**}, D_{\tanh}^*)$, there exists a unique traveling wave front $U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U > \theta$ on $(-\infty, 0)$ and $U < \theta$ on $(0, \infty)$. Additionally,

$$\lim_{z \rightarrow -\infty} U(z) = \beta_{\tanh}, \quad \lim_{z \rightarrow \infty} U(z) = 0, \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0,$$

where $\beta_{\tanh} > \theta$ is a constant such that $\frac{1}{D} \tanh(D\beta_{\tanh}) = \alpha$. (II-3) The wave speed enjoys the limit

$$\lim_{D \rightarrow D_{\tanh}^*} \mu_0(D) = 0.$$

For any $D \in (0, D_{\tanh}^{**}) \cup \left(\frac{1}{\alpha}, \infty\right)$, there exists no traveling wave front.

Case Two: If

$$\theta < \frac{\alpha}{2},$$

then for all $D \in \left(0, \frac{1}{\alpha}\right)$, there exists a unique traveling wave front

$U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$.

Additionally,

$$\lim_{z \rightarrow -\infty} U(z) = 0, \quad \lim_{z \rightarrow \infty} U(z) = \beta_{\tanh}, \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0,$$

where $\beta_{\tanh} > \theta$ is a constant such that $\frac{1}{D} \tanh(D\beta_{\tanh}) = \alpha$. Moreover, the wave speed enjoys the limit

$$\lim_{D \rightarrow 0} \mu_0(D) = \mu_0, \quad \lim_{D \rightarrow 1/\alpha} \mu_0(D) = c,$$

where $\mu_0 > 0$ is the wave speed of the front of equation (6) with $f_0(u) = u$. If $D \in \left(\frac{1}{\alpha}, \infty\right)$, then there exists no traveling wave front.

(III) Consider the integral differential equation (2.10) with the sodium current function $f_{\text{cubic}}(u) = u(u-1)(Du-1)$. Given the positive constants $\alpha > 0$ and $0 < \theta < 1$, there exist two real numbers D_{cubic}^* and D_{cubic}^{**} , $D_{\text{cubic}}^{**} < D_{\text{cubic}}^*$, defined by

$$\theta(\theta-1)(D_{\text{cubic}}^*\theta-1) = \frac{\alpha}{2},$$

and

$$\theta(\theta-1)(D_{\text{cubic}}^{**}\theta-1) = \alpha.$$

(III-1) For any $D \in (D_{\text{cubic}}^*, \infty)$, we have

$$\theta(\theta-1)(D\theta-1) < \frac{\alpha}{2}.$$

There exists a positive number $\beta = \beta_{\text{cubic}}$, such that $\beta_{\text{cubic}} > 1$ and

$$\beta(\beta-1)(D\beta-1) = \alpha.$$

There exists a unique traveling wave front $U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U < \theta$ on $(-\infty, 0)$ and $U > \theta$ on $(0, \infty)$. Additionally, we have

$$\lim_{z \rightarrow -\infty} U(z) = 0, \quad \lim_{z \rightarrow \infty} U(z) = \beta_{\text{cubic}}, \quad \lim_{z \rightarrow \infty} U'(z) = 0.$$

(III-2) For any $D \in (D_{\text{cubic}}^{**}, D_{\text{cubic}}^*)$, we have

$$\frac{\alpha}{2} < \theta(\theta-1)(D\theta-1) < \alpha.$$

There exists a positive number $\beta = \beta_{\text{cubic}}$, such that $\beta_{\text{cubic}} > 1$ and

$$\beta(\beta-1)(D\beta-1) = \alpha.$$

There exists a unique traveling wave front $U = U_{\text{front}}(\cdot)$, such that $U(0) = \theta$, $U > \theta$ on $(-\infty, 0)$ and $U < \theta$ on $(0, \infty)$. Additionally, we have

$$\lim_{z \rightarrow -\infty} U(z) = \beta_{\text{cubic}}, \quad \lim_{z \rightarrow \infty} U(z) = 0, \quad \lim_{z \rightarrow \infty} U'(z) = 0.$$

The wave speed enjoys the limits

$$\lim_{D \rightarrow D_{\text{cubic}}^*} \mu_0(D) = 0, \quad \lim_{D \rightarrow \infty} \mu_0(D) = c.$$

Proof. The existence and uniqueness of each of the numbers D_{sinh}^* , D_{sinh}^{**} , D_{tanh}^* , D_{tanh}^{**} , D_{cubic}^* and D_{cubic}^{**} are obviously true. Let $D \in (0, D_{\text{sinh}}^*)$. Then

$$0 < \frac{1}{D} \sinh(D\theta) < \frac{\alpha}{2},$$

and the equation

$$\frac{1}{D} \sinh(Du) = \alpha,$$

has a unique solution $\beta_{\text{sinh}} > \theta$. The existence and uniqueness of the traveling wave front may be proved very easily. Let $D \in (D_{\text{sinh}}^*, D_{\text{sinh}}^{**})$. Then

$$\frac{\alpha}{2} < \frac{1}{D} \sinh(D\theta) < \alpha,$$

and the equation

$$\frac{1}{D} \sinh(Du) = \alpha,$$

has a unique solution $\beta_{\text{sinh}} > \theta$. The existence and uniqueness of the traveling wave front may be proved very easily. Let $D \in (D_{\text{sinh}}^{**}, \infty)$. Suppose that there exists a solution $\beta_{\text{sinh}} \geq \theta$ to the equation $\frac{1}{D} \sinh(Du) = \alpha$, so that there exists a traveling wave front connecting the fixed point $U = 0$ at $z = -\infty$ to the fixed point $U = \beta_{\text{sinh}}$ at $z = \infty$. Note that $\frac{1}{D} \sinh(D\theta)$ is an increasing function of D and $\frac{1}{D} \sinh(Du)$ is an increasing function of u if $D > 0$ is fixed. Therefore, we get

$$\alpha = \frac{1}{D} \sinh(D\beta_{\text{sinh}}) \geq \frac{1}{D} \sinh(D\theta) > \frac{1}{D_{\text{sinh}}^{**}} \sinh(D_{\text{sinh}}^{**}\theta) = \alpha.$$

This is a contradiction. Hence there is no traveling wave front to equation (2.10) with $f_+(u) = \frac{1}{D} \sinh(Du)$ if $D \in (D_{\text{sinh}}^{**}, \infty)$.

The mathematical analysis of the traveling wave front of equation (2.10) with $f_-(u) = \frac{1}{D} \tanh(Du)$ is very similar to the above analysis and is omitted.

Let $D \in (D_{\text{cubic}}^*, \infty)$. Then

$$\theta(\theta - 1)(D\theta - 1) < \frac{\alpha}{2},$$

and the equation

$$u(u - 1)(Du - 1) = \alpha$$

has a solution $\beta_{\text{cubic}} > 1$. The existence and uniqueness of the traveling wave front may be proved very easily. Let $D \in (D_{\text{cubic}}^{**}, D_{\text{cubic}}^*)$. Then

$$\frac{\alpha}{2} < \theta(\theta - 1)(D\theta - 1) < \alpha,$$

and the equation

$$u(u - 1)(Du - 1) = \alpha$$

has a solution $\beta_{\text{cubic}} > 1$. The existence and uniqueness of the traveling wave front may be proved very easily.

Based on Theorem 2.2.10, for the traveling wave front of equation (2.10) with $f_+(u) = \frac{1}{D} \sinh(Du)$, $f_-(u) = \frac{1}{D} \tanh(Du)$, $f_c(u) = u(u - 1)(Du - 1)$, respectively, the wave speed $\mu_0(D)$ is a decreasing, an increasing, an increasing function of D . Note

that $0 < \mu_0(D) < c$ for all cases. Therefore, the limits

$$L_{\sinh}^* := \lim_{D \rightarrow 0} \mu_0(D), \text{ and}$$

$$L_{\sinh}^{**} := \lim_{D \rightarrow D_{\sinh}^*} \mu_0(D), \text{ for } f_+(u) = \frac{1}{D} \sinh(Du),$$

$$L_{\tanh}^* := \lim_{D \rightarrow 0} \mu_0(D), \text{ and}$$

$$L_{\tanh}^{**} := \lim_{D \rightarrow D_{\tanh}^*} \mu_0(D), \text{ for } f_-(u) = \frac{1}{D} \tanh(Du),$$

$$L_{\text{cubic}}^* := \lim_{D \rightarrow \infty} \mu_0(D), \text{ and}$$

$$L_{\text{cubic}}^{**} := \lim_{D \rightarrow D_{\text{cubic}}^*} \mu_0(D), \text{ for } f_c(u) = u(u-1)(Du-1),$$

exist. Let us find these limits. Note that $f(0) = 0$ and $f'(0) = 1$. First of all, we have the traveling wave front representation

$$\begin{aligned} U(z) &= \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left(\frac{x-z}{\mu_0}\right) \frac{c}{c+s(x)\mu_0} K\left(\frac{cx}{c+s(x)\mu_0}\right) dx \\ &\quad + \frac{1}{\mu_0} \int_{-\infty}^z \exp\left(\frac{x-z}{\mu_0}\right) [U(x) - f(U(x))] dx. \end{aligned}$$

Recall that the speed is determined by the condition $U(0) = \theta$, namely, the equation

$$\begin{aligned} &\alpha \int_{-\infty}^0 K(x) dx - \alpha \int_{-\infty}^0 \exp\left(\frac{c-\mu_0}{c\mu_0}x\right) K(x) dx \\ &\quad + \frac{1}{\mu_0} \int_{-\infty}^0 \exp\left(\frac{x}{\mu_0}\right) [U(x) - f(U(x))] dx = \theta. \end{aligned}$$

Note that

$$\lim_{D \rightarrow 0} \frac{1}{D} \sinh(Du) = u, \quad \lim_{D \rightarrow 0} \frac{1}{D} \tanh(Du) = u.$$

Letting $D \rightarrow 0$ for $f_+(u) = \frac{1}{D} \sinh(Du)$, letting $D \rightarrow 0$ for $f_-(u) = \frac{1}{D} \tanh(Du)$, letting $D \rightarrow \infty$ for $f_c(u) = u(u-1)(Du-1)$, we find that

$$\frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_{\sinh}}{c\mu_{\sinh}}x\right) K(x)dx = \theta.$$

$$\frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_{\tanh}}{c\mu_{\tanh}}x\right) K(x)dx = \theta.$$

$$\frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(\frac{c - \mu_{\text{cubic}}}{c\mu_{\text{cubic}}}x\right) K(x)dx = 0.$$

Therefore, we find that $L_{\sinh}^* = \mu_0$, $L_{\tanh}^* = \mu_0$ and $L_{\text{cubic}}^* = c$. For equation (2.10), note that the traveling wave front also satisfies

$$\mu_0 U' + f(U) = \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x)dx.$$

Letting $z = 0$ and $U = \theta$, we obtain

$$\mu_0 U'(0) + f(\theta) = \frac{\alpha}{2}.$$

Letting $D \rightarrow D_{\sinh}^*$ for $f_+(u) = \frac{1}{D} \sinh(Du)$, letting $D \rightarrow D_{\tanh}^*$ for $f_-(u) = \frac{1}{D} \tanh(Du)$, letting $D \rightarrow D_{\text{cubic}}^*$ for $f_c(u) = u(u-1)(Du-1)$, respectively, we find $L_{\sinh}^{**} = 0$, $L_{\tanh}^{**} = 0$, $L_{\text{cubic}}^{**} = 0$. The proof of Theorem 2.2.16 is finished. \blacksquare

Corollary 2.2.17. *There hold the results*

$$\begin{array}{ll} \inf_{K \in \text{class (A)}} \mu_0(K) = 0, & \sup_{K \in \text{class (A)}} \mu_0(K) = c, \\ \inf_{K \in \text{class (B)}} \mu_0(K) = 0, & \sup_{K \in \text{class (B)}} \mu_0(K) = c, \\ \inf_{K \in \text{class (C)}} \mu_0(K) = 0, & \sup_{K \in \text{class (C)}} \mu_0(K) = c. \end{array}$$

Proof. It is not difficult to prove these results by using Theorem 2.2.2 through Theorem 2.2.16. ■

Please see Figure 2.2.8, Figure 2.2.8 and Figure 2.2.8 for the dependence of the wave speed on the parameters D and ρ for the three sodium current functions $f_+(u) = \frac{1}{D} \sinh(Du)$, $f_-(u) = \frac{1}{D} \tanh(Du)$ and $f_c(u) = u(u-1)(Du-1)$.

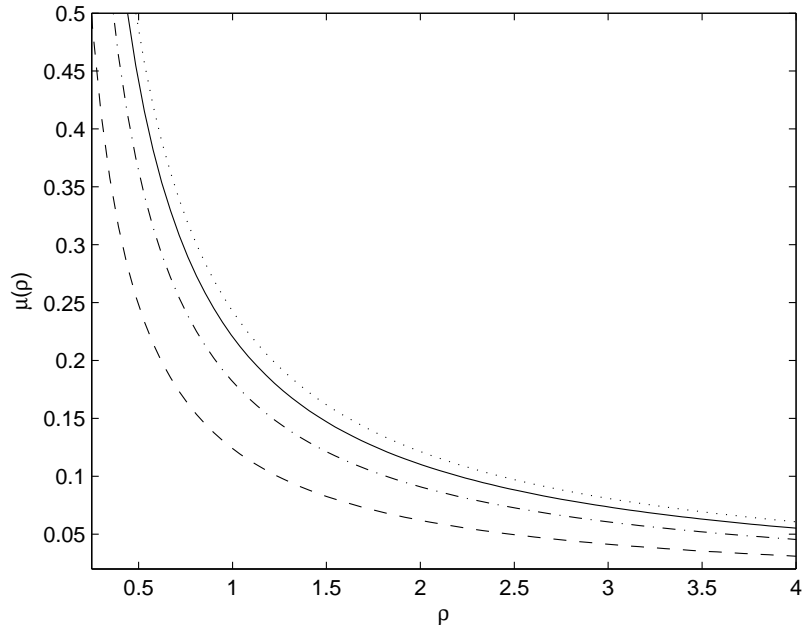


Figure 2.8: Influence of D and ρ on wave speed: $\mu = \mu(D, \rho)$, where $D \in [0, 0.5]$ and $\rho \in [0.25, 4]$. The wave speed μ is a decreasing function of D and it is also a decreasing function of ρ . In this graph, we use $c = \infty$, $\alpha = 5$ and $\theta = 2$. Let $f_+(u) = \frac{1}{D} \sinh(Du)$ and $K(x) = \frac{\rho}{2} \exp(-\rho|x|)$. For the dotted curve, $D = 0.125$. For the solid curve, $D = 0.25$. For the dash-dotted curve, $D = 0.375$. For the dashed curve, $D = 0.5$.

2.2.9 Influence of synaptic couplings on wave speeds (some numerical calculations)

Derivation of a speed formula. Let $A > 0$, $B > 0$, $a > 0$ and $b > 0$ be constants, such

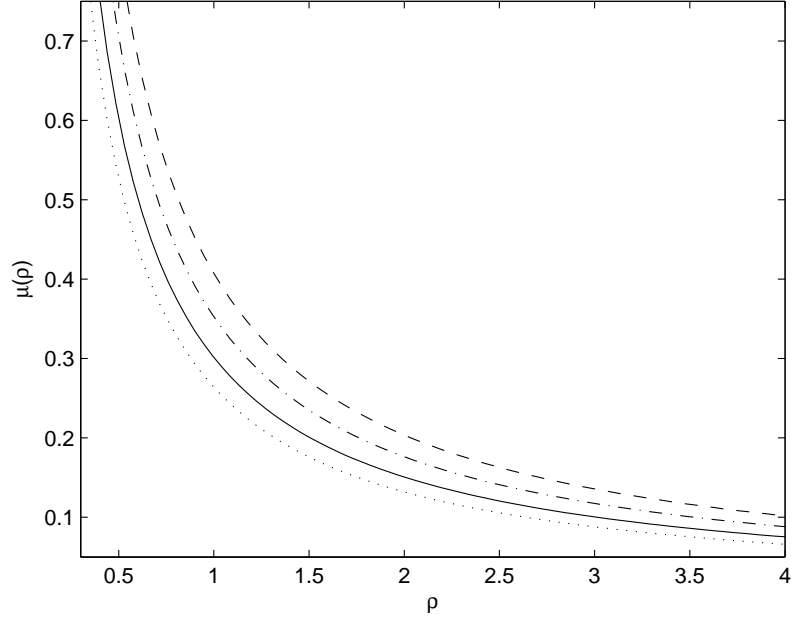


Figure 2.9: Influence of D and ρ on wave speed: $\mu = \mu(D, \rho)$, where $D \in [0, 0.5]$ and $\rho \in [0.35, 4]$. The wave speed μ is an increasing function of D and it is a decreasing function of ρ . In this graph, we use $c = \infty$, $\alpha = 5$ and $\theta = 2$. Let $f_-(u) = \frac{1}{D} \tanh(Du)$ and $K(x) = \frac{\rho}{2} \exp(-\rho|x|)$. For the dotted curve, $D = 0.125$. For the solid curve, $D = 0.25$. For the dash-dotted curve, $D = 0.375$. For the dashed curve, $D = 0.5$.

that

$$\frac{A}{a} - \frac{B}{b} = \frac{1}{2}.$$

Define $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$. Then

$$\int_{\mathbb{R}} K(x) dx = 1.$$

The speed of the traveling wave front of (2.5) with $f(u) = u$ satisfies

$$\alpha \int_{-\infty}^0 \exp\left(\frac{x}{\nu}\right) K(x) dx = \frac{\alpha}{2} - \theta.$$

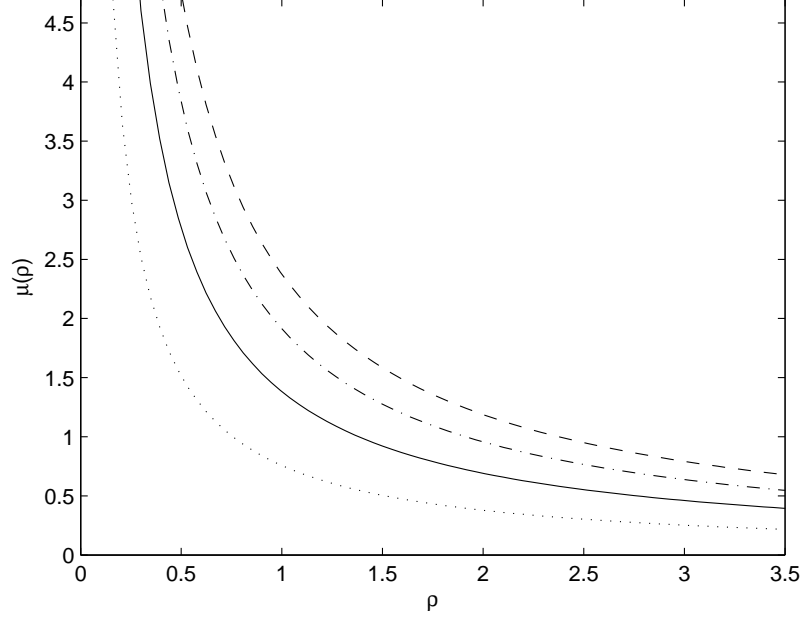


Figure 2.10: Influence of D and ρ on speed: $\mu = \mu(D, \rho)$, where $D \in [2, 4]$ and $\rho \in [0, 3.5]$. The wave speed μ is an increasing function of D and it is a decreasing function of ρ . In this graph, $c = 10$, $\alpha = 5$, $\theta = 0.5$ and $\rho = 1$. Let the sodium current function $f_c(u) = u(u - 1)(Du - 1)$ and $K(x) = \frac{\rho}{2} \exp(-\rho|x|)$. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

In another word

$$\alpha \left\{ \frac{A}{a + \frac{1}{\nu}} - \frac{B}{b + \frac{1}{\nu}} \right\} = \frac{\alpha}{2} - \theta.$$

This equation is equivalent to

$$\alpha [A\nu(1 + b\nu) - B\nu(1 + a\nu)] = \left(\frac{\alpha}{2} - \theta \right) (1 + a\nu)(1 + b\nu),$$

or

$$\begin{aligned} \left[(Ab - Ba) - ab \left(\frac{1}{2} - \frac{\theta}{\alpha} \right) \right] \nu^2 &- \left[(a + b) \left(\frac{1}{2} - \frac{\theta}{\alpha} \right) - (A - B) \right] \nu \\ &- \left(\frac{1}{2} - \frac{\theta}{\alpha} \right) \\ &= 0. \end{aligned}$$

Let

$$\begin{aligned} P &= (Ab - Ba) - ab \left(\frac{1}{2} - \frac{\theta}{\alpha} \right) = ab \frac{\theta}{\alpha}, \\ Q &= (a + b) \left(\frac{1}{2} - \frac{\theta}{\alpha} \right) - (A - B), \\ R &= \frac{1}{2} - \frac{\theta}{\alpha}. \end{aligned}$$

Then the wave speed of equation (2.5) with $f(u) = u$ is given by

$$\nu = \frac{Q + \sqrt{Q^2 + 4PR}}{2P}.$$

Here we investigate the influence of synaptic couplings on wave speeds by using a series of numerical calculations, instead of rigorous mathematical analysis.

Please see Figure 2.2.9, Figure 2.2.9, Figure 2.2.9, Figure 2.2.9 and Figure 2.2.9 for the dependence of the wave speed on the parameters A , B , a , b , ρ and σ , where $\rho = A = a$ and $\sigma = 2B = b$ for the kernel function used in Figure 2.2.9.

2.2.10 Applications to real biology

In this part, we try to find connections of our results to real wave speeds.

Pinto and Ermentrout [46] and Pinto, Patrick, Huang and Connors [16] have obtained some hard numbers on wave speeds through experiments: $\mu = 4\text{m/s}$ (meters per second). To match this speed, let $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$. If A , B , a , b ,

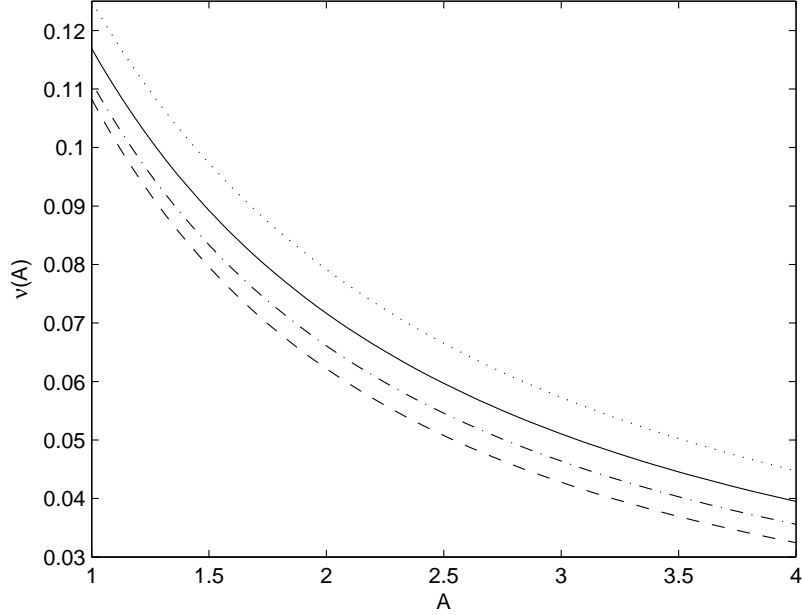


Figure 2.11: Influence of a and A on speed: $\mu = \mu(a, A)$, where $a \in [2, 4.2]$ and $A \in [1, 4]$. The wave speed μ is an increasing function of a and it is a decreasing function of A . In this graph, we use $c = \infty$, $\alpha = 5$, $\theta = 2$ and $f(u) = u$. Let the synaptic coupling $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$, where $b = 1$ is fixed and $B = \frac{A}{a} - \frac{1}{2}$ is a dependent variable. For the dotted curve, $a = 2.0$. For the solid curve, $a = 2.4$. For the dash-dotted curve, $a = 3.0$. For the dashed curve, $a = 4.2$.

α and θ are constants such that

$$\alpha \left\{ \frac{A}{a + \frac{1}{4}} - \frac{B}{b + \frac{1}{4}} \right\} = \frac{\alpha}{2} - \theta,$$

then the wave speed $\nu = 4\text{m/s}$. In particular, we may choose $A = 4$, $B = 1$, $a = 4$, $b = 2$, $\alpha = 17$ and $\theta = \frac{1}{18}$, and we find the speed $\nu_0 = 4\text{m/s}$. We may also choose $A = 4$, $B = 1$, $D = 3$, $a = 4$, $b = 2$, $c = 10$, $\alpha = \frac{8360}{27} \approx 309.63$, $\theta = 2$, $f(u) = u(u - 1)(Du - 1)$ and $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$, we also find the speed $\mu_0 = 4\text{m/s}$. Of course we are not sure if these parameters are reasonably close to real biological data. If yes, then we may be able to find real applications to biology.

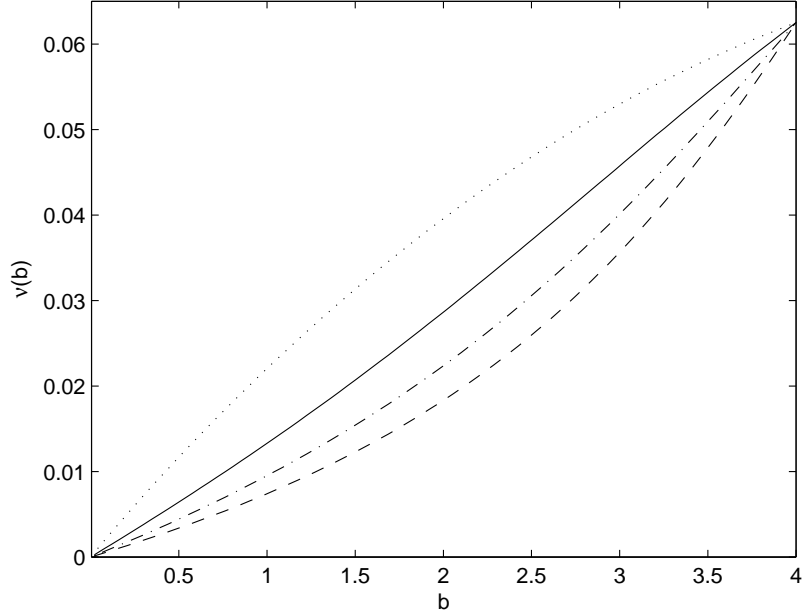


Figure 2.12: Influence of B and b on speed: $\mu = \mu(b, B)$, where $b \in [0, 4]$ and $B \in [1, 4]$. The wave speed μ is an increasing function of b and it is a decreasing function of B . In this graph, we use $c = \infty$, $\alpha = 5$, $\theta = 2$ and $f(u) = u$. Let the synaptic coupling $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$, where $a = 4$ is fixed and $A = 4 \left(\frac{1}{2} + \frac{B}{b} \right)$ is a dependent variable. For the dotted curve, $B = 1.0$. For the solid curve, $B = 2.0$. For the dash-dotted curve, $B = 3.0$. For the dashed curve, $B = 4.0$.

