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Regular Functionals on Seaweed Lie Algebras

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by

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Regular Functionals on Seaweed Lie Algebras

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Abstract

The index of a Lie algebra $\mathfrak{g}$ is defined by ind $\mathfrak{g} = \min_{f\in\mathfrak{g}^*} \dim(\ker(B_f))$, where f is an element of the linear dual $\mathfrak{g}^*$ and $B_f(x,y) = f([x,y])$ is the associated skew-symmetric Kirillov form. We develop a broad general framework for the explicit construction of regular (index realizing) functionals for seaweed subalgbras of $\mathfrak{gl}(n)$ and the classical Lie algebras in type-A and type-C. (Type-B and type-D are also considered - though subtle cases remain open).) Until now, this significant problem has remained open in all cases.

Chapter 1

Introduction

Notation: All Lie algebras $(\mathfrak{g}, [\cdot, \cdot])$ will be finite dimensional over the complex numbers.

The *index* of a Lie algebra $\mathfrak{g}$ is an important algebraic invariant which was first formally introduced by Dixmier ([**15**], 1974). It is defined by

$$\text{ind } \mathfrak{g} = \min_{f \in \mathfrak{g}^*} \dim(\ker(B_f)),$$

where f is an element of the linear dual $\mathfrak{g}^*$ and B_f is the associated skew-symmetric *Kirillov form* defined by

$$B_f(x, y) = f([x, y]), \text{ for all } x, y \in \mathfrak{g},$$

and

$$\ker B_f = \{x \in \mathfrak{g} \mid f([x, y]) = 0, \text{ for all } \ y \in \mathfrak{g}\}.$$

In this thesis, we focus on a class of matrix algebras called *seaweed algebras*, or simply "seaweeds". These algebras were first introduced by Dergachev and A. Kirillov in ([**13**], 2000), where they defined such algebras as subalgebras of $\mathfrak{gl}(n)$ preserving certain flags of subspaces developed from two compositions of n (see Definition 2.1.1). The passage to the classical types is accomplished by requiring that elements of the seaweed subalgebra of $\mathfrak{gl}(n)$ satisfy additional algebraic conditions. For example, the Type-A case ($A_{n-1} = \mathfrak{sl}(n)$) is defined by a vanishing trace condition.

Here, while we have more than a passing interest in the index theory of seaweed subalgebras of the classical families of Lie algebras, our main focus is on the production of index-realizing functionals in these types. On a given $\mathfrak{g}$, such index-realizing functionals are called *regular* and exist in profusion, being dense in both the Zariski and Euclidean topolgies of $\mathfrak{g}^*$ (see **[24]**). Even so, methods for explicitly constructing regular functionals are few.

One such method is due indirectly to Kostant. In 1960, and after the fashion of the Gram-Schmidt orthogonalization process, Kostant developed an algorithm, called a *cascade*[1], that produces a set of strongly orthogonal roots from a root system which defines a Lie algebra. In 2004, Tauvel and Yu **[27]** noted that, in many cases, and as a byproduct of this process, a regular functional could be constructed using representative elements in $\mathfrak{g}$ of the root spaces for the highest roots generated by the cascade. For *Frobenius* (index zero) seaweeds, the cascade will always produce a regular (Frobenius) functional (see **[21]**, p. 19), but in the non-Frobenius case, the cascade will often fail (see Examples B.2.6, B.2.9, B.2.10, and B.2.12). Beyond the Kostant cascade and prior to our work here, we knew of no algorithmic procedure that will produce a regular functional – and Joseph has noted that the resolution of this gap is a significant open problem noted by Joseph in even the Type-A case (see **[21]**, p. 774).

Why the cascade fails to produce a regular functional, or rather for what seaweeds it fails, is the starting point for our investigation. We find that the obstruction to the cascade producing a regular functional is linked to the component structure of a certain planar graph, called a *meander*, which is associated with a given seaweed.

[1] See (**[20]**, 1976) for an early description of the cascade by Anthony Joseph. This paper cites Bertram Kostant and Jacques Tits as discovering this process independently. Tits is cited as (**[28]**, 1960), although the author could not find the original article, and Kostant is cited as private communication with no specified year. For a more recent paper on the cascade by Kostant, see (**[22]**, 2012).

Meanders were introduced in **[13]** by Dergachev and A. Kirillov, where they showed that the index of a seaweed subalgebra of $\mathfrak{gl}(n)$ or $\mathfrak{sl}(n)$ could be computed by an elementary combinatorial formula based on the number and type of the connected components of the meander associated with the seaweed. Even so, significant computational complexity persists. This complexity can be can be mitigated somewhat by "winding-down" the meander through a sequence of deterministic graph-theoretic moves ("winding-down moves") which yields the meander's essential configuration which we call the meander's *homotopy type*. (Reversing the winding-down moves yields "winding-up" moves from which any meander (and so any seaweed), of any size or configuration, can be constructed.)[2]

The strategy for producing a regular functional on a seaweed $\mathfrak{g}$ in the various classical types is to first develop an explicit regular functional F_n on $\mathfrak{gl}(n)$. At this juncture, we make critical use of $\mathfrak{g}$'s meander by showing how the meander's components identify a configuration of admissible positions in $\mathfrak{g}$. Each of these configurations contains certain square matrix blocks $C_{c\times c}$, called *core blocks of the configuration.* The union of all core blocks over all configurations constitutes $\mathfrak{C}$ - the *core of the seaweed* $\mathfrak{g}$. To build a regular functional on $\mathfrak{g}$, one inserts a copy of F_c into $C_{c\times c}$ for all C in $\mathfrak{C}$, and zeros elsewhere in the admissible locations of $\mathfrak{g}$ – with some exceptions based on aspects of the seaweed's "shape". The regularity of the adjusted functional is established by Theorem 3.2.8. This not only resolves the open problem for seaweed subalgebras of $\mathfrak{gl}(n)$, but (in conjunction with Theorem 3.3.1) more broadly delivers an algorithmic procedure for constructing a regular functional on a seaweed in the classical Type-A and Type-C - as well as a large class of those in Type-B and Type-D.

The organization of this dissertation is as follows. In Chapter 2, we define the fundamental objects necessary for our study of seaweed subalgebras $\mathfrak{g}$ of $\mathfrak{gl}(n)$. In

[2]In **[9]**, Coll et al extended this formulaic construction to the Type-C and Type-B cases (cf., **[26]**). More recently, Cameron (in his 2019 Ph.D. thesis at Lehigh University **[4]**) has extended these results to the Type-D case, thus completing the combinatorial classification of seaweeds in the classical types (cf., **[19]**).

particular, we review the construction of meanders and discuss homotopy types and the associated winding-up and winding-down machinery.

In Chapter 3, we develop a broad analytic framework for producing regular functionals on a given seaweed $\mathfrak{g}$. An important first step is the development of a *relations matrix* B associated with the space $\ker(B_F)$ for $F \in \mathfrak{g}^*$. A relations matrix is a bookkeeping device which encodes, among other things, the degrees of freedom in the system of the equations which define $\ker(B_F)$.

Associated with a relations matrix B is a non-unique set P of matrix positions (i,j). The entries in the remaining positions of B are explicitly determined as linear combinations of the entries in the positions in P. Each assignment $\vec{b} = (b_{i,j} \mid (i,j) \in P)$ of complex numbers to the positions in P yields an element of $\ker(B_F)$, so $\dim\ker(B_F) = |P|$, and the resulting kernel elements span $\ker(B_F)$.

Now, for each winding up move m of Lemma 2.3.4 (which can be construed as yielding an algebra $m(\mathfrak{g})$), we define maps $\mathfrak{f}_m : \mathfrak{g}^* \to m(\mathfrak{g})^*$ which send a functional F to a functional $\mathfrak{f}_m(F) \in m(\mathfrak{g})^*$. If m is not a Component Creation move, we will have that $\dim\ker(B_{\mathfrak{f}_m(F)}) = \dim\ker(B_F)$. When m is a Component Creation $C(c)$ we utilize a new functional $F_c \in \mathfrak{gl}(c)^*$ and $F \mapsto F_C \oplus F$. In this case, we will have $\dim\ker(B_{\mathfrak{f}_{C(c)}(F)}) = \dim\ker(B_F) - c$.

The inductive proof method accomplishes more than just showing that $\ker(B_F)$ has the "right" dimension as long as $F \in \mathfrak{gl}(c)^*$ was regular. it provides an explicit relations matrix of $\ker(B_F)$. The chapter closes by providing an explicit regular functional $F_n \in \mathfrak{gl}(n)^*$ (see Theorem 3.3.1). In Section 3.4 we develop several other regular functionals in $\mathfrak{gl}(n)^*$.

Chapter 4 begins by reviewing the classification of simple Lie algebras by their root systems. Of particular interest are the seaweed subalgebras of the classical Lie algebras $A_n = \mathfrak{sl}(n+1)$, $B_n = \mathfrak{so}(2n+1)$, $C_n = \mathfrak{sp}(2n)$, and $D_n = \mathfrak{so}(2n)$. We note that the Kostant cascade can fail to produce a regular functional on a given seaweed subalgebra of classical type.[3] The obstruction to a successful cascade is a

[3]In **[3]**, Baur and Moreau remark that the cascade successfully produces a regular functional for any seaweed of classical type. This is NOT true (see Examples B.2.6, 4.3, B.2.9, B.2.10, and B.2.12).

certain homotopy type (see Conjecture 5.2.5). The main result of Chapter 4 (and of this thesis) is to show how the framework of Chapter 3 can be leveraged to create a regular functional on seaweed subalgebras of classical type (i.e., a subalgebra of A_n, B_n, C_n, or D_n.)

Chapter 5 presents directions for future work and details questions which arise from our current study.

Chapter 2

Preliminaries

2.1 Seaweed Subalgebras of $\mathfrak{gl}(n)$

Definition 2.1.1. *A **Lie Algebra** over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ together with a skew-symmetric bilinear operator*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \text{ for all } x,y,z \in \mathfrak{g}.$$

Ongoing, we tacitly assume that all Lie algebras $\mathfrak{g}$ are finite-dimensional over the complex numbers and by Ado's theorem (see **[1]**) may therefore be assumed to be subalgebras of $\mathfrak{gl}(n)$.

We assume that $\mathfrak{g}$ comes equipped with a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{h} \oplus \mathfrak{u}_-,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{u}_+$ and $\mathfrak{u}_-$ consist of the upper and lower triangular matrices, respectively. Let $e_{i,j}$ be the matrix in $\mathfrak{g}$ with a 1 in position (i,j) and 0's elsewhere, and let e_i represent the vector in $\mathbb{C}^n$ with a 1 in the i^{th} position and 0's elsewhere. Denote by $\mathfrak{g}^*$ the linear dual of $\mathfrak{g}$ and let $e^*_{i,j}$ be the element in $\mathfrak{g}^*$ which chooses the field entry in position (i,j) from a matrix in $\mathfrak{g}$.

The basic objects of our study are the evocatively named "seaweed" Lie algebras first introduced by Dergacev and A. Kirillov in [**13**] and defined as follows.

Definition 2.1.2. *Let* $(a_1, \cdots, a_m)$ *and* $(b_1, \cdots, b_t)$ *be two compositions of* n*, and let* $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{C}^n$*, and* $\mathbb{C}^n = W_0 \supset W_1 \supset \cdots \supset W_t = \{0\}$*, where* $V_i = span\{e_1, \cdots, e_{a_1+\cdots+a_i}\}$ *and* $W_j = span\{e_{b_1+\cdots+b_j+1}, \cdots, e_n\}$*. The* ***standard***[1] ***seaweed***[2] $\mathfrak{g}$ *of* ***type*** $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ *is the subalgebra of* $\mathfrak{gl}(n)$ *which preserves the spaces* V_i *and* W_j *(see Example 2.1.3).*

Example 2.1.3. *We construct the seaweed* $\mathfrak{g}$ *of type* $\frac{4|1}{2|1|2}$*. Let* $X = [x_{i,j}] \in \mathfrak{g}$*. Definition 2.1.2 yields the following vector spaces.*

1. $V_0 = \{0\}$,
2. $V_1 = span\{e_1, e_2, e_3, e_4\}$,
3. $V_2 = \mathbb{C}^5$,
4. $W_0 = \mathbb{C}^5$,
5. $W_1 = span\{e_3, e_4, e_5\}$,
6. $W_2 = span\{e_4, e_5\}$,
7. $W_3 = \{0\}$.

Requiring that $Xv_i \in V_i$ *and* $Xw_j \in W_j$ *for all* i, j *forces that any nonzero entries of* X *can only occur in starred locations illustrated in Figure 2.1 (left). In general, the seaweed of type* $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ *can only have nonzero entries illustrated in the grey region of Figure 2.1 (right).*

[1]Standard is defined with respect to a specific Borel (maximal solvable) subalgebra. We use the upper triangular matrices.

[2]Joseph elsewhere calls these algebras *biparabolic*(cf., Definition 4.1.6).

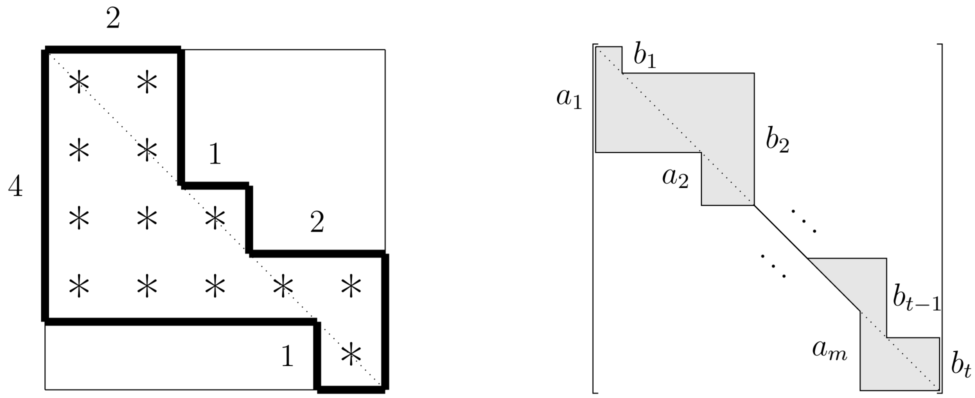


Figure 2.1: Seaweeds of type $\frac{4|1}{2|1|2}$ (left) and type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ (right)

To each seaweed of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ we associate a planar graph called a ***meander***, constructed as follows. First, place n vertices v_1 through v_n in a horizontal line. Next, create two partitions of the vertices by forming ***top*** and ***bottom blocks*** of vertices of size a_1, a_2, $\cdots$, a_m, and b_1, b_2, $\cdots$, b_t, respectively. Place edges in each top (likewise bottom) block in the same way. Add an edge from the first vertex of the block to the last vertex of the same block. Repeat this edge addition on the second vertex and the second to last vertex within the same block and so on within each block of both partitions. Top edges are drawn concave down and bottom edges are drawn concave up. We say that the meander is of **type** $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ (see Example 2.1.4).

Example 2.1.4. *Here, we construct the meander of type* $\frac{4|1}{2|1|2}$ *(cf., Example 2.1.3). The actual meander appears in Figure 2.2 (right). The bars and numbers in Figure 2.2 (left) are visual aids for the construction of the edges.*

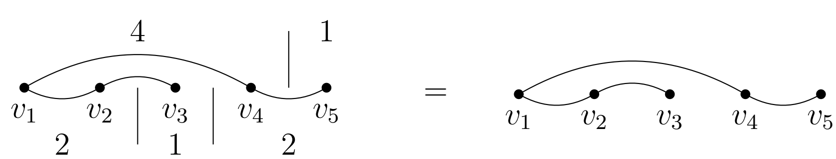


Figure 2.2: Meander of type $\frac{4|1}{2|1|2}$

Remark 2.1.5. *A meander can be visualized inside its associated seaweed* $\mathfrak{g}$ *if one views the diagonal entries* $e_{i,i}$ *of* $\mathfrak{g}$ *as the* n *vertices* v_i *of the meander and reckons the top edges* (v_i, v_j) *with* $i < j$ *of the meander as the unions of line segments connecting the matrix locations* $(i,i) \to (j,i) \to (j,j)$ *and the bottom edges* (v_i, v_j) *with* $i < j$ *of the meander as the unions of line segments connecting the matrix locations* $(i,i) \to (i,j) \to (j,j)$*. See Figure 2.3.*

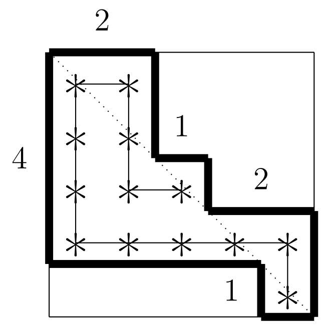


Figure 2.3: Meander of type $\frac{4|1}{2|1|2}$ visualized in its seaweed

2.2 The Index of a Seaweed Algebra

Recall from the Introduction that the index of a Lie algebra $\mathfrak{g}$ is defined by

$$\text{ind } \mathfrak{g} = \min_{f \in \mathfrak{g}^*} \dim \ker B_f.$$

Using the meander associated with a Lie algebra, Dergachev and A. Kirillov provide a combinatorial formula for the index of $\mathfrak{g}$ in terms of the number and type of the meander's connected components.

Theorem 2.2.1 (Dergachev and A. Kirillov [**13**])**.** *If* $\mathfrak{g}$ *is a seaweed subalgebra of* $\mathfrak{gl}(n)$ *and* M *is its associated meander, then*

$$\textit{ind } \mathfrak{g} = 2C + P,$$

where C *is the number of cycles and* P *is the number of paths in* M*.*

We have the following immediate Corollary.

Theorem 2.2.2. *The Lie algebra* $\mathfrak{gl}(n)$ *has index* n.

Proof. The Lie algebra $\mathfrak{gl}(n)$ is a seaweed of type $\frac{n}{n}$. If n is even, then the meander associated with $\mathfrak{gl}(n)$ consists of $\frac{n}{2}$ nested cycles (see Figure 2.4). By Theorem 2.2.1, the index is $2 \cdot \frac{n}{2} + 0 = n$. If n is odd, then the meander of type $\frac{n}{n}$ consists of $\lfloor \frac{n}{2} \rfloor$ nested cycles with a vertex (degenerate path) in the interior (see Figure 2.4). Therefore, by Theorem 2.2.1, the index of $\mathfrak{gl}(n)$ is $2 \cdot \lfloor \frac{n}{2} \rfloor + 1 = (n-1) + 1 = n$.

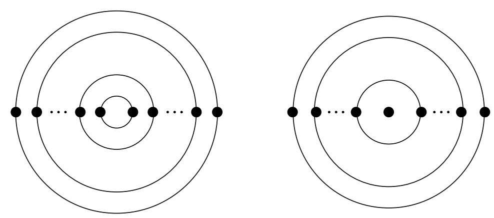


Figure 2.4: Meander of type $\frac{n}{n}$ for n even (left) and n odd (right)

□

Note that in Example 2.1.4, the meander of $\mathfrak{g}$ consists of exactly one path, so $\text{ind}\,\mathfrak{g} = 1$ for $\mathfrak{g}$ of type $\frac{4|1}{2|1|2}$. The combinatorial formula in Theorem 2.2.1 is elegant, but computational complexity persists (see Example 2.2.3).

Example 2.2.3. *Let* $\mathfrak{g}$ *be the seaweed of type* $\frac{2|6|18|6|11}{5|18|6|14}$ *(see Figure 2.5).*

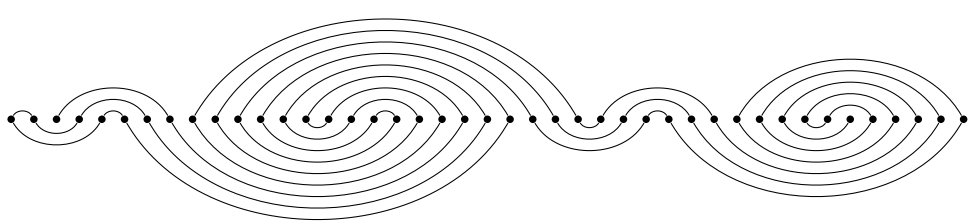


Figure 2.5: Meander of type $\frac{2|6|18|6|11}{5|18|6|14}$

This meander consists of one path and one cycle, so by Theorem 2.2.1, $\mathfrak{g}$ is of index 3. These components of the meander are colored in Figure 2.6.

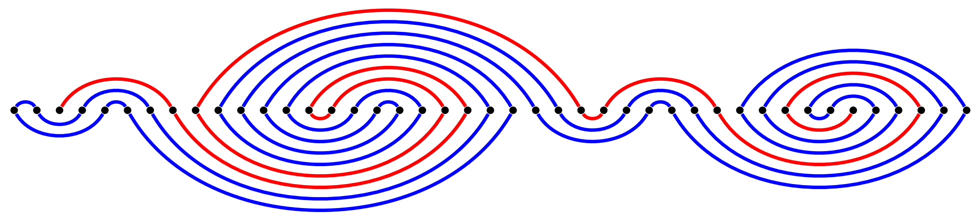


Figure 2.6: Cycle (blue) and path (red) of the meander of type $\frac{2|6|18|6|11}{5|18|6|14}$

2.3 The Homotopy Type of a Seaweed Algebra

To aid in the application of Theorem 2.2.1, and following Coll et al, [**7**], we note that any meander can be contracted or "wound down" to the empty meander through a sequence of graph-theoretic moves; each of which is uniquely determined by the structure of the meander at the time of the move application. In [**7**], the authors established that there are five such moves, only one of which affects the component structure of the graph and is therefore the only move capable of modifying the index of the meander which we define as the index of the associated seaweed. Since the sequence of moves which contracts a meander to the empty meander uniquely identifies the graph, we call the sequence the meander's signature. Although developed independently in [**12**], the authors in find that the signature is essentially a graph theoretic recasting of Panyushev's reduction algorithm, which in [**25**] was used to develop inductive formulas for the index of seaweeds in $\mathfrak{gl}(n)$. There, these inductive formulas are expressed in terms of elementary functions, which are laid plain by the explicit nature of the signature.

Lemma 2.3.1 (Coll, Hyatt, and Magnant [**11**])**.** *Let $\mathfrak{g}$ be a seaweed of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ with associated meander M. Create a meander M' by one of the following moves.*

1. ***Block Elimination (B):*** *If* $a_1 = 2b_1$*, then* $M \mapsto M'$ *of type* $\frac{b_1|a_2|\cdots|a_m}{b_2|b_3|\cdots|b_t}$.

2. ***Rotation Contraction (R):*** *If* $b_1 < a_1 < 2b_1$*, then* $M \mapsto M'$ *of type* $\frac{b_1|a_2|\cdots|a_m}{(2b_1-a_1)|b_2|\cdots|b_t}$.

3. ***Pure Contraction (P):*** *If* $a_1 > 2b_1$*, then* $M \mapsto M'$ *of type* $\frac{(a_1-2b_1)|b_1|a_2|\cdots|a_m}{b_2|b_3|\cdots|b_t}$.

4. ***Flip (F):*** *If* $a_1 < b_1$*, then* $M \mapsto M'$ *of type* $\frac{b_1|b_2|\cdots|b_t}{a_1|\cdots|a_m}$.

5. ***Component Deletion (C(c)):*** *If* $a_1 = b_1 = c$*, then* $M \mapsto M'$ *of type* $\frac{a_2|\cdots|a_m}{b_2|\cdots|b_t}$.

These moves are called ***winding-down moves****. For all moves, except the component deletion move,* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *(the seaweed with meander* M'*) have the same index.*

Given a meander M, there exists a unique sequence of moves (elements of the set $\{B, R, P, F, C(c)\}$) which reduces M to the empty meander. This sequence is called the ***signature*** of M. If $C(c_1), \cdots, C(c_h)$ are the component deletion moves which appear (in order) in the signature of M, then M's ***homotopy type***, denoted $H(c_1, \cdots, c_h)$, is the meander of type $\frac{c_1|\cdots|c_h}{c_1|\cdots|c_h}$. The individual meanders of type $\frac{c_i}{c_i}$ for each i are the ***components*** of the homotopy type, with the numbers c_i referred to as the ***sizes*** of the respective components. We similarly refer to the homotopy type, signature, and components of a seaweed $\mathfrak{g}$ to be the corresponding object of the meander associated with $\mathfrak{g}$. Analogously, the index of a meander is taken to be the index of its associated seaweed.

Example 2.3.2. *Let* M *be the meander of type* $\frac{17|3}{10|4|6}$*. By repeated applications of Lemma 2.3.1,* M *has signature* $RPC(4)FBC(3)$*. The unwinding of* M *is demonstrated in Figure 2.7, and the homotopy type of* M *is* $H(4, 3)$*. See Figure 2.8.*

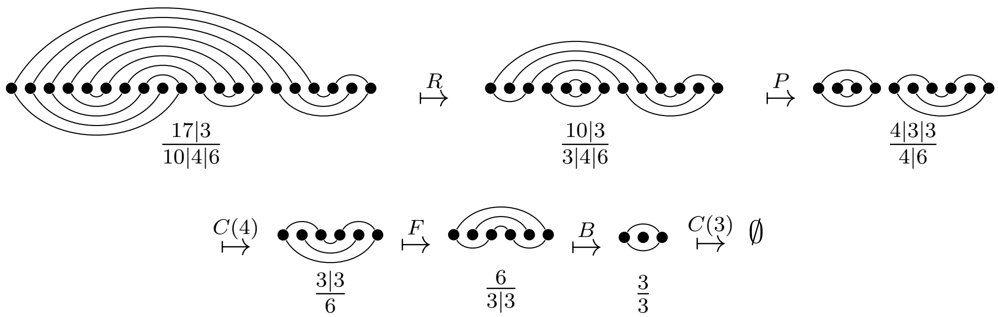


Figure 2.7: Winding down the meander of type $\frac{17|3}{10|4|6}$

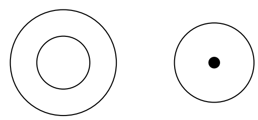


Figure 2.8: The homotopy type of $\frac{17|3}{10|4|6}$ is $H(4,3)$.

Theorem 2.3.3. *If $\mathfrak{g}$ is a seaweed with homotopy type $H(c_1, \cdots, c_h)$, then*

$$\text{ind}\,(\mathfrak{g}) = \sum_{i=1}^{h} c_i.$$

Proof. Let M be the meander associated with $\mathfrak{g}$. The only winding-down move which alters the index of M is the component deletion move. To apply $C(c_i)$, the meander M must be of type $\frac{c_i|a_1|\cdots|a_m}{c_i|b_1|\cdots|b_t}$. In this case, $\mathfrak{g} = \mathfrak{gl}(c_i) \oplus \mathfrak{g}'$ where $\mathfrak{g}'$ is of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$. By Theorem 2.2.2, the index of $\mathfrak{gl}(c_i)$ is c_i, so the application of $C(c_i)$ produces a new meander M' whose index is exactly c_i less than the index of $\mathfrak{g}$. Once the vertex set is of size zero, the meander has been completely "wound down", and the index is zero. Therefore, the index of $\mathfrak{g}$ is $\sum_{i=1}^{h} c_i$, as claimed. □

In the following Lemma 2.3.4, we introduce the ***winding-up moves*** which reverse the winding-down moves of Lemma 2.3.1. Note that not every winding-up move can be applied to an arbitrary meander of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$. For example, a Pure Expansion move can only be applied if the given meander has at least two top blocks;

the only moves that can be applied after a Component Creation move are a Block Creation move or another Component Creation move; and, per force, we disallow consecutive Flip moves.

Lemma 2.3.4 (Coll, Hyatt, and Magnant [**11**]). *Every meander is the result of a sequence of the following moves starting with $C(c)$ for some $c \in \mathbb{N}$. Given a meander M of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$, create a meander M' by one of the following moves.*

1. ***Block Creation:*** $M \mapsto \frac{2a_1|a_2|\cdots|a_m}{a_1|b_1|\cdots|b_t}$,
2. ***Rotation Expansion:*** $M \mapsto \frac{(2a_1-b_1)|a_2|\cdots|a_m}{a_1|b_2|\cdots|b_t}$,
3. ***Pure Expansion:*** $M \mapsto \frac{(a_1+2a_2)|a_3|\cdots|a_m}{a_2|b_1|\cdots|b_t}$,
4. ***Flip:*** $M \mapsto \frac{b_1|\cdots|b_t}{a_1|\cdots|a_m}$,
5. ***Component Creation:*** $M \mapsto \frac{c|a_1|\cdots|a_m}{c|b_1|\cdots|b_t}$, *for* $c \in \mathbb{N}$.

These moves are called ***winding-up moves****. For all moves, except the Component Creation move, M and M' have the same index.*

Using the the winding-up moves of Lemma 2.3.4, we now define another type of meander associated with a seaweed.

Definition 2.3.5. *Given a seaweed $\mathfrak{g}$ with signature S, the* ***component meander*** *CM of $\mathfrak{g}$ is the meander with the same signature as $\mathfrak{g}$ except that the component deletions are all of size 1.*

The component meander is created to highlight the path of each meander component inside the associated seaweed. The component meander is created using Lemma 2.3.4 (see Example 2.3.6).

Example 2.3.6. *Consider $\mathfrak{g}$ of type $\frac{10|2|4}{16}$. The signature of $\mathfrak{g}$ is $FRPC(2)C(4)$, and $\mathfrak{g}$ has homotopy type $H(2,4)$. The component meander of $\mathfrak{g}$ has signature $FRPC(1)C(1)$. Using the winding-up moves of Lemma 2.3.4 (see Figure 2.9, where* *the path which results from the component of size four is red and the path which results from the component of size two is blue), we construct the component meander of* $\mathfrak{g}$.

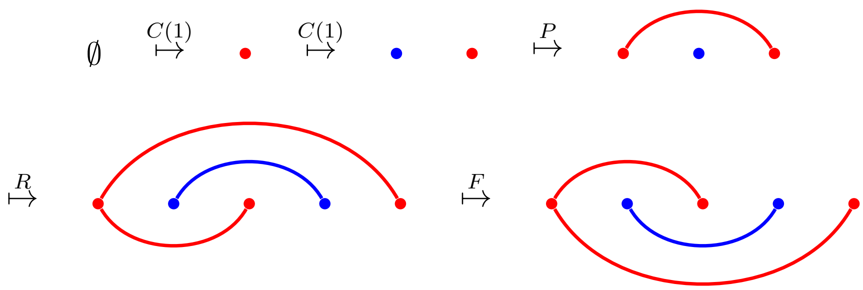


Figure 2.9: Winding-up of the component meander for $\mathfrak{g}$ of type $\frac{10|2|4}{16}$

See Figure 2.10 (left) which illustrates $\mathfrak{g}$*'s meander with the component of size two in blue and the component of size four in red. Figure 2.10 (right) illustrates the component meander associated with* $\mathfrak{g}$ *with the resulting paths from the components of size four and two in red and blue, respectively.*

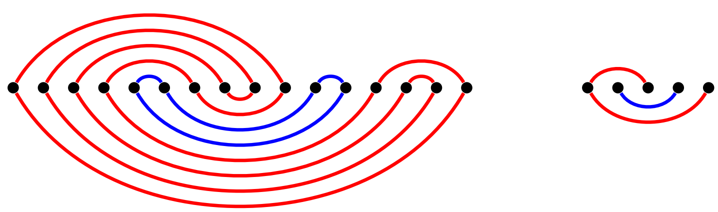


Figure 2.10: Meander (left) and component meander (right) of $\mathfrak{g}$ of type $\frac{10|2|4}{16}$

The vertices of CM are $v_{A_1}, \cdots, v_{A_t}$, where A_i is the set of indices for the adjacent vertices that were merged into one vertex in CM from M. The size of the subscript for v_{A_i} is equal to c_j for its corresponding component in the homotopy type of $\mathfrak{g}$. The labels for the new vertices can be tracked inductively under the winding up moves as follows.

1. For each component deletion $C(c)$ replaced from S, replace it with $C_c(1)$ as a bookkeeping method.

2. For the initial Component Creation $C_c(1)$, place the vertex $v_{\{1,\cdots,c\}}$.

3. Assume CM is of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ with vertices $v_{A_1},\cdots,v_{A_n}$. Let $N=\max(A_1)$.

   (a) **Component Creation ($C_c(1)$):** Place the vertex $v_{\{N+1,\cdots,N+c\}}$ on the left of the meander.

   (b) **Block Creation (B):** Place a_1 vertices on the left of the meander labeled $v_{B_{a_1}},\cdots,v_{B_1}$ with $B_1=\{N+1,\cdots,N+|A_1|\}$ and $B_i=\{\max(B_{i-1})+1,\cdots,\max(B_{i-1})+|A_i|\}$.

   (c) **Rotation Expansion (R):** Place a_1-b_1 vertices on the left of the meander labeled $v_{R_{a_1-b_1}},\cdots,v_{R_1}$ with $R_1=\{N+1,\cdots,N+|A_{a_1+1}|\}$ and $R_i=\{\max(R_{i-1})+1,\cdots,\max(R_{i-1})+|A_{a_1+i}|\}$.

   (d) **Pure Expansion (P):** Place a_2 vertices on the left of the meander labeled $v_{P_{a_1}},\cdots,v_{P_1}$ such that $P_1=\{N+1,\cdots,N+|A_{a_1+1}|\}$ and $P_i=\{\max(P_{i-1})+1,\cdots,\max(P_{i-1})+|A_{a_1+i}|\}$.

   (e) **Flip move (F):** Leave the vertices fixed.

4. Finally, reverse the elements in the labels of the vertices $v_{A_1},\cdots,v_{A_h}$. In other words, replace each vertex label A_i with

$$A_i'=\left\{\sum_{s=1}^{h}|A_i|+1-a\;\middle|\;a\in A_i\right\}.$$

Example 2.3.7. *Consider $\mathfrak{g}$ of Example 2.3.6. The vertex labels for CM are*

$$v_{\{1,2,3,4\}},\qquad v_{\{5,6\}},\qquad v_{\{7,8,9,10\}},\qquad v_{\{11,12\}},\qquad v_{\{13,14,15,16\}}.$$

2.4 Distinguished Subsets of a Seaweed Algebra

In this section, we highlight several important subsets of a seaweed. These subsets are defined by configurations of positions cut out of the seaweed by the components of the meander associated with the seaweed.

Definition 2.4.1. *Let $\mathfrak{g}$ be a seaweed, and let M be the meander associated with $\mathfrak{g}$ with homotopy type $H(c_1, \cdots, c_h)$. Let $\mathscr{I}_{c_i}$ represent the index set of the component c_i in M. In other words, visualize the meander in the matrix form of $\mathfrak{g}$ (see Example 2.4.2); $\mathscr{I}_{c_i}$ consists of each index (j,k) "covered" by an edge in the component c_i of M. Denote by $\mathfrak{g}|_{c_i}$ the **configuration of positions** in the component c_i, to be the set of all matrices generated by $e_{j,k}$ such that $(j,k) \in \mathscr{I}_{c_i}$.*

Example 2.4.2. *Let $\mathfrak{g}$ be the seaweed from the running Example 2.3.6, and let M be its associated meander. See Figure 2.11 (left). By Lemma 2.3.1, the homotopy type of $\mathfrak{g}$ is $H(2,4)$. As before (see Figure 2.3), we can visualize M inside of $\mathfrak{g}$ (see Figure 2.11 (right)).*

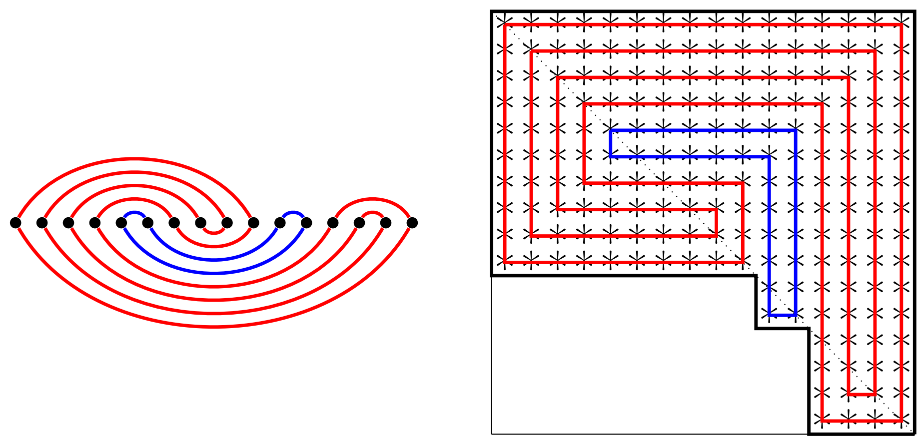


Figure 2.11: Meander of type $\frac{10|2|4}{16}$ (left), visualized in the seaweed (right)

The restriction of $\mathfrak{g}$ to its individual components is illustrated in Figure 2.12.

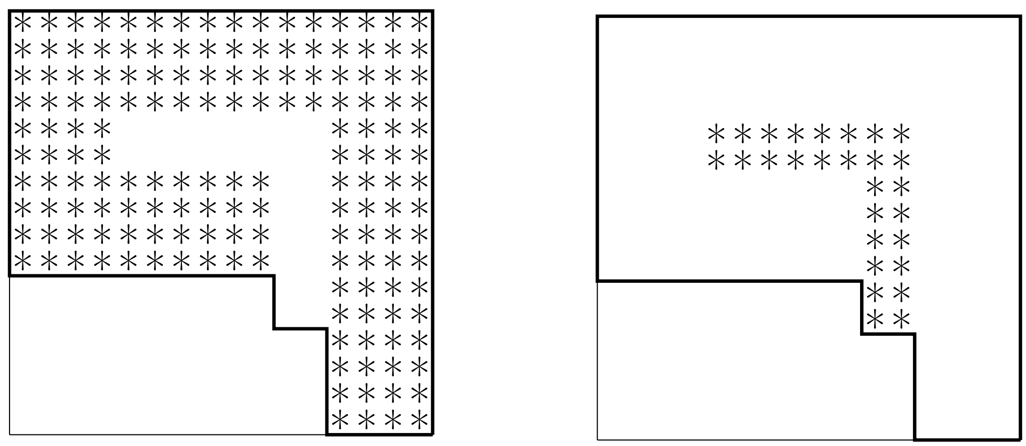


Figure 2.12: Restriction of $\mathfrak{g}$ of type $\frac{10|2|4}{16}$ to its component of size four (left) and its component of size two (right)

Remark 2.4.3. *A seaweed might have multiple components of the same size. Further, the restriction of a seaweed to one of its components often has no algebraic structure; it may simply be a subspace of* $\mathfrak{g}$.

The following Theorem 2.4.4 is a trivial consequence of Definition 2.4.1.

Theorem 2.4.4. *If* $\mathfrak{g}$ *is a seaweed with homotopy type* $H(c_1, \cdots, c_h)$*, then*

$$\mathfrak{g} = \sum_{i=1}^{h} \mathfrak{g}|_{c_i}.$$

We now highlight two other important subsets of $\mathfrak{g}$, called the *core* of $\mathfrak{g}$ and *peak set* of $\mathfrak{g}$.

Definition 2.4.5. *Let* $\mathfrak{g}$ *be a seaweed with homotopy type* $H(c_1, \cdots, c_h)$ *and component meander* CM*. Consider one component* c_i*. Define the sets*

$$V_{c_i} = \{A_j \mid v_{A_j} \text{is a vertex in } CM \text{ on the path of } c_i\},$$

$$\mathfrak{C}_{c_i} = \{A_I \times A_I \mid A_I \in V_{c_i}\}.$$

The set $\mathfrak{C}_{c_i}$ *is the* ***core*** *of* $\mathfrak{g}|_{c_i}$ *– the set of* $c_i \times c_i$ *blocks on the diagonal of* $\mathfrak{g}$ *contained in* $\mathfrak{g}|_{c_i}$.

Fix a vertex v_{A_I} on the path of c_i in CM. Partition V_{c_i} into two sets:

$$\mathscr{A}_{c_i} = \{A_j \mid \text{ the path from } v_{A_I} \text{ to } v_{A_j} \text{ has odd length}\},$$

$$\mathscr{B}_{c_i} = \{A_j \mid \text{ the path from } v_{A_I} \text{ to } v_{A_j} \text{ has even or zero length}\}.$$

Note: The choice of partitioning by distance from V_{A_I} is arbitrary.

