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Structural Considerations for Interval Orders with Length Constraints

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But all things must be done properly and in an orderly manner.
1 Corinthians 14:40

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Abstract

A partially ordered set (poset), $P=(X,\prec)$, is a set X together with a relation, $\prec$, that is irreflexive and transitive. An interval order is a poset which has an interval representation: an assignment of a closed interval, I_x, in the real number line to each $x \in X$ so that $x \prec y$ if and only if I_x is completely to the left of I_y. Wiener and Fishburn characterized interval orders as posets which do not contain a $\mathbf{2+2}$ as an induced suborder [5, 22]. Define $\mathcal{P}[p,q]$ to be posets for which there exists an interval representation with interval lengths in $[p,q]$. We will consider p and q to be positive integers. Scott and Suppes characterize $\mathcal{P}[1,1]$ as posets which do not contain a $\mathbf{2+2}$ or a $\mathbf{3+1}$ as induced suborders, and Fishburn generalizes this result to characterize $\mathcal{P}[1,q]$ as posets which do not contain a $\mathbf{2+2}$ or a $(\mathbf{q+2})+\mathbf{1}$ as induced suborders [20, 8]. We use the weighted digraph techniques of [2] to develop complete lists of minimal forbidden substructures for $\mathcal{P}[2,q]$ and $\mathcal{P}[3,q]$ and partial lists for $\mathcal{P}[p,q]$. We also relate $\mathcal{P}[p,kp+1]$ and $\mathcal{P}[p+1,k(p+1)+1]$ and give a list of relationships for structures in $\mathcal{P}[p,q]$.

Chapter 1

Introduction

1.1 Problem Description

Suppose a hospital needs to schedule a set of surgeries on a given day. Some surgeries must occur before others: surgeries on the same patient or surgeries performed by the same surgical team. There could also be surgeries that must overlap: a patient that requires multiple surgeries but cannot be sedated more than once. The hospital administrators must decide which surgeries to schedule on which days to maximize healthy outcomes and minimize cost. How can a hospital know which sets of surgeries can be scheduled in the desired way? Mathematics and computer science of course! This is an applied example of what mathematicians call the Interval Order Problem. We will focus on a variation of the Interval Order Problem, but before we discuss the variation, we must first understand its origins.

Wiener was first interested in interval orders in 1914 [22]. However, the term interval order was introduced by Fishburn in 1970 [5]. Our work in the following chapters was inspired by Fishburn's book *Interval Orders and Interval Graphs*, and so we will often use his definitions. With the goal of defining an interval order, we first define a partially ordered set (poset).

Definition 1.1.1. A **partially ordered set** (poset), $P=(X,R)$, is a set X together with a relation, R, that is irreflexive (not(xRx)) and transitive (if xRy and yRz then

xRz).

Many authors define the poset relation to be reflexive (xRx), antisymmetric (if xRy, then not(yRx)), and transitive. Note that the relation in Definition 1.1.1 is antisymmetric since if xRy and yRx, transitivity says xRx, which breaks irreflexivity. Thus, the only difference is whether or not xRx.

We will denote our relation, $\prec$, and we read $x \prec y$ as x precedes y or y succeeds x. We write $x \cap y$ to indicate that $x \not\prec y$ and $y \not\prec x$. If $x \cap y$, we will say that x and y are incomparable. We note that $\cap$ is not the standard notation. Often $\|$ or $\sim$ is used in place of $\cap$. However, since we are working with intervals, using set intersection is natural when discussing elements whose intervals overlap. Our definition of interval order will be based on the following definition of interval representation.

Definition 1.1.2. An **interval representation** of a poset, $P = (X, \prec)$, is an assignment of a closed interval, I_x, in the real number line, $\mathbb{R}$, to each $x \in X$ so that $x \prec y$ if and only if I_x is completely to the left of I_y.

We can now present our definition on interval order.

Definition 1.1.3. An **interval order**, $P = (X, R)$, is a poset which has an interval representation.

If we can decide if a poset is an interval order, then we could decide if a hospital can schedule a given set of surgeries since surgery time slots are just intervals in time. The following characterization (credited to Fishburn, but inspired by Wiener [10]) for interval orders provides a way to decide if a poset has an interval representation or not.

Theorem 1.1.4 ([5, 22])**.** A poset, P, has an interval representation if and only if it does not contain a **2+2** (see Figure 1.1) as an induced sub-poset.

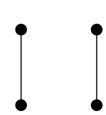


Figure 1.1: Structure that cannot appear in an interval order

Figure 1.3 shows the Hasse diagram of the minimal order that does not have an interval representation. In Hasse diagrams, if a variable, x, is not connected to a variable, y, then $x \cap y$ and if x is connected to y and below y, then $x \prec y$ [21]. We call $x_1 \succ x_2 \succ \cdots \succ x_n$ a **chain of length n**. The notation $\mathbf{m+n}$ refers to a poset consisting of a chain of length m and a chain of length n such that each pair of elements from different chains is incomparable.

In an interval representation, we do not restrict the length of the intervals. However, this might not be practical. For example, a surgeon cannot preform surgery for 10 hours without adding complications. Thus, we might want to add further restrictions to the interval order problem in the form of interval length restrictions. Fishburn calls adding the restriction that the length of each interval must be between p and q, inclusive,

$$\mathcal{P}[p,q] = \left\{ \begin{array}{ll} (X,\prec): & (X,\prec) \text{ is a finite interval order some} \\ & \text{representation of which has } \rho(X) \subseteq [p,q] \end{array} \right\},$$

where $p, q \in \mathbb{R}^+$, $p \leq q$, and $\rho(X)$ is the set of lengths of intervals in a representation [8].

The first investigation into $\mathcal{P}[p,q]$ was $\mathcal{P}[1,1]$- meaning that all intervals have length one. By scaling, this is equivalent to all intervals having the same length. These posets are called unit interval orders or semiorders. Semiorders were introduced by Luce in 1956 in the context of utility theory [15]. In 1958, Scott and Suppes mathematically defined a semiorder to be a poset, $P = (X, \prec)$, for which there is a function, $f : X \to \mathbb{R}$, such that $x \prec y$ if and only if $f(y) > f(x) + 1$ [20]. This definition makes it clear that semiorders and unit interval orders are equivalent. Theorem 1.1.5 of Scott and Suppes gives a complete list of minimal forbidden induced suborders (i.e., subposets) which prevent a poset from having a unit interval representation.

Theorem 1.1.5 ([20])**.** A poset, P, has a unit interval representation if and only if it does not contain a $\mathbf{2+2}$ or a $\mathbf{3+1}$ (see Figure 1.2) as an induced suborder.

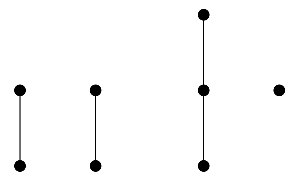


Figure 1.2: Structures that cannot appear in a unit interval order

We will refer to **2+2** and **3+1** as the minimal forbidden substructures for unit interval orders. We offer the following definition.

Definition 1.1.6. A **minimal forbidden substructure** for a certain criteria is a poset, P, which does not satisfy the desired criteria, but when a single element is removed from P, the resulting poset satisfies the criteria.

For example, **4+1** does not have a unit interval representation, but removing one element in the chain of four elements does not create a poset with a unit interval representation. Thus, **4+1** is a forbidden substructure but not a minimal forbidden substructure.

Definition 1.1.6 implies that if a poset, P, contains a minimal forbidden substructure, P', as an induced subposet, then P also does not satisfy the criteria. We will often interchange substructure and suborder with substructure used mostly in reference to the Hasse diagram of an order. In our context, the criteria will always be that the poset can be represented on the real line by closed intervals with lengths between positive integers, p and q. Beyond unit interval orders, Fishburn also considered posets with interval representations with lengths between 1 and positive integer, q. Theorem 1.1.7 gives the list of minimal forbidden substructures for these length restrictions. Note that the notation $(\cap^2)(\prec^2)\subseteq\prec$ means that if $a\cap b\cap c\prec d\prec e$, then $a\prec e$.

Theorem 1.1.7 ([8])**.** Suppose $(X,\prec)$ is a poset and $q\in\mathbb{Z}^+$. Then, $(X,\prec)\in\mathcal{P}[1,q]$ if and only if $(\cap)(\prec^{q+1})\subseteq\prec$.

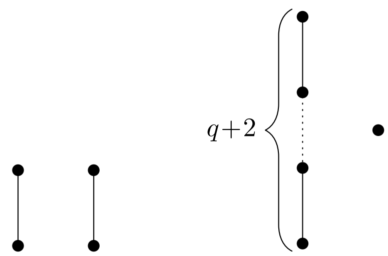


Figure 1.3: Structures that cannot appear in a $[1,q]$ representable interval order

Figure 1.3 shows the Hasse diagram of the minimal interval order such that $(\cap)(\prec^{q+1}) \not\subseteq \prec$. It is a $(\mathbf{q}+\mathbf{2})+\mathbf{1}$. There are only two minimal forbidden suborders for $\mathcal{P}[1,q]$. When the lower bound on length is greater than one, the minimal forbidden substructures list is longer. We will develop similar lists of minimal forbidden substructures for other values of p and q. We will use the following notation for these minimal lists.

Definition 1.1.8. Let $\mathcal{F}_p^q$ be the set of minimal forbidden substructures for $\mathcal{P}[p,q]$.

Our lists will be minimal in two senses. First, each structure is minimal as in Definition 1.1.6. This implies the second sense of minimality stated as Fact 1.1.9.

Fact 1.1.9. If any structure is removed from $\mathcal{F}_p^q$, then the list no longer characterizes the posets with interval representations with the desired lengths.

Thus, both the list and its structures are minimal.

Fishburn's work will be used as a basis for our inquiry. First, by scaling, $\mathcal{P}\,[1,q/p] = \mathcal{P}[p,q]$. For example $\mathcal{P}[1,2] = \mathcal{P}[2,4]$. Also, if q/p is irrational, then there is not a finite list of minimal forbidden suborders, and the infinite list is not efficiently enumerable [8]. Thus, we will only consider relatively prime p,q with $p,q \in \mathbb{Z}^+$ [8]. The following theorem of Fishburn will also help to focus our approach. This result implies that $\mathcal{F}_p^q$ is finite when q/p is rational.

Theorem 1.1.10 ([8])**.** Suppose p and q are positive integers with $p \leq q$ that are relatively prime. Suppose also that $(X,\prec)$ is an interval order. Then, $(X,\prec) \in \mathcal{P}[p,q]$

if and only if $(X,\prec)$ satisfies $A[p,q]_n$ for $n=1,\ldots,p$ where $A[p,q]_n$ says:
For all $(\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n)\geq(2,2,\ldots,2,1)$ with $\sum_{i=1}^n \alpha_i=q+n$ and $\sum_{i=1}^n \beta_i=p+n-1$, we have $\prec^{\alpha_1}\cap^{\beta_1}\cdots\prec^{\alpha_n}\cap^{\beta_n}\subseteq\prec$ and $\cap^{\beta_n}\prec^{\alpha_n}\cdots\cap^{\beta_1}\prec^{\alpha_1}\subseteq\prec$.

Theorem 1.1.10 is very useful in that it gives necessary and sufficient conditions for an interval order to have a representation with interval lengths in $[p,q]$, but the conditions do not directly yield minimal forbidden substructures. Fishburn only gives the complete structures for $\mathcal{P}[1,q]$ as in Theorem 1.1.7. The conditions could be used to narrow the search for minimal forbidden substructures, but the notation can be challenging to follow. Instead we will use different, more accessible methods to first recreate these conditions (Chapter 2) and then find minimal forbidden substructures in some cases (Chapters 3, 4, and 5). Fishburn notes that we technically only need to consider one of $\prec^{\alpha_1}\cap^{\beta_1}\cdots\prec^{\alpha_n}\cap^{\beta_n}\subseteq\prec$ and $\cap^{\beta_n}\prec^{\alpha_n}\cdots\cap^{\beta_1}\prec^{\alpha_1}\subseteq\prec$. Our approach will also only require one analogous condition.