Below we offer a sketch of how we calculated the speed in the second case. Given the differential equation

$$\frac{\partial u}{\partial t} + u(u-1)(Du-1) = \alpha \int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) dy,$$

as before, we can find the approximate equation

$$\frac{\partial u}{\partial t} + m(u-n) = \alpha \int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) dy,$$

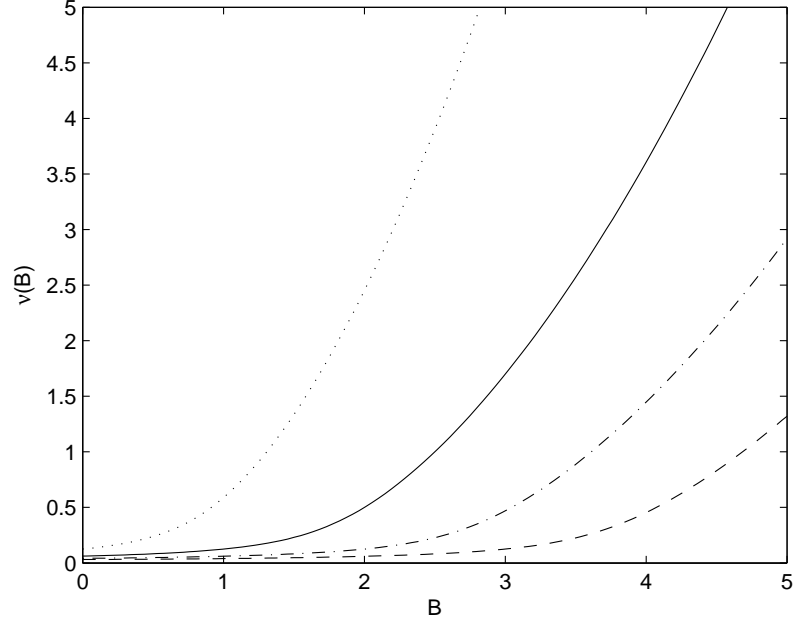


Figure 2.13: Influence of A and B on speed: $\mu = \mu(A, B)$, where $A \in [1, 4]$ and $B \in [0, 5]$. The wave speed μ is an increasing function of A and it is a decreasing function of B . In this graph, we use $c = \infty$, $\alpha = 5$, $\theta = 2$ and $f(u) = u$. Let $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$, where $b = 2$ is fixed and $a = \frac{2A}{1+B}$ is a dependent variable. For the dotted curve, $A = 1.0$. For the solid curve, $A = 2.0$. For the dash-dotted curve, $A = 3.0$. For the dashed curve, $A = 4.0$.

on $(0, \theta)$, where

$$\begin{aligned}
 m &= (\theta - 1)(D\theta - 1), \\
 n &= \frac{\theta}{2} - \frac{1}{\theta(\theta - 1)(D\theta - 1)} \left(\frac{D}{4}\theta^4 - \frac{1+D}{3}\theta^3 + \frac{1}{2}\theta^2 \right), \\
 mn &= \frac{1}{2}\theta(\theta - 1)(D\theta - 1) - \frac{D}{4}\theta^3 + \frac{1+D}{3}\theta^2 - \frac{1}{2}\theta.
 \end{aligned}$$

Moreover, we can transform the approximate equation to

$$v_\tau + v = \alpha \int_{\mathbb{R}} K(x - y) H \left(v \left(y, \tau - \frac{1}{C}|x - y| \right) - \Theta \right) dy,$$

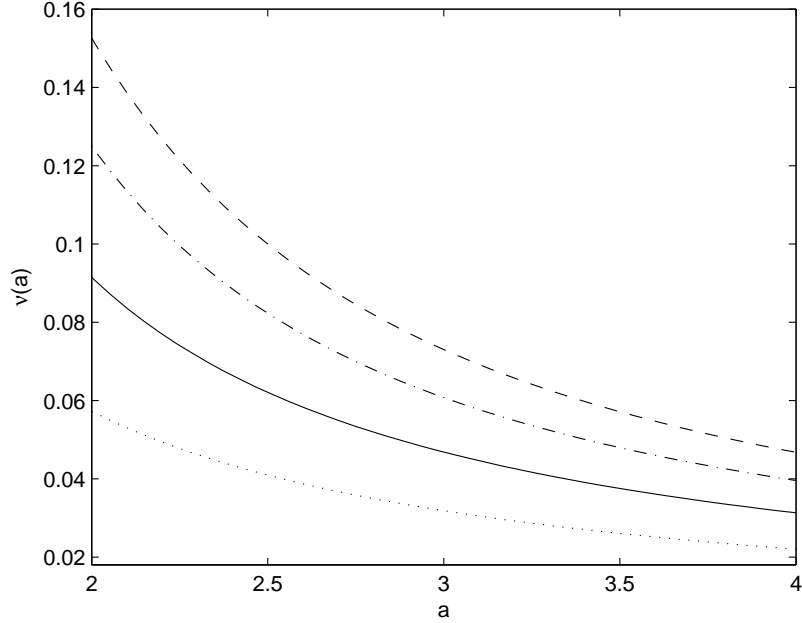


Figure 2.14: Influence of a and b on speed: $\mu = \mu(a, b)$, where $a \in [2, 4]$ and $b \in [1, 3]$. The wave speed μ is an increasing function of b and it is a decreasing function of a . In this graph, we use $c = \infty$, $\alpha = 5$, $\theta = 2$ and $f(u) = u$. Let $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$, where $B = 1$ is fixed and $A = a \left(\frac{1}{2} + \frac{1}{b} \right)$ is a dependent variable. For the dotted curve, $b = 1.0$. For the solid curve, $b = 1.5$. For the dash-dotted curve, $b = 2.0$. For the dashed curve, $b = 2.5$.

where

$$\tau = mt, \quad v(x, \tau) = m[u(x, t) - n], \quad C = \frac{c}{m}, \quad \Theta = m\theta - mn.$$

Let the kernel function $K(x) = A \exp(-a|x|) - B \exp(-b|x|)$, where $A > B > 0$ and $a > b > 0$ are constants, satisfying the conditions

$$\frac{A}{a} - \frac{B}{b} = \frac{1}{2}, \quad \frac{A}{a^2} \geq \frac{B}{b^2}.$$

Therefore, if the model parameters A, B, D, a, b, c, α and θ satisfy

$$\frac{1}{4} = \frac{1}{c} + \frac{1}{\nu}, \quad \nu = m\tilde{\nu},$$

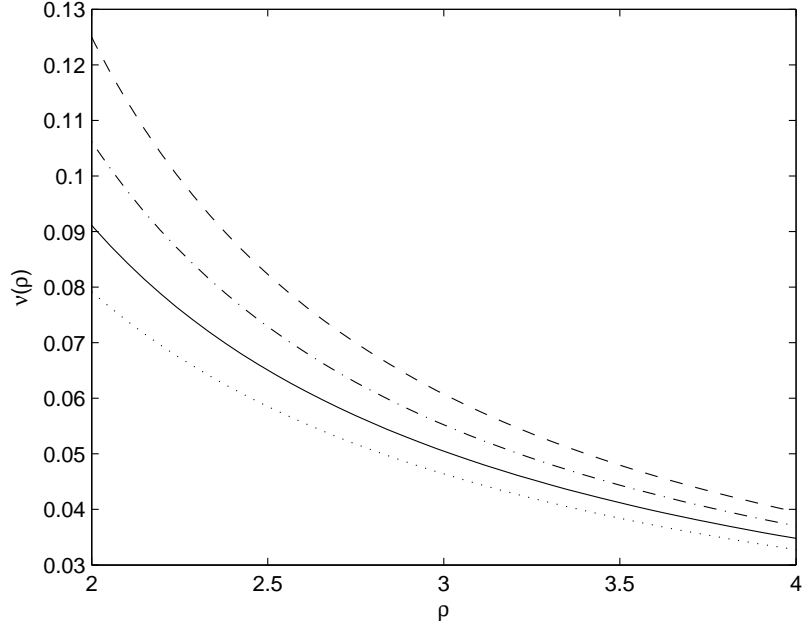


Figure 2.15: Influence of excitation ρ and inhibition σ on speed: $\mu = \mu(\rho, \sigma)$, where $\rho \in [2, 4]$ and $\sigma \in [1, 2]$. The wave speed μ is an increasing function of ρ and it is a decreasing function of σ . In this graph, we use $c = \infty$, $\alpha = 5$, $\theta = 2$ and $f(u) = u$. Let $K(x) = \rho \exp(-\rho|x|) - \frac{\sigma}{2} \exp(-\sigma|x|)$. For the dotted curve, $\sigma = 1.0$. For the solid curve, $\sigma = 1.333$. For the dash-dotted curve, $\sigma = 1.666$. For the dashed curve, $\sigma = 2.0$.

$$\alpha \left\{ \frac{A}{a + \frac{1}{v}} - \frac{B}{b + \frac{1}{v}} \right\} = \frac{\alpha}{2} - \Theta,$$

where we recall that

$$\begin{aligned} \Theta &= m\theta - mn, \\ m &= (\theta - 1)(D\theta - 1), \\ n &= \frac{\theta}{2} - \frac{1}{\theta(\theta - 1)(D\theta - 1)} \left(\frac{D}{4}\theta^4 - \frac{1+D}{3}\theta^3 + \frac{1}{2}\theta^2 \right), \\ mn &= \frac{1}{2}\theta(\theta - 1)(D\theta - 1) - \frac{D}{4}\theta^3 + \frac{1+D}{3}\theta^2 - \frac{1}{2}\theta, \end{aligned}$$

then $\mu = 4\text{m/s}$. In particular, if we choose $A = 4$, $B = 1$, $D = 3$, $a = 4$, $b = 2$, $c = 10$, $\alpha = 8360/27 \approx 309.63$ and $\theta = 2$, then we find $m = 5$, $mn = 10/3$, $\Theta = 20/3$,

$\tilde{\nu} = 4/3$, $\nu = 20/3$ and the real wave speed $\mu = 4\text{m/s}$.

Chapter 3

Two Delay Model

3.1 Introduction

There have been many very interesting research results on traveling wave solutions of these reduced model equations. Based on different biophysical interpretations, by using concrete examples, through mathematical analysis and numerical simulations, important properties of waves, such as their wave speeds as well as their dependence on the parameters and degree of homogeneity of the networks, have been established for the reduced equations.

Our main goal is to use mathematical analysis to offer positive solutions to the open problems. We will investigate how the biological pairs (a, b) , (α, β) , (θ, Θ) , (ξ, η) and (K, W) influence the wave speeds. We will derive new lower bound and upper bound for the wave speeds.

We will introduce speed index functions which, again, is very helpful in the study of wave speeds. Through this we will be able to investigate how the speed depends on various parameters as well as the synaptic couplings. Many estimates and asymptotic behaviors of the speed as the parameters approach certain numbers can be investigated very clearly. By using properties of the speed index functions, we are able to prove simple but elegant identities, which connects the speed of the front of the model where

there is a delay to the speed of the front where there is no delay. See Theorem 3.2.5, for such a relationship.

We are going to investigate how various neurobiological mechanisms (in particular, synaptic couplings, threshold and synaptic rate constant) influence traveling wave speed. We will be concerned with the three classes of synaptic couplings. We are concerned with asymptotic behaviors of the speed as various parameters approach certain numbers or infinity.

3.1.1 The model equations

Consider the following integral differential equation arising from synaptically coupled neuronal networks

$$\begin{aligned} & \frac{\partial u}{\partial t} + f(u) \\ = & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) dy \right] dc \\ + & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \end{aligned} \quad (3.1)$$

This model equation is motivated by several previous models appearing in Atay and Hutt [6]-[7], Coombes, Lord and Owen [13], Hutt and Atay [32]-[33], Magpantay and Zou [44], Pinto and Ermentrout [46], Wilson and Cowan [56]-[57], Yanagida and Zhang [58], Zhang [61], [62], [64], Zhang and Hutt [65]-[66], Zhang, Wu and Stoner [67].

The parameters of the model are consistent with the description in earlier chapters and the general assumptions are true. In addition, we note that the transmission speed distribution ξ and the feedback delay distribution η are probability density functions. Typical examples of delayed functions are

$$\xi(c) = \frac{p}{c^{p+1}} H(c-1), \quad \eta(\tau) = \frac{1}{q!} \tau^q \exp(-\tau),$$

where $p \geq 1$ and $q \geq 0$ are integers, and

$$\xi(c) = \frac{1}{p} \sum_{i=1}^p \delta(c - c_i), \quad \eta(\tau) = \frac{1}{q} \sum_{i=1}^q \delta(\tau - \tau_i),$$

where δ represents the Dirac delta impulse function, $0 < c_1 < c_2 < \dots < c_p < \infty$ and $0 < \tau_1 < \tau_2 < \dots < \tau_q < \infty$ are parameters. The kernel functions K and W represent synaptic couplings between neurons in the neuronal networks. Typical examples of synaptic couplings are

$$K(x) = \frac{1}{2m!} \rho^{m+1} |x|^m \exp(-\rho|x|), \quad W(x) = \frac{1}{2n!} \sigma^{n+1} |x|^n \exp(-\sigma|x|),$$

and

$$K(x) = \frac{1}{2m} \sum_{i=1}^m [\delta(x - \rho_i) + \delta(x + \rho_i)],$$

$$W(x) = \frac{1}{2n} \sum_{i=1}^n [\delta(x - \sigma_i) + \delta(x + \sigma_i)],$$

where $0 < \rho < \infty$, $0 < \rho_1 < \rho_2 < \dots < \rho_m < \infty$, $0 < \sigma < \infty$, $0 < \sigma_1 < \sigma_2 < \dots < \sigma_n < \infty$ are parameters, $m \geq 0$ and $n \geq 0$ are integers. Here we collect some known results closely related to our general model equation. For the particular case $a = 0$, $b = 0$, $\beta = 0$, $f(u) = u$, and $\xi(c) = \delta(c - c_0)$, where $c_0 \in (0, \infty)$ is a parameter, Coombes, Lord and Owen [13] derived a speed equation for equation (3.1); Pinto and Ermentrout [46] derived a speed equation and discussed the influence of c_0 and θ on the speed for equation (3.1); Zhang [64] investigated the influence of α , θ and c_0 on the speed for equation (3.1). For the special case, $a = 0$, $b = 0$ and $f(u) = u$, Zhang and Hutt [65] investigated the influence of α , β , θ , Θ , ξ , η , K and W on the speed for equation (3.1). For the case $a = 0$, $b = 0$, $\beta = 0$ and $\xi(c) = \delta(c - c_0)$, where $c_0 \in (0, \infty)$ is a parameter, in chapter 2, we studied the influence of f , c_0 , α and θ on the speed for equation (3.1). However, the influence of the neurobiological mechanisms (a, b) , (α, β) ,

(θ, Θ) , (ξ, η) and (K, W) on the wave speeds has not been solved completely. The general equation (3.1) contains many important integral differential equations arising from synaptically coupled neuronal networks. The model may be reduced to previous equations

- (I) if $a = 0$, $b = 0$, $\beta = 0$ and $\xi(c) = \delta(c - c_0)$, where $c_0 \in (0, \infty)$ is a parameter; then (3.1) becomes the integral differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + f(u) \\ &= (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) dy \right] dc. \end{aligned}$$

See chapter 2 for details on this equation.

- (II) if $a = 0$, $b = 0$, $\alpha = 0$, $f(u) = u$ and $\eta(\tau) = \delta(\tau - \tau_0)$, where $\tau_0 \in (0, \infty)$ is a parameter; then (3.1) becomes

$$\frac{\partial u}{\partial t} + f(u) = (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau.$$

See Coombes, Lord and Owen [13] and Hutt and Atay [33] for this equation.

- (III) if $a = 0$, $b = 0$ and $f(u) = u$; then (3.1) reduces to

$$\begin{aligned} \frac{\partial u}{\partial t} + u &= \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) dy \right] dc \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \end{aligned}$$

See Zhang and Hutt [65],[66] for this equation.

- (IV) if $\xi(c) = \delta(c - c_0)$ and $\eta(\tau) = \delta(\tau - \tau_0)$, where $c_0 \in (0, \infty)$ and $\tau_0 \in (0, \infty)$ are two positive parameters; then (3.1) becomes

$$\begin{aligned} \frac{\partial u}{\partial t} + f(u) &= (\alpha - au) \int_{\mathbb{R}} K(x-y) H \left(u \left(y, t - \frac{1}{c_0}|x-y| \right) - \theta \right) dy \\ &+ (\beta - bu) \int_{\mathbb{R}} W(x-y) H(u(y, t - \tau_0) - \Theta) dy. \end{aligned}$$

(V) if $\xi(c) = \delta(c - \infty)$ and $\eta(\tau) = \delta(\tau)$. then (3.1) reduces to

$$\begin{aligned} \frac{\partial u}{\partial t} + f(u) &= (\alpha - au) \int_{\mathbb{R}} K(x-y) H(u(y,t) - \theta) dy \\ &+ (\beta - bu) \int_{\mathbb{R}} W(x-y) H(u(y,t) - \Theta) dy. \end{aligned}$$

See [62] for a reduced model of this equation, where $f(u) = u$, $a = 0$ and $b = 0$. For each of these reduced integral differential equations, under certain assumptions on the model parameters and functions, there exists a traveling wave front with a positive wave speed. See [7], [13], [14], [18], [32], [46], [61], [64] and [65], [66].

3.1.2 Assumptions

The function $w = f(u)$ is smooth, such that the equation $f(u) = 0$ has a unique solution $U_0 = n < \theta$ and the equation $au + bu + f(u) = \alpha + \beta$ has a unique solution $U_1 = U_1(a, b, \alpha, \beta, f) > \Theta$. Moreover $m = f'(n) > 0$ and $f'(U(a, b, \alpha, \beta, f)) > 0$.

If $b = 0$, $\beta = 0$ and $f(u) = m(u - n)$, then $U_0 = n < \theta$ and $U_1 = \frac{\alpha + mn}{a + m} > \theta$.
If $a = 0$, $\alpha = 0$ and $f(u) = m(u - n)$, then $U_1 = \frac{\beta + mn}{b + m} > \Theta$. If $a > 0$, $b > 0$ and $f(u) = m(u - n)$, then $U_1 = \frac{\alpha + \beta + mn}{a + b + m} > \Theta$.

Suppose that the parameters satisfy the following conditions

$$n < \theta \leq \Theta, \quad a\theta + 2f(\theta) < \alpha, \quad (3.2)$$

$$b\Theta + 2f(\Theta) < \beta, \quad a\theta + b\theta + 2f(\theta) < \alpha + \beta, \quad (3.3)$$

$$(\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m\tau) d\tau \right] > 0. \quad (3.4)$$

We assume that the distributed delay functions satisfy the conditions

$$\begin{aligned} \xi \geq 0 \quad \text{and} \quad \eta \geq 0 \quad \text{on } \mathbb{R}^+, \\ \int_0^\infty \xi(c) dc = 1, \quad \int_0^\infty \eta(\tau) d\tau = 1, \quad \int_0^\infty \eta(\tau) \exp(m\tau) d\tau < \infty. \end{aligned}$$

Also, we assume that there exists a positive number $\varepsilon_0 > 0$, such that $\xi = 0$ on $[0, \varepsilon_0]$.

We assume that the synaptic coupling K is at least piecewise smooth on the entire real line \mathbb{R} , satisfying the conditions

$$\int_{\mathbb{R}} K(x)dx = 1, \quad \int_{-\infty}^0 K(x)dx = \frac{1}{2}, \quad \int_0^{\infty} K(x)dx = \frac{1}{2}, \quad (3.5)$$

$$\int_{-\infty}^0 |x|K(x)dx = \int_0^{\infty} |x|K(x)dx, \quad (3.6)$$

$$|K(x)| \leq C \exp(-\rho|x|) \quad \text{on} \quad \mathbb{R}, \quad (3.7)$$

for two positive constants C and ρ .

There exists a unique stable traveling wave front $U = U(z)$ to equation (3.1), where $z = x + \mu_0 t$ and $\mu_0 = \mu_0(a, b, \alpha, \beta, \theta, \Theta, \xi, \eta, K, W)$ represents the wave speed. It also satisfies the traveling wave equation

$$\begin{aligned} & \mu_0 U' + f(U) \\ &= (\alpha - aU) \int_0^{\infty} \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y|\right) - \theta\right) dy \right] dc \\ &+ (\beta - bU) \int_0^{\infty} \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(U(y - \mu_0 \tau) - \Theta) dy \right] d\tau, \end{aligned} \quad (3.8)$$

and the boundary conditions

$$\lim_{z \rightarrow -\infty} U(z) = n, \quad \lim_{z \rightarrow \infty} U(z) = U(a, b, \alpha, \beta, f), \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0. \quad (3.9)$$

3.2 Linear Speed Analysis

In this section, we will focus on the speed analysis of equation (3.1) with the linear sodium current function $f(u) = m(u - n)$, where $m > 0$ and n are real constants. We may interpret the constant m as the sodium conductance and the constant n as the sodium reversal potential.

The general assumptions (3.2)-(3.4) become

$$\begin{aligned} n < \theta \leq \Theta, \quad (a + 2m)\theta < \alpha + 2mn, \quad (b + 2m)\Theta < \beta + 2mn, \\ (a + b + 2m)\theta < \alpha + \beta + 2mn, \\ (\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m\tau) d\tau \right] > 0. \end{aligned}$$

First of all, we consider the particular case $b = 0$ and $\beta = 0$.

Definition 3.2.1. Define the sign function $s = s(x)$ by $s(x) = -1$ for all $x < 0$, $s(0) = 0$ and $s(x) = 1$ for all $x > 0$. Define the following four auxiliary functions

$$\begin{aligned} \omega_1(z) &= \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx, \\ \omega_2(z) &= m + a \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx, \\ \omega_3(z) &= mz + az \int_{-\infty}^z K(x) dx - a \int_z^0 |x|K(x) dx, \\ \omega_4(z) &= \exp \left\{ \frac{1}{\mu} \left[mz + az \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right. \right. \\ &\quad \left. \left. - a \int_{-\infty}^z \frac{cx}{c+s(x)\mu} K \left(\frac{cx}{c+s(x)\mu} \right) dx \right] \right\}. \end{aligned}$$

Then, on $(-\infty, 0)$, we have

$$\begin{aligned} \omega_1(0) &= \frac{1}{2}, \quad \omega_2(0) = m + \frac{a}{2}, \quad \omega_3(0) = 0, \\ \omega_1'(z) &= \frac{c}{c+s(z)\mu} K \left(\frac{c}{c+s(z)\mu} z \right), \\ \omega_2'(z) &= \frac{ca}{c+s(z)\mu} K \left(\frac{c}{c+s(z)\mu} z \right), \\ \omega_3'(z) &= m + a \int_{-\infty}^z K(x) dx + azK(z) + a|z|K(z) \\ &= m + a \int_{-\infty}^z K(x) dx > 0. \end{aligned}$$

Additionally, on $(-\infty, 0)$, we get

$$\begin{aligned} \omega_4(z) = & \exp \left\{ \frac{c-\mu}{c\mu} \left[\frac{cmz}{c+s(z)\mu} + a \frac{cz}{c+s(z)\mu} \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right. \right. \\ & \left. \left. + a \int_{-\infty}^{cz/(c+s(z)\mu)} |x|K(x) dx \right] \right\}. \end{aligned}$$

Moreover, on $(-\infty, 0)$, we find

$$\begin{aligned} \lim_{z \rightarrow -\infty} \omega_1(z) &= 0, & \lim_{z \rightarrow -\infty} \omega_2(z) &= m, & \lim_{z \rightarrow -\infty} \omega_4(z) &= 0, \\ \omega_4'(z) &= \frac{m + a\omega_1(z)}{\mu} \omega_4(z) = \frac{1}{\mu} \omega_2(z) \omega_4(z), \\ \omega_4(0) &= \exp \left[a \frac{c-\mu}{c\mu} \int_{-\infty}^0 |x|K(x) dx \right], \end{aligned}$$

and

$$\begin{aligned} \frac{\omega_4(z)}{\omega_4(0)} &= \exp \left\{ \frac{1}{\mu} \left[mz + az \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right. \right. \\ & \left. \left. + a \frac{c-\mu}{c} \int_{cz/(c+s(z)\mu)}^0 xK(x) dx \right] \right\} \\ &= \exp \left\{ \frac{c-\mu}{c\mu} \left[\frac{cmz}{c+s(z)\mu} + \frac{acz}{c+s(z)\mu} \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right. \right. \\ & \left. \left. + a \int_{cz/(c+s(z)\mu)}^0 xK(x) dx \right] \right\} \\ &= \exp \left[\frac{c-\mu}{c\mu} \omega_3 \left(\frac{cz}{c+s(z)\mu} \right) \right]. \end{aligned}$$

All of these auxiliary functions $\omega_1(z)$, $\omega_2(z)$, $\omega_3(z)$ and $\omega_4(z)$ will help us find the traveling wave front of the integral differential equation (3.1).

Definition 3.2.2. Define the speed index function ϕ by

$$\phi(\mu) = m(\alpha - an) \int_{-\infty}^0 \exp \left[\frac{c-\mu}{c\mu} \omega_3(z) \right] \frac{K(z)}{\left[m + a \int_{-\infty}^z K(x) dx \right]^2} dz.$$

Theorem 3.2.3. Let $b = 0$ and $\beta = 0$. Let $f(u) = m(u - n)$ in equation (3.1), where $m > 0$ and n are real constants. Suppose that

$$n < \theta, \quad 0 < (2m + a)(\theta - n) < \alpha - an,$$

and

$$m + a \int_{-\infty}^z K(x)dx > 0, \text{ on } (-\infty, 0).$$

Suppose that the traveling wave front satisfies the conditions $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$ and $U > \theta$ on $(0, \infty)$. Then, there exists a unique positive wave speed $\mu_0 = \mu_0(a, m, n, \alpha, \theta, \xi, K)$, determined by the equation

$$(\alpha - an) \int_{-\infty}^0 \exp \left\{ \frac{c - \mu_0}{c\mu_0} \left[mz + az \int_{-\infty}^z K(x)dx - a \int_z^0 |x|K(x)dx \right] \right\} \cdot \frac{K(z)}{\left[m + a \int_{-\infty}^z K(x)dx \right]^2} dz = \frac{\alpha - an}{m(2m + a)} - \frac{\theta - n}{m}.$$

Proof. Let $\mu \in (0, c)$ represent the wave speed and let $z = x + \mu t$. Suppose that $u(x, t) = U(x + \mu t)$ is a traveling wave front of (3.1), then

$$\begin{aligned} & \mu U'(z) + m[U(z) - n] \\ &= [\alpha - aU(z)] \int_{\mathbb{R}} K(z - y) H \left(U \left(y - \frac{\mu}{c}|z - y| \right) - \theta \right) dy. \end{aligned}$$

Let

$$\omega = y - \frac{\mu}{c}|z - y|.$$

Then

$$z - y = \frac{c}{c + s(z - \omega)\mu} (z - \omega),$$

and the traveling wave equation becomes

$$\begin{aligned} & \mu U'(z) + m[U(z) - n] \\ &= [\alpha - aU(z)] \int_{\mathbb{R}} \frac{c}{c + s(z - \omega)\mu} K \left(\frac{c(z - \omega)}{c + s(z - \omega)\mu} \right) H(U(\omega) - \theta) d\omega. \end{aligned}$$

Suppose that $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$ and $U > \theta$ on $(0, \infty)$. Then we have the simpler equation

$$\begin{aligned} & \mu U'(z) + m[U(z) - n] \\ = & [\alpha - aU(z)] \int_0^\infty \frac{c}{c + s(z - \omega)\mu} K\left(\frac{c(z - \omega)}{c + s(z - \omega)\mu}\right) d\omega \\ = & [\alpha - aU(z)] \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx, \end{aligned}$$

where in the last step

$$x = \frac{c}{c + s(z - \omega)\mu}(z - \omega).$$

Rewriting this equation as a nonhomogeneous, first order, linear differential equation

$$\begin{aligned} & \mu[U(z) - n]' + \left[m + a \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] [U(z) - n] \\ = & (\alpha - an) \int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx. \end{aligned}$$

That is

$$\mu[U(z) - n]' + \omega_2(z)[U(z) - n] = (\alpha - an)\omega_1(z).$$

The integrating factor of this equation is exactly equal to the last auxiliary function $\omega_4(z)$. Now we have

$$\mu \{ \omega_4(z) [U(z) - n] \}' = (\alpha - an)\omega_1(z)\omega_4(z).$$

Integrating this equation with respect to z over $(-\infty, z)$, we get

$$\mu\omega_4(z)[U(z) - n] = (\alpha - an) \int_{-\infty}^z \omega_1(x)\omega_4(x) dx.$$

Therefore, the solution subject to the homogeneous boundary condition $U(-\infty) = n$ is

given by

$$\begin{aligned}
U(z) &= n + \frac{\alpha - an}{\mu\omega_4(z)} \int_{-\infty}^z \omega_1(x)\omega_4(x)dx \\
&= n + \frac{\alpha - an}{\omega_4(z)} \int_{-\infty}^z \frac{\omega_1(x)}{\omega_2(x)}\omega_4'(x)dx \\
&= n + \frac{(\alpha - an)\omega_1(z)}{\omega_2(z)} - \frac{m(\alpha - an)}{\omega_4(z)} \int_{-\infty}^z \frac{\omega_1'(x)\omega_4(x)}{[\omega_2(x)]^2}dx.
\end{aligned}$$