Now orient CM counter-clockwise (i.e., top edges are oriented from right to left, bottom edges are oriented left to right). Let E_{CM} be the set of edges in the oriented CM. Define the ***peak set*** *of $\mathfrak{g}|_{c_i}$ as*

$$\mathfrak{P}_{c_i} = \{A_I \times A_J \mid A_I, A_J \in V_{c_i} \text{ with } (A_I, A_J) \in E_{CM}\}.$$

We define the ***core of*** *$\mathfrak{g}$ and the* ***peak set of*** *$\mathfrak{g}$ as the union over the components. In other words,*

$$\mathfrak{C}_{\mathfrak{g}} = \bigcup_{i=1}^{h} \mathfrak{C}_{c_i}, \qquad \text{and} \qquad \mathfrak{P}_{\mathfrak{g}} = \bigcup_{i=1}^{h} \mathfrak{P}_{c_i}.$$

Example 2.4.6. *Consider once again $\mathfrak{g}$ from our running Example 2.3.6. Table 2.1 lists V_{c_i}, $\mathfrak{C}_{c_i}$, $\mathscr{A}_{c_i}$, $\mathscr{B}_{c_i}$, and $\mathfrak{P}_{c_i}$ for $c_1 = 2$ and $c_2 = 4$. In the second and third columns of Table 2.1, $\mathfrak{g}|_{c_i}$ is shaded to better highlight which configuration of $\mathfrak{g}|_{c_i}$ is being identified. Further, $\mathfrak{C}_{c_i}$ and $\mathfrak{P}_{c_i}$ are represented as matrices with an asterisk to represent every index (i, j) which could appear in $\bigcup_{C \in \mathfrak{C}_{c_i}} C$ or $\bigcup_{P \in \mathfrak{P}_{c_i}} P$. Each individual $c_i \times c_i$ block is a set of indices in the corresponding core or peak set.*

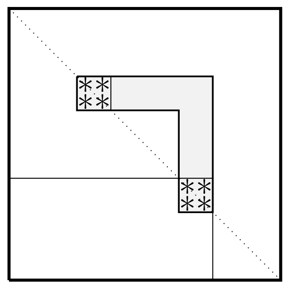
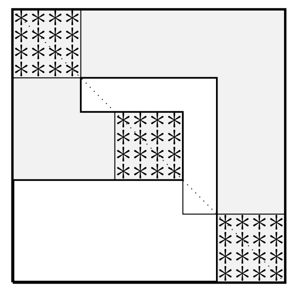
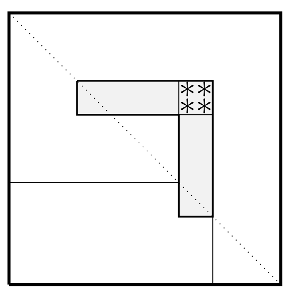
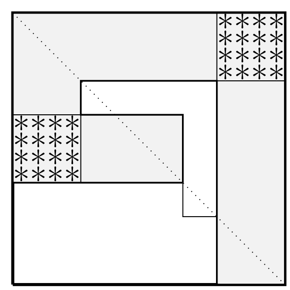
	$c_1 = 2$	$c_2 = 4$
V_{c_i}	$\{\{5,6\},\ \{11,12\}\}$	$\{\{1,2,3,4\},\ \{7,8,9,10\},\ \{13,14,15,16\}\}$
$\mathfrak{C}_{c_i}$		
$\mathscr{A}_{c_i}$	$\{\{11,12\}\}$	$\{\{7,8,9,10\},\ \{13,14,15,16\}\}$
$\mathscr{B}_{c_i}$	$\{\{5,6\}\}$	$\{\{1,2,3,4\}\}$
$\mathfrak{P}_{c_i}$		

Table 2.1: V_{c_i}, $\mathfrak{C}_{c_i}$, $\mathscr{A}_{c_i}$, $\mathscr{B}_{c_i}$, and $\mathfrak{P}_{c_i}$ in $\mathfrak{g}$ of type $\frac{10|2|4}{16}$

Chapter 3

Regular Functionals

In this chapter, we construct a regular functional F on a seaweed $\mathfrak{g}$ with homotopy type $H(c_1, \cdots, c_h)$. We do this by developing a broad analytic framework (see Section 3.1) which relies on the choices of regular functionals $F_{c_i} \in \mathfrak{gl}(c_i)^*$. The construction of the functional F involves embedding copies of F_{c_i} into the core of $\mathfrak{g}$ in such a way that the constructed functional F satisfies

$$\dim\ker(B_F) = \sum_{i=1}^{h} \dim\ker(B_{F_{c_i}}).$$

The regularity of the constructed F is assured by Theorem 3.2.8, the proof for which is an induction on the winding-up moves of Lemma 2.3.4. The induction makes heavy use of a ***relations matrix*** (see Section 3.1), a bookkeeping device which encodes, among other things, the degrees of freedom in the system of the equations which define $\ker(B_F)$.

Associated with a relations matrix B is a non-unique set P of matrix positions (i, j). The entries in the remaining positions of B are explicitly determined as linear combinations of the entries in the positions in P. Each assignment $\vec{b} = (b_{i,j} \mid (i, j) \in P)$ of complex numbers to the positions in P yields an element of $\ker(B_F)$, so $\dim\ker(B_F) = |P|$, and the resulting kernel elements span $\ker(B_F)$.

In Section 3.2 we develop a framework for the construction of a regular functional on $\mathfrak{gl}(n)$. The explicit functional is built in Section 3.3. We close the Chapter with

Section 3.4, where several more explicit regular functionals on $\mathfrak{gl}(n)$ are established.

3.1 A Relations Matrix of a Space

Let $\mathfrak{g}$ be a seaweed of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$. Every $F \in \mathfrak{g}^*$ is defined in terms of the functionals $e^*_{i,j}$. We may therefore write F in the form $F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j}e^*_{i,j}$, with $c_{i,j} \in \mathbb{C}$ and $\mathscr{I}_F \subseteq \{1,\ldots,n\}\times\{1,\ldots,n\}$ the ***index set*** of F. For any matrix B, denote by B^t the ***transpose*** of B. Similarly, if $F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j}e^*_{i,j}$, define by F^t the transpose of F (i.e., $F^t = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j}e^*_{j,i}$ and $\mathscr{I}_{F^t} = \{(j,i) \mid (i,j) \in \mathscr{I}_F\}$). We call F (and similarly $\mathscr{I}_F$) ***symmetric*** with respect to the main diagonal if $F = F^t$ (i.e., $\mathscr{I}_F = \mathscr{I}_{F^t}$). Using the same terminology, we call $\mathfrak{g}$ symmetric if $\mathfrak{g}^t := \{X^t \mid X \in \mathfrak{g}\} = \mathfrak{g}$. This happens if and only if $\overline{a} = \overline{b}$, or equivalently if and only if $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{gl}(a_i)$. Denote by $\mathscr{I}_\mathfrak{g}$ the set of all admissible locations in $\mathfrak{g}$ (i.e., $\mathfrak{g}$ is the linear span of $\{e_{i,j} \mid (i,j) \in \mathscr{I}_\mathfrak{g}\}$). If $F \in \mathfrak{g}^*$ then we assume $\mathscr{I}_F \subseteq \mathscr{I}_\mathfrak{g}$. We will use the superscript $\widehat{t}$ to represent transposition across the antidiagonal (i.e., if F is defined on $\mathfrak{gl}(n)$, then $F^{\widehat{t}} = \sum_{(i,j)\in\mathscr{I}_F} e^*_{n+1-j,n+1-i}$, etc.), and we have analogous definitions with respect to the antidiagonal. We will also use the superscript R to represent rotation of a matrix twice (i.e., $B^R = (A_n)^{-1}B(A_n)$, where $A_n = \sum_{i=1}^n e_{i,n+1-i}$), and we have the analogous definitions.

The *relations matrix* of a space of matrices is defined in Definition 3.1.1.

Definition 3.1.1. *Given a subspace* $\mathfrak{q} \subseteq \mathfrak{gl}(n)$ *with* $\dim \mathfrak{q} = m$*, fix a basis* $\{q_1,\cdots,q_m\}$ *of* $\mathfrak{q}$*. Define a linear transformation* $f : \mathbb{C}^m \to \mathfrak{q}$ *by* $f(e_i) = q_i$ *for each* i*. Given variables* $b_1,\cdots,b_m$*, the matrix form*

$$B := f(b_1,\cdots,b_m) = \sum_{i=1}^m b_iq_i$$

is a ***relations matrix*** *of* $\mathfrak{q}$*. By substitution of the variables* b_i *with complex coefficients,* B *satisfies the following statement:* $\mathfrak{q} = \{B \mid b_i \in \mathbb{C}\}$*.*

Remark 3.1.2. *A relations matrix is defined (up to a relabeling of the variables* b_i*) by a choice of basis for a space, and so it is defined up to conjugation by* $G \in$

$GL(n;\mathbb{C})$, a change of basis. Our purpose in constructing relations matrices for spaces is to infer the number of degrees of freedom of a space from it, so actual form does not matter.

Example 3.1.3. *Let $\mathfrak{q} \subseteq \mathfrak{gl}(2)$ be the space of matrices $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ subject to the constraints $x_1 = x_2 + x_4$ and $x_3 = x_2$. A basis for the space is*

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The matrix

$$B = b_1 q_1 + b_2 q_2 = b_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 + b_2 & b_1 \\ b_1 & b_2 \end{pmatrix}$$

is a relations matrix of $\mathfrak{q}$.

To facilitate the construction of a relations matrix of $\ker(B_F)$, we make use of the following technical lemmas. The first, Lemma 3.1.4, is used to shorten the calculations by any existent symmetry in the seaweed and functional.

Lemma 3.1.4. *Let $F \in \mathfrak{g}^*$ for a seaweed $\mathfrak{g}$ such that F and $\mathfrak{g}$ are symmetric with respect to the main diagonal (or the antidiagonal). Let $B = [b_{i,j}]$ be a relations matrix of $\ker(B_F)$. Let $\mathscr{B} = \{b_{i,j}\}$ be the set of free variables in B – i.e., if $\mathscr{I}$ is the set of indices in $\mathscr{B}$, then for each $(i,j) \in \mathscr{I}_{\mathfrak{g}}$, there exist complex coefficients $c_{\alpha,\beta}$ such that $b_{i,j} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha,\beta} b_{\alpha,\beta}$. For all $(i,j) \in \mathscr{I}_{\mathfrak{g}}$, if $b_{i,j} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha,\beta} b_{\alpha,\beta}$ with $c_{\alpha,\beta} \in \mathbb{C}$, then $b_{j,i} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha,\beta} b_{\beta,\alpha}$ (respectively, $b_{n+1-j,n+1-i} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha_\beta} b_{n+1-\beta,n+1-\alpha}$).*

Proof. We establish the theorem for symmetry across the main diagonal (the antidiagonal proof is similar.) For each $(i,j) \in \mathscr{I}_{\mathfrak{g}}$, there exist coefficients $c_{\alpha,\beta} \in \mathbb{C}$ such that

$$b_{i,j} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha,\beta} b_{\alpha,\beta}$$

by definition of $\mathscr{B}$. For each $B \in \ker(B_F)$, consider $B^t = [b'_{i,j}]$ and note that $b'_{i,j} = b_{j,i}$. Evidently, $B^t \in \ker(B_{F^t})$, where F^t is defined on $\mathfrak{g}^t$. However, $\mathfrak{g}^t = \mathfrak{g}$ and

$F^t = F$ by assumption. Therefore, $B^t \in \ker(B_F)$, and

$$b_{j,i} = b'_{i,j} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha,\beta} b'_{\alpha,\beta} = \sum_{(\alpha,\beta)\in\mathscr{I}} c_{\alpha,\beta} b_{\beta,\alpha}.$$

□

We have the following easy corollary to Lemma 3.1.4.

Lemma 3.1.5. *If $F \in \mathfrak{g}^*$ for a seaweed $\mathfrak{g}$ and F and $\mathfrak{g}$ are symmetric with respect to the main diagonal (or the antidiagonal), then for any relations matrix B of* $\ker(B_F)$, *$b_{i,j} = 0$ if and only if $b_{j,i} = 0$ (respectively, $b_{n+1-j,n+1-i} = 0$).*

To prove that a matrix B is in $\ker(B_F)$ for $F \in \mathfrak{g}^*$ amounts to showing that the entries $b_{i,j}$ in B satisfy a specific system of equations. This system is developed in Lemma 3.1.6.

Lemma 3.1.6. *Let $\mathfrak{g}$ be a seaweed, and let $F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j} e^*_{i,j} \in \mathfrak{g}^*$ with $c_{i,j} \in \mathbb{C}$. The space* $\ker(B_F)$ *is spanned by all matrices $B = [b_{i,j}]$ whose entries $b_{i,j}$ form a solution to the two sets of equations*

1. $\sum_{(s,j)\in\mathscr{I}_F} c_{s,j} b_{s,i} = \sum_{(i,s)\in\mathscr{I}_F} c_{i,s} b_{j,s}$, *for all $(i,j) \in \mathscr{I}_{\mathfrak{g}}$;*
2. $b_{i,j} = 0$, *for all $(i,j) \notin \mathscr{I}_{\mathfrak{g}}$.*

Proof. Let $B = [b_{i,j}] \in \ker(B_F)$. Since $B \in \mathfrak{g}$, we know $b_{i,j} = 0$ for all $(i,j) \notin \mathscr{I}_{\mathfrak{g}}$ by definition. This handles the equations in (2).

By definition, $\ker(B_F) = \{B \in \mathfrak{g} \mid B_F(B,X) = 0 \text{ for all } X \in \mathfrak{g}\}$. Therefore, to show $B \in \ker(B_F)$, it is necessary and sufficient to require $B_F(B, e_{i,j}) = 0$ for all $(i,j) \in \mathscr{I}_{\mathfrak{g}}$. Consider the image of $e_{i,j}$ under $B_F(B,\cdot)$. Recall that

$$B_F(B,X) = F([B,X]) = F(BX - XB),$$

and

$$[B, e_{i,j}] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & b_{1,i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{2,i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{3,i} & 0 & \cdots & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1,i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{n,i} & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_{j,1} & b_{j,2} & b_{j,3} & \cdots & b_{j,n-2} & b_{j,n-1} & b_{j,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

where the above matrices are $n \times n$ matrices with a potentially non-zero column j and row i respectively. It follows that

$$F([B, e_{i,j}]) = \sum_{(s,j)\in\mathscr{I}_F} c_{s,j}b_{s,i} - \sum_{(i,s)\in\mathscr{I}_F} c_{i,s}b_{j,s}. \tag{3.1}$$

Upon evaluating (3.1) at zero, the result in 1. follows. □

3.2 A Framework for Building Regular Functionals on Seaweed Algebras

To describe how we will construct a functional on a seaweed $\mathfrak{g}$, first assume that F_c represents a functional (not necessarily regular) on $\mathfrak{gl}(c)$ for any $c > 0$. The functionals F_{c_i} will be our building blocks for any seaweed of homotopy type $H(c_1, \cdots, c_h)$. Several explicit examples of regular functionals are provided in Sections 3.3 and 3.4. The indices in $\mathscr{I}_{F_c}$ will be fixed as a subset of $c \times c$.

We then describe a set of winding-up moves on the space $\mathfrak{g}^*$ so that, when applied to some $F \in \mathfrak{g}^*$, the produced functional F' is such that $\dim\ker(B_{F'}) = \dim\ker(B_F)$. The basis step will be the Component Creation moves, which will be a direct sum of functionals to produce a new functional defined on the direct sum of their domains. We first introduce several technical lemmas to accomplish this. Then, we define what are intuitively two choices of winding-up moves on $\mathfrak{g}^*$ in Definition 3.2.5. The proof is inductive in nature on the winding-up of the meander associated with $\mathfrak{g}$ to show that the kernels of the Kirillov forms are of the same dimension. The proof does something stronger in that it develops an explicit relations matrix of $\ker(B_F)$ which can be tracked under these moves (see Theorem 3.2.10).

Definition 3.2.1. *Given a seaweed $\mathfrak{g}$, a functional $F \in \mathfrak{g}^*$, and $a \in \mathbb{N}$, define the **shift of** F **by** a as the new functional*

$$F^a := \sum_{(i,j)\in\mathscr{I}_F} e^*_{i+a,j+a}. \tag{3.2}$$

Note: The right-hand side of (3.2) is defined only when the indices are admissible indices in the seaweed.

Given two algebras $\mathfrak{g}_1 \subseteq \mathfrak{gl}(n_1)$ and $\mathfrak{g}_2 \subseteq \mathfrak{gl}(n_2)$ with $F_i \in \mathfrak{g}_i^*$, we define the functional $F_1 \oplus F_2$ in $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^*$ by $F_1 + F_2^{n_1}$.

Lemma 3.2.2. *Let $\mathfrak{g}$ be a seaweed and assume that there exist $\mathfrak{g}_i \subseteq \mathfrak{gl}(n_i)$ such that $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ (it must be true that $n = \sum_{i=1}^k n_i$). Let $F_i \in \mathfrak{g}_i^*$ for all i and define $F = \bigoplus_{i=1}^k F_i$. A matrix B is a relations matrix of $\ker(B_F)$ if and only if*

$$B = \bigoplus_{i=1}^{k} B_i,$$

where B_i is a relations matrix of $\ker(B_{F_i})$ for each i. It follows that

$$\dim\ker(B_F) = \sum_{i=1}^{k} \dim\ker(B_{F_i}).$$

Proof. Let $\mathbf{0}_{m\times n}$ represent the $m \times n$ zero matrix.

By induction, it suffices to prove the claim for $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i \subseteq \mathfrak{gl}(n_i)$. Let $F_i \in \mathfrak{g}_i^*$ for each i, and define $F = F_1 \oplus F_2$. By construction,

$$\ker(B_F) = \ker(B_{F_1}) \oplus \ker(B_{F_2}).$$

Pick a basis $\mathscr{B} = \{k_1, \cdots, k_m\}$ of $\ker(B_F)$. Relabeling if necessary, $\{k_1, \cdots, k_{m_1}\}$ is a basis for $\ker(B_{F_1}) \oplus \mathbf{0}_{n_2\times n_2}$ and $\{k_{m_1+1}, \cdots, k_{m_1+m_2}\}$ is a basis for $\mathbf{0}_{n_1\times n_1} \oplus \ker(B_{F_2})$. Note that $m = m_1 + m_2$. Now, if $B = [b_{i,j}]$ is a relations matrix of $\ker(B_F)$ with respect to $\mathscr{B}$, then there exist $n_i \times n_i$ matrices B_i such that $B = B_1 \oplus B_2$. In this case,

$$B_1 \oplus \mathbf{0}_{n_2\times n_2} = f(b_1, \cdots, b_{m_1}, 0, \cdots, 0)$$

is a relations matrix of $\ker(B_{F_1}) \oplus \mathbf{0}_{n_2\times n_2}$. By isomorphism, B_1 is a relations matrix of $\ker(B_{F_1})$. Similarly,

$$\mathbf{0}_{n_1\times n_1} \oplus B_2 = f(0, \cdots, 0, b_{m_1+1}, \cdots, b_m)$$

is a relations matrix of $\mathbf{0}_{n_1\times n_1} \oplus \ker(B_{F_2})$, so by the natural isomorphism, B_2 is a relations matrix of $\ker(B_{F_2})$.

For the reverse direction, assume B_i a relations matrix of $\ker(B_{F_i})$. By definition, there exist linear transformations $f_i : \mathbb{C}^{m_i} \to \ker(B_{F_i})$ appropriately defined so that $B_i = f(b_1^i, \cdots, b_{m_i}^i)$. Define $f : \mathbb{C}^m \to \ker(B_F)$ by

$$f(x_1, \cdots, x_m) = f_1(x_1, \cdots, x_{m_1}) \oplus f_2(x_{m_1+1}, \cdots, x_m).$$

The matrix $B = f(b_1, \cdots, b_m) = f_1(b_1, \cdots, b_{m_1}) \oplus f_2(b_{m_1+1}, \cdots, b_m) = B_1 \oplus B_2$ is a relations matrix of $\ker(B_F)$. The dimension result follows. □

Since we are ultimately interested in constructing regular functionals on $\mathfrak{g}$ the following Theorem 3.2.3 is essential.

Theorem 3.2.3. *If $\mathfrak{g} \subseteq \mathfrak{gl}(n)$ is such that $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ for $\mathfrak{g}_i \subseteq \mathfrak{gl}(n_i)$, and $F \in \mathfrak{g}^*$, then F is regular if and only if $F = \bigoplus_{i=1}^k F_i$ with F_i regular on $\mathfrak{g}_i$ for each i.*

Proof. Fix $F_i \in \mathfrak{g}_i^*$ such that $F = \bigoplus_{i=1}^k F_i$.

Assume that F is regular and, towards a contradiction, that there exists j such that F_j is not regular on $\mathfrak{g}_j$. Fix F_j' regular on $\mathfrak{g}_j$ and define

$$F' = \bigoplus_{i=1}^{j-1} F_i \oplus F_j' \bigoplus_{i=j+1}^{k} F_i.$$

By definition, $\dim\ker(B_{F_j'}) < \dim\ker(B_{F_j})$, and by Lemma 3.2.2

$$\begin{aligned}
\dim\ker(B_{F'}) &= \sum_{\substack{i=1 \\ i\neq j}}^{k} \dim\ker(B_{F_i}) + \dim\ker(B_{F_j'}) \\
&< \sum_{\substack{i=1 \\ i\neq j}}^{k} \dim\ker(B_{F_i}) + \dim\ker(B_{F_j}) \\
&= \dim\ker(B_F).
\end{aligned}$$

This contradicts the regularity of F.

Now, assume that F_i is regular for all i. Again, if F is not regular fix a regular $F' \in \mathfrak{g}^*$. Let $F_i' \in \mathfrak{g}_i^*$ be such that $F' = \bigoplus_{i=1}^k F_i'$. We have

$$\sum_{i=1}^{k} \dim\ker(B_{F_i}) = \dim\ker(B_F) > \dim\ker(B_{F'}) = \sum_{i=1}^{k} \dim\ker(B_{F_i'}).$$

Let j be the first index such that $\dim\ker(B_{F_i}) > \dim\ker(B_{F_i'})$ (such an index must exist). Then F_i is not regular on $\mathfrak{g}_i$, a contradiction. $\square$

Lemma 3.2.4. *Let $\mathfrak{g}$ be a seaweed. If $\mathfrak{g} = \mathfrak{g}^t$, then*

$$\dim\ker(B_F) = \dim\ker(B_{F^t}).$$

Similarly, if $\mathfrak{g} = \mathfrak{g}^{\widehat{t}}$, then

$$\dim\ker(B_F) = \dim\ker(B_{F^{\widehat{t}}}).$$

It follows that F is regular if and only if F^t (respectively $F^{\widehat{t}}$) is regular on $\mathfrak{g}$.

Proof. We know that if $F \in \mathfrak{g}^*$, then $F^t \in (\mathfrak{g}^t)^* = \mathfrak{g}^*$. It is easy to see that

$$B \in \ker(B_F) \quad \Leftrightarrow \quad B^t \in \ker(B_{F^t}).$$

Since transposition is an isomorphism of spaces, the result follows. The argument is identical for transposition across the antidiagonal, and regularity follows from the definition of the index. $\square$

Now, using the component meander associated with a seaweed $\mathfrak{g}$, we describe a method for building a functional $F \in \mathfrak{g}^*$ using functionals $F_{c_i} \in \mathfrak{gl}(c_i)^*$ over the components c_i of $\mathfrak{g}$'s homotopy type $H(c_1, \cdots, c_h)$.

Definition 3.2.5. *Let $\mathfrak{g}$ be a seaweed with homotopy type $H(c_1, \cdots, c_h)$. Let $\mathscr{A}_{c_i}$ and $\mathscr{B}_{c_i}$ be defined as in Definition 2.4.5. Let F_c^R represent the functional obtained through a rotation of the indices in $\mathscr{I}_{F_c}$ (i.e., $F_c^R = \sum_{(i,j)\in\mathscr{I}_{F_c}} e^*_{c+1-i,c+1-j}$). Define sets*

$$\mathscr{D}^a_{c_i} = \bigcup_{P\in\mathfrak{P}}\{(I'-s, J'+s) \mid s \in [0, c_i-1],\ \ I' = \max A_I,\ \ J' = \min A_J,\ \mathit{for}\ P = A_I \times A_J\}.$$

and

$$\mathscr{D}_{c_i} = \bigcup_{P\in\mathfrak{P}}\{(I'+s, J'+s) \mid s \in [0, c_i-1],\ \ I' = \min A_I,\ \ J' = \min A_J,\ \mathit{for}\ P = A_I \times A_J\}.$$

The sets $\mathscr{D}^a_{c_i}$ *and* $\mathscr{D}_{c_i}$ *are just the entries on the antidiagonal and main diagonal (respectively) of each* $c_i \times c_i$ *square* $A_I \times A_J$ *in* $\mathfrak{P}_{c_i}$*. Given a functional* $F_{c_i} \in \mathfrak{gl}(c_i)^*$*, define functionals* $\overline{F}^a_{c_i}$ *and* $\overline{F}_{c_i}$ *in* $\mathfrak{g}^*$ *as follows:*

$$\overline{F}^a_{c_i} := \sum_{A \in \mathscr{A}_{c_i}} (F^R_{c_i})^{\min(A)-1} + \sum_{A \in \mathscr{B}_{c_i}} (F_{c_i})^{\min(A)-1} + \sum_{(i,j) \in \mathscr{D}^a_{c_i}} e^*_{i,j},$$

$$\overline{F}_{c_i} := \sum_{A \in \mathscr{A}_{c_i} \cup \mathscr{B}_{c_i}} (F_{c_i})^{\min(A)-1} + \sum_{(i,j) \in \mathscr{D}_{c_i}} e^*_{i,j}.$$

Define two functionals $\overline{F}^a, \overline{F} \in \mathfrak{g}^*$ *by*

$$\overline{F}^a := \sum_{i=1}^{h} \overline{F}^a_{c_i}, \qquad \text{and} \qquad \overline{F} := \sum_{i=1}^{h} \overline{F}_{c_i}.$$

Remark 3.2.6. *Intuitively, the functionals defined in Definition 3.2.5 are a way of keeping track of a set of indices under the winding-up moves of each configuration of positions* $\mathfrak{g}|_{c_i}$*. The method is as follows. Fix* $F_{c_i} \in \mathfrak{gl}(c_i)^*$ *and embed this functional to a core block* $C \in \mathfrak{C}_{c_i}$*. Let* P *be the matrix embedded into a peak block adjacent to* C *(i.e.* $P = I_{c_i}$ *or* $P = A_{c_i}$ *– the matrix with one's on the antidiagonal). If* I_F *is the coefficient matrix of* F *(i.e., if* $F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j} e^*_{i,j}$*, then* $I_F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j} e_{i,j}$*), embed the functional with coefficient matrix* $P^{-1} I_F P$ *to the block* $C' \in \mathfrak{C}_{c_i}$ *such that* P *is the peak block on the edge between* C *and* C' *in the component meander associated with* $\mathfrak{g}$*.*

We denote by $\overline{F}$ *and* $\overline{F}^a$ *the functionals where the same method is used throughout (i.e., we always embed functionals along the main diagonal of the peak blocks for* $\overline{F}$ *and we always embed functionals along the antidiagonal of the peak blocks for* $\overline{F}^a$*). The proof of Theorem 3.2.8 deals explicitly with* $\overline{F}$ *and* $\overline{F}^a$*, but no aspect of the proof requires the consistent choice of functionals in the peak blocks. Therefore, it follows that these methods may be mixed.*

Example 3.2.7. *Let* $\mathfrak{g}$ *be the seaweed of our running Example 2.3.6. Recall that* $\mathfrak{g}$ *has type* $\frac{10|4|2}{16}$ *and homotopy type* $H(2,4)$*. The function* $\overline{F}^a$ *constructed using*

$F_2 \in \mathfrak{gl}(2)^*$ *and* $F_4 \in \mathfrak{gl}(4)^*$ *of Theorem 3.3.1 is*

$$\begin{aligned}\overline{F}^a = & e^*_{1,1} + e^*_{1,2} + e^*_{1,3} + e^*_{1,4} + e^*_{1,16} + e^*_{2,1} + e^*_{2,2} + e^*_{2,3} + e^*_{2,15} + e^*_{3,1} + e^*_{3,2} + e^*_{3,14} \\ & + e^*_{4,1} + e^*_{4,13} + e^*_{5,5} + e^*_{5,6} + e^*_{5,12} + e^*_{6,5} + e^*_{6,11} + e^*_{7,4} + e^*_{7,10} + e^*_{8,3} + e^*_{8,9} + e^*_{8,10} + \\ & e^*_{9,2} + e^*_{9,8} + e^*_{9,9} + e^*_{9,10} + e^*_{10,1} + e^*_{10.7} + e^*_{10,8} + e^*_{10,9} + e^*_{10,10} + e^*_{11,12} + e^*_{12,11} \\ & + e^*_{12,12} + e^*_{13,16} + e^*_{14,15} + e^*_{14,16} + e^*_{15,14} + e^*_{15,15} + e^*_{15,15} + e^*_{16,13} + e^*_{16,14} + e^*_{16,15} \\ & + e^*_{16,16}.\end{aligned}$$

The functional $\overline{F}$ *constructed using the same* F_2 *and* F_4 *is*

$$\begin{aligned}\overline{F} = & e^*_{1,1} + e^*_{1,2} + e^*_{1,3} + e^*_{1,4} + e^*_{1,13} + e^*_{2,1} + e^*_{2,2} + e^*_{2,3} + e^*_{2,14} + e^*_{3,1} + e^*_{3,2} + e^*_{3,15} \\ & + e^*_{4,1} + e^*_{4,16} + e^*_{5,5} + e^*_{5,6} + e^*_{5,11} + e^*_{6,5} + e^*_{6,12} + e^*_{7,1} + e^*_{7,7} + e^*_{7,8} + e^*_{7,9} + e^*_{7,10} + \\ & e^*_{8,2} + e^*_{8,7} + e^*_{8,8} + e^*_{8,9} + e^*_{9,3} + e^*_{9,7} + e^*_{9,8} + e^*_{10,4} + e^*_{10,7} + e^*_{11,11} + e^*_{11,12} \\ & + e^*_{12,11} + e^*_{13,13} + e^*_{13,14} + e^*_{13,15} + e^*_{13,16} + e^*_{14,13} + e^*_{14,14} + e^*_{14,15} + e^*_{15,13} + e^*_{15,14} \\ & + e^*_{16,13}.\end{aligned}$$

We illustrate the sets $\mathscr{I}_{\overline{F}^a}$ *and* $\mathscr{I}_{\overline{F}}$ *by placing a black dot in each entry* $(i,j) \in \mathscr{I}_{\overline{F}}$ *and* $(i,j) \in \mathscr{I}_{\overline{F}^a}$ *in the matrix form of* $\mathfrak{g}$ *in Figure 3.1 (left and right, respectively). The configuration of positions* $\mathfrak{g}|_4$ *is left shaded in grey to emphasize the embedding of the functionals* F_2 *and* F_4 *into the core and how the peak dots affect this choice. The functionals* $\overline{F}_4$ *and* $\overline{F}^a_4$ *are the sum of* $e^*_{i,j}$ *where* (i,j) *is in the shaded region, while the functionals* $\overline{F}_2$ *and* $\overline{F}^a_2$ *are the sum of* $e^*_{i,j}$ *over the indices* (i,j) *outside the shaded region.*

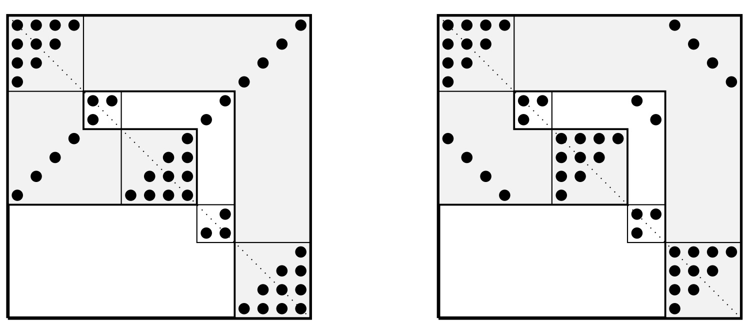


Figure 3.1: Constructed functionals $\overline{F}^a$ and $\overline{F}$ on $\mathfrak{g}$ of type $\frac{10|4|2}{16}$

Theorem 3.2.8. *Let $\mathfrak{g}$ be a seaweed with homotopy type $H(c_1, \cdots, c_h)$, and let $F_{c_i} \in \mathfrak{gl}(c_i)^*$, for each i. The functionals $\overline{F}, \overline{F}^a \in \mathfrak{g}^*$ of Definition 3.2.5 are such that*

$$\dim\ker(B_{\overline{F}^a}) = \dim\ker(B_{\overline{F}}) = \sum_{i=1}^{t} \dim\ker(B_{F_{c_i}}). \tag{3.3}$$

Assuming Theorem 3.2.8 for the moment, we have the following immediate Corollary.

Theorem 3.2.9. *The functionals $\overline{F}$ and $\overline{F}^a$ ini Definition 3.2.5 are regular if and only if F_{c_i} is regular on $\mathfrak{gl}(c_i)$, for each i.*

Proof of Theorem 3.2.8. The proof is an induction on the winding-up moves of Lemma 2.3.4. We first show how the winding-up moves affect the index set of F, where F is either $\overline{F}$ or $\overline{F}^a$. Consider the seaweed $\mathfrak{g}'$ obtained by a Component Creation move $C(c)$ applied to the meander associated with $\mathfrak{g}$. The new functional constructed by Definition 3.2.5 is $F_c \oplus F$, and the dimension result of equation (3.3) follows trivially. Further, if B is a relations matrix of $\ker(B_F)$ and B_c is a relations matrix of $\ker(B_{F_c})$, then $B_c \oplus B$ is a relations matrix of $\ker(B_{F_c \oplus F})$. The Flip move is trivial, $F \mapsto F^t$, and B^t is a relations matrix of $\ker(B_{F^t})$.

Now, assume that $\mathfrak{g}$ is of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$. Consider the seaweed $\mathfrak{g}'$ constructed by applying a Block Creation move to $\mathfrak{g}$ and, without loss of generality, assume that the block a_1 in the meander M associated with $\mathfrak{g}$ is part of a single component of size a_1 – the argument for multiple components is a finite number of arguments identical to the following argument. Under Definition 3.2.5, there are two choices for F' on $\mathfrak{g}'$. If $F_{[a,b]} = \sum_{\substack{(i,j)\in\mathscr{I}_F \\ i,j\in[a,b]}} e^*_{i,j}$ is the functional defined on $\mathfrak{gl}(b+1-a)$, then let $F'_1 = (F|_{[1,a_1]} \oplus F) + \sum_{i=1}^{a_1} e^*_{a_1+i,i}$ (the direct copying of the indices to the new $a_1 \times a_1$ block in $\mathfrak{g}'$ and the addition of functionalsalong the main diagonal of the peak block), and let $F'_2 = ((F|_{[1,a_1]})^R \oplus F) + \sum_{i=1}^{a_1} e^*_{a_1+i,a_1+1-i}$ (the rotation of the indices being copied to the new $a_1 \times a_1$ block in $\mathfrak{g}'$ and the addition of the antidiagonal functionals in the peak block). Let B' be a relations matrix of F'_1 and consider the following division of B' into four quadrants, whose indices (i,j) are relabeled as indicated.

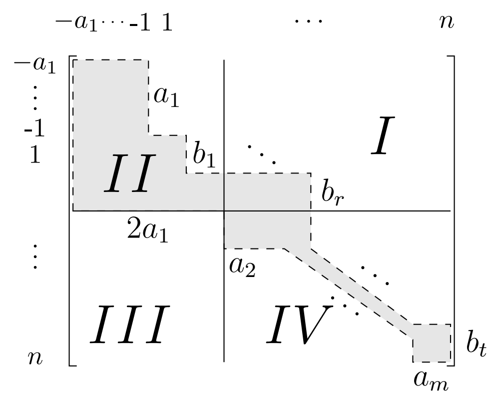


Figure 3.2: The four quadrants of B'.

Now, fix $i, j \in [1, n]$ and consider the images of the basis elements $e_{-i,j}$, $e_{i,j}$, and $e_{i,-j}$ (basis elements in quadrant II). We get the following three expressions under the map $B_{F_1'}([B, \cdot])$.

$$e_{-i,-j} \mapsto \left(\sum_{(s,-j)\in \mathscr{I}_{F_1'}} b_{s,-i} - \sum_{(-i,s)\in \mathscr{I}_{F_1'}} b_{-j,s} \right) \tag{3.4}$$

$$e_{i,-j} \mapsto \left(\sum_{(s,-j)\in \mathscr{I}_{F_1'}} b_{s,i} - \sum_{(i,s)\in \mathscr{I}_{F_1'}} b_{-j,s} \right) \tag{3.5}$$

$$e_{a_1+1-i,a_1+1-j} \mapsto \left(\sum_{(s,a_1+1-j)\in \mathscr{I}_{F_1'}} b_{s,a_1+1-i} - \sum_{(a_1+1-i,s)\in \mathscr{I}_{F_1'}} b_{a_1+1-j,s} \right) \tag{3.6}$$

Consider the equations provided by setting the right hand side of (3.5) equal to zero. Note that $(s, -j) \in \mathscr{I}_{F_1'}$ if and only if $s = a_1 + 1 - j$ (i.e., $e^*_{s,-j}$ is the unique functional in the new peak block in column $-j$) or $s < 0$. If $s < 0$, then $(s, i) \notin \mathscr{I}_{\mathfrak{g}'}$, so $b_{s,i} = 0$. Similarly, $(i, s) \in \mathscr{I}_{F_1'}$ if and only if $s = i - a_1 - 1$ or $s > 0$. If $s > 0$, then $b_{-j,s} = 0$ as $(-j, s) \notin \mathscr{I}_{\mathfrak{g}'}$. Therefore, the system of equations resulting from (3.5) reduces to

$$b_{a_1+1-j,i} = b_{-j,i-a_1-1}. \tag{3.7}$$

That is, the top $a_1 \times a_1$ block of B' is equal to the second $a_1 \times a_1$ block of B'. An identical argument on the basis elements $e_{i,-j}$ mapped under $B_{F_2'}([B,\cdot])$ shows that $b_{-i,-j} = b_{i,j}$ for all $i, j \leq n$, meaning the top $a_1 \times a_1$ block of B' is the rotation of the second $a_1 \times a_1$ block of B'. To show that the application of a Block Creation move does not change the index of the seaweed, it suffices to show that elements in the peak block $[1, a_1] \times [-1, -a_1]$ indicated in Figure 3.2 are zero. We will show that the elements in the peaks must be a direct copy of previous peak blocks (if any). Therefore, by recursion it will suffice to consider a seaweed of the form $\frac{2n+m}{n|a_1|\cdots|a_k|n}$, with $\sum a_i = m$ (i.e., the outer most peak block created in a component of size n in $\mathfrak{g}$). The recursion on the peak blocks is justified by evaluating the right hand side of (3.4) and (3.6) at zero and summing. Notice that $(s, -j) \in \mathscr{I}_{F_1'}$ if and only if $s = a_1 + 1 - j$ or $s < 0$ (i.e. $(s, -j)$ is one of the copied indices), and $(-i, s) \in \mathscr{I}_{F_1'}$ implies $s < 0$. Therefore, the equation given by evaluating (3.4) at zero combined with (3.7) is equivalent to teh system

$$\sum_{\substack{(s,-j)\in\mathscr{I}_{F_1'} \\ s<0}} b_{s,-i} + b_{a_1+1-j,-i} = \sum_{\substack{(-i,s)\in\mathscr{I}_{F_1'} \\ s<0}} b_{-j,s}$$

$$\Leftrightarrow \sum_{\substack{(s,a_1+1-j)\in\mathscr{I}_{F_1'} \\ s>0}} b_{s,a_1+1-i} + b_{a_1+1-j,-i} = \sum_{\substack{(a_1+1-i,s)\in\mathscr{I}_{F_1'} \\ s>0}} b_{a_1+1-j,s}. \tag{3.8}$$

When evaluated at zero, the right hand side of (3.6) is equivalent to system

$$\sum_{\substack{(s,a_1+1-j)\in\mathscr{I}_{F_1'} \\ s>0}} b_{s,a_1+1-i} = \sum_{\substack{(a_1+1-i,s)\in\mathscr{I}_{F_1'} \\ 0<s\leq a_1}} b_{a_1+1-j,s} + \sum_{\substack{(a_1+1-i,s)\in\mathscr{I}_{F_1'} \\ a_1<s}} b_{a_1+1-j,s} + b_{a_1+1-j,-i}. \tag{3.9}$$

Combining equations (3.8) and (3.9) so that the appropriate summations cancel, we have the following equation:

$$b_{a_1+1-j,-i} = \frac{-1}{2} \sum_{\substack{(a_1+1-i,s)\in\mathscr{I}_{F_1'} \\ a_1<s}} b_{a_1+1-j,s}. \tag{3.10}$$

The same argument for F_2' yields an equation similar to equation (3.10). The Rotation Expansion move and the Pure Expansion move only require an appropriate relabeling of indices. Hence, by recursion it suffices to compute what the first peak block must be (which will be a zero matrix), and to show that any indices $b_{i,j}$ which occur outside the peak and core blocks are equal to zero.

Without loss of generality, consider the seaweed $\mathfrak{g}$ of type $\frac{2n+m}{n|m|n}$. Let G_m be a functional on $\mathfrak{gl}(m)$, define $F_1' = (F_n \oplus G_m \oplus F_n) + \sum_{i=1}^{n} e^*_{n+m+i,i}$, and let $F_2' = (F_c \oplus G_m \oplus F_c^R) + \sum_{i=1}^{n} e^*_{2n+m+1-i,i}$. The indices in these functionals are pictured in Figure 3.3.