Any introduction to interval orders should mention their relationship to interval graphs. A graph, $G=(V,E)$, is an interval graph if and only if there is an assignment of a closed interval, I_v, in the real number line to each $v\in V$ so that $uv\in E$ if and only if I_u and I_v intersect. This definition is similar to interval orders except that when two vertices are comparable, there is no precedence between the two. This means that the interval representation of an interval graph can correspond to the interval representation of many different interval orders which Fishburn calls *agreeing* interval orders. Fishburn refers to a graph with an interval representation with interval lengths in $[p,q]$ as $\mathcal{I}[p,q]$ [8]. He states the following theorem:

Theorem 1.1.11 ([8])**.** An interval graph, G, is in $\mathcal{I}[p,q]$ if and only if every interval order that agrees with G is in $\mathcal{P}[p,q]$.

Theorem 1.1.11 implies that any results stated for $\mathcal{P}[p,q]$ have implications for $\mathcal{I}[p,q]$. Since edges in the graph correspond to incomparability, and edges in a Hasse diagram correspond to comparability, we look at structures in $\mathcal{F}_p^q$ to get the complements ($xy\in E$ if and only if $x\not\prec y$ and $y\not\prec x$ in P) of their agreeing graphs. For example, a $(\mathbf{q+2})+\mathbf{1}$ becomes a graph with one vertex of degree $q+2$ and $q+2$

vertices of degree one adjacent to it (i.e., a $K_{1,q+2}$). This correspondence between the graph and interval versions of a problem does not always exist. For example, if the lower bound on length can be different for each element of the set, the interval problem is polynomial solvable while the graph problem is NP-hard [18].

The field of reasoning about time is rich with areas of study. One could investigate adding length constraints specific to each element of a poset [13], providing a set of values into which the interval lengths must fit [1, 7], or using other relation sets (subalgebras of Allen's algebra) [14]. These areas could also be analyzed for minimal forbidden substructures. There are similar areas of study for length constrained interval graphs, but the results are not always as related to interval orders as they are in the case of $\mathcal{I}[p, q]$ and $\mathcal{P}[p, q]$ [9, 18, 12]. In fact Reasoning about time can be particularly interesting because of its applications to scheduling. Knowing that a set of events is an interval order is wonderful, but the interval representation could require that one event or task be ten times as long as another which might not be practical. Being able to set restrictions on the lengths of the intervals improves the usefulness of scheduling algorithms.

1.2 Organization

Chapter 2 presents the model and methods we will use. Chapter 3 defines $\mathcal{F}_2^q$ giving the minimal forbidden substructures of $\mathcal{P}[2, q]$: partial orders which have interval representations with lengths between 2 and an odd integer q, and Chapter 4 defines $\mathcal{F}_3^q$ for $\mathcal{P}[3, q]$: partial orders which have interval representations with lengths between 3 and an integer q not divisible by 3. Lastly, Chapter 5 presents partial results that apply for all values of p and discusses the challenges of large p values. Some of the results of Chapters 3 and 4 are implied by Proposition 5.2.4. However, the earlier chapters provide the details of the specific structures and their Hasse Diagram representations necessitating their inclusion.

Chapter 2

Digraph model for $\mathcal{P}[p, q]$

2.1 Preliminaries

For a poset, P, we add additional constraints to the interval order problem statement in the form of minimum and maximum interval lengths. We seek to create a list of minimal forbidden suborders which prevent P from having an interval representation with lengths in $[p, q]$. Our method involves translating the partial order into a weighted, directed graph and then searching for negative cycles in this associated digraph. We will first consider the translation of our problem into a system of linear inequalities. The flows in the associated digraph will then correspond to these inequalities. This technique of using potentials in a digraph to model an interval representation was used first by Doignon in [3, 4], Isaak in [11], and more recently to give a simple proof of Theorem 1.1.7 in [2]. We seek to extend the work of [2] to larger minimum interval lengths. We note that Fishburn's work with picycles for $\mathcal{P}[p.q]$ also uses inequalities but not in the context of digraphs [6].

Let $P{=}(X, \prec)$ be a partial order. If P has an interval representation, $\mathcal{I}{=}\{I_x\}_{x\in X}{=}\{[L(x), R(x)]\}_{x\in X}$, the endpoints must satisfy the following inequalities for some $\epsilon{>}0$:

1. $R(x){\leq}L(y){-}\epsilon$ for all $x, y{\in}X$ with $x{\prec}y$,

2. $L(y){\leq}R(x)$ for all $x, y{\in}X$ with $x{\cap}y$ or $x{=}y$.

Adding the restriction that the length of each interval must be between p and q adds the following inequalities for all $x \in X$:

3. $L(x) \leq R(x) - p$,

4. $R(x) \leq L(x) + q$.

We now explain how to translate an instance of the problem into a weighted digraph generalizing the model in [2]. We provide an upper bound for ϵ to assure that when we are later calculating cycle weights, the number of weight ϵ arcs will not impact whether or not the cycle is negative.

Definition 2.1.1. Let $P = (X, \prec)$. Let $0 < \epsilon < \frac{1}{2|X|}$. Let $D_p^q(P)$ be the digraph defined as follows: For each variable, $x \in X$, add two vertices: x_ℓ and x_r, and add arc, $x_\ell \rightarrow x_r$, with weight q and arc, $x_r \rightarrow x_\ell$, with weight $-p$.
Additionally, for $x \prec y$ add the arc, $y_\ell \rightarrow x_r$, with weight $-\epsilon$, where ϵ is an arbitrarily small positive constant, and for $x \cap y$ add $x_r \rightarrow y_\ell$ and $y_r \rightarrow x_\ell$ each with weight $-\epsilon$. See Figure 2.1.

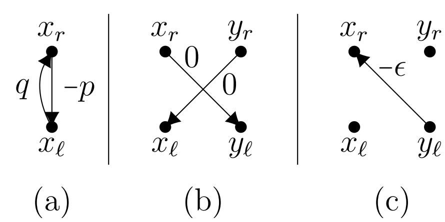


Figure 2.1: Digraph representations of (a) x, (b) $x \cap y$, and (c) $x \prec y$

Fact 2.1.2. Since $D_p^q(P)$ contains $2|X|$ vertices, a negative cycle in $D_p^q(P)$ contains at most $2|X|$ arcs. Since at least one of these arcs does not have weight $-\epsilon$ the total weight contributed by the $-\epsilon$ weight arcs is less than -1.

Once we translate the problem to a digraph, we use a well known result from graph theory on potentials in digraphs defined as follows.

Definition 2.1.3. A **potential function** on a weighted digraph, $D=(V,A)$, is a function, $f:V\to\mathbb{R}$, satisfying $f(v)-f(u)\leq w_{uv}$ for all $(u,v)\in A$.

Potential functions will be useful in the proof of Theorem 3.2.2 due to the following theorem relating potential functions and the negative cycles which we seek to characterize. A cycle in a digraph is a sequence of arcs, $C=(u_1u_2),(u_2u_3),(u_3u_4),\ldots,(u_{n-1},u_n)$, such that each u_i is unique except $u_1=u_n$. We will often denote C as $u_1,u_2,\ldots,u_{n-1},u_1$. Cycle C has length $n-1$, and the weight of C is the sum of its arc weights. Later we will use **shortest cycle** to refer to a cycle with the shortest length.

Theorem 2.1.4 (see Chapter 8 of [19])**.** A weighted digraph has a potential function if and only if it contains no negative cycles.

The following result holds for all positive integer values of p and q and provides the basis for the use of the digraph model. We will always assume that our posets are finite and that p and q are positive integers. We note that all results in Chapter 2 except for Lemma 2.2.10 hold for all positive values of p and q, but we will only use them in the context of integer p and q.

Theorem 2.1.5. Let $P=(X;\prec)$ be a partial order. The following are equivalent:

1. Poset, P, has an interval representation with lengths between integer p and q (inclusive).
2. The weighted digraph $D_p^q(P)$ contains no negative cycles.

Proof. (1) $\Rightarrow$ (2) Suppose P has an interval representation $I=\{I_x\}_{x\in X}$, where $I_x=[L(x),R(x)]$, with lengths between p and q. Then, the endpoints satisfy the following inequalities:

1. $R(x)-L(y)\leq\text{-}\epsilon$ for all $x,y\in X$ with $x\prec y$,
2. $L(y)-R(x)\leq 0$ for all $x,y\in X$ with $x\cap y$,
3. $L(x)-R(x)\leq\text{-}p$ for all $x\in X$,

4. $R(x)-L(x)\leq q$ for all $x\in X$.

Define $f:V\left(D_p^q(P)\right)\to\mathbb{R}$ by $f(y)=\begin{cases} L(x) & \text{if } y=x_\ell \text{ for some } x\in X \\ R(x) & \text{if } y=x_r \text{ for some } x\in X \end{cases}$. Then, f satisfies

1. $f(x_r)-f(y_\ell)\leq -\epsilon$ for all $x,y\in X$ with $x\prec y$,

2. $f(y_\ell)-f(x_r)\leq 0$ for all $x,y\in X$ with $x\cap y$,

3. $f(x_\ell)-f(x_r)\leq -p$ for all $x\in X$,

4. $f(x_r)-f(x_\ell)\leq q$ for all $x\in X$.

Thus, for all $u,v\in V\left(D_p^q(P)\right)$, we have $f(v)-f(u)\leq w_{uv}$, so by Definition 2.1.3, f is a potential function on $D_p^q(P)$. Then, by Theorem 2.1.4, $D_p^q(P)$ contains no negative cycles.

(2) $\Rightarrow$ (1) If $D_p^q(P)$ contains no negative cycles, then by Theorem 2.1.4, there exists a potential function f on $D_p^q(P)$. For each $x\in X$, let $L(x)=f(x_\ell), R(x)=f(x_r)$, and $I_x=[L(x),R(x)]$. As above, we can show that the inequalities f needs to satisfy as a potential function on $D_p^q(P)$ can be rewritten in terms of $L(x)$ and $R(x)$, which then guarantees that $\{I_x\}_{x\in X}$ forms a valid interval representation of P with lengths between p and q. □

We will use the second equivalence to determine lists of minimal forbidden suborders. Thus, we will be considering negative cycles in the digraph model. The following fact will be useful in later the proofs.

Fact 2.1.6. Since each arc of $D_p^q(P)$ connects an ℓ vertex to an r vertex, $D_p^q(P)$ is bipartite, and thus all cycles have even length.

To simply our language, we offer the following definition.

Definition 2.1.7. We call a shortest negative cycle in $D_p^q(P)$ with the least negative weight a **minimal negative cycle**.

In the following sections we develop a set of lemmas on the structure of a minimal negative cycle in ${D_p}^q(P)$. Our negative cycles are analogous to Fishburn's picycles [6], but we use common graph theoretical language.

2.2 Minimal negative cycle structure

The first lemma simply shows that the digraph model confirms that P cannot contain an induced **2+2** and have an interval representation.

Lemma 2.2.1. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. All the arcs of C have weight $-\epsilon$ or 0 if and only if P contains an induced **2+2**.

Proof. If P contains an induced **2+2**, say $x_1 \succ x_2$, $x_3 \succ x_4$, $x_1 \cap x_3$, $x_1 \cap x_4$, $x_2 \cap x_3$ and $x_2 \cap x_4$, then $D_p^q(P)$ contains the cycle $x_{1\ell}, x_{2r}, x_{3\ell}, x_{4r}, x_{1\ell}$. Call it C'. Cycle C' has weight -2ϵ. We claim that C' is a minimal cycle. By Fact 2.1.6, the length of any cycle in $D_p^q(P)$ is even. Thus, a cycle cannot be shorter than length two. Now, a length two cycle would have the form $y_{1\ell}, y_{2r}, y_{1\ell}$. If $y_1 = y_2$, then the cycle has weight $p - q > 0$. If $y_1 \neq y_2$, then $y_1 \succ y_2$ and $y_1 \cap y_2$ which is a contradiction. Thus, C' is a shortest negative cycle. Since $-\epsilon$ is the least negative weight of a left right arc, any other negative cycle of length four would have weight at most -2ϵ. Therefore C' is a minimal negative cycle and it contains only arcs of weight $-\epsilon$ or 0.