Now, we have

$$\begin{aligned}
U(0) &= n + \frac{(\alpha - an)\omega_1(0)}{\omega_2(0)} - \frac{m(\alpha - an)}{\omega_4(0)} \int_{-\infty}^0 \frac{\omega_1'(x)\omega_4(x)}{[\omega_2(x)]^2}dx \\
&= n + \frac{\alpha - an}{2m + a} \\
&\quad - m(\alpha - an) \int_{-\infty}^0 \exp\left[\frac{c - \mu}{c\mu}\omega_3(z)\right] \frac{K(z)}{\left[m + a \int_{-\infty}^z K(x)dx\right]^2}dz,
\end{aligned}$$

where

$$\begin{aligned}
&\int_{-\infty}^0 \frac{\omega_1'(x)}{[\omega_2(x)]^2} \frac{\omega_4(x)}{\omega_4(0)}dx \\
&= \int_{-\infty}^0 \frac{\frac{c}{c-\mu}K\left(\frac{cx}{c-\mu}\right) \exp\left[\frac{c-\mu}{c\mu}\omega_3\left(\frac{cx}{c-\mu}\right)\right]}{\left[m + a \int_{-\infty}^{cx/(c-\mu)} K(\xi)d\xi\right]^2}dx \\
&= \int_{-\infty}^0 \frac{K(y) \exp\left[\frac{c-\mu}{c\mu}\omega_3(y)\right]}{\left[m + a \int_{-\infty}^y K(\xi)d\xi\right]^2}dy.
\end{aligned}$$

Therefore, the wave speed μ is determined by the speed equation

$$m(\alpha - an) \int_{-\infty}^0 \exp\left[\frac{c - \mu}{c\mu}\omega_3(z)\right] \frac{K(z)}{\left[m + a \int_{-\infty}^z K(x)dx\right]^2}dz = n + \frac{\alpha - an}{2m + a} - \theta.$$

Note that

$$\begin{aligned}
m(\alpha - an) \int_{-\infty}^0 \frac{K(z)}{\left[m + a \int_{-\infty}^z K(\xi) d\xi \right]^2} dz &= \left. \frac{-\frac{m(\alpha - an)}{a}}{\left[m + a \int_{-\infty}^z K(\xi) d\xi \right]} \right|_{-\infty}^0 \\
&= \frac{m(\alpha - an)}{am} - \frac{m(\alpha - an)}{a \left(m + \frac{1}{2}a \right)} \\
&= \frac{m(\alpha - an)}{m(2m + a)} = \frac{\alpha - an}{2m + a}.
\end{aligned}$$

Now we can easily verify that

$$\begin{aligned}
\lim_{\mu \rightarrow 0} \phi(\mu) &= 0 < n + \frac{\alpha - an}{2m + a} - \theta, & \lim_{\mu \rightarrow c} \phi(\mu) &= \frac{\alpha - an}{2m + a} > n + \frac{\alpha - an}{2m + a} - \theta, \\
\phi'(\mu) &= -\frac{m(\alpha - an)}{\mu^2} \int_{-\infty}^0 \omega_3(z) \exp \left[\frac{c - \mu}{c\mu} \omega_3(z) \right] \frac{K(z)}{\left[m + a \int_{-\infty}^z K(x) dx \right]^2} dz.
\end{aligned}$$

The derivative $\phi'(\mu) > 0$ on $(0, c)$ if the kernel function K is in class (A) or class (B). However, the derivative $\phi'(\mu) < 0$ on $(0, c_*)$ and $\phi'(\mu) > 0$ on (c_*, c) if the kernel function K is in class (C), where $c_* \in (0, c)$ is a constant, depending on K . Therefore, the existence and uniqueness of the speed are guaranteed. The proof of Theorem 3.2.3 is completed. \blacksquare

Theorem 3.2.4. In equation (3.1), let $b = 0$, $\beta = 0$, $(a + 2m)(\theta - n) < \alpha - an$, let $f(u) = m(u - n)$, and let

$$\xi(c) = \delta(c - c_0), \quad K(x) = \frac{1}{2} [\delta(x + \rho) + \delta(x - \rho)],$$

where $m > 0$, $n, \rho > 0$ are parameters. Then the wave speed is given by

$$\frac{1}{\mu_0(a, c_0, m, n, \alpha, \theta)} = \frac{1}{c_0} - \frac{2}{(2m + a)\rho} \ln \left[\frac{(4m + a)^2}{8m(\alpha - an)} \left(n + \frac{\alpha - an}{2m + a} - \theta \right) \right].$$

Proof. Note that on $(-\infty, 0)$, we have

$$\int_{-\infty}^z K(x) dx = \frac{1}{2} H(z + \rho), \quad \int_z^0 |x| K(x) dx = \frac{\rho}{2} [1 - H(z + \rho)],$$

$$\begin{aligned}
m + a \int_{-\infty}^z K(x)dx &= m + \frac{a}{2}H(z + \rho), \\
\omega_3(z) &= mz + az \int_{-\infty}^z K(x)dx - a \int_z^0 |x|K(x)dx \\
&= mz + \frac{1}{2}azH(z + \rho) - \frac{1}{2}a\rho[1 - H(z + \rho)].
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1}{2}m(\alpha - an) \int_{-\infty}^0 \exp \left\{ \frac{c - \mu}{c\mu} \left[mz + \frac{1}{2}azH(z + \rho) - \frac{1}{2}a\rho(1 - H(z + \rho)) \right] \right\} \\
&\cdot \frac{\delta(z + \rho) + \delta(z - \rho)}{\left[m + \frac{1}{2}aH(z + \rho) \right]^2} dz \\
&= n + \frac{\alpha - an}{2m + a} - \theta.
\end{aligned}$$

Thus

$$\frac{8m(\alpha - an)}{(4m + a)^2} \exp \left\{ -\frac{c - \mu}{c\mu} \left[m\rho + \frac{1}{2}a\rho \right] \right\} = n + \frac{\alpha - an}{2m + a} - \theta.$$

Finally, we obtain the speed formula

$$\frac{1}{\mu} = \frac{1}{c} - \frac{2}{(2m + a)\rho} \ln \left[\frac{(4m + a)^2}{8m(\alpha - an)} \left(n + \frac{\alpha - an}{2m + a} - \theta \right) \right].$$

The proof of Theorem 3.2.4 is completed. ■

Theorem 3.2.5. *Suppose that $a \geq 0$, $c > 0$, $\alpha > 0$ and $\theta > 0$, such that $0 < (a + 2m)(\theta - n) < \alpha - an$. Suppose also that*

$$m + a \int_{-\infty}^z K(x)dx > 0, \quad \text{on } (-\infty, 0).$$

Let $\mu_0(a, c, m, n, \alpha, \theta)$ and $\mu_0(a, m, n, \alpha, \theta)$ represent the wave speeds of the traveling wave fronts of the integral differential equation

$$\frac{\partial u}{\partial t} + m(u - n) = (\alpha - au) \int_{\mathbb{R}} K(x - y)H \left(u \left(y, t - \frac{1}{c}|x - y| \right) - \theta \right) dy, \quad (3.10)$$

and the integral differential equation

$$\frac{\partial u}{\partial t} + m(u - n) = (\alpha - au) \int_{\mathbb{R}} K(x - y)H(u(y, t) - \theta)dy, \quad (3.11)$$

respectively, where there is a spatial temporal delay in (3.10) and where there is no delay in (3.11). Then the wave speeds satisfy the relationship

$$\frac{1}{\mu_0(a, c, m, n, \alpha, \theta)} = \frac{1}{c} + \frac{1}{\mu_0(a, m, n, \alpha, \theta)}.$$

Proof. Note that the wave speeds $\mu_0(a, c, m, n, \alpha, \theta)$ and $\mu_0(a, m, n, \alpha, \theta)$ satisfy the equations

$$\begin{aligned} \alpha \int_{-\infty}^0 \exp \left\{ \frac{c - \mu_0(a, c, m, n, \alpha, \theta)}{c\mu_0(a, c, m, n, \alpha, \theta)} \left[z + az \int_{-\infty}^z K(x)dx - a \int_z^0 |x|K(x)dx \right] \right\} \\ \times \frac{K(z)}{\left[1 + a \int_{-\infty}^z K(x)dx \right]^2} dz = \frac{\alpha}{2 + a} - \theta, \end{aligned}$$

and

$$\begin{aligned} \alpha \int_{-\infty}^0 \exp \left\{ \frac{1}{\mu_0(a, m, n, \alpha, \theta)} \left[z + az \int_{-\infty}^z K(x)dx - a \int_z^0 |x|K(x)dx \right] \right\} \\ \times \frac{K(z)}{\left[1 + a \int_{-\infty}^z K(x)dx \right]^2} dz = \frac{\alpha}{2 + a} - \theta, \end{aligned}$$

respectively. By uniqueness, we find that

$$\frac{c - \mu_0(a, c, m, n, \alpha, \theta)}{c\mu_0(a, c, m, n, \alpha, \theta)} = \frac{1}{\mu_0(a, m, n, \alpha, \theta)}.$$

The proof of Theorem 3.2.5 is completed. ■

Now we consider more general cases $a \geq 0, b \geq 0, \alpha \geq 0$ and $\beta \geq 0$.

Theorem 3.2.6. (I) Let $\alpha > 0$ and $\beta = 0$. Let $c_0 \in (0, \infty)$ be a parameter and let

$\xi(c) = \delta(c - c_0)$. Then

$$\begin{aligned}\lim_{c_0 \rightarrow 0} \mu_0(\alpha, c_0, K, \theta) &= 0, \\ \lim_{c_0 \rightarrow 0} \left\{ \frac{1}{c_0} \mu_0(\alpha, c_0, K, \theta) \right\} &= 1, \\ \lim_{c_0 \rightarrow \infty} \mu_0(\alpha, c_0, K, \theta) &= \mu_0(\alpha, K, \theta), \\ \lim_{c_0 \rightarrow \infty} \left\{ c_0 [\mu_0(\alpha, K, \theta) - \mu_0(\alpha, c_0, K, \theta)] \right\} &= [\mu_0(\alpha, K, \theta)]^2.\end{aligned}$$

(II) Let $\alpha = 0$ and $\beta > 0$. Let $\tau_0 \in (0, \infty)$ be a parameter and let $\eta(\tau) = \delta(\tau - \tau_0)$.

Then

$$\begin{aligned}\lim_{\tau_0 \rightarrow 0} \mu_0(\beta, \tau_0, W, \Theta) &= \mu_0(\beta, W, \Theta), \\ \lim_{\tau_0 \rightarrow 0} \frac{\mu_0(\beta, W, \Theta) - \mu_0(\beta, \tau_0, W, \Theta)}{\tau_0} &= \frac{[\mu_0(\beta, W, \Theta)]^2 \left\{ \int_{-\infty}^0 \exp \left[\frac{x}{\mu_0(\beta, W, \Theta)} \right] W(x) dx \right\}}{\int_{-\infty}^0 |x| \exp \left[\frac{x}{\mu_0(\beta, W, \Theta)} \right] W(x) dx}, \\ \lim_{\tau_0 \rightarrow \infty} \mu_0(\beta, \tau_0, W, \Theta) &= 0, \\ \lim_{\tau_0 \rightarrow \infty} \left\{ \tau_0 \mu_0(\beta, \tau_0, W, \Theta) \right\} &= \Gamma_0,\end{aligned}$$

where $\Gamma_0 > 0$ is a constant, such that

$$\int_{-\Gamma_0}^0 W(x) dx = \frac{1}{2} - \frac{\Theta}{\beta}.$$

Proof. For the first case, we have $0 < \mu_0 < c_0$ and

$$\frac{1}{\mu_0(\alpha, c_0, K, \theta)} = \frac{1}{\mu_0(\alpha, K, \theta)} + \frac{1}{c_0}.$$

From the estimate $0 < \mu_0 < c_0$, we find

$$\lim_{c_0 \rightarrow 0} \mu_0(\alpha, c_0, K, \theta) = 0.$$

From the equation

$$\frac{1}{\mu_0(\alpha, c_0, K, \theta)} = \frac{1}{\mu_0(\alpha, K, \theta)} + \frac{1}{c_0},$$

we have

$$\lim_{c_0 \rightarrow 0} \frac{\mu_0(\alpha, c_0, K, \theta)}{c_0} = 1.$$

Again, from the equation

$$\frac{1}{\mu_0(\alpha, c_0, K, \theta)} = \frac{1}{\mu_0(\alpha, K, \theta)} + \frac{1}{c_0},$$

we have

$$\lim_{c_0 \rightarrow \infty} \mu_0(\alpha, c_0, K, \theta) = \mu_0(\alpha, K, \theta).$$

Moreover, we have

$$\begin{aligned} & \lim_{c_0 \rightarrow \infty} \{c_0 [\mu_0(\alpha, K, \theta) - \mu_0(\alpha, c_0, K, \theta)]\} \\ &= \lim_{c_0 \rightarrow \infty} [\mu_0(\alpha, c_0, K, \theta) \mu_0(\alpha, K, \theta)] = [\mu_0(\alpha, K, \theta)]^2. \end{aligned}$$

From the speed equation

$$\int_{-\mu_0 \tau_0}^0 W(x) dx + e^{\tau_0} \int_{-\infty}^{-\mu_0 \tau_0} \exp\left(\frac{x}{\mu_0}\right) W(x) dx = \frac{1}{2} - \frac{\Theta}{\beta},$$

we get the limit

$$\lim_{\tau_0 \rightarrow 0} \mu_0(\beta, \tau_0, W, \Theta) = \mu_0(\beta, W, \Theta),$$

where $\mu_0(\beta, W, \Theta)$ is a unique solution of

$$\int_{-\infty}^0 \exp\left[\frac{x}{\mu_0(\beta, W, \Theta)}\right] W(x) dx = \frac{1}{2} - \frac{\Theta}{\beta}.$$

Moreover

$$\begin{aligned}
& \lim_{\tau_0 \rightarrow 0} \frac{\mu_0(\beta, W, \Theta) - \mu_0(\beta, \tau_0, W, \Theta)}{\tau_0} \\
&= - \lim_{\tau_0 \rightarrow 0} \frac{\partial \mu_0(\beta, \tau_0, W, \Theta)}{\partial \tau_0} \\
&= \lim_{\tau_0 \rightarrow 0} \frac{[\mu_0(\beta, \tau_0, W, \Theta)]^2 \int_{-\infty}^{-\mu_0 \tau_0} \exp\left[\frac{x}{\mu_0(\beta, \tau_0, W, \Theta)}\right] W(x) dx}{\int_{-\infty}^{-\mu_0 \tau_0} |x| \exp\left[\frac{x}{\mu_0(\beta, \tau_0, W, \Theta)}\right] W(x) dx} \\
&= \frac{[\mu_0(\beta, W, \Theta)]^2 \left\{ \int_{-\infty}^0 \exp\left[\frac{x}{\mu_0(\beta, W, \Theta)}\right] W(x) dx \right\}}{\int_{-\infty}^0 |x| \exp\left[\frac{x}{\mu_0(\beta, W, \Theta)}\right] W(x) dx}.
\end{aligned}$$

It is not difficult to show that

$$\lim_{\tau_0 \rightarrow \infty} \mu_0(\beta, \tau_0, W, \Theta) = 0.$$

Additionally, we have

$$\begin{aligned}
& \left| e^{\tau_0} \int_{-\infty}^{-\mu_0 \tau_0} \exp\left(\frac{x}{\mu_0}\right) W(x) dx \right| \\
&= \left| \mu_0 e^{\tau_0} \int_{-\infty}^{-\tau_0} \exp(y) W(\mu_0 y) dy \right| \\
&\leq \left| \mu_0 e^{\tau_0} \int_{-\infty}^{-\tau_0} \exp(y) dy \right| \sup_{x \in \mathbb{R}} |W(x)| \\
&= \mu_0 \sup_{x \in \mathbb{R}} |W(x)|.
\end{aligned}$$

Therefore

$$\lim_{\tau_0 \rightarrow \infty} \left\{ e^{\tau_0} \int_{-\infty}^{-\mu_0 \tau_0} \exp\left[\frac{x}{\mu_0(\beta, W, \Theta)}\right] W(x) dx \right\} = 0.$$

It is easy to see from the speed equation

$$\int_{-\mu_0 \tau_0}^0 W(x) dx + e^{\tau_0} \int_{-\infty}^{-\mu_0 \tau_0} \exp\left(\frac{x}{\mu_0}\right) W(x) dx = \frac{1}{2} - \frac{\Theta}{\beta},$$

that

$$\lim_{\tau_0 \rightarrow \infty} (\mu_0 \tau_0) = \Gamma_0,$$

where $\Gamma_0 > 0$ is a constant, such that

$$\int_{-\Gamma_0}^0 W(x)dx = \frac{1}{2} - \frac{\Theta}{\beta}.$$

The proof of Theorem 3.2.6 is finished. ■

Theorem 3.2.7. *Suppose that $f(u) = u$. Let $\tilde{\xi}(c) = \rho\xi(\rho c)$, $\tilde{K}(x) = \rho K(\rho x)$ and $\tilde{W}(x) = \sigma W(\sigma x)$, where $\rho > 0$ and $\sigma > 0$ are parameters.*

(I) *Let $\alpha > 0$ and $\beta = 0$. Then $\mu_0(\rho) = \frac{1}{\rho}\mu_0(1)$.*

(II) *Let $\alpha = 0$ and $\beta > 0$. Then $\mu_0(\sigma) = \frac{1}{\sigma}\mu_0(1)$.*

(III) *Let $\rho = \sigma$, $\alpha > 0$ and $\beta > 0$. Then $\mu_0(\rho) = \frac{1}{\rho}\mu_0(1)$.*

Proof. (I) In the speed equation, if we replace ξ , K and μ_0 by $\tilde{\xi}$, \tilde{K} and $\tilde{\mu}_0$, respectively, then we have

$$\alpha \int_0^\infty \rho\xi(\rho c) \left[\int_{-\infty}^0 \exp\left(\frac{c - \tilde{\mu}_0}{c\tilde{\mu}_0}x\right) \rho K(\rho x)dx \right] dc = \frac{\alpha}{2} - \theta.$$

Making a change of variables $d = \rho c$ and $y = \rho x$ and then replace d and y by c and x , respectively, we find

$$\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c - \rho\tilde{\mu}_0}{c\rho\tilde{\mu}_0}x\right) K(x)dx \right] dc = \frac{\alpha}{2} - \theta.$$

Recall that ξ , K and μ_0 satisfy

$$\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c - \mu_0}{c\mu_0}x\right) K(x)dx \right] dc = \frac{\alpha}{2} - \theta.$$

Therefore, we obtain $\rho\tilde{\mu}_0 = \mu_0$. Hence $\tilde{\mu}_0 = \frac{\mu_0}{\rho}$.

(II) In the speed equation, if we replace W and μ_0 by \widetilde{W} and $\widetilde{\mu}_0$, respectively, then we get

$$\begin{aligned} & \beta \int_0^\infty \eta(\tau) \left[\int_{-\widetilde{\mu}_0\tau}^0 \sigma W(\sigma x) dx \right] d\tau \\ + & \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\widetilde{\mu}_0\tau} \exp\left(\frac{x}{\widetilde{\mu}_0}\right) \sigma W(\sigma x) dx \right] d\tau \\ = & \frac{\beta}{2} - \Theta. \end{aligned}$$

Making a change of variable $y = \sigma x$ and then replace y by x , we find

$$\begin{aligned} & \beta \int_0^\infty \eta(\tau) \left[\int_{-\sigma\widetilde{\mu}_0\tau}^0 W(x) dx \right] d\tau \\ + & \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\sigma\widetilde{\mu}_0\tau} \exp\left(\frac{x}{\sigma\widetilde{\mu}_0}\right) W(x) dx \right] d\tau \\ = & \frac{\beta}{2} - \Theta. \end{aligned}$$

Recall that W and μ_0 satisfy

$$\begin{aligned} & \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu_0\tau}^0 W(x) dx \right] d\tau \\ + & \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu_0\tau} \exp\left(\frac{x}{\mu_0}\right) W(x) dx \right] d\tau \\ = & \frac{\beta}{2} - \Theta. \end{aligned}$$

By uniqueness of the wave speed, we find $\sigma\widetilde{\mu}_0 = \mu_0$.

(III) In the speed equation, if we replace ξ , K , W and μ_0 by $\widetilde{\xi}$, \widetilde{K} , \widetilde{W} and $\widetilde{\mu}_0$, respectively, where $\sigma = \rho$, then we find

$$\begin{aligned} & \alpha \int_0^\infty \rho\xi(\rho c) \left[\int_{-\infty}^0 \exp\left(\frac{c - \widetilde{\mu}_0}{c\widetilde{\mu}_0} x\right) \rho K(\rho x) dx \right] dc \\ + & \beta \int_0^\infty \eta(\tau) \left[\int_{-\widetilde{\mu}_0\tau}^0 \rho W(\rho x) dx \right] d\tau \\ + & \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\widetilde{\mu}_0\tau} \exp\left(\frac{x}{\widetilde{\mu}_0}\right) \rho W(\rho x) dx \right] d\tau = \frac{\alpha + \beta}{2} - \theta. \end{aligned}$$

Making the change of variables $d = \rho c$ and $y = \rho x$ and then replace d and y with c and x , respectively, we see

$$\begin{aligned} & \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c - \rho\tilde{\mu}_0}{c\rho\tilde{\mu}_0}x\right) K(x)dx \right] dc \\ & + \beta \int_0^\infty \eta(\tau) \left[\int_{-\rho\tilde{\mu}_0\tau}^0 W(x)dx \right] d\tau \\ & + \beta \int_0^\infty \eta(\tau)e^\tau \left[\int_{-\infty}^{-\rho\tilde{\mu}_0\tau} \exp\left(\frac{x}{\rho\tilde{\mu}_0}\right) W(x)dx \right] d\tau = \frac{\alpha + \beta}{2} - \theta. \end{aligned}$$

Recall that ξ , K , W and μ_0 satisfy

$$\begin{aligned} & \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c - \mu_0}{c\mu_0}x\right) K(x)dx \right] dc \\ & + \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu_0\tau}^0 W(x)dx \right] d\tau \\ & + \beta \int_0^\infty \eta(\tau)e^\tau \left[\int_{-\infty}^{-\mu_0\tau} \exp\left(\frac{x}{\mu_0}\right) W(x)dx \right] d\tau = \frac{\alpha + \beta}{2} - \theta. \end{aligned}$$

By uniqueness of the wave speed, we find $\rho\tilde{\mu}_0 = \mu_0$. The proof of Theorem 3.2.7 is finished. \blacksquare

Theorem 3.2.8. (I) Let μ_0 be the wave speed of the traveling wave front of the integral differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + m(u - n) \\ & = (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x - y) H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right) dy \right] dc. \end{aligned}$$

Then

$$0 < \frac{m\alpha - a\theta - 2m\theta + 2mn}{(\alpha - a\theta)K(0)} < \mu_0 \leq \frac{m}{\ln \frac{\alpha}{\alpha - 2m\theta + 2mn}} \int_{\mathbb{R}} |x|K(x)dx,$$

(II) Let μ_0 be the wave speed of the traveling wave front of the integral differential

equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + m(u - n) \\ = & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \end{aligned}$$

Then

$$\begin{aligned} \mu_0 & < \frac{m}{\ln \frac{\beta}{\beta - 2m\theta + 2mn}} \int_{\mathbb{R}} |x| W(x) dx, \\ \mu_0 & > \frac{\frac{m}{2}(\beta - b\theta - 2m\theta + 2mn)}{(\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m\tau) d\tau \right]} > 0. \end{aligned}$$

(III) Let μ_0 be the wave speed of the traveling wave front of the integral differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + m(u - n) \\ = & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x - y) \left(u \left(y, t - \frac{1}{c}|x - y| \right) - \theta \right) dx \right] dc \\ + & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(u(y, t - \tau) - \Theta) dy \right] d\tau. \end{aligned}$$

Then

$$\begin{aligned} \mu_0 & \leq \frac{m}{(\alpha + \beta) \ln \frac{\alpha + \beta}{\alpha + \beta - 2m\theta + 2mn}} \int_{\mathbb{R}} [\alpha|x|K(x) + \beta|x|W(x)] dx, \\ \mu_0 & > \frac{m}{2}(\alpha + \beta - a\theta - b\theta - 2m\theta + 2mn) \\ & / \left\{ (\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m\tau) d\tau \right] \right\} > 0. \end{aligned}$$

Proof. The proof of Theorem 3.2.8 may be completed by setting $m_1 = m_2 = m > 0$ and $n_1 = n_2 = n$ in Theorem 3.3.6. ■

3.3 Nonlinear Analysis

3.3.1 Representation of the traveling wave front of (3.1)

Definition 3.3.1. Define the following auxiliary functions $A(z) = A(\mu, z)$, $B(z) = B(\mu, z)$, $C(z) = C(\mu, z)$ and $D(z) = D(\mu, z)$ by

$$\begin{aligned} A(z) &= (\alpha - an) \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \\ &+ (\beta - bn) \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z_0} W(x) dx \right] d\tau, \end{aligned}$$

$$\begin{aligned} B(z) &= m + a \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \\ &+ b \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z_0} W(x) dx \right] d\tau, \end{aligned}$$

$$C(z) = \exp \left\{ \left[\int_0^\infty \xi(c) \frac{c-\mu}{c\mu} dc \right] \left[mz + az \int_{-\infty}^z K(x) dx - a \int_z^0 |x| K(x) dx \right] \right\},$$

$$\begin{aligned} D(z) &= \exp \left\{ \frac{m}{\mu} z + \frac{a}{\mu} z \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \right. \\ &- \frac{a}{\mu} \int_0^\infty \xi(c) \left[\int_{-\infty}^z \frac{cx}{c+s(x)\mu} K \left(\frac{cx}{c+s(x)\mu} \right) dx \right] dc \\ &+ \frac{b}{\mu} z \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z_0} W(x) dx \right] d\tau \\ &\left. - \frac{b}{\mu} \int_0^\infty \eta(\tau) \left[\int_{-\infty}^z xW(x-\mu\tau-Z_0) dx \right] d\tau \right\}. \end{aligned}$$

Note that the auxiliary functions A , B , C and D depend on a , b , m , n , α , β , ξ , η , K , W and μ . These functions will help us study the dependence of the wave speed μ_0 on the parameters a , b , m , n , α , β , θ and the kernel functions (ξ, η) and (K, W) .

Let us investigate the basic properties of the four auxiliary functions. First of all, we

have

$$\begin{aligned}
A(0) &= \frac{\alpha - an}{2} + (\beta - bn) \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{-\mu\tau - Z_0} W(x) dx \right] d\tau, \\
B(0) &= m + \frac{a}{2} + b \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{-\mu\tau - Z_0} W(x) dx \right] d\tau, \\
C(0) &= 1, \\
D(0) &= \exp \left\{ a \left[\int_0^\infty \xi(c) \frac{c - \mu}{c\mu} dc \right] \left[\int_{-\infty}^0 |x| K(x) dx \right] \right. \\
&\quad \left. + \frac{b}{\mu} \int_0^\infty \eta(\tau) \left[\int_{-\infty}^0 |x| W(x - \mu\tau - Z_0) dx \right] d\tau \right\}.
\end{aligned}$$

Second, we get

$$\lim_{z \rightarrow -\infty} A(z) = 0, \quad \lim_{z \rightarrow -\infty} B(z) = m, \quad \lim_{z \rightarrow -\infty} C(z) = 0, \quad \lim_{z \rightarrow -\infty} D(z) = 0.$$

Third, we obtain

$$\begin{aligned}
A'(z) &= (\alpha - an) \int_0^\infty \xi(c) \left[\frac{c}{c + s(z)\mu} K \left(\frac{cz}{c + s(z)\mu} \right) \right] dc \\
&\quad + (\beta - bn) \int_0^\infty \eta(\tau) W(z - \mu\tau - Z_0) d\tau, \\
B'(z) &= a \int_0^\infty \xi(c) \left[\frac{c}{c + s(z)\mu} K \left(\frac{cz}{c + s(z)\mu} \right) \right] dc \\
&\quad + b \int_0^\infty \eta(\tau) W(z - \mu\tau - Z_0) d\tau, \\
C'(z) &= \left\{ \int_0^\infty \xi(c) \frac{c - \mu}{c\mu} \left[m + a \int_{-\infty}^z K(x) dx \right] dc \right\} C(z), \text{ if } z < 0, \\
D'(z) &= \left\{ \frac{m}{\mu} + \frac{a}{\mu} \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \right. \\
&\quad \left. + \frac{b}{\mu} \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu\tau - Z_0} W(x) dx \right] d\tau \right\} D(z) \\
&= \frac{1}{\mu} B(z) D(z).
\end{aligned}$$

Fourth, if $z < 0$, then

$$\begin{aligned} \frac{D(z)}{D(0)} &= \exp \left\{ \frac{m}{\mu} z + \frac{a}{\mu} z \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \right. \\ &+ a \int_0^\infty \xi(c) \frac{c-\mu}{c\mu} \left[\int_{cz/(c+s(z)\mu)}^0 xK(x) dx \right] dc \\ &+ \frac{b}{\mu} z \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z_0} W(x) dx \right] d\tau \\ &\left. + \frac{b}{\mu} \int_0^\infty \eta(\tau) \left[\int_z^0 xW(x-\mu\tau-Z_0) dx \right] d\tau \right\}. \end{aligned}$$

If $b = 0$ and $z < 0$, then

$$\begin{aligned} \frac{D(z)}{D(0)} &= \exp \left\{ \left[\int_0^\infty \xi(c) \frac{c-\mu}{c\mu} dc \right] \left[m\tilde{z} + a\tilde{z} \int_{-\infty}^{\tilde{z}} K(x) dx + a \int_{\tilde{z}}^0 xK(x) dx \right] \right\} \\ &= C(\tilde{z}), \quad \tilde{z} = \frac{c}{c-\mu} z. \end{aligned}$$

Definition 3.3.2. Define the speed index function ϕ by

$$\phi(\mu) = \int_{-\infty}^0 \left[\frac{A(x)}{B(x)} \right]' \frac{D(x)}{D(0)} dx - \frac{A(0)}{B(0)},$$

where the auxiliary functions A , B , C and D have been defined in Definition 3.