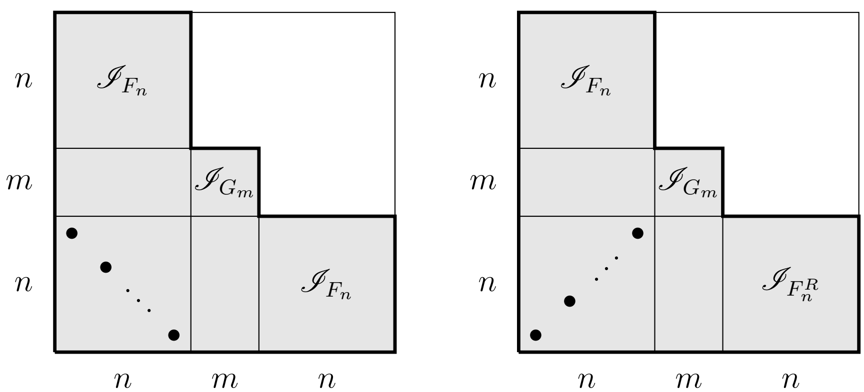


Figure 3.3: Indices in $\mathscr{I}_{F_1'}$ (left) and $\mathscr{I}_{F_2'}$ (right)

For F_1', consider the images of the basis elements under $B_{F_1'}(B, \cdot)$:

1. $e_{i,j} \mapsto \sum_{(s,j)\in\mathscr{I}_{F_n}} b_{s,i} + b_{n+m+j,i} - \sum_{(i,s)\in\mathscr{I}_{F_n}} b_{j,s}$,
   for $i, j \in [1, n]$,

2. $e_{i,j} \mapsto \sum_{(s,j)\in\mathscr{I}_{F_n}} b_{s,i} + b_{n+m+j,i} - \sum_{(i-n,s-n)\in\mathscr{I}_{G_m}} b_{j,s}$,
   for $i \in [n+1, m]$, $j \in [1, n]$,

3. $e_{i,j} \mapsto \sum_{(s-n,j-n)\in\mathscr{I}_{G_m}} b_{s,i} - \sum_{(i-n,s-n)\in\mathscr{I}_{G_m}} b_{j,s}$,
   for $i, j \in [n+1, m]$,

4. $e_{i,j} \mapsto \sum_{(s,j)\in\mathscr{I}_{F_n}} b_{s,i} + b_{n+m+j,i} - \sum_{(i-n,m,s-n-m)\in\mathscr{I}_{F_n}} b_{j,s} - b_{j,i-n-m}$,
   for $i \in [n+m+1, 2n+m]$, $j \in [1, n]$,

5. $e_{i,j} \mapsto \sum_{(s-n,j-n)\in \mathscr{I}_{G_m}} b_{s,i} - \sum_{(i-n,m,s-n-m)\in \mathscr{I}_{F_n}} b_{j,s} - b_{j,i-n-m}$,
   for $i \in [n+m+1, 2n+m]$, $j \in [n+1, m]$,

6. $e_{i,j} \mapsto \sum_{(s-n-m,j-n-m)\in \mathscr{I}_{F_n}} b_{s,i} - \sum_{(i-n,m,s-n-m)\in \mathscr{I}_{F_n}} b_{j,s} - b_{j,i-n-m}$,
   for $i, j \in [n+m+1, 2n+m]$.

The expressions in (1.), when evaluated at zero, combine with the expressions in (6.) evaluated at zero to show that $b_{j,i} = 0$ for all $j[n+m+1, 2n+m]$, $i \in [1, n]$. The equations generated by evaluating (4.) at zero yield that $b_{i,j} = b_{i+n+m,j+n+m}$ for all $i, j \in [1, n]$ (since both summations in the expression must evaluate to zero as $b_{s,i} = b_{j,s} = 0$ on their given domains due to the indices not being in $\mathscr{I}_{\mathfrak{g}}$). The expressions in (1.), knowing now that $b_{n+m+j,i} = 0$, solve to a relations matrix B_1 of $\ker(B_{F_n})$.

In (2.), both summations are zero since the indices are not in $\mathscr{I}_{\mathfrak{g}}$, and so the equation which results from evaluating the right hand side at zero simplifies to $b_{n+m+j,i} = 0$ for all $j \in [1, n]$, and $i \in [n+1, m]$. This is the $n \times m$ rectangle in the bottom row of Figure 3.3 (left). The same argument on the expressions in (5.) generates the $m \times n$ rectangle in the first column of Figure 3.3 (left) must also be a zero matrix. Finally, the expressions in (3.) when evaluated at zero will yield a relations matrix B_2 of $\ker(B_{G_m})$.

The final evaluation of the system of equations given by mapping the basis elements to zero under $B_{F_1'}([B, \cdot])$ will be that a relations matrix B of $\ker(B_{F_1'})$ is

$$B = B_1 \oplus B_2 \oplus B_1.$$

Hence, $\dim \ker(B_{F_1'}) = \dim \ker(B_{F_n}) + \dim \ker(B_{G_m})$. A similar argument on F_2' shows that a relations matrix B' is of the form

$$B' = B_1 \oplus B_2 \oplus B_1^R,$$

and the dimension argument holds. □

We have the following immediate Corollary from the proof of Theorem 3.2.8.

Theorem 3.2.10. *Let $\mathfrak{g}$ be a seaweed with homotopy type $H(c_1, \cdots, c_h)$, and let $F_{c_i} \in \mathfrak{gl}(c_i)^*$ for each i. Any functional F constructed via the methods of Definition 3.2.5 using the functionals F_{c_i} is such that, if $B \in \ker(B_F)$ and $b_{i,j} \neq 0$, then (i, j) is in the core of $\mathfrak{g}$. Further, if $C_{c_{i_1}} \oplus C_{c_{i_2}} \oplus \cdots \oplus C_{c_{i_k}}$ is the core of $\mathfrak{g}$ (i.e., $i_j \in \{1, \cdots, h\}$ for all j), then a relations matrix of the space $\ker(B_F)$ is*

$$B_{i_1}^r \oplus \cdots \oplus B_{i_k}^r,$$

where the superscript r represents the appropriate rotation of the matrix where necessary.

Example 3.2.11. *Consider the functionals $\overline{F}$ and $\overline{F}^a$ of Example 3.2.7 on the seaweed $\mathfrak{g}$ of type $\frac{10|2|4}{16}$. By direct computation (as demonstrated in the proof of Theorem 3.3.1), F_2 and F_4 are regular, and they have the respective relations matrices:*

$$B_4 = \begin{pmatrix} b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 \\ b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 \\ b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_5+b_6 & b_5 \\ b_5 & b_6 \end{pmatrix}.$$

A relations matrix of $\ker(B_{\overline{F}})$ is the matrix $B_4 \oplus B_2 \oplus B_4 \oplus B_2 \oplus B_4$,

$$\left(\begin{array}{cccccccccccccccc}
b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_5+b_6 & b_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_5 & b_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & b_5+b_6 & b_5 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & b_5 & b_6 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 \\
 & & & & & & & & & & & & b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 \\
 & & & & & & & & & & & & b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 \\
 & & & & & & & & & & & & b_1 & b_2 & b_3 & b_4
\end{array}\right),$$

and a relations matrix of $\ker(B_{\overline{F}^a})$ is the matrix $B_4 \oplus B_2 \oplus B_4^R \oplus B_2^R \oplus B_4^R$,

$$
\begin{pmatrix}
b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_5+b_6 & b_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_5 & b_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_4 & b_3 & b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_3 & b_2+b_4 & b_1+b_3 & b_1+b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_2 & b_1+b_3 & b_1+b_2+b_4 & b_1+b_2+b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_1+b_2 & b_1+b_2+b_3 & b_1+b_2+b_3+b_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & b_6 & b_5 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & b_5 & b_5+b_6 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & b_4 & b_3 & b_2 & b_1 \\
 & & & & & & & & & & & & b_3 & b_2+b_4 & b_1+b_3 & b_1+b_2 \\
 & & & & & & & & & & & & b_2 & b_1+b_3 & b_1+b_2+b_4 & b_1+b_2+b_3 \\
 & & & & & & & & & & & & b_1 & b_1+b_2 & b_1+b_2+b_3 & b_1+b_2+b_3+b_4
\end{pmatrix}.
$$

Example 3.2.12. *In accordance with Remark 3.2.6, the functional F whose indices $\mathscr{I}_F$ are shown in Figure 3.4 is also regular on $\mathfrak{g}$ of type $\frac{10|4|2}{16}$.*

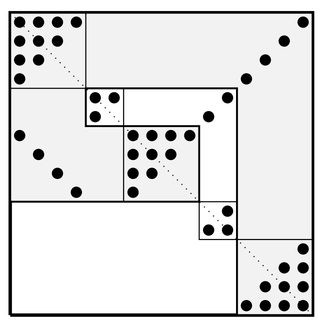


Figure 3.4: Indices in $\mathscr{I}_F$

A relations matrix of F is $B_4 \oplus B_2 \oplus B_4 \oplus B_2^R \oplus B_4^R$,

$$
\begin{pmatrix}
b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_5+b_6 & b_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_5 & b_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & b_6 & b_5 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & b_5 & b_5+b_6 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & b_4 & b_3 & b_2 & b_1 \\
 & & & & & & & & & & & & b_3 & b_2+b_4 & b_1+b_3 & b_1+b_2 \\
 & & & & & & & & & & & & b_2 & b_1+b_3 & b_1+b_2+b_4 & b_1+b_2+b_3 \\
 & & & & & & & & & & & & b_1 & b_1+b_2 & b_1+b_2+b_3 & b_1+b_2+b_3+b_4
\end{pmatrix}.
$$

3.3 Explicit Regular Functionals on $\mathfrak{gl}(n)$

The purpose of this section is to provide an explicit regular functional F_n on $\mathfrak{gl}(n)$. We will leverage this construction through Definition 3.2.5 to construct regular fuctionals on any seaweed subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(n)$ with homotopy type $H(c_1, \cdots, c_h)$ by embedding the functionals F_{c_i} appropriately.

Theorem 3.3.1. *The functional* $F_n = \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} e_{i,j}^*$ *is regular on* $\mathfrak{gl}(n)$*.*

The indices in $\mathscr{I}_{F_n}$ are illustrated in Figure 3.5.

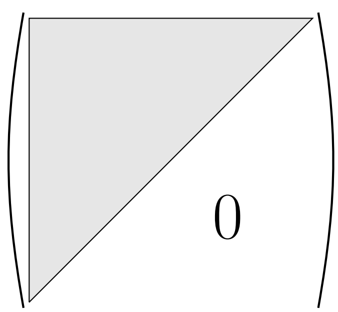


Figure 3.5: Indices in $\mathscr{I}_{F_n}$

Proof. Let $B = [b_{i,j}]$ be a relations matrix of $\ker(B_{F_n})$. It follows from Theorem 2.2.2 that the minimum dimension of $\ker(B_F)$ over all $F \in \mathfrak{g}^*$ is n. Therefore, to show that F_n is regular it suffices to determine n degrees of freedom b_i in B and show that every other entry in B is defined as a linear combination the elements b_i. The proof will follow from the verification of the following two claims:

Claim 1 For each $(i, j) \in (n-1) \times (n-1)$, $b_{i,j} = \sum_{s=1}^{n} c_s b_{s,n} + \sum_{s=1}^{n-1} c_s' b_{n,s}$ for suitable coefficients $c_s \in \mathbb{C}$,

and

Claim 2 $b_{n,s} = b_{s,n}$, for all $s \in [1, n]$.

To understand why these claims are sufficient, we argue as follows. Claim 1 will define the top $(n-1) \times (n-1)$ matrix in terms of the elements in the last row/column of B, determining there are at most $2n-1$ degrees of freedom in B. Claim 2 will then prove that there are exactly (as it cannot possibly be smaller due to minimality of the index) n degrees of freedom in these $2n-1$ positions.

Proof of Claim 1

The proof is by induction. We proceed according to the following steps.

1. An application of Lemma 3.1.4 ("symmetry lemma") halves the work by allowing us to only consider indices (i,j) illustrated in Figure 3.6.

2. The system of equations $F([B, e_{i,j}]) = 0$ is developed explicitly, along with two formulas which will be needed in the inductive step.

3. For the base case, we show that the first and last row are explicit sums of elements $b_{n,s}$. Proceeding by induction on pairs of rows (first and last) moving towards the center of B in the halved domain, we show that for $i \in \left[1, \lceil \frac{n}{2} \rceil\right]$, any elements $b_{i,j}$ and $b_{n+1-i,j}$ can be defined in terms of the previous row defined. More specifically, $b_{i,j} = b_{i-1,j-1} + b_{n,s}$ for some $s \in [1,n]$ and $b_{n+1-i,j} = b_{n+2-i,j-1} + b_{n,r}$ for some $r \in [1,n]$.

Step 1: For ease of notation, let $b_s = b_{n,s}$ and $b'_s = b_{s,n}$ for all $s \in [1,n]$ – note that $b_n = b'_n$. By invoking Lemma 3.1.4 (and making use of our convenient choice for b_s and b'_s being symmetric across the diagonal), it suffices to show the claim for all elements $b_{i,j}$ such that

$$(i,j) \in \mathscr{I} = \left\{ (i,j) \,\middle|\, i \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right],\ j \in [i, n+1-i] \right\}$$
$$\cup \left\{ (i,j) \,\middle|\, i \in \left(\left\lceil \frac{n}{2} \right\rceil, n\right],\ j \in (n+1-i, i] \right\}.$$

The indices in $\mathscr{I}$ are illustrated in Figure 3.6.

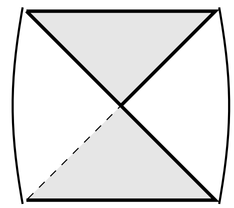


Figure 3.6: Indices in $\mathscr{I}$.

We will define $b_{i,j}$ in terms of elements b_s over all $(i,j) \in \mathscr{I}$, and it will follow that every $(i,j) \notin \mathscr{I}$ (and all (i,i) on the diagonal) are defined in terms of elements b'_s.

Step 2: To begin, observe

$$\mathscr{I}_{F_n} = \{(i,j) \mid 1 \le i \le n,\ \ 1 \le j \le n{+}1{-}i\} = \{(i,j) \mid 1 \le j \le n,\ \ 1 \le i \le n{+}1{-}j\}$$

and refer to Lemma 3.1.6 to see that B must satisfy n^2 conditions of the form

$$\sum_{s=1}^{n+1-j} b_{s,i} = \sum_{s=1}^{n+1-i} b_{j,s} \tag{3.11}$$

over $(i,j) \in \mathscr{I}_{\mathfrak{gl}(n)}$. There are no additional requirements on B as there are no forced zeroes in $\mathfrak{gl}(n)$. For all $i \in [2, n-1]$ consider the following equation obtained by applying equation (3.11) as follows:

$$\begin{aligned} b_{i,j} &= \sum_{s=1}^{i} b_{s,j} - \sum_{s=1}^{i-1} b_{s,j} \\ &= \sum_{s=1}^{n+1-(n+1-i)} b_{s,j} - \sum_{s=1}^{n+1-(n+1-(i-1))} b_{s,j} \\ &= \sum_{s=1}^{n+1-j} b_{n+1-i,s} - \sum_{s=1}^{n+1-j} b_{n+1-(i-1),s} \end{aligned}$$

$$\begin{aligned}
\Rightarrow b_{n+2-i,n+1-j} &= \left(\sum_{s=1}^{n-j} b_{n+1-i,s} - \sum_{s=1}^{n-j} b_{n+1-(i-1),s}\right) + b_{n+1-i,n+1-j} - b_{i,j} \\
&= \left(\sum_{s=1}^{n+1-(j+1)} b_{n+1-i,s} - \sum_{s=1}^{n+1-(j+1)} b_{n+1-(i-1),s}\right) + b_{n+1-i,n+1-j} - b_{i,j} \\
&= \left(\sum_{s=1}^{n+1-(n+1-i)} b_{s,j+1} - \sum_{s=1}^{n+1-(n+1-(i-1))} b_{s,j+1}\right) + b_{n+1-i,n+1-j} - b_{i,j} \\
&= \left(\sum_{s=1}^{i} b_{s,j+1} - \sum_{s=1}^{i-1} b_{s,j+1}\right) + b_{n+1-i,n+1-j} - b_{i,j} \\
&= b_{i,j+1} + b_{n+1-i,n+1-j} - b_{i,j}.
\end{aligned}$$

Thus, we get the following formula:

$$b_{n+2-i,n+1-j} = b_{i,j+1} + b_{n+1-i,n+1-j} - b_{i,j}. \tag{3.12}$$

By expressing $b_{i,j}$ as $\sum_{s=1}^{j} b_{i,s} - \sum_{s=1}^{j-1} b_{i,s}$ instead and applying equation (3.11), we get a second formula:

$$b_{n+1-i,n+1-j} = b_{i,j} + b_{n+2-i,n+1-j} - b_{i,j+1}. \tag{3.13}$$

Step 3: We proceed by induction. The base of the induction will be filling in the first and last rows of B for positions in $\mathscr{I}$ in terms of b_s. From there, assuming we have defined $b_{i,j}$ appropriately for all $(i,j) \in \mathscr{I}$ with $i \in [1, I] \cup [n+1-I, n]$ (i.e., the first and last I rows of B), we will define $b_{I+1,j}$ and $b_{n+1-(I+1),j}$ for $(I+1, j), (n+1-(I+1), j) \in \mathscr{I}$ in terms of elements b_s.

The last row is already filled by $b_{n,i} = b_i$ for $i \in [1, n]$. The first row comes from equation (3.11) evaluated for $j = n$:

$$b_{1,i} = \sum_{s=1}^{n+1-n} b_{s,i} = \sum_{s=1}^{n+1-i} b_{n,s} = \sum_{s=1}^{n+1-i} b_s. \tag{3.14}$$

Now, assume that for some $I \in [1, \lfloor \frac{n}{2} \rfloor)$, $b_{i,j}$ and $b_{n+1-i,j}$ are defined in terms of elements b_s for all $(i,j), (n+1-i, j) \in \mathscr{I}$ with $i \leq I$ (some care is needed if $I = \lfloor \frac{n}{2} \rfloor$

– we handle this separately for each claim depending on whether n is even or odd). We assert the following claims about indices

$(i,j),(n+1-i,j) \in \mathscr{I}$:

$$b_{i,j} = b_{i-1,j-1} - b_{n+3-i-j} \tag{3.15}$$

for $(i,j) \in \mathscr{I}$ such that $i \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right]$ if n is odd and $i \in \left[1, \frac{n}{2}+1\right]$ if n is even,

$$b_{n+1-i,j} = b_{n+2-i,j+1} + b_{j-i+1} \tag{3.16}$$

for $(i,j) \in \mathscr{I}$ such that $i \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor + 1\right]$ if n is odd and $i \in \left[1, \frac{n}{2}\right]$ if n is even. The need for the domains is due to the necessary conditions that $n+3-i-j, j-i+1 > 0$.

Assuming Equations (3.15) and (3.16) are true, consider $b_{I+1,j}$ and $b_{n+1-(I+1),j}$ (since $I < \lfloor \frac{n}{2} \rfloor$, there is no domain issue with these equations). By the inductive hypothesis and definition of $\mathscr{I}$, if $(I+1,j) \in \mathscr{I}$ and $(n+1-(I+1),j) \in \mathscr{I}$, then $(I,j-1) \in \mathscr{I}$ and $(n+1-I,j-1) \in \mathscr{I}$. Moreover, $b_{I,j-1}$ and $b_{n+1-I,j+1}$ are defined in terms of elements b_s, so $b_{I+1,j}$ and $b_{n+1-I,j}$ are also defined in terms of elements b_s for appropriate indices. Therefore, it suffices to prove these equations and then address $I = \lfloor \frac{n}{2} \rfloor$. We proceed by induction.

Assume that Equations (3.15) and (3.16) are true for all $b_{i,j}$ and $b_{n+1-i,j}$ with $i \in [2, I]$ and $(i,j),(n+1-i,j) \in \mathscr{I}$. We will show it's true for all appropriate $(I+1,j)$ and $(n+1-(I+1),j)$ and then we will show the basis step. Let $i = n+1-I$ and $s = n+1-j$. We invoke the formula in equation (3.12) twice below,

$$\begin{aligned} b_{I+1,j} &= b_{n+2-i,n+1-s} \\ &= b_{i,s+1} + b_{n+1-i,n+1-s} - b_{i,s} \\ &= b_{n+1-I,s+1} + b_{I,n+1-s} - b_{n+1-I,s}, \end{aligned}$$

and

$$\begin{aligned} b_{I,j-1} &= b_{n+2-(i+1),n+1-(s+1)} \\ &= b_{i+1,s+2} + b_{n+1-(i+1),n+1-(s+1)} - b_{i+1,s+1} \\ &= b_{n+2-I,s+2} + b_{I-1,n-s} - b_{n+2-I,s+1}. \end{aligned}$$

Therefore, by the induction hypotheses on $(I, s+1)$, $(I, n+1-s)$, and (I, s), we get

$$\begin{aligned} b_{I,j-1} - b_{I+1,j} &= (b_{n+2-I,s+2} - b_{n+1-I,s+1}) + (b_{I-1,n-s} - b_{I,n+1-s}) - (b_{n+2-I,s+1} - b_{n+1-I,s}) \\ &= -b_{s+2-I} + b_{n+3-I-(n+1-s)} + b_{s+1-I} \\ &= -b_{s+2-I} + b_{s+2-I} + b_{s+1-I} \\ &= b_{s+1-I} \\ &= b_{n+3-j-(I+1)}. \end{aligned}$$

It is easy to see this is the desired result for the formula in (3.15) on $(I+1, j)$. Now, if n is even we can repeat the above steps for $I = \frac{n}{2}$ since all of the above indices will still be defined (i.e., since $i = I+1$ and $(i,j) \in \mathscr{I}$ forces $j = \frac{n}{2}$ or $j = \frac{n}{2}+1$, we know

$$n + 3 - j - (I+1) = n + 3 - j - \frac{n+2}{2} = 1 + \frac{n+2}{2} - j = \frac{n}{2} + 2 - j \in \{1, 2\}$$

which gives the extended domain for the formula in (3.15) if n is even; this does not hold if n is odd).

In a similar manner, we invoke Formula 3.13 twice below,

$$\begin{aligned} b_{n+1-(I+1),j} &= b_{n+1-(I+1),n+1-s} \\ &= b_{I+1,s} + b_{n+2-(I+1),n+1-s} - b_{I+1,s+1} \\ &= b_{I+1,s} + b_{n+1-I,n+1-s} - b_{I+1,s+1}, \end{aligned}$$

and

$$\begin{aligned} b_{n+2-(I+1),j+1} &= b_{n+1-I,n+1-(s-1)} \\ &= b_{I,s-1} + b_{n+2-I,n+1-(s-1)} - b_{I,s} \\ &= b_{I,s-1} + b_{n+2-I,n+2-s} - b_{I,s}. \end{aligned}$$

By the inductive hypotheses and equation (3.15), with the result above being proven for $I+1$, $(I+1, s)$, $(I, n+1-s)$, and $(I+1, s+1)$, we get

$$
\begin{aligned}
b_{n+1-(I+1),j} - b_{n+2-(I+1),j+1} &= (b_{I+1,s} - b_{I,s-1}) + (b_{n+1-I,n+1-s} - b_{n+2-I,n+2-s}) - (b_{I+1,s+1} - b_{I,s}) \\
&= -b_{n+3-(I+1)-s} + b_{n+1-s-I+1} + b_{n+3-(I+1)-(s+1)} \\
&= -b_{n+2-I-s} + b_{n+2-I-s} + b_{n+1-I-s} \\
&= b_{n+1-I-s} \\
&= b_{n+1-I-(n+1-j)} \\
&= b_{j-(I+1)+1}.
\end{aligned}
$$

It is easy to see that this is the desired result for the formula in (3.16) on $(I+1, j)$. Further, if n is odd the above calculations for $I = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ may be repeated since all of the above indices will still be defined (i.e., since $i = n+1-(I+1) = \frac{n+1}{2}$ and $(i,j) \in \mathscr{I}$ forces $j = \frac{n+1}{2}$, we know

$$
j - (I+1) + 1 = \frac{n+2}{2} - \frac{n+2}{2} + 1 = 1
$$

which gives the extended domain for Formula (3.16) if n is odd; this does not hold if n is even). Therefore, To complete Claim 1 it suffices show that this result holds as a relation between the indices of $\mathscr{I}$ for $I = 2$ (the basis step).

Recall from equation (3.14), that $b_{1,j} = \sum_{s=1}^{n+1-j} b_s$ for all $j \in [1, n]$. By the formula in (3.12),

$$
\begin{aligned}
b_{2,j} &= b_{n,n+2-j} + b_{1,j} - b_{n,n+1-j} \\
&= b_{n+2-j} + \left(\sum_{s=1}^{n+1-j} b_s \right) - b_{n+1-j} \\
&= \left(\sum_{s=1}^{n+2-j} b_s \right) - b_{n+2-j} + b_{n+2-j} - b_{n+1-j} \\
&= b_{1,j-1} + b_{n+1-j}.
\end{aligned}
$$

This proves the formula in (3.15) for $I = 2$. Now, by the formula in (3.13) and the equation justified immediately above,

$$\begin{aligned}
b_{n-1,j} &= b_{2,n+1-j} + b_{n,j} - b_{2,n+2-j} \\
&= (b_{1,n-j} - b_j) + b_j - (b_{1,n+1-j} - b_{j-1}) \\
&= \left(\sum_{s=1}^{n+1-(n-j)} b_s \right) - \left(\sum_{s=1}^{n+1-(n+1-j)} b_s \right) + b_{j-1} \\
&= \left(\sum_{s=1}^{j+1} b_s \right) - \left(\sum_{s=1}^{j} b_s \right) + b_{j-1} \\
&= b_{j+1} + b_{j-1} \\
&= b_{n,j+1} + b_{(j-2)+1}.
\end{aligned}$$

This establishes Claim 1.

<u>Proof of Claim 2</u>

The proof is by induction along the diagonal of B through the application of Lemma 3.1.4, as every element $b_{i,i}$ will be defined explicitly in terms of elements $b_{n,s}$ and $b_{s,n}$ through Claim 1. Odd and even values of n are addressed separately to carefully cross the center of B.

Assume that n is even.

Claim 2.1.1: For $k \in \left[0, \frac{n}{2} - 1\right]$,

$$b_{n-k,n-k} = \sum_{s=0}^{k} b_{n-2s} \qquad \text{and} \qquad b_{n-2k} = b'_{n-2k}.$$

Claim 2.1.1 will yield that for all even indices s, $b_s = b'_s$ – which is half of Claim 2, provided that n is even. The proof is by induction. Trivially, $b_{n,n} = b_n = b'_n$. Further, by Formula (3.16),

$$b_{n-1,n-1} = b_{n,n} + b_{n-2}$$

(this is seen using $i = 2$, $j = n-1$). Now, by Lemma 3.1.4, since $b_{n-1,n-1} = b_{n-1,n-1}$ trivially we get

$$b'_n + b'_{n-2} = b_n + b_{n-2} \qquad \Rightarrow \qquad b_{n-2} = b'_{n-2}.$$

Now, fix $k < \frac{n}{2} - 1$ assume for all $K \leq k$ the claim holds. By Formula (3.16), we have

$$b_{n-(k+1),n-(k+1)} = b_{n-k,n-k} + b_{n-2k-2} = \sum_{s=0}^{k} b_{n-2s} + b_{n-2(k+1)}$$

(this is seen using $i = k+2$, $j = n-(k+1)$). By Lemma 3.1.4, we have

$$\sum_{s=0}^{k} b_{n-2s} + b_{n-2(k+1)} = \sum_{s=0}^{k} b'_{n-2s} + b'_{n-2(k+1)}.$$

By induction, $b_{n-2(k+1)} = b'_{n-2(k+1)}$ and $b_{n-(k+1),n-(k+1)} = \sum_{s=0}^{k+1} b_{n-2s}$. This proves Claim 2.1.1.

Claim 2.1.2: For $k \in \left[0, \frac{n}{2} - 1\right]$,

$$b_{\frac{n}{2}-k,\frac{n}{2}-k} = b_{\frac{n}{2}+1,\frac{n}{2}+1} + \sum_{s=0}^{k} b_{1+2s} \qquad \text{and} \qquad b_{1+2k} = b'_{1+2k}$$

Claim 2.1.2 will yield that for all odd indices s, $b_s = b'_s$ which is the second half of Claim 2 provided n is even. The proof is by induction.

Since n is even, by Formula (3.15) and the fact that $n - (\frac{n}{2} - 1) = \frac{n}{2} + 1$, we can see that

$$b_{\frac{n}{2},\frac{n}{2}} = b_{\frac{n}{2}+1,\frac{n}{2}+1} + b_{n+3-(\frac{n}{2}+1)-(\frac{n}{2}+1)} = b_{\frac{n}{2}+1,\frac{n}{2}+1} + b_1.$$

By Lemma 3.1.4 and Claim 2.1.1, $b_1 = b'_1$. Now, assume for some $k < \frac{n}{2} - 1$ we have that the above claim WHICH ONE? holds for all $K \leq k$. Then by Formula (3.15), we have

$$b_{\frac{n}{2}-(k+1),\frac{n}{2}-(k+1)} = b_{\frac{n}{2}-k,\frac{n}{2}-k} + b_{3+2k} = \left(b_{\frac{n}{2}+1,\frac{n}{2}+1} + \sum_{s=0}^{k} b_{1+2s}\right) + b_{1+2(k+1)}.$$

By Lemma 3.1.4, $b_{1+2(k+1)} = b'_{1+2(k+1)}$ and $b_{\frac{n}{2}-(k+1),\frac{n}{2}-(k+1)} = b_{\frac{n}{2}+1,\frac{n}{2}+1} + \sum_{s=0}^{k+1} b_{1+2s}$, as desired. This proves the Claim 2.1.2 and suffices to establish Claim 2 if n is even.

Assume that n is odd.

We will take an identical approach to that taken when n was even.

Claim 2.2.1: For $k \in \left[0, \frac{n-1}{2}\right]$,

$$b_{n-k,n-k} = \sum_{s=0}^{k} b_{n-2s} \qquad \text{and} \qquad b_{n-2k} = b'_{n-2k}.$$

Establishing Claim 2.2.1 will give us that all odd indices s, $b_s = b'_s$, which is half of Claim 2 provided n is odd. Because of the extended domain for Formula (3.16) when n is odd we can induction step further than in Claim 2.1.1, but otherwise the induction is identical to that used in the proof of Claim 2.1.1.

Claim 2.2.2: For $k \in \left[1, \frac{n-1}{2}\right]$,

$$b_{\frac{n+1}{2}-k,\frac{n+1}{2}-k} = b_{\frac{n+1}{2},\frac{n+1}{2}} + \sum_{s=1}^{k} b_{2s} \qquad \text{and} \qquad b_{2k} = b'_{2k}.$$

Establishing Claim 2.2.2 will complete the proof of Claim 2 by showing that for all even indices s, $b_s = b'_s$. The proof of Claim 2.2.2 is inductive.

For the base case (showing Claim 2.2.2 for $k = 1$), by the formula in (3.15), we have

$$b_{\frac{n+1}{2}-1,\frac{n+1}{2}-1} = b_{\frac{n+1}{2},\frac{n+1}{2}} + b_2,$$

as claimed. Further, by Lemma 3.1.4 we get $b_2 = b'_2$. Now, assume that for some $k \in [1, \frac{n-1}{2})$ the Claim 2.2.2 is true on all $K \leq k$. Then by Formula (3.15) and the induction hypotheses,

$$b_{\frac{n+1}{2}-(k+1),\frac{n+1}{2}-(k+1)} = b_{\frac{n+1}{2}-k,\frac{n+1}{2}-k} + b_{2+2k} = \left(b_{\frac{n+1}{2},\frac{n+1}{2}} + \sum_{s=0}^{k} b_{2s}\right) + b_{2(k+1)}.$$

By Lemma 3.1.4, we have that $b_{\frac{n+1}{2}-(k+1),\frac{n+1}{2}-(k+1)} = b_{\frac{n+1}{2},\frac{n+1}{2}} + \sum_{s=0}^{k+1} b_{2s}$ and $b_{2(k+1)} = b'_{2(k+1)}$, as desired. □

Example 3.3.2. *Consider $\mathfrak{gl}(5)$, and let F_5 be as defined in Theorem 3.3.1. A relations matrix B described in the proof of Theorem 3.3.1 is*

$$B = \begin{pmatrix} b_1+b_2+b_3+b_4+b_5 & b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 \\ b_1+b_2+b_3+b_4 & b_1+b_2+b_3+b_5 & b_1+b_2+b_4 & b_1+b_3 & b_2 \\ b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3+b_5 & b_2+b_4 & b_3 \\ b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3+b_5 & b_4 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix},$$

and we get the following basis for $\ker(B_{F_5})$*:*

$$\left\{ \begin{pmatrix} 1&1&1&1&1 \\ 1&1&1&1&0 \\ 1&1&1&0&0 \\ 1&1&0&0&0 \\ 1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&0 \\ 1&1&1&0&1 \\ 1&1&0&1&0 \\ 1&0&1&0&0 \\ 0&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0 \\ 1&1&0&1&0 \\ 1&0&1&0&1 \\ 0&1&0&1&0 \\ 0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \right\}.$$

By Lemma 3.2.4, we know that $F_n^{\widehat{t}}$ is regular on $\mathfrak{gl}(n)$.

3.4 Three More Regular Functionals

We define the ***size*** of a functional F to be equal to $|\mathscr{I}_F|$. Computationally, smaller is better. In this section we develop seven additional regular functionals H_n, H_n', K_n, K_n', G_n, G_n', and F_n' all of which are based on F_n. Their relative sizes are:

$$|\mathscr{I}_{F_n'}| \leq |\mathscr{I}_{G_n}| = |\mathscr{I}_{G_n'}| \leq |\mathscr{I}_{H_n}| = |\mathscr{I}_{H_n'}| \leq |\mathscr{I}_{K_n}| = |\mathscr{I}_{K_n'}| \leq |\mathscr{I}_{F_n}|.$$

The smallest of these functionals, $F_n' = (0) \oplus F_{n-2} \oplus (0)$, is smaller than F_n by $2n-1$ degrees of freedom (i.e., $|\mathscr{I}_{F_n}| = |\mathscr{I}_{F_n'}| + 2n - 1$). The proofs of these seven functionals are closely related, starting with G_n which is developed in Theorem 3.4.1 through an induction on the images of $e_{i,1}$ for $i \in [2, n]$ under $B_{G_n}([B, \cdot])$ to show that $b_{1,i} = 0$ (and a similar argument to show that $b_{i-1,n} = 0$). This will show that $\ker(B_{G_n})$ is a subalgebra of the seaweed of type $\frac{1|n-2|1}{1|n-2|1}$. Therefore, by Lemma 3.2.3 and the regularity of F_{n-2}, we obtain G_n's regularity.

Theorem 3.4.1. *The functional* $G_n = e^*_{1,1} + \sum_{i=2}^{n-1} \sum_{j=2}^{n+1-i} e^*_{i,j} = e^*_{1,1} \oplus F_{n-2} \oplus (0)$ *is regular on* $\mathfrak{gl}(n)$ *for* $n \geq 4$.

As a visual aid, the indices of the functional of G_n Theorem 3.4.1 is the functional such that $\mathscr{I}_{G_n}$ is the set of indices illustrated by the grey region in Figure 3.7.

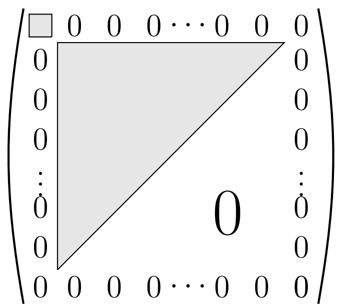


Figure 3.7: Indices in $\mathscr{I}_{G_n}$

Proof. Let $B = [b_{i,j}] \in \ker(B_{G_n})$. We first prove the following.

Claim: For $i \in [2, n]$, $b_{1,i} = b_{i,1} = b_{i-1,n} = b_{n,i-1} = 0$.

By establishing the claim, we will have shown that $\ker(B_{G_n})$ is a subalgebra of the seaweed $\mathfrak{gl}(1) \oplus \mathfrak{gl}(n-2) \oplus \mathfrak{gl}(1)$. We will then move on to a direct sum argument. By Lemma 3.1.5, to establish the Claim it suffices to show $b_{1,i} = b_{i-1,n} = 0$ for all $i \in [2, n]$. Note that

$$\mathscr{I}_{G_n} = \{(1,1)\} \cup \{(i,j) \mid 2 \leq i \leq n-1,\ 2 \leq j \leq n+1-i\}.$$

We start by showing $b_{1,i} = 0$ for all $i \in [2, n]$. By Lemma 3.1.6, $e_{n,1} \mapsto b_{1,n}$ since there is only one element in $\mathscr{I}_{G_n}$ which is of the form $(s, 1)$ and none of the form (n, s). Similarly,

$$e_{n,i} \mapsto \sum_{j=1}^{n-i} b_{j+1,n},$$

for $i \in [2, n-1]$, as there is are no indices in $\mathscr{I}_{G_n}$ of the form (n, s) and the only indices of the form (s, i) in $\mathscr{I}_{G_n}$ are indexed by $s \in [2, n+1-i]$. By evaluating

$B_{G_n}([B, e_{n,i}]) = 0$, we have immediately that $b_{1,n} = 0$ and

$$b_{n+1-i,n} = -\sum_{j=1}^{n-i-1} b_{j+1,n} \tag{3.17}$$

for all $i \in [2, n-1]$. For the basis step, $b_{1,2} = 0$ by evaluating equation (3.17) at $i = n$. Now, if $b_{s,n} = 0$ for all $s < k$, then from equation (3.17) evaluated at $i = n+1-k$, it follows that $b_{k,n} = -\sum_{j=1}^{k-2} b_{j+1,n} = 0$. By induction, $b_{i,n} = 0$ for all $i \in [1, n-1]$.

Now, by Lemma 3.1.6 we have for all $i \in [2, n-1]$, that

$$e_{1,i} \mapsto \left(\sum_{j=1}^{n-i} b_{j+1,1}\right) - b_{i,1} \tag{3.18}$$

since the only indices $(1, s)$ in $\mathscr{I}_{G_n}$ is $(1, 1)$ and the only indices (s, i) in $\mathscr{I}_{G_n}$ are indexed by $s \in [2, n-i+1]$. To show that $b_{i,1} = 0$ for all $i \in [2, n-1]$, we address n even and n odd separately. The problem is translated to a linear algebra problem by solving the matrix for the system of equations and showing that the vectors formed by the coefficients of $b_{i,1}$ are linearly independent. Hence, we will show the following set of vectors in $\mathbb{R}^{n-2}$ is linearly independent:

$$\mathscr{V} = \left\{ \begin{bmatrix} 1\\0\\0\\0\\0\\\vdots\\0\\0\\0\\0\\-1 \end{bmatrix}, \begin{bmatrix} 1\\1\\0\\0\\0\\\vdots\\0\\0\\0\\-1\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\1\\0\\0\\\vdots\\0\\0\\-1\\0\\0 \end{bmatrix}, \cdots, \begin{bmatrix} 1\\1\\0\\1\\1\\\vdots\\1\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\1\\1\\\vdots\\1\\1\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\1\\1\\\vdots\\1\\1\\1\\1\\1 \end{bmatrix} \right\}.$$

This set of vectors is formed by considering the order of the elements $b_{i,1}$ to be in reverse order (i.e., position i of a vector v is for the coefficient of $b_{n-i,1}$ in each equation), then the equations as row vectors for the positive coefficients (i.e., before subtracting $b_{i,1}$ from each equation) are

$$\begin{array}{c}
\langle 1 \;\; 1 \;\; 1 \;\; 1 \;\; \cdots \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rangle, \\
\langle 0 \;\; 1 \;\; 1 \;\; 1 \;\; \cdots \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rangle, \\
\langle 0 \;\; 0 \;\; 1 \;\; 1 \;\; \cdots \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rangle, \\
\langle 0 \;\; 0 \;\; 0 \;\; 1 \;\; \cdots \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rangle, \\
\vdots \\
\langle 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rangle, \\
\langle 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1 \;\; 1 \;\; 1 \rangle, \\
\langle 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; 1 \;\; 1 \rangle, \\
\langle 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; 0 \;\; 1 \rangle.
\end{array}$$

By subtracting off $b_{i,1}$ from equation i, we get the defined set of vectors $\mathscr{V}$. This $(n-2) \times (n-2)$ matrix is illustrated in Figure 3.8, for n even (left) and n odd (right).

$$\begin{array}{c}
R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{\frac{m}{2}} \\ v_{\frac{m}{2}} \\ \vdots \\ v_4 \\ v_3 \\ v_2 \\ v_1
\end{array}
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 0 & 1 & 1 & 1 \\
 & & & & \vdots & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & & \vdots & & & & \\
0 & 0 & 0 & -1 & \cdots & 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
0 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}
\qquad
\begin{array}{c}
R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{\frac{m-1}{2}} \\ V \\ v_{\frac{m-1}{2}} \\ \vdots \\ v_4 \\ v_3 \\ v_2 \\ v_1
\end{array}
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 \\
 & & & & \vdots & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & & \vdots & & & & & \\
0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}$$

Figure 3.8: Matrix of image coefficients for $e_{1,i}$ with $i \in [2, n-1]$ for n even (left) and n odd (right).