Now, assume C is a minimal negative cycle that contains only arcs of weight $-\epsilon$ or 0. Cycle C can be written in the form $x_{1\ell}, x_{2r}, x_{3\ell}, x_{4r}, \ldots, x_{mr}, x_{1\ell}$. We have $x_1 \succ x_2 \cap x_3 \succ x_4$, so $x_1 \neq x_2, x_3$. Now, If $x_1 = x_4$, then $x_3 \succ x_4 = x_1 \succ x_2$, and by transitivity, $x_3 \succ x_2$ which is a contradiction. Also, $x_2 \neq x_3, x_4$ and $x_3 \neq x_4$, so our four elements are distinct.

Now consider x_1 and x_4. If $x_1 \prec x_4$, then $x_2 \prec x_1 \prec x_4 \prec x_3 \cap x_2$ which contradicts transitivity. If $x_1 \succ x_4$, removing x_{2r} and $x_{3\ell}$ from C creates a shorter negative cycle which contradicts our assumption of minimality. Thus, $x_1 \cap x_4$.

Next, consider the relationship between x_1 and x_3. If $x_1 \prec x_3$, then $x_2 \prec x_1 \prec x_3 \cap x_2$ which contradicts transitivity. If $x_1 \succ x_3$, then $x_1 \succ x_3 \succ x_4 \cap x_1$, which contradicts transitivity. Thus, $x_1 \cap x_3$.

Finally, if $x_2 \prec x_4$, then $x_3 \succ x_4 \succ x_2 \cap x_3$ contradicting transitivity. If $x_2 \succ x_4$, then $x_1 \succ x_2 \succ x_4 \cap x_1$ again contradicting transitivity. Thus, $x_2 \cap x_4$. Therefore, x_1, x_2, x_3, and x_4 form an induced **2+2** in $D_p^q(P)$. $\square$

Corollary 2.2.2. If the digraph of an interval order, P, contains a negative cycle, C, then C contains at least one arc of weight $-p$.

Proof. By Fact 2.1.2, if all negative arcs in C have weight $-\epsilon$, then C cannot contain any arcs of weight q or the cycle would not have negative weight. Thus, C contains only arcs of weight $-\epsilon$ or 0. By Lemma 2.2.1, P is not an interval order because it contains an induced **2+2**. $\square$

The next lemma uses transitivity to show that negative cycles in the digraph of an interval order must contain at least one positive weight arc.

Lemma 2.2.3. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. If C contains an arc of weight $-p$, then C contains an arc of weight q.

Proof. (By contradiction)
Assume C contains an arc of weight $-p$ but no positive weight arcs.

Then, C contains a sequence of vertices of the form x_ℓ, y_r, y_ℓ, z_r. Now, $w(y_r y_\ell) = -p$ and $x \succ y \succ z$. By transitivity, $x \succ z$ and so (x_ℓ, z_ℓ) is an arc of $D_p^q(P)$. Thus, replacing x_ℓ, y_r, y_ℓ, z_r with $x_\ell z_r$ in C yields a shorter negative weight cycle. $\square$

The following lemma further restricts the structure of a minimal negative cycle in the digraph.

Lemma 2.2.4. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$.

(a) If C contains a weight $-p$ arc and a weight q arc with only weight ϵ and weight 0 arcs in between them, then they are separated by exactly one weight $-\epsilon$/weight 0 pair.

(b) If C contains two weight q arcs with only weight ϵ and weight 0 arcs in between them, then they are separated by a single weight 0 arc.

(c) If C contains two weight $-p$ arcs with only weight ϵ and weight 0 arcs in between them, then they are separated by a single weight $-\epsilon$ arc.

Proof. (by contradiction) If we assume that (a), (b), or (c) is not true, then we have a minimal negative cycle, C, which contains the path $Q{=}x_{1\ell}, x_{2r}, x_{3\ell}, x_{4r}$ or the path $Q'{=}y_{1r}, y_{2\ell}, y_{3r}, y_{4\ell}$. If $x_1 \prec x_4$, then $x_2 \prec x_1 \prec x_4 \prec x_3 \cap x_2$ contradicting transitivity. If $x_1 \succ x_4$ then replacing Q in C with $x_{1\ell} \to x_{4r}$ creates a shorter negative cycle. If $x_1 \cap x_4$, then $\{x_1, x_2, x_3, x_4\}$ is a $\mathbf{2+2}$ contradicting minimality. Thus, there is no relationship between x_1 and x_4 which preserves minimality, and our minimality assumption was false. Similarly, $y_1 \cap y_4$ creates a shorter negative cycle (using the arc $y_{1r} \to y_{4\ell}$) and $y_1 \prec y_4$ creates a $\mathbf{2+2}$. If $y_1 \succ y_4$, then consider the vertex before y_{1r} in C. If it is $y_{1\ell}$, then replacing Q with $y_{1\ell} \to y_{4r} \to y_{4\ell}$ creates a shorter negative cycle. If it is z_ℓ, then $z_\ell, y_{1r}, y_{2\ell}, y_{3r}$ is a path like Q in C, so there is no relationship between z and y_4 which preserves minimality. Thus, the conclusions of (a), (b), and (c) hold.

□

Definition 2.2.5. The term **adjacent** will refer to pairs of weight $-p$ and/or weight q arcs separated only by the minimum number of weight $-\epsilon$/weight 0 arcs as dictated by Lemma 2.2.4.

Thus, Lemma 2.2.4 says that our minimal cycles are sequences of adjacent sets of adjacent weight q arcs and adjacent weight $-p$ arcs. We will always consider our cycles to start with a set of weight $-p$ arcs and thus end with a set of weight q arcs.

The next lemma determines the number of weight $-p$ arcs based on the number of weight q arcs in a minimal cycle in the digraph.

Lemma 2.2.6. *Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. If C has contains α weight q arcs, then C contains $\left\lceil \frac{\alpha q}{p} \right\rceil$ arcs of weight $-p$.*

Proof. Let C be a minimal negative cycle in $D_p^q(P)$, and let α be the number of weight q arcs in C. Let β be the number of weight $-p$ arcs in C. Since C has negative weight, $\beta \geq \left\lceil \frac{\alpha q}{p} \right\rceil$. Assume $\beta > \left\lceil \frac{\alpha q}{p} \right\rceil$. Since $q > p$, $\beta > \alpha$ and in C there are at least two adjacent weight $-p$ arcs giving us the path $x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}$ in C (after relabeling). This gives $x_1 \succ x_2 \succ x_3$. Now, by transitivity, $x_1 \succ x_3$, but then replacing $x_{1\ell} \to x_{3r}$ in C gives a cycles with α weight q arcs and $\beta - 1$ weight $-p$ arc. Now, $\beta - 1 \geq \left\lceil \frac{\alpha q}{p} \right\rceil$ so this is a shorter negative cycle which is a contradiction. $\square$

In Lemma 2.2.7, we consider the possibility of there existing elements of P whose right and left vertices are both on a minimal negative cycle, but are not adjacent on the cycle. The only possible elements in this category are those not represented by a weight q edge or a weight $-p$ edge. In Fishburn's terminology, a cycle with all distinct elements is called pure [6].

Lemma 2.2.7. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. Let α_i be the number of weight $-p$ arcs in the i^{th} set of adjacent weight $-p$ arcs and let β_i be the number of weight q arcs in the i^{th} set of adjacent weight $-p$ arcs. Let N be the number of such sets. Let x_{ir} be the vertex immediately following the i^{th} set of adjacent weight $-p$ arcs and let $y_{i\ell}$ be the vertex immediately following the i^{th} set of adjacent weight q arcs. We have the following:

(a) If

(i) $\sum_{j=i+1}^{i+k-1} \beta_j \geq \left\lceil \frac{\left(1+\sum_{j=i}^{i+k} \alpha_j\right) q}{p} \right\rceil$, or

(ii) $1+\sum_{j=1}^{i} \beta_j + \sum_{j=i+k+1}^{N} \beta_j \geq \left\lceil \frac{\left(\sum_{j=1}^{i-1} \alpha_j + \sum_{j=i+k+1}^{N} \alpha_j\right) q}{p} \right\rceil$,

then x_i and y_{i+k} are distinct elements in P.

(b) If

(i) $1+\sum_{j=i+1}^{i+k} \beta_j \geq \left\lceil \frac{\left(\sum_{j=i+1}^{i+k-1} \alpha_j\right) q}{p} \right\rceil$, or

(ii) $\sum_{j=1}^{i} \beta_j + \sum_{j=i+k+1}^{N} \beta_j \geq \left\lceil \frac{\left(1+\sum_{j=1}^{i} \alpha_j + \sum_{j=i+k}^{N} \alpha_j\right) q}{p} \right\rceil$,

then y_i and x_{i+k}, $k \geq 1$ are distinct elements in P.

Proof. If $x_i = y_{i+k}$, then the cycle, C' created by following C from $x_{ir} \to y_{i+k\ell} = x_{i\ell}$ and then taking the arc $(x_{i\ell}, x_{ir})$ has $1 + \sum_{j=i}^{i+k} \alpha_j$ arcs of weight q and $\sum_{j=i+1}^{i+k-1} \beta_j$ arcs of weight $-p$. If $\sum_{j=i+1}^{i+k-1} \beta_j \geq \left\lceil \frac{\left(1+\sum_{j=i}^{i+k} \alpha_j\right) q}{p} \right\rceil$, then C' is shorter negative cycle than C.

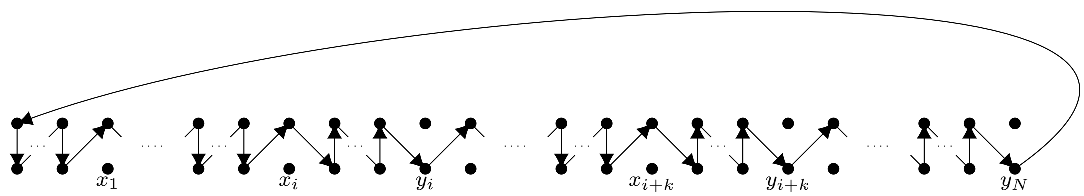


Figure 2.2: Generic minimal negative cycle in $D_p^q(P)$: Each arc directed vertically downward has weight $-p$, the arc directed vertically upward has weight q, arcs directed diagonally downward have weight 0, and arcs directed diagonally upward have weight $-\epsilon$.

Also, if $x_i = y_{i+k}$, the cycle C'' that is C with $x_{ir} \to y_{i+k\ell} = x_{i\ell}$ replaced by the arc $(x_{ir}, x_{i\ell})$. Cycle C'' has $1 + \sum_{j=1}^{i} \beta_j + \sum_{j=i+k+1}^{N} \beta_j$ arcs of weight $-p$ and $\left(\sum_{j=1}^{i-1} \alpha_j + \sum_{j=i+k+1}^{N} \alpha_j\right)$ arcs of weight q. If
$1 + \sum_{j=1}^{i} \beta_j + \sum_{j=i+k+1}^{N} \beta_j \geq \left\lceil \frac{\left(\sum_{j=1}^{i-1} \alpha_j + \sum_{j=i+k+1}^{N} \alpha_j\right) q}{p} \right\rceil$, C'' is a negative cycle that is shorter than C.

Next, if $y_i = x_{i+k}$, then the cycle C' which follows C from $y_{i\ell} \to x_{i+k_r} = y_{ir}$ and then takes arc $(y_{ir}, y_{i\ell})$ contains $1 + \sum_{j=i+1}^{i+k} \beta_j$ arcs of weight $-p$ and $\left(\sum_{j=i+1}^{i+k-1} \alpha_j\right)$ arcs of weight q, so if $1 + \sum_{j=i+1}^{i+k} \beta_j \geq \left\lceil \frac{\left(\sum_{j=i+1}^{i+k-1} \alpha_j\right) q}{p} \right\rceil$, then C' is a shorter negative cycle than C.