The wave speed is a solution of the equation

$$\phi(\mu) = n - \theta. \quad (3.12)$$

This definition is motivated by the condition $U(0) = \theta$ if $f(u) = m(u - n)$ and $\theta = \Theta$.

Theorem 3.3.3. (I) Suppose that the traveling wave front satisfies the condition $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$ and $U > \theta$ on $(0, \infty)$. Suppose also that $U < \Theta$ on $(-\infty, Z_0)$, $U(Z_0) = \Theta$ and $U > \Theta$ on (Z_0, ∞) , for some constant $Z_0 \geq 0$. Then there holds the following representation for the traveling wave front of the integral

differential equation (3.1)

$$\begin{aligned}
U(z) &= n + \frac{A(z)}{B(z)} - \frac{1}{D(z)} \int_{-\infty}^z \left[\frac{A(x)}{B(x)} \right]' D(x) dx \\
&+ \frac{1}{\mu_0 D(z)} \int_{-\infty}^z D(x) \{ m[U(x) - n] - f(U(x)) \} dx,
\end{aligned}$$

where $z = x + \mu_0 t$, μ_0 is a positive number, representing the wave speed. The wave speed μ_0 satisfies the following equations

$$\begin{aligned}
\theta &= n + \frac{A(0)}{B(0)} - \frac{1}{D(0)} \int_{-\infty}^0 \left[\frac{A(x)}{B(x)} \right]' D(x) dx \\
&+ \frac{1}{\mu_0 D(0)} \int_{-\infty}^0 D(x) \{ m[U(x) - n] - f(U(x)) \} dx, \\
\Theta &= n + \frac{A(Z_0)}{B(Z_0)} - \frac{1}{D(Z_0)} \int_{-\infty}^{Z_0} \left[\frac{A(x)}{B(x)} \right]' D(x) dx \\
&+ \frac{1}{\mu_0 D(Z_0)} \int_{-\infty}^{Z_0} D(x) \{ m[U(x) - n] - f(U(x)) \} dx.
\end{aligned}$$

(II) Let $f(u) = m(u - n)$, for two constants m and n . There exists a unique pair (μ_0, Z_0) , such that $U(0) = \theta$ and $U(Z_0) = \Theta$.

(III) Suppose that f is a nonlinear smooth function, such that $f(n) = 0$ and $m = f'(n) > 0$, for two constants m and n . There exists a unique pair (μ_0, Z_0) , such that $U(0) = \theta$ and $U(Z_0) = \Theta$.

Proof. (I) The traveling wave equation

$$\begin{aligned}
&\mu U' + f(U) \\
&= (\alpha - aU) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z - y) H \left(U \left(y - \frac{\mu}{c} |z - y| \right) - \theta \right) dy \right] dc \\
&+ (\beta - bU) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - y) H (U(y - \mu\tau) - \Theta) dy \right] d\tau
\end{aligned}$$

may be written as

$$\begin{aligned}
& \mu U' + m(U - n) \\
= & (\alpha - aU) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z - y) H \left(U \left(y - \frac{\mu}{c} |z - y| \right) - \theta \right) dy \right] dc \\
+ & (\beta - bU) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - y) H(U(y - \mu\tau) - \Theta) dy \right] d\tau \\
+ & m(U - n) - f(U).
\end{aligned}$$

Making the following change of variables for these two integrals:

$$\omega = y - \frac{\mu}{c} |z - y|, \quad r = y - \mu\tau,$$

we find that

$$z - y = \frac{c(z - \omega)}{c + s(z - \omega)\mu}, \quad z - y = z - \mu\tau - r,$$

respectively. Suppose that the traveling wave front satisfies the conditions $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$ and $U > \theta$ on $(0, \infty)$. Similarly, suppose that the traveling wave front satisfies the conditions $U < \Theta$ on $(-\infty, Z_0)$, $U(Z_0) = \Theta$ and $U > \Theta$ on (Z_0, ∞) ,

for some constant $Z_0 \geq 0$. Then we have the traveling wave equation

$$\begin{aligned}
& \mu U' + m(U - n) \\
= & (\alpha - aU) \\
& \cdot \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} \frac{c}{c + s(z - \omega)\mu} K \left(\frac{c(z - \omega)}{c + s(z - \omega)\mu} \right) H(U(\omega) - \theta) d\omega \right] dc \\
+ & (\beta - bU) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - \mu\tau - r) H(U(r) - \Theta) dr \right] d\tau \\
+ & m(U - n) - f(U) \\
= & (\alpha - aU) \int_0^\infty \xi(c) \left[\int_0^\infty \frac{c}{c + s(z - \omega)\mu} K \left(\frac{c(z - \omega)}{c + s(z - \omega)\mu} \right) d\omega \right] dc \\
+ & (\beta - bU) \int_0^\infty \eta(\tau) \left[\int_{Z_0}^\infty W(z - \mu\tau - r) dr \right] d\tau + m(U - n) - f(U) \\
= & (\alpha - aU) \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \\
+ & (\beta - bU) \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu\tau - Z_0} W(x) dx \right] d\tau + m(U - n) - f(U).
\end{aligned}$$

Rewriting this equation as a nonhomogeneous, first order, linear differential equation

$$\begin{aligned}
& \mu(U - n)' + \left\{ m + a \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \right. \\
+ & \left. b \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu\tau - Z_0} W(x) dx \right] d\tau \right\} (U - n) \\
= & (\alpha - an) \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x) dx \right] dc \\
+ & (\beta - bn) \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu\tau - Z_0} W(x) dx \right] d\tau + m(U - n) - f(U).
\end{aligned}$$

That is

$$\mu[U(z) - n]' + B(z)[U(z) - n] = A(z) + m[U(z) - n] - f(U(z)).$$

The integrating factor of this equation is exactly equal to $D(z)$. Multiplying the differential equation by the integrating factor $D(z)$ and integrating with respect to z over

$(-\infty, z)$, we get

$$\begin{aligned}\{\mu D(z)[U(z) - n]\}' &= A(z)D(z) + D(z)\{m[U(z) - n] - f(U(z))\}, \\ \mu D(z)[U(z) - n] &= \int_{-\infty}^z A(x)D(x)dx \\ &\quad + \int_{-\infty}^z D(x)\{m[U(x) - n] - f(U(x))\}dx.\end{aligned}$$

Therefore, the traveling wave solution may be represented as

$$\begin{aligned}U(z) - n &= \frac{1}{\mu D(z)} \int_{-\infty}^z A(x)D(x)dx \\ &\quad + \frac{1}{\mu D(z)} \int_{-\infty}^z D(x)\{m[U(x) - n] - f(U(x))\}dx \\ &= \frac{A(z)}{B(z)} - \frac{1}{D(z)} \int_{-\infty}^z \left[\frac{A(x)}{B(x)}\right]' D(x)dx \\ &\quad + \frac{1}{\mu D(z)} \int_{-\infty}^z D(x)\{m[U(x) - n] - f(U(x))\}dx,\end{aligned}$$

where

$$\begin{aligned}&\frac{1}{\mu D(z)} \int_{-\infty}^z A(x)D(x)dx \\ &= \frac{1}{D(z)} \int_{-\infty}^z \frac{A(x)}{B(x)} \left[\frac{1}{\mu} B(x)D(x)\right] dx \\ &= \frac{1}{D(z)} \int_{-\infty}^z \frac{A(x)}{B(x)} D'(x)dx \\ &= \frac{A(z)}{B(z)} - \frac{1}{D(z)} \int_{-\infty}^z \left[\frac{A(x)}{B(x)}\right]' D(x)dx.\end{aligned}$$

Now letting $z = 0$ and $Z = Z_0$, respectively, we have

$$\begin{aligned}U(0) - n &= \frac{A(0)}{B(0)} - \frac{1}{D(0)} \int_{-\infty}^0 \left[\frac{A(x)}{B(x)}\right]' D(x)dx \\ &\quad + \frac{1}{\mu D(0)} \int_{-\infty}^0 D(x)\{m[U(x) - n] - f(U(x))\}dx, \\ U(Z_0) - n &= \frac{A(Z_0)}{B(Z_0)} - \frac{1}{D(Z_0)} \int_{-\infty}^{Z_0} \left[\frac{A(x)}{B(x)}\right]' D(x)dx \\ &\quad + \frac{1}{\mu D(Z_0)} \int_{-\infty}^{Z_0} D(x)\{m[U(x) - n] - f(U(x))\}dx.\end{aligned}$$

The wave speed μ_0 and the constant Z_0 are determined by the system of equations $U(0) = \theta$ and $U(Z_0) = \Theta$, that is

$$\begin{aligned}\theta - n &= \frac{A(0)}{B(0)} - \frac{1}{D(0)} \int_{-\infty}^0 \left[\frac{A(x)}{B(x)} \right]' D(x) dx \\ &+ \frac{1}{\mu_0 D(0)} \int_{-\infty}^0 D(x) \{ m[U(x) - n] - f(U(x)) \} dx, \\ \Theta - n &= \frac{A(Z_0)}{B(Z_0)} - \frac{1}{D(Z_0)} \int_{-\infty}^{Z_0} \left[\frac{A(x)}{B(x)} \right]' D(x) dx \\ &+ \frac{1}{\mu_0 D(Z_0)} \int_{-\infty}^{Z_0} D(x) \{ m[U(x) - n] - f(U(x)) \} dx.\end{aligned}$$

By using fixed point theorem, we can establish the existence and uniqueness of the wave speed μ_0 and the traveling wave front U , respectively. The proofs of (II) and (III) are omitted. The proof of Theorem 3.3.3 is finished. ■

3.3.2 Estimates on the wave speeds

To derive the upper bound and the lower bound of the wave speed, we need to build some technical lemmas.

Lemma 3.3.4. *Let $p > 0$ and $q > 0$ be constants. Suppose that $\psi \geq 0$ on $(-\infty, 0)$ and*

$$\begin{aligned}\int_{-\infty}^0 \psi(x) dx &> 0, & \int_{-\infty}^0 |x| \psi(x) dx &> 0, \\ \lim_{x \rightarrow -\infty} \left[x \int_{-\infty}^x \psi(\xi) d\xi \right] &= 0, & \int_{-\infty}^0 \exp(px) \psi(x) dx &= q.\end{aligned}$$

Then, there holds the following estimate

$$q > \left[\int_{-\infty}^0 \psi(x) dx \right] \exp \left\{ -p \left[\int_{-\infty}^0 |x| \psi(x) dx \right] / \left[\int_{-\infty}^0 \psi(x) dx \right] \right\}.$$

Proof. See [65] for the proof of Lemma 3.3.4. ■

Lemma 3.3.5. *Suppose that $p > 0$, $q > 0$ and $\mu > 0$ are positive constants. Suppose that the function ψ satisfies the conditions $\psi' \geq 0$ on $(-\infty, 0)$ and $\psi(0) > 0$. Suppose*

also that

$$\lim_{x \rightarrow -\infty} \left[\exp\left(\frac{x}{\mu}\right) \psi(x) \right] = 0, \quad p \int_{-\infty}^0 \exp\left(\frac{x}{\mu}\right) \psi(x) dx \geq q.$$

Then we have the estimate

$$\mu > \frac{q}{p\psi(0)}.$$

Proof. It is simple to see that

$$\begin{aligned} \frac{q}{p\mu} &\leq \int_{-\infty}^0 \left[\exp\left(\frac{x}{\mu}\right) \right]' \psi(x) dx \\ &= \exp\left(\frac{x}{\mu}\right) \psi(x) \Big|_{-\infty}^0 - \int_{-\infty}^0 \exp\left(\frac{x}{\mu}\right) \psi'(x) dx \\ &= \psi(0) - \int_{-\infty}^0 \exp\left(\frac{x}{\mu}\right) \psi'(x) dx < \psi(0). \end{aligned}$$

Therefore, $\frac{q}{p\mu} < \psi(0)$, and we get the estimate

$$\mu > \frac{q}{p\psi(0)}.$$

The proof of Lemma 3.3.5 is finished. ■

Theorem 3.3.6. Let m_1, m_2, n_1 and n_2 be constants, with $m_1 > 0$ and $m_2 > 0$. Suppose that

$$m_1(u - n_1) \leq f(u) \leq m_2(u - n_2),$$

for all $u \in \mathbb{R}$. Suppose also that

$$\begin{aligned} \alpha + \beta - 2m_1\theta + 2m_1n_1 &> 0, \quad \alpha + \beta - a\theta - b\theta - 2m_2\theta + 2m_2n_2 > 0, \\ (\alpha - a\theta)K(0) + (\beta - b\theta)W(0) &\left[\int_0^\infty \eta(\tau) \exp(m_2\tau) d\tau \right] > 0. \end{aligned}$$

Then the wave speed μ_0 satisfies the estimates

$$\mu_0 \leq \frac{m_1}{(\alpha + \beta) \ln \frac{\alpha + \beta}{\alpha + \beta - 2m_1\theta + 2m_1n_1}} \left\{ \int_{\mathbb{R}} [\alpha|x|K(x) + \beta|x|W(x)] dx \right\},$$

$$\begin{aligned}\mu_0 &> \frac{\frac{m_2}{2}(\alpha + \beta - a\theta - b\theta - 2m_2\theta + 2m_2n_2)}{(\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m_2\tau) d\tau \right]} \\ &> 0.\end{aligned}$$

Proof. Recall that the traveling wave equation is

$$\begin{aligned}&\mu_0 U' + f(U) \\ &= (\alpha - aU) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z - y) H \left(U \left(y - \frac{\mu_0}{c} |z - y| \right) - \theta \right) dy \right] dc \\ &+ (\beta - bU) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - y) H(U(y - \mu_0\tau) - \Theta) dy \right] d\tau.\end{aligned}$$

Note that

$$m_1(U - n_1) \leq f(U) \leq m_2(U - n_2),$$

and

$$\alpha - a\theta \leq \alpha - aU \leq \alpha, \quad \beta - b\theta \leq \beta - bU \leq \beta,$$

on $(-\infty, 0)$. Now we get the following differential inequalities

$$\begin{aligned}&\mu_0 U' + m_1(U - n_1) \\ &\leq \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z - y) H \left(U \left(y - \frac{\mu_0}{c} |z - y| \right) - \theta \right) dy \right] dc \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - y) H(U(y - \mu_0\tau) - \Theta) dy \right] d\tau \\ &= \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \right] dc \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu_0\tau - Z_0} W(x) dx \right] d\tau,\end{aligned}$$

and

$$\begin{aligned}
& \mu_0 U' + m_2(U - n_2) \\
\geq & (\alpha - a\theta) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z - y) H\left(U\left(y - \frac{\mu_0}{c}|z - y|\right) - \theta\right) dy \right] dc \\
+ & (\beta - b\theta) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - y) H(U(y - \mu_0\tau) - \Theta) dy \right] d\tau \\
= & (\alpha - a\theta) \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \right] dc \\
+ & (\beta - b\theta) \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu_0\tau - Z_0} W(x) dx \right] d\tau.
\end{aligned}$$

Solving these differential inequalities, we obtain

$$\begin{aligned}
U(z) \leq & n_1 + \frac{\alpha}{m_1} \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \right] dc \\
+ & \frac{\beta}{m_1} \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu_0\tau - Z_0} W(x) dx \right] d\tau \\
- & \frac{\alpha}{m_1} \int_0^\infty \xi(c) \left[\int_{-\infty}^z \exp\left(m_1 \frac{x - z}{\mu_0}\right) \frac{c}{c + s(x)\mu_0} K\left(\frac{cx}{c + s(x)\mu_0}\right) dx \right] dc \\
- & \frac{\beta}{m_1} \exp\left(m_1 \frac{Z_0}{\mu_0}\right) \\
\cdot & \int_0^\infty \eta(\tau) \exp(m_1\tau) \left[\int_{-\infty}^{z - \mu_0\tau - Z_0} \exp\left(m_1 \frac{x - z}{\mu_0}\right) W(x) dx \right] d\tau,
\end{aligned}$$

and

$$\begin{aligned}
U(z) \geq & n_2 + \frac{\alpha - a\theta}{m_2} \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \right] dc \\
+ & \frac{\beta - b\theta}{m_2} \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z - \mu_0\tau - Z_0} W(x) dx \right] d\tau \\
- & \frac{\alpha - a\theta}{m_2} \\
\cdot & \int_0^\infty \xi(c) \left[\int_{-\infty}^z \exp\left(m_2 \frac{x - z}{\mu_0}\right) \frac{c}{c + s(x)\mu_0} K\left(\frac{cx}{c + s(x)\mu_0}\right) dx \right] dc \\
- & \frac{\beta - b\theta}{m_2} \exp\left(m_2 \frac{Z_0}{\mu_0}\right) \\
\cdot & \int_0^\infty \eta(\tau) \exp(m_2\tau) \left[\int_{-\infty}^{z - \mu_0\tau - Z_0} \exp\left(m_2 \frac{x - z}{\mu_0}\right) W(x) dx \right] d\tau.
\end{aligned}$$

Setting $z = 0$ and $U(0) = \theta$, we find

$$\begin{aligned}
\theta &\leq n_1 + \frac{\alpha + \beta}{2m_1} - \frac{\alpha}{m_1} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \right] dc \\
&- \frac{\beta}{m_1} \int_0^\infty \eta(\tau) \left[\int_{-\mu_0\tau - Z_0}^0 W(x) dx \right] d\tau \\
&- \frac{\beta}{m_1} \exp\left(m_1 \frac{Z_0}{\mu_0}\right) \\
&\cdot \int_0^\infty \eta(\tau) \exp(m_1\tau) \left[\int_{-\infty}^{-\mu_0\tau - Z_0} \exp\left(m_1 \frac{x}{\mu_0}\right) W(x) dx \right] d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\theta &\geq n_2 + \frac{\alpha + \beta - a\theta - b\theta}{2m_2} \\
&- \frac{\alpha - a\theta}{m_2} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \right] dc \\
&- \frac{\beta - b\theta}{m_2} \int_0^\infty \eta(\tau) \left[\int_{-\mu_0\tau - Z_0}^0 W(x) dx \right] d\tau \\
&- \frac{\beta - b\theta}{m_2} \exp\left(m_2 \frac{Z_0}{\mu_0}\right) \\
&\cdot \int_0^\infty \eta(\tau) \exp(m_2\tau) \left[\int_{-\infty}^{-\mu_0\tau - Z_0} \exp\left(m_2 \frac{x}{\mu_0}\right) W(x) dx \right] d\tau.
\end{aligned}$$

Rearranging terms, we have

$$\begin{aligned}
&\frac{\alpha}{m_1} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \right] dc \\
&+ \frac{\beta}{m_1} \int_0^\infty \eta(\tau) \left[\int_{-\mu_0\tau - Z_0}^0 W(x) dx \right] d\tau \\
&+ \frac{\beta}{m_1} \exp\left(m_1 \frac{Z_0}{\mu_0}\right) \int_0^\infty \eta(\tau) \exp(m_1\tau) \left[\int_{-\infty}^{-\mu_0\tau - Z_0} \exp\left(m_1 \frac{x}{\mu_0}\right) W(x) dx \right] d\tau \\
&\leq n_1 + \frac{\alpha + \beta}{2m_1} - \theta,
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\alpha - a\theta}{m_2} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_0}{c\mu_0} x\right) K(x) dx \right] dc \\
& + \frac{\beta - b\theta}{m_2} \int_0^\infty \eta(\tau) \left[\int_{-\mu_0\tau - Z_0}^0 W(x) dx \right] d\tau \\
& + \frac{\beta - b\theta}{m_2} \exp\left(m_2 \frac{Z_0}{\mu_0}\right) \\
& \cdot \int_0^\infty \eta(\tau) \exp(m_2\tau) \left[\int_{-\infty}^{-\mu_0\tau - Z_0} \exp\left(m_2 \frac{x}{\mu_0}\right) W(x) dx \right] d\tau \\
& \geq n_2 + \frac{\alpha + \beta - a\theta - b\theta}{2m_2} - \theta.
\end{aligned}$$

Let

$$c_0 = \sup \{d > 0 : \xi(c) = 0 \text{ on } (0, d)\}.$$

Now we have

$$\int_{-\infty}^0 \exp\left(m_1 \frac{x}{\mu_0}\right) \left[\frac{\alpha}{m_1} K(x) + \frac{\beta}{m_1} W(x) \right] dx \leq n_1 + \frac{\alpha + \beta}{2m_1} - \theta,$$

and

$$\begin{aligned}
& \frac{\alpha - a\theta}{m_2} \int_{-\infty}^0 \exp\left(m_2 \frac{x}{\mu_0}\right) \exp\left(m_2 \frac{|x|}{c_0}\right) K(x) dx \\
& + \frac{\beta - b\theta}{m_2} \exp\left(m_2 \frac{Z_0}{\mu_0}\right) \int_0^\infty \eta(\tau) \exp(m_2\tau) d\tau \left[\int_{-\infty}^0 \exp\left(m_2 \frac{x}{\mu_0}\right) W(x) dx \right] \\
& \geq n_2 + \frac{\alpha + \beta - a\theta - b\theta}{2m_2} - \theta.
\end{aligned}$$

By applying Lemma 3.3.4 and Lemma 3.3.5, we obtain the estimates on the wave speed

$$\mu_0 \leq \frac{m_1}{(\alpha + \beta) \ln \frac{\alpha + \beta}{\alpha + \beta - 2m_1\theta + 2m_1n_1}} \left\{ \int_{\mathbb{R}} [\alpha|x|K(x) + \beta|x|W(x)] dx \right\},$$

and

$$\begin{aligned}
\mu_0 & > \frac{\frac{m_2}{2} (\alpha + \beta - a\theta - b\theta - 2m_2\theta + 2m_2n_2)}{(\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \exp\left(m_2 \frac{Z_0}{\mu_0}\right) \left[\int_0^\infty \eta(\tau) \exp(m_2\tau) d\tau \right]} \\
& > 0.
\end{aligned}$$

If $\theta = \Theta$, then $Z_0 = 0$ and

$$\begin{aligned}\mu_0 &> \frac{\frac{m_2}{2}(\alpha + \beta - a\theta - b\theta - 2m_2\theta + 2m_2n_2)}{(\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m_2\tau) d\tau \right]} \\ &> 0.\end{aligned}$$

The proof of Theorem 3.3.6 is finished. ■

3.3.3 Several change of variables

Theorem 3.3.7. *Make the following changes of variables*

$$\begin{aligned}a_0 &= \frac{a}{m}, & b_0 &= \frac{b}{m}, & t_0 &= mt, \\ \alpha_0 &= \alpha - an, & \beta_0 &= \beta - bn, \\ \theta_0 &= m\theta - mn, & \Theta_0 &= m\Theta - mn, \\ c_0 &= \frac{c}{m}, & \tau_0 &= m\tau, \\ \xi_0(c_0) &= m\xi(mc_0), & \eta_0(\tau_0) &= \frac{1}{m}\eta\left(\frac{1}{m}\tau_0\right), \\ v(x, t_0) &= m[u(x, t) - n].\end{aligned}$$

Then the integral differential equation

$$\begin{aligned}&\frac{\partial u}{\partial t} + m(u - n) \\ &= (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x - y) H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right) dy \right] dc \\ &+ (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(u(y, t - \tau) - \Theta) dy \right] d\tau\end{aligned}$$

is equivalent to the following integral differential equation

$$\begin{aligned}&\frac{\partial v}{\partial t_0} + v \\ &= (\alpha_0 - a_0v) \int_0^\infty \xi_0(c_0) \left[\int_{\mathbb{R}} K(x - y) H\left(v\left(y, t_0 - \frac{1}{c_0}|x - y|\right) - \theta_0\right) dy \right] dc_0 \\ &+ (\beta_0 - b_0v) \int_0^\infty \eta_0(\tau_0) \left[\int_{\mathbb{R}} W(x - y) H(v(y, t_0 - \tau_0) - \Theta_0) dy \right] d\tau_0.\end{aligned}$$

Proof. The proof of Theorem 3.3.7 is simple and is omitted. ■

3.4 Numerical Analysis

In this section, we will focus on the numerical analysis of equation (3.1) with a nonlinear sodium current function $f(u)$, such that $f(n) = 0$ and $m = f'(n) > 0$, for two real parameters m and n .

Given a nonlinear smooth function $w = f(u)$, we may find a linear function $f_l(u) = m_0(u - n_0)$, where $m_0 > 0$ and n_0 are real constants, such that

$$\max_{[0,\theta]} |f(u) - m_0(u - n_0)| = \min_{m,n} \left\{ \max_{[0,\theta]} |f(u) - m(u - n)| \right\}.$$

In another word, $f_l(u) = m_0(u - n_0)$ is the best function to approximate the nonlinear function $w = f(u)$. This kind of optimal approximation has a great influence on the delicate estimate of the wave speed. As we see, the wave speed of the traveling wave front of the integral differential equation (3.1) with a linear function satisfy a nice equation, see Theorem 3.2.3 and Theorem 3.2.4 We can show that the wave speed of the traveling wave front of the integral differential equation (3.1) with the nonlinear equation is very close to the wave speed of the traveling wave front of the integral differential equation (3.1) with the particular linear function $f_l(u) = m_0(u - n_0)$.

Theorem 3.4.1. *Let μ_0 and μ_{appr} represent the real speed and the approximate speed of (3.1) with the nonlinear function $w = f(u)$ and the linear function $f_l(u)$, respectively. Then, there holds the following estimates*

$$|\mu_0 - \mu_{\text{appr}}| \leq \ln \left\{ 1 + \max_{u \in [0,\theta]} |f(u) - m_0(u - n_0)| \right\}. \quad (3.13)$$

Proof. The approximate wave speed μ_{appr} is determined by the equation

$$\theta - n_0 = \frac{A_0(0)}{B_0(0)} - \frac{1}{D_0(0)} \int_{-\infty}^0 \left[\frac{A_0(x)}{B_0(x)} \right]' D_0(x) dx.$$

By an intermediate value theorem, there exists a real number κ , such that the real speed μ_0 satisfies

$$\begin{aligned}\theta - n &= \frac{A(0)}{B(0)} - \frac{1}{D(0)} \int_{-\infty}^0 \left[\frac{A(x)}{B(x)} \right]' D(x) dx \\ &+ \frac{1}{\mu_0 D(0)} \int_{-\infty}^0 D(x) \{ m[U(x) - n] - f(U(x)) \} dx \\ &= \frac{A(0)}{B(0)} - \frac{1}{D(0)} \int_{-\infty}^0 \left[\frac{A(x)}{B(x)} \right]' D(x) dx,\end{aligned}$$

where

$$|\kappa| \leq \ln \left\{ 1 + \max_{[0, \theta]} |f(u) - m(u - n)| \right\}.$$

By uniqueness, we find

$$\frac{c - \mu_0}{c\mu_0} + \kappa = \frac{c - \mu_{\text{appr}}}{c\mu_{\text{appr}}}.$$

The proof of Theorem 3.4.1 is finished. ■

3.4.1 Numerical simulations

We perform some numerical simulations of (3.1) with a nonlinear function $f(u)$. See Figure 3.4.1 to Figure 3.4.1.