Assume that n is even. To show that the columns are linearly independent (and hence $b_{i,1} = 0$ for all $i \in [2, n-1]$), it suffices to show that there are $m = n - 2$ leading 1's. In other words, if v_i is the vector with a -1 in position i and 1's in positions $m+1-i$ through m over $i \in [1, \frac{m}{2}]$ (i.e., v_i enumerates the last half of the rows in the matrix above), we must show that we can get the leading 1's into the final $\frac{m}{2}$ positions of these vectors. For each $i < \frac{m}{2}$, we can get the leading 1 into position $m-i$ and then we can get the leading one for the vector $v_{\frac{m}{2}}$ into position m. For each i, let R_i represent the respective row in the top half of the matrix. Consider the vector $v_i' = v_i + R_i - R_{i+1}$ for each $i < \frac{m}{2}$. This yields the vector

$$\langle 0, 0, \cdots, 0, 1, 0, 1, 1, \cdots, 1 \rangle,$$

where the first 1 is in position $m-i$ ($v_i + R_i$ cancels the -1 in position i, puts a 1 in every zero between position $i+1$ and position $m-i$, position $m+1-i$ had a zero from R_i so it remains as a 1 from v_i, but everything after is now a 2, gaining 1 from each vector; when you subtract off R_{i+1}, everything from position $i+1$ to the end goes down by 1 except position $m-i$). Therefore, from vectors $v_1, \cdots, v_{\frac{m}{2}-1}$ we get the leading 1's in positions $\frac{m}{2}+1, \cdots, m-1$ which row-reduces the matrix in Figure 3.8 (left) to the matrix depicted in Figure 3.9.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{\frac{m}{2}} \\ v_{\frac{m}{2}} \\ v'_{\frac{m}{2}-1} \\ v'_{\frac{m}{2}-2} \\ v'_{\frac{m}{2}-3} \\ \vdots \\ v'_3 \\ v'_2 \\ v'_1 \end{array}
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 \\
 & & & & \vdots & & & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{pmatrix}$$

Figure 3.9: Reduced matrix of image coefficients for $e_{1,i}$ with $i \in [2, n-1]$ for even n

Now, consider the vector $v_{\frac{m}{2}}$ of the form $\langle 0, 0, \cdots, -1, 1, \cdots, 1\rangle$. Define

$$v'_{\frac{m}{2}} = v_{\frac{m}{2}} + R_{\frac{m}{2}} = \langle 0, \cdots, 0, 1, 2, \cdots, 2\rangle$$

with the 1 in position $\frac{m}{2}+1$. Consider the following six row operations applied to $v'_{\frac{m}{2}}$.

$$\begin{aligned}
v'_{\frac{m}{2}} - v'_{\frac{m}{2}-1} &= \langle 0, \cdots, 0, 2, 1, 1, \cdots, 1\rangle, && \text{2 in position } \frac{m}{2}+2;\\
v'_{\frac{m}{2}} - v'_{\frac{m}{2}-1} - 2v'_{\frac{m}{2}-2} &= \langle 0, \cdots, 0, 0, 1, -1, \cdots, -1\rangle, && \text{1 in position } \frac{m}{2}+3;\\
v'_{\frac{m}{2}} - v'_{\frac{m}{2}-1} - 2v'_{\frac{m}{2}-2} - v'_{\frac{m}{2}-3} &= \langle 0, \cdots, 0, -1, -2, \cdots, -2\rangle && \text{-1 in position } \frac{m}{2}+4.
\end{aligned}$$

Define $V := v'_{\frac{m}{2}} - v'_{\frac{m}{2}-1} - 2v'_{\frac{m}{2}-2} - v'_{\frac{m}{2}-3}$.

$$\begin{aligned}
V + v'_{\frac{m}{2}-4} &= \langle 0, \cdots, 0, -2, -1, -1, \cdots, -1\rangle, && \text{-2 in position } \frac{m}{2}+5;\\
V + v'_{\frac{m}{2}-4} + 2v'_{\frac{m}{2}-5} &= \langle 0, \cdots, 0, 0, -1, 1, \cdots, 1\rangle, && \text{-1 in position } \frac{m}{2}+6;\\
V + v'_{\frac{m}{2}-4} + 2v'_{\frac{m}{2}-5} + v_{\frac{m}{2}-6} &= \langle 0, \cdots, 0, 1, 2, \cdots, 2\rangle && \text{1 in position } \frac{m}{2}+7.
\end{aligned}$$

A result of the above calculation is that recursively, the matrix in Figure 3.9 can be row-reduced into a form which shows the column vectors are linearly independent, and thus $b_{i,1} = 0$ for $i \in [2, n-1]$ if n is even.

Now, assume that n is odd and refer to the matrix in Figure 3.8 (right). Again, let $m = n - 2$. We must ensure that there are leading 1's in the last $\frac{m+1}{2}$ columns to ensure that the matrix in Figure 3.8 is invertible. As was done for even n, define v_i as the vector with a -1 in position i and 1's in positions $m+1-i$ through m over $i \in [1, m-1/2]$ (the strict bottom half of the rows enumerated going up). Define $v'_i = v_i + R_i - R_{i+1}$ where R_i represents row i of Figure 3.8 (right). This puts a leading 1 in position $m - i$ for each i. In this way, the only column still in need of a leading 1 is the last column. The matrix in Figure 3.8 (right) is now row-reduced to the matrix shown in Figure 3.10.

$$
\begin{array}{c}
R_1 \\ R_2 \\ R_3 \\ R_4 \\ \vdots \\ R_{\frac{m-1}{2}} \\ V \\ v'_{\frac{m-1}{2}} \\ v'_{\frac{m-1}{2}-1} \\ v'_{\frac{m-1}{2}-2} \\ \vdots \\ v'_3 \\ v'_2 \\ v'_1
\end{array}
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 \\
 & & & & \vdots & & & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{pmatrix}
$$

Figure 3.10: Reduced matirx of image coefficients for $e_{1,i}$ with $i \in [2, n-1]$ for odd n

Now, let $V = \langle 0, \cdots, 0, 1, \cdots, 1\rangle$ where the first 1 is in position $\frac{m+1}{2} + 1$ (V is the center row of the matrix). Consider the following four row operations applied to V:

$$V - v'_{\frac{m-1}{2}-1} = \langle 0, \cdots, 0, 1, 0, \cdots, 0\rangle, \qquad \text{1 in position } \frac{m+1}{2} + 1;$$

$$V - v'_{\frac{m-1}{2}-1} - v'_{\frac{m-1}{2}-2} = \langle 0, \cdots, 0, -1, -1, \cdots, -1\rangle, \quad \text{-1 in position } \frac{m+1}{2} + 2.$$

Define $V' := V - v'_{\frac{m-1}{2}-1} - v'_{\frac{m-1}{2}-2}$.

$$V' + v'_{\frac{m-1}{2}-3} = \langle 0, \cdots, 0, -1, 0, \cdots, 0\rangle, \qquad \text{-1 in position } \frac{m+1}{2} + 3;$$

$$V' + v'_{\frac{m-1}{2}-3} + v'_{\frac{m-1}{2}-4} = \langle 0, \cdots, 0, 1, 1, \cdots, 1\rangle, \qquad \text{1 in position } \frac{m+1}{2} + 4.$$

Hence, we have row reduced the matrix in Figure 3.8 (right) to an invertible matrix. Therefore, the column vectors are linearly independent, and so $b_{i,1} = 0$ for $i \in [2, n-1]$ if n is odd. This establishes the claim.

By the claim, $\ker(B_{G_n})$ is a subalgebra of the seaweed $\mathfrak{gl}(1) \oplus \mathfrak{gl}(n-2) \oplus \mathfrak{gl}(1)$, and by Theorem 3.2.3, G_n is regular if and only if the restriction of G_n to its three components is regular in each case. We know $F = e^*_{1,1}$ and $F = 0$ are regular on $\mathfrak{gl}(1)$. Further, $\mathscr{I}_{G_n} \cap [2, n-2] \times [2, n-2]$ is the functional F_{n-2} from Theorem 3.3.1 defined on $\mathfrak{gl}(n-2)$. Hence, G_n is regular. Further, a relations matrix of $\ker(B_{G_n})$ is $(b_1) \oplus B \oplus (b_n)$, where B is a relations matrix of $\ker(B_{F_{n-2}})$ on $\mathfrak{gl}(n-2)$ which uses the variables $\{b_2, \cdots, b_{n-1}\}$. $\square$

Remark 3.4.2. *It is necessary to specify that $n \geq 4$ to use G_n, as it is not defined on $n = 1$ or $n = 2$, and for $n = 3$ there are not enough conditions to force $b_{1,2} = b_{2,1} = 0$. Instead, $e_{1,2} \mapsto b_{2,1} - b_{2,1} = 0$ and $e_{2,1} \mapsto b_{1,2} - b_{1,2} = 0$ under B_{G_3} and we lose those conditions. This adds 2 to the dimension of the kernel.*

As immediate corollaries to the proof of Theorem 3.4.1, we have two more regular functionals on $\mathfrak{gl}(n)$.

Theorem 3.4.3. *The functional $H_n = \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} e^*_{i,j} = F_{n-1}$ is regular on $\mathfrak{gl}(n)$.*

As a visual aid, the indices in $\mathscr{I}_{H_n}$ are illustrated in Figure 3.11.

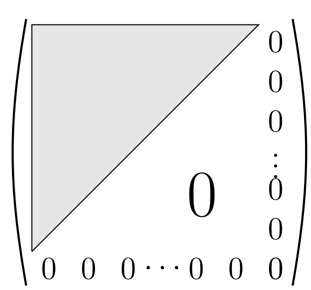


Figure 3.11: Indices in $\mathscr{I}_{H_n}$

Theorem 3.4.4. *The functional $K_n = e^*_{1,1} + \sum_{i=2}^{n} \sum_{j=2}^{n+2-i} e_{i,j} = G_{n+1}$ is regular on $\mathfrak{gl}(n)$.*

As a visual aid, the indices in $\mathscr{I}_{K_n}$ are illustrated in Figure 3.12.

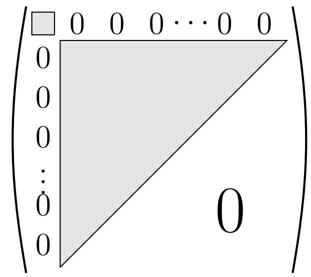


Figure 3.12: Indices in $\mathscr{I}_{K_n}$

Proof. The proof for Theorems 3.4.3 and 3.4.4 will follow from Lemma 3.2.3. Recall that $G_n = e_{1,1}^* \oplus F_{n-2} \oplus (0)$ is such that $\ker(B_{G_n})$ is a subalgebra of $gl(1) \oplus \mathfrak{gl}(n-2) \oplus \mathfrak{gl}(1)$ (see proof of theorem 3.4.1). Then $G_n = (e_{1,1}^* \oplus F_{n-2}) \oplus (0)$ is such that $\ker(B_{G_n})$ is a subalgebra of $\mathfrak{gl}(n-1) \oplus \mathfrak{gl}(1)$. By Lemma 3.2.3, $e_{1,1}^* \oplus F_{n-1}$ must be regular on $\mathfrak{gl}(n-1)$. This is K_n. Similarly, $G_n = e_{1,1}^* \oplus (F_{n-2} \oplus (0))$ is such that $\ker(B_{G_n})$ is a subalgebra of $\mathfrak{gl}(1) \oplus \mathfrak{gl}(n-1)$, so $F_{n-2} \oplus (0)$ is regular on $\mathfrak{gl}(n-1)$. These are the functionals proposed in Theorems 3.4.3 and 3.4.4. a relations matrix of $\ker(B_{H_n})$ is $B \oplus (b_n)$, where B is a relations matrix of $\ker(B_{F_{n-1}})$, and a relations matrix of $\ker(B_{K_n})$ is $(b_1) \oplus B'$, where B' is a relations matrix of $\ker(B_{F_{n-1}})$ defined on variables $\{b_2, \cdots, b_n\}$. □

Through an identical linear algebra argument to the one constructed in the proof of Theorem 3.4.1, we get four more functionals.

Theorem 3.4.5. *The functionals*

$$G'_n = 0 \oplus F_{n-2} \oplus e_{1,1}^*, \qquad K'_n = F_{n-1} \oplus e_{1,1}^*,$$

$$H'_n = 0 \oplus F_{n-1}, \qquad \text{and} \qquad F'_n = 0 \oplus F_{n-2} \oplus 0$$

are regular on $\mathfrak{gl}(n)$.

Let f be a functional of Theorem 3.4.5. A relations matrices for $\ker(B_f)$ is an appropriate direct sum of relations matrices of $\ker(B_{F_{n-1}})$, $\ker(B_{F_{n-2}})$, and (b_i).

As in Section 3.3, the transposition across the antidiagonal of any functional defined in this section is also regular on $\mathfrak{gl}(n)$. Note that F_n' is the smallest of the eight regular functionals thus far constructed.

Chapter 4

Semisimple Lie Algebras

In this chapter, we transition from building regular functionals on seaweed subalgebras of $\mathfrak{gl}(n)$ to building regular functionals on seaweed subalgebras of the classical Lie algebras: $A_n = \mathfrak{sl}(n+1)$, $B_n = \mathfrak{so}(2n+1)$, $C_n = \mathfrak{sp}(2n)$, and $D_n = \mathfrak{so}(2n)$. We note that the Kostant cascade, in general, fails to produce a regular functional on a given seaweed subalgebra of classical type. The obstruction to a successful cascade is a certain homotopy type (see Conjecture 5.2.5).

The main result of this Chapter is to show how the framework of Chapter 3 can be leveraged to create a regular functional on any seaweed subalgebra of classical type.

Due to the fundamental work by Killing, Ado, Cartan, Dynkin, and others, the simple Lie algebras are defined in terms of root systems, which we review in the following Definition 4.0.1.

Definition 4.0.1. *A subset R of a real inner-product space $(E, (\cdot,\cdot))$ is a* ***root system*** *if it satisfies the following axioms.*

1. *R is finite, it spans E, and it does not contain 0.*
2. *If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.*
3. *If $\alpha \in R$, then the reflection $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ permutes the elements of R.*
4. *If $\alpha, \beta \in R$, then $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.*

4.1 The Classification

Recall from Chapter 2 that each Lie algebra $\mathfrak{g}$ comes equipped with a triangular decomposition $\mathfrak{g} = \mathfrak{u}_- + \mathfrak{h} + \mathfrak{u}_+$, where $\mathfrak{u}_+$ and $\mathfrak{u}_-$ are the upper and lower triangular matrices respectively and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (specifically it is the diagonal matrices in $\mathfrak{g}$). Every Lie algebra $\mathfrak{g}$ has an associated root system, based on functionals $\alpha \in \mathfrak{g}^*$.

Definition 4.1.1. *For each $\alpha \in \mathfrak{g}^*$, define*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \ \forall H \in \mathfrak{h}\}.$$

*Define Φ as the set of nonzero α such that $\mathfrak{g}_\alpha \neq \emptyset$. The set Φ is the **root system** associated with $\mathfrak{g}$ and we call each $\alpha \in \Phi$ a **root** of $\mathfrak{g}$. The associated set $\mathfrak{g}_\alpha$ is called the **root space** of α in $\mathfrak{g}$. By definition, Φ is finite. Therefore, there exists a vector d which is not orthogonal to any $\alpha \in \Phi$. Allow the hyperplane through the origin defined by this vector to split the real inner-product space into two parts, where the positive roots are defined as those on the same side of the hyperplane as d and the negative roots are on the opposite side. The positive roots which cannot be written as a sum of positive roots are called simple. The set of simple roots is denoted by Π.*

Remark 4.1.2. *Note that the choice of positive roots and negative roots is not unique.*

A Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is called ***simple*** if it contains no nontrivial ***ideals*** (a subspace which is absorbing under $[\cdot,\cdot]$), and ***semisimple*** if it is a direct sum of simple Lie algebras. The methods developed in this chapter and in Appendix B.1 require the use of a *Chevalley basis* for $\mathfrak{g}$, (see Definition 4.1.3).

Definition 4.1.3. *Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Φ, and set of simple roots Π. For each $\alpha \in \Phi$, choose $h_\alpha \in \mathfrak{h}$ so that $h_\alpha \in [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$, and $\alpha(h_\alpha) = 2$. Further, choose $x_\alpha \in \mathfrak{g}_\alpha$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$ and $[x_\alpha, x_\beta] = \pm(p+1)e_{\alpha+\beta}$, where p is the greatest integer for which $\beta + p\alpha \in \Phi$. The set*

$$\{h_\alpha \mid \alpha \in \Pi\} \cup \{x_\alpha \mid \alpha \in \Phi\}$$

is a basis for $\mathfrak{g}$*, called a* ***Chevalley basis****.*

Example 4.1.4. *Consider the Lie algebra* $\mathfrak{g} = \mathfrak{sl}(4)$*. Its root system is*

$$\Pi_{A_3} = \{\alpha_i = e_i - e_{i+1} \mid i = 1, 2, 3\}$$

$$\Phi_{A_3} = \left\{ \pm\beta_{i,j} = \pm \sum_{s=i}^{j} \alpha_s = \pm(e_i - e_{j+1}) \;\middle|\; i < j \text{ and } i, j \in [1, 3] \right\}$$

(see Table B.2 and Appendix B.1 for more details on this). The Cartan subalgebra $\mathfrak{h}$ *is generated by* $\{e_{i,i} - e_{i+1,i+1} \mid i \in [1, 3]\}$*. Fix* $H = diag(h_1, h_2, h_3, h_4) \in \mathfrak{g}$ *an arbitrary element of* $\mathfrak{h}$ *(so* $\sum_{i=1}^{4} h_i = 0$*). Given a root* $\alpha = e_a - e_b \in \Phi$*, this corresponds to the functional* $e^*_{a,a} - e^*_{b,b} \in \mathfrak{g}^*$*. Fix* $X = [x_{i,j}] \in \mathfrak{g}$*, and note that* $[H, X] = K$*, where* K *is the matrix* $[(h_i - h_j)x_{i,j}]$*. Furthermore,* $\alpha(H) = (h_a - h_b)$*. To require* $K = \alpha(H)X$ *for all* $H \in \mathfrak{h}$ *(which must be true for* $X \in \mathfrak{g}_\alpha$*, by definition), we can see that it must be the case that* $x_{i,j} = 0$ *for any* $(i, j) \neq (a, b)$*, and therefore* $\mathfrak{g}_\alpha$ *is generated by* $e_{a,b}$*, and by a similar computation,* $\mathfrak{g}_{-\alpha}$ *is generated by* $e_{b,a}$*. Now,* $[e_{a,b}, e_{b,a}]$ *is the matrix with a 1 in position* (a, a) *and a -1 in position* (b, b)*. This element of* $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$ *is* h_α*, it satisfies that* $\alpha(h_\alpha) = 2$*, and we allow* $x_\alpha = e_{a,b}$*. It is easy to verify that the rest of the conditions in Definition 4.1.3 are satisfied, and the basis for* $\mathfrak{sl}(4)$ *is*

$$\{e_{i,i} - e_{i+1,i+1} \mid i = 1, 2, 3\} \cup \{e_{i,j} \mid i \neq j \text{ and } i, j \in [1, 4]\}.$$

Due to work of Dynkin [**16**], Cartan [**6**], and Ado [**1**], we know that every finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 can be represented as a Lie algebra of square matrices over $\mathbb{C}$ with the commutator bracket, and further if the algebra is simple then it is isomorphic to one of the following:

Classic Lie algebras: $A_n = \mathfrak{sl}(n+1)$, $B_n = \mathfrak{so}(2n+1)$, $C_n = \mathfrak{sp}(2n)$, $D_n = \mathfrak{so}(2n)$,

Exceptional Lie algebras: E_6, E_7, E_8, F_4, and G_2.

The first four algebra families are referred to as the ***classical*** Lie algebras, while the remaining five algebras are called the ***exceptional*** Lie algebras. The subscripts in each case refer to the number of simple roots (or the ***rank***, the dimension of the root space) associated with the algebra. In the typical way, we define $s_\alpha(\beta)$ for $\alpha, \beta \in \Phi$ to be the reflection of β across the hyperplane defined by α. In other words,

$$s_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

We define also

$$\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)}.$$

Lemma 4.1.5. *For any roots $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$,*

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

Proof. By definition, $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for any two roots, so it suffices to show there are only four possibilities. Recall that the angle θ between α and β is such that

$$(\alpha, \beta)^2 = (\alpha, \alpha)(\beta, \beta) \cos^2 \theta.$$

This gives

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2 \theta \leq 4. \tag{4.1}$$

The only way to have equality in (4.1) is if $\cos^2 \theta = 1$, meaning θ is an integer multiple of π. This would contradict our assumption that $\beta \neq \pm\alpha$, satisfying our claim. □

We detail one choice of simple roots of each of the simple Lie algebras above. They are listed in Table 4.1. The simple Lie algebras are defined by their *Dynkin diagrams*, a multi-graph which places a node for each simple root α_i and draws $d_{i,j} = \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges between α_i and α_j. In the case that $d_{i,j} > 1$, then α_i and α_j have different lengths, so an arrow is added pointing from the longer root to the shorter root. The nine Dynkin diagrams corresponding to the simple Lie algebras are illustrated in Table 4.2.

Algebra	Set of Simple Roots α_i
A_ℓ, $\ell \geq 1$	$\{\alpha_i := e_i - e_{i+1} \mid 1 \leq i \leq n\}$
B_ℓ, $\ell \geq 2$	$\{\alpha_i := e_i - e_{i+1} \mid 1 \leq i < n\} \cup \{\alpha_n := e_n\}$
C_ℓ, $\ell \geq 3$	$\{\alpha_i := e_i - e_{i+1} \mid 1 \leq i < n\} \cup \{\alpha_n := 2e_n\}$
D_ℓ, $\ell \geq 4$	$\{\alpha_n := e_i - e_{i+1} \mid 1 \leq i < n\} \cup \{\alpha_n := e_{n-1} + e_n\}$
G_2	$\{\alpha_1 := -2e_1 + e_2 + e_3,\ \alpha_2 := e_1 - e_2\}$
F_4	$\{\alpha_i := e_i - e_{i+1} \mid 1 \leq i < 3\} \cup \left\{\alpha_3 := e_3,\ \alpha_4 := \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)\right\}$
E_6	$\left\{\alpha_1 := \frac{1}{2}(-e_1 - e_8 + \sum_{i=2}^{7} e_i)\right\} \cup \{\alpha_2 := -e_1 - e_2\} \cup \{\alpha_i := e_{i-2} - e_{i-1} \mid 3 \leq i \leq 6\}$
E_7	$\left\{\alpha_1 := \frac{1}{2}(-e_1 - e_8 + \sum_{i=2}^{7} e_i)\right\} \cup \{\alpha_2 := -e_1 - e_2\} \cup \{\alpha_i := e_{i-2} - e_{i-1} \mid 3 \leq i \leq 7\}$
E_8	$\left\{\alpha_1 := \frac{1}{2}(-e_1 - e_8 + \sum_{i=2}^{7} e_i)\right\} \cup \{\alpha_2 := -e_1 - e_2\} \cup \{\alpha_i := e_{i-2} - e_{i-1} \mid 3 \leq i \leq 8\}$

Table 4.1: Simple roots for the simple Lie algebras

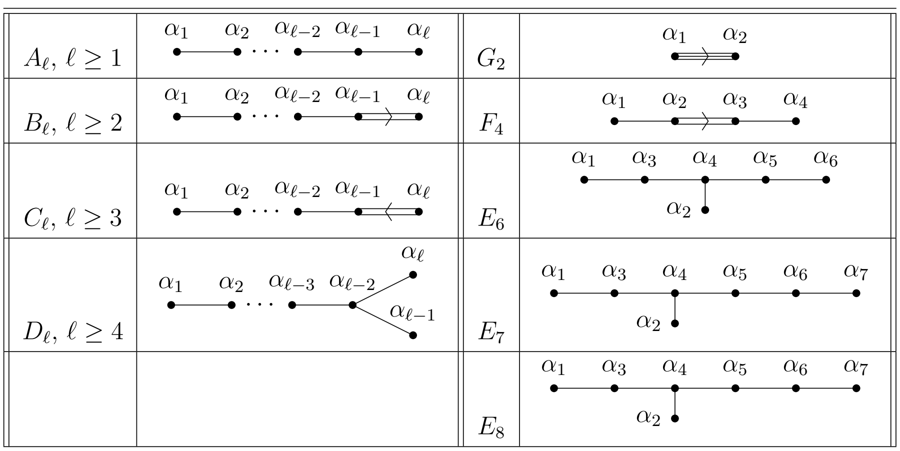


Table 4.2: Dynkin diagrams of the simple Lie algebras

Definition 4.1.6. *A* ***biparabolic*** *(or* ***seaweed****) subalgebra of a matrix Lie algebra* $\mathfrak{g}$ *is the intersection of two* ***parabolic*** *subalgebras* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *(subalgebras which contain a Borel subalgebra of* $\mathfrak{g}$*, such as the upper or lower triangular matrices) such that* $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$*. We call* $\mathfrak{p} \cap \mathfrak{p}'$ *standard if* $\mathfrak{p} \supseteq \mathfrak{h} \oplus \mathfrak{u}_+$ *and* $\mathfrak{p}' \supseteq \mathfrak{h} \oplus \mathfrak{u}_-$*. If* $\mathfrak{p} \cap \mathfrak{p}'$ *is a standard seaweed, let* Π *be the set of simple roots in the root system associated with* $\mathfrak{g}$*, and define* $\Psi = \{\alpha \in \Pi \mid \mathfrak{g}_{-\alpha} \not\subseteq \mathfrak{p}\}$*, and* $\Psi' = \{\alpha \in \Pi \mid \mathfrak{g}_{\alpha} \not\subseteq \mathfrak{p}'\}$*. It is customary to write* $\mathfrak{p} \cap \mathfrak{p}' = \mathfrak{p}_m(\Psi \mid \Psi')$*, where* $m = |\Pi|$*.*

As addressed by Joseph in **[21]**, every seaweed Lie algebra $\mathfrak{g} = \mathfrak{p} \cap \mathfrak{p}'$ is conjugate to a seaweed $\mathfrak{g}' = \mathfrak{p}_m(\Psi \mid \Psi')$ which is standard. It will be a standing assumption for the rest of this paper that $\mathfrak{g}$ refers to a seaweed in standard form. The rest of this chapter deals with semisimple Lie algebras of the classical variety.

It was previously noted (see **[3]**) that, given any seaweed $\mathfrak{g} = \mathfrak{p}_m(\Psi \mid \Psi')$, Kostant's cascade would be one method of producing a regular functional on $\mathfrak{g}$. This is NOT always the case. The cascade is explained in detail in Appendix B.1, and explicit counter examples to the regularity of the produced functional are provided therein.

4.2 Type-A Seaweeds

Recall that the simple Lie algebra A_n is $\mathfrak{sl}(n+1)$. It has simple roots

$$\alpha_i = e_i - e_{i+1} \text{ for } i \in [1, n].$$

We denote seaweed subalgebras of A_n by $\mathfrak{p}_n^A(\Psi \mid \Psi')$. By direct computation, the root system Φ_{A_n} is the set of roots $\pm\beta_{i,j} = \pm\sum_{s=i}^{j} \alpha_s$ over $1 \le i \le j \le n$. Further,

$$\mathfrak{g}_{\beta_{i,j}} = \text{span}\{e_{i,j+1}\} \qquad \text{and} \qquad \mathfrak{g}_{-\beta_{i,j}} = \text{span}\{e_{j+1,i}\}.$$

We fix the Chevalley basis for A_n to be

$$\{x_{\beta_{i,j}} = e_{i,j+1},\ x_{-\beta_{i,j}} = e_{j+1,i} \mid 1 \le i \le j \le n\} \cup \{h_{\alpha_i} = e_{i,i} - e_{i+1,i+1} \mid \alpha_i \in \Pi\}.$$

See Section B.1.1 and Example 4.1.4 for example computations of the root system and its root spaces for A_4 and A_3, respectively.

Example 4.2.1. *Consider $\mathfrak{g} = A_4$. The root spaces $\mathfrak{g}_\beta$ for $\beta \in \Phi_{A_4}$ are indicated in the root space matrix of Figure 4.1.*

$$\begin{pmatrix} * & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\ \mathfrak{g}_{-\alpha_1} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \\ \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{\alpha_3+\alpha_4} \\ \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_3} & * & \mathfrak{g}_{\alpha_4} \\ \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_4} & * \end{pmatrix}$$

Figure 4.1: Root space matrix for A_4

Now, to construct a parabolic subalgebra of A_n, one selects a set of simple roots to omit. The effect of this omission is demonstrated in the following example.

Example 4.2.2. *Consider the seaweed* $\mathfrak{g} = \mathfrak{p}_4^A(\{\alpha_4\} \mid \{\alpha_2, \alpha_3\})$*. By excluding* $\mathfrak{g}_{-\alpha_4}$ *from* $\mathfrak{p}$*, we eliminate the root spaces for all roots*

$$\{-\alpha_4,\ -\alpha_3 - \alpha_4,\ -\alpha_2 - \alpha_3 - \alpha_4,\ -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\}$$

from the algebra. This is pictured in Figure 4.2 (left). Similarly, by excluding $\mathfrak{g}_{\alpha_2}$ *and* $\mathfrak{g}_{\alpha_3}$ *from* $\mathfrak{p}'$*, we eliminate the root spaces for the eight roots which include* α_2 *or* α_3 *in their sum. This is illustrated in Figure 4.2 (center). These are parabolic subalgebras of* $\mathfrak{sl}(5)$*, and their intersection is shown in Figure 4.2 (right). It is easy to see that this is a subalgebra of the seaweed of type* $\frac{4|1}{2|1|2}$ *contained in* $\mathfrak{gl}(5)$ *(cf., Example 2.1.3).*

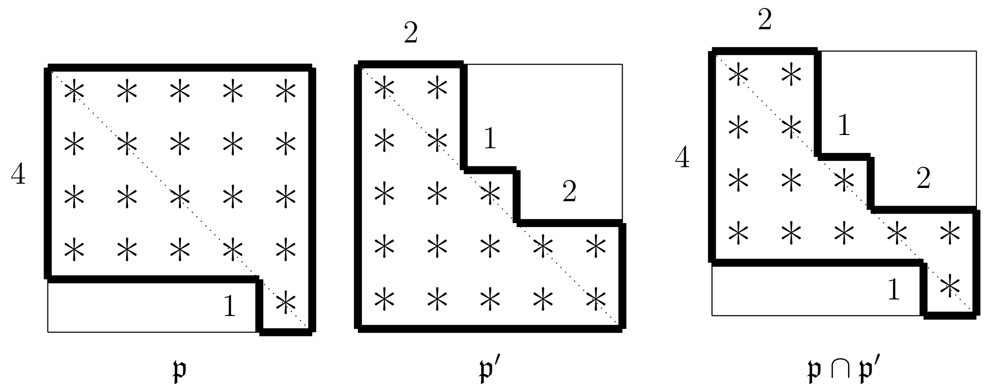


Figure 4.2: Construction of $\mathfrak{p}_4^A(\{\alpha_4\} \mid \{\alpha_2, \alpha_3\})$

In general, the seaweed $\mathfrak{p}_n^A(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_\ell}\})$ with $i_s < i_{s+1}$ and $j_s < j_{s+1}$ is a subalgebra of the seaweed $\mathfrak{g} \subseteq \mathfrak{gl}(n+1)$ of type $\frac{i_1|i_2-i_1|\cdots|i_k-i_{k-1}|n+1-i_k}{j_1|j_2-j_1|\cdots|j_\ell-j_{\ell-1}|n+1-j_\ell}$.

Notation: As we will often refer to the block compositions for the seaweed, we use the notation $\mathfrak{p}_n^A\ \frac{i_1|i_2-i_1|\cdots|i_k-i_{k-1}|n+1-i_k}{j_1|j_2-j_1|\cdots|j_\ell-j_{\ell-1}|n+1-j_\ell}$ to mean $\mathfrak{p}_n^A(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_\ell}\})$. We will adopt a similar naming convention in the other classical types. We define the meander, signature, and homotopy type of $\mathfrak{p}_n^A(\Psi \mid \Psi')$ according to Chapter 2.

4.2.1 Type-A Regular Functionals

To begin, we must know how restricting to algebras of trace zero affects the index of a seaweed.

Theorem 4.2.3 (Dergachev and A. Kirillov, [**13**])**.** *If $\mathfrak{g} = \mathfrak{p}_n^A(\Psi \mid \Psi')$, then*

$$\text{ind } \mathfrak{g} = 2C + P - 1,$$

where C is the number of cycles and P is the number of paths and isolated points in the meander associated with $\mathfrak{g}$.

We have the following immediate Corollary.

Theorem 4.2.4. *If $\mathfrak{g} = A_n$, then*

$$\text{ind } \mathfrak{g} = n.$$

Proof. Recall that if $\mathfrak{g} = A_n$, then $\mathfrak{g}$ is a seaweed subalgebra of the seaweed of type $\frac{n+1}{n+1} = \mathfrak{gl}(n+1)$ which has index $n+1 = 2C + P$ (since the meander for $\mathfrak{g}$ and the seaweed of type $\frac{n+1}{n+1}$ are the same). The result follows. □

Theorem 4.2.5. *The functional $F_n = \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} e_{i,j}^*$ of Theorem 3.3.1 is regular on A_n.*

Proof. Our proof will be through the equivalence of the systems of equations for $B_{F_n}(B, b)$ and $B_{F_n}(B, b')$ with $\{b\}$ a basis for A_n and $\{b'\}$ a basis for $\mathfrak{gl}(n)$.

Consider the Chevalley basis for A_n as our chosen basis. For all $i \neq j$ with $i, j \leq n$, we have

$$e_{i,j} \mapsto \left(\sum_{s=1}^{n+1-j} b_{s,i} - \sum_{s=1}^{n+1-i} b_{j,s} \right). \tag{4.2}$$

The system of equations which results when the expressions in (4.2) are evaluated at zero is identical to the system of equations for the image under B_{F_n} of basis elements $e_{i,j}$ with $i \neq j$ on $\mathfrak{gl}(n)$. Now, consider the basis elements $e_{i,i} - e_{i+1,i+1}$

with $i \leq n$. By requiring $B_{F_n}(B, e_{i,i} - e_{i+1,i+1}) = 0$, we get the weaker condition that $B_{F_n}(B, e_{i,i}) = B_{F_n}(B, e_{i+1,i+1})$ for all i. However, we have

$$B_{F_n}(B, e_{n+1,n+1}) = 0$$

as there are no indices $(i,j) \in \mathscr{I}_{F_n}$ with $i = n+1$ or $j = n+1$. Therefore, the system of equations for the n^2 basis elements of A_n mentioned thus far is isomorphic to the system of equations on $\mathfrak{gl}(n)$. It suffices to address the last $2n$ basis elements $e_{i,n+1}$ and $e_{n+1,i}$ for $i \neq n+1$. Consider the image of the first n basis elements under $B_{F_n}([B, \cdot])$:

$$e_{i,n+1} \mapsto -\sum_{s=1}^{n+1-i} b_{n+1,s}. \tag{4.3}$$

Setting the right hand side of (4.3) equal to zero and inducting from $i = n$ down to $i = 1$, we get $b_{n+1,i} = 0$ for all $i \in [1, n]$. The argument is similar for $b_{i,n+1} = 0$ for all $i \in [1, n]$, or Lemma 3.1.5 may be applied. The resulting relations matrix of $\ker(B_{F_n})$ on A_n is $B \oplus (b)$, where B is a relations matrix of $\ker(B_{F_n})$ defined on $\mathfrak{gl}(n)$ and $b = -\sum_{i=1}^{n} b_{i,i}$. $\square$

Remark 4.2.6. *Note that F_n defined on A_n is the sum of functionals $e_{i,j}^*$* ***strictly above*** *the antidiagonal, as A_n is a subalgebra of $\mathfrak{gl}(n+1)$, and not $\mathfrak{gl}(n)$.*

Remark 4.2.7. *The other functionals of Section 3.4 when defined on $\mathfrak{gl}(n+1)$ will also be regular on A_n, but we omit spending further time on the proving of regular functionals here.*

Now, we address proper seaweed subalgebras of A_n.

Theorem 4.2.8. *If $\mathfrak{p}_n^A \frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ is a seaweed with homotopy type $H(c_1, \cdots, c_h)$, any funcitonal $\overline{F}$ built using Definition 3.2.5 with functionals f_{c_i} embedded into the components of size c_i is such that*

$$\dim\ker(B_{\overline{F}}) = -1 + \sum_{i=1}^{h} \dim\ker(B_{f_{c_i}}),$$

where $\dim\ker(B_{\overline{F}})$ *is over $\mathfrak{sl}(n+1)$, but* $\dim\ker(B_{f_{c_i}})$ *is the dimension of the kernel in $\mathfrak{gl}(c_i)$.*

Proof. The proof is similar to the proof of Theorem 3.2.8. We apply induction along the winding-up moves of Lemma 2.3.4 to place appropriately adjusted copies of $\ker(B_{f_{c_i}})$ into the core of $\mathfrak{g}$. The difference in Type-A is that in the first Component Creation move in the winding-up of the meander associated with $\mathfrak{g}$, there is an index t such that $b_{t,t}$ is the negative sum of the diagonal to ensure the vanishing trace condition of A_n (this was $b_{n+1,n+1}$ for B a relations matrix of F_n in Theorem 4.2.5). Under the winding up, we will map the functional as we did in Definition 3.2.5, but the kernel adjustments must be such that the sum of all instances of $b_{t,t}$ on the diagonal maintains the vanishing trace condition. $\square$

This completely resolves the problem of naming regular functionals for seaweed subalgebras of Type-A. See Example 4.2.9.

Example 4.2.9. *Consider $\mathfrak{g} = \mathfrak{p}_7^A \frac{4|4}{8}$. The functional described in Theorem 4.2.8 by embedding F_3 of Theorem 4.2.5 yields*

$$\overline{F}^a = e_{1,1}^* + e_{1,2}^* + e_{1,3}^* + e_{1,8}^* + e_{2,1}^* + e_{2,2}^* + e_{2,7}^* + e_{3,1}^* + e_{3,6}^* + e_{4,5}^* + e_{6,8}^* + e_{7,7}^* + e_{7,8}^* + e_{8,6}^* + e_{8,7}^* + e_{8,8}^*.$$

The indices for $\overline{F}^a$ are shown in Figure 4.3.

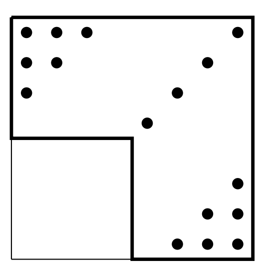


Figure 4.3: Indices $\mathscr{I}_{\overline{F}^a}$ on $\mathfrak{p}_7^A(\{\alpha_4\} \mid \emptyset)$

A messy calculation yields a relations matrix B of $\ker(B_{\overline{F}^a})$. Note that blank spaces are inadmissible locations for the seaweed, and are, per force, filled with zeroes.

This relations matrix B is given by

$$\begin{pmatrix} b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 & 0 \\ b_1+b_2 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2b_1-b_2-3b_3 & 0 & 0 & 0 & 0 \\ & & & & -2b_1-b_2-3b_3 & 0 & 0 & 0 \\ & & & & 0 & b_3 & b_2 & b_1 \\ & & & & 0 & b_2 & b_1+b_3 & b_1+b_2 \\ & & & & 0 & b_1 & b_1+b_2 & b_1+b_2+b_3 \end{pmatrix}.$$

It is apparent that, since $\operatorname{ind}\mathfrak{g}=3$, $\overline{F}^a$ *is regular.*

4.3 Type-C

Recall that the simple Lie algebra C_n is $\mathfrak{sp}(2n)$. It has simple roots

$$\alpha_i = e_i - e_{i+1} \text{ for } i \in [1, n-1], \quad \alpha_n = 2e_n.$$

We denote seaweed subalgebras of C_n by $\mathfrak{p}_n^C(\Psi \mid \Psi')$. Note that

$$\begin{aligned} \Phi_{C_n} &= \left\{ \pm\beta_{i,j} = \pm\sum_{s=i}^{j}\alpha_s \;\middle|\; 1 \le i \le j < n \right\} \\ &\cup \left\{ \pm\rho_i = \pm\alpha_n \pm 2\sum_{s=i}^{n-1}\alpha_s \;\middle|\; i \in [1,n] \right\} \\ &\cup \left\{ \pm\delta_{i,j} = \pm\sum_{s=i}^{n-1}\alpha_s \pm \sum_{s=j}^{n-1}\alpha_s + \alpha_n \;\middle|\; 1 \le i < j \le n \right\}. \end{aligned}$$

We now compute the Chevalley basis and the root spaces $\mathfrak{g}_\alpha$ for $\mathfrak{g} = C_n$. Consider an element $H = \operatorname{diag}(h_1, \cdots, h_{2n})$ of the Cartan subalgebra $\mathfrak{h}$ of C_n. It must be true that $h_i = -h_{2n+1-i}$ for all i. Recall from Example 4.1.4 that for any $X = [x_{i,j}] \in \mathfrak{g}$, $[H, X] = K$, where K is the matrix $[(h_i - h_j)x_{i,j}]$. As elements of the dual, we have $\beta_{i,j} = e^*_{i,i} - e^*_{j+1,j+1}$, $\rho_i = 2e^*_{i,i}$, and $\delta_{i,j} = e^*_{i,i} + e^*_{j,j}$. Therefore, $\beta_{a,b}(H)X = (h_a - h_{b+1})X$, and $K = \beta_{a,b}(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, b+1), (2n-b, 2n+1-a)$. Thus,

$$\mathfrak{g}_{\beta_{i,j}} = \operatorname{span}\{e_{i,j+1} - e_{2n-j,2n+1-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\beta_{i,j}} = \operatorname{span}\{e_{j+1,i} - e_{2n+1-i,2n-j}\}.$$

Similarly, $\rho_a(H)X = 2h_aX = (h_a - h_{2n+1-a})X$, and $K = \rho_a(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, 2n+1-a)$. Thus,

$$\mathfrak{g}_{\rho_i} = \operatorname{span}\{e_{i,2n+1-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\rho_i} = \operatorname{span}\{e_{2n+1-i,i}\}.$$

Finally, $\delta_{a,b}(H)X = (h_a + h_b)X = (h_a - h_{2n+1-b})X$, and $K = \delta_{a,b}(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, 2n+1-b), (b, 2n+1-a)$. Hence,

$$\mathfrak{g}_{\delta_{i,j}} = \operatorname{span}\{e_{i,2n+1-j} + e_{j,2n+1-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\delta_{i,j}} = \operatorname{span}\{e_{2n+1-j,i} + e_{2n+1-i,j}\}.$$

We fix the Chevalley basis for C_n to be following union:

$$\begin{aligned}
&\{x_{\beta_{i,j}} = e_{i,j+1} - e_{2n-j,2n+1-i},\ x_{-\beta_{i,j}} = e_{j+1,i} - e_{2n+1-i,2n-j} \mid 1 \leq i \leq j < n\} \\
&\cup \{x_{\rho_i} = 2e_{i,2n+1-i},\ x_{-\rho_i} = 2e_{2n+1-i,i} \mid 1 \leq i \leq n\} \\
&\cup \{x_{\delta_{i,j}} = e_{i,2n+1-j} + e_{j,2n+1-i},\ x_{-\delta_{i,j}} = e_{2n+1-j,i} + e_{2n+1-i,j} \mid 1 \leq i < j \leq n\} \\
&\cup \{h_{\alpha_i} = e_{i,i} - e_{2n+1-i,2n+1-i} \mid i \in [1,n]\}.
\end{aligned}$$

See Table A.3 for the computation of the root system of C_4.