Also, if $y_i = x_{i+k}$, then the cycle C'' which is C with $y_{i\ell} \to x_{i+k_r} = y_{ir}$ replaced with the arc $(y_{i\ell}, y_{ir})$ contains $\sum_{j=1}^{i} \beta_j + \sum_{j=i+k+1}^{N} \beta_j$ arcs of weight $-p$ and
$\left(1 + \sum_{j=1}^{i} \alpha_j + \sum_{j=i+k}^{N} \alpha_j\right)$ arcs of weight q. If
$\sum_{j=1}^{i} \beta_j + \sum_{j=i+k+1}^{N} \beta_j \geq \left\lceil \frac{\left(1+\sum_{j=1}^{i} \alpha_j + \sum_{j=i+k}^{N} \alpha_j\right) q}{p} \right\rceil$, then C'' is a shorter negative cycle than C. □

Our main use of Lemma 2.2.7 will be in the form of the following corollary which states that elements on opposite sides of a set of either weight q or weight $-p$ arcs must be distinct. This is particularly useful when a minimal negative cycle contains at most two sets of each type because it implies that all elements of the cycle are distinct.

Corollary 2.2.8. Using the notation of Lemma 2.2.7, y_i and x_i/x_{i+1} are distinct.

Proof. Let α and β be the numbers of weight q and weight $-p$ arcs, respectively, in a minimal negative cycle.
First, consider x_i and y_i. Using Lemma 2.2.6, we have
$1+\sum_{j=1}^{i}\beta_j+\sum_{j=i+1}^{N}\beta_j=1+\beta>\left\lceil\frac{\alpha q}{p}\right\rceil>\left\lceil\frac{(\sum_{j=1}^{i-1}\alpha_j+\sum_{j=i+1}^{N}\alpha_j)q}{p}\right\rceil$, so by Lemma 2.2.7.a, x_i and y_i are distinct.
Next, consider y_i and x_{i+1}. We have $1+\sum_{j=i+1}^{i+1}\beta_j>0=\left\lceil\frac{(\sum_{j=i+1}^{i}\alpha_j)q}{p}\right\rceil$, so by Lemma 2.2.7.b, y_i and x_{i+1} are distinct. □

Lemma 2.2.9 determines the minimal forbidden substructure which corresponds to a poset whose minimal negative cycle contains only one positive weight arc. This result holds for $p\geq 1$.

Lemma 2.2.9. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. If C has exactly one weight q arc, then P contains an induced $\left\lceil\mathbf{\frac{q+2p}{p}}\right\rceil\mathbf{+1}$.

Proof. By Lemma 2.2.4, C can be written with $k-3$ arcs of weight $-p$ at the beginning of the cycle as

$$x_{1r},x_{1\ell},x_{2r},x_{2\ell},x_{3r},x_{3\ell},\ldots,x_{k-3r},x_{k-3\ell},x_{k-2r},x_{k-1\ell},x_{k-1r},x_{k\ell},x_{1r}.$$

Since C is a cycle, all of the x_i's are distinct except possibly x_{k-2} and x_k, but by Corollary 2.2.8, they are distinct. Thus, all elements represented in C are distinct. See Figure 2.3 for a diagram of C.

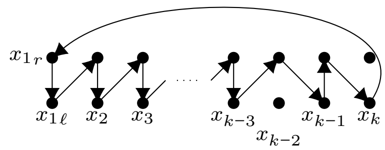


Figure 2.3: Cycle in $D_p^q(P)$ with one weight q arc: Each arc directed vertically downward has weight $-p$, the arc directed vertically upward has weight q, arcs directed diagonally downward have weight 0, and arcs directed diagonally upward have weight $-\epsilon$.

Now, by Lemma 2.2.6, $k-3=\left\lceil\frac{q}{p}\right\rceil$. Thus, we have $x_{\left\lceil\frac{q}{p}\right\rceil+3}\succ x_1\succ x_2\succ\cdots\succ x_{\left\lceil\frac{q}{p}\right\rceil}\succ x_{\left\lceil\frac{q}{p}\right\rceil+1}\cap x_{\left\lceil\frac{q}{p}\right\rceil+2}\cap x_{\left\lceil\frac{q}{p}\right\rceil+3}$, and there are $\left\lceil\frac{q}{p}\right\rceil+3$ elements in this forbidden structure.

Next, consider the relationship between $x_{\left\lceil\frac{q}{p}\right\rceil+2}$ and x_i for $i\neq\left\lceil\frac{q}{p}\right\rceil+3$ or $\left\lceil\frac{q}{p}\right\rceil+1$. By transitivity $x_{\left\lceil\frac{q}{p}\right\rceil+2}\not\prec x_i$ and $x_{\left\lceil\frac{q}{p}\right\rceil+2}\not\succ x_i$. Thus, $x_{\left\lceil\frac{q}{p}\right\rceil+2}\cap x_i$ for each i, and $D_p^q(P)$ corresponds to a $\left(\left\lceil\mathbf{\frac{q}{p}}\right\rceil+\mathbf{2}\right)+\mathbf{1}$. □

Our final lemma determines the largest number of positive weight arcs that a minimal negative cycle can contain based on the value of p. This is also a consequence of Fishburn's work on picycles as Theorem 1.1.10 only considers $A[p,q]_n$ for $n=1,\ldots,p$ [8].

Lemma 2.2.10. Let P be a poset, and let C be a minimal negative cycle in $D_p^q(P)$. If α is the number of weight q arcs in C, then $\alpha\leq p$.

Proof. (by contradiction)
Let C and α be as in the statement of the lemma. Assume that $\alpha>p$. By Lemma 2.2.6, C contains $\beta:=\left\lceil\frac{\alpha q}{p}\right\rceil$ arcs of weight $-p$.

We wish to show that somewhere in C there is a path, Q, containing p weight q arcs and q weight p arcs. Toward a contradiction assume that no such Q occurs in C.

Let β_i be the number of $-p$ weight arcs in the set immediately following the i^{th} weight q arc in C. Then, $0\leq\beta_i\leq\beta$, $\forall i\in[\alpha]$, and $\beta=\sum_{i=1}^{\alpha}\beta_i$. We will consider the subscripts to be cyclic, so when we reach α, we return to 1.

Case 1. We have $\exists i \in [\alpha]$ such that $\sum_{j=0}^{p-2} \beta_{i+j} = q$.
Then, we have a path in C that contains exactly p weight q arcs and q weight $-p$ arcs. In fact, the path starts and ends with a weight q arc.

Case 2. We have $\nexists i \in [\alpha]$ such that $\sum_{j=0}^{p-2} \beta_{i+j} = q$, but $\exists i \in [\alpha]$ such that $\sum_{j=0}^{p-2} \beta_{i+j} < q$. Then, there are two possibilities. If $\sum_{j=-1}^{p-1} \beta_{i+j} \geq q$, then we have a Q path in C. However, if $\sum_{j=-1}^{p-1} \beta_{i+j} < q$, then moving around the cycle, either we find a path with our desired arc weights, or $\forall i$, we have $\sum_{j=-1}^{p-1} \beta_{i+j} < q$. Then, summing over the positive weight arcs, we have $(p+1)\beta = \sum_{i=1}^{\alpha} \left(\sum_{j=-1}^{p-1} \beta_{i+j} \right) < \alpha q$.

Dividing by $p+1$, this gives $\beta < \frac{\alpha q}{p+1} < \frac{\alpha q}{p} \leq \left\lceil \frac{\alpha q}{p} \right\rceil$. This contradicts Lemma 2.2.6.

Case 3. We have $\sum_{j=0}^{p-2} \beta_{i+j} > q$, $\forall i \in [\alpha]$.
Since each β_i is an integer, $\sum_{j=0}^{p-2} \beta_{i+j} \geq q+1$ for each i. Summing over each i, we have $(p-1)\beta = \sum_{i=1}^{\alpha} \left(\sum_{j=0}^{p-2} \beta_{i+j} \right) \geq \alpha(q+1)$. Dividing by $p-1$, this gives $\beta \geq \frac{\alpha(q+1)}{p-1} > \frac{\alpha(q+1)}{p} = \frac{\alpha q}{p} + \frac{\alpha}{p} > \frac{\alpha q}{p} + 1 > \left\lceil \frac{\alpha q}{p} \right\rceil$, a contradiction.

Thus, we can always find a path, Q, in C that contains exactly p weight q arcs and q weight $-p$ arcs. If the first arc in Q has weight $-p$, then extend Q back one edge along C. If the last arc in Q has weight $-p$, then extend Q forward one edge along C. Call this (possibly) extended path Q'. This extension forces Q' to start at a left vertex and end at a right vertex. See Figure 2.4 for the possible starting and ending configurations of Q'. Path Q' still contains exactly p weight q arcs and q weight $-p$ arcs and $C - Q'$ contains $(\alpha - p)$ weight q arcs and $\left\lceil \frac{\alpha q}{p} \right\rceil - q = \left\lceil \frac{\alpha q}{p} - \frac{pq}{p} \right\rceil = \left\lceil \frac{(\alpha-p)q}{p} \right\rceil$ weight $-p$ arcs.

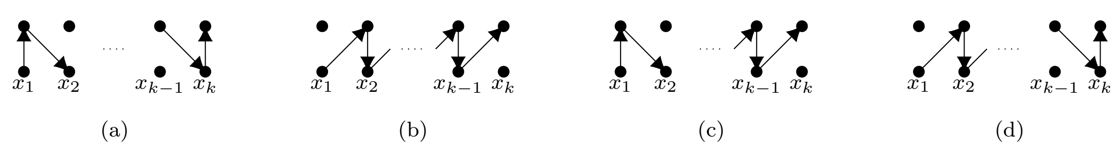


Figure 2.4: Portion of C in $D_p^q(P)$ with p weight q arcs and q weight $-p$ arcs: The four possible ending configurations of Q' are shown.

Let x_1 and x_k be the elements in P represented by the first and last vertices of Q' respectively. If $x_1 \succ x_k$, then replacing the path $x_{1\ell} \to x_{kr}$ in C with the arc $(x_{1\ell}, x_{kr})$, creates a cycle with $(\alpha - p)$ weight q arcs and $\left\lceil \frac{(\alpha-p)q}{p} \right\rceil$ weight $-p$ arcs and some $-\epsilon$ weight arcs for an overall negative weight cycle.

Similarly, if $x_1 \cap x_k$, then replacing the path $x_{kr} \to x_{1\ell}$ in C with the arc $(x_{kr}, x_{1\ell})$, creates a cycle with p weight q arcs and q weight $-p$ arcs and some $-\epsilon$ weight arcs for an overall negative weight cycle.

Next, if $x_1 \prec x_k$, then we encounter a few cases. If x_{1r} and $x_{k\ell}$ are on Q' ((a) of Figure 2.4), then replacing $x_{k\ell} \to x_{1r}$ on C with the arc $(x_{k\ell}, x_{1r})$ creates a cycle with $p-2$ weight q arcs and q weight $-p$ arcs which gives a negative cycle. If x_{1r} and $x_{k\ell}$ are not on Q' ((b) of Figure 2.4), then the cycle $Q' x_{k\ell} x_{1r} x_{1\ell}$ is a cycle with p weight q edges and $q+2$ weight $-p$ edges, yielding a shorter negative cycle than C. The other two combinations of x_{1r} and $x_{k\ell}$ on/not on C produce similar negative cycles.

Lastly, in case (b) of Figure 2.4, if $x_1 = x_k$, then Q', x_1 is a cycle with p weight q arcs and $q+1$ weight $-p$ arcs: a shorter negative cycle.