Summary from numerical simulations of the wave speed:

- * The wave speed μ_0 is a decreasing function of the parameter a .
- * The wave speed μ_0 is a decreasing function of the parameter b .
- * The wave speed μ_0 is an increasing function of the parameter ρ if the synaptic coupling $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$.
- * The wave speed μ_0 is a decreasing function of the parameter ρ if the synaptic coupling

$$W(x) = \frac{\rho}{2} \exp(-\rho|x|).$$

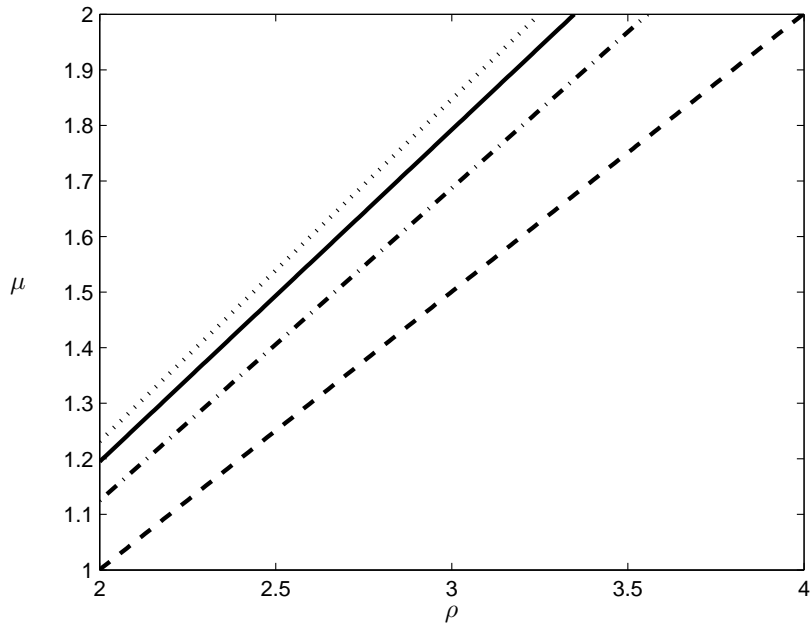


Figure 3.1: Let $a = 0$, $b = 0$, $c_0 = \infty$, $\alpha = 5$, $\beta = 0$, $\theta = 2$. Let $f(u) = \frac{1}{D} \sinh(Du)$, $\xi(c) = \delta(c - c_0)$ and $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$. The graph of the wave speed $\mu_0 = \mu_0(\rho, D)$, where $\rho > 0$ and $D > 0$ are parameters. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

- * The wave speed μ_0 is a decreasing function of the parameter D if the sodium current function is modeled with the nonlinear function $f(u) = \frac{1}{D} \sinh(Du)$.
- * The wave speed μ_0 is an increasing function of the parameter D if the sodium current function is modeled with the nonlinear function $f(u) = \frac{1}{D} \tanh(Du)$.
- * The wave speed μ_0 is an increasing function of the parameter D if the sodium current function is modeled with the nonlinear function $f(u) = u(u - 1)(Du - 1)$.

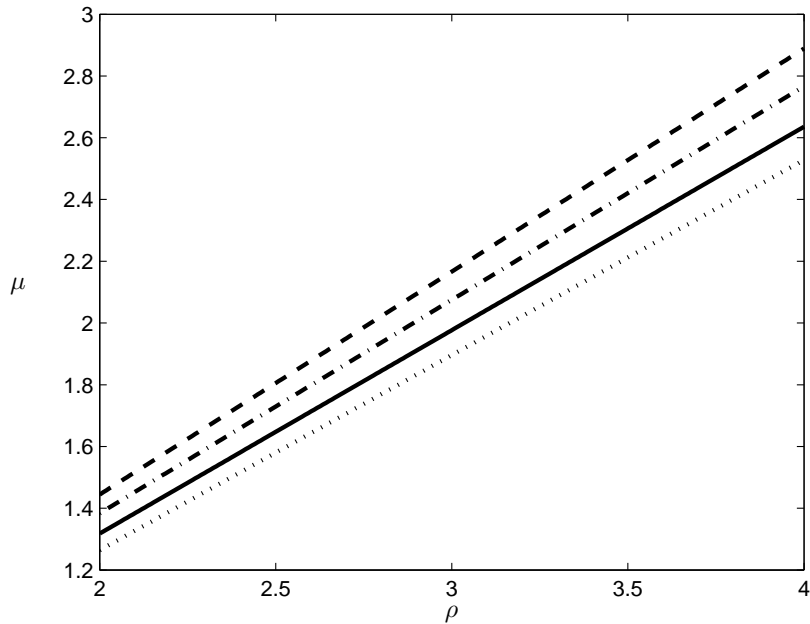


Figure 3.2: Let $a = 0$, $b = 0$, $c_0 = \infty$, $\alpha = 5$, $\beta = 0$, $\theta = 2$. Let $f(u) = \frac{1}{D} \tanh(Du)$, $\xi(c) = \delta(c - c_0)$ and $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$. The graph of the wave speed $\mu_0 = \mu_0(\rho, D)$, where $\rho > 0$ and $D > 0$ are parameters. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

3.5 Discussions

Various models describing neuronal activities have been proposed and analyzed in recent years. As we continue to develop the model equations, we have been able to incorporate more biological mechanisms of neuronal networks in an effort to increase the accuracy of the model equation and discover further impacts of these mechanisms. It is in this light that our work expands upon recent models by incorporating not only two time delay factors, but also the refractory terms. To add another layer of accuracy to the model, we consider not only linear representations of sodium currents but also nonlinear terms which allow us to capture more realistic behaviors of the sodium current. With these

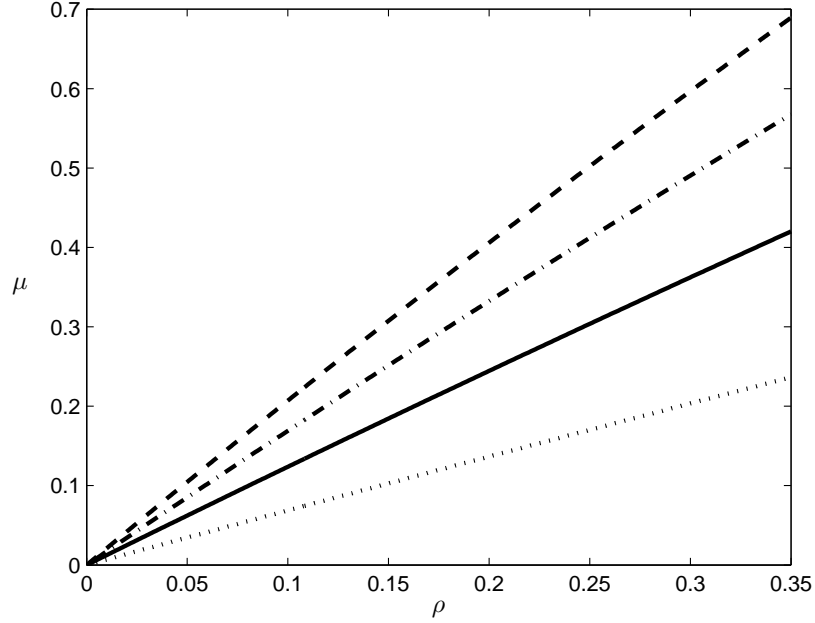


Figure 3.3: Let $a = 0, b = 0, c_0 = \infty, \alpha = 5, \beta = 0, \theta = 2$. Let $f(u) = u(u - 1)(Du - 1)$, $\xi(c) = \delta(c - c_0)$ and $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$. The graph of the wave speed $\mu_0 = \mu_0(\rho, D)$, where $\rho > 0$ and $D > 0$ are parameters. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

changes to the model equation, we are considering a very general model that generalizes most models in current literature.

We use the general integral differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + f(u) \\ = & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c}|x - y| \right) - \theta \right) dy \right] dc \\ + & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(u(y, t - \tau) - \Theta) dy \right] d\tau, \end{aligned}$$

arising from synaptically coupled neuronal networks to investigate the influence of biological mechanisms on speeds of traveling wave fronts. In particular, we investigated

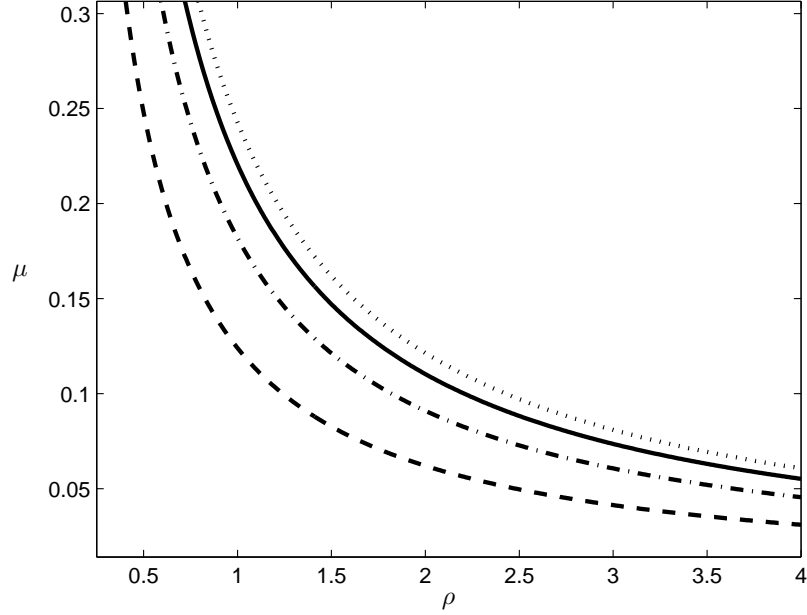


Figure 3.4: Let $a = 0$, $b = 0$, $\tau_0 = 0$, $\alpha = 0$, $\beta = 5$, $\Theta = 2$. Let $f(u) = \frac{1}{D} \sinh(Du)$, $\eta(\tau) = \delta(\tau - \tau_0)$ and $W(x) = \frac{\rho}{2} \exp(-\rho|x|)$. The graph of the wave speed $\mu_0 = \mu_0(\rho, D)$, where $\rho > 0$ and $D > 0$ are parameters. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

how synaptic couplings, sodium conductance, sodium reversal potential, synaptic conductance and synaptic reversal potential influence the wave speeds. The mathematical methods of analysis of this model equation are similar to methods used in our previous work. Using these techniques, we were able to establish the existence and uniqueness of traveling wave fronts to the model equations, despite the added complexity of the additional time delay and refractory term. The wave speed is a decreasing function with respect to parameters a and b . The wave speed can be increasing or decreasing based upon parameters within the kernel functions K and W . The same is true depending on the sodium current function $f(u)$ as illustrated by various examples. These results

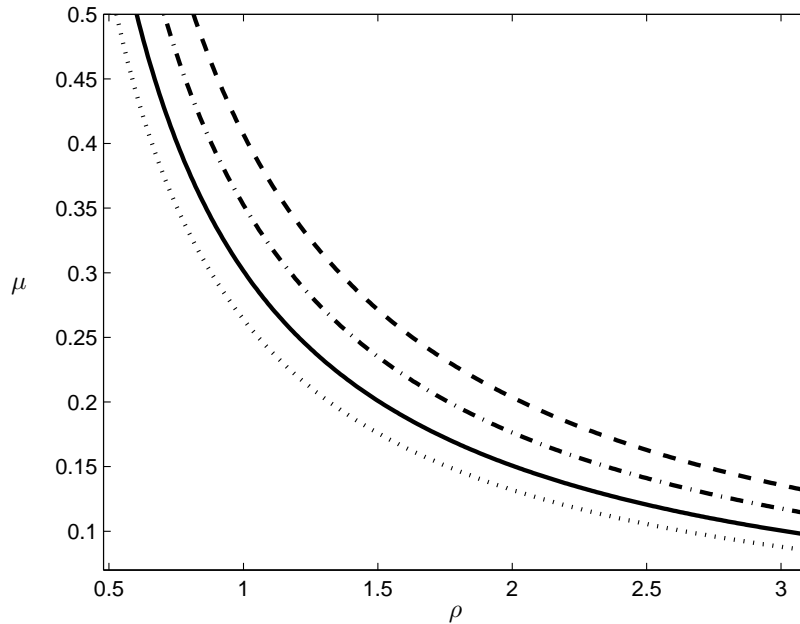


Figure 3.5: Let $a = 0$, $b = 0$, $\tau_0 = 0$, $\alpha = 0$, $\beta = 5$, $\Theta = 2$. Let $f(u) = \frac{1}{D} \tanh(Du)$, $\eta(\tau) = \delta(\tau - \tau_0)$ and $W(x) = \frac{\rho}{2} \exp(\rho|x|)$. The graph of the wave speed $\mu_0 = \mu_0(\rho, D)$, where $\rho > 0$ and $D > 0$ are parameters. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

generalize and partly improve our previous results. We hope to find real applications to biological problems and that this work will continue to shed light on the behavior of traveling wave fronts and wave pulses in the brain and their behavior consequential to changes in the neuronal networks. There are still many important open problems to be solved. For example, can we improve the upper bound and lower bound of the wave speed?

The traveling wave front has a unique wave speed and we are able to establish an upper bound and a lower bound on this wave speed dependent upon parameters of the neuronal networks. Let μ_0 be the wave speed of the traveling wave front of the simpler

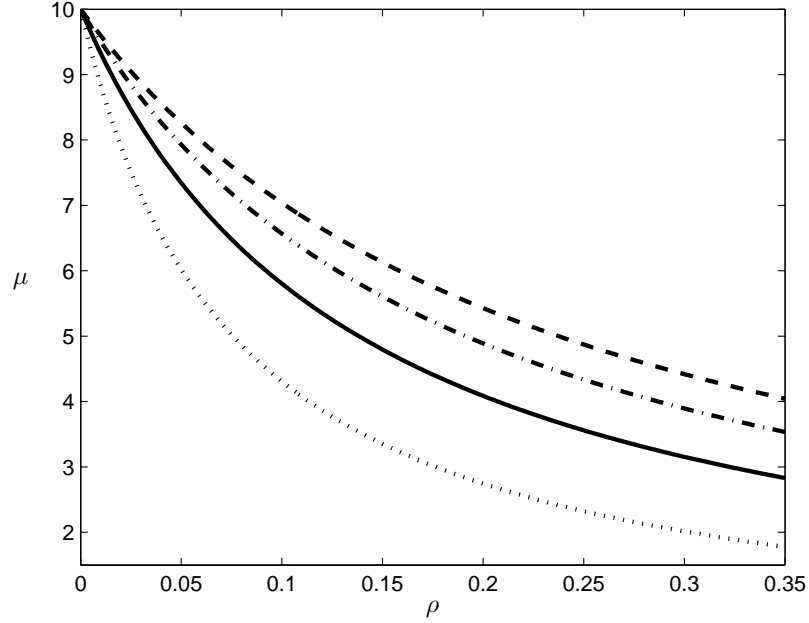


Figure 3.6: Let $a = 0, b = 0, \tau_0 = 0, \alpha = 0, \beta = 5, \Theta = 2$. Let $f(u) = u(u - 1)(Du - 1), \eta(\tau) = \delta(\tau - \tau_0)$ and $W(x) = \frac{\rho}{2} \exp(-\rho|x|)$. The graph of the wave speed $\mu_0 = \mu_0(\rho, D)$, where $\rho > 0$ and $D > 0$ are parameters. For the dotted curve, $D = 2.5$. For the solid curve, $D = 3.0$. For the dash-dotted curve, $D = 3.5$. For the dashed curve, $D = 4.0$.

integral differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} + m(u - n) \\ = & (\alpha - au) \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c}|x - y| \right) - \theta \right) dy \right] dc \\ + & (\beta - bu) \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x - y) H(u(y, t - \tau) - \Theta) dy \right] d\tau u. \end{aligned}$$

Then

$$\mu_0 < \frac{m}{(\alpha + \beta) \ln \frac{\alpha + \beta}{\alpha + \beta - 2m\theta + 2mn}} \int_{\mathbb{R}} [\alpha|x|K(x) + \beta|x|W(x)] dx,$$

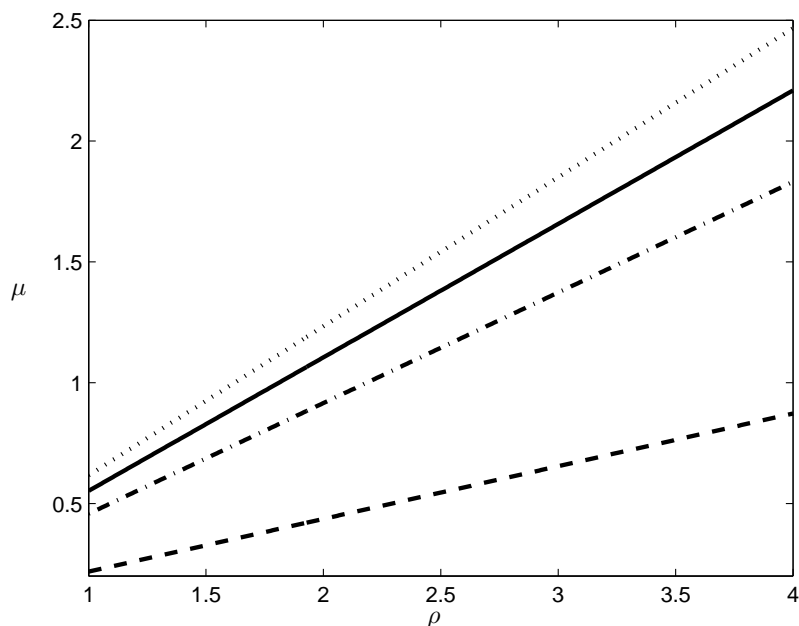


Figure 3.7: Let $b = 0$, $c_0 = \infty$, $\alpha = 5$, $\beta = 0$, $\theta = 2$ and $\rho = 1$. Let $f(u) = \frac{1}{D} \sinh(Du)$, $\xi(c) = \delta(c - c_0)$ and $K(x) = \frac{1}{2} [\delta(x + \rho) + \delta(x - \rho)]$. The graph of the wave speed $\mu_0 = \mu_0(a, D)$, where $a > 0$ and $D > 0$ are parameters. For the dotted curve, $a = 0.01$. For the solid curve, $a = 0.13$. For the dash-dotted curve, $a = 0.24$. For the dashed curve, $a = 0.36$.

$$\begin{aligned} \mu_0 &> \frac{\frac{m}{2}(\alpha + \beta - a\theta - b\theta - 2m\theta + 2mn)}{(\alpha - a\theta)K(0) + (\beta - b\theta)W(0) \left[\int_0^\infty \eta(\tau) \exp(m\tau) d\tau \right]} \\ &> 0. \end{aligned}$$

Once we establish boundaries of the wave speed, the need to determine the behavior of the wave speed based upon changes in the neuronal networks becomes biologically relevant. We have been able to obtain the increasing and decreasing behavior of the wave speed based on changes in specific biological parameters.

The speed index functions are very interesting and important concept in mathematical neuroscience. It has potential applications and impacts in applied mathematics. With the introduction of the speed index functions, we can do much more analysis on

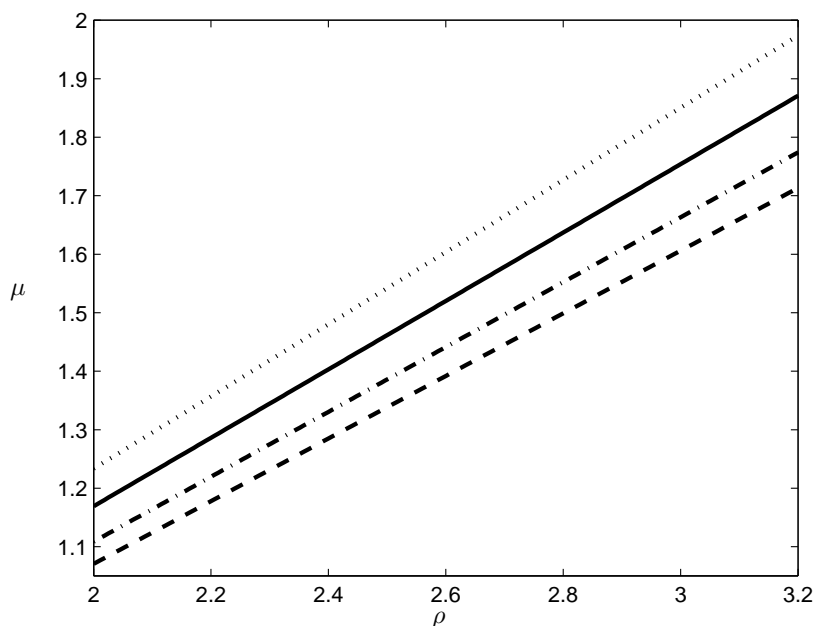


Figure 3.8: Let $b = 0$, $c_0 = \infty$, $\alpha = 5$, $\beta = 0$, $\theta = 2$ and $\rho = 1$. Let $f(u) = \frac{1}{D} \tanh(Du)$, $\xi(c) = \delta(c - c_0)$ and $K(x) = \frac{1}{2}[\delta(x + \rho) + \delta(x - \rho)]$. The graph of the wave speed $\mu_0 = \mu_0(a, D)$, where $a > 0$ and $D > 0$ are parameters. For the dotted curve, $a = 0.01$. For the solid curve, $a = 0.09$. For the dash-dotted curve, $a = 0.20$. For the dashed curve, $a = 0.36$.

the speed than previously. One interesting point is that we may define a stability index function that utilizes the speed index function. By using this relationship, the stability of the traveling wave can be analyzed easily. This shows us that adding complexity to the model to account for more accurate conditions in neuronal networks does not affect the stability of the wave front. The speed index functions may play very important roles in rigorous mathematical analysis of traveling waves of nonlinear singularly perturbed systems of integral differential equations. Moreover, the analysis and results on the speeds and the speed index functions can be applied to computational neuroscience.

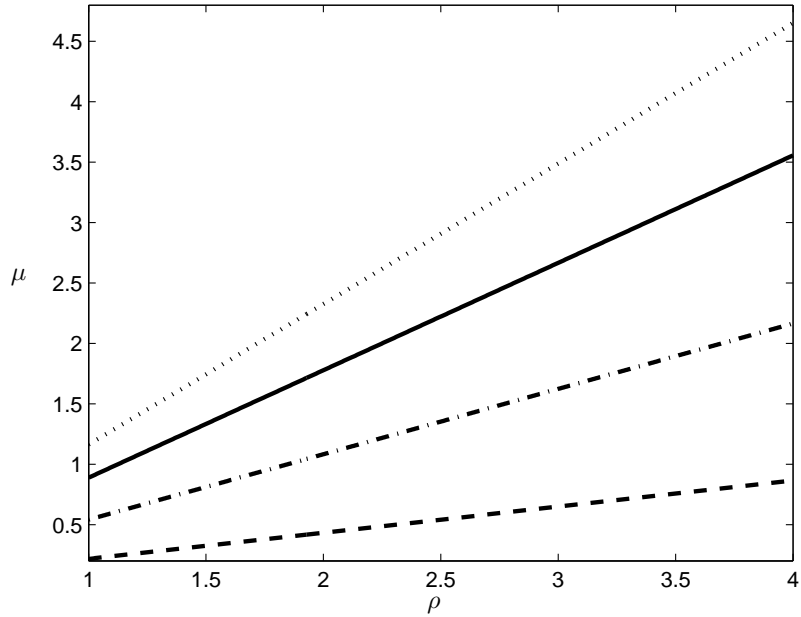


Figure 3.9: Let $b = 0$, $c_0 = \infty$, $\alpha = 5$, $\beta = 0$, $\theta = 2$ and $\rho = 1$. Let $f(u) = u(u-1)(Du-1)$, $\xi(c) = \delta(c-c_0)$ and $K(x) = \frac{1}{2}[\delta(x+\rho) + \delta(x-\rho)]$. The graph of the wave speed $\mu_0 = \mu_0(a, D)$, where $a > 0$ and $D > 0$ are parameters. For the dotted curve, $a = 0.01$. For the solid curve, $a = 0.16$. For the dash-dotted curve, $a = 0.30$. For the dashed curve, $a = 0.36$.

Chapter 4

Traveling Waves with Lateral Inhibition Kernel Functions

4.1 Introduction

As we described in the introduction, there are three classes of kernel function representing pure excitation, lateral inhibition, and lateral excitation. Most work regarding the existence and uniqueness of traveling wave solutions and traveling pulse solutions has been done by considering class (A) kernel functions, mainly to ease the mathematical analysis as those kernel functions are always positive. While they are nice to work with, neuronal networks with pure excitation are rare. Neuronal networks with lateral inhibition are much more common in the body. So in this work we expand upon previous work by Zhang [61] [62], Pinto and Ermentrout [46], and Terman [53].

In this chapter we focus on the model equation proposed by Pinto and Ermentrout in 2001 [46]. We let the kernel function be of class (B) and establish the existence and uniqueness of the traveling wave solution. The goal is then to show that the unique solution is exponentially stable. Use use complex analytic functions as in previous chapters along with the linearized stability criterion to establish the stability of the wave.

4.1.1 Model equation and biological background

Consider the following integral-differential equations to represent the membrane potential of a neuron arising from synaptically coupled neuronal networks

$$u_t + u + w = \alpha \int_{\mathbb{R}} K(x - y) H(u(y, t) - \theta) dy \quad (4.1)$$

$$w_t = \epsilon(u - \gamma w), \quad (4.2)$$

when $u = u(x, t)$ represents the membrane potential at position x and time t , $w = w(x, t)$ represents the leaking current. In this model we choose the gain function to be the Heaviside step function: $H(u - \theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$, and $H(u - \theta) = 1$ for all $u > \theta$. We represent the interactions between neurons by convoluting the kernel function with the gain function. In addition, we use the assumptions on the class (B) kernel function K as described in the introduction.

4.1.2 Mathematical Assumptions

We begin our discussion of mathematical assumptions by considering restrictions to the kernel functions representing the synaptic coupling in the network. Suppose that the kernel function K in class (B) and which satisfy all assumptions described in the

introduction along with the following

$$\int_{-\infty}^0 g(x)K(x - Z)dx < 0 \quad (4.3)$$

$$\int_{-\infty}^0 |x|K(x)dx \leq 0 \quad (4.4)$$

$$2\theta(\gamma + 1) > \alpha\gamma \quad (4.5)$$

$$\frac{\alpha\gamma}{\gamma + 1} \int_{-M}^0 K(x) dx > \theta \quad (4.6)$$

4.2 Existence

4.2.1 System (4.1)-(4.2)

We begin by establishing the existence of the traveling wave solution to equations (4.1)-(4.2) with class (B) kernel function $K(x)$.