Example 4.3.1. *Consider $\mathfrak{g} = C_3$. The root spaces $\mathfrak{g}_\beta$ for $\beta \in \Phi_{C_3}$ are indicated in the root space matrix of Figure 4.4.*

$$\begin{pmatrix}
* & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3} & \mathfrak{g}_{2\alpha_1+2\alpha_2+\alpha_3} \\
\mathfrak{g}_{-\alpha_1} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{2\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3} \\
\mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \\
\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_3} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2} \\
\mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3} & \mathfrak{g}_{-2\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_1} \\
\mathfrak{g}_{-2\alpha_1-2\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_1} & *
\end{pmatrix}$$

Figure 4.4: Root space matrix for C_3

Now, to construct a parabolic subalgebra of C_n, one selects a set of simple roots to omit. The effect of such an omission is demonstrated in the following example.

Example 4.3.2. *Consider the seaweed* $\mathfrak{g} = \mathfrak{p}_3^C(\{\alpha_3\} \mid \{\alpha_2\})$*. As was done in Type-A, we eliminate any root space from* $\mathfrak{p}$ *which relied on the root* $-\alpha_3$*, and we eliminate any root space from* $\mathfrak{p}'$ *which relied on* α_2*. The parabolic subalgebras* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *are illustrated in Figure 4.3.4 below (left and center, respectively). These are parabolic subalgebras of* $\mathfrak{sp}(6)$*, and the seaweed* $\mathfrak{g}$ *(the intersection of* $\mathfrak{p}$ *and* $\mathfrak{p}'$*) is then displayed in Figure 4.3.4 (right). It is easy to see that this is a subalgebra of the seaweed* $\mathfrak{g}' \subseteq \mathfrak{gl}(6)$ *of type* $\frac{3|3}{2|2|2}$*.*

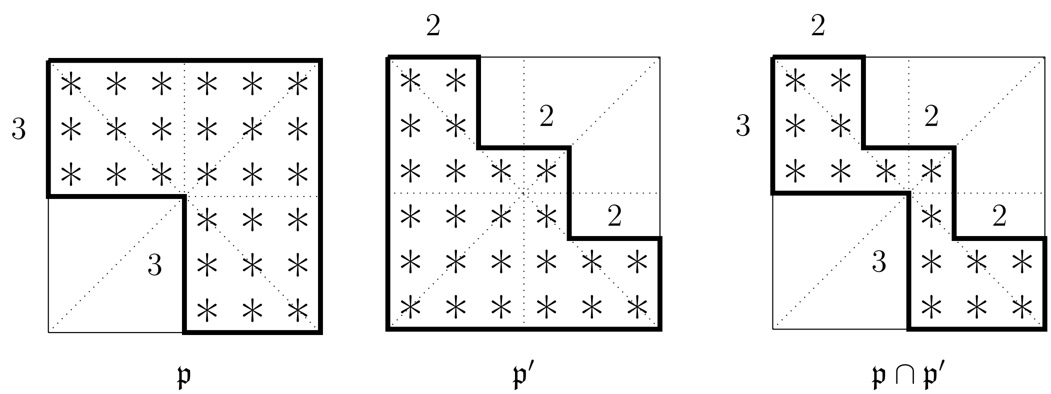


Figure 4.5: Construction of $\mathfrak{p}_3^C(\{\alpha_3\} \mid \{\alpha_2\})$

In general, the seaweed $\mathfrak{p}_n^C(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$ with $i_s < i_{s+1}$ and $j_s < j_{s+1}$ is a subalgebra of the seaweed $\mathfrak{g} \subseteq \mathfrak{gl}(2n)$ of type

$$\frac{i_1|i_2 - i_1|\cdots|i_k - i_{k-1}|2n - 2i_k|i_k - i_{k-1}|\cdots|i_2 - i_1|i_1}{j_1|j_2 - j_1|\cdots|j_t - j_{t-1}|2n - 2j_t|j_t - j_{t-1}|\cdots|j_2 - j_1|j_1}. \tag{4.4}$$

To ease computations, we can leverage the symmetry across the antidiagonal of a seaweed subalgebra subalgebra of C_n, and make use of a meander on n vertices instead of the full $2n$ vertices that a seaweed subalgebra of $\mathfrak{gl}(2n)$ would normally require.

Definition 4.3.3. *Let* $\mathfrak{g} = \mathfrak{p}_n^C(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$ *with* $i_s < i_{s+1}$ *and* $j_s < j_{s+1}$*. The meander associated with* $\mathfrak{g}$ *(denoted* M_n^C *to indicate that it has* *been shortened from the meander of the full type), is constructed as follows. Place n vertices v_1 through v_n in a line. Create two partitions (top and bottom) of the vertices based on the partial compositions of n,*

$$\overline{a} = (i_1, i_2 - i_1, \cdots, i_k - i_{k-1})$$

and

$$\overline{b} = (j_1, j_2 - j_1, \cdots, j_t - j_{t-1}).$$

Draw arcs in the first k top blocks and the first t bottom blocks as you would a meander in $\mathfrak{gl}(n)$. There may be vertices left over. We define the following sets: $T_a = \{v_i \mid i > i_k\}$ and $T_b = \{v_i \mid i > j_t\}$. The ***tail*** *$T_\mathfrak{g}$ of the meander M_n^C is the symmetric difference of T_a and T_b (i.e., $(T_a \cup T_b) \backslash (T_a \cap T_b)$). The* ***aftertail*** *$T_\mathfrak{g}^a$ of the meander is $T_a \cap T_b$. (The aftertail is so named because it consists of the vertices which occur to the right of the tail in M_n^C). (See Example 4.3.4).*

Example 4.3.4. *Consider the seaweed $\mathfrak{g} = \mathfrak{p}_7^C(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$. The meander M_n^C associated with $\mathfrak{g}$ is illustrated in Figure 4.6 (right). The bars and numbers in Figure 4.6 (left) are visual aids for the construction of the meander and not a part of the meander itself.*

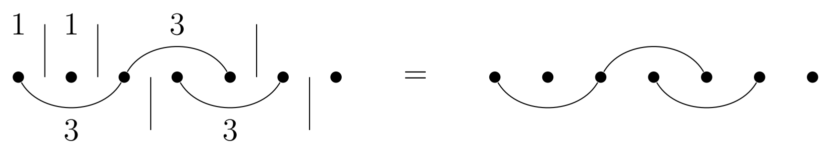


Figure 4.6: Meander M_7^C associated with $\mathfrak{p}_7^C(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$

By definition, we have

$$T_a = \{v_6, v_7\}, \qquad T_b = \{v_7\}, \qquad T_\mathfrak{g} = \{v_6\}, \qquad \text{and} \qquad T_\mathfrak{g}^a = \{v_7\}.$$

As in $\mathfrak{gl}(n)$, we can visualize the meander within the seaweed by mapping v_i to $e_{i,i}$ (see Figure 4.7). We color the tail vertices blue and the aftertail vertices red for illustration purposes.

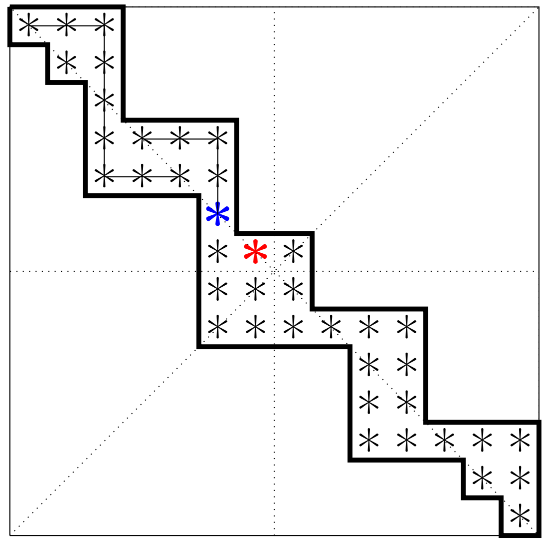


Figure 4.7: Meander M_7^C visualized in the seaweed $\mathfrak{p}_7^C(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$

When describing how to construct a regular functional in Section 4.3.1, we first address any component of the meander M_n^C which interacts with the tail, and we next address the aftertail. From there, the embedding of functionals F_n will be as it is in $\mathfrak{gl}(n)$ for any part of the meander which remains unaddressed.

4.3.1 Type-C Regular Functionals

To begin, we must know how restricting to algebras in $\mathfrak{sp}(2n)$ affects the index of a seaweed.

Theorem 4.3.5 (Coll, Hyatt, and Magnant [**9**])**.** *If $\mathfrak{g} = \mathfrak{p}_n^C(\Psi_1 \mid \Psi_2)$ is a seaweed of Type-C, then*

$$\text{ind}\,\mathfrak{g} = 2C + \tilde{P},$$

where C is the number of cycles and $\tilde{P}$ is the number of paths with zero or two endpoints in the tail of the meander M_n^C associated with $\mathfrak{g}$.

We have the following immediate Corollary.

Theorem 4.3.6. *If $\mathfrak{g} = C_n$, then*

$$\text{ind } \mathfrak{g} = n.$$

Proof. If $\mathfrak{g} = C_n$, then the meander M_n^C associated with $\mathfrak{g}$ consists of n vertices v_i all of which occur in the aftertail. This means M_n^C consists of n isolated points outside of the tail, contributing n to the index. □

Just as we used a meander half the size of the full meander in $\mathfrak{gl}(2n)$, when there is symmetry across the antidiagonal, it suffices to consider functional with indices on or above the antidiagonal only. The functional $F' = F + \sum_{(i,j)\in \mathscr{I}_F} c_{i,j} e^*_{2n+1-j,2n+1-i}$, where $c_{i,j}$ is the negative coefficient of $e^*_{i,j}$ in F for $i, j \in [1, n]$, and equal to the coefficient of $e^*_{i,j}$ in F otherwise, has the same kernel of the Kirillov form.

Theorem 4.3.7. *The Functional $F_n = \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} e^*_{i,j}$ of Theorem 3.3.1 is regular on C_n.*

Proof. Consider the Chevalley basis for C_n as our chosen basis. For $i, j \leq n$, we have

$$e_{i,j} - e_{2n+1-j,2n+1-i} \mapsto \left(\sum_{s=1}^{n+1-j} b_{s,i} - \sum_{s=1}^{n+1-i} b_{j,s} \right). \tag{4.5}$$

This system of equations which results from evaluating the n^2 expressions on the right hand side of (4.5) at zero is equivalent to the system of equations for the image of the basis elements $e_{i,j}$ for F_n defined on $\mathfrak{gl}(n)$. For (i, j) with $i + j \leq 2n + 1$, $i \leq n$, and $j > n$ we have

$$e_{i,j} + e_{2n+1-j,2n+1-i} \mapsto \left(-\sum_{s=1}^{n+1-i} b_{j,s} - \sum_{s=1}^{j-n} b_{2n+1-i,s} \right). \tag{4.6}$$

Through a linear algebra argument similar to those in Chapter 3, the solution to the system of equations which results from evaluating the right hand side of (4.6) at zero is $b_{j,i} = 0$ for all i, j defined. By Lemma 3.1.5, this implies $b_{i,j} = 0$. A relations matrix of $\ker(B_{F_n})$ is $B \oplus (-B^{\widehat{t}})$, where B is a relations matrix of $\ker(B_{F_n})$ on $\mathfrak{gl}(n)$. □

Example 4.3.8. *Consider the Lie algebra C_4. A relations matrix B of* $\ker(B_{F_4})$ *is*

$$B = \begin{pmatrix} b_1+b_2+b_3+b_4 & b_1+b_2+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 & 0 \\ b_1+b_2+b_3 & b_1+b_2+b_4 & b_1+b_3 & b_2 & 0 & 0 & 0 & 0 \\ b_1+b_2 & b_1+b_3 & b_2+b_4 & b_3 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_4 & -b_3 & -b_2 & -b_1 \\ 0 & 0 & 0 & 0 & -b_3 & -b_2-b_4 & -b_1-b_3 & -b_1-b_2 \\ 0 & 0 & 0 & 0 & -b_2 & -b_1-b_3 & -b_1-b_2-b_4 & -b_1-b_2-b_3 \\ 0 & 0 & 0 & 0 & -b_1 & -b_1-b_2 & -b_1-b_2-b_3 & -b_1-b_2-b_3-b_4 \end{pmatrix}$$

Now, we address proper seaweed subalgebras of C_n. We describe the adjustments needed from Definition 3.2.5 to account for the aftertail and tail in Theorem 4.3.9.

Theorem 4.3.9. *Let $\mathfrak{g}$ be a seaweed of type C with associated meander M_n^C and full meander M defined on $2n$ vertices whose homotopy type is $H(c_1, \cdots, c_h)$. Let f_c represent a functional on $\mathfrak{gl}(c)$ for all c. Let $A = \mathscr{I}_{T_\mathfrak{g}} \times \mathscr{I}_{T_\mathfrak{g}}$ be the indices in the square block on the diagonal of $\mathfrak{g}$ which contains the tail and $B = \mathscr{I}_{T^a_\mathfrak{g}} \times \mathscr{I}_{T^a_\mathfrak{g}}$ be the indices in the square block on the diagonal of $\mathfrak{g}$ which contains the aftertail. For each c_i such that $\mathfrak{C}_{c_i} \cap A \neq \emptyset$ (i.e., each component whose core interacts with the tail of the meander M_n^C), define $\overline{F}_{c_i}$ as in Definition 3.2.5 by embedding a functional $f_{\lfloor c_i/2 \rfloor}$, except only sum over $e^*_{i,j}$ with $i+j < 2n+1$ (i.e., strictly above the antidiagonal). For each c_i such that $\mathfrak{C}_{c_i} \cap A = \mathfrak{C}_{c_i} \cap B = \emptyset$, define $\overline{F}_{c_i}$ as in Definition 3.2.5 except only sum over $e^*_{i,j}$ with $i+j < 2n+1$. As in Definition 3.2.5, in both these embeddings we allow for the choice to rotate the indices or not by adding the appropriate functionals in the peak blocks for any peak block which occurs strictly above the antidiagonal of $\mathfrak{g}$. The only difference is that, when crossing the antidiagonal, we require the choice of functionals over the main diagonal of the peak block (which occur on or above the antidiagonal). Finally, if $t = |T^a_\mathfrak{g}|$, then the final functional*

$$F = \sum \overline{F}_{c_i} + f_t^{n-t}$$

is such that

$$\dim \ker(B_F) = \sum \dim \ker(B_{f_{c_i}}) + \sum \dim \ker(B_{f_{\lfloor c_i/2 \rfloor}}) + \dim \ker(B_{f_t}),$$

where $\dim\ker(B_F)$ *is over* $\mathfrak{g}$*, for each* i $\dim\ker(B_{f_{c_i}})$ *over* $\mathfrak{gl}(c_i)$ *and* $\dim\ker(B_{f_{\lfloor c_i/2\rfloor}})$ *is over* $\mathfrak{p}^C_{c_i}(\{c_i\} \mid \emptyset)$*, and* $\dim\ker(B_{f_t})$ *is over* C_t*.*

As before, the constructed functional is regular if and only if we embed regular functionals in each component.

We first introduce the following nontrivial example which demonstrates Theorem 4.3.9 and highlights the differences with the tail and after tail.

Example 4.3.10. *Consider* $\mathfrak{g} = \mathfrak{p}^C_{18}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$*. This is a subalgebra of the seaweed of type* $\frac{5|10|6|10|5}{2|4|3|1|1|14|1|1|3|4|2}$*. The meanders* M^C_{18} *and* M *are shown in Figures 4.8 and 4.9, respectively, with the tail vertices and components colored blue and the aftertail vertices and component colored red. It follows from Theorem 4.3.5 that* $\operatorname{ind}\mathfrak{g} = 7$*.*

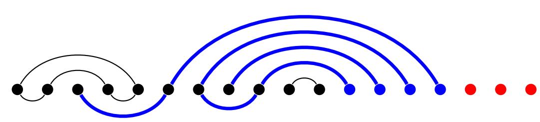


Figure 4.8: Meander M^C_{18} associated with $\mathfrak{p}^C_{18}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$

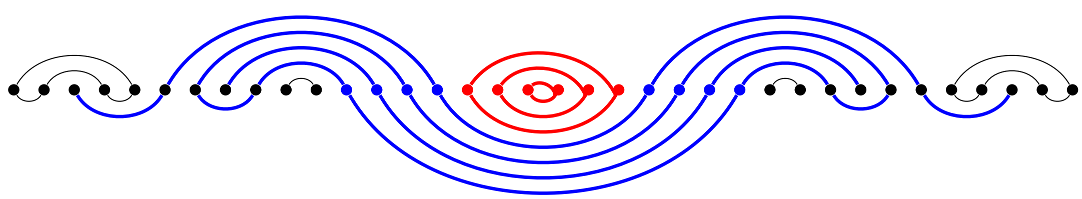


Figure 4.9: Meander M associated with $\mathfrak{p}^C_{18}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$

A functional described by Theorem 4.3.9 by embedding functionals F_c *from Theorem 3.3.1 is*

$$\begin{aligned}\overline{F} =& e^*_{1,1} + e^*_{1,2} + e^*_{2,1} + e^*_{3,6} + e^*_{4,2} + e^*_{4,5} + e^*_{5,1} + e^*_{5,4} + e^*_{5,5} + e^*_{7,7} \\ &+ e^*_{11,10} + e^*_{12,9} + e^*_{12,23} + e^*_{13,8} + e^*_{13,24} + e^*_{14,7} + e^*_{14,14} + e^*_{15,6} + e^*_{15,22} \\ &+ e^*_{16,16} + e^*_{16,17} + e^*_{16,18} + e^*_{17,16} + e^*_{17,17} + e^*_{18,16}.\end{aligned}$$

The seaweed $\mathfrak{g}$ is illustrated in Figure 4.10, where the indices in the aftertail component are colored red, the indices in the tail components are colored blue, and a black dot is placed in each index of $\mathscr{I}_{\overline{F}}$. We have added lines to emphasize the core and components of $\mathfrak{g}$.

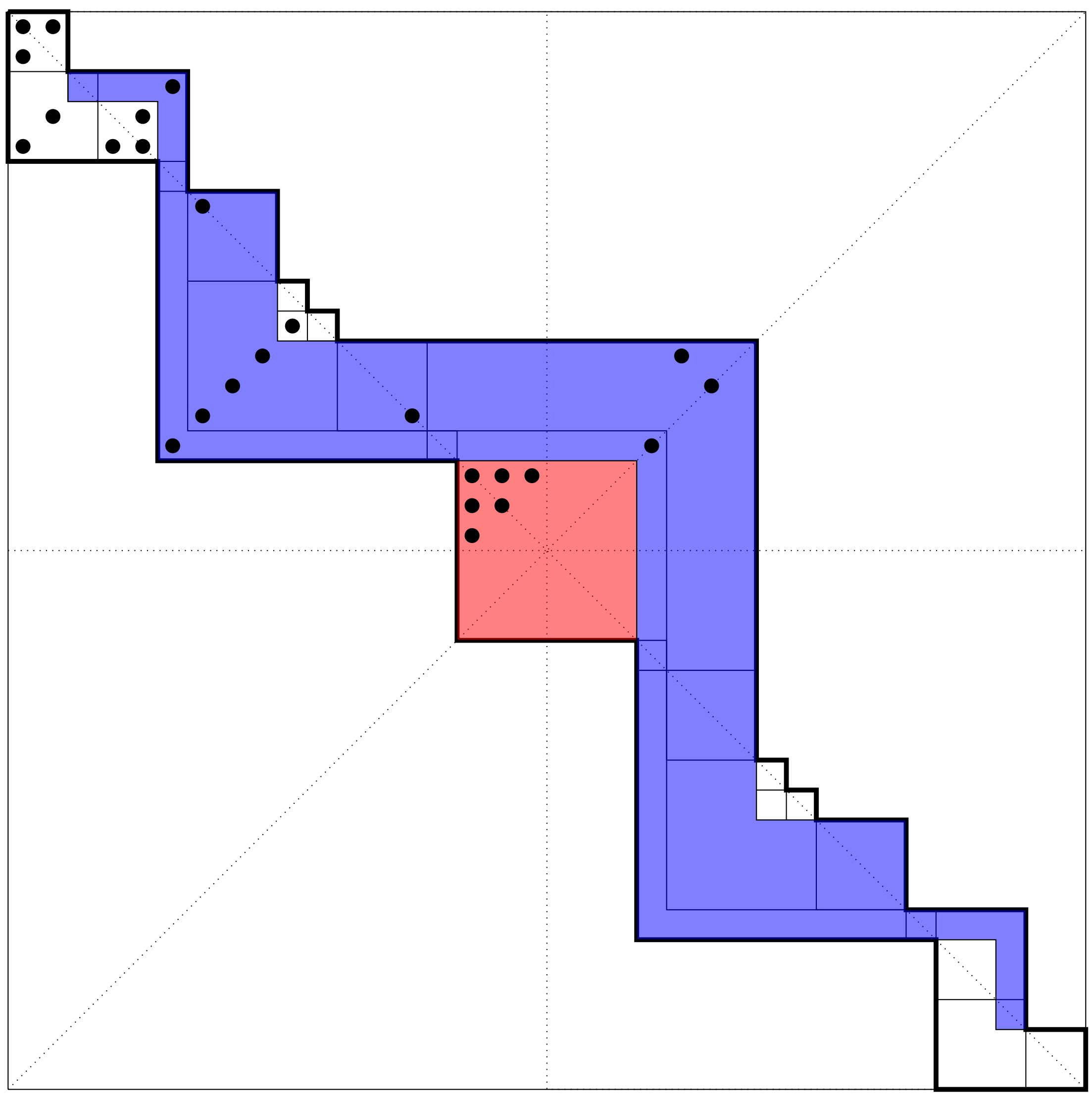


Figure 4.10: Indices in $\mathscr{I}_{\overline{F}}$ on $\mathfrak{p}_{18}^{C}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$

Figure 4.10 illustrates the embedding of F_3 to the aftertail component (red) in the center of the seaweed, the embedding of $F_{\lfloor 3/2 \rfloor} = F_1$ to the component of size three which interacts with the tail (blue) and that in the 3×3 block where this component crosses the diagonal, the functionals over the main diagonal are used. Further, the embedding for every component independent of the tail and aftertail are embedded as expected.

A messy computation yields the following portion of a relations matrix of $\ker(B_{\overline{F}})$ *(see Figure 4.11). Only the entries on or above the antidiagonal are shown; the remaining positions should be filled in according to the symmetry in* $\mathfrak{sp}(36)$. *Further, blank spaces are left for inadmissible positions of the seaweed, and are, per force, filled with zeroes.*

$$\left(\begin{array}{ccccccccccccccccccccccccc}
b_1+b_2 & b_1 & & & & & & & & & & & & & & & & & & & & & & & \\
b_1 & b_2 & & & & & & & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & b_2 & b_1 & 0 & & & & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & b_1 & b_1+b_2 & 0 & & & & & & & & & & & & & & & & & & & \\
 & & & & & 0 & & & & & & & & & & & & & & & & & & & \\
 & & & & & 0 & b_3 & 0 & 0 & & & & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & -b_3 & & & & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & b_4 & & & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & b_4 & & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & -b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & & & & & & & & & b_5+b_6+b_7 & b_5+b_6 & b_5 & 0 & 0 & 0 & & & & \\
 & & & & & & & & & & & & & & & b_5+b_6 & b_5+b_7 & b_6 & 0 & 0 & & & & & \\
 & & & & & & & & & & & & & & & b_5 & b_6 & b_7 & 0 & & & & & & \\
 & & & & & & & & & & & & & & & 0 & 0 & 0 & & & & & & & \\
 & & & & & & & & & & & & & & & 0 & 0 & & & & & & & & \\
 & & & & & & & & & & & & & & & 0 & & & & & & & & &
\end{array}\right)$$

Figure 4.11: Top half of a relations matrix of $\ker(B_{\overline{F}})$ on $\mathfrak{p}_{18}^{C}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$

Proof of Theorem 4.3.9. The proof follows almost entirely from the proof of Theorem 3.2.8. Some care is needed to address the tail and aftertail (Component Creation in Type-C is, once again, a direct sum unlike in Type-A). By definition, the aftertail is a self-contained component (a set of nested cycles which is not wound-up) and, therefore, the proof follows from the proof of Theorem 3.2.8. For the components which have cores that intersect the tail nontrivially, the induction is the same as in the proof of Theorem 3.2.8, except that a separate base case is needed.

Note that a component of size c_i no longer contributes c_i to the index of $\mathfrak{g}$, but rather $\lfloor \frac{c_i}{2} \rfloor$. For the base case on tail components, consider the seaweed $\mathfrak{p}^C_{c_i}(\{c_i\} \mid \emptyset)$ and the functional

$$F = F_{\lfloor c_i/2 \rfloor} + \sum_{i=1}^{\lceil c_i/2 \rceil} e^*_{i,c_i+i}.$$

Let B be a $\lfloor \frac{c_i}{2} \rfloor \times \lfloor \frac{c_i}{2} \rfloor$ relations matrix of $\ker(B_{F_{\lfloor c_i/2 \rfloor}})$ on $\mathfrak{gl}(\lfloor \frac{c_i}{2} \rfloor)$. By direct computation, if c_i is even then $\ker(B_F)$ has a relations matrix

$$B \oplus (-\widehat{B^t}) \oplus B \oplus (-\widehat{B^t}).$$

If n is odd, then a relations matrix of $\ker(B_F)$ is

$$B \oplus (0) \oplus (-\widehat{B^t}) \oplus B \oplus (0) \oplus (-\widehat{B^t}).$$

□

We introduce the following *reduced homotopy type* for seaweeds of Type-C to construct the analogue of Theorem 2.3.3. We use the word "reduced" as some of the c_i's are omitted from the full homotopy type $H(c_1, \cdots, c_h)$.

Definition 4.3.11. *Let $\mathfrak{g}$ be a seaweed subalgebra of C_n with reduced meander M^C_n and full meander M. Color the aftertail component (if any) of M red and the tail components (if any) of M blue. Eliminate any arcs and vertices to the right of v_n in M which are not colored red or blue. This produces a meander M' on I vertices with $I \in [n, 2n]$. Apply Lemma 2.3.1 to M' to unwind it, and in each Component Elimination move $C(c)$, color c the color of the component removed. Then $H_C(c_1, \cdots, c_h)$ is the* ***reduced homotopy type*** *of a Type-C seaweed.*

Example 4.3.12. *Let* $\mathfrak{g} = \mathfrak{p}_{18}^C(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$ *(cf., Example 4.3.10). The meander* M *is in Figure 4.9. The meander* M' *of Definition 4.3.11 is shown in Figure 4.12.*

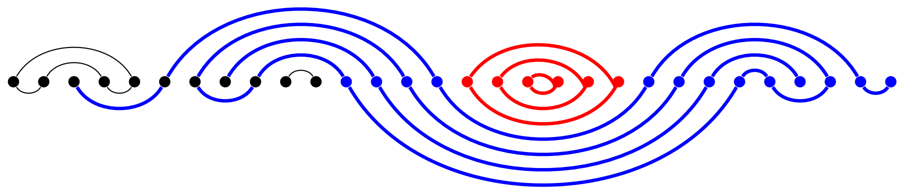


Figure 4.12: Reduced meander M' of $\mathfrak{p}_{18}^C(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_2, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}\})$

The reduced homotopy type of $\mathfrak{g}$ *is* $H_C(2, 1, \boldsymbol{1}, \boldsymbol{3}, \boldsymbol{6})$*, shown in Figure 4.13.*

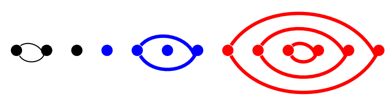


Figure 4.13: Reduced homotopy type $H_C(2, 1, \mathbf{1}, \mathbf{3}, \mathbf{6})$

The following theorem is the Type-C analogue of the theorem in Type-A and $\mathfrak{gl}(n)$ (cf., Theorem 2.3.3). Note that in Type-A, there is no tail or aftertail.

Theorem 4.3.13. *If* $\mathfrak{g}$ *is a seaweed of Type-C with reduced homotopy type*

$$H_C(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}}),$$

then

$$\text{ind } \mathfrak{g} = \sum_{i=1}^{h_1} c_i + \sum_{i=h_1+1}^{h_2} \left\lfloor \frac{c_i}{2} \right\rfloor + \frac{c_{h_2+1}}{2}.$$

4.4 Type-B

While Type-B and Type-D seaweeds are both subalgebras of $\mathfrak{so}(n)$ (for odd and even n, respectively), their root systems are significantly different and, therefore, must be addressed separately.

Recall that the simple Lie algebra B_n is $\mathfrak{so}(2n+1)$. It has simple roots

$$\alpha_i = e_i - e_{i+1} \text{ for } i \in [1, n-1], \quad \alpha_n = e_n.$$

We denote seaweed subalgebras of B_n by $\mathfrak{p}_n^B(\Psi \mid \Psi')$. By direct computation, the root system Φ_{B_n} is the set of roots

$$\left\{ \pm\beta_{i,j} = \pm\sum_{s=i}^{j} \alpha_s \;\middle|\; 1 \le i \le j < n \right\}$$
$$\cup \left\{ \pm\rho_i = \pm\sum_{s=i}^{n} \alpha_s \;\middle|\; i \in [1, n] \right\}$$
$$\cup \left\{ \pm\delta_{i,j} = \pm\sum_{s=i}^{n} \alpha_s \pm \sum_{s=j}^{n} \alpha_s \;\middle|\; 1 \le i < j \le n \right\}.$$

We now compute the Chevalley basis and the root spaces $\mathfrak{g}_\alpha$ for $\mathfrak{g} = B_n$. Consider an element $H = \mathrm{diag}(h_1, \cdots, h_{2n+1})$ of the Cartan subalgebra $\mathfrak{h}$ of B_n. It must be true that $h_i = -h_{2n+2-i}$ for all i, and therefore $h_{n+1} = 0$. Recall from Example 4.1.4 that for any $X \in \mathfrak{g}$, $[H, X] = K$, where K is the matrix $[(h_i - h_j)x_{i,j}]$. As elements of the dual, we have $\beta_{i,j} = e^*_{i,i} - e^*_{j+1,j+1}$, $\rho_i = e^*_{i,i}$, and $\delta_{i,j} = e^*_{i,i} + e^*_{j,j}$. Therefore, $\beta_{a,b}(H)X = (h_a - h_{b+1})X$, and $K = \beta_{a,b}(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, b+1), (2n+1-b, 2n+2-a)$. Thus,

$$\mathfrak{g}_{\beta_{i,j}} = \mathrm{span}\{e_{i,j+1} - e_{2n+1-j,2n+2-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\beta_{i,j}} = \mathrm{span}\{e_{j+1,i} - e_{2n+2-i,2n+1-j}\}.$$

Similarly, $\rho^B_a(H)X = h_a X = (h_a - h_{n+1})X$, and $K = \rho^B_a(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, n+1), (n+1, 2n+2-a)$. Thus,

$$\mathfrak{g}_{\rho_i} = \mathrm{span}\{e_{i,n+1} - e_{n+1,2n+2-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\rho_i} = \mathrm{span}\{e_{n+1,i} - e_{2n+2-i,n+1}\}.$$

Finally, $\delta_{a,b}(H)X = (h_a + h_b)X = (h_a - h_{2n+2-b})X$, and $K = \delta_{a,b}(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, 2n+2-b), (b, 2n+2-a)$. Thus,

$$\mathfrak{g}_{\delta_{i,j}} = \mathrm{span}\{e_{i,2n+2-j} - e_{j,2n+2-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\delta_{i,j}} = \mathrm{span}\{e_{2n+2-j,i} - e_{2n+2-i,j}\}.$$

We fix the Chevalley basis for B_n to be

$$\begin{aligned}
&\{x_{\beta_{i,j}} = e_{i,j+1} - e_{2n+1-j,2n+2-i},\ x_{-\beta_{i,j}} = e_{j+1,i} - e_{2n+2-i,2n+1-j} \mid 1 \le i \le j < n\} \\
\cup\, &\{x_{\rho_i} = e_{i,n+1} - e_{n+1,2n+2-i},\ x_{-\rho_i} = e_{n+1,i} - e_{2n+2-i,n+1} \mid 1 \le i \le n\} \\
\cup\, &\{x_{\delta_{i,j}} = e_{i,2n+2-j} - e_{j,2n+2-i},\ x_{-\delta_{i,j}} = e_{2n+2-j,i} - e_{2n+2-i,j} \mid i \le i < j \le n\} \\
\cup\, &\{h_{\alpha_i} = e_{i,i} - e_{2n+2-i,2n+2-i} \mid i \in [1,n]\}.
\end{aligned}$$

See Table A.2 for the computation of the root system Φ_{B_4}.

Example 4.4.1. *Consider $\mathfrak{g} = B_3$. The matrices in $\mathfrak{g}$ have the following form with respect to the root spaces $\mathfrak{g}_\beta$ for $\beta \in \Phi_{B_3}$:*

$$\begin{pmatrix}
* & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+2\alpha_3} & \mathfrak{g}_{\alpha_1+2\alpha_2+2\alpha_3} & 0 \\
\mathfrak{g}_{-\alpha_1} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_2+2\alpha_3} & 0 & \mathfrak{g}_{\alpha_1+2\alpha_2+2\alpha_3} \\
\mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_3} & 0 & \mathfrak{g}_{\alpha_2+2\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+2\alpha_3} \\
\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_3} & 0 & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \\
\mathfrak{g}_{-\alpha_1-\alpha_2-2\alpha_3} & \mathfrak{g}_{-\alpha_2-2\alpha_3} & 0 & \mathfrak{g}_{-\alpha_3} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2} \\
\mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3} & 0 & \mathfrak{g}_{-\alpha_2-2\alpha_3} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_1} \\
0 & \mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3} & \mathfrak{g}_{-\alpha_1-\alpha_2-2\alpha_3} & \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_1} & *
\end{pmatrix}.$$

To construct a seaweed subalgebra of B_n, one selects a set of simple roots to omit. The effect of this omission is demonstrated in the following example.

Example 4.4.2. *Consider the seaweed $\mathfrak{g} = \mathfrak{p}_3^B(\{\alpha_3\} \mid \{\alpha_2\})$. As was done in Type-C, we eliminate any root space from $\mathfrak{p}$ which relied on the root $-\alpha_3$, and we eliminate any root space from $\mathfrak{p}'$ which needed α_2. The parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ are demonstrated in Figure 4.14 (left and center, respectively). These are parabolic subalgebras of $\mathfrak{so}(7)$, and the seaweed $\mathfrak{g}$ (the intersection) is displayed in Figure 4.14 (right). It is easy to see that $\mathfrak{g}$ is a subalgebra of the seaweed $\mathfrak{g}' \subseteq \mathfrak{gl}(7)$ of type $\frac{3|1|3}{2|3|2}$.*

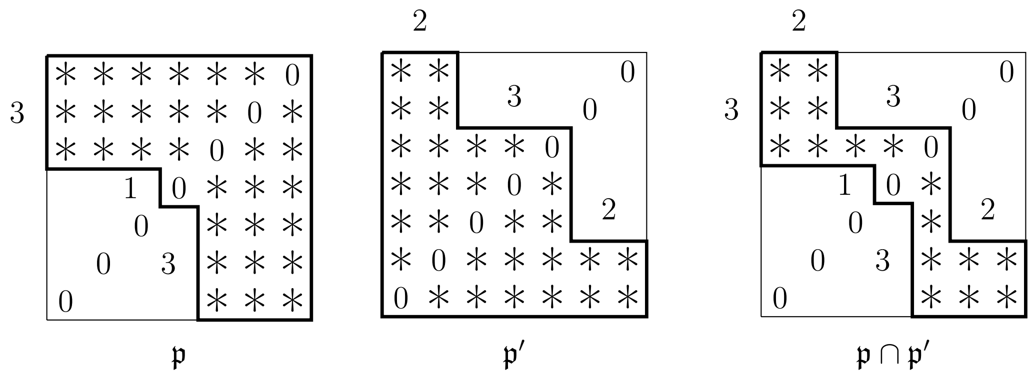


Figure 4.14: Construction of $\mathfrak{p}_3^B(\{\alpha_3\} \mid \{\alpha_2\})$

In general, the seaweed $\mathfrak{p}_n^B(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$ with $i_s < i_{s+1}$ and $j_s < j_{s+1}$ is a subalgebra of the seaweed $\mathfrak{g} \subseteq \mathfrak{gl}(2n+1)$ of type

$$\frac{i_1|i_2 - i_1|\cdots|i_k - i_{k-1}|2n+1-2i_k|i_k - i_{k-1}|\cdots|i_2 - i_1|i_1}{j_1|j_2 - j_1|\cdots|j_t - j_{t-1}|2n+1-2j_t|j_t - j_{t-1}|\cdots|j_2 - j_1|j_1}. \tag{4.7}$$

As in C_n, we introduce the meander M_n^B defined on n vertices instead of the full meander on $2n+1$ vertices to capitalize on the symmetry of $\mathfrak{so}(2n+1)$.

Definition 4.4.3. *Let* $\mathfrak{g} = \mathfrak{p}_n^B(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$ *with* $i_s < i_{s+1}$ *and* $j_s < j_{s+1}$. *The meander* M_n^B *is defined identically to the meander* M_n^C *associated with* $\mathfrak{p}_n^C(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$. *The tail and aftertail in* M_n^B *are defined the same as for* M_n^C.

Example 4.4.4. *Let* $\mathfrak{g} = \mathfrak{p}_9^B(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6, \alpha_7\})$. *The meander* M_n^B *associated with* $\mathfrak{g}$ *is illustrated in Figure 4.15 (right). The bars and numbers in Figure 4.15 (left) are not part of the meander itself, rather are visual aids for the meander construction.*

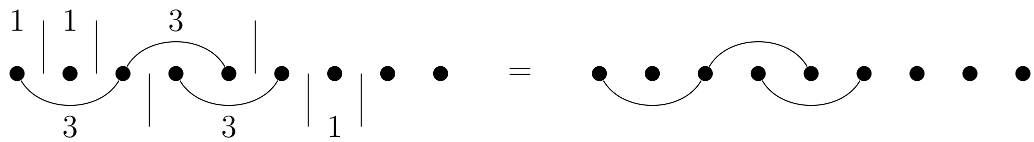


Figure 4.15: Meander M_9^B associated with $\mathfrak{p}_9^B(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6, \alpha_7\})$

By Definition 4.4.14, we have

$$T_a = \{v_6, v_7, v_8, v_9\}, \qquad T_b = \{v_8, v_9\}, \qquad T_{\mathfrak{g}} = \{v_6, v_7\}, \qquad \textit{and} \qquad T_{\mathfrak{g}}^a = \{v_8, v_9\}.$$

As in $\mathfrak{gl}(n)$, we can visualize the meander within the seaweed by mapping v_i to $e_{i,i}$ (see Figure 4.23). The tail vertices are colored blue and the aftertail vertices are colored red.

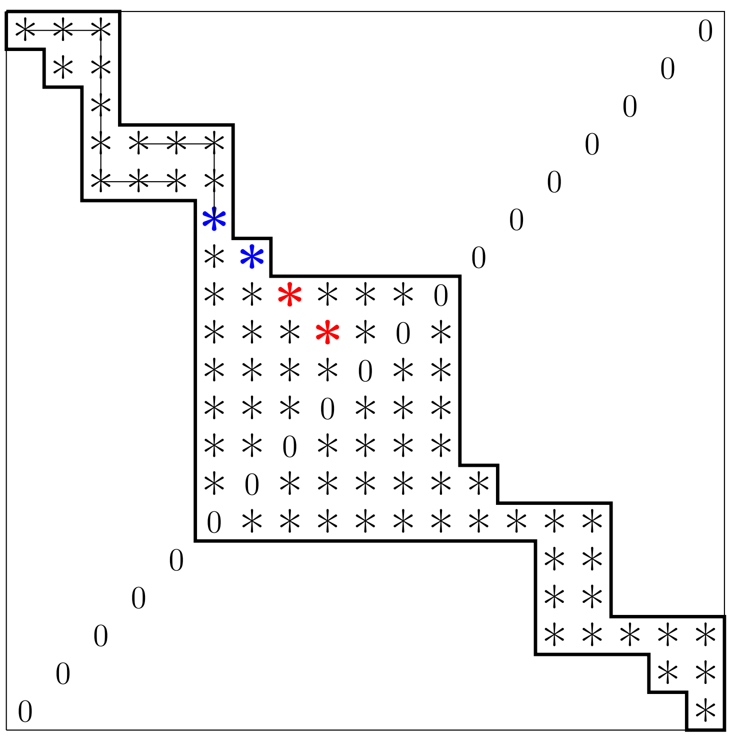


Figure 4.16: Meander M_9^B visualized within the seaweed $\mathfrak{p}_9^B(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6, \alpha_7\})$

The approach used to construct a regular functional in Type-B is exactly as it is in Type-C: to take a sum over a functional defined on the aftertail component, functionals defined on the tail components, and the typical functionals $\overline{F}_{c_i}$ on components of M which do not interact with the tail.