Thus, any relationship between x_1 and x_k yields a shorter negative cycle than C contradicting minimality. □

Lemma 2.2.10 creates a manageable list of negative cycle possibilities for $p=2$ and $p=3$. Lemmas 2.2.4, 2.2.6, and 2.2.10 are analogous to Theorem 1.1.10.

In the remaining chapters, we will present lists of minimal forbidden substructures for interval lengths in $[p, q]$ where p and q are positive integers. We present the following notation for these lists. Recall that we refer to these lists as $\mathcal{F}_p^q$.

2.3 Algorithm

We conclude with a proposition demonstrating how we could use $D_p^q(P)$ to algorithmically construct an interval representation of a poset P with lengths between p and q or determine that no such representation exists.

Proposition 2.3.1. Let $P = (X, \prec)$ be a poset, and let p and q be relatively prime numbers. In polynomial time, we can either construct an interval representation of P in which all interval lengths are between p and q or determine that no such representation exists.

Proof. Given a partial order $P=(X,\prec)$, p and q, construct the associated weighted digraph $D_p^q(P)$ using Definition 2.1. Use a standard shortest-paths algorithm that can handle negative arc weights and which finds minimal negative cycles, such as the Bellman-Ford algorithm, on $D_p^q(P)$ to compute the minimum weight of a path between each pair of vertices or detect a negative cycle. If a negative cycle is detected, then by Theorem 2.1.5, there is no interval representation of P in which all interval lengths are between p and q. Also, a minimal negative cycle will be detected, so a minimal forbidden substructure can be determined by the structure of the negative cycle. If the digraph contains no negative cycles, then the function $f:V\left(D_p^q(P)\right)\to\mathbb{R}$, where $f(y)$ is the minimum weight of a path in $D_p^q(P)$ ending at y, is a potential function on $D_p^q(P)$. Then, as we showed in the proof of (2) $\Rightarrow$ (1) of Theorem 2.1.5, we can construct an interval for each element of the poset such that this collection of intervals forms an interval representation of P with lengths between p and q. Note that each step in this process takes at most polynomial time, so the entire construction can be carried out in polynomial time. Thus, we have a polynomial-time certifying algorithm. $\square$

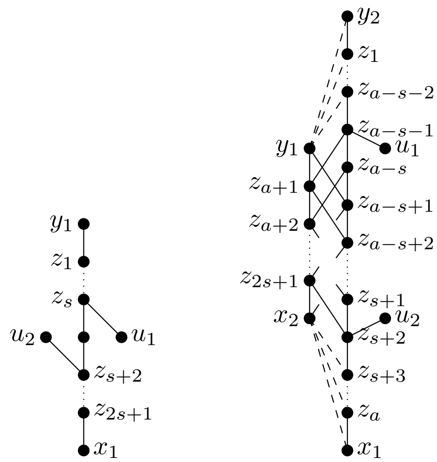
Chapter 3

Interval orders with lengths in $[2, q]$: $\mathcal{P}[2, q]$

In this chapter, we will focus on determining $\mathcal{F}_p^q$ for p=2 and integer q (Theorem 3.2.2). We will then illustrate the result for small values of q. Completely determining the minimal forbidden substructures goes beyond the work of Fishburn. First, we prove another new result which applies for all values of p.

3.1 Minimal negative cycles with two weight q arcs

Proposition 3.1.2 characterizes structures in $\mathcal{F}_p^q$ which correspond to minimal negative cycles in the digraph which contain exactly two weight q arcs. The smallest p value for which this proposition is useful is p=2. Figure 3.1 contains modified Hasse diagrams. They are Hasse diagrams except that the dashed lines indicate optional precedence. For example, in structure (ii) either $y_1 \prec z_1$ or $y_1 \cap z_1$. If $y_1 \prec z_1$, $y_1 \cap y_2$ is no longer an option due to transitivity. If a diagram contains one dashed precedence at the top and one at the bottom, then it represents four different posets: one where no precedences are chosen, one where both are chosen, one where the top is a precedence but the bottom is incomparable, and one where the bottom is a precedence but the top is incomparable. If a diagram contains two dashed precedences at the top and two at the bottom, then it represents nine different posets as in Figures 3.13 and 3.14.



(i) (ii) $a \in \{s+1, \ldots, 2s\}$

Figure 3.1: Labeling for posets in $\mathcal{F}_p^q(2)$ with $s = \left\lfloor \frac{q}{p} \right\rfloor$

We use Figure 3.1 to make the following definition.

Definition 3.1.1. Let $\mathcal{F}_p^q(2)$ be the posets labeled as in Figure 3.1 with the following relationships where $s = \left\lfloor \frac{q}{p} \right\rfloor$ and $q(\text{mod } p) \leq \frac{p}{2}$:

Subfigure (i) 1. u_1,

(a) $\prec y_1$,

(b) $\prec z_i$ for $i \in \{1, 2, \ldots, s\}$,

(c) $\cap z_i$ for $i \in \{s+1, s+2, \ldots, 2s+1\}$,

(d) $\cap x_1$,

2. u_2,

(a) $\cap y_1$,

(b) $\cap z_i$ for $i \in \{1, 2, \ldots, s+1\}$,

(c) $\succ z_i$ for $i \in \{s+2, s+3, \ldots, 2s+1\}$,

(d) $\succ x_1$,

3. $y_1 \succ z_1 \succ z_2 \succ \cdots \succ z_{\lceil 2q/p \rceil} \succ x_1$,

Subfigure (ii) 1. u_1,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \{1, 2, \ldots, a-s-1\}$,

(c) $\cap z_i$ for $i \in \{a-s, \ldots, a\}$,

(d) $\cap z_j$ for $j \in \{a+1, \ldots, 2s+1\}$,

(e) $\cap x_2$,

(f) $\cap u_2$,

(g) $\cap x_1$,

(h) $\cap y_1$,

2. u_2,

(a) $\cap z_i$ for $i \in \{1, 2, \ldots, s+1\}$,

(b) $\succ z_i$ for $i \in \{s+2, \ldots, a\}$,

(c) $\succ x_1$,

(d) $\cap y_1$,

(e) $\cap z_j$ for $j \in \{a+1, \ldots, 2s+1\}$,

(f) $\cap x_2$,

(g) $\cap y_2$,

3. y_1,

(a) $\begin{cases} \cap y_2 & a = s+1 \\ \prec \cap y_2 & a > s+1 \end{cases}$,

(b) $\prec \cap z_i$ for $i \in \{1, 2, \ldots, a-s-2\}$,

(c) $\cap z_i$ for $i \in \{a-s-1, a-s\}$,

(d) $\succ z_i$ for $i \in \{a-s+1, \ldots, a\}$,

(e) $\succ x_1$,

4. x_2,
   (a) $\prec y_2$,
   (b) $\prec z_i$ for $i \in \{1, 2, \ldots, s\}$,
   (c) $\cap z_i$ for $i \in \{s+1, s+2\}$,
   (d) $\cap\!\succ z_i$ for $i \in \{s+3, \ldots, a\}$,
   (e) $\begin{cases} \cap x_1 & a = s+1 \\ \cap\!\succ x_1 & a > s+1 \end{cases}$,
5. z_j for $j \in \{a+1, \ldots, 2s+1\}$,
   (a) $\prec y_2$,
   (b) $\prec z_i$ for $i \in \{1, 2, \ldots, j-s-2\}$,
   (c) $\cap z_i$ for $i \in \{j-s-1, j-s\}$,
   (d) $\succ z_i$ for $i \in \{j-s+1, \ldots, a\}$,
   (e) $\succ x_1$,
6. $y_2 \succ z_1 \succ z_2 \succ \cdots \succ z_a \succ x_1$,
7. $y_1 \succ z_{a+1} \succ z_{a+2} \succ \cdots \succ z_{2s+1} \succ x_2$.

Posets in family (ii) of Definition 3.1.1 will contain $2s+1+6 = \left\lceil \frac{2q}{p} \right\rceil + 6$ elements. Since $a \in \{s+1, \ldots, 2s\}$, there are s structures in this family when we disregard the dashed lines. Now, taking the dashed lines into consideration, the family contains $\sum_{i=1}^{s} i^2$ posets. We explain this calculation for $p=2$ in Proposition 3.2.3. Next, we use Definition 3.1.1 to state Proposition 3.1.2.

Proposition 3.1.2. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. If C has exactly two weight q arcs, then P contains an induced subposet isomorphic to one of the posets in $\mathcal{F}_p^q(2)$.

Proof. In what follows, when calculating cycle weight, we will disregard the contribution of the weight $-\epsilon$ arcs. Thus, a cycle with weight zero below is actually a negative cycle since each cycle we consider will contain at least one $-\epsilon$ weight arc.

By Lemma 2.2.6, we have that $\beta=\left\lceil\frac{2q}{p}\right\rceil$. Now, $q=ps+d$ where $d\in\{1,2,\ldots,p-1\}$ with $gcd(p,d)=1$. Then, $\beta=\left\lceil\frac{2q}{p}\right\rceil=\left\lceil\frac{2(ps+d)}{p}\right\rceil=2s+\left\lceil\frac{2d}{p}\right\rceil$. Now, $\beta=\begin{cases}2s+1 & d\leq\frac{p}{2}\\ 2s+2 & d>\frac{p}{2}\end{cases}$

We will consider two cases: when the weight q arcs are adjacent on the cycle and when they are not.

Case 1. The two weight q arcs are adjacent on C.
Cycle C can be written as

$$z_{1r}, z_{1\ell}, z_{2r}, z_{2\ell}, \ldots, z_{\beta_r}, z_{\beta_\ell}, x_{1r}, u_{1\ell}, u_{1r}, u_{2\ell}, u_{2r}, y_{1\ell}, x_{1r}.$$

Here, since we have a cycle, all vertices must be distinct and so all elements must be distinct except possibly z_1 and y_1, but by Corollary 2.2.8 they are distinct.

If $\beta=2s+2$, consider the relationship between z_{s+1} and u_2. Transitivity eliminates $u_1\succ z_{s+1}$. If $u_1\prec z_{s+1}$, then replacing $z_{s+1\ell}\rightarrow u_{1r}$ in C with the arc $(z_{s+1\ell}, u_{1r})$ creates a cycle, C' with $s+1$ weight -p arcs and one weight q arc for a total weight less than $q-p(s+1)=ps+d-ps-p=d-p<0$. Thus, C' is a shorter negative cycle than C. If $u_1\cap z_{s+1}$, then replacing $u_{1r}\rightarrow x_{s+1\ell}$ in C with the arc $(u_{1r}, z_{s+1\ell})$ creates a cycle C'' with $2s+2-(s+1)=s+1$ weight -p arcs and one weight q arc. As above, C'' is a shorter negative cycle than C. Since all relationships between z_{s+1} and u_1 yield shorter negative cycles, C is not minimal when $d>\frac{p}{2}$.

For the remainder of this case we will assume $d\leq\frac{p}{2}$. Thus, $\beta=2s+1$.

Cycle C can be drawn as in Figure 3.2.

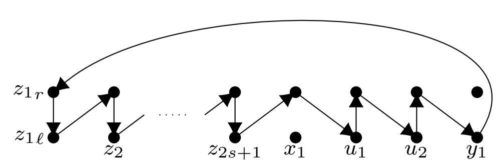


Figure 3.2: Cycle in $D_p^q(P)$ with two weight q arcs that are adjacent on C: Each arc directed vertically downward has weight -p, each arc directed vertically upward has weight q, arcs directed diagonally downward have weight 0, and arcs directed diagonally upward have weight -ϵ.

This cycle gives $y_1 \succ z_1 \succ \cdots \succ z_{2s+1} \succ x_1 \cap u_1 \cap u_2 \cap y_1$ (relationships i(1)d, i(2)a, and i3 of Definition 3.1.1). The relationships between u_1 and y_1 or z_i for $i \in \{1,2,\ldots,2s+1\}$ and between u_2 and z_j for $j=1,2,\ldots,2s+1$ or x_1 need to be determined. The possibilities to consider for the z_i's are $u_1 \prec \cap z_i$ and $u_2 \cap \succ z_j$ for the z_j's due to transitivity.