Theorem 4.2.1. *Suppose that $\alpha \geq 0$, $\varepsilon > 0$, $\theta > 0$ are real constants and choose $K(x)$ to be lateral inhibition. Then there exists traveling wave solutions $(U(z), W(z))$ to the*

system (4.1)-(4.2), namely

$$\begin{aligned}
U(\varepsilon, z) &= \frac{\alpha\gamma}{\gamma+1} \int_{z-Z}^z K(\xi) d\xi \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x - Z)] dx
\end{aligned}$$

$$\begin{aligned}
W(\varepsilon, z) &= \frac{\alpha}{\gamma+1} \int_{z-Z}^z K(\xi) d\xi \\
&+ \frac{\alpha\varepsilon}{\omega_1 - \omega_2} \int_{-\infty}^z \left\{ \frac{1}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x - Z)] dx
\end{aligned}$$

Proof: We begin to solve the system (4.1)-(4.2) for the speed of the traveling wave. We begin by letting $(u(x, t), w(x, t)) = (U(z), W(z))$ where $(U(z), W(z))$ is the traveling wave solution to the system (4.1)-(4.2) with $z = x + \nu t$. The traveling wave $U(z)$ has the following properties, $U(z) < \theta$ on $(-\infty, 0) \cup (Z, \infty)$, $U(0) = \theta$, and $U(z) > \theta$ on $(0, Z)$ Then the system reduces to

$$\nu U'(z) + U(z) + W(z) = \alpha \int_{\mathbb{R}} K(z - y) H(U(z) - \theta) dy \quad (4.7)$$

$$\nu W'(z) = \varepsilon (U(z) - \gamma W(z)), \quad (4.8)$$

Note that

$$\begin{aligned}
\alpha \int_{\mathbb{R}} K(z - y) H(U(z) - \theta) dy &= \alpha \int_0^Z K(z - y) dy \\
&= \alpha \int_{z-Z}^z K(x) dx
\end{aligned}$$

So the system simplifies to

$$\nu U'(z) + U(z) + W(z) = \alpha \int_{z-Z}^z K(x) dx \quad (4.9)$$

$$\nu W'(z) - \varepsilon U(z) + \gamma \varepsilon W(z) = 0, \quad (4.10)$$

To further simplify the system, we can write the system as a matrix equation, namely

$$\nu \begin{pmatrix} U \\ V \end{pmatrix}' + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \gamma \varepsilon \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \alpha \int_{z-Z}^z K(x) dx \\ 0 \end{pmatrix}$$

To begin solving the new matrix equation we solve the homogeneous equation,

$$\nu \begin{pmatrix} U \\ V \end{pmatrix}' + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \gamma \varepsilon \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 0,$$

by finding its eigenvalues and eigenvectors to establish a fundamental solution.

$$\begin{aligned} \begin{vmatrix} 1 - \lambda & 1 \\ -\varepsilon & \gamma \varepsilon - \lambda \end{vmatrix} &= (1 - \lambda)(\gamma \varepsilon - \lambda) + \varepsilon \\ &= \lambda^2 - (\gamma \varepsilon + 1)\lambda + \varepsilon(\gamma + 1) \end{aligned}$$

Then we have the two eigenvalues, ω_1 and ω_2 as follows;

$$\begin{aligned} \omega_1 &= \frac{1 + \gamma \varepsilon + \sqrt{(\gamma \varepsilon + 1)^2 - 4\varepsilon(\gamma + 1)}}{2} \\ &= \frac{1 + \gamma \varepsilon + \sqrt{(\gamma \varepsilon - 1)^2 - 4\varepsilon}}{2} \end{aligned}$$

$$\begin{aligned}\omega_2 &= \frac{1 + \gamma\varepsilon - \sqrt{(\gamma\varepsilon + 1)^2 - 4\varepsilon(\gamma + 1)}}{2} \\ &= \frac{1 + \gamma\varepsilon - \sqrt{(\gamma\varepsilon - 1)^2 - 4\varepsilon}}{2}\end{aligned}$$

Remark 4.2.2.

$$\omega_1 + \omega_2 = \gamma\varepsilon + 1$$

$$\begin{aligned}\omega_1\omega_2 &= \frac{(\gamma\varepsilon + 1)^2 - [(\gamma\varepsilon - 1)^2 - 4\varepsilon]}{4} \\ &= \varepsilon(1 + \gamma)\end{aligned}$$

The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ \omega_1 - 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ \omega_2 - 1 \end{pmatrix}$.

Now we can construct the fundamental solution to the matrix equation, namely

$$\Phi(z) = \begin{pmatrix} e^{-\frac{\omega_1}{\nu}z} & e^{-\frac{\omega_2}{\nu}z} \\ e^{-\frac{\omega_1}{\nu}z}(\omega_1 - 1) & e^{-\frac{\omega_2}{\nu}z}(\omega_2 - 1) \end{pmatrix}. \quad (4.11)$$

Now we solve the inhomogeneous equation, by finding

$$\begin{aligned}&\frac{1}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^z \Phi(z)\Phi^{-1}(x)Bdx, \\ &\text{where } B = \begin{pmatrix} \alpha \int_{z-z}^z K(x)dx \\ 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\Phi^{-1}(x) &= \begin{pmatrix} e^{-\frac{\omega_2}{\nu}z}(\omega_2 - 1) & -e^{-\frac{\omega_2}{\nu}z} \\ e^{-\frac{\omega_1}{\nu}z}(1 - \omega_1) & e^{-\frac{\omega_1}{\nu}z} \end{pmatrix} \cdot \frac{e^{\frac{\omega_1 + \omega_2}{\nu}x}}{\omega_2 - \omega_1} \\ &= \frac{1}{\omega_2 - \omega_1} \begin{pmatrix} e^{\frac{\omega_1}{\nu}z}(\omega_2 - 1) & -e^{\frac{\omega_1}{\nu}z} \\ e^{\frac{\omega_2}{\nu}z}(1 - \omega_1) & e^{\frac{\omega_2}{\nu}z} \end{pmatrix} \\ &= \frac{1}{\omega_1 - \omega_2} \begin{pmatrix} e^{\frac{\omega_1}{\nu}z}(1 - \omega_2) & e^{\frac{\omega_1}{\nu}z} \\ e^{\frac{\omega_2}{\nu}z}(\omega_1 - 1) & -e^{\frac{\omega_2}{\nu}z} \end{pmatrix}\end{aligned}$$

$$\Phi(z)\Phi^{-1}(x) = \frac{1}{\omega_1 - \omega_2} \begin{pmatrix} (1 - \omega_2)e^{\frac{\omega_1}{\nu}(x-z)} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}(x-z)} & e^{\frac{\omega_1}{\nu}(x-z)} - e^{\frac{\omega_2}{\nu}(x-z)} \\ -\varepsilon e^{\frac{\omega_1}{\nu}(x-z)} + \varepsilon e^{\frac{\omega_2}{\nu}(x-z)} & (\omega_1 - 1)e^{\frac{\omega_1}{\nu}(x-z)} - (\omega_2 - 1)e^{\frac{\omega_2}{\nu}(x-z)} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} U \\ V \end{pmatrix} &= \frac{1}{\omega_1 - \omega_2} \int_{-\infty}^z \Phi(z)\Phi^{-1}(x)Bdx \\ &= \frac{1}{\omega_1 - \omega_2} \int_{-\infty}^z \Phi(z)\Phi^{-1}(x) \begin{pmatrix} \alpha \int_{x-Z}^x K(\xi)d\xi \\ 0 \end{pmatrix} dx \end{aligned}$$

Hence we have the solutions for $U(z)$ and $W(z)$:

$$U(\varepsilon, z) = \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^z \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}(x-z)} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}(x-z)} \right\} \int_{x-Z}^x K(\xi)d\xi dx$$

$$W(\varepsilon, z) = \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^z \left\{ -\varepsilon e^{\frac{\omega_1}{\nu}(x-z)} + \varepsilon e^{\frac{\omega_2}{\nu}(x-z)} \right\} \int_{x-Z}^x K(\xi)d\xi dx$$

Using integration by parts we arrive at the solution,

$$\begin{aligned}
U(\varepsilon, z) &= \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \left[\int_{x-Z}^x K(\xi) d\xi \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} \right] \Big|_{-\infty}^z \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x - Z)] dx \\
&= \frac{\alpha}{\omega_1 - \omega_2} \int_{z-Z}^z K(\xi) d\xi \left(\frac{1 - \omega_2}{\omega_1} - \frac{1 - \omega_1}{\omega_2} \right) \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x - Z)] dx \\
&= \frac{\alpha\gamma}{\gamma + 1} \int_{z-Z}^z K(\xi) d\xi \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x - Z)] dx
\end{aligned}$$

$$\begin{aligned}
W(\varepsilon, z) &= \frac{\alpha}{\omega_1 - \omega_2} \left[\int_{x-Z}^x K(\xi) d\xi \left\{ \frac{-\varepsilon}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} + \frac{\varepsilon}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} \right] \Big|_{-\infty}^z \\
&- \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^z \left\{ \frac{-\varepsilon}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} + \frac{\varepsilon}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x-Z)] dx \\
&= \frac{\alpha}{\omega_1 - \omega_2} \int_{z-Z}^z K(\xi) d\xi \left(\frac{-\varepsilon}{\omega_1} + \frac{\varepsilon}{\omega_2} \right) \\
&- \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^z \left\{ \frac{-\varepsilon}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} + \frac{\varepsilon}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x-Z)] dx \\
&= \frac{\alpha}{\gamma + 1} \int_{z-Z}^z K(\xi) d\xi \\
&+ \frac{\alpha\varepsilon}{\omega_1 - \omega_2} \int_{-\infty}^z \left\{ \frac{1}{\omega_1} e^{\frac{\omega_1}{\nu}(x-z)} - \frac{1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-z)} \right\} [K(x) - K(x-Z)] dx
\end{aligned}$$

■

Since we are looking for the speed of the traveling wave solution for the membrane potential, we focus on the solution for $U(\varepsilon, z)$. We use the two initial conditions to attempt to solve the system, $U(0) = \theta$ and $U(Z) = \theta$. By letting $\tilde{x} = x - Z$ and $d\tilde{x} = dx$ in the second case, we obtain the following two equations, which we will use to generate speed index functions.

$$\begin{aligned}
U(0) &= \theta \\
&= \frac{\alpha\gamma}{\gamma + 1} \int_{-Z}^0 K(\xi) d\xi \\
&- \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x-Z)] dx
\end{aligned}$$

$$\begin{aligned}
U(Z) &= \theta \\
&= \frac{\alpha\gamma}{\gamma+1} \int_0^Z K(\xi) d\xi \\
&\quad - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^Z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-Z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-Z)} \right\} \\
&\quad \cdot [K(x) - K(x - Z)] dx
\end{aligned}$$

Remark 4.2.3. $K(x)$ is symmetric, so $\int_0^Z K(x) dx = \int_0^{-Z} K(x) dx$.

4.2.2 Speed Index Functions

To establish the uniqueness of the wave speed, we begin by constructing speed index functions.

Definition 4.2.4. We define two speed index functions,

$$\begin{aligned}
\varphi_1(\nu) &= \frac{\alpha\gamma}{\gamma+1} \int_{-Z}^0 K(\xi) d\xi \\
&\quad - \frac{\alpha}{\omega_1 - \omega_2} \\
&\quad \cdot \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)] dx \quad (4.12)
\end{aligned}$$

$$\begin{aligned}
\varphi_2(\nu) &= \frac{\alpha\gamma}{\gamma+1} \int_0^Z K(\xi) d\xi \\
&\quad - \frac{\alpha}{\omega_1 - \omega_2} \\
&\quad \cdot \int_{-\infty}^Z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-Z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-Z)} \right\} [K(x) - K(x - Z)] dx, \\
&= \frac{\alpha\gamma}{\gamma+1} \int_0^Z K(\xi) d\xi \\
&\quad - \frac{\alpha}{\omega_1 - \omega_2} \\
&\quad \cdot \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}x} \right\} [K(x + Z) - K(x)] dx. \quad (4.13)
\end{aligned}$$

We want to show that $\varphi_1(\nu_1, Z) = \varphi_2(\nu_2, Z) = \theta$ has a unique solution. To establish this, we need a unique solution to the equation $\varphi_1(\nu) = \varphi_2(\nu)$. We begin by looking at the behavior of $\varphi_1(\nu)$ and $\varphi_2(\nu)$. More specifically, we look at the derivatives of $\varphi_1(\nu)$ and $\varphi_2(\nu)$. It is easy to find

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \nu} &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-\infty}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)] dx, \end{aligned}$$

$$\begin{aligned} \frac{\partial \varphi_2}{\partial \nu} &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-\infty}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x + Z) - K(x)] dx. \end{aligned}$$

For simplicity we make the following identifications,

$$g(x) = (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x},$$

$$h(x) = \left(\frac{1 - \omega_2}{\omega_1} \right) e^{\frac{\omega_1}{\nu}x} - \left(\frac{1 - \omega_1}{\omega_2} \right) e^{\frac{\omega_2}{\nu}x},$$

$$K_1(x) = K(x) - K(x - Z),$$

$$K_2(x) = K(x) - K(x + Z).$$

We also define the following notation, there exist constants $N, N_i, M, M_i > 0$ such

that

$$K(M) = K(-M) = 0,$$

$$K_i(-M_i) = 0,$$

$$g(-N, \nu) = 0,$$

$$g_i(-N_i, \nu_i) = 0.$$

$$g'(-P, \nu) = 0,$$

$$h(-N_h, \nu) = 0,$$

$$h'(-P_h, \nu) = 0,$$

Remark 4.2.5. We define $-M_i$ as the first zero of K_i to the left of the origin for the corresponding kernel function.

We consider the behavior of the functions K , K_i , g , and h . The kernel function $K(x)$ is lateral inhibition, meaning that $K(x) > 0$ on $(-M, M)$ and $K(x) < 0$ on $(-\infty, -M) \cup (M, \infty)$. We consider $g(x)$ and find $N(\nu) = \frac{\nu}{\omega_1 - \omega_2} \ln \left(\frac{1 - \omega_2}{1 - \omega_1} \right)$. Then $g(x) > 0$ on $(-N, \infty)$ and $g(x) < 0$ on $(-\infty, -N)$. Similarly, $\nu g'(x) = \omega_1(1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - \omega_2(1 - \omega_1)e^{\frac{\omega_2}{\nu}x}$, and $P = \frac{\nu}{\omega_1 - \omega_2} \ln \left(\frac{\omega_1(1 - \omega_2)}{\omega_2(1 - \omega_1)} \right)$. So $g(x)$ is increasing on $(-P, \infty)$ and decreasing on $(-\infty, -P)$. Note $N < P$. Lastly, we consider $h(x)$. As

h varies from g by a few constants, structurally, they are similar. So we have $N_h(\nu) = \frac{\nu}{\omega_1 - \omega_2} \ln \left(\frac{\omega_2(1 - \omega_2)}{\omega_1(1 - \omega_1)} \right)$. Then $h(x) > 0$ on $(-N_h, \infty)$ and $h(x) < 0$ on $(-\infty, -N_h)$. Similarly, $\nu h'(x) = g(x)$, and $P_h = N = \frac{\nu}{\omega_1 - \omega_2} \ln \left(\frac{1 - \omega_2}{1 - \omega_1} \right)$. So $h(x)$ is increasing on $(-N, \infty)$ and decreasing on $(-\infty, -N)$. We also note that $N > N_h$.

We first consider

$$\Psi(x, \nu) = \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} K(x) dx,$$

which reduces to

$$\Psi(x, \nu) = \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^0 |x| g(x) K(x) dx. \quad (4.14)$$

Lemma 4.2.6. *Suppose $-N < -M$, then*

$$\Psi(x, \nu) = \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^0 |x| g(x) K(x) dx > 0. \quad (4.15)$$

Proof: As noted above, $g(x) > 0$ and $g(x)$ is increasing on $(-N, 0)$. So g is positive and increasing on $(-M, 0)$. Also we have the assumption that $\int_{-\infty}^0 |x| K(x) dx \geq 0$. We note that on $(-\infty, -N) \cup (-N, -M)$ $K(x) < 0$, so $\int_{-N}^0 |x| K(x) dx \geq \int_{-\infty}^0 |x| K(x) dx \geq 0$.

$$\begin{aligned} \int_{-N}^0 |x| g(x) K(x) dx &= \int_{-N}^{-M} |x| g(x) K(x) dx + \int_{-M}^0 |x| g(x) K(x) dx \\ &> g(-M) \int_{-N}^{-M} |x| K(x) dx + g(M) \int_{-M}^0 |x| K(x) dx \\ &= g(-M) \int_{-N}^0 |x| K(x) dx \\ &\geq 0 \end{aligned}$$

Since $K(x) < 0$ and $g(x) < 0$ on $(-\infty, -N)$, then

$$\begin{aligned}\Psi(x, \nu) &= \int_{-\infty}^0 |x|g(x)K(x)dx \\ &= \int_{-\infty}^{-N} |x|g(x)K(x)dx + \int_{-N}^0 |x|g(x)K(x)dx \\ &> 0\end{aligned}$$

■

Since $|x|g(x)K(x)$ goes to zero dramatically for $x < -M$, $\int_{-\infty}^{-M} |x|g(x)K(x)dx$ is negligible. Hence $\int_{-\infty}^0 |x|g(x)K(x)dx \approx \int_{-M}^0 |x|g(x)K(x)dx$. As the slow pulse does not have biological relevance, we consider only the fast pulse, so we restrict our speed ν away from zero. We shall only consider $\nu > \nu_0$ where ν_0 is such that $\int_{-M}^0 |x|g(x)K(x)dx = 0$, then notationally, we say that $g_0(x) = g(x, \nu_0)$.

Lemma 4.2.7. *Suppose $-P < -M < -N < -N_0$, then*

$$\Psi(x, \nu) = \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^0 |x|g(x)K(x)dx > 0. \quad (4.16)$$

Proof: Now $K(x) < 0$ and $g(x) < 0$ on $(-\infty, -M)$ so again we have

$$\int_{-\infty}^{-M} |x|g(x)K(x)dx > 0, \text{ so it remains to show that } \int_{-M}^0 |x|g(x)K(x)dx > 0.$$

Since $\frac{\partial g}{\partial \nu} = \frac{|x|}{\nu^2} \left[\omega_1(1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - \omega_2(1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right]$ which for

$x > -N = \frac{\nu}{\omega_1 - \omega_2} \ln \left(\frac{1 - \omega_1}{1 - \omega_2} \right)$ is positive, we have that $g(x) > g_0(x)$ on $(-N_0, 0)$

and $K(x) > 0$ on $(-M, 0)$ so we have

$$\begin{aligned}\int_{-M}^0 |x|g(x)K(x)dx &> \int_{-M}^0 |x|g_0(x)K(x)dx \\ &= 0\end{aligned}$$

$$\begin{aligned}
\Psi(x, \nu) &= \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^0 |x|g(x)K(x)dx \\
&= \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^{-M} |x|g(x)K(x)dx + \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-M}^0 |x|g(x)K(x)dx \\
&> 0
\end{aligned}$$

■

We define three kernel functions in terms of the original kernel function $K(x)$ and define their zeros.

Remark 4.2.8. *Then we define $-M_i$ as the first zero to the left of zero for the corresponding kernel function. Hence $K_1(-M_1) = 0$ and $K_2(-M_2) = 0$.*

Remark 4.2.9. *We note that $Z = \frac{C}{\varepsilon}$ for some value of C . and $0 < \varepsilon \ll 1$ so Z is large compared to other values in the network.*

Since Z is large, we can assume that Z is significantly larger than M . Hence $K(x) \approx K(x) - K(x - Z)$ on $(-\infty, 0)$ and $K(x) \approx K(x) - K(x + Z)$ on $(-\frac{Z}{2}, 0)$. Hence we know that $-M \approx -M_1 \approx -M_2$. Then let $\nu = \nu_i$ be such that $\int_{-M_i}^0 |x|g(x)K_i(x)dx = 0$ for $i = 1, 2$ and as before $\nu = \nu_0$ be such that $\int_{-M}^0 |x|g(x)K(x)dx = 0$. Then $K_1(x) > 0$ on $(-M_1, M_1)$ and $K_1(x) < 0$ on $(-\infty, -M_1) \cup (M_1, \infty)$. Since $K_1(x) < 0$ and $g(x) < 0$ on $(-\infty, -M_1)$, then

$$\int_{-M_1}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)]dx > 0.$$

Now we go back and consider $\frac{\partial \varphi_1}{\partial \nu}$ and $\frac{\partial \varphi_2}{\partial \nu}$.

Theorem 4.2.10. *Suppose $\nu > \max\{\nu_1, \nu_2\}$, then*

(I)

$$\begin{aligned}\frac{\partial\varphi_1}{\partial\nu} &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-\infty}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)] dx \\ &< 0.\end{aligned}$$

(II)

$$\begin{aligned}\frac{\partial\varphi_2}{\partial\nu} &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-\infty}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x + Z) - K(x)] dx \\ &> 0.\end{aligned}$$

Proof of I: For $\nu > \max(\nu_1, \nu_2)$ we have

$$\begin{aligned}\frac{\partial\varphi_1}{\partial\nu} &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-\infty}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)] dx \\ &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-\infty}^{-M_1} |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)] dx \\ &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\ &\cdot \int_{-M_1}^0 |x| \left\{ (1 - \omega_2)e^{\frac{\omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\omega_2}{\nu}x} \right\} [K(x) - K(x - Z)] dx \\ &< 0.\end{aligned}$$

Proof of II: So we consider the integral $\int_{-\infty}^0 |x|g(x)K_2(x)dx$ in three parts and show that it is positive.

$$\begin{aligned}\int_{-\infty}^0 |x|g(x)K_2(x)dx &= \int_{-\infty}^{-Z} |x|g(x)K_2(x)dx + \int_{-Z}^{-\frac{Z}{2}} |x|g(x)K_2(x)dx \\ &+ \int_{-\frac{Z}{2}}^0 |x|g(x)K_2(x)dx\end{aligned}$$

We consider the three parts of the integral and name them as follows:

$$\begin{aligned} I_1 &= \int_{-\frac{Z}{2}}^0 |x|g(x)K_2(x)dx \\ I_2 &= \int_{-Z}^{-\frac{Z}{2}} |x|g(x)K_2(x)dx \\ I_3 &= \int_{-\infty}^{-Z} |x|g(x)K_2(x)dx \end{aligned}$$

Then it suffices to show that I_1, I_3 , and $I_1 + I_2 > 0$ are each positive:

Lemma 4.2.11. For $\nu > \max\{\nu_1, \nu_2\}$,

$$(I) \quad I_1 = \int_{-\frac{Z}{2}}^0 |x|g(x)K_2(x)dx > 0.$$

$$(II) \quad I_1 + I_2 = \int_{-Z}^{-\frac{Z}{2}} |x|g(x)K_2(x)dx + \int_{-\frac{Z}{2}}^0 |x|g(x)K_2(x)dx > 0.$$

$$(III) \quad I_3 = \int_{-\infty}^{-Z} |x|g(x)K_2(x)dx > 0.$$

Proof of I: Now on $(-\frac{Z}{2}, -M_2)$, we have $K_2(x) < 0$, $g(x) < 0$ and $|x| > 0$. So $|x|g(x)K_2(x) > 0$ on $(-\frac{Z}{2}, -M_2)$, hence $\int_{-\frac{Z}{2}}^{-M_2} |x|g(x)K_2(x)dx > 0$. Now we are considering the fast pulse so we take only values of ν away from zero, mainly $\nu > \max(\nu_0, \nu_1, \nu_2)$. So $\int_{-M_2}^0 |x|g(x)K_2(x)dx > \int_{-M_2}^0 |x|g(x, \nu_3)K_2(x)dx = 0$. Hence,

$$\begin{aligned} I_1 &= \int_{-\frac{Z}{2}}^0 |x|g(x)K_2(x)dx \\ &= \int_{-\frac{Z}{2}}^{-M_2} |x|g(x)K_2(x)dx + \int_{-M_2}^0 |x|g(x)K_2(x)dx \\ &> 0 \end{aligned}$$

Proof of II: If $I_2 > 0$, then the proof is complete. If not, then $-I_2 > 0$ and $I_1 + I_2 = I_1 - (-I_2)$. We look at $-I_2$ and make a change of variables, $-y = x + Z$.

$$\begin{aligned}
-I_2 &= - \int_{-Z}^{-\frac{Z}{2}} |x|g(x) [K(x) - K(x+Z)] dx \\
&= - \int_0^{-\frac{Z}{2}} |-y-Z|g(-(y+Z)) [K(-(y+Z)) - K(-y)] (-dy) \\
&= \int_0^{-\frac{Z}{2}} |y+Z|g(-(y+Z)) [K(y+Z) - K(y)] dy \\
&= \int_0^{-\frac{Z}{2}} |x+Z|g(-(x+Z)) [K(x+Z) - K(x)] dx \\
&= \int_{-\frac{Z}{2}}^0 |x+Z|g(-(x+Z)) [K(x) - K(x+Z)] dx \\
&= \int_{-\frac{Z}{2}}^0 |x+Z|g(-(x+Z))K_2(x)dx
\end{aligned}$$

Then, we note that $|x+Z|g(-(x+Z)) < 0$ and $|x+Z||g(-(x+Z))| \ll 1$ on $(-\frac{Z}{2}, 0)$, then $|x|g(x) - |x+Z|g(-(x+Z)) > |x|g(x)$.

$$\begin{aligned}
I_1 - (-I_2) &= \int_{-\frac{Z}{2}}^0 |x+Z|g(-(x+Z))K_2(x)dx \\
&= \int_{-\frac{Z}{2}}^0 |x|g(x)K_2(x)dx - \int_{-\frac{Z}{2}}^0 |x+Z|g(-(x+Z))K_2(x)dx \\
&= \int_{-\frac{Z}{2}}^0 \{|x|g(x) - |x+Z|g(-(x+Z))\} K_2(x)dx \\
&= \int_{-\frac{Z}{2}}^{-M_2} \{|x|g(x) - |x+Z|g(-(x+Z))\} K_2(x)dx \\
&\quad + \int_{-M_2}^0 \{|x|g(x) - |x+Z|g(-(x+Z))\} K_2(x)dx \\
&> \int_{-\frac{Z}{2}}^{-M_2} \{|x|g(x) - |x+Z|g(-(x+Z))\} K_2(x)dx \\
&\quad + \int_{-M_2}^0 |x|g(x)K_2(x)dx \\
&> 0
\end{aligned}$$

Proof of III: Let $R > 0$ be such that $-\infty < -R < -Z$ and $K_2(-R) = 0$. Now on $(-\infty, -R)$, we have $K_2(x) > 0$, $|x|g(x) < 0$ and $|x|g(x)$ is decreasing and on $(-R, -Z)$, we have $K_2(x) < 0$, $|x|g(x) < 0$ and $|x|g(x)$ is decreasing.

$$\begin{aligned}
I_3 &= \int_{-\infty}^{-Z} |x|g(x)K_2(x)dx \\
&= \int_{-\infty}^{-R} |x|g(x)K_2(x)dx + \int_{-R}^{-Z} |x|g(x)K_2(x)dx \\
&> Rg(-R) \int_{-\infty}^{-R} K_2(x)dx + Rg(-R) \int_{-R}^{-Z} K_2(x)dx \\
&= Rg(-R) \int_{-\infty}^{-Z} K_2(x)dx \\
&= -R|g(-R)| \int_{-\infty}^{-Z} K_2(x)dx \\
&= -R|g(-R)| \int_{-\infty}^{-Z} K(x) - K(x+Z)dx \\
&= -R|g(-R)| \int_{-\infty}^0 K(x-Z) - K(x)dx \\
&= R|g(-R)| \int_{-\infty}^0 K(x) - K(x-Z)dx \\
&\geq R|g(-R)| \int_{-\infty}^0 K(x)dx \\
&= \frac{R|g(-R)|}{2} \\
&> 0.
\end{aligned}$$

This completes the proof of Lemma 4.2.11. ■

Hence,

$$\begin{aligned}
\frac{\partial \varphi_2}{\partial \nu} &= -\frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \\
&\cdot \int_{-\infty}^0 |x| \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} [K(x + Z) - K(x)] dx \\
&= \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} \int_{-\infty}^0 |x| g(x) K_2(x) dx \\
&= \frac{\alpha}{\nu^2(\omega_1 - \omega_2)} [I_1 + I_2 + I_3] \\
&> 0
\end{aligned}$$

which completes the proof of Theorem 4.2.10. ■

We now consider the derivatives with respect to Z of our speed index functions

$$\begin{aligned}
\varphi_1(\nu) &= \frac{\alpha\gamma}{\gamma + 1} \int_{-Z}^0 K(\xi) d\xi \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} [K(x) - K(x - Z)] dx
\end{aligned}$$

$$\begin{aligned}
\varphi_2(\nu) &= \frac{\alpha\gamma}{\gamma + 1} \int_0^Z K(\xi) d\xi \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^Z \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}(x-Z)} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}(x-Z)} \right\} [K(x) - K(x - Z)] dx, \\
&= \frac{\alpha\gamma}{\gamma + 1} \int_0^Z K(\xi) d\xi \\
&- \frac{\alpha}{\omega_1 - \omega_2} \\
&\cdot \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} [K(x + Z) - K(x)] dx.
\end{aligned}$$

It is easy to find

$$\begin{aligned}\frac{\partial \varphi_1}{\partial Z} &= \frac{\alpha \gamma}{\gamma + 1} K(-Z) - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K'(x - Z) dx \\ &= \frac{\alpha \gamma}{\gamma + 1} K(-Z) - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h(x) K'(x - Z) dx,\end{aligned}$$

$$\begin{aligned}\frac{\partial \varphi_2}{\partial Z} &= \frac{\alpha \gamma}{\gamma + 1} K(Z) - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K'(x + Z) dx \\ &= \frac{\alpha \gamma}{\gamma + 1} K(Z) - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h(x) K'(x + Z) dx.\end{aligned}$$

Theorem 4.2.12. *Suppose $\nu > \max\{\nu_1, \nu_2\}$, then*

(I)

$$\begin{aligned}\frac{\partial \varphi_1}{\partial Z} &= \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) K(x - Z) dx \\ &< 0.\end{aligned}$$

(II)

$$\begin{aligned}\frac{\partial \varphi_2}{\partial Z} &= \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) K(x + Z) dx \\ &< 0.\end{aligned}$$

Proof of I:

$$\begin{aligned}
\frac{\partial \varphi_1}{\partial Z} &= \frac{\alpha \gamma}{\gamma + 1} K(-Z) \\
&- \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K'(x - Z) dx \\
&= \frac{\alpha \gamma}{\gamma + 1} K(-Z) - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h(x) K'(x - Z) dx \\
&= \frac{\alpha \gamma}{\gamma + 1} K(-Z) - \frac{\alpha}{\omega_1 - \omega_2} h(x) K(x - Z) \Big|_{-\infty}^0 \\
&+ \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h'(x) K(x - Z) dx \\
&= \frac{\alpha \gamma}{\gamma + 1} K(-Z) - \frac{\alpha}{\omega_1 - \omega_2} h(0) K(-Z) + \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h'(x) K(x - Z) dx \\
&= \frac{\alpha \gamma}{\gamma + 1} K(-Z) - \frac{\alpha \gamma}{\gamma + 1} K(-Z) + \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h'(x) K(x - Z) dx \\
&= \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) K(x - Z) dx
\end{aligned}$$

It remains to show that $\frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) K(x - Z) dx < 0$.

$$\begin{aligned}
&- \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) K(x - Z) dx \\
&= \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) [-K(x - Z)] dx \\
&= \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^{-N} g(x) [-K(x - Z)] dx \\
&+ \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-N}^0 g(x) [-K(x - Z)] dx \\
&> \frac{\alpha}{\nu(\omega_1 - \omega_2)} [-K(-N - Z)] \int_{-\infty}^{-N} g(x) dx \\
&+ \frac{\alpha}{\nu(\omega_1 - \omega_2)} [-K(-N - Z)] \int_{-N}^0 g(x) dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha}{\omega_1 - \omega_2} [-K(-N - Z)] \int_{-\infty}^0 \frac{1}{\nu} g(x) dx \\
&= \frac{\alpha}{\omega_1 - \omega_2} [-K(-N - Z)] h(x) \Big|_{-\infty}^0 \\
&= \frac{\alpha}{\omega_1 - \omega_2} [-K(-N - Z)] h(0) \\
&= \frac{\alpha\gamma}{\gamma + 1} [-K(-N - Z)] \\
&> 0
\end{aligned}$$

Then, $\frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x)K(x - Z)dx < 0$ which completes the proof of I.