4.4.1 Type-B Regular Functionals

To begin, we must once again know how restricting to algebras in $\mathfrak{so}(2n+1)$ affects the index of a seaweed. The effect is identical to that which occurs in Type-C.

Theorem 4.4.5 (Coll, Hyatt, and Magnant [**12**]; Panyushev and Yakimova [**26**])**.** *If $\mathfrak{g} = \mathfrak{p}_n^B(\Psi_1 \mid \Psi_2)$ is a seaweed of Type-B, then*

$$\text{ind}\,\mathfrak{g} = 2C + \tilde{P},$$

where C is the number of cycles and $\tilde{P}$ is the number of paths and isolated points with zero or two endpoints in the tail of the meander M_n^B associated with $\mathfrak{g}$.

Theorem 4.4.5 yields the following immediate Corollary. The proof carries over *mutatis mutandis* from the Type-C case (see Theorem 4.3.6).

Theorem 4.4.6. *If $\mathfrak{g} = B_n$, then*

$$\text{ind}\,\mathfrak{g} = n.$$

As in Section 4.3, we only consider Type-B functionals $F = \sum_{(i,j)\in\mathscr{I}_F} e_{i,j}^*$ with $i + j < 2n + 2$ (i.e., strictly above the antidiagonal), a smaller functional than the full functional $F' = \sum_{(i,j)\in\mathscr{I}_F}(e_{i,j}^* - e_{2n+2-j,2n+2-i}^*)$ such that $\ker(B_F) = \ker(B_{F'})$.

Theorem 4.4.7. *The functional F_n of Theorem 3.3.1 is regular on B_n.*

Proof. The systems of equations generated by $B_{F_n}(B, e_{i,j} - e_{2n+2-j,2n+2-i}) = 0$ on $\mathfrak{so}(2n+1)$ and $B_{F_n}(B, e_{i,j}) = 0$ on $\mathfrak{gl}(n)$ for $i, j \leq n$ are equivalent. To prove F_n is regular on B_n, it suffices to show that $b_{i,j} = 0$ for all $i \in [1, n]$, $j > n$. Note that

$$e_{i,n+1} - e_{n+1,2n+1-i} \mapsto \sum_{s=1}^{n+1-i} b_{n+1,s}. \tag{4.8}$$

By setting the expressions on the right hand side of (4.8) equal to zero, we get a system of equations n whose solution is $b_{n+1,i} = 0$ for $s \in [1, n]$ (this is seen by induction, the base case is $i = n$ and the induction goes down to $i = 1$). We get $b_{i,j} = 0$ for all $(i, j) \in [1, n] \times [n + 2, 2n + 1]$ through a linear algebra argument similar to that in the proof of Theorem 3.4.1 on the set of equations

$$B_{F_n}(B, e_{j,i} - e_{2n+2-i,2n+2-j}) = \sum_{s=1}^{n+1-i} b_{s,j} + \sum_{s=1}^{j-(n+1)} b_{s,2n+2-i} = 0.$$

In conclusion, a relations matrix of $\ker(B_{F_n})$ on B_n will be

$$B \oplus (0) \oplus \left(-B^{\hat{t}}\right),$$

where B is an $n \times n$ relations matrix of $\ker(B_{F_n})$ on $\mathfrak{gl}(n)$. $\square$

The reduced homotopy type $H_B(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}})$ on $\mathfrak{g}$ is defined the same as in Type-C. We have the immediate analogue of Theorem 4.3.13.

Theorem 4.4.8. *If $\mathfrak{g}$ is a seaweed of Type-B with reduced homotopy type*

$$H_B(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}}),$$

then

$$\text{ind } \mathfrak{g} = \sum_{i=1}^{h_1} c_i + \sum_{i=1}^{h_2} \left\lfloor \frac{c_i}{2} \right\rfloor + \frac{c_{h_2+1} - 1}{2}.$$

Theorem 4.4.9. *Let $\mathfrak{g}$ be a Type-B seaweed with reduced homotopy type*

$$H_B(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}})$$

such that $\boldsymbol{c_i}$ is even for all $i \in [h_1 + 1, h_2 - 1]$. Let F be the functional constructed as in Theorem 4.3.9 except that if $\boldsymbol{c_{c_{h_2}}}$ is odd, instead of adding the the functional $e^*_{i,j}$ on the antidiagonal of $\mathfrak{g}$ (note that $\mathfrak{g}$ must have zeroes on the antidiagonal), add the corresponding functional $e^*_{i,n+1}$. Then the index of the constructed functional F is equal to $\sum \dim \ker(B_{F_{c_i}})$ as in Theorem 4.3.9.

The proof requires a modification to the base case on the tail components for the odd component.

Modification of the tail base case. Denote by $\mathbf{0}_m$ the zero functional on $\mathfrak{gl}(m)$. Consider the seaweed $\mathfrak{p}_n^B(\{c_i\} \mid \emptyset)$, and let $F_{c_i} \in \mathfrak{gl}(\lfloor \frac{c_i}{2} \rfloor)^*$ and $F_c \in \mathfrak{gl}(n - c_i)^*$. Let $F'_{c_i} = F_{c_i} \oplus \mathbf{0}_{\lceil c_i/2 \rceil}$, and let B and B' be relations matrices of $\ker(B_{F_{c_i}})$ on $\mathfrak{gl}(\lfloor \frac{c_i}{2} \rfloor)$ and $\ker(B_{F_c})$ on $\mathfrak{gl}(n - c_i)$, respectively.

Assume that n is even, and let

$$F = (F'_{c_i} \oplus F_c) + \sum_{i=1}^{c_i/2} e^*_{i,2n-c_i+1+i}.$$

By direct computation, $B \oplus (-\widehat{B^t}) \oplus B' \oplus (0) \oplus (-\widehat{(B')^t}) \oplus B \oplus (-\widehat{B^t})$ is a relations matrix of $\ker(B_F)$.

Assume that n is odd, and let

$$F = (F'_{c_i} \oplus F_c) + \sum_{i=1}^{\lfloor c_i/2 \rfloor} e^*_{i,2n-c_i+1+i} + e^*_{\lceil \frac{c_i}{2} \rceil, n+1}.$$

By direct computation, $B \oplus (0) \oplus (-\widehat{B^t}) \oplus B' \oplus (0) \oplus (-\widehat{(B')^t}) \oplus B \oplus (0) \oplus (-\widehat{B^t})$ is a relations matrix of $\ker(B_F)$. □

We conclude this section with the following nontrivial example which demonstrates the changes necessary in Theorem 4.3.9.

Example 4.4.10. *Consider* $\mathfrak{g} = \mathfrak{p}_{15}^B(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_1, \alpha_8, \alpha_{10}\})$. *This is a subalgebra of the seaweed of type* $\frac{5|10|5|10|5}{1|7|2|15|2|7|1}$. *The meanders* M_{15}^B *and* M *are shown in Figures 4.17 and 4.18, respectively, with the tail vertices and components colored blue. It follows from Theorem 4.4.5 that* ind $\mathfrak{g} = 3$.

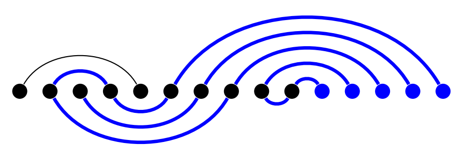


Figure 4.17: Meander M_{15}^B associated with $\mathfrak{p}_{15}^B(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_1, \alpha_8, \alpha_{10}\})$

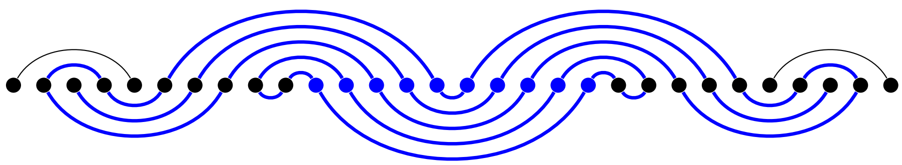


Figure 4.18: Meander M associated with $\mathfrak{p}_{15}^{B}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_1, \alpha_8, \alpha_{10}\})$

A functional described by Theorem 4.3.9 adapted to Type-B (using functionals F_c from Theorem 3.3.1) is

$$\begin{aligned}\overline{F} =& e^*_{2,2} + e^*_{2,8} + e^*_{3,7} + e^*_{4,6} + e^*_{5,1} + e^*_{8,8} + e^*_{9,9} + e^*_{11,10} + e^*_{11,20} \\ &+ e^*_{12,9} + e^*_{12,12} + e^*_{13,9} + e^*_{13,13} + e^*_{13,17} + e_{14,8} + e^*_{14,16} + e^*_{15,7}\end{aligned}$$

The seaweed $\mathfrak{g}$ is illustrated in Figure 4.19, where the indices in the tail components are colored blue, and a black dot is placed over each index in $\mathscr{I}_{\overline{F}}$. We have added lines to emphasize the core components of $\mathfrak{g}$.

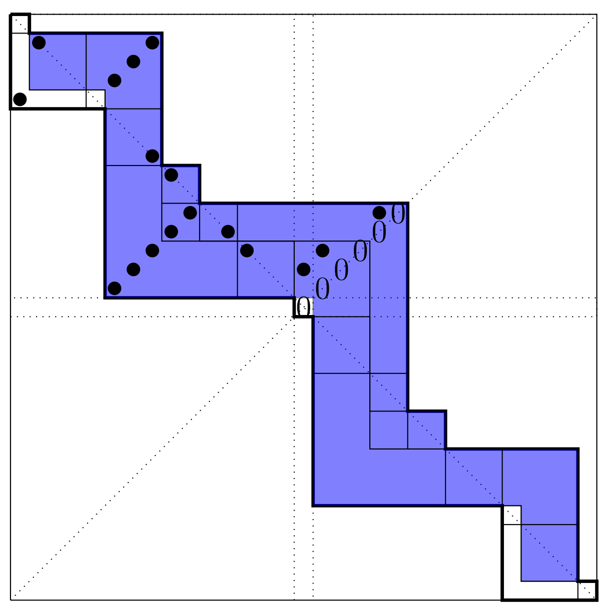


Figure 4.19: Indices in $\mathscr{I}_{\overline{F}}$ on $\mathfrak{p}_{15}^{B}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_1, \alpha_8, \alpha_{10}\})$

A messy computation yields the following portion of a relations matrix for $\ker(B_{\overline{F}})$ *(see Figure 4.20). Only the entries on or above the antidiagonal are shown; the remaining positions should be filled in according to the symmetry in* $\mathfrak{so}(31)$*. Further, inadmissible positions of* $\mathfrak{g}$ *are left blank. They are, per force, filled with zeroes.*

$$B = \begin{pmatrix}
b_1 &&&&&&&&&&&&&&&&&&&& \\
0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&&&&& \\
0 & 0 & 0 & -b_2 & 0 & 0 & 0 & 0 &&&&&&&&&&&&& \\
0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 &&&&&&&&&&&&& \\
&&&&& -b_2 & 0 & 0 &&&&&&&&&&&&& \\
&&&&& 0 & 0 & 0 &&&&&&&&&&&&& \\
&&&&& 0 & 0 & b_2 &&&&&&&&&&&&& \\
&&&&& 0 & 0 & 0 & b_3 & 0 &&&&&&&&&&& \\
&&&&& 0 & 0 & 0 & 0 & -b_3 &&&&&&&&&&& \\
&&&&& 0 & 0 & 0 & 0 & 0 & -b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&& 0 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
&&&&& 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 && \\
&&&&& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&& \\
&&&&& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_2 & 0 & 0 &&&&
\end{pmatrix}.$$

Figure 4.20: Top half of a relations matrix of $\ker(B_{\overline{F}})$ on $\mathfrak{p}_{15}^{B}(\{\alpha_5, \alpha_{15}\} \mid \{\alpha_1, \alpha_8, \alpha_{10}\})$

4.4.2 Type-D

The difference between the root spaces Φ_{B_n} and Φ_{D_n} results in Type-D seaweeds not always having seaweed "shape". This problem is discussed in greater detail in Section 5.1.1. For the remainder of this section, we assume that each seaweed $\mathfrak{p}_n^D(\Psi \mid \Psi')$ is such that if α_n (or α_{n-1}) is in Ψ, then α_{n-1} (respectively, α_n) is *not* in Ψ'.

Recall that the simple Lie algebra D_n is $\mathfrak{so}(2n)$. It has simple roots

$$\alpha_i = e_i - e_{i+1} \text{ for } i \in [1, n-1], \quad \alpha_n = e_{n-1} + e_n.$$

We denote seaweed subalgebras of D_n by $\mathfrak{p}_n^D(\Psi \mid \Psi')$. By direct computation, the root system Φ_{D_n} is the set of roots

$$\left\{ \pm\beta_{i,j} = \pm \sum_{s=i}^{j} \alpha_s \;\middle|\; 1 \le i \le j < n \right\}$$
$$\cup \left\{ \pm\delta_{i,j} = \pm \sum_{s=i}^{n-2} \alpha_s \pm \sum_{s=j}^{n} \alpha_s \;\middle|\; 1 \le i < j \le n \right\}.$$

We now compute the Chevalley basis and the root spaces $\mathfrak{g}_\alpha$ for $\mathfrak{g} = D_n$. Consider an element $H = \operatorname{diag}(h_1, \cdots, h_{2n})$ of the Cartan subalgebra $\mathfrak{h}$ of D_n. It must be true that $h_i = -h_{2n+1-i}$ for all i. Recall from Example 4.1.4 that for any $X \in \mathfrak{g}$, $[H, X] = K$, where K is the matrix $[(h_i - h_j)x_{i,j}]$. As elements of the dual, we have $\beta_{i,j} = e^*_{i,i} - e^*_{j+1,j+1}$ and $\delta_{i,j} = e^*_{i,i} + e^*_{j,j}$. Therefore, $\beta_{a,b}(H)X = (h_a - h_{b+1})X$, and $K = \beta_{a,b}(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, b+1), (2n-b, 2n+1-a)$. Thus,

$$\mathfrak{g}_{\beta_{i,j}} = \operatorname{span}\{e_{i,j+1} - e_{2n-j,2n+1-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\beta_{i,j}} = \operatorname{span}\{e_{j+1,i} - e_{2n+1-i,2n-j}\}.$$

Similarly, $\delta_{a,b}(H)X = (h_a + h_b)X = (h_a - h_{2n+1-b})X$, and $K = \delta_{a,b}(H)X$ for all $H \in \mathfrak{h}$ if and only if $x_{i,j} = 0$ for all $(i,j) \neq (a, 2n+1-b), (b, 2n+1-a)$. Thus,

$$\mathfrak{g}_{\delta_{i,j}} = \operatorname{span}\{e_{i,2n+1-j} - e_{j,2n+1-i}\} \qquad \text{and} \qquad \mathfrak{g}_{-\delta_{i,j}} = \operatorname{span}\{e_{2n+1-j,i} - e_{2n+1-i,j}\}.$$

We fix the Chevalley basis for D_n to be

$$\{x_{\beta_{i,j}} = e_{i,j+1} - e_{2n-j,2n+1-i},\ x_{-\beta_{i,j}} = e_{j+1,i} - e_{2n+1-i,2n-j} \mid 1 \le i \le j < n\}$$
$$\cup \{x_{\delta_{i,j}} = e_{i,2n+1-j} - e_{j,2n+1-i},\ x_{-\delta_{i,j}} = e_{2n+1-j,i} - e_{2n+1-i,j} \mid i \le i < j \le n\}$$
$$\cup \{h_{\alpha_i} = e_{i,i} - e_{2n+1-i,2n+1-i} \mid i \in [1,n]\}.$$

See Table A.4 for the computation of the root system Φ_{D_4}.

Example 4.4.11. *Consider $\mathfrak{g} = D_4$. The matrices in $\mathfrak{g}$ have the following form* *with respect to the root spaces $\mathfrak{g}_\beta$ for $\beta \in \Phi_{D_4}$:*

$$\begin{pmatrix}
* & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_4} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} & \mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} & 0 \\
\mathfrak{g}_{-\alpha_1} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_2+\alpha_4} & \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} & 0 & \mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
\mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{\alpha_4} & 0 & \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_3} & * & 0 & \mathfrak{g}_{\alpha_4} & \mathfrak{g}_{\alpha_2+\alpha_4} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_4} \\
\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} & \mathfrak{g}_{-\alpha_2-\alpha_4} & \mathfrak{g}_{-\alpha_4} & 0 & * & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{\alpha_2+\alpha_3} & \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \\
\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4} & 0 & \mathfrak{g}_{-\alpha_4} & \mathfrak{g}_{-\alpha_3} & * & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_1+\alpha_2} \\
\mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4} & 0 & \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_2-\alpha_4} & \mathfrak{g}_{-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_2} & * & \mathfrak{g}_{\alpha_1} \\
0 & \mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} & \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} & \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} & \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_1} & *
\end{pmatrix}.$$

Now, to construct a parabolic subalgebra of D_n, one selects a set of simple roots to omit. The effect of such an omission is demonstrated in the following example.

Example 4.4.12. *Consider the seaweed $\mathfrak{g} = \mathfrak{p}_4^D(\{\alpha_2\} \mid \emptyset)$. As in Type-B, we eliminate any root space from $\mathfrak{p}$ which is reliant on the root $-\alpha_2$. The algebra $\mathfrak{g}$ is isomorphic to the parabolic $\mathfrak{p}$ (see Figure 4.21). Evidently, $\mathfrak{g}$ is a subalgebra of the seaweed of type $\frac{2|4|2}{8}$ contained in $\mathfrak{gl}(8)$.*

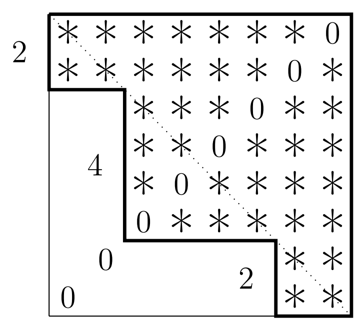


Figure 4.21: Construction of $\mathfrak{p}_4^D(\{\alpha_2\} \mid \emptyset)$

In general, the seaweed $\mathfrak{p}_n^D(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$ with $i_s < i_{s+1}$ and $j_s < j_{s+1}$ is a subalgebra of the seaweeds $\mathfrak{g} \subseteq \mathfrak{gl}(2n)$ of type

$$\frac{i_1|i_2-i_1|\cdots|i_k-i_{k-1}|2n-2i_k|i_k-i_{k-1}|\cdots|i_2-i_1|i_1}{j_1|j_2-j_1|\cdots|j_t-j_{t-1}|2n-2j_t|j_t-j_{t-1}|\cdots|j_2-j_1|j_1}. \tag{4.9}$$

Remark 4.4.13. *If $i_k = n-1$ (or $j_t = n-1$), then in Type-D we will assume the corresponding center block is actually two blocks of size one, and $\mathfrak{g}$ is of the form*

$$\frac{i_1|i_2-i_1|\cdots|i_k-i_{k-1}|1|1|i_k-i_{k-1}|\cdots|i_2-i_1|i_1}{j_1|j_2-j_1|\cdots|j_t-j_{t-1}|2n-2j_t|j_t-j_{t-1}|\cdots|j_2-j_1|j_1}. \tag{4.10}$$

Definition 4.4.14. *Let* $\mathfrak{g} = \mathfrak{p}_n^D(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$*, with* $i_s < i_{s+1}$ *and* $j_s < j_{s+1}$*. The meander* M_n^D *is identical to the meander* M_n^C *associated with* $\mathfrak{p}_n^C(\{\alpha_{i_1}, \cdots, \alpha_{i_k}\} \mid \{\alpha_{j_1}, \cdots, \alpha_{j_t}\})$*, but the tail is defined different. Let* $T_{\mathfrak{g}}$ *represent the tail per Definition 4.3.3. Per the work of Cameron ([4]), the tail for* M_n^D *is defined as:*

1. $T_{\mathfrak{g}}$ *if* $|i_k - j_t|$ *is even or zero,*
2. $T_{\mathfrak{g}} \cup \{v_{\max\{i_k,j_t\}+1}\}$ *if* $|i_k - j_t|$ *is odd and* $\max\{i_k, j_t\} \notin \{n-1, n\}$,
3. $T_{\mathfrak{g}} \backslash \{v_n\}$ *if* $|i_k - j_t|$ *is odd and* $\max\{i_k, j_t \in \{n-1, n\}$.

The aftertail is defined the same, and so there is the possibility to have a vertex in both the tail and aftertail and it is possible to have a vertex between the tail and aftertail which is in neither.

Example 4.4.15. *Let* $\mathfrak{g} = \mathfrak{p}_9^D(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$*. The meander* M_9^D *associated with* $\mathfrak{g}$ *is illustrated in Figure 4.22 (right).*

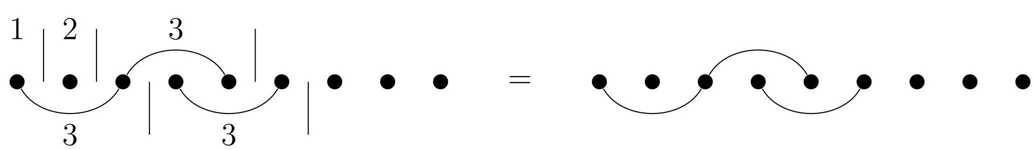


Figure 4.22: Meander M_9^D associated with $\mathfrak{p}_9^D(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$

By Definition 4.4.14, we have

$$T_a = \{v_6, v_7, v_8, v_9\}, \quad T_b = \{v_7, v_8, v_9\}, \quad T_{\mathfrak{g}} = \{v_6, v_7\}, \quad \textit{and} \quad T_{\mathfrak{g}}^a = \{v_7, v_8, v_9\}.$$

As in $\mathfrak{gl}(n)$*, we can visualize the meander within the seaweed by mapping* v_i *to* $e_{i,i}$ *(see Figure 4.23). We color the tail vertices blue, the aftertail vertices red, and the vertex in both purple.*

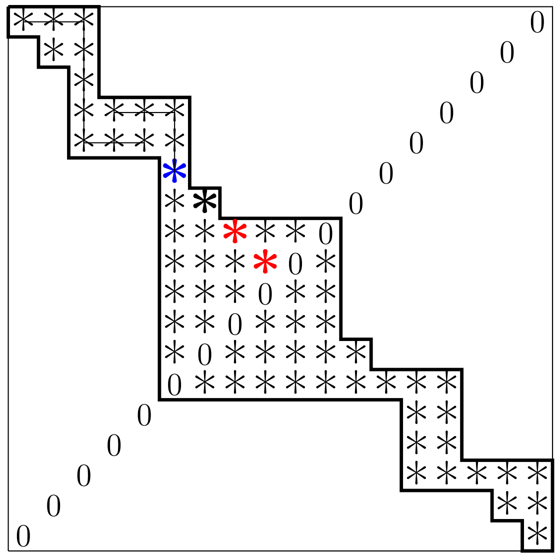


Figure 4.23: Meander M_9^D visualized within the seaweed $\mathfrak{p}_9^D(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$

When constructing regular functionals in D_n, the approach is similar to B_n. We first address the aftertail and tail components. As in Type-B, special attention is needed for the tail components of odd size, but type-D also requires additional definition for any component which might have a vertex in both the tail and after-tail and any component which might not be contained entirely in the tail. Every component which does not interact with the tail is embedded as in Definition 3.2.5.

4.4.3 Type-D Regular Functionals

To begin, we must know how restricting to algebras in $\mathfrak{so}(2n)$ affects the index of a seaweed.

Theorem 4.4.16 (Cameron [4]). *If $\mathfrak{g} = \mathfrak{p}_n^D(\Psi_1 \mid \Psi_2)$ is a seaweed of Type-D, then*

$$\text{ind}\,\mathfrak{g} = 2C + \tilde{P},$$

where C is the number of cycles and $\tilde{P}$ is the number of paths (and isolated points) with zero or two endpoints in the tail of the meander M_n^D associated with $\mathfrak{g}$.

Theorem 4.4.16 yields the following immediate Corollary. The proof carries over *mutatis mutandis* from the Type-C case (See Theorem 4.3.6).

Theorem 4.4.17. *If $\mathfrak{g} = D_n$, then*

$$\text{ind } \mathfrak{g} = n.$$

As in Section 4.3, we only consider Type-D functionals $F = \sum_{(i,j)\in\mathscr{I}_F} e^*_{i,j}$ with $i + j < 2n + 2$ (i.e., strictly above the antidiagonal), a smaller functional than the full functional $F' = \sum_{(i,j)\in\mathscr{I}_F}(e^*_{i,j} - e^*_{2n+2-j,2n+2-i})$ such that $\ker(B_F) = \ker(B_{F'})$. The following theorem is the analogue of theorem 4.4.7.

Theorem 4.4.18. *The functional F_n of Theorem 3.3.1 is regular on D_n.*

We have the immediate analogue of Definition 4.3.11 in Type-D to construct a reduced homotopy type $H_D(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, c_{h_2+1}, c_{h_2+2}, \boldsymbol{c_{h_2+3}})$ on $\mathfrak{g}$.

Definition 4.4.19. *Let $\mathfrak{g}$ be a seaweed subalgebra of D_n with reduced meander M^D_n and full meander M. Color the vertex in both the tail and aftertail (if present) purple, color the vertex after the tail but before the aftertail (if any) green, color any remaining tail vertices blue and any remaining aftertail vertices red. Consider every component whose vertex set contains a colored vertex. Color the component containing the purple vertex purple, the component containing the green vertex green, and then any other components in the tail or aftertail blue or red, respectively. Eliminate any arcs and vertices ot the right of v_n in M which are not colored. This produces a meander M' on I vertices with $I \in [n, 2n]$. Apply Lemma 2.3.1 to M' to unwind it, and in each Component Elimination move $C(c)$, color c the color of the component removed. Then $H_D(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, c_{h_2+1}, c_{h_2+2}, \boldsymbol{c_{h_2+3}})$ is the **reduced homotopy type** of a Type-D seaweed.*

Remark 4.4.20. *Note that at most one of c_{h_2+1}, c_{h_2+2}, and $\boldsymbol{c_{h_2+3}}$ can be nonzero, depending on the tail adjustments (if any) made to M^D_n.*

Example 4.4.21. *Consider $\mathfrak{p}^D_9(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$ of Example 4.4.15. The colored meander M is illustrated in Figure 4.24 and the reduced homotopy type is* $H_D(1, 1, \mathbf{1}, \mathbf{6})$.

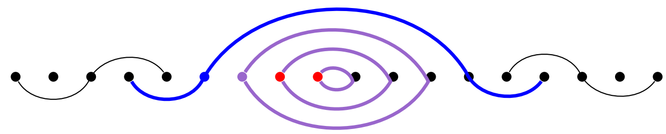


Figure 4.24: Colored meander M of $\mathfrak{p}_9^D(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$

Example 4.4.22. *Consider $\mathfrak{p}_9^D(\{\alpha_3, \alpha_9\} \mid \{\alpha_1, \alpha_2, \alpha_6\})$. The colored meander M is illustrated in Figure 4.25 and the reduced homotopy type is $H_D(1, \mathbf{1}, \mathbf{2})$.*

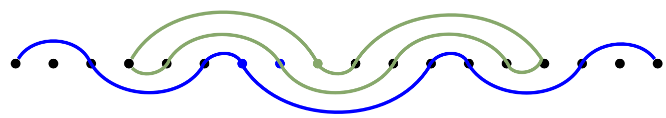


Figure 4.25: Colored meander M of $\mathfrak{p}_9^D(\{\alpha_3, \alpha_9\} \mid \{\alpha_1, \alpha_2, \alpha_6\})$

We have the following analogue of Theorem 4.4.8.

Theorem 4.4.23. *If $\mathfrak{g}$ is a seaweed of Type-D with reduced homotopy type*

$$H_D(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}}, \boldsymbol{c_{h_2+2}}, \boldsymbol{c_{h_2+3}}),$$

then

$$\text{ind } \mathfrak{g} = \sum_{i=1}^{h_1} c_i + \sum_{i=1}^{h_2} \left\lfloor \frac{c_i}{2} \right\rfloor + \frac{c_{h_2+1} - 2}{2} + \left\lfloor \frac{c_{h_2+2} - 2}{2} \right\rfloor + \frac{c_{h_2+3}}{2}.$$

Theorem 4.4.24. *Let $\mathfrak{g}$ be a Type-D seaweed with reduced homotopy type*

$$H_D(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}}, \boldsymbol{c_{h_2+2}}, \boldsymbol{c_{h_2+3}})$$

such that $\boldsymbol{c_i}$ is even for all $i \in [h_1 + 1, h_2]$ and $\boldsymbol{c_{h_2+1}} = \boldsymbol{c_{h_2+2}} = \boldsymbol{c_{h_2+3}} = 0$. The functional constructed as in Theorem 4.3.9 is regular.

The proof is exactly as in Type-B. □

Chapter 5

Future Work

5.1 Closing the Problem

The most natural next step is to close the problem of naming regular functionals for the few seaweeds in Type-B and Type-D which have not been handled. We have the following conjectures and next steps.

Conjecture 5.1.1. *Let $\mathfrak{g}$ be a Type-B seaweed with reduced homotopy type*

$$H_B(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}}).$$

Fix functionals $F_{c_i} \in \mathfrak{gl}(c_i)^*$ for all $i \in [1, h_1]$, $F_{c_i} \in \mathfrak{gl}(\lfloor \frac{c_i}{2} \rfloor)$ for $i \in [h_1 + 1, h_2]$, and $F_{c_{h_2+1}} \in \mathfrak{gl}(\frac{c_{h_2+1}-1}{2})$. For $i \in [h_1 + 1, h_2]$, define recursively

$$D_{h_2+1-i} = D_{h_2+1-(i-1)} \cup \left\{ n + 1 + \sum_{s=1}^{i-1} c_{h_2+1-s} + \left(\left\lfloor \frac{c_{h_2+1-i}}{2} \right\rfloor + 1 \right) \right\}$$

with $D_{h_2} = \{n + 1\}$. The adjusted framework to construct a functional F on $\mathfrak{g}$ is such that anytime the framework in Theorem 4.3.9 requires the use of a functional $e^*_{i,j}$ on the antidiagonal of $\mathfrak{g}$, replace $e^*_{i,j}$ with $\sum_{s \in D_{h_2+1-t}} e^*_{i,s}$, where c_{h_2+1-t} is the tail component for which $e^*_{i,j}$ is in $\mathfrak{P}_{c_{h_2+1-t}}$. Then $\dim \ker(B_F) = \sum i = 1^{h_2+1} \dim \ker(B_{F_{c_i}})$ over the reduced homotopy type components c_i.

Remark 5.1.2. *This framework is* ***not*** *one such that $\ker(B_F)$ is a subalgebra of the core $\mathfrak{C}$ associated with $\mathfrak{g}$.*

Example 5.1.3. *Consider $\mathfrak{p}_{20}^B(\{\alpha_3, \alpha_7, \alpha_{14}, \alpha_{16}, \alpha_{19}, \alpha_{20}\} \mid \emptyset)$. The functional described in Conjecture 5.1.1 using the functionals F_n of Theorem 3.3.1 is $F = \sum_{(i,j)\in\mathscr{I}_F} e_{i,j}^*$, where $\mathscr{I}_F$ is illustrated in Figure 5.1. This functional is regular. Only the portion of $\mathfrak{g}$ on or above the antidiagonal is illustrated in Figure 5.1.*

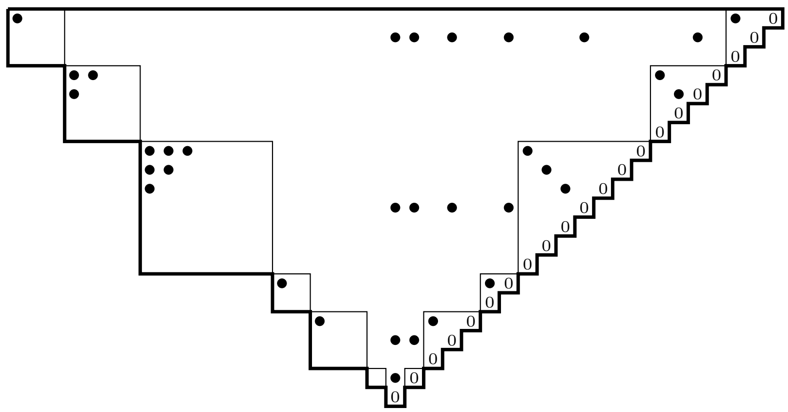


Figure 5.1: Indices in $\mathscr{I}_F$ for $\mathfrak{p}_{20}^B(\{\alpha_3, \alpha_7, \alpha_{14}, \alpha_{16}, \alpha_{19}, \alpha_{20}\} \mid \emptyset)$

Conjecture 5.1.4. *Let $\mathfrak{g}$ be a Type-D seaweed with reduced homotopy type*

$$H_D(c_1, \cdots, c_{h_1}, \boldsymbol{c_{h_1+1}}, \cdots, \boldsymbol{c_{h_2}}, \boldsymbol{c_{h_2+1}}, \boldsymbol{c_{h_2+2}}, \boldsymbol{c_{h_2+3}})$$

such that $c_{h_2+1} = 0$ and if $c_{h_2+2} \neq 0$, c_{h_2+2} is odd. Fix functionals $F_{c_i} \in \mathfrak{gl}(c_i)^*$ over $i \in [1, h_1] \cup \{t \in [h_1+1, h_2] \mid c_t$ is odd $\}$ and $F_{c_i} \in \mathfrak{gl}(\frac{c_i}{2})^*$ over $i \in [h_1+1, h_2]$ with c_i even. If $c_{h_2+3} \neq 0$, fix $F_{c_{h_2+3}} \in \mathfrak{gl}(\frac{c_{h_2+3}}{2})^*$; if $c_{h_2+2} \neq 0$, fix $F_{c_{h_2+2}} \in \mathfrak{gl}(c_{h_2+2}-2)^*$. Define D_{h_2+1-i} over $i \in [1, h_2]$ as in Conjecture 5.1.1 with $D_{h_2} = \emptyset$. Construct a functional $F \in \mathfrak{g}^*$ such according to the framework in Theorem 4.3.9, replacing any $e_{i,j}^*$ on the antidiagonal of $\mathfrak{g}$ with $e_{i,I}^*$ for $I \in D_{h_2+1-i} \backslash D_{h_2+1-(i-1)}$ for any component in the tail, and embed the functional $F_{c_{h_2+2}} + e_{c_{h_2+2},1}^*$ to the core blocks of c_{h_2+2}. The functional F constructed in this way is such that $\dim\ker(B_F) = \sum_{i=1}^{h_2+3} \dim\ker(B_{F_{c_i}})$ over the reduced homotopy type components c_i.

Remark 5.1.5. *As in Conjecture 5.1.1, the functional F constructed according to Conjecture 5.1.4 is not necessarily such that $\ker(B_F)$ is a subalgebra of the core $\mathfrak{C}$ associated with $\mathfrak{g}$.*

Example 5.1.6. *Consider* $\mathfrak{g} = \mathfrak{p}_{11}^D(\{\alpha_2, \alpha_5, \alpha_6, \alpha_{11}\} \mid \emptyset)$*. The reduced homotopy type of* $\mathfrak{g}$ *is* $H_D(\mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{5})$*. The functional described by Conjecture 5.1.4 using the functionals* F_n *of Theorem 3.3.1 is* $F = \sum_{(i,j)\in\mathscr{I}_F} e_{i,j}^*$*, where* $\mathscr{I}_F$ *is illustrated in Figure 5.2. This functional is regular. Note that only the portion of* $\mathfrak{g}$ *on or above the antidiagonal is illustrated in Figure 5.2.*

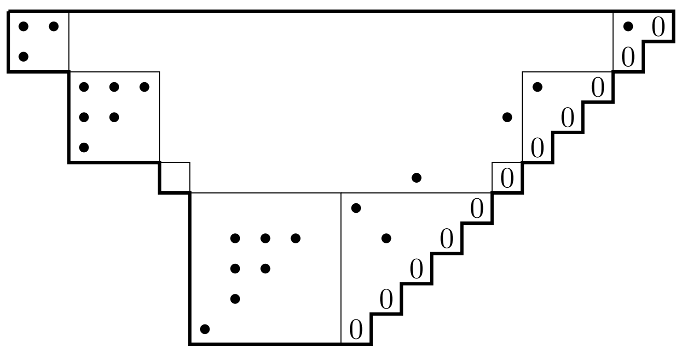


Figure 5.2: Indices in $\mathscr{I}_F$ for $\mathfrak{p}_{11}^D(\{\alpha_2, \alpha_5, \alpha_6, \alpha_{11}\} \mid \emptyset)$.

To close the problem for Type-D seaweeds, it suffices to determine how to embed functionals to components c_{h_2+1} for odd and even sizes and to address how to embed functionals to components c_{h_2+2} of even size. These are the new basis steps for the framework in Type-B and Type-D. The proofs for the basis steps will be inductive on components of the tail. Further, there are two more classes of seaweeds we aim to develop regular functionals on:

1. seaweed subalgebras of D_n which do not have seaweed "shape" (i.e. $\mathfrak{p}_n^D(\Psi \mid \Psi')$ with α_n in either Ψ or Ψ' and α_{n-1} in the other, but neither is in both – see Section 5.1.1), and

2. seaweed subalgebras of the exceptional Lie algebras, for which the meandric machinery has not yet been developed.

The final thing to consider is an optimization of this procedure. In Section 3.4, we provide seven additional regular functionals on $\mathfrak{gl}(n)$ which are all smaller than

F_n, as this is computationally better. It would be interesting to explore whether or not there is a "smallest" (minimal) functional S_n on $\mathfrak{gl}(n)$ and if there is a way to prove a minimum number of indices needed in $\mathscr{I}_{S_n}$ for S_n to be regular on $\mathfrak{gl}(n)$.

5.1.1 Discussion of Seaweeds without Seaweed "Shape"

By examining the root space matrix in Example 4.4.11, the natural question arises: what happens when we exclude α_3 but not α_4? The matrix form of $\mathfrak{p}_4^D(\emptyset \mid \{\alpha_3\})$ is illustrated in Figure 5.3.

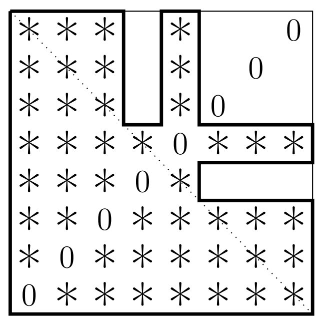


Figure 5.3: Matrix form of $\mathfrak{p}_4^D(\emptyset \mid \{\alpha_3\})$

Due to work by Cameron **[4]**, and Panyushev and Yakimova (see **[19]**), we understand that there is only one situation when the resulting seaweed is not isomorphic to a standard seaweed with seaweed shape (as defined in Definition **??**. Assume $\alpha_{n-1} \in \Psi$. One of the following four cases must be true:

1. $\alpha_{n-1}, \alpha_n \in \Psi'$,

2. $\alpha_{n-1} \in \Psi'$ and $\alpha_n \notin \Psi'$,

3. $\alpha_n \in \Psi'$ and $\alpha_{n-1} \notin \Psi'$,

4. $\alpha_{n-1}, \alpha_n \notin \Psi'$.

Through examination of the Dynkin diagrams, under the isomorphism which switches α_{n-1} and α_n, the seaweeds in cases (1), (2), and (4) are isomorphic to a seaweed with seaweed shape. Therefore, the only situation which produces a seaweed which is not isomorphic to a standard seaweed of type $\frac{a_1|\cdots|a_m}{b_1|\cdots|b_t}$ in $\mathfrak{gl}(2n)$ occurs in case (3).

5.2 More Work with Functionals

The framework described in Chapter 3 was inspired by observing how the cascade functional changed under the winding-up of a Type-A seaweed. In Type-A the cascade creates a functional which sums over functionals on the indices of the peaks of the blocks in a seaweed $\mathfrak{g}$. While this is not demonstrated in either method outlined in Chapter 3, it seems to point to a third method of constructing functionals.

Definition 5.2.1. *Let $\mathfrak{g}$ be a seaweed with homotopy type $H(c_1, \cdots, c_h)$, and let $F_{c_i} \in \mathfrak{gl}(c_i)^*$, for all i. Define the functionals*

$$F_{c_i}^+ = \sum_{\substack{(i,j)\in\mathscr{I}_F \\ i\leq j}} e_{i,j}^*$$

and

$$F_{c_i}^- = \sum_{\substack{(i,j)\in\mathscr{I}_F \\ i\geq j}} e_{i,j}^*.$$

Embed $F_{c_i}^+$ into a $c_i \times c_i$ block in the core of $\mathfrak{g}$ such that the vertex v_{A_I} corresponds to an endpoint in the component meander on the path c_i. Embed $(F_{c_i}^-)^r$ into the $c_i \times c_i$ block which corresponds to the other endpoint for appropriate rotation based on main diagonal or antidiagonal functionals in peaks.