Relationship i(1)a: If $y_1 \succ u_1$, then replacing $y_{1\ell} \to u_{1r}$ in C with the arc $(y_{1\ell}, u_{1r})$ creates a cycle with weight $q>0$. If $y_1 \cap u_1$, then replacing $u_{1r} \to y_{1\ell}$ in C with the arc $(u_{1r}, y_{1\ell})$ creates a cycle with weight $q-p(2s+1)<0$. **Thus, $\boldsymbol{y_1 \succ u_1}$.**

Relationships i(1)b and i(1)c: If $u_1 \prec z_i$, then replacing $z_{i\ell} \to u_{1r}$ in C with the arc $(z_{i\ell}, u_{1r})$ creates a cycle with weight $q-p(i)$ which is positive for $i \in \{1,2,\ldots \left\lceil \frac{q}{p} \right\rceil -1\} = \{1,2,\ldots,s\}$. If $u_1 \cap z_i$, then replacing $z_{ir} \to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $2q-p(i-1)>0$ and replacing $u_{1r} \to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $q-p(2s+1-i)$ which is positive for $i \in \{s+1, s+2, \ldots, 2s+1\}$. **Thus, $\boldsymbol{u_1 \prec z_i}$ for $\boldsymbol{i \in \{1,2,\ldots,s\}}$ and $\boldsymbol{u_1 \cap z_i}$ for $\boldsymbol{i \in \{s+1, s+2, \ldots, 2s+1\}}$.**

Relationship i(2)d: If $x_1 \prec u_2$, then replacing $u_{2\ell} \to x_{1r}$ in C with the arc $(u_{2\ell}, x_{1r})$ creates a cycle with weight $q>0$. If $x_1 \cap u_2$, then replacing $x_{1r} \to u_{2\ell}$ in C with the arc $(x_{1r}, u_{2\ell})$ creates a cycle with weight $q-p(2s+1)<0$. **Thus, $\boldsymbol{x_1 \prec u_2}$.**

Relationships i(2)b and i(2)c: If $u_2 \succ z_j$, then replacing $u_{2\ell} \to z_{j_r}$ in C with the arc $(u_{2\ell}, z_{j_r})$ creates a cycle with weight $q-p(2s+1-(j-1))$ which is positive for $j \in \{s+2, s+3, \ldots, 2s+1\}$. If $u_2 \cap z_j$, then replacing $z_{j_r} \to u_{2\ell}$ in C with the arc $(z_{j_r}, u_{2\ell})$ creates a cycle with weight $q-p(j-1)$ which is positive for $i \in \{1,2,\ldots,s+1\}$ and replacing $u_{2r} \to z_{j_\ell}$ in C with the arc (u_{2r}, z_{j_ℓ}) creates a cycle with weight $2q-p(2s+1-j)>0$. **Thus, $\boldsymbol{u_2 \cap z_j}$ for $\boldsymbol{j \in \{1,2,\ldots,s+1\}}$ and $\boldsymbol{u_2 \succ z_j}$ for $\boldsymbol{j \in \{s+2, s+3, \ldots, 2s+1\}}$.**

The preceding relationships correspond to structure (i) of Definition 3.1.1.

Case 2. The two weight q arcs are not adjacent on C.

Here, C can be written as

$$z_{1_r}, z_{1_\ell}, z_{2_r}, z_{2_\ell}, \ldots, z_{a_r}, z_{a_\ell}, \mathbf{x_{1r}}, u_{1_\ell}, u_{1_r}, \mathbf{y_{1}}_\ell, z_{a+1_r}, z_{a+1_\ell}, z_{a+2_r}, z_{a+2_\ell}, \cdots, z_{\beta_r}, x_{\beta_\ell}, \mathbf{x_{2r}},$$

$$u_{2_\ell}, u_{2_r}, \mathbf{y_{2}}_\ell, z_{1_r}.$$

Now, $a \in \{1,2,\ldots,2s\} = \{1,2,\ldots,\beta-1\}$. However, by symmetry, $a=i$ will produce the same structure as $a=\beta-i$. Thus, we only address $a \in \{s+1, s+2, \ldots, 2s\}$.

Next, consider when $\beta=2s+2$. Consider the relationship between $z_{a-(s+1)}$ and u_1. Transitivity excludes $z_{a-(s+1)} \prec u_1$. If $z_{a-(s+1)} \succ u_1$, then the cycle, C' created by replacing $z_{a-(s+1)_\ell} \to u_{1_r}$ in C with the arc $(z_{a-(s+1)_\ell}, u_{1_r})$ contains $2s+2-(s+1)= s+1$ arcs of weight -p and one weight $q=ps+d$ arc. Thus, C' has weight less than $(ps+d)(1)-p(s+1)=d-p<0$ and is a shorter negative cycle than C. If $z_{a-(s+1)} \cap u_1$, then the cycle, C'', created by replacing $u_{1_r} \to z_{a-(s+1)_\ell}$ in C with the arc $(u_{1_r}, z_{a-(s+1)_\ell})$ contains $s+1$ arcs of weight -p and one weight $q=ps+d$ arc. Thus, C' has weight less than $(ps+d)(1)-p(s+1)=d-p<0$ and is a shorter negative cycle than C. Since all relationships between z_{s+1} and u_1 yield shorter negative cycles, C is not minimal when $\beta=2s+2$.

For the rest of this case, we will assume that $\beta=2s+1$. See Figure 3.3 for a digraph representation of C. We can think of C as a set of a weight -p arcs, followed by a weight q arc, followed by a set of $2s+1-a$ weight -p arcs and finally the remaining weight q arc with weight -ϵ and 0 arcs interspersed as needed.

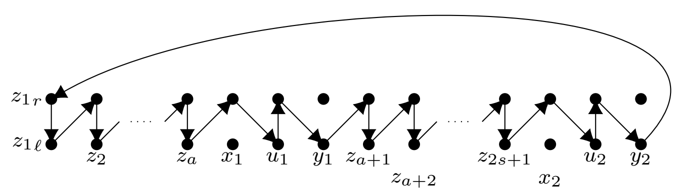


Figure 3.3: Cycle in $D_3^q(P)$ with two nonadjacent weight q arcs: Each arc directed vertically downward has weight -3, the arc directed vertically upward has weight q, arcs directed diagonally downward have weight 0, and arcs directed diagonally upward have weight -ϵ.

First, we need to confirm that each element is distinct. The only possible repeats are x_1, y_1, x_2, and y_2.

By Corollary 2.2.8, x_1 is distinct from y_1 and y_2, and x_2 is distinct from y_1 and y_2. Thus, all of the elements represented in the cycle are distinct.

Cycle C gives $y_2 \succ z_1 \succ z_2 \succ \cdots \succ z_a \succ x_1 \cap u_1 \cap y_1 \succ z_{a+1} \succ \cdots \succ x_{2s+1} \succ \cap u_2 \cap y_2$ (relationships ii(1)g, ii(1)h, ii(2)f, ii(2)g, ii6, and ii7 of Definition 3.1.1).

The relationships between u_1 and u_2, x_2, y_2, and $z_1, z_2, \ldots, x_{2s+1}$; between u_2 and x_1, y_1, and $z_1, z_2 \ldots, z_{2s+1}$; and among the elements in the chains are not directly determined by C. A diagram of the information given by C is shown in Figure 3.4. We must determine the relationships between each pair of elements not "connected" (in the Hasse diagram sense) in the diagram.

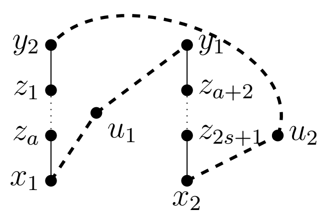


Figure 3.4: Modified Hasse diagram of the relationships defined by C: Here, solid lines function as in a Hasse diagram but the thicker dashed lines indicate incomparability. No line indicates an unknown relationship. Also, note that the thinner dotted lines within the chains indicate the possibility of more elements not pictured within the chain.

Relationships in ii1:

We start with u_1. Let $i \in \{1, 2, \ldots, a\}$. If $u_1 \prec z_i$, then replacing $z_{i\ell} \to u_{1r}$ in C with arc $(z_{i\ell}, u_{1r})$ creates cycle with weight $q - p(2s+1-a+i)$ which is positive for $i \in \{1, 2, \ldots, a-s-1\}$. If $u_1 \cap z_i$, then replacing $u_{1r} \to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $q - p(a-i)$ which is positive for $i \in \{a-s, \ldots a\}$, and replacing $z_{ir} \to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $2q - p(i-1+2s+1-a) > 0$. By transitivity, $u_1 \not\succ z_i$, and $u_1 \not\succ y_2$. If $u_1 \prec y_2$, then replacing $y_{2\ell} \to u_{1r}$

in C with the arc $(y_{2\ell}, u_{1r})$ creates a cycle with weight $q-p(2s+1-a)>q-p(s)>0$. If $u_1 \cap y_2$, then replacing $u_{1r} \to y_{2\ell}$ in C with the arc $(u_{1r}, y_{2\ell})$ creates a cycle with weight $q-pa<0$.

Let $j \in \{a+1, \ldots, 2s+1\}$. If $u_1 \succ z_j$, then replacing $u_{1\ell} \to z_{j_r}$ in C with $(u_{1\ell}, z_{j_r})$ creates cycle with weight $q-p(2s+1-(j-1)+a)<0$. If $u_1 \cap z_j$, then replacing $u_{1r} \to z_{j_\ell}$ in C with the arc (u_{1r}, z_{j_ℓ}) creates a cycle with weight $2q-p(2s+1-j+a)>0$, and replacing $z_{j_r} \to u_{1\ell}$ in C with the arc $(z_{j_r}, u_{1\ell})$ creates a cycle with weight $q-p(j-1-a)>0$. By transitivity, $u_1 \not\prec x_2$ and $u_1 \not\prec z_i$ for $i \in \{a+1, \ldots, 2s+1\}$. If $u_1 \succ x_2$, then replacing $u_{1\ell} \to x_{2r}$ in C with the arc $(u_{1\ell}, x_{2r})$ creates a cycle with weight $q-p(a)<0$. If $u_1 \cap x_2$, then replacing $x_{2r} \to u_{1\ell}$ in C with the arc $(x_{2r}, u_{1\ell})$ creates a cycle with weight $q-p(2s+1-a)>0$, and replacing $u_{1r} \to x_{2r}$ in C with the path $u_{1r}, x_{2\ell}, x_{2r}$ creates a cycle with weight $3q-p(a)>0$.

If $u_1 \prec u_2$, then replacing $u_{2\ell} \to u_{1r}$ in C with the arc $(u_{2\ell}, u_{1r})$ creates a cycle with weight $-p(2s+1-a)<0$. If $u_1 \succ u_2$, then replacing $u_{1\ell} \to u_{2r}$ in C with the arc $(u_{1\ell}, u_{2r})$ creates a cycle with weight $-p(a)<0$. If $u_1 \cap u_2$ then replacing $u_{1r} \to u_{2\ell}$ in C with the arc $(u_{1r}, u_{2\ell})$ creates a cycle with weight $2q-p(a)>0$ and replacing $u_{2r} \to u_{1\ell}$ in C with the arc $(u_{2r}, u_{1\ell})$ creates a cycle with weight $2q-p(2s+1-a)>0$.

Thus, $\boldsymbol{u_1 \prec y_2}$, $\boldsymbol{u_1 \prec z_i}$ for $\boldsymbol{i \in \{1, 2, \ldots, a-s-1\}}$, $\boldsymbol{u_1 \cap z_i}$ for $\boldsymbol{i \in \{a-s, \ldots a\}}$, $\boldsymbol{u_1 \cap z_j}$ for $\boldsymbol{j \in \{a+1, \ldots, 2s+1\}}$, $\boldsymbol{u_1 \cap x_2}$, and $\boldsymbol{u_1 \cap u_2}$.