Proof of II:

$$\begin{aligned}
\frac{\partial\varphi_2}{\partial Z} &= \frac{\alpha\gamma}{\gamma + 1} K(Z) \\
&- \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu}x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu}x} \right\} K'(x + Z) dx \\
&= \frac{\alpha\gamma}{\gamma + 1} K(Z) - \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h(x) K'(x + Z) dx \\
&= \frac{\alpha\gamma}{\gamma + 1} K(Z) - \frac{\alpha}{\omega_1 - \omega_2} h(x) K(x + Z) \Big|_{-\infty}^0 \\
&+ \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h'(x) K(x + Z) dx \\
&= \frac{\alpha\gamma}{\gamma + 1} K(Z) - \frac{\alpha}{\omega_1 - \omega_2} h(0) K(Z) + \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h'(x) K(x + Z) dx \\
&= \frac{\alpha\gamma}{\gamma + 1} K(Z) - \frac{\alpha\gamma}{\gamma + 1} K(Z) + \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 h'(x) K(x + Z) dx \\
&= \frac{\alpha}{\nu(\omega_1 - \omega_2)} \int_{-\infty}^0 g(x) K(x + Z) dx
\end{aligned}$$

Then for all $K(x)$ that satisfy $\int_{-\infty}^0 g(x)K(x - Z)dx < 0$. We have $\frac{\partial\varphi_2}{\partial Z} < 0$. This completes the proof of Theorem 4.2.12

Thus by implicit function theorem and by the equations $\varphi_1(\nu, Z) = \theta$ and as $\varphi_2(\nu, Z) = \theta$, respectively, we find that the functions $\nu = \mathcal{A}(Z)$ and as $\nu = \mathcal{B}(Z)$

are well defined, such that $\varphi_1(\mathcal{A}(Z), Z) = \theta$ and $\varphi_2(\mathcal{B}(Z), Z) = \theta+$. We also note $\mathcal{A}(\varepsilon) = \mathcal{A}_\varepsilon$, $\mathcal{A}(+\infty) = \mathcal{A}_+$, $\mathcal{B}(\varepsilon) = \mathcal{B}_\varepsilon$ and $\mathcal{B}(+\infty) = \mathcal{B}_+$, where \mathcal{A}_ε , \mathcal{A}_+ , \mathcal{B}_ε and \mathcal{B}_+ , satisfy

$$\begin{aligned} & \theta - \frac{\alpha\gamma}{\gamma+1} \int_{-M}^0 K(\xi) d\xi & (4.17) \\ = & -\frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{A}_M} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{A}_M} x} \right\} [K(x) - K(x - M)] dx \end{aligned}$$

$$\begin{aligned} & \theta - \frac{\alpha\gamma}{2(\gamma+1)} & (4.18) \\ = & -\frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{A}_+} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{A}_+} x} \right\} K(x) dx \end{aligned}$$

$$\begin{aligned} & \theta - \frac{\alpha\gamma}{\gamma+1} \int_0^M K(\xi) d\xi & (4.19) \\ = & -\frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{B}_M} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{B}_M} x} \right\} [K(x + M) - K(x)] dx. \end{aligned}$$

$$\begin{aligned} & \theta - \frac{\alpha\gamma}{2(\gamma+1)} & (4.20) \\ = & \frac{\alpha}{\omega_1 - \omega_2} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{B}_+} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{B}_+} x} \right\} K(x) dx. \end{aligned}$$

The existence and uniqueness of each of the numbers \mathcal{A}_M , \mathcal{A}_+ , \mathcal{B}_M and \mathcal{B}_+ are guaranteed by the assumptions on the parameters. Differentiating the equations $\varphi_1(\nu, Z) = \theta$ and as $\varphi_2(\nu, Z) = \theta$ with respect to Z , we obtain

$$\frac{\partial \varphi_1}{\partial \nu}(\mathcal{A}(Z), Z)\mathcal{A}'(Z) + \frac{\partial \varphi_1}{\partial Z}(\mathcal{A}(Z), Z) = 0$$

$$\frac{\partial \varphi_2}{\partial \nu}(\mathcal{B}(Z), Z)\mathcal{A}'(Z) + \frac{\partial \varphi_1}{\partial Z}(\mathcal{B}(Z), Z) = 0.$$

Thus,

$$\mathcal{A}'(Z) = -\frac{\partial_Z \varphi_1(\mathcal{A}(Z), Z)}{\partial_\nu \varphi_1(\mathcal{A}(Z), Z)} < 0, \quad \mathcal{B}'(Z) = -\frac{\partial_Z \varphi_2(\mathcal{B}(Z), Z)}{\partial_\nu \varphi_2(\mathcal{B}(Z), Z)} > 0.$$

It remains to show that $\mathcal{A}_+ < \mathcal{B}_+$ and $\mathcal{A}_M > \mathcal{B}_M$.

Lemma 4.2.13. *For \mathcal{A}_+ and \mathcal{B}_+ satisfying equations (4.17)-(4.20), and if $2\theta(\gamma + 1) > \alpha\gamma$, then $\mathcal{A}_+ < \mathcal{B}_+$.*

Proof: From the equations 4.18 and 4.20 and the symmetric nature of $K(x)$, we can say that if $2\theta(\gamma + 1) > \alpha\gamma$, then

$$\begin{aligned} 0 &< \theta - \frac{\alpha\gamma}{\gamma + 1} \int_0^{\frac{1}{\sqrt{\varepsilon}}} K(\xi) d\xi \\ &= - \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{A}_+} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{A}_+} x} \right\} K(x) dx \\ &= \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{B}_+} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{B}_+} x} \right\} K(x) dx. \end{aligned}$$

So we know that,

$$\begin{aligned} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{B}_+} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{B}_+} x} \right\} K(x) dx &> 0 \\ \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\mathcal{A}_+} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\mathcal{A}_+} x} \right\} K(x) dx &< 0. \end{aligned}$$

Now we look at $\int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx$. We know the following,

$$\begin{aligned} \lim_{\nu \rightarrow 0^+} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx &= 0 \\ \lim_{\nu \rightarrow \infty} \int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx &> 0. \end{aligned}$$

We also know from previous lemmas, that $\int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx$ as a function of ν is decreasing on an initial interval $(0, \nu_0)$ and increasing on the remaining interval (ν_0, ∞) . Hence there exists a number $\nu_{0h} \in (\nu_0, \infty)$ such that $\int_{-\infty}^0 \left\{ \frac{1 - \omega_2}{\omega_1} e^{\frac{\omega_1}{\nu_{0h}} x} - \frac{1 - \omega_1}{\omega_2} e^{\frac{\omega_2}{\nu_{0h}} x} \right\} K(x) dx = 0$. Thus $\mathcal{A}_+ < \nu_{0h} < \mathcal{B}_+$. ■

Lemma 4.2.14. For \mathcal{A}_M and \mathcal{B}_M satisfying equations 4.17-4.20, and if

$$\theta < \frac{\alpha\gamma}{\gamma+1} \int_{-M}^0 K(x) dx, \text{ then } \mathcal{A}_M > \mathcal{B}_M.$$

Proof: Consider the graphs of $K(x) - K(x - M)$ and $K(x + M) - K(x)$. We note that $K(x) - K(x - M)$ is positive on $(-M - k, 0)$ for some small $k > 0$ and negative on $(-\infty, -M - k)$ since $K(x) - K(x - M)$ is a Mexican Hat kernel function and $K(x) - K(x - M) \approx K(x)$. Also we note that $K(x + M) - K(x)$ is positive on $(-2M - k, -\frac{M}{2})$ for some small $k > 0$ and negative on $(-\infty, -2M - k) \cup (-\frac{M}{2}, 0)$. $\theta < \frac{\alpha\gamma}{\gamma+1} \int_{-M}^0 K(x) dx$ implies that the integrals are positive. Based on earlier lemmas, we know that \mathcal{A}_M must be large enough so that $\int_{-\infty}^0 g(x, \nu)[K(x) - K(x - M)] dx > 0$. We can also see from the behavior of $g(x)$ and $K(x + M) - K(x)$ that \mathcal{B}_M must be small enough that $\int_{-\infty}^0 g(x, \nu)[K(x + M) - K(x)] dx > 0$. Hence we can conclude that $\mathcal{B}_M < \mathcal{A}_M$. This completes the proof of Lemma 4.2.14. ■

Now for all networks that satisfy the assumptions in Section 4.1.2, we have established that there is a unique solution ν_0, Z_0 that solves the equation $\varphi_1(\nu) = \varphi_2(\nu)$. Hence, the wave speed is unique.

4.3 Stability

We approach stability conditions and results by first deriving the eigenvalue problem and creating an Evans function. After establishing the Evans function we look at the zeros of the function and compare results to the linearized stability criterion to determine exponential stability. We begin here by linearizing the system and deriving the eigenvalue problem.

We rewrite the system in moving coordinates, mainly $z \equiv x + \nu t$ and set $(P(z, t), Q(z, t)) \equiv (u(x - \nu t, t), w(x - \nu t, t))$, then the system becomes

$$\nu P_z + P_t + P + Q = \alpha \int_{\mathbb{R}} K(z - y) H(P(y, t) - \theta) dy \quad (4.21)$$

$$\nu Q_z + Q_t = \varepsilon(P - \gamma Q). \quad (4.22)$$

Now consider the difference between the system above and the traveling wave solution $(U(z), W(z))$ and let $(p(z, t), q(z, t)) = (P(z, t) - U(z), Q(z, t) - W(z))$ which yields the new system

$$\begin{aligned} \nu p_z + p_t + p + q &= \alpha \int_{\mathbb{R}} K(z - y) [H(P(y, t) - \theta) - H(U(y) - \theta)] dy \\ \nu q_z + q_t &= \varepsilon(p - \gamma q). \end{aligned}$$

Using the same strategy as in [65], namely lemma 5, we obtain the linearization

$$\nu p_z + p_t + p + q = \alpha \frac{K(z)}{U'(0)} p(0, t) - \alpha \frac{K(z - Z)}{U'(Z)} p(Z, t) \quad (4.23)$$

$$\nu q_z + q_t = \varepsilon(p - \gamma q). \quad (4.24)$$

Let $\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in BC^1(R, C^2) \equiv [BC^1(R, C)]^2$ and define

$$\mathcal{N} = \alpha \frac{K(z)}{U'(0)} \xi(0, t) - \alpha \frac{K(z - Z)}{U'(Z)} \xi(Z, t) = a(z) \xi(\lambda, \varepsilon, 0) + b(z) \xi(\lambda, \varepsilon, Z).$$

Now define a linear operator and derive the eigenvalue problem,

$$\mathcal{L}(\varepsilon) = \begin{pmatrix} -\nu\xi_z - \xi - \eta + \mathcal{N} \\ -\nu\eta_z + \varepsilon(\xi - \gamma\eta) \end{pmatrix} = -\nu\frac{\partial\psi}{\partial z} - \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix} \psi + \begin{pmatrix} \mathcal{N} & 0 \\ 0 & 0 \end{pmatrix}.$$

The associated eigenvalue problem is $\mathcal{L}(\varepsilon)\psi = \lambda\psi$, more explicitly we have

$$\nu\frac{\partial\psi}{\partial z} + \left[\lambda I + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix} \right] \psi = \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}.$$

We now prove a series of lemmas that will help to verify the linearized stability criterion for the solution.

Lemma 4.3.1. *One solution of the eigenvalue problem $\mathcal{L}(\varepsilon)\psi = \lambda\psi$ is*

$$\begin{aligned} \Psi(\lambda, \varepsilon, z) &= \Phi(\lambda, \varepsilon, z) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} \\ &+ \frac{1}{\nu} \int_{-\infty}^z \Phi(\lambda, \varepsilon, z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N} dx, \end{aligned}$$

where c and d are complex functions of λ and ε .

Proof: From previous calculations, we know that the eigenvalues for the matrix

$\begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix}$ are $\omega_1(\varepsilon)$ and $\omega_2(\varepsilon)$, therefore we have the eigenvalues for this operator are $\lambda + \omega_1$ and $\lambda + \omega_2$ with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ \omega_1 - 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ \omega_2 - 1 \end{pmatrix}$. Hence when we consider the homogeneous system of differential equations

$$\nu\psi' + \left[\lambda I + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix} \right] \psi = 0,$$

we obtain a fundamental solution matrix

$$\Phi(\lambda, \varepsilon, z) = \begin{pmatrix} e^{-\frac{\lambda+\omega_1}{\nu}z} & e^{-\frac{\lambda+\omega_2}{\nu}z} \\ (\omega_1 - 1)e^{-\frac{\lambda+\omega_1}{\nu}z} & (\omega_2 - 1)e^{-\frac{\lambda+\omega_2}{\nu}z} \end{pmatrix}.$$

In order to solve the inhomogeneous case, we need the inverse of the fundamental solution matrix which we find to be

$$\Phi^{-1}(\lambda, \varepsilon, z) = \frac{1}{\omega_1 - \omega_2} \begin{pmatrix} (1 - \omega_2)e^{\frac{\lambda + \omega_1}{\nu}z} & e^{\frac{\lambda + \omega_1}{\nu}z} \\ (\omega_1 - 1)e^{\frac{\lambda + \omega_2}{\nu}z} & -e^{\frac{\lambda + \omega_2}{\nu}z} \end{pmatrix}. \quad (4.25)$$

We use the method of variation of parameters to solve the system of differential equations. In the process we compute the product of the fundamental solution matrix and its inverse as follows

$$\Phi(\lambda, \varepsilon, z)\Phi^{-1}(\lambda, \varepsilon, x) = \frac{1}{\omega_1 - \omega_2} \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$$

where

$$\psi_1 = \begin{pmatrix} (1 - \omega_2)e^{\frac{\lambda + \omega_1}{\nu}(x-z)} - (1 - \omega_1)e^{\frac{\lambda + \omega_2}{\nu}(x-z)} \\ -\varepsilon e^{\frac{\lambda + \omega_1}{\nu}(x-z)} + \varepsilon e^{\frac{\lambda + \omega_2}{\nu}(x-z)} \end{pmatrix}.$$

and

$$\psi_2 = \begin{pmatrix} e^{\frac{\lambda + \omega_1}{\nu}(x-z)} - e^{\frac{\lambda + \omega_2}{\nu}(x-z)} \\ (\omega_2 - 1)e^{\frac{\lambda + \omega_1}{\nu}(x-z)} - (\omega_1 - 1)e^{\frac{\lambda + \omega_2}{\nu}(x-z)} \end{pmatrix}.$$

The general solution of the linear system of differential equations is

$$\begin{aligned} \Psi(\lambda, \varepsilon, z) &= \Phi(\lambda, \varepsilon, z) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} \\ &+ \frac{1}{\nu} \int_{-\infty}^z \Phi(\lambda, \varepsilon, z)\Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N} dx \end{aligned}$$

■

We now attempt to find $\begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix}$ explicitly. We begin by defining

$$R(\lambda, \varepsilon, z) = \begin{pmatrix} -\frac{\lambda + \omega_1}{\nu} & 0 \\ e^{-\frac{\lambda + \omega_1}{\nu}z} & e^{-\frac{\lambda + \omega_2}{\nu}z} \end{pmatrix}.$$

Then we let $\Psi = \begin{pmatrix} \xi(\lambda, \varepsilon, z) \\ \eta(\lambda, \varepsilon, z) \end{pmatrix}$, $z = 0$ and multiplying both sides of the solution to the eigenvalue problem by the vector $(1 \ 0)$, we obtain the following equation

$$\begin{aligned} (1 \ 0) \begin{pmatrix} \xi(\lambda, \varepsilon, 0) \\ \eta(\lambda, \varepsilon, 0) \end{pmatrix} &= (1 \ 0) \Phi(\lambda, \varepsilon, 0) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} \\ &+ (1 \ 0) \frac{1}{\nu} \int_{-\infty}^0 \Phi(\lambda, \varepsilon, 0) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N} dx \end{aligned}$$

$$\begin{aligned} \xi(\lambda, \varepsilon, 0) &= R(\lambda, \varepsilon, 0) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} \\ &+ \frac{1}{\nu} \xi(\lambda, \varepsilon, 0) \int_{-\infty}^0 R(\lambda, \varepsilon, 0) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} a(x) \\ 0 \end{pmatrix} dx \\ &+ \frac{1}{\nu} \xi(\lambda, \varepsilon, Z) \int_{-\infty}^0 R(\lambda, \varepsilon, 0) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} b(x) \\ 0 \end{pmatrix} dx \end{aligned}$$

$$\begin{aligned} R(\lambda, \varepsilon, 0) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} &= \left[1 - \frac{1}{\nu} \int_{-\infty}^0 R(\lambda, \varepsilon, 0) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} a(x) \\ 0 \end{pmatrix} dx \right] \\ &\cdot \xi(\lambda, \varepsilon, 0) \\ &- \frac{1}{\nu} \xi(\lambda, \varepsilon, Z) \int_{-\infty}^0 R(\lambda, \varepsilon, 0) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} b(x) \\ 0 \end{pmatrix} dx \\ &= 1 - M_{11} \xi(\lambda, \varepsilon, 0) - M_{12} \xi(\lambda, \varepsilon, Z). \end{aligned}$$

Now we complete the same calculation, however, we use the second initial value $z = Z$ and multiply both sides of the solution to the eigenvalue problem by the vector $(1 \ 0)$, we obtain the following equation

$$\begin{aligned} (1 \ 0) \begin{pmatrix} \xi(\lambda, \varepsilon, Z) \\ \eta(\lambda, \varepsilon, Z) \end{pmatrix} &= (1 \ 0) \Phi(\lambda, \varepsilon, Z) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} \\ &+ (1 \ 0) \frac{1}{\nu} \int_{-\infty}^Z \Phi(\lambda, \varepsilon, Z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N} dx \end{aligned}$$

$$\begin{aligned}
\xi(\lambda, \varepsilon, Z) &= R(\lambda, \varepsilon, Z) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} \\
&+ \frac{1}{\nu} \xi(\lambda, \varepsilon, 0) \int_{-\infty}^Z R(\lambda, \varepsilon, Z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} a(x) \\ 0 \end{pmatrix} dx \\
&+ \frac{1}{\nu} \xi(\lambda, \varepsilon, Z) \int_{-\infty}^Z R(\lambda, \varepsilon, Z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} b(x) \\ 0 \end{pmatrix} dx
\end{aligned}$$

$$\begin{aligned}
R(\lambda, \varepsilon, Z) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} &= \left[1 - \frac{1}{\nu} \int_{-\infty}^Z R(\lambda, \varepsilon, Z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} b(x) \\ 0 \end{pmatrix} dx \right] \\
&\cdot \xi(\lambda, \varepsilon, Z) \\
&- \frac{1}{\nu} \xi(\lambda, \varepsilon, 0) \int_{-\infty}^Z R(\lambda, \varepsilon, Z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} a(x) \\ 0 \end{pmatrix} dx \\
&= -M_{21} \xi(\lambda, \varepsilon, 0) + (1 - M_{22}) \xi(\lambda, \varepsilon, Z).
\end{aligned}$$

Putting these two equations together we form a new equation and solve for the constant solution, $\begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix}$.

$$\begin{aligned}
\begin{pmatrix} R(\lambda, \varepsilon, 0) \\ R(\lambda, \varepsilon, Z) \end{pmatrix} \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} &= \left[I - \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right] \begin{pmatrix} \xi(\lambda, \varepsilon, 0) \\ \xi(\lambda, \varepsilon, Z) \end{pmatrix} \\
&= (I - M) \begin{pmatrix} \xi(\lambda, \varepsilon, 0) \\ \xi(\lambda, \varepsilon, Z) \end{pmatrix}.
\end{aligned}$$

$$\begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} = \begin{pmatrix} R(\lambda, \varepsilon, 0) \\ R(\lambda, \varepsilon, Z) \end{pmatrix}^{-1} (I - M) \begin{pmatrix} \xi(\lambda, \varepsilon, 0) \\ \xi(\lambda, \varepsilon, Z) \end{pmatrix}.$$

Hence our solution to the eigenvalue problem is

$$\begin{aligned}
\Psi(\lambda, \varepsilon, z) &= \Phi(\lambda, \varepsilon, z) \begin{pmatrix} R(\lambda, \varepsilon, 0) \\ R(\lambda, \varepsilon, Z) \end{pmatrix}^{-1} (I - M) \begin{pmatrix} \xi(\lambda, \varepsilon, 0) \\ \xi(\lambda, \varepsilon, Z) \end{pmatrix} \\
&+ \frac{1}{\nu} \int_{-\infty}^z \Phi(\lambda, \varepsilon, z) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{N} dx.
\end{aligned}$$

For the solution to the eigenvalue problem to be bounded on $(-\infty, \infty)$, then

$$\begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

However, $\begin{pmatrix} \xi(\lambda, \varepsilon, 0) \\ \xi(\lambda, \varepsilon, Z) \end{pmatrix} \neq 0$ so the solution is bounded iff $\det(I - M) = 0$. Hence we define the Evans function as

$$\mathcal{E}(\lambda, \varepsilon) = \det I - M$$

We define four functions as follows:

$$A(x) = \frac{1}{\nu(\omega_1 - \omega_2)} \left[(1 - \omega_2)e^{\frac{\lambda + \omega_1}{\nu}x} - (1 - \omega_1)e^{\frac{\lambda + \omega_2}{\nu}x} \right]$$

$$B(x) = \frac{1}{\nu(\omega_1 - \omega_2)} \left[-e^{\frac{\lambda + \omega_1}{\nu}x} + e^{\frac{\lambda + \omega_2}{\nu}x} \right]$$

$$C(x) = \frac{1}{\omega_1 - \omega_2} \left[\frac{1 - \omega_2}{\lambda + \omega_1} e^{\frac{\lambda + \omega_1}{\nu}x} - \frac{1 - \omega_1}{\lambda + \omega_2} e^{\frac{\lambda + \omega_2}{\nu}x} \right]$$

$$D(x) = \frac{1}{\omega_1 - \omega_2} \left[-\frac{1}{\lambda + \omega_1} e^{\frac{\lambda + \omega_1}{\nu}x} + \frac{1}{\lambda + \omega_2} e^{\frac{\lambda + \omega_2}{\nu}x} \right]$$

$$M(x) = \begin{pmatrix} \int_{-\infty}^0 A(x)a(x)dx & \int_{-\infty}^0 A(x)b(x)dx \\ \int_{-\infty}^Z A(x - Z)a(x)dx & \int_{-\infty}^Z A(x - Z)b(x)dx \end{pmatrix}$$

Hence,

$$\begin{aligned}
\mathcal{E}(\lambda, \varepsilon) &= \det(I - M) \\
&= \det \left[I - \begin{pmatrix} \int_{-\infty}^0 A(x)a(x)dx & \int_{-\infty}^0 A(x)b(x)dx \\ \int_{-\infty}^Z A(x-Z)a(x)dx & \int_{-\infty}^Z A(x-Z)b(x)dx \end{pmatrix} \right] \\
&= \det \left[I - \begin{pmatrix} \int_{-\infty}^0 A(x)a(x)dx & \int_{-\infty}^0 A(x)b(x)dx \\ \int_{-\infty}^0 A(x)a(x+Z)dx & \int_{-\infty}^0 A(x)b(x+Z)dx \end{pmatrix} \right] \\
&= \det \begin{pmatrix} 1 - \int_{-\infty}^0 A(x)a(x)dx & - \int_{-\infty}^0 A(x)b(x)dx \\ - \int_{-\infty}^0 A(x)a(x+Z)dx & 1 - \int_{-\infty}^0 A(x)b(x+Z)dx \end{pmatrix} \\
&= \begin{vmatrix} 1 - \frac{\alpha}{U'(0)} \int_{-\infty}^0 A(x)K(x)dx & - \frac{\alpha}{|U'(Z)|} \int_{-\infty}^0 A(x)K(x-Z)dx \\ - \frac{\alpha}{U'(0)} \int_{-\infty}^0 A(x)K(x+Z)dx & 1 - \frac{\alpha}{|U'(Z)|} \int_{-\infty}^0 A(x)K(x)dx \end{vmatrix} \\
&= \begin{vmatrix} \mathcal{E}_1(\lambda, \varepsilon) & \mathcal{E}_2(\lambda, \varepsilon) \\ \mathcal{E}_3(\lambda, \varepsilon) & \mathcal{E}_4(\lambda, \varepsilon) \end{vmatrix} \\
&= \mathcal{E}_1(\lambda, \varepsilon)\mathcal{E}_4(\lambda, \varepsilon) - \mathcal{E}_2(\lambda, \varepsilon)\mathcal{E}_3(\lambda, \varepsilon) \\
&= \left[1 - \frac{\alpha}{|U'(0)|} \int_{-\infty}^0 A(x)K(x)dx \right] \left[1 - \frac{\alpha}{|U'(Z)|} \int_{-\infty}^0 A(x)K(x)dx \right] \\
&\quad - \frac{\alpha}{|U'(Z)|} \frac{\alpha}{|U'(0)|} \int_{-\infty}^0 A(x)K(x-Z)dx \int_{-\infty}^0 A(x)K(x+Z)dx
\end{aligned}$$

We now investigate the behavior of each $\mathcal{E}_i(\lambda, \varepsilon)$. Let $C = \frac{\alpha}{\nu(\omega_1 - \omega_2)} > 0$, then

$$\mathcal{E}_1(\lambda, \varepsilon) = 1 - \frac{C}{|U'(0)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x) dx,$$

$$\mathcal{E}_2(\lambda, \varepsilon) = -\frac{C}{|U'(Z)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x - Z) dx,$$

$$\mathcal{E}_3(\lambda, \varepsilon) = -\frac{C}{|U'(0)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x + Z) dx,$$

$$\mathcal{E}_4(\lambda, \varepsilon) = 1 - \frac{C}{|U'(Z)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x) dx.$$

Lemma 4.3.2. $\mathcal{E}(\lambda, \varepsilon) \neq 0$, for all $0 < \varepsilon \ll 1$ and all $\lambda \neq 0$ with $\operatorname{Re} \lambda \geq 0$.