Conjecture 5.2.2. *The functional F constructed according to Definition 5.2.1 is such that* $\dim\ker(B_F) = \sum_{i=1}^h \dim\ker(B_{F_{c_i}})$.

Remark 5.2.3. *It is important to note (see Examples B.2.7 and B.2.8) that the framework described in Definition 5.2.1 is not necessarily one for which* $\ker(B_F)$ *is a subalgebra of the core of* $\mathfrak{g}$.

If Conjecture 5.2.2 is true, then the cascade on seaweeds of Type-A and Type-C is constructed via Definition 5.2.1 through the use of the functionals

$$F_{c_i} = \sum_{s=1}^{\lfloor c_i/2 \rfloor} (e^*_{i,c_i+1-i} + e^*_{c_i+1-i,i}).$$

Substantial empirical evidence suggests the following conjectures.

Conjecture 5.2.4. *The cascade functional $\varphi_{\mathfrak{g}}$ succeeds on any classical simple Lie algebra $\mathfrak{g}$ if and only if $\mathfrak{g} = A_1$ or $\mathfrak{g} = A_2$. In fact, the dimension of $\ker(B_{\varphi_{\mathfrak{g}}})$ is as follows.*

(1) In Type-A,

$$\begin{aligned}\ker(B_{\varphi_{A_n}}) =& span(\{e_{i,j} + e_{n+2-i,n+2-j} \mid i \in [1,n], i \neq \lfloor n/2 \rfloor,\ j \in [1, \lfloor n/2 \rfloor],\ i \neq j\} \\ & \cup\ \{e_{i,i} + e_{n+2-i,n+2-i} - P \mid i \in [1, \lfloor n/2 \rfloor]\}),\end{aligned}$$

where $P = e_{\frac{n}{2},\frac{n}{2}} + e_{\frac{n}{2}+1,\frac{n}{2}+1}$ if n is even and $P = 2e_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ if n is odd. It follows that $\dim \ker(B_{\varphi_{A_n}}) = 2\left(\lfloor \frac{n}{2} \rfloor\right)^2 - (n \ mod\ 2)$. Therefore, φ_{A_n} is regular on A_n for $n \in \{1, 2\}$. This is verified through direct computation as a relations matrix B_n for $\ker(B_{\varphi_{A_n}})$ for $n \in \{1, 2\}$ is

$$B_1 = \begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix}, \qquad \text{and} \qquad B_2 = \begin{pmatrix} b_1 & 0 & b_2 \\ 0 & -2b_1 & 0 \\ b_2 & 0 & b_1 \end{pmatrix}.$$

(2) In Type-C,

$$\begin{aligned}\ker(B_{\varphi_{C_n}}) =& span(\{e_{i,j} - e_{j,i} - e_{2n+1-j,2n+1-i} + e_{2n+1-i,2n+1-j} \mid i,j \in [1,n],\ i \neq j\} \\ & \cup\ \{e_{i,j} + e_{j,i} + e_{2n+1-i,2n+1-j} + e_{2n+1-j,2n+1-i} \\ & \qquad \mid i \in [1,n],\ j > n,\ i+j \leq 2n+1\}).\end{aligned}$$

It follows that $\dim \ker(B_{\varphi_{C_n}}) = n^2$.

(3) In Type-B,

$$\begin{aligned}\ker(B_{B_n}) =& span(\{e_{i,j}+e_{j,i}-e_{2n+2-j,2n+2-i}-e_{2n+2-i,2n+2-j} \\ & \qquad \mid i\in[1,n]\cup[n+2,2n+1],\ i+j<2n+2,\ i\neq j\} \\ & \cup\ \{e_{i,n+1}-e_{n+1,2n+2-i}+(-1)^{i+1}(e_{n+1,i}-e_{2n+2-i,n+1})\mid i\in[1,n]\}).\end{aligned}$$

It follows that $\dim\ker(B_{\varphi_{B_n}})=n^2$.

(4) In Type-D,

$$\begin{aligned}\ker(B_{\varphi_{D_n}}) =& span\{e_{i,j}+e_{j,i}-e_{2n+1-i,2n+1-j}-e_{2n+1-j,2n+1-i} \\ & \qquad \mid i\in[1,2n],\ i+j<2n+1,\ i\neq j\}.\end{aligned}$$

It follows that $\dim\ker(B_{\varphi_{D_n}})=n^2-n$.

Conjecture 5.2.5. *(Obstruction Theory)*

(1) If $\mathfrak{g}$ is a Type-A seaweed with homotopy type $H(c_1,\cdots,c_h)$, then the cascade functional fails if there exists i with $c_i\geq 4$.

(2) If $\mathfrak{g}$ is a Type-BCD seaweed with reduced homotopy type

$$H_C(c_1,\cdots,c_{h_1},\boldsymbol{c_{h_1+1}},\cdots,\boldsymbol{c_{h_2}},\boldsymbol{c_{h_2+1}}),$$

then the cascade functional fails if there exists $i\in[1,h_1]$ with $c_i\geq 4$, or if there exists $i\in[h_1+1,h_2]$ with $c_i\geq 5$, or if $c_{h_2+1}\geq 2$.

From the fact that the zero functional is regular on $\mathfrak{gl}(1)$, we have the following Theorem.

Theorem 5.2.6. *Let $\mathfrak{g}$ be a Type-A or Type-C seaweed with homotopy type $H(1,\cdots,1)$. Then $\varphi_{\mathfrak{g}}$ is regular on $\mathfrak{g}$.*

The proof of Theorem 5.2.6 follows from the fact that the functional $\varphi_{\mathfrak{g}}$ is equal to the functional constructed in Chapter 3. By Theorem 4.2.3, a Type-A seaweed $\mathfrak{g}$ is Frobenius if and only if the meander associated with $\mathfrak{g}$ consists of exactly one path (i.e., is of homotopy type $H(1)$). By Theorems 4.3.5, 4.4.5, and 4.4.16, a Type-BCD seaweed $\mathfrak{g}$ is Frobenius if and only if the meander M_n^{BCD} associated with $\mathfrak{g}$ is a forest rooted in the tail (i.e., is of reduced homotopy type $H_C(\mathbf{1}, \cdots, \mathbf{1})$). By Theorem 5.2.6, we have the following immediate corollary.

Theorem 5.2.7. *If $\mathfrak{g}$ is a Frobenius seaweed subalgebra of a classic Lie algebra, then the cascade functional $\varphi_{\mathfrak{g}}$ is regular on $\mathfrak{g}$.*

Of further interest to investigate is the method in which functionals are embedded in the frameworks developed in Chapter 3. Given $F \in \mathfrak{gl}(c)^*$, if $F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j}e^*_{i,j}$, let $I_F = \sum_{(i,j)\in\mathscr{I}_F} c_{i,j}e_{i,j}$ be the $c \times c$ ***coefficient matrix*** whose entry in position (i, j) is the coefficient of $e^*_{i,j}$ in F.

Conjecture 5.2.8. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(n)$ be a seaweed and let $F \in \mathfrak{g}^*$ have coefficient matrix I_F. For any $P \in GL(n)$, the functional $F' \in (P^{-1}\mathfrak{g}P)^*$ with coefficient matrix $P^{-1}I_FP$ is such that $\dim\ker(B_F) = \dim\ker(B_{F'})$.*

Let $A_c = \sum_{i=1}^{c} e_{i,2n+1-i}$ and I_c be the identity matrix. When embedding functionals in Chapter 3, the funcitonal F_c whose coefficients form the coefficient matrix $I_cI_FI_c = (I_c)^{-1}I_FI_c$ is embedded in the core if the main diagonal functionals (i.e., I_c) are added, and the functional F_c^R whose coefficients form the matrix $A_cI_FA_c = (A_c)^{-1}I_FA_c$ is embedded in the core if the antidiagonal funtionals (i.e., A_c) are added. This may point to a more general approach to building a framework which represents the winding-up of functionals. It may also explain why the method outlined in Definition 5.2.1 is such that $\ker(B_F)$ is not a subalgebra of the core of $\mathfrak{g}$.

Similarly, while we were not interested in an explicit basis for $\ker(B_F)$ in any part of this paper, the methods outlined in Chapter 3 name a basis. Exploration of this basis may provide insight to other problems in the field.

5.3 Quasireductive Lie Algebras

A ***quasi-reductive Lie algebra*** (see Baur and Moreau [**3**], and Moreau and Yakimova [**23**]) is a Lie algebra for which there exists $f \in \mathfrak{g}^*$ of ***reductive type*** (i.e., a functional such that $\mathfrak{g}(f)/\mathfrak{z}$ is a reductive Lie algebra whose center consists of semisimple elements of $\mathfrak{g}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and

$$\mathfrak{g}(f) = \{X \in \mathfrak{g} \mid ad_X^* \circ f = 0\},$$

where ad^* is the coadjoint representation of $\mathfrak{g}$).[1] Although they use different terminology, Moreau and Yakimova (see [**23**]) explicitly compute the ***Maximal Reductive Stabilizer*** ("MRS"), which is the stabilizer of f, of any quasi-reductive seaweed $\mathfrak{g}$ using a meander-type mechanism. They note that when it exists, the MRS is essentially unique and is a conjugation invariant. (The conjugation takes place over the associated algebraic group.) The homotopy type of a quasi-reductive seaweed can be seen explicitly in the matrix representation of the MRS. It is of important note that seaweeds in Type-A and Type-C are quasi-reductive. It follows that the homotopy type is also a conjugation invariant in Type-A and Type-C. We conjecture that this is true more generally – i.e., in Type-B and Type-D.

Conjecture 5.3.1. *The homotopy type is a conjugation invariant in all the classical cases.*

5.4 Unbroken Spectrum of a Frobenius Seaweed

Of keen current interest is the ***spectrum*** of a Frobenius Lie algebra $\mathfrak{g}$ (see [**18**] and [**14**]). This is the spectrum of eigenvalues associated with $ad_X \in \mathfrak{g}^*$ (where ad_X is the adjoint endomorphism, see Definition B.2.1), where X is a *principal element* of $\mathfrak{g}$ (i.e., an element of $\mathfrak{g}$ such that $F \circ ad_X = F$). Generally, the spectral values of a Frobenius Lie algebra take on any value in the ground field. (See [**14**], for examples.) But when $\mathfrak{g}$ is also a seaweed, the eigenvalues must be integers (see [**21**]). Topical work on the spectrum has established that in the classical cases, the

[1] Although the definition of quasi-reductive may seem overly technical and abstruse, it captures a great deal of tangible information about the Lie algebra. For example, recent work by Ammari [**2**] establishes that the rigidity of a seaweed is equivalent to the quasi-reductivity of the seaweed. Although this does not preclude the existence of Lie cohomolgy it is suggestive that such Lie algebras are cohomologically inert – which is known to be true in the Type-A case (see [17]).

spectrum is, in fact, an unbroken sequence of integers centered about one-half for seaweed algebras in Type-A (see [**10**]), Type-B and Type-C (see [**8**]), and Type-D (see [**5**]). Extensive simulations suggest that this sequence is unimodal (see [**12**]). The techniques of building up a functional via the winding up moves may be useful for *analytically* tracking what happens to the spectrum of a Frobenius algebra as the moves are applied - and may give insight into what geometric property of the associated Lie group the unbroken spectrum of the Lie algebra is manifesting.

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Appendix A

Root System Calculations

We include some example calculations of the full root system for the classical Lie algebras A_n, B_n, C_n, and D_n. The positive roots of every system are generated through reflections in the hyperplane about the simple roots – our chosen basis for the system. Recall that we already provide the simple roots for each algebra in Table 4.1. The calculations for A_4, B_4, C_4, and D_4 are detailed below.

Root β	Simple Root Coefficients	Vector	$s_{\alpha_1}(\beta)$	$s_{\alpha_2}(\beta)$	$s_{\alpha_3}(\beta)$	$s_{\alpha_4}(\beta)$
α_1	(1,0,0,0)	(1,-1,0,0,0)	$-\alpha_1$	$\beta_1 := \alpha_1 + \alpha_2$	α_1	α_1
α_2	(0,1,0,0)	(0,1,-1,0,0)	β_1	$-\alpha_2$	$\beta_2 := \alpha_2 + \alpha_3$	α_2
α_3	(0,0,1,0)	(0,0,1,-1,0)	α_3	β_2	$-\alpha_3$	$\beta_3 := \alpha_3 + \alpha_4$
α_4	(0,0,0,1)	(0,0,0,1,-1)	α_4	α_4	β_3	$-\alpha_4$
β_1	(1,1,0,0)	(1,0,-1,0,0)	α_2	α_1	$\beta_4 := \beta_1 + \alpha_3$	β_1
β_2	(0,1,1,0)	(0,1,0,-1,0)	β_4	α_3	α_2	$\beta_5 := \beta_2 + \alpha_4$
β_3	(0,0,1,1)	(0,0,1,0,-1)	β_3	β_5	α_4	α_3
β_4	(1,1,1,0)	(1,0,0,-1,0)	β_2	β_4	β_1	$\beta_6 := \beta_4 + \alpha_4$
β_5	(0,1,1,1)	(0,1,0,0,-1)	β_6	β_3	β_5	β_2
β_6	(1,1,1,1)	(1,0,0,0,-1)	β_5	β_6	β_6	β_4

Table A.1: Positive roots in Φ_{A_4}

Root β	Simple Root Coefficients	Vector	$s_{\alpha_1}(\beta)$	$s_{\alpha_2}(\beta)$	$s_{\alpha_3}(\beta)$	$s_{\alpha_4}(\beta)$
α_1	(1,0,0,0)	(1,-1,0,0)	$-\alpha_1$	$\beta_1 := \alpha_1 + \alpha_2$	α_1	α_1
α_2	(0,1,0,0)	(0,1,-1,0)	β_1	$-\alpha_2$	$\beta_2 := \alpha_2 + \alpha_3$	α_2
α_3	(0,0,1,0)	(0,0,1,-1)	α_3	β_2	$-\alpha_3$	$\beta_3 := \alpha_3 + 2\alpha_4$
α_4	(0,0,0,1)	(0,0,0,1)	α_4	α_4	$\beta_4 := \alpha_3 + \alpha_4$	$-\alpha_4$
β_1	(1,1,0,0)	(1,0,-1,0)	α_2	α_1	$\beta_5 := \beta_1 + \alpha_3$	β_1
β_2	(0,1,1,0)	(0,1,0,-1)	β_5	α_3	α_2	$\beta_6 := \beta_2 + 2\alpha_4$
β_3	(0,0,1,2)	(0,0,1,1)	β_3	β_6	β_3	α_3
β_4	(0,0,1,1)	(0,0,1,0)	β_4	$\beta_7 := \beta_4 + \alpha_2$	α_4	β_4
β_5	(1,1,1,0)	(1,0,0,-1)	β_2	β_5	α_3	$\beta_8 := \beta_5 + 2\alpha_4$
β_6	(0,1,1,2)	(0,1,0,1)	β_8	β_3	$\beta_9 := \beta_6 + \alpha_3$	β_2
β_7	(0,1,1,1)	(0,1,0,0)	$\beta_{10} := \beta_7 + \alpha_1$	β_4	β_7	β_7
β_8	(1,1,1,2)	(1,0,0,1)	β_6	β_8	$\beta_{11} := \beta_8 + \alpha_3$	β_5
β_9	(0,1,2,2)	(0,1,1,0)	β_{11}	β_9	β_6	β_9
β_{10}	(1,1,1,1)	(1,0,0,0)	β_7	β_{10}	β_{10}	β_{10}
β_{11}	(1,1,2,2)	(1,0,1,0)	β_9	$\beta_{12} := \beta_{11} + \alpha_2$	β_8	β_{11}
β_{12}	(1,2,2,2)	(1,1,0,0)	β_{12}	β_{11}	β_{12}	β_{12}

Table A.2: Positive roots in Φ_{B_4}

Root β	Simple Root Coefficients	Vector	$s_{\alpha_1}(\beta)$	$s_{\alpha_2}(\beta)$	$s_{\alpha_3}(\beta)$	$s_{\alpha_4}(\beta)$
α_1	(1,0,0,0)	(1,-1,0,0)	$-\alpha_1$	$\beta_1 := \alpha_1 + \alpha_2$	α_1	α_1
α_2	(0,1,0,0)	(0,1,-1,0)	β_1	$-\alpha_2$	$\beta_2 := \alpha_2 + \alpha_3$	α_2
α_3	(0,0,1,0)	(0,0,1,-1)	α_3	β_2	$-\alpha_3$	$\beta_3 := \alpha_3 + \alpha_4$
α_4	(0,0,0,1)	(0,0,0,2)	α_4	α_4	$\beta_4 := 2\alpha_3 + \alpha_4$	$-\alpha_4$
β_1	(1,1,0,0)	(1,0,-1,0)	α_2	α_1	$\beta_5 := \beta_1 + \alpha_3$	β_1
β_2	(0,1,1,0)	(0,1,0,-1)	β_5	α_3	α_2	$\beta_6 := \beta_2 + \alpha_4$
β_3	(0,0,1,1)	(0,0,1,1)	β_3	β_6	β_3	α_3
β_4	(0,0,2,1)	(0,0,2,0)	β_4	$\beta_7 := \beta_4 + 2\alpha_2$	α_4	β_4
β_5	(1,1,1,0)	(1,0,0,-1)	β_2	β_5	β_1	$\beta_8 := \beta_5 + \alpha_4$
β_6	(0,1,1,1)	(0,1,0,1)	β_8	β_3	$\beta_9 := \beta_6 + 2\alpha_3$	β_2
β_7	(0,2,2,1)	(0,2,0,0)	$\beta_{10} := \beta_7 + 2\alpha_1$	β_4	β_7	β_7
β_8	(1,1,1,1)	(1,0,0,1)	β_6	β_8	$\beta_{11} := \beta_8 + \alpha_3$	β_5
β_9	(0,1,2,1)	(0,1,1,0)	β_{11}	β_9	β_6	β_9
β_{10}	(2,2,2,1)	(2,0,0,0)	β_7	β_{10}	β_{10}	β_{10}
β_{11}	(1,1,2,1)	(1,0,1,0)	β_9	$\beta_{12} := \beta_{11} + \alpha_2$	β_8	β_{11}
β_{12}	(1,2,2,1)	(1,1,0,0)	β_{12}	β_{11}	β_{12}	β_{12}

Table A.3: Positive roots in Φ_{C_4}

Root β	Simple Root Coefficients	Vector	$s_{\alpha_1}(\beta)$	$s_{\alpha_2}(\beta)$	$s_{\alpha_3}(\beta)$	$s_{\alpha_4}(\beta)$
α_1	(1,0,0,0)	(1,-1,0,0)	$-\alpha_1$	$\beta_1 := \alpha_1 + \alpha_2$	α_1	α_1
α_2	(0,1,0,0)	(0,1,-1,0)	β_1	$-\alpha_2$	$\beta_2 := \alpha_2 + \alpha_3$	$\beta_3 := \alpha_2 + \alpha_4$
α_3	(0,0,1,0)	(0,0,1,-1)	α_3	β_2	$-\alpha_3$	α_3
α_4	(0,0,0,1)	(0,0,1,1)	α_4	β_3	α_4	$-\alpha_4$
β_1	(1,1,0,0)	(1,0,-1,0)	α_2	α_1	$\beta_4 := \beta_1 + \alpha_3$	$\beta_5 := \beta_1 + \alpha_4$
β_2	(0,1,1,0)	(0,1,0,-1)	β_4	α_3	α_2	$\beta_6 := \beta_2 + \alpha_4$
β_3	(0,1,0,1)	(0,1,0,1)	β_5	α_4	β_6	α_2
β_4	(1,1,1,0)	(1,0,0,-1)	β_2	β_4	β_1	$\beta_7 := \beta_4 + \alpha_4$
β_5	(1,1,0,1)	(1,0,0,1)	β_3	β_5	β_7	β_1
β_6	(0,1,1,1)	(0,1,1,0)	β_7	β_6	β_3	β_2
β_7	(1,1,1,1)	(1,0,1,0)	β_6	$\beta_8 := \beta_7 + \alpha_2$	β_5	β_4
β_8	(1,2,1,1)	(1,1,0,0)	β_8	β_7	β_8	β_8

Table A.4: Positive roots in Φ_{D_4}

In general, we have the following.

Theorem A.0.1. *The root systems for the classical Lie algebras are as follows.*

(1) The root system for A_n is a subset of the real inner product space $\mathbb{R}^{n+1}$. If $\{\alpha_i\}_{i=1}^n$ is the set of simple roots for A_n, then Φ_{A_n} has $\frac{n(n+1)}{2}$ positive roots, each of the form $e_i - e_j = \sum_{s=i}^{j-1} \alpha_s$ over $i, j \in [1, n+1]$ with $i < j$.

(2) The root system for B_n is a subset of the real inner product space $\mathbb{R}^n$. If $\{\alpha_i\}_{i=1}^n$ is the set of simple roots for B_n, then Φ_{B_n} has n^2 positive roots, each of one of the following three forms over $i, j \in [1, n]$ with $i < j$:

1. $e_i - e_j = \sum_{s=i}^{j-1} \alpha_s$,

2. $e_i = \sum_{s=i}^{n} \alpha_s$, *and*

3. $e_i + e_j = \sum_{s=i}^{n} \alpha_s + \sum_{s=j}^{n} \alpha_s$.

(3) The root system for C_n is a subset of the real inner product space $\mathbb{R}^n$. If $\{\alpha_i\}_{i=1}^n$ is the set of simple roots for C_n, then Φ_{C_n} has n^2 positive roots, each of one of the following three forms over $i, j \in [1, n]$ with $i < j$:

1. $e_i - e_j = \sum_{s=i}^{j-1} \alpha_s$,
2. $2e_i = \alpha_n + 2\sum_{s=i}^{n-1} \alpha_s$, *and*
3. $e_i + e_j = \sum_{s=i}^{n-1} \alpha_s + \sum_{s=j}^{n} \alpha_s$.

(4) The root system for D_n is a subset of the real inner product space $\mathbb{R}^n$. If $\{\alpha_i\}_{i=1}^n$ is the set of simple roots for D_n, then Φ_{D_n} has $n(n-1)$ positive roots, each of one of the following forms for $i, j \in [1, n]$ with $i < j$:

1. $e_i - e_j = \sum_{s=i}^{j-1} \alpha_s$, *and*
2. $e_i + e_j = \sum_{s=i}^{n-2} \alpha_s + \sum_{s=j}^{n} \alpha_s$.

Appendix B

The Cascade

It has previously been noted (see [**3**]) that Kostant's cascade can be used to construct a regular functional on any seaweed. In this appendix, we outline how the cascade works, provide example calculations using it, and detail explicit counterexamples to the cascade producing a regular functional in certain cases.

B.1 The Cascade

The cascade is a method of constructing a subset $\Gamma \subseteq \Phi$ of ***strongly orthogonal roots*** (i.e. a subset of orthogonal roots such that for $\alpha, \beta \in \Gamma$, if $\alpha \neq \beta$, then $\alpha \pm \beta \notin \Phi \cup \{0\}$). Given a subset Π' of the simple roots Π, the *cascade* $\mathscr{K}_{\Pi'}$ is defined by induction on the cardinality of Π' as follows:

1. $\mathscr{K}_{\emptyset} := \emptyset$

2. If $\pi'_1, \cdots, \pi'_r$ are the connected components of Π' (connected according to the Dynkin diagram), then
$$\mathscr{K}_{\Pi'} = \bigcup_{i=1}^{r} \mathscr{K}_{\pi'_r}$$

3. If Π' is connected, then $\mathscr{K}_{\Pi'} := \{\Pi'\} \cup \mathscr{K}_T$, where $T := \{\alpha \in \Pi' \mid \langle \alpha, \epsilon^{\vee}_{\Pi'} \rangle = 0\}$, where $\epsilon^{\vee}_{\Pi'}$ is the highest root generated by Π' (i.e. the root with the largest sum of coefficients when expressed as a sum of simple roots).

Remark B.1.1. *It is important to note that $\mathscr{K}_{\Pi'}$ is a set of sets of simple roots, not a set of roots. The set of strongly orthogonal roots the cascade points to is the set $\{\beta_K \mid K \in \mathscr{K}_{\Pi'}\}$, where β_K is the highest root generated by the set K in $\mathscr{K}_{\Pi'}$.*

Note that $\langle \alpha, \beta \rangle = 0$ is equivalent to saying $(\alpha, \beta) = 0$.

A table of the highest roots in each progressive step of the cascade for the simple Lie algebras (acting on all of Π and not a proper subset) is provided by Baur and Moreau in [**3**], and included here in Tables B.1 and B.2.

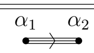
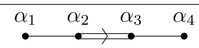
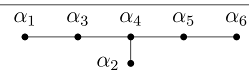
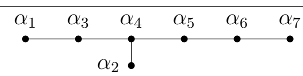
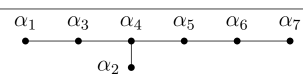
G_2		$\epsilon_1 := 2\alpha_1 + 3\alpha_2$, $\epsilon_2 := \alpha_2$
F_4		$\epsilon_1 := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\epsilon_2 := \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\epsilon_3 := \alpha_2 + 2\alpha_3$, $\epsilon_4 := \alpha_2$
E_6		$\epsilon_1 := \alpha_1 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_2$, $\epsilon_2 := \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\epsilon_3 := \alpha_3 + \alpha_4 + \alpha_5$, $\epsilon_4 := \alpha_4$
E_7		$\epsilon_1 := 2\alpha_1 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_2$, $\epsilon_2 := \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_2$, $\epsilon_3 := \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_2$, $\epsilon_4 := \alpha_7$, $\epsilon_5 := \alpha_2$, $\epsilon_6 := \alpha_3$, $\epsilon_7 := \alpha_5$
E_8		$\epsilon_1 := 2\alpha_1 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 + \alpha_2$, $\epsilon_2 := 2\alpha_1 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 1\alpha_7 + 2\alpha_2$, $\epsilon_3 := \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_2$, $\epsilon_4 := \alpha_3 + \alpha_4 + \alpha_5 + \alpha_2$, $\epsilon_5 := \alpha_7$, $\epsilon_6 := \alpha_2$, $\epsilon_7 := \alpha_3$, $\epsilon_8 := \alpha_5$

Table B.1: Highest roots in the cascade calculation for the exceptional Lie algebras

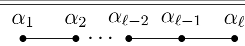
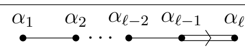
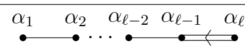
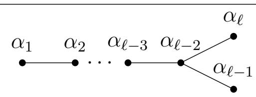
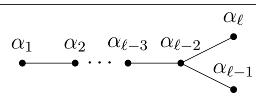
A_ℓ, $\ell \geq 1$		$\left\{\epsilon_i := \alpha_i + \cdots + \alpha_{i+(\ell-2i+1)} \mid i \leq \left\lfloor \frac{\ell+1}{2} \right\rfloor\right\}$
B_ℓ, $\ell \geq 2$		$\{\epsilon_i := \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_\ell \mid i \text{ even },\ i \leq \ell - 1\}$ $\cup\{\epsilon_i := \alpha_i \mid i \text{ odd },\ i \leq \ell\}$
C_ℓ, $\ell \geq 3$		$\{\epsilon_i := 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell \mid i \text{ even },\ i \leq \ell\} \cup \{\epsilon_\ell := \alpha_\ell\}$
D_ℓ, ℓ even, $\ell \geq 4$		$\{\epsilon_i := \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell \mid i \text{ even },\ i < \ell - 1\}$ $\cup\{\epsilon_i := \alpha_i \mid i \text{ odd },\ i < \ell\} \cup \{\epsilon_\ell := \alpha_\ell\}$
D_ℓ, ℓ odd, $\ell \geq 4$		$\{\epsilon_i := \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell \mid i \text{ even },\ i < \ell - 1\}$ $\cup\{\epsilon_i := \alpha_i \mid i \text{ odd },\ i < \ell\} \cup \{\epsilon_{\ell-1} := \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell\}$

Table B.2: Highest roots in the cascade calculation for the classical Lie algebras

B.1.1 Examples

In this section, we demonstrate how the cascade operates on several example sets of simple roots. First, we address the simple roots for the Lie algebras A_4, B_4, C_4, D_4, and D_5. These examples will be referenced in section B.2. We then demonstrate running the cascade on proper subsets of the simple roots for the simple Lie algebras.

Example B.1.2. *Consider Π_{A_4} (the simple roots in Φ_{A_4}). The positive roots of Φ_{A_4} are calculated in Table A.1 of Appendix A. Evidently, β_6 is the highest root in the table. Now, we have*

$$(\beta_6, \alpha_1) = 1, \qquad (\beta_6, \alpha_2) = 0, \qquad (\beta_6, \alpha_3) = 0, \qquad (\beta_6, \alpha_4) = 1.$$

Therefore, the cascade is equal to $\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\} \cup \mathscr{K}_{\{\alpha_2,\alpha_3\}}$. Referencing Table A.1 once more, the highest root generated by $\{\alpha_2, \alpha_3\}$ is β_2. To repeat the process,

$$(\beta_2, \alpha_2) = (\beta_2, \alpha_3) = 1.$$

Therefore, the cascade is completed and gives the following:

$$\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_2, \alpha_3\}\}. \tag{B.1}$$

The set of strongly orthogonal roots the cascade points to is the set of highest roots for each set in (B.1). In other words,

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \ \ \alpha_2 + \alpha_3\}.$$

It is easy to verify that this agrees with Table B.2.

Example B.1.3. *Consider Π_{B_4}. Referencing Table A.2 in Appendix A, we find that β_{12} is the highest positive root in the system. The next step of the cascade is to determine which simple roots are orthogonal to β_{12}. As α_2 is the only simple root not orthogonal to β_{12}, the cascade will be $\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\} \cup \mathscr{K}_{\{\alpha_1,\alpha_3,\alpha_4\}}$, and it is now sufficient to determine the highest root generated by $\{\alpha_1, \alpha_3, \alpha_4\}$. Since $\{\alpha_1, \alpha_3, \alpha_4\}$ is not connected in the Dynkin diagram for B_4, by condition (2) of the cascade construction, the cascade splits into a union of the cascade on the connected components $\{\alpha_1\}$ and $\{\alpha_3, \alpha_4\}$. The cascade on $\{\alpha_1\}$ is complete. By referencing Table A.2, the highest root generated by $\{\alpha_3, \alpha_4\}$ is β_3. Only α_3 is orthogonal to β_3. Therefore, the cascade yields the following set*

$$\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_1\}, \{\alpha_3, \alpha_4\}, \{\alpha_3\}\},$$

with set of highest roots

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \ \ \alpha_3 + 2\alpha_4, \ \ \alpha_1, \ \ \alpha_3\}.$$

Again, this agrees with Table B.2.

Example B.1.4. *Consider Π_{C_4}. Refering to Table A.3, the highest positive root in Φ_{C_4} is β_{10}. Note that α_1 is the only simple root not orthogonal to β_{10}. The new subset $\{\alpha_2, \alpha_3, \alpha_4\}$ generates β_7 as its highest root. The simple root α_2 is not orthogonal to β_7, and the highest root generated by $\{\alpha_3, \alpha_4\}$ is β_4. Finally, α_3 is not orthogonal to β_4 and the cascade gives the following results in agreement with Table B.2:*

$$\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_2, \alpha_3, \alpha_4\}, \{\alpha_3, \alpha_4\}, \{\alpha_4\}\},$$

$$\{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \ 2\alpha_2 + 2\alpha_3 + \alpha_4, \ 2\alpha_3 + \alpha_4, \alpha_4\}.$$

Example B.1.5. *Consider* Π_{D_4}. *The highest positive root in* Φ_{D_4} *is* β_8. *The roots* $\{\alpha_1, \alpha_3, \alpha_4\}$ *are orthogonal to* β_8. *By examining the Dynkin diagram for* D_4 *(see Table 4.2), it is apparent that this set is three separate connected singleton components, and so the cascade is complete. The result is*

$$\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_1\}, \{\alpha_3\}, \{\alpha_4\}\},$$

with set of highest roots

$$\{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_1, \quad \alpha_3, \quad \alpha_4\}.$$

This agrees with Table A.4.

To highlight why n even and odd are listed separately in Table B.2, consider the following Example B.1.6.

Example B.1.6. *Consider* Π_{D_5}. *The highest positive root in* Φ_{D_5} *(cf., Table B.2) is*

$$\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = (1, 1, 0, 0, 0).$$

The only simple root which is not orthogonal to β_1 *is* α_2. *The next step of the cascade splits over the two connected subsets of simple roots which result from the elimination of* α_2. *These sets are* $\{\alpha_1\}$ *and* $\{\alpha_3, \alpha_4, \alpha_5\}$. *Now, there are two ways to consider the latter set. With a list of the positive roots in* Φ_{D_5}, *it becomes obvious that the highest root generated by these three roots is*

$$\beta_2 = \alpha_3 + \alpha_4 + \alpha_5 = (0, 0, 1, 1, 0).$$

However, this can also be seen from the fact that the portion of the Dynkin diagram generated by these three roots is isomorphic (with appropriate relabeling of simple roots) to the Dynkin diagram for A_3. *The only root in* $\{\alpha_3, \alpha_4, \alpha_5\}$ *orthogonal to* β_2 *is* α_3. *Hence, the cascade on* Π_{D_5} *is*

$$\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \{\alpha_1\}, \{\alpha_3, \alpha_4, \alpha_5\}, \{\alpha_3\}\},$$

with set of highest roots

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, \quad \alpha_1, \quad \alpha_3 + \alpha_4 + \alpha_5, \quad \alpha_3\}.$$

This agrees with Table B.2 for D_n *with* n *odd.*

Now, consider the following examples of subsets of the simple root systems.

Example B.1.7 ($\Pi' := \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\} \subset \Pi_{A_5}$)**.** *Consider the proper subset of the simple roots for A_5 designated by Π'. There are two connected components of the Dynkin diagram generated by Π':*

$$\{\alpha_1, \alpha_2, \alpha_3\} \qquad \textit{and} \qquad \{\alpha_5\}.$$

The cascade for the second component is done. The first component is isomorphic to the Dynkin diagram of A_3, and we know the cascade on Π_{A_3} will yield $\{\{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_2\}\}$. The final set of highest roots is

$$\{\alpha_1 + \alpha_2 + \alpha_3, \ \ \alpha_2, \ \ \alpha_5\}.$$

Example B.1.8 ($\Pi' := \{\alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_7\} \subset \Pi_{D_7}$)**.** *By considering the Dynkin diagram for D_7, it is evident that there are four separate connected components generated by Π'. They are $\{\alpha_1, \alpha_2\}$, $\{\alpha_4\}$, $\{\alpha_6\}$, and $\{\alpha_7\}$. Hence, it suffices to determine the cascade for the first connected component. This portion of the Dynkin diagram is isomorphic to A_2, and the cascade on Π_{A_2} is $\{\{\alpha_1, \alpha_2\}\}$. Therefore, the cascade yields $\mathscr{K}(\Pi') = \{\{\alpha_1, \alpha_2\}, \{\alpha_4\}, \{\alpha_6\}, \{\alpha_7\}\}$, and the set of highest roots is*

$$\{\alpha_1 + \alpha_2, \ \ \alpha_4, \ \ \alpha_6, \ \ \alpha_7\}.$$

Example B.1.9 ($\Pi' := \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\} \subset \Pi_{E_7}$)**.** *Again, we first split Π' into a union of the connected components:*

$$\{\alpha_1\} \qquad \textit{and} \qquad \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}.$$

The cascade for the latter component is equal to (with appropriate relabeling of simple roots) the cascade on Π_{A_4}, so we compute $\mathscr{K}(\Pi') = \{\{\alpha_1\}, \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}, \{\alpha_4, \alpha_5\}\}$. The resulting set of highest roots is

$$\{\alpha_1, \ \ \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \ \ \alpha_4 + \alpha_5\}.$$

Example B.1.10 ($\Pi' := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9\} \subset \Pi_{D_9}$)**.** *The first connected component of the Dynkin diagram generated by Π' is isomorphic to A_4, and so the* *cascade on this set is* $\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_2, \alpha_3\}\}$*. The second connected component is isomorphic to* D_4*, so referencing the calculations in Example B.1.5, the final cascade calculation is*

$$\mathscr{K}(\Pi') = \{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_2, \alpha_3\}, \{\alpha_6, \alpha_7, \alpha_8, \alpha_9\}, \{\alpha_6\}, \{\alpha_8\}, \{\alpha_9\}\},$$

with set of highest roots

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \;\; \alpha_2 + \alpha_3, \;\; \alpha_6 + 2\alpha_7 + \alpha_8 + \alpha_9, \;\; \alpha_6, \;\; \alpha_8, \;\; \alpha_9\}.$$

B.2 Creating a Functional

Through the use of the *Killing form* (see Definition B.2.1), the dual of a seaweed $\mathfrak{g}$ is identified with $\mathfrak{g}^t$ (see Lemma B.2.2). This information is necessary to describe how a set of roots (such as the set generated by the cascade) points to a specific functional in $\mathfrak{g}^*$.

Definition B.2.1. *Let* $\mathfrak{g}$ *be a Lie algebra. For any* $X \in \mathfrak{g}$*, the* ***adjoint endormorphism*** $ad_X \in \mathfrak{g}^*$ *is defined by* $ad_X(Y) = [X, Y]$*. The symmetric, bilinear form* κ *defined by*

$$\kappa(X, Y) = trace(ad_X \circ ad_Y)$$

for all $X, Y \in \mathfrak{g}$ *is the* ***Killing form****.*

Lemma B.2.2. *If* $\mathfrak{g} \subseteq \mathfrak{gl}(n)$ *is a matrix Lie algebra, then* $\mathfrak{g}^*$ *can be identified with* $\mathfrak{g}^t$ *via the map* $f : \mathfrak{g}^t \to \mathfrak{g}^*$ *such that*

$$f(e_{j,i}) = \kappa(e_{j,i}, \cdot) = c_{i,j} e^*_{i,j},$$

for an appropriate nonzero constant $c_{i,j} \in \mathbb{C}$*.*

Proof. Through direct computation, we show $\kappa(e_{b,a}, \cdot) = c_{a,b} e^*_{a,b}$ for some nonzero constant $c_{a,b} \in \mathbb{C}$. Fix $X = [x_{i,j}] \in \mathfrak{g}$. To compute trace($ad_{e_{j,i}} \circ ad_X$), we identify the space of $n \times n$ matrices with $\mathbb{C}^{n^2}$ and add up the coefficients of the s^{th} basis element

e_s in the image of e_s under this map. In other words, we consider the coefficient of $e_{r,s}$ in $ad_{e_{j,i}} \circ ad_X(e_{r,s}) = [e_{j,i}, [X, e_{r,s}]]$. This gives

$$[X, e_{r,s}] = Xe_{r,s} - e_{r,s}X = \begin{pmatrix} & & x_{1,r} & \\ & & x_{2,r} & \\ & & \vdots & \\ -x_{s,1} & -x_{s,2} & \cdots & -x_{s,n} \\ & & x_{n,r} & \end{pmatrix},$$

where the entries $-x_{s,t}$ occur in row r and $x_{t,r}$ occur in column s. Let $A = [X, e_{r,s}]$. Then, in a similar fashion, we have

$$[e_{j,i}, A] = e_{j,i}A - Ae_{j,i} = \begin{pmatrix} & & -A_{1,j} & \\ & & -A_{2,j} & \\ & & \vdots & \\ A_{i,1} & A_{i,2} & \cdots & A_{i,n} \\ & & -A_{n,j} & \end{pmatrix},$$

where the entries $A_{i,t}$ occur in row j and $-A_{t,j}$ occur in column i. If $r \neq i$ and $s \neq j$, then

$$[e_{j,i}, [X, e_{r,s}]] = x_{i,r}e_{j,s} + x_{s,j}e_{r,i}.$$

If $r = i$ but $s \neq j$, then

$$[e_{j,i}, [X, e_{r,s}]] = \left(\sum_{t=1}^{n} -x_{s,t}e_{j,t} \right) + x_{r,r}e_{j,s} + x_{s,j}e_{r,i}.$$

If $r \neq i$ but $s = j$, then

$$[e_{j,i}, [X, e_{r,s}]] = x_{i,r}e_{j,s} + \left(\sum_{t=1}^{n} -x_{t,r}e_{t,i} \right) + x_{s,s}e_{r,i}.$$

Finally, if $(r, s) = (i, j)$, then

$$[e_{j,i}, [X, e_{r,s}]] = \left(\sum_{t=1}^{n} -x_{s,t}e_{j,t} \right) + x_{i,r}e_{j,s} + \left(\sum_{t=1}^{n} -x_{t,r}e_{t,i} \right) + x_{s,s}e_{r,i}.$$

Assume that $i \neq j$.