Relationships in ii2:

Next, consider u_2. Let $i \in \{1, 2, \ldots, a\}$. If $u_2 \succ z_i$, then replacing $u_{2\ell} \to z_{ir}$ in C with the arc $(u_{2\ell}, z_{ir})$ creates a cycle with weight $q-p(2s+1-(i-1))$ which is positive for $i \in \{s+2, \ldots, a\}$. If $u_2 \cap z_i$, then replacing $u_{2r} \to z_{i\ell}$ in C with the arc $(u_{2r}, z_{i\ell})$ creates a cycle with weight $2q-p(2s+1-i)>0$, and replacing $z_{ir} \to u_{2\ell}$ in C with the arc $(z_{ir}, u_{2\ell})$ creates a cycle with weight $q-p(i-1)$ which is positive for $i \in \{1, 2, \ldots, s+1\}$. By transitivity, $u_2 \not\prec z_i$ for $i \in \{1, 2, \ldots, a\}$ and $u_2 \not\prec x_1$. If $u_2 \succ x_1$, then replacing $u_{2\ell} \to x_{1r}$ in C with the arc $(u_{2\ell}, x_{1r})$ creates a cycle with weight $q-p(2s+1-a)>0$. If $u_2 \cap x_1$, then replacing $x_{1r} \to u_{2\ell}$ in C with the arc $(x_{1r} \to u_{2\ell})$ creates a cycle with weight $q-p(a)<0$.

Let $j \in \{a+1, \ldots, 2s+1\}$. If $u_2 \prec z_j$, then replacing $z_{j_\ell} \to u_{2r}$ in C with arc

(z_{j_ℓ}, u_{2r}) creates cycle with weight $q-p(j)<0$. If $u_2 \cap z_j$, then replacing $u_{2r} \to z_{j_\ell}$ in C with the arc (u_{2r}, z_{j_ℓ}) creates a cycle with weight $q-p(2s+1-j)>0$, and replacing $z_{j_r} \to u_{2\ell}$ in C with the arc $(z_{j_r}, u_{2\ell})$ creates a cycle with weight $2q-p(j-1)>0$. By transitivity, $u_2 \not\succ z_j$, and $u_2 \not\succ y_1$. If $u_2 \prec y_1$, then replacing $y_{1\ell} \to u_{2r}$ in C with the arc $(y_{1\ell}, u_{2r})$ creates a cycle with weight $q-p(a)<0$. If $u_2 \cap y_1$, then replacing $u_{2r} \to y_{1\ell}$ in C with the arc $(u_{2r}, y_{1\ell})$ creates a cycle with weight $q-p(2s+1-a)>0$, and replacing $y_{1\ell} \to u_{2\ell}$ in C with the path $y_{1\ell}, y_{1r}, u_{2\ell}$ creates a cycle with weight $3q-p(a)>0$.

Thus, $\boldsymbol{u_2 \cap z_i}$ for $\boldsymbol{i \in \{1, 2, \ldots, s+1\}}$, $\boldsymbol{u_2 \succ z_i}$ for $\boldsymbol{i \in \{s+2, \ldots, a\}}$, $\boldsymbol{u_2 \succ x_1}$, $\boldsymbol{u_2 \cap y_1}$, and $\boldsymbol{u_2 \cap z_j}$ for $\boldsymbol{j \in \{a+1, \ldots, 2s+1\}}$.

Now, consider pairs of elements with one element in each chain, starting with y_1.

Relationships in ii3:

If $y_1 \prec y_2$, then replacing $y_{2\ell} \to y_{1\ell}$ in C with the path $y_{2\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $q-p(2s+1-a+1)$ which is positive when $a>s+1$. If $y_1 \succ y_2$, and replacing $y_{1\ell} \to y_{2\ell}$ in C with the path $y_{1\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q-p(a+1)<0$. If $y_1 \cap y_2$, then replacing $y_{2\ell} \to y_{1\ell}$ in C with the path $y_{2\ell}, y_{2r}, y_{1\ell}$ creates a cycle with weight $2q-p(2s+1-a)>0$, and replacing $y_{1\ell} \to y_{2\ell}$ in C with the path $y_{1\ell}, y_{1r}, y_{2\ell}$ creates a cycle with weight $2q-p(a)>0$.

Let $i \in \{1, 2, \ldots, a\}$. If $y_1 \succ z_i$, then replacing $y_{1\ell} \to z_{ir}$ in C with the arc $(y_{1\ell}, z_{ir})$ creates a cycle with weight $q-p(a-(i-1))$ which is positive for $i \in \{a-s+1, \ldots, a\}$. If $y_1 \prec z_i$, then replacing $z_{i\ell} \to y_{1\ell}$ in C with the path $z_{i\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $q-p(i+1+2s+1-a)$ which is positive for $i \in \{1, 2, \ldots, a-s-2\}$. If $y_1 \cap z_i$, then replacing $z_{ir} \to y_{1\ell}$ in C with the arc $(z_{ir}, y_{1\ell})$ creates a cycle with weight $q-p(i-1+2s+1-a)$ which is positive for $i \in \{1, 2, \ldots, a-s\}$, and replacing $y_{1\ell} \to z_{i\ell}$ in C with the path $y_{1\ell}, y_{1r}, z_{i\ell}$ creates a cycle with weight $2q-p(a-i)>0$.

If $y_1 \succ x_1$, then replacing $y_{1\ell} \to x_{1r}$ in C with the arc $(y_{1\ell}, x_{1r})$ creates a cycle with weight $q>0$. If $y_1 \prec x_1$, then replacing $x_{1\ell} \to y_{1\ell}$ in C with the path $x_{1\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $q-p(2s+1)<0$. If $y_1 \cap x_1$, then replacing $x_{1r} \to y_{1\ell}$ in C with the arc $(x_{1r}, y_{1\ell})$ creates a cycle with weight $q-p(2s+1)<0$.

Thus, $\boldsymbol{\begin{cases} y_1 \cap y_2 & a=s+1 \\ y_1 \prec\cap y_2 & a>s+1 \end{cases}}$**,** $\boldsymbol{y_1 \prec\cap z_i}$ **for** $\boldsymbol{i\in\{1,2,\dots,a-s-2\}}$**,** $\boldsymbol{y_1\cap z_i}$ **for** $\boldsymbol{i\in\{a-s-1,a-s\}}$**,** $\boldsymbol{y_1\succ z_i}$ **for** $\boldsymbol{i\in\{a-s+1,\dots,a\}}$**, and** $\boldsymbol{y_1\succ x_1}$**.**

Relationships in ii4:

Next, consider x_2. If $x_2\succ y_2$, then replacing $z_{2s+1\ell}\to y_{2\ell}$ in C with the path $z_{2s+1\ell}, y_{2r}$, $y_{2\ell}$ creates a cycle with weight $q-p(2s+2)<0$. If $x_2\prec y_2$, and replacing $y_{2\ell}\to x_{2r}$ in C with the arc $(y_{2\ell},x_{2r})$ creates a cycle with weight $q>0$. If $x_2\cap y_2$, then replacing $x_{2r}\to y_{2\ell}$ in C with the arc $(x_{2r},y_{2\ell})$ creates a cycle with weight $q-p(2s+1)<0$.

Let $i\in\{1,2,\dots,a\}$. If $x_2\succ z_i$, then replacing $x_{2r}\to z_{ir}$ in C with the path $x_{2r},x_{2\ell},z_{ir}$ creates a cycle with weight $q-p(2s+2-(i-1))$ which is positive for $i\in\{s+3,\dots,a\}$. If $x_2\prec z_i$, then replacing $z_{i\ell}\to x_{2r}$ in C with the arc $(z_{i\ell},x_{2r})$ creates a cycle with weight $q-p(i)$ which is positive for $i\in\{1,2,\dots,s\}$. If $x_2\cap z_i$, then replacing $x_{2r}\to z_{i\ell}$ in C with the arc $(x_{2r},z_{i\ell})$ creates a cycle with weight $q-p(2s+1-i)$ which is positive for $i\in\{s+1,\dots,a\}$, and replacing $z_{ir}\to x_{2r}$ in C with the path $z_{ir},x_{2\ell},x_{2r}$ creates a cycle with weight $2q-p(i-1)>0$.

If $x_2\prec x_1$, then replacing $x_{1r}\to x_{2r}$ in C with the path $x_{1r},x_{1\ell},x_{2r}$ creates a cycle with weight $q-p(a+1)<0$. If $x_2\succ x_1$, then replacing $x_{2r}\to x_{1r}$ in C with the path $x_{2r},x_{2\ell},x_{1r}$ creates a cycle with weight $q-p(2s+1-a+1)$ which is positive when $a>s+1$. If $x_2\cap x_1$, then replacing $x_{1r}\to x_{2r}$ in C with the path $x_{1r},x_{2\ell},x_{2r}$ creates a cycle with weight $2q-p(a)>0$, and replacing $x_{2r}\to x_{1r}$ in C with the path $x_{2r},x_{1\ell},x_{1r}$ creates a cycle with weight $2q-p(2s+1-a)>0$.

Thus, $\boldsymbol{x_2\prec y_2}$**,** $\boldsymbol{x_2\prec z_i}$ **for** $\boldsymbol{i\in\{1,2,\dots,s\}}$**,** $\boldsymbol{x_2\cap z_i}$ **for** $\boldsymbol{i\in\{s+1,s+2\}}$**,** $\boldsymbol{x_2\cap\succ z_i}$ **for** $\boldsymbol{i\in\{s+3,\dots,a\}}$**, and** $\boldsymbol{\begin{cases} x_2\cap x_1 & a=s+1 \\ x_2\cap\succ x_1 & a>s+1 \end{cases}}$**.**

Relationships in ii5:

Let $j\in\{a+1,\dots,2s+1\}$. If $z_j\prec y_2$, then replacing $y_{2\ell}\to z_{j_r}$ in C with the arc $(y_{2\ell},z_{j_r})$ creates a cycle with weight $q-p(2s+1-(j-1))$ which is positive when $j>s+1$ which is always true. If $z_j\cap y_2$, then replacing $z_{j_r}\to y_{2\ell}$ in C with the arc $(z_{j_r},y_{2\ell})$ creates a cycle with weight $q-p(j-1)$ which is positive when $j<s+2$ but

$j \geq s+2$.If $z_j \succ y_2$, then replacing $z_{j_\ell} \to y_{2\ell}$ in C with the path $z_{j_\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q-p(j+1)$ which is positive when $j<s$, but $j \geq s+2$.

If $z_j \prec x_1$, then replacing $x_{1r} \to z_{j_r}$ in C with the path $x_{1r}, x_{1\ell}, z_{j_r}$ creates a cycle with weight $q-p(a+1+2s+1-(j-1))$ which is positive when $j>a+s+2 \geq 2s+3$ which is not possible. If $z_j \cap x_1$, then replacing $x_{1r} \to z_{j_\ell}$ in C with the arc (x_{1r}, z_{j_ℓ}) creates a cycle with weight $q-p(a+2s+1-j)$ which is positive when $j>a+s \geq 2s+1$ but $j \leq 2s+1$. If $z_j \succ x_1$, then replacing $z_{j_\ell} \to x_{1r}$ in C with the arc (z_{j_ℓ}, x_{1r}) creates a cycle with weight $q-p(j-a)$ which is positive when $j<s+1+a$ which is always true.

Let $i \in \{1,2,\ldots,a\}$. If $z_j \prec z_i$, then replacing $z_{i\ell} \to z_{j_r}$ in C with the arc $(z_{i\ell}, z_{j_r})$ creates a cycle with weight $q-p(i+2s+1-(j-1))$ which is positive $i \in \{1,2,\ldots,j-s-2\}$. If $z_j \succ z_i$, then replacing $z_{j_\ell} \to z_{ir}$ in C with the arc (z_{j_ℓ}, z_{ir}) creates a cycle with weight $q-p(j-(i-1))$ which is positive $i \in \{j-s+1,\ldots,a\}$. If $z_j \cap z_i$, then replacing $z_{j_r} \to z_{i\ell}$ in C with the arc $(z_{j_r}, z_{i\ell})$ creates a cycle with weight $q-p(j-1-i)$ which is positive for $i \in \{j-s-1,\ldots,a\}$, and replacing $z_{ir} \to z_{j_\ell}$ in C with the arc (z_{ir}, z_{j_ℓ}) creates a cycle with weight $q-p(i-1+2s+1-j)$ which is positive for $i \in \{1,2,\ldots,j-s\}$.

Thus, for $\boldsymbol{j \in \{a+1,\ldots,2s+1\}}$, $\boldsymbol{z_j \prec y_2}$, $\boldsymbol{z_j \prec z_i}$ for $\boldsymbol{i \in \{1,2,\ldots,j-s-2\}}$, $\boldsymbol{z_j \cap z_i}$ for $\boldsymbol{i \in \{j-s-1, j-s\}}$, and $\boldsymbol{z_j \succ z_i}$ for $\boldsymbol{i \in \{j-s+1,\ldots,a\}}$, and $\boldsymbol{z_j \succ x_1}$.

The relationships in bold text above are listed below:

1. u_1,
    (a) $\prec y_2$
    (b) $\prec z_i$ for $i \in \{1,2,\ldots,a-s-1\}$,
    (c) $\cap z_i$, for $i \in \{a-s,\ldots,a\}$,
    (d) $\cap z_j$, for $j \in \{a+1,\ldots,2s+1\}$,
    (e) $\cap x_2$,
    (f) $\cap u_2$,

2. u_2,

   (a) $\cap z_i$ for $i \in \{1,2,\ldots,s+1\}$,

   (b) $\succ z_i$ for $i \in \{s+2,\ldots,a\}$,

   (c) $\succ x_1$,

   (d) $\cap y_1$,

   (e) $\cap z_j$ for $j \in \{a+1,\ldots,2s+1\}$,

3. y_1,

   (a) $\begin{cases} \cap y_2 & a=s+1 \\ \prec\cap y_2 & a>s+1 \end{cases}$,

   (b) $\prec\cap z_i$ for $i \in \{1,2,\ldots,a-s-2\}$,

   (c) $\cap z_i$ for $i \in \{a-s-1, a-s\}$,

   (d) $\succ z_i$ for $i \in \{a-s+1,\ldots,a\}$,

   (e) $\succ x_1$,

4. x_2,

   (a) $\prec y_2$,

   (b) $\prec z_i$ for $i \in \{1,2,\ldots,s\}$,

   (c) $\cap z_i$ for $i \in \{s+1, s+2\}$,

   (d) $\cap\succ z_i$ for $i \in \{s+3,\ldots,a\}$,

   (e) $\begin{cases} \cap x_1 & a=s+1 \\ \cap\succ x_1 & a>s+1 \end{cases}$,

5. z_j, $j \in \{a+1,\ldots,2s+1\}$,

   (a) $\prec y_2$,

   (b) $\prec z_i$ for $i \in \{1,2,\ldots,j-s-2\}$,

   (c) $\cap z_i$ for $i \in \{j-s-1, j-s\}$,

(d) $\succ z_i$ for $i \in \{j-s+1, \ldots, a\}$,

(e) $\succ x_1$.

These relationships along with the relationships defined by the cycle are exactly the posets in (ii) of Definition 3.1.1 for each value of $a \in \{s+1, \ldots, 2s\}$. Note that the uncertainty of 3.(a) and 4.(a) are represented by the dashed lines in these diagrams.

□

Corollary 3.1.3. *Let $q=ps+d$. If $d>\frac{p}{2}$, then a negative cycle with exactly two weight q arcs cannot be minimal.*

Proof. As in the proof of Proposition 3.1.2, $\beta = \begin{cases} 2s+1 & d \le \frac{p}{2} \\ 2s+2 & d > \frac{p}{2} \end{cases}$ and in both cases of the proof, $\beta=2s+2$ allowed a shorter negative cycle to be found. □

As an example of Corollary 3.1.3, consider $p=4$ and $q=11$. We have $11=4(2)+3$, so $d=3$. Now, $\frac{p}{2}=\frac{4}{2}=2$. Thus, $d>\frac{p}{2}$. By Corollary 3.1.3, $\mathcal{F}_4^{11}$ does not contain any structures corresponding to a minimal negative cycle with exactly two weight 11 arcs. Such a cycle would require six weight −4 arcs, but this cycle could always be split into two negative cycles each containing one weight 11 arc and three weight −4 arcs. Thus, the forbidden structure corresponding to a negative cycle with two weight 11 arcs would contain a $\mathbf{5}+\mathbf{1}$ which is the minimal structure which corresponds to a minimal negative cycle with one weight 11 arc and three weight −4 arcs (Lemma 2.2.9).

3.2 Structural result for lengths in $[2,q]$, q odd

Definition 3.2.1. Let $\mathcal{F}_2^q$ be the collection of posets shown in Figure 3.5 where (a) is a $\mathbf{2}+\mathbf{2}$, (b) is a $\mathbf{\frac{q+5}{2}}+\mathbf{1}$, (c) is structure (i) of Definition 3.1.1 when $p=2$, and (d) is family (ii) of Definition 3.1.1 when $p=2$. (Theorem 3.2.2 will show that this notation is appropriate.)

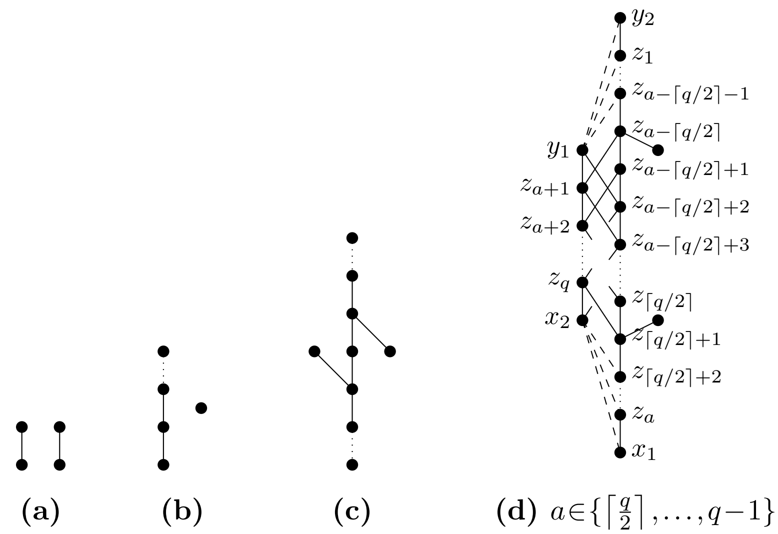


Figure 3.5: Collection of minimal forbidden substructures, $\mathcal{F}_2^q$, which prevent a poset from being representable as intervals with lengths between 2 and odd q: Dashed lines indicate optional precedence. (a) is a $\mathbf{2}+\mathbf{2}$. (b) is a $\mathbf{\frac{q+5}{2}}+\mathbf{1}$, (c) is structure (i) of Definition 3.1.1 when $p=2$, and (d) is family (ii) of Definition 3.1.1 when $p=2$. Posets in (c), and (d) contain $q+4$ and $q+6$ elements, respectively.

In Figure 3.5, posets (a), (b), and (c) are horizontally symmetric (i.e., replacing each $\prec$ with $\succ$ results in the same poset): (a) is the forbidden sub-poset for interval orders, (b) is a $\mathbf{\frac{q+5}{2}}+\mathbf{1}$, and (c) contains $q+4$ elements. Poset (b) will be generalized to higher values of p in Lemma 5.2.1. In family (d), each poset contains $q+6$ elements, and when $a=\lceil\frac{q}{2}\rceil$, there will be no dashed lines and $y_1=z_{a-\lceil q/2\rceil}$ and $x_1=z_{\lceil q/2\rceil+1}$ in the diagram. Disregarding the dashed line precedences, posets in (d) are also horizontally symetric. However, we note that certain selections for the dashed line precedences will create posets that are no longer horizontally symmetric. Section 3.3 illustrates $\mathcal{F}_p^q$ for small values of q.

The following is the main structural theorem of this chapter.

Theorem 3.2.2. Let $P=(X;\prec)$ be a partial order and let $q=2s+1$, with $s\in\mathbb{Z}_{\geq 1}$. The following are equivalent:

1. Poset, P, has an interval representation with lengths between 2 and q,
2. The weighted digraph $D_2^q(P)$ contains no negative cycles.
3. Poset, P, contains no induced sub-poset from $\mathcal{F}_2^q$.

Proof. (1) $\Leftrightarrow$ (2) This is a special case of Theorem 2.1.5.

(2) $\Rightarrow$ (3) (by contrapositive) Recall that in $D_2^q(P)$, an edge $x_\ell \to x_r$ has weight q and the reverse edge has weight -2. All other edges have weight $-\epsilon$ or 0. If P contains an induced $\mathbf{2}+\mathbf{2}$ (poset 3.5a), say $x \prec y$ and $u \prec v$ with $x \cap u, x \cap v, y \cap u$, and $y \cap v$, then $y_\ell, x_r, v_\ell, u_r, y_\ell$ is a cycle of weight -2ϵ. If P contains an induced $\mathbf{\frac{q+5}{2}}+\mathbf{1}$ (poset 3.5b), say $(x_1 \succ x_2 \succ \cdots \succ x_{(q+5)/2}) \cap y$, then, the cycle $x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}, x_{3\ell}, x_{4r}, \cdots, x_{(q+5)/2_r}, y_\ell, y_r, x_{1\ell}$ has weight $-2\left(\frac{q+1}{2}\right)+q-\epsilon\left(\frac{q+3}{2}\right)<0$. If P contains an induced poset isomorphic to poset 3.5c of $\mathcal{F}_2^q$, say $x_1 \succ x_2 \succ \ldots, \succ x_{q+2} \cap y_1 \cap y_2$ with $y_1 \succ x_{(q+1)/2}$ and $y_2 \prec x_{(q+3)/2}$. Then, the cycle

$$x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}, x_{3\ell}, x_{4r}, \cdots, x_{q+2_r}, y_{1\ell}, y_{1r}, y_{2\ell}, y_{2r}, x_{1\ell}$$

has weight $-2(q)+q(2)-\epsilon(q+1)<0$. Next, consider the posets in family 3.5d labeled as in Figure 3.6.

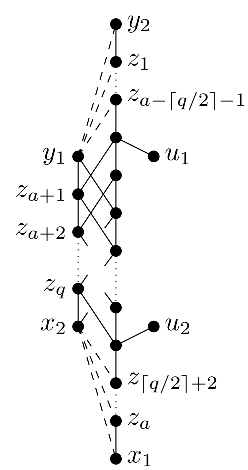


Figure 3.6: Labeling for the family (d) of posets from Figure 3.5

Now, the cycle

$$y_{1\ell}, z_{1r}, z_{1\ell}, z_{2r}, \cdots, z_{ar}, z_{a\ell}, x_{1r}, u_{1\ell}, u_{1r}, y_{1\ell}, z_{a+1r},$$

$$z_{a+1\ell}, z_{a+2r}, z_{a+2\ell}, \cdots, z_{qr}, z_{q\ell}, x_{2r}, u_{2\ell}, u_{2r}, y_{1\ell}$$

has weight $-2(q+2-a)-2(a-2)+2q-\epsilon(q+2)=-\epsilon(q+6)<0$. Thus, if a poset P contains an induced poset in $\mathcal{F}_2^q$, then $D_2^q(P)$ contains a negative cycle.

(3) $\Rightarrow$ (2) (By contrapositive) Assume $D_2^q(P)$ contains a negative cycle. We will show that P contains an element of $\mathcal{F}_2^q$ as an induced suborder.

Let C be a minimal negative cycle in $D_2^q(P)$.

Case 1. All arcs of C have weight $-\epsilon$ or 0.

By Lemma 2.2.1, P contains an induced $\mathbf{2}+\mathbf{2}$.

Case 2. Cycle C contains an