Proof: Using the fact that

$$\left| \int_{-\infty}^0 e^{\frac{1+\lambda}{\nu} x} K(x) dx \right| < \int_{-\infty}^0 e^{\frac{x}{\nu}} K(x) dx$$

from [64], we have the following results

$$\begin{aligned}
|\mathcal{E}_1(\lambda, \varepsilon)| &= \left| 1 - \frac{C}{|U'(0)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x) dx \right| \\
&= 1 - \frac{C}{|U'(0)|} \left| \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x) dx \right| \\
&> 1 - \frac{C}{|U'(0)|} \left| \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx \right| \\
&= \left| 1 - \frac{C}{|U'(0)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx \right| \\
&= |\mathcal{E}_1(0, \varepsilon)|
\end{aligned}$$

$$\begin{aligned}
|\mathcal{E}_2(\lambda, \varepsilon)| &= \left| \frac{C}{|U'(Z)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x - Z) dx \right| \\
&< \left| \frac{C}{|U'(Z)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} K(x - Z) dx \right| \\
&= |\mathcal{E}_2(0, \varepsilon)|
\end{aligned}$$

$$\begin{aligned}
|\mathcal{E}_3(\lambda, \varepsilon)| &= \left| \frac{C}{|U'(0)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x + Z) dx \right| \\
&< \left| \frac{C}{|U'(0)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} K(x + Z) dx \right| \\
&= |\mathcal{E}_3(\lambda, \varepsilon)|
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_4(\lambda, \varepsilon) &= \left| 1 - \frac{C}{|U'(Z)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x) dx \right| \\
&= 1 - \frac{C}{|U'(Z)|} \left| \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1 + \lambda}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2 + \lambda}{\nu} x} \right\} K(x) dx \right| \\
&> 1 - \frac{C}{|U'(Z)|} \left| \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx \right| \\
&= \left| 1 - \frac{C}{|U'(Z)|} \int_{-\infty}^0 \left\{ (1 - \omega_2) e^{\frac{\omega_1}{\nu} x} - (1 - \omega_1) e^{\frac{\omega_2}{\nu} x} \right\} K(x) dx \right| \\
&= |\mathcal{E}_4(0, \varepsilon)|
\end{aligned}$$

So then we have

$$\begin{aligned}\mathcal{E}(\lambda, \varepsilon) &= \begin{vmatrix} \mathcal{E}_1(\lambda, \varepsilon) & \mathcal{E}_2(\lambda, \varepsilon) \\ \mathcal{E}_3(\lambda, \varepsilon) & \mathcal{E}_4(\lambda, \varepsilon) \end{vmatrix} \\ &= \mathcal{E}_1(\lambda, \varepsilon)\mathcal{E}_4(\lambda, \varepsilon) - \mathcal{E}_2(\lambda, \varepsilon)\mathcal{E}_3(\lambda, \varepsilon)\end{aligned}$$

$$\begin{aligned}|\mathcal{E}(\lambda, \varepsilon)| &= |\mathcal{E}_1(\lambda, \varepsilon)| |\mathcal{E}_4(\lambda, \varepsilon)| - |\mathcal{E}_2(\lambda, \varepsilon)| |\mathcal{E}_3(\lambda, \varepsilon)| \\ &> |\mathcal{E}_1(0, \varepsilon)| |\mathcal{E}_4(0, \varepsilon)| - |\mathcal{E}_2(0, \varepsilon)| |\mathcal{E}_3(0, \varepsilon)| \\ &= |\mathcal{E}(0, \varepsilon)| = 0\end{aligned}$$

for all $\lambda \geq 0$. Hence there are no nonzero solutions of $\mathcal{E}(\lambda, \varepsilon) = 0$ inside $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$. This complete the proof of Lemma 4.3.2. ■

Lemma 4.3.3. *The asymptotic behavior for the Evans function is as follows $\mathcal{E}(\lambda, \varepsilon) \rightarrow 1$, as $|\lambda| \rightarrow \infty$, inside $\Omega(\varepsilon)$.*

Proof: We use Lebesgue's Dominated Convergence Theorem to find that

$$\lim_{|\lambda| \rightarrow \infty} M_{ij}(\lambda, \varepsilon) = 0$$

so we have that

$$\lim_{|\lambda| \rightarrow \infty} \mathcal{E}(\lambda, \varepsilon) = \lim_{|\lambda| \rightarrow \infty} \det(I - M(\lambda, \varepsilon)) = \lim_{|\lambda| \rightarrow \infty} \det(I) = 1$$

■

Lemma 4.3.4. *There exists a positive but small constant κ , which may depend on ε , such that there is no eigenvalue of $\mathcal{L}(\varepsilon)$ in the region $\{\lambda : \operatorname{Re} \lambda > -\kappa, \lambda \neq 0\}$.*

Proof: There exists a positive constant M where M may depend on ε such that there is no eigenvalue of the linear operator $\mathcal{L}(\varepsilon)$ outside the circle $|\lambda| = M$ in the complex

plane. Then consider the interior of the circle, $|\lambda| \leq M$ in the complex plane. Since the region inside the circle is compact it has at most finitely many eigenvalues of $\mathcal{L}(\varepsilon)$ contained in it. As we established in Lemma 4.3.2, there are no eigenvalues in the right half plane $\text{Re } \lambda \geq 0$ except $\lambda = 0$. Since there are finitely many eigenvalues in $|\lambda| \leq M$ and they all have $\text{Re } \lambda < 0$ except $\lambda = 0$, then choose $-\kappa$ as the real part of the eigenvalue with the smallest real component. This completes the proof of Lemma 4.3.4. ■

Lemma 4.3.5. *The eigenvalue $\lambda = 0$ is simple.*

Proof: If there is no bounded solution on $(-\infty, +\infty)$ to the following variation equations, then the traveling wave solution is exponentially stable. Consider

$$\nu \frac{\partial \psi}{\partial z} + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon \gamma \end{pmatrix} \psi + \begin{pmatrix} \varphi_z \\ \varphi_z \end{pmatrix} = \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}, \quad (4.26)$$

where

$$\mathcal{N}(0, \varepsilon, z) = \alpha \frac{K(z)}{U'(0)} \xi(0, \varepsilon, 0) - \alpha \frac{K(z-Z)}{U'(Z)} \xi(0, \varepsilon, Z) = a(z) \xi(0, \varepsilon, 0) + b(z) \xi(0, \varepsilon, Z).$$

Again using the method of variation of parameters, we obtain the solution

$$\begin{aligned} \Psi(0, \varepsilon, z) &= \Phi(0, \varepsilon, z) \begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} \\ &+ \frac{1}{\nu} \int_{-\infty}^z \Phi(0, \varepsilon, z) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} a(x) & 0 \\ b(x) & 0 \end{pmatrix} \begin{pmatrix} \xi(0, \varepsilon, 0) \\ \xi(0, \varepsilon, Z) \end{pmatrix} dx \\ &- \frac{1}{\nu} \int_{-\infty}^z \Phi(0, \varepsilon, z) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx, \end{aligned}$$

Now we take $z = 0$ and $z = Z$ in the first component of the solution. For $z = 0$ we get

$$\begin{aligned}
\xi(0, \varepsilon, 0) &= R(0, \varepsilon, 0) \begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} \\
&+ \frac{1}{\nu} \xi(0, \varepsilon, 0) \int_{-\infty}^0 R(0, \varepsilon, 0) \Phi^{-1}(0, \varepsilon, x) \\
&\cdot \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi(0, \varepsilon, 0) \\ \xi(0, \varepsilon, Z) \end{pmatrix} dx \\
&- \frac{1}{\nu} \int_{-\infty}^0 R(0, \varepsilon, 0) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx \\
R(0, \varepsilon, 0) \begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} &= \left[1 - \frac{1}{\nu} \int_{-\infty}^0 R(0, \varepsilon, 0) \Phi^{-1}(\lambda, \varepsilon, x) \begin{pmatrix} a(x) \\ 0 \end{pmatrix} dx \right] \\
&\cdot \xi(\lambda, \varepsilon, 0) \\
&- \frac{1}{\nu} \xi(0, \varepsilon, Z) \int_{-\infty}^0 R(0, \varepsilon, 0) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} b(x) \\ 0 \end{pmatrix} dx \\
&- \frac{1}{\nu} \int_{-\infty}^0 R(0, \varepsilon, 0) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx \\
&= 1 - M_{11} \xi(0, \varepsilon, 0) - M_{12} \xi(0, \varepsilon, Z) - T_1(0, \varepsilon).
\end{aligned}$$

And for $z = Z$ we get

$$\begin{aligned}
\xi(0, \varepsilon, Z) &= R(0, \varepsilon, Z) \begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} \\
&+ \frac{1}{\nu} \xi(\lambda, \varepsilon, 0) \int_{-\infty}^Z R(0, \varepsilon, Z) \Phi^{-1}(0, \varepsilon, x) \\
&\cdot \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi(0, \varepsilon, 0) \\ \xi(0, \varepsilon, Z) \end{pmatrix} dx \\
&- \frac{1}{\nu} \int_{-\infty}^Z R(0, \varepsilon, Z) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx \\
R(0, \varepsilon, Z) \begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} &= \left[1 - \frac{1}{\nu} \int_{-\infty}^Z R(0, \varepsilon, Z) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} b(x) \\ 0 \end{pmatrix} dx \right] \\
&\cdot \xi(0, \varepsilon, Z) \\
&- \frac{1}{\nu} \xi(0, \varepsilon, 0) \int_{-\infty}^Z R(0, \varepsilon, Z) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} a(x) \\ 0 \end{pmatrix} dx \\
&- \frac{1}{\nu} \int_{-\infty}^Z R(0, \varepsilon, Z) \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx \\
&= -M_{21} \xi(0, \varepsilon, 0) + (1 - M_{22}) \xi(0, \varepsilon, Z) - T_2(0, \varepsilon).
\end{aligned}$$

We now put the to equations together in matrix notation as follows

$$\begin{pmatrix} R(0, \varepsilon, 0) \\ R(0, \varepsilon, Z) \end{pmatrix} \begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} = \left[I - \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right] \begin{pmatrix} \xi(0, \varepsilon, 0) \\ \xi(0, \varepsilon, Z) \end{pmatrix} - \begin{pmatrix} T_1(0, \varepsilon) \\ T_2(0, \varepsilon) \end{pmatrix}$$

$$R(0, \varepsilon) \begin{pmatrix} c(\lambda, \varepsilon) \\ d(\lambda, \varepsilon) \end{pmatrix} = (I - M) \begin{pmatrix} \xi(0, \varepsilon, 0) \\ \xi(0, \varepsilon, Z) \end{pmatrix} - T(0, \varepsilon).$$

Note that

$$\begin{aligned}
\det R(0, \varepsilon) &= \det \begin{vmatrix} 1 & 1 \\ e^{-\frac{\omega_1}{\nu} Z} & e^{-\frac{\omega_2}{\nu} Z} \end{vmatrix} \\
&= e^{-\frac{\omega_2}{\nu} Z} - e^{-\frac{\omega_1}{\nu} Z} \\
&\neq 0.
\end{aligned}$$

Also we examine the components of $T(0, \varepsilon)$,

$$\begin{aligned} T_1(0, \varepsilon) &= R(0, \varepsilon, 0) \int_{-\infty}^0 \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx \\ &= O(1) \\ T_2(0, \varepsilon) &= R(0, \varepsilon, Z) \int_{-\infty}^Z \Phi^{-1}(0, \varepsilon, x) \begin{pmatrix} U_x(\varepsilon, x) \\ W_x(\varepsilon, x) \end{pmatrix} dx \\ &= O(1)e^{\frac{1}{\varepsilon}} \end{aligned}$$

Since $T(0, \varepsilon) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and it is not perpendicular to the null space of the matrix $[I - M(0, \varepsilon)]^T$, and $\det [I - M(0, \varepsilon)] = 0$, hence $\begin{pmatrix} c(0, \varepsilon) \\ d(0, \varepsilon) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This implies that there exists no bounded solution to the variation system 4.26. Hence, we have established the simplicity of the eigenvalue $\lambda = 0$. ■

Theorem 4.3.6. *The traveling pulse solution of the singularly perturbed system 4.1-4.2 of integral differential equations is exponentially stable as $t \rightarrow +\infty$.*

Proof: The proof follows from the linearized stability criterion and Lemmas 4.3.1 to 4.3.5. ■

We have now established that there is a traveling wave solution to the system of integral differential equations with a unique speed for the fast pulse and that the solution is exponentially stable.

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Appendix

The results in the Appendix have been used many times in Section 2, either explicitly or implicitly.

Appendix A

Representation of traveling wave front: linear sodium current

Let $m > 0$, $n \in \mathbb{R}$, $c > 0$, $\alpha > 0$ and $\theta > 0$ be real constants, where n is not necessarily positive, such that $n < \theta < n + \frac{\alpha}{2m}$. Then the integral differential equation

$$\frac{\partial u}{\partial t} + m(u - n) = \alpha \int_{\mathbb{R}} K(x - y) H \left(u \left(y, t - \frac{1}{c}|x - y| \right) - \theta \right) dy,$$

has a unique traveling wave front $U(z) = U(x + \mu_0 t)$:

$$\begin{aligned} U(z) &= n + \frac{\alpha}{m} \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \\ &\quad - \frac{\alpha}{m} \int_{-\infty}^z \exp \left[\frac{m}{\mu_0}(x - z) \right] \frac{c}{c + s(x)\mu_0} K \left(\frac{cx}{c + s(x)\mu_0} \right) dx. \end{aligned}$$

Recall that $s(x) = -1$ for $x < 0$, $s(0) = 0$ and $s(x) = 1$ for $x > 0$. The front satisfies

$$U(0) = \theta, \quad U < \theta \text{ on } (-\infty, 0), \quad U > \theta \text{ on } (0, \infty),$$

and the following boundary conditions

$$\lim_{z \rightarrow -\infty} U(z) = n, \quad \lim_{z \rightarrow \infty} U(z) = n + \frac{\alpha}{m}, \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0.$$

The wave speed μ_0 is a positive number and it is the unique solution of the equation

$$\phi(\mu) = \alpha \int_{-\infty}^0 \exp\left(m \frac{c - \mu}{c\mu} x\right) K(x) dx = \frac{\alpha}{2} - m\theta + mn.$$

Proof. There are two fixed points $U_0 = n$ and $U_1 = n + \frac{\alpha}{m}$ with $U_0 < \theta < U_1$. ■

Appendix B

Representation of traveling wave front: nonlinear sodium currents

Suppose that there is a real number n , such that $f(n) = 0$ and $f'(n) > 0$. The traveling wave front of (2.10) satisfies

$$\begin{aligned} U(z) &= n + \frac{\alpha}{f'(n)} \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx \\ &- \frac{\alpha}{f'(n)} \int_{-\infty}^z \exp\left[\frac{f'(n)}{\mu_0}(x-z)\right] \frac{c}{c+s(x)\mu_0} K\left(\frac{cx}{c+s(x)\mu_0}\right) dx \\ &+ \frac{1}{\mu_0} \int_{-\infty}^z \exp\left[f'(n)\frac{x-z}{\mu_0}\right] [f'(n)(U(x)-n) - f(U(x))] dx. \end{aligned}$$

Proof. Using simple techniques from differential equations, we can easily obtain the representation of the front. ■

Appendix C

Speed estimate - I

Let $\mu_1 > 0$, $m_1 > 0$ and $n_1 \in \mathbb{R}$ be real constants. Suppose that U is a solution of the differential inequality

$$\mu_1 U' + m_1(U - n_1) \leq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_1)} K(x) dx, \quad U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = n_1.$$

Then

$$\alpha \int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_1}{c\mu_1} x\right) K(x) dx \leq \frac{\alpha}{2} - m_1\theta + m_1 n_1.$$

Proof. Solving the given differential inequality, we find

$$\begin{aligned} m_1[U(z) - n_1] &\leq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_1)} K(x)dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left[\frac{m_1}{\mu_1}(x-z)\right] \frac{c}{c+s(x)\mu_1} K\left(\frac{cx}{c+s(x)\mu_1}\right) dx. \end{aligned}$$

Letting $z = 0$, we have

$$m_1(\theta - n_1) \leq \frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m_1 \frac{c - \mu_1}{c\mu_1} x\right) K(x)dx.$$

Therefore, the estimate follows. ■

Appendix D

Speed estimate - II

Let $\mu_2 > 0$, $m_2 > 0$ and $n_2 \in \mathbb{R}$ be real constants. Suppose that U is a solution of the differential inequality

$$\mu_2 U' + m_2(U - n_2) \geq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_2)} K(x)dx, \quad U(0) = \theta, \quad \lim_{z \rightarrow -\infty} U(z) = n_2.$$

Then

$$\frac{\alpha}{2} - m_2\theta + m_2n_2 \leq \alpha \int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_2}{c\mu_2} x\right) K(x)dx.$$

Proof. Solving the given differential inequality, we find

$$\begin{aligned} m_2[U(z) - n_2] &\geq \alpha \int_{-\infty}^{cz/(c+s(z)\mu_2)} K(x)dx \\ &\quad - \alpha \int_{-\infty}^z \exp\left[\frac{m_2}{\mu_2}(x-z)\right] \frac{c}{c+s(x)\mu_2} K\left(\frac{cx}{c+s(x)\mu_2}\right) dx. \end{aligned}$$

Letting $z = 0$, we have

$$m_2(\theta - n_2) \geq \frac{\alpha}{2} - \alpha \int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_2}{c\mu_2} x\right) K(x)dx.$$

Finally, we have the estimate

$$\frac{\alpha}{2} - m_2\theta + m_2n_2 \leq \alpha \int_{-\infty}^0 \exp\left(m_2 \frac{c - \mu_2}{c\mu_2} x\right) K(x) dx.$$

Therefore, the estimate follows. ■

Appendix E

Are the speeds of the fast traveling pulse solutions of (2.1)-(2.2) and (2.3)-(2.4) singular perturbations of the speeds of the fronts of (2.5) and (2.7), respectively?

In synaptically coupled neuronal networks, studying traveling wave fronts is basically a preparation for studying traveling pulse solutions. In this paper, we investigated how the speeds of fronts are influenced by sodium currents. It will make sense to investigate how wave speeds of pulses are influenced by sodium currents. For that purpose, we demonstrate that the wave speeds of the traveling pulse solutions of (2.1)-(2.2) and (2.3)-(2.4) are singular perturbations of the corresponding wave speeds of the traveling wave fronts of (2.5) and (2.7), respectively, namely, $|\mu_{\text{pulse}}(\varepsilon) - \mu_{\text{front}}| \leq C(\varepsilon)$, where $C(\varepsilon) > 0$ for $\varepsilon > 0$ and $C(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$.

Proof. Let us consider the system

$$\begin{aligned} \frac{\partial u}{\partial t} + f(u) + w &= \alpha \int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) dy, \\ \frac{\partial w}{\partial t} &= \varepsilon(g(u) - \gamma w), \end{aligned}$$

and the scalar equation

$$\frac{\partial u}{\partial t} + f(u) = \alpha \int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) dy.$$

These are general equations of (2.1)-(2.2) and (2.5) (by taking $c = \infty$), respectively, and of (2.3)-(2.4) and (2.7) (by taking $0 < c < \infty$), respectively. The traveling pulse

solution $(u(x, t), w(x, t)) = (U_{\text{pulse}}(\varepsilon, z), W_{\text{pulse}}(\varepsilon, z))$, where $z = x + \mu(\varepsilon)t$ and $\mu(\varepsilon)$ represents the wave speed, and the traveling wave front $u(x, t) = U_{\text{front}}(z)$, where $z = x + \mu_0 t$ and μ_0 represents the wave speed, satisfy

$$\begin{aligned}\mu(\varepsilon)U' + f(U) + W &= \alpha \int_{\mathbb{R}} K(z - y)H \left(U \left(y - \frac{\mu(\varepsilon)}{c}|z - y| \right) - \theta \right) dy, \\ \mu(\varepsilon)W' &= \varepsilon(g(U) - \gamma W),\end{aligned}$$

and

$$\mu_0 U' + f(U) = \alpha \int_{\mathbb{R}} K(z - y)H \left(U \left(y - \frac{\mu_0}{c}|z - y| \right) - \theta \right) dy,$$

respectively. These traveling waves are translation invariant. Without loss of generality, suppose that

$$\begin{aligned}U_{\text{pulse}}(\varepsilon, 0) &= U_{\text{pulse}}(\varepsilon, \mathcal{Z}(\varepsilon)) = \theta, \\ U_{\text{pulse}} &< \theta \text{ on } (-\infty, 0) \cup (\mathcal{Z}(\varepsilon), \infty), \quad U_{\text{pulse}} > \theta \text{ on } (0, \mathcal{Z}(\varepsilon)), \\ U_{\text{front}}(0) &= \theta, \quad U_{\text{front}} < \theta \text{ on } (-\infty, 0), \quad U_{\text{front}} > \theta \text{ on } (0, \infty).\end{aligned}$$

Let

$$\eta = y - \frac{\mu}{c}|z - y|.$$

Then

$$z - y = \frac{c}{c + s(z - \eta)\mu}(z - \eta).$$

By a series of change of variables, we find that the traveling waves satisfy the equations

$$\begin{aligned}\mu(\varepsilon)U' + f(U) + W &= \alpha \int_{c(z - \mathcal{Z}(\varepsilon))/(c + s(z - \mathcal{Z}(\varepsilon))\mu(\varepsilon))}^{cz/(c + s(z)\mu(\varepsilon))} K(x)dx, \\ \mu(\varepsilon)W' &= \varepsilon(g(U) - \gamma W),\end{aligned}$$

and

$$\mu_0 U' + f(U) = \alpha \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x) dx,$$

respectively. Letting $z = 0$ in these equations, we have

$$\begin{aligned} \mu(\varepsilon)U'(0) + f(\theta) + W(0) &= \alpha \int_{-c\mathcal{Z}(\varepsilon)/(c-\mu(\varepsilon))}^0 K(x) dx, \\ \mu(\varepsilon)W'(0) &= \varepsilon(g(\theta) - \gamma W(0)), \end{aligned}$$

and

$$\mu_0 U'(0) + f(\theta) = \frac{\alpha}{2}.$$

It is not difficult to show that $\mathcal{Z}(\varepsilon) = \frac{1}{\varepsilon}\mathcal{O}(1)$ as $\varepsilon \rightarrow 0$, see [61]. The traveling pulse solution also satisfies the estimate

$$|U_{\text{pulse}}(\varepsilon, z)| + |W_{\text{pulse}}(\varepsilon, z)| \leq C \exp(-\rho|z|), \text{ for all } z < 0,$$

for some positive constants C and ρ . Thus

$$\begin{aligned} W_{\text{pulse}}(\varepsilon, 0) &= \int_{-\infty}^0 \frac{\partial}{\partial z} W_{\text{pulse}}(\varepsilon, z) dz \\ &= \frac{\varepsilon}{\mu(\varepsilon)} \int_{-\infty}^0 [g(U_{\text{fast}}(\varepsilon, z)) - \gamma W_{\text{pulse}}(\varepsilon, z)] dz = \varepsilon \mathcal{O}(1). \end{aligned}$$

By a geometric singular perturbation theory, we also find

$$\left| \frac{\partial}{\partial z} U_{\text{pulse}}(\varepsilon, 0) - \frac{\partial}{\partial z} U_{\text{front}}(0) \right| \leq C\varepsilon.$$

Finally, we obtain the desired estimate

$$|\mu(\varepsilon) - \mu_0| \leq C\varepsilon.$$

Therefore, the speeds of the fast traveling pulse solutions of systems (2.1)-(2.2) and (2.3)-(2.4) are perturbations of speeds of traveling wave fronts of the scalar equations (2.5) and (2.7), respectively. ■

Appendix F

The Fitzhugh-Nagumo and Hodgkin-Huxley equations

To provide biophysical explanation of the nonlinear function $f(u)$ in the model equations, we review the famous Fitzhugh-Nagumo equations and the well-known Hodgkin-Huxley equations. The Fitzhugh-Nagumo equations ([4], [1], [22], [23] and [37]) are

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) - w, \\ \frac{\partial w}{\partial t} &= \varepsilon(u - \gamma w),\end{aligned}$$

where $0 < a < \frac{1}{2}$, $0 < \gamma < \frac{4}{(1-a)^2}$, and $0 < \varepsilon \ll 1$ are constants. The Hodgkin-Huxley equations ([34], [21] and [30]) are

$$\begin{aligned}C_M \frac{\partial u}{\partial t} &= D_M \frac{\partial^2 u}{\partial x^2} - G_{\text{Na}} m^3 h (u - u_{\text{Na}}) - G_{\text{K}} n^4 (u - u_{\text{K}}) - G_{\text{L}} (u - u_{\text{L}}), \\ \frac{\partial m}{\partial t} &= \alpha_m (1 - m) - \beta_m m, \\ \frac{\partial n}{\partial t} &= \alpha_n (1 - n) - \beta_n n, \\ \frac{\partial h}{\partial t} &= \alpha_h (1 - h) - \beta_h h,\end{aligned}$$

where

$$\begin{aligned}\alpha_m &= a_m \frac{V + b_m}{\exp\left(\frac{V+b_m}{b_m}\right) - 1}, & \beta_m &= c_m \exp\left(\frac{V}{d_m}\right), \\ \alpha_n &= a_n \frac{V + b_n}{\exp\left(\frac{V+b_n}{b_n}\right) - 1}, & \beta_n &= c_n \exp\left(\frac{V}{d_n}\right), \\ \alpha_h &= a_h \exp\left(\frac{V}{b_h}\right), & \beta_h &= \frac{1}{\exp\left(\frac{V+c_h}{d_h}\right) + 1},\end{aligned}$$

where (C_M, D_M) , $(G_{\text{Na}}, G_{\text{K}}, G_{\text{L}}, u_{\text{Na}}, u_{\text{K}}, u_{\text{L}})$, (a_m, b_m, c_m, d_m) , (a_n, b_n, c_n, d_n) , (a_h, b_h, c_h, d_h) are five groups of parameters. In general, the values of these parameters are different for different neurons. The empirical model which Hodgkin and Huxley

developed is not a physiological model based on physical laws or biological theory, but it is a model based on curve fitting by using exponential functions, see [30]. The Fitzhugh-Nagumo equations are singularly perturbed, simplified version of the Hodgkin-Huxley equations, see [37]. These reaction diffusion equations are more or less related to the integral differential equations mentioned in the Introduction.

As in the Fitzhugh-Nagumo equations and the Hodgkin-Huxley equations, $f(u)$ usually describes the sodium currents. Recall that the maximum of sodium conductance is G_{Na} and the probability that a sodium channel is open is m^3h . Therefore the sodium conductance is $G_{\text{Na}}m^3h$. The number u_{Na} is called the sodium reversal potential and the driving potential is $u - u_{\text{Na}}$. Using Ohm's law, the sodium current is modeled by $I_{\text{Na}} = G_{\text{Na}}m^3h(u - u_{\text{Na}})$, see [1] and [21]. The sodium reversal potential is the value of the membrane potential when the sodium concentration which produces an inward flux of sodium through the sodium channel, is balanced by the electrical potential gradient tending to move sodium ions in the channel in the opposite direction. The sodium channel depends on two variables m and u . There are two gates to each sodium channel. The function $f(u)$ characterizes when the gates are open so that the sodium ions may flow through, see [1]. As is well known, sodium currents may be influenced by many factors, such as the conductance of sodium channels, the distribution of sodium channels across the membrane, interior concentration and exterior concentration of sodium ions, activation m and deactivation h of sodium channels. Notice that the activation and deactivation of sodium channels may be influenced by many things, see the differential equations satisfied by m and h in the Hodgkin-Huxley equations. The investigation of influence of sodium currents on wave speeds should be of interest to biologists and mathematicians.

Vita

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Education

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