When $r \neq i$ and $s \neq j$, then $e_{r,s}$ shows up in $[e_{j,i}, [X, e_{r,s}]]$ if $j = r$ or $i = s$. When $r = j$ but $i \neq s$ (which occurs for $n-1$ indices (r,s)), the coefficient of $e_{r,s}$ is $x_{i,j}$. This contributes $(n-1)x_{i,j}$ to the trace calculation. Similarly, when $s = i$ but $r \neq j$ (which occurs for $n-1$ new indices (r,s)), the coefficient of $e_{r,s}$ is $x_{i,j}$. This contributes $(n-1)x_{i,j}$ to the calculation of the trace. Finally, if $(r,s) = (j,i)$, then the coefficient of $e_{r,s}$ is $2x_{i,j}$. The total trace calculation for all the indices of the appropriate form is $2nx_{i,j}$. When $r = i$ and $s \neq j$, the basis element $e_{r,s}$ appears in $[e_{j,i}, [X, e_{r,s}]]$ if $s = i$. The coefficient of $e_{r,s}$ will be $x_{i,j}$. Similarly, when $s = j$ and $r \neq i$, the basis element $e_{r,s}$ appears in $[e_{j,i}, [X, e_{r,s}]]$ if $r = j$. The coefficient of $e_{r,s}$ will be $x_{i,j}$. Finally, if $(r,s) = (i,j)$, then the coefficient of $e_{r,s}$ is zero. Putting this all together: if $i \neq j$, then

$$\kappa(e_{j,i}, X) = (2n+2)x_{i,j},$$

making $\kappa(e_{j,i}, \cdot) = (2n+2)e^*_{i,j}$.

Assume that $i = j$.

If $r \neq i$ and $s \neq i$, then $e_{r,s}$ appears in $[e_{i,i}, [X, e_{r,s}]]$ when $i = r$ or $i = s$. However, both of these scenarios contradict the standing assumption, so the coefficient of $e_{r,s}$ must always be zero in this situation. If $r = i$ but $s \neq i$, then $e_{r,s}$ has coefficient $-x_{s,s} + x_{r,r}$ in $[e_{i,i}, [X, e_{r,s}]]$ for all $(n-1)$ choices of s. If $r \neq i$ and $s = i$, then $e_{r,s}$ has coefficient $-x_{r,r} + x_{s,s}$ in $[e_{i,i}, [X, e_{r,s}]]$ for all $(n-1)$ choices of r. If $(r,s) = (i,i)$, then the basis element $e_{r,s}$ has coefficient $-6x_{i,i}$. Putting this all together: if $i = j$, then

$$\kappa(e_{i,i}, X) = -4x_{i,i},$$

making $\kappa(e_{i,i}, \cdot) = -4e^*_{i,i}$. □

Consider the functional φ_{ψ_1,ψ_2} defined as follows.

Definition B.2.3. *Let $\mathfrak{p}_m(\psi_1 \mid \psi_2)$ be a seaweed subalgebra of a Lie algebra $\mathfrak{g}$ with root system Φ, simple roots Π, and Chevalley basis $\{h_\alpha \mid \alpha \in \Pi\} \cup \{x_\alpha \mid \alpha \in \Phi\}$. Let K_1 and K_2 be the set of highest roots formed from running the cascade on $\Pi \backslash \psi_1$*

and $\Pi\backslash\psi_2$ respectively. Fix vectors $\overline{a} \in \mathbb{C}^{|K_1|}$ and $\overline{b} \in \mathbb{C}^{|K_2|}$ with with $a_i, b_j \neq 0$ for all i, j, and let

$$u^{\overline{a},\overline{b}}_{\psi_1,\psi_2} = \sum_{\alpha\in K_1} a_i x_\alpha + \sum_{\alpha\in K_2} b_j x_{-\alpha}.$$

Define a functional $\varphi_{u^{\overline{a},\overline{b}}_{\psi_1,\psi_2}} \in \mathfrak{p}_m(\psi_1 \mid \psi_2)^*$ by

$$\varphi_{u^{\overline{a},\overline{b}}_{\psi_1,\psi_2}}(x) = \kappa(u, x).$$

*The set functionals formed over all choices $\overline{a}$ and $\overline{b}$ are the **cascade functionals** on a given seaweed $\mathfrak{p}_m(\Psi_1 \mid \Psi_2)$.*

Remark B.2.4. *For the purposes of this chapter, we refer to **the** cascade functional, meaning the functional defined by the vectors $\overline{a}$ and $\overline{b}$ so that the image of $a_i e_{r,s}$ and $b_j e_{r,s}$ under the killing form is exactly $e^*_{s,r}$. We denote this functional by $\varphi_{\mathfrak{g}}$ on the given seaweed $\mathfrak{g}$.*

B.2.1 Examples

We introduce the following examples of cascade functionals on seaweed subalgebras of the classic Lie algebras, and we calculate explicitly the subalgebras $\ker(B_{\varphi_{\mathfrak{g}}})$ for each one. For calculations of the Chevalley basis, see Sections 4.2, 4.3, 4.4, and 4.4.2 of Chapter 4.

Example B.2.5. *Consider the seaweed $\mathfrak{g} = \mathfrak{p}^A_4(\{\alpha_4\} \mid \{\alpha_2, \alpha_3\})$ of Example 4.2.2. The meander associated with $\mathfrak{g}$, previously constructed in Example 2.1.4 and displayed in Figure 2.2, consists of exactly one path. Therefore, by Theorem 4.2.3 the seaweed $\mathfrak{g}$ has index zero, and is Frobenius.*

To construct $\varphi_{\mathfrak{g}}$, we must first run the cascade on $\{\alpha_1, \alpha_2, \alpha_3\}$, and $\{\alpha_1, \alpha_4\}$. The set of highest roots K_1 is $\{\alpha_1 + \alpha_2 + \alpha_3,\ \ \alpha_2\}$, and the set of highest roots K_2 is $\{\alpha_1,\ \ \alpha_4\}$. Therefore, we have

$$\begin{aligned} u &= \sum_{\alpha\in K_1} a_i x_\alpha + \sum_{\alpha\in K_2} b_i x_{-\alpha} \\ &= a_1 x_{\alpha_1+\alpha_2+\alpha_3} + a_2 x_{\alpha_2} + b_1 x_{-\alpha_1} + b_2 x_{-\alpha_4} \\ &= a_1 e_{1,4} + a_2 e_{2,3} + b_1 e_{2,1} + b_2 e_{5,4}, \end{aligned}$$

where the vectors $\overline{a}$ and $\overline{b}$ are chosen according to Remark B.2.4. We then have

$$\varphi_{\mathfrak{g}} = \kappa(u, \cdot) = e^*_{1,2} + e^*_{4,5} + e^*_{4,1} + e^*_{3,2}.$$

The indices of $\mathscr{I}_{\varphi_{\mathfrak{g}}}$ are displayed in Figure B.1.

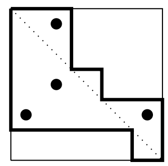


Figure B.1: Indices in $\mathscr{I}_{\varphi_{\mathfrak{g}}}$ for $\mathfrak{g} = \mathfrak{p}_4^A(\{\alpha_4\} \mid \{\alpha_2, \alpha_3\})$

We now compute $\dim\ker(B_{\varphi_{\mathfrak{g}}})$*. The image of $B_{\varphi_{\mathfrak{g}}}(B, b)$ for $B = [b_{i,j}]$ over the basis elements b of $\mathfrak{g}$ is calculated in Table B.3.*

Basis Element b	$B_{\varphi_{\mathfrak{g}}}(B, b)$	Basis Element b	$B_{\varphi_{\mathfrak{g}}}(B, b)$	Basis Element b	$B_{\varphi_{\mathfrak{g}}}(B, b)$
$e_{1,1} - e_{2,2}$	$b_{4,1} - b_{1,2} - b_{1,2} - b_{3,2}$	$e_{1,2}$	$b_{1,1} + b_{3,1} - b_{2,2}$	$e_{4,1}$	$b_{4,4} - b_{1,1} - b_{1,5}$
$e_{2,2} - e_{3,3}$	$b_{1,2} + b_{3,2} + b_{3,2}$	$e_{2,1}$	$b_{4,2}$	$e_{4,2}$	$b_{1,4} + b_{3,4} - b_{2,1} - b_{2,5}$
$e_{3,3} - e_{4,4}$	$b_{4,1} + b_{4,5} - b_{3,2}$	$e_{3,1}$	$b_{4,3} - b_{1,2}$	$e_{4,3}$	$-b_{3,1} - b_{3,5}$
$e_{4,4} - e_{5,5}$	$-b_{4,1} - b_{4,5} - b_{4,5}$	$e_{3,2}$	$b_{1,3} + b_{3,3} - b_{2,2}$	$e_{4,5}$	$b_{4,4} - b_{5,1} - b_{5,5}$

Table B.3: Image of $B_{\varphi_{\mathfrak{g}}}(B, b)$ over basis elements b of $\mathfrak{p}_4^A(\{\alpha_4\} \mid \{\alpha_2, \alpha_3\})$

If $B \in$ $\ker(B_{\varphi_{\mathfrak{g}}})$, we require that $B_{\varphi_{\mathfrak{g}}}(B, b) = 0$ for all basis elements b of $\mathfrak{g}$. Setting the expressions in columns two, four, and six of Table B.3 equal to zero yields twelve equations which, together with the assumption that $B \in \mathfrak{g}$, requires that B can only be the zero matrix. Hence, $\dim\ker(B_{\varphi_{\mathfrak{g}}}) = 0$*, and $\varphi_{\mathfrak{g}}$ is regular.*

In Example B.2.5, $\varphi_{\mathfrak{g}}$ is regular on $\mathfrak{g}$. We now include an example in Type-A where $\varphi_{\mathfrak{g}}$ is not regular (see Example B.2.6).

Example B.2.6. *Consider the seaweed* $\mathfrak{p}_3^A(\emptyset \mid \emptyset) = A_3$*. The cascade on* Π_{A_3} *produces a set of two roots:* $\{\alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_2\}$*, and the cascade functional for* A_3 *is*

$$\varphi_{A_3} = e^*_{1,4} + e^*_{2,3} + e^*_{3,2} + e^*_{4,1}.$$

The indices of $\mathscr{I}_{\varphi_{A_3}}$ *are illustrated in Figure B.2.*

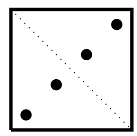


Figure B.2: Indices in $\mathscr{I}_{\varphi_{A_3}}$

The image of $B_{\varphi_{A_3}}(B, b)$ *for* $B = [b_{i,j}]$ *over all basis elements* b *of* A_3 *is calculated in Table B.4.*

Basis Element b	$B_{\varphi_{A_3}}(B, b)$	Basis Element b	$B_{\varphi_{A_3}}(B, b)$	Basis Element b	$B_{\varphi_{A_3}}(B, b)$
$e_{1,1} - e_{2,2}$	$b_{4,1} + b_{2,3} - b_{3,2} - b_{1,4}$	$e_{1,4}$	$b_{1,1} - b_{4,4}$	$e_{3,2}$	$b_{3,3} - b_{2,2}$
$e_{2,2} - e_{3,3}$	$b_{3,2} + b_{3,2} - b_{2,3} - b_{2,3}$	$e_{2,1}$	$b_{4,2} - b_{1,3}$	$e_{3,4}$	$b_{1,3} - b_{4,2}$
$e_{3,3} - e_{4,4}$	$b_{2,3} + b_{4,1} - b_{1,4} - b_{3,2}$	$e_{2,3}$	$b_{2,2} - b_{3,3}$	$e_{4,1}$	$b_{4,4} - b_{1,1}$
$e_{1,2}$	$b_{3,1} - b_{2,4}$	$e_{2,4}$	$b_{1,2} - b_{4,3}$	$e_{4,2}$	$b_{3,4} - b_{2,1}$
$e_{1,3}$	$b_{2,1} - b_{3,4}$	$e_{3,1}$	$b_{4,3} - b_{1,2}$	$e_{4,3}$	$b_{2,4} - b_{3,1}$

Table B.4: Image of $B_{\varphi_{\mathfrak{g}}}(B, b)$ over basis elements b of A_3

If $B \in \ker(B_{\varphi_{A_3}})$*, then* $B_{\varphi_{A_3}}(B, b) = 0$ *for all basis elements* b *of* A_3*. Setting the expressions in columns two, four, and six of Table B.4 equal to zero yields fifteen equations which, together with the assumption that* $B \in \mathfrak{sl}(4)$*, requires that* B *must* *be of the following form:*

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & -b_1 & b_6 & b_7 \\ b_7 & b_6 & -b_1 & b_5 \\ b_4 & b_3 & b_2 & b_1 \end{pmatrix}.$$

Therefore, $\dim \ker(B_{\varphi_{A_3}}) = 7$*, but we know* $\operatorname{ind} A_3 = 3$*. Hence,* φ_{A_3} *is **not** regular on* A_3*.*

Example B.2.7. *Consider the seaweed* $\mathfrak{g} = \mathfrak{p}_7^A(\{\alpha_4\} \mid \emptyset)$*. The cascade on* $\Pi_{A_7} \backslash \{\alpha_4\}$ *yields the set of roots*

$$\{\alpha_1 + \alpha_2 + \alpha_3, \ \ \alpha_2, \ \ \alpha_5 + \alpha_6 + \alpha_7, \ \ \alpha_6\}.$$

The cascade on Π_{A_7} *yields the set of roots*

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \ \ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \ \ \alpha_3 + \alpha_4 + \alpha_5, \ \ \alpha_4\}.$$

Hence,

$$\varphi_{\mathfrak{g}} = e^*_{4,1} + e^*_{3,2} + e^*_{8,5} + e^*_{7,6} + e^*_{1,8} + e^*_{2,7} + e^*_{3,6} + e^*_{4,5}.$$

The indices in $\mathscr{I}_{\varphi_{\mathfrak{g}}}$ *are illustrated in Figure B.3.*

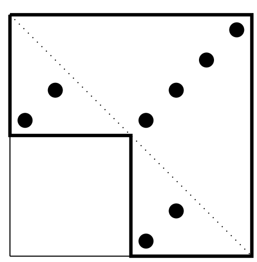


Figure B.3: Indices in $\mathscr{I}_{\varphi_{\mathfrak{g}}}$ for $\mathfrak{g} = \mathfrak{p}_7^A(\{\alpha_4\} \mid \emptyset)$

Through direct computation, we get that a relations matrix B of $\ker(B_{\varphi_{\mathfrak{g}}})$ *is*

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & 0 & 0 & b_3 & b_4 \\ b_5 & -b_1 & b_6 & b_7 & 0 & 0 & b_6 & b_7 \\ b_7 & b_6 & -b_1 & b_5 & -b_7 & -b_6 & 0 & 0 \\ b_4 & b_3 & b_2 & b_1 & -b_4 & -b_3 & 0 & 0 \\ & & & & b_1 & b_2 & b_3 & b_4 \\ & & & & b_5 & -b_1 & b_6 & b_7 \\ & & & & b_7 & b_6 & -b_1 & b_5 \\ & & & & b_4 & b_3 & b_2 & b_1 \end{pmatrix}.$$

Blank spaces are inadmissible locations of $\mathfrak{g}$ *and are, per force, filled with zeroes. Hence,* $\dim\ker(B_{\varphi_{\mathfrak{g}}}) = 7$*, but* $\operatorname{ind}\mathfrak{g} = 3$ *by Theorem 4.2.3. Therefore,* $\varphi_{\mathfrak{g}}$ *is* ***not*** *regular.*

The conjecture this example seems to point to is that if there is a component c_i in the homotopy type $H(c_1, \cdots, c_h)$ of a Type-A seaweed $\mathfrak{g}$ such that $\varphi_{A_{c_i-1}}$ is not regular on $\mathfrak{sl}(c_i)$, then $\varphi_{\mathfrak{g}}$ is not regular on $\mathfrak{g}$. This is discussed in greater detail in Chapter 5.

Example B.2.8. *Consider the seaweed* $\mathfrak{g} = \mathfrak{p}_7^C(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$ *of Example 4.3.4. The meander associated with* $\mathfrak{g}$ *consists of a path rooted in the tail, a vertex in the aftertail, a path disjoint from the tail, and an isolated point disjoint from both the tail and aftertail. Hence,* $\operatorname{ind}\mathfrak{g} = 3$*. To construct* $\varphi_{\mathfrak{g}}$*, we must first run the cascade on* $\{\alpha_3, \alpha_4, \alpha_6, \alpha_7\}$ *and* $\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_7\}$*. The set of highest roots* K_1 *is* $\{\alpha_3 + \alpha_4,\ 2\alpha_6 + \alpha_7,\ \alpha_7\}$*, and the set of highest roots* K_2 *is* $\{\alpha_1 + \alpha_2,\ \alpha_4 + \alpha_5,\ \alpha_7\}$*. Therefore, we have*

$$\begin{aligned} u &= \sum_{\alpha \in K_1} a_i x_\alpha + \sum_{\alpha \in K_2} b_i x_{-\alpha} \\ &= a_1 x_{\alpha_3+\alpha_4} + a_2 x_{2\alpha_6+\alpha_7} + a_3 x_{\alpha_7} + b_1 x_{-\alpha_1-\alpha_2} + b_2 x_{-\alpha_4-\alpha_5} + b_3 x_{-\alpha_7} \\ &= a_1(e_{3,5} - e_{10,12}) + a_2 e_{6,9} + a_3 e_{7,8} + b_1(e_{3,1} - e_{14,12}) + b_2(e_{6,4} - e_{11,9}) + b_3 e_{8,7}, \end{aligned}$$

where the vectors $\overline{a}$ *and* $\overline{b}$ *are chosen according to Remark B.2.4. We then have*

$$\varphi_{\mathfrak{g}} = \kappa(u, \cdot) = e^*_{5,3} - e^*_{12,10} + e^*_{6,9} + e^*_{8,7} + e^*_{1,3} - e^*_{12,14} + e^*_{4,6} + e^*_{7,8}.$$

Due to the symmetry in C_7, it suffices to use

$$\varphi'_{\mathfrak{g}} = e^*_{5,3} + e^*_{6,9} + e^*_{8,7} + e^*_{1,3} + e^*_{4,6} - e^*_{9,11} + e^*_{7,8}.$$

The indices in $\mathscr{I}_{\varphi'_{\mathfrak{g}}}$ are illustrated in Figure B.4.

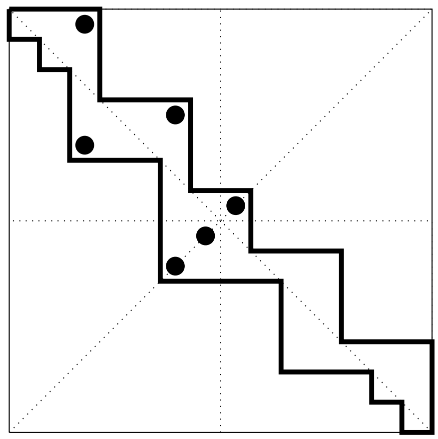


Figure B.4: Indices in $\mathscr{I}_{\varphi'_{\mathfrak{g}}}$ on $\mathfrak{g} = \mathfrak{p}_7^C(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$

Let $B = [b_{i,j}] \in \ker(B_{\varphi'_{\mathfrak{g}}})$. Consider how the Kirillov form $B_{\varphi'_{\mathfrak{g}}}(B, \cdot)$ acts on the basis of $\mathfrak{g}$. These calculations are shown in Table B.5.

Basis Element b	$B_{\varphi'_{\mathfrak{g}}}(B,b)$	Basis Element b	$B_{\varphi'_{\mathfrak{g}}}(B,b)$	Basis Element b	$B_{\varphi'_{\mathfrak{g}}}(B,b)$
$e_{1,1} - e_{14,14}$	$b_{1,3}$	$e_{4,4} - e_{11,11}$	$-b_{4,6}$	$e_{6,6} - e_{9,9}$	$b_{4,6} + b_{9,6}$
$e_{1,2} - e_{13,14}$	$-b_{2,1} - b_{2,3}$	$e_{4,5} - e_{10,11}$	$-b_{5,6}$	$e_{7,6} - e_{9,8}$	$b_{4,7} + b_{9,7} + b_{8,6} - b_{7,9} - b_{6,8}$
$e_{1,3} - e_{12,14}$	$b_{1,1} + b_{5,1} - b_{3,1} - b_{3,3}$	$e_{4,6} - e_{9,11}$	$b_{4,4} + b_{9,4} - b_{6,6}$	$e_{7,7} - e_{8,8}$	$2b_{8,7} - 2b_{7,8}$
$e_{2,2} - e_{13,13}$	0	$e_{5,3} - e_{12,10}$	$b_{1,5} + b_{5,5} - b_{3,3}$	$e_{7,8}$	$b_{7,7} - b_{8,8}$
$e_{2,3} - e_{12,13}$	$b_{1,2} + b_{5,2}$	$e_{5,4} - e_{11,10}$	$-b_{4,3}$	$e_{8,6} + e_{9,7}$	$b_{4,8} + b_{9,8} + b_{8,9} - b_{6,7} - b_{7,6}$
$e_{3,3} - e_{12,12}$	$b_{1,3} - b_{5,3}$	$e_{5,5} - e_{10,10}$	$-b_{5,3}$	$e_{8,7}$	$b_{8,8} - b_{7,7}$
$e_{4,3} - e_{12,11}$	$b_{1,4} + b_{5,4} - b_{3,6}$	$e_{5,6} - e_{9,10}$	$b_{4,5} + b_{9,5} - b_{6,3}$	$e_{9,6}$	$b_{4,9} + b_{9,9} - b_{6,6}$

Table B.5: Image of basis for $\mathfrak{p}_7^C(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6\})$ under $B_{\varphi'_{\mathfrak{g}}}(B, \cdot)$

As $B \in \ker(B_{\varphi'_{\mathfrak{g}}})$, we require that $B_{\varphi'_{\mathfrak{g}}}(B, b) = 0$ for all basis elements b. Setting the expressions in columns two, four, and six equal to zero yields twenty-one equations which, coupled with teh requirement that $B \in \mathfrak{g}$, produces the following relations matrix of $\ker(B_{\varphi'_{\mathfrak{g}}})$:

$$B = \left(\begin{array}{cccccccccccccc}
b_1 & 0 & 0 & & & & & & & & & & & \\
 & b_2 & 0 & & & & & & & & & & & \\
 & & b_1 & & & & & & & & & & & \\
 & & 0 & 0 & 0 & 0 & & & & & & & & \\
 & & 0 & 0 & b_1 & 0 & & & & & & & & \\
 & & & & & 0 & & & & & & & & \\
 & & & & & 0 & 0 & b_3 & & & & & & \\
 & & & & & 0 & b_3 & 0 & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & & & -b_1 & 0 & & & \\
 & & & & & & & & & 0 & 0 & & & \\
 & & & & & & & & & 0 & 0 & -b_1 & 0 & 0 \\
 & & & & & & & & & & & & -b_2 & 0 \\
 & & & & & & & & & & & & & -b_1
\end{array}\right).$$

Consistent with our previous examples, the blank spaces are inadmissible locations of $\mathfrak{g}$ and are, per force, filled with zeroes. We conclude that $\varphi'_{\mathfrak{g}}$ is regular on $\mathfrak{g}$.

In Example B.2.8, $\varphi'_{\mathfrak{g}}$ is regular on $\mathfrak{g}$. We now present an example in Type-C where $\varphi_{\mathfrak{g}}$ is not regular (see Example B.2.9).

Example B.2.9. *Consider the simple Lie algebra C_3. The functional φ_{C_3} described by the cascade is*

$$\varphi_{C_3} = e^*_{1,6} + e^*_{2,5} + e^*_{3,4} + e^*_{4,3} + e^*_{5,2} + e^*_{6,1}.$$

The indices in $\mathscr{I}_{\varphi_{C_3}}$ are illustrated in Figure B.5.

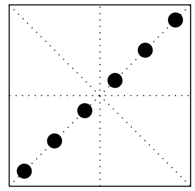


Figure B.5: Indices in $\mathscr{I}_{\varphi_{C_3}}$ on C_3

Let $B = [b_{i,j}] \in \ker(B_{\varphi_{C_3}})$*. Consider how the Kirillov form* $B_{\varphi_{C_3}}(B, \cdot)$ *acts on the basis of* C_3*. These calculations are shown in Table B.6.*

Basis Element b	$B_{\varphi_{C_3}}(B,b)$	Basis Element b	$B_{\varphi_{C_3}}(B,b)$	Basis Element b	$B_{\varphi_{C_3}}(B,b)$
$e_{1,1}-e_{6,6}$	$2b_{6,1}-2b_{1,6}$	$e_{3,2}-e_{5,4}$	$b_{5,3}-b_{3,5}-b_{2,4}+b_{4,2}$	$e_{5,1}+e_{6,2}$	$b_{6,5}+b_{5,6}-b_{1,2}-b_{2,1}$
$e_{1,2}-e_{5,6}$	$b_{5,1}-b_{1,5}-b_{2,6}+b_{6,2}$	$e_{3,3}-e_{4,4}$	$2b_{4,3}-2b_{3,4}$	$e_{1,6}$	$b_{1,1}-b_{6,6}$
$e_{1,3}-e_{4,6}$	$b_{4,1}-b_{1,4}-b_{3,6}+b_{6,3}$	$e_{1,4}+e_{3,6}$	$b_{3,1}+b_{1,3}-b_{4,6}-b_{6,4}$	$e_{2,5}$	$b_{2,2}-b_{5,5}$
$e_{2,1}-e_{6,5}$	$b_{6,2}-b_{2,6}-b_{1,5}+b_{5,1}$	$e_{1,5}+e_{2,6}$	$b_{2,1}+b_{1,2}-b_{5,6}-b_{6,5}$	$e_{3,4}$	$b_{3,3}-b_{4,4}$
$e_{2,2}-e_{5,5}$	$2b_{5,2}-2b_{2,5}$	$e_{2,4}+e_{3,5}$	$b_{3,2}+b_{2,3}-b_{4,5}-b_{5,4}$	$e_{4,3}$	$b_{4,4}-b_{3,3}$
$e_{2,3}-e_{4,5}$	$b_{4,2}-b_{2,4}-b_{3,5}+b_{5,3}$	$e_{4,1}+e_{6,3}$	$b_{6,4}+b_{4,6}-b_{1,3}-b_{3,1}$	$e_{5,2}$	$b_{5,5}-b_{2,2}$
$e_{3,1}-e_{6,4}$	$b_{6,3}-b_{3,6}-b_{1,4}+b_{4,1}$	$e_{4,2}+e_{5,3}$	$b_{5,4}+b_{4,5}-b_{2,3}-b_{3,2}$	$e_{6,1}$	$b_{6,6}-b_{1,1}$

Table B.6: Image of basis for C_3 under $B_{\varphi_{C_3}}(B, \cdot)$

Since $B \in \ker(B_{\varphi_{C_3}})$*, we require that* $B_{\varphi_{C_3}}(B, b) = 0$ *for all basis elements* b*. Setting the expressions in columns two, four, and six equal to zero yields twenty-one equations which, together with the requirement that* $B \in \mathfrak{sp}(6)$*, produces a relations matrix* B *of* $\ker(B_{\varphi_{C_3}})$ *of following form:*

$$B = \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ -b_1 & 0 & b_6 & b_7 & b_8 & b_4 \\ -b_2 & -b_6 & 0 & b_9 & b_7 & b_3 \\ b_3 & b_7 & b_9 & 0 & -b_6 & -b_2 \\ b_4 & b_8 & b_7 & b_6 & 0 & -b_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 \end{pmatrix}.$$

It follows that $\dim\ker(B_{\varphi_{C_3}}) = 9$. *However,* $\operatorname{ind} C_3 = 3$ *(see Theorem 4.3.6), and so the functional* φ_{C_3} *is* ***not*** *regular.*

It is also true that $\varphi_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{p}_4^C(\{\alpha_4\} \mid \emptyset)$ is not regular (we omit showing this explicitly, the calculation is similar to the one done in Example B.2.7). Therefore, as in Type-A, we conjecture that the cascade functional $\varphi_{\mathfrak{g}}$ fails to be regular in direct correspondence with some aspect of the homotopy type associated with $\mathfrak{g}$. This is discussed in further detail in Chapter 5.

Example B.2.10. *Consider the seaweed* $\mathfrak{g} = \mathfrak{p}_9^B(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6, \alpha_7\})$ *of Example 4.4.4. The meander associated with* $\mathfrak{g}$ *consists of a path and an isolated point (a degenerate path) rooted in the tail, a path and an isolated point disjoint from the tail and the aftertail, along with two points in the aftertail. It follows from Thoerem 4.4.5 that* $\operatorname{ind}\mathfrak{g} = 4$. *To construct* $\varphi_{\mathfrak{g}}$*, we must first run the cascade on* $\{\alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9\}$ *and* $\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_8, \alpha_9\}$. *The set of highest roots* K_1 *is* $\{\alpha_3 + \alpha_4,\ \ \alpha_6 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9,\ \ \alpha_8 + 2\alpha_9,\ \ \alpha_6,\ \ \alpha_8\}$ *and the set of highest roots* K_2 *is* $\{\alpha_1 + \alpha_2,\ \ \alpha_4 + \alpha_5,\ \ \alpha_8 + 2\alpha_9,\ \ \alpha_9\}$. *Therefore, we have*

$$
\begin{aligned}
u = &\sum_{\alpha \in K_1} a_i x_\alpha + \sum_{\alpha \in K_2} b_i x_{-\alpha} \\
=&a_1 x_{\alpha_3+\alpha_4} + a_2 x_{\alpha_6+2\alpha_7+2\alpha_8+2\alpha_9} + a_3 x_{\alpha_8+2\alpha_9} + a_4 x_{\alpha_6} + a_5 x_{\alpha_8} \\
&+ b_1 x_{-\alpha_1-\alpha_2} + b_2 x_{-\alpha_4-\alpha_5} + b_3 x_{-\alpha_8-2\alpha_9} + b_4 x_{-\alpha_9} \\
=&a_1(e_{3,5} - e_{15,17}) + a_2(e_{6,13} - e_{7,14}) + a_3(e_{8,11} - e_{9,12}) \\
&+ a_4(e_{6,7} - e_{13,14}) + a_5(e_{8,9} - e_{11,12}) \\
&+ b_1(e_{3,1} - e_{19,17}) + b_2(e_{6,4} - e_{16,14}) + b_3(e_{11,8} - e_{12,9}) + b_4(e_{9,8} - e_{12,11}),
\end{aligned}
$$

where the vectors $\overline{a}$ *and* $\overline{b}$ *are chosen according to Remark B.2.4. We then have*

$$
\begin{aligned}
\varphi_{\mathfrak{g}} = \kappa(j, \cdot) =&e^*_{5,3} - e^*_{17,15} + e^*_{13,6} - e^*_{14,7} + e^*_{11,8} - e^*_{12,9} + e^*_{7,6} - e^*_{14,13} + e^*_{9,8} - e^*_{12,11} \\
&+ e^*_{1,3} - e^*_{17,19} + e^*_{4,6} - e^*_{14,16} + e^*_{8,11} - e^*_{9,12} + e^*_{8,9} - e^*_{11,12}.
\end{aligned}
$$

As in Type-C, it suffices to use

$$
\varphi'_{\mathfrak{g}} = e^*_{5,3} + e^*_{13,6} + e^*_{11,8} + e^*_{7,6} + e^*_{9,8} + e^*_{1,3} + e^*_{4,6} + e^*_{8,11} + e^*_{8,9}.
$$

The indices in $\mathscr{I}_{\varphi'_{\mathfrak{g}}}$ are displayed in Figure B.6.

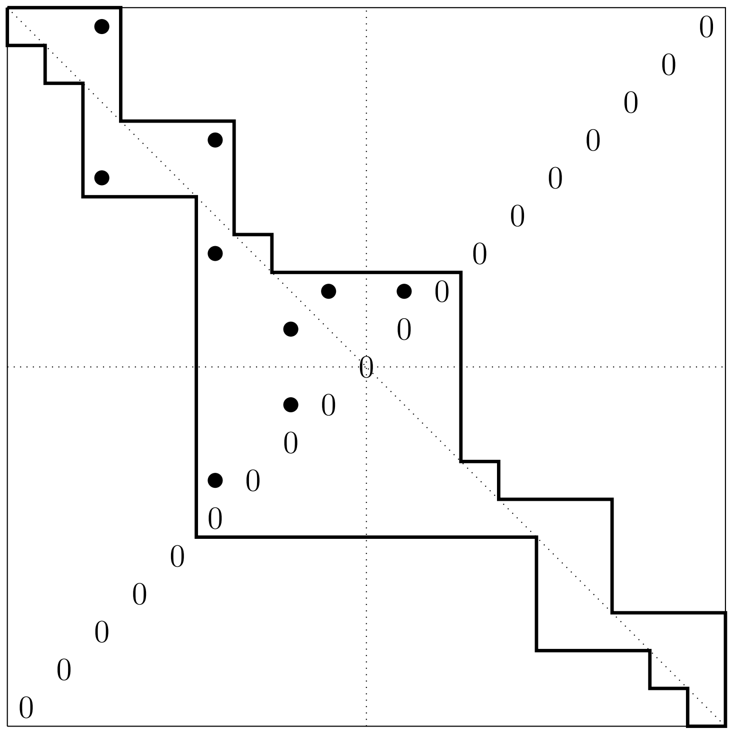


Figure B.6: Indices in $\mathscr{I}_{\varphi'_{\mathfrak{g}}}$ on $\mathfrak{g} = \mathfrak{p}_9^B(\{\alpha_1, \alpha_2, \alpha_5\} \mid \{\alpha_3, \alpha_6, \alpha_7\})$

We omit the table of images under $B_{\varphi'_{\mathfrak{g}}}(B, \cdot)$ of the basis elements for $\mathfrak{g}$, but by a similar computation to those carried out previously in this section we find an explicit relations matrix for $\ker(B_{\varphi'_{\mathfrak{g}}})$ to be

$$
B = \left(\begin{array}{ccccccccccccccccccc}
b_1 & 0 & 0 & & & & & & & & & & & & & & & & \\
 & b_2 & 0 & & & & & & & & & & & & & & & & \\
 & & b_1 & & & & & & & & & & & & & & & & \\
 & & 0 & 0 & 0 & 0 & & & & & & & & & & & & & \\
 & & 0 & 0 & b_1 & 0 & & & & & & & & & & & & & \\
 & & & & & 0 & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & & & & & & & & & & & & \\
 & & & & & 0 & 0 & 0 & b_3 & b_4 & b_5 & 0 & & & & & & & \\
 & & & & & 0 & 0 & b_3 & 0 & b_6 & 0 & -b_5 & & & & & & & \\
 & & & & & 0 & 0 & -b_4 & b_6 & 0 & -b_6 & -b_4 & & & & & & & \\
 & & & & & 0 & 0 & b_5 & 0 & -b_6 & 0 & -b_3 & & & & & & & \\
 & & & & & 0 & 0 & 0 & -b_5 & b_4 & -b_3 & 0 & & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & & & & & & & & -b_1 & 0 & & & \\
 & & & & & & & & & & & & & & 0 & 0 & & & \\
 & & & & & & & & & & & & & & 0 & 0 & -b_1 & 0 & 0 \\
 & & & & & & & & & & & & & & & & & -b_2 & 0 \\
 & & & & & & & & & & & & & & & & & & -b_1
\end{array}\right).
$$

Consistent with our previous examples, the blank spaces are inadmissible locations of $\mathfrak{g}$ and are, per force, filled with zeroes. Therefore, $\operatorname{ind}\ker(B_{\varphi'_{\mathfrak{g}}}) = 6$*, and* $\varphi'_{\mathfrak{g}}$ *is **not** regular.*

Example B.2.11. *Consider the simple Lie algebra B_2. The functional φ'_{B_2} described by the cascade and reduced by the symmetry of the algebra is*

$$\varphi'_{B_2} = e^*_{1,2} + e^*_{1,4} + e^*_{2,1} + e^*_{4,1}.$$

The indices in $\mathscr{I}_{\varphi'_{B_2}}$ are illustrated in Figure B.7.

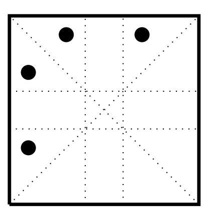


Figure B.7: Indices in $\mathscr{I}_{\varphi'_{B_2}}$

Let $B = [b_{i,j}] \in \ker(B_{\varphi'_{B_2}})$. Consider how the Kirillov form $B_{\varphi'_{B_2}}(B, \cdot)$ acts on the basis of B_2. These calculations are shown in Table B.7.

Basis Element b	$B_{\varphi'_{B_2}}(B,b)$	Basis Element b	$B_{\varphi'_{B_2}}(B,b)$	Basis Element b	$B_{\varphi'_{B_2}}(B,b)$
$e_{1,1}-e_{5,5}$	$b_{2,1}+b_{4,1}-b_{1,2}-b_{1,4}$	$e_{2,1}-e_{5,4}$	$b_{2,2}+b_{4,2}-b_{1,5}-b_{1,1}$	$e_{3,1}-e_{5,3}$	$b_{2,3}+b_{4,3}$
$e_{1,2}-e_{4,5}$	$b_{1,1}+b_{5,1}-b_{2,2}-b_{2,4}$	$e_{2,2}-e_{4,4}$	$b_{1,2}+b_{4,1}-b_{1,4}-b_{2,1}$	$e_{3,2}-e_{4,3}$	$b_{1,3}+b_{3,1}$
$e_{1,3}-e_{3,5}$	$-b_{3,2}-b_{3,4}$	$e_{2,3}-e_{3,4}$	$-b_{1,3}-b_{3,1}$	$e_{4,1}-e_{5,2}$	$b_{2,4}+b_{4,4}-b_{1,5}-b_{1,1}$
$e_{1,4}-e_{2,5}$	$b_{1,1}+b_{5,1}-b_{4,2}-b_{4,4}$				

Table B.7: Image of basis for B_2 under $B_{\varphi'_{B_2}}(B, \cdot)$

Since $B \in \ker(B_{\varphi'_{B_2}})$, we require that $B_{\varphi'_{B_2}}(B,b) = 0$ for all basis elements b. Evaluating the expressions in columns two, four, and six at zero yields ten equations which, coupled with the requirement that $B \in \mathfrak{so}(5)$, produces a relations matrix B of $\ker(B_{\varphi'_{B_2}})$ of the following form:

$$B = \begin{pmatrix} 0 & b_1 & b_2 & b_3 & 0 \\ b_1 & 0 & b_4 & 0 & -b_3 \\ -b_2 & b_4 & 0 & -b_4 & -b_2 \\ b_3 & 0 & -b_4 & 0 & -b_1 \\ 0 & -b_3 & b_2 & -b_1 & 0 \end{pmatrix}.$$

*It follows that $\dim\ker(B_{\varphi'_{B_2}}) = 4$. However, $\operatorname{ind} B_2 = 2$ (see Theorem 4.4.5), and so the functional φ'_{B_2} is **not** regular on B_2.*

Example B.2.12. *Consider the simple Lie algebra D_4. The functional φ'_{D_4} described by the cascade and reduced by the symmetry of the algebra is*

$$\varphi'_{D_4} = e^*_{1,2} + e^*_{,7} + e^*_{2,1} + e^*_{3,4} + e^*_{3,5} + e^*_{4,3} + e^*_{5,3} + e^*_{7,1}.$$

The indices in $\mathscr{I}_{\varphi'_{D_4}}$ are illustrated in Figure B.8.

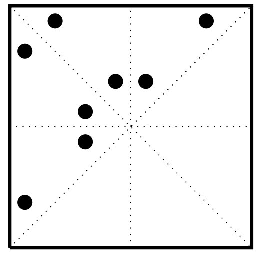


Figure B.8: Indices in $\mathscr{I}_{\varphi'_{D_4}}$

We omit the table of images under $B_{\varphi'_{D_4}}(B, \cdot)$ *of the basis elements for* $\mathfrak{g}$*, but by a similar computation to those carried out previously in this section we find an explicit relations matrix for* $\ker(B_{\varphi'_{D_4}})$ *to be*

$$B = \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_1 & 0 & b_7 & b_8 & b_9 & b_{10} & 0 & -b_6 \\ b_2 & b_7 & 0 & b_{11} & b_{12} & 0 & -b_{10} & -b_5 \\ b_3 & b_8 & b_{11} & 0 & 0 & -b_{12} & -b_9 & -b_4 \\ b_4 & b_9 & b_{12} & 0 & 0 & -b_{11} & -b_8 & -b_3 \\ b_5 & b_{10} & 0 & -b_{12} & -b_{11} & 0 & -b_7 & -b_2 \\ b_6 & 0 & -b_{10} & -b_9 & -b_8 & -b_7 & 0 & -b_1 \\ 0 & -b_6 & -b_5 & -b_4 & -b_3 & -b_2 & -b_1 & 0 \end{pmatrix}.$$

It follows that $\dim\ker(B_{\varphi'_{D_4}}) = 12$*. However,* $\operatorname{ind} D_4 = 4$ *(see 4.4.16). Therefore,* φ'_{D_4} *is* ***not*** *regular on* D_4*.*

Vita

My name is Aria Lynn Dougherty, I am a driven individual with a wide range of interests and hobbies. My natural academic inclination has always been towards mathematics, it's a GOD given gift that has been with me as long as I can remember, very much like a second language I have always understood. I participated in regional and state level competitions in middle school, I became a member of MENSA in high school, and I received early admittance to college at the age of fifteen. My career goals in mathematics are to continue doing research.

Away from the realms of academia, I am an active member of my church. I help organize youth events, participate in musical features, and sing in the choir every week. Music has been a hobby of mine since I was eight when I took up singing and playing the trumpet. While completing my undergraduate degree, I studied music with an emphasis in vocal performance during my spare time. Currently, I am learning to play the piano with the hopes of playing the organ for my church one day. In my spare time outside of the church, I am invested in many of the other arts as well. Primarily, these include drawing/graphic design, crocheting, painting, baking, and jewelry making. I aspire to open my own business one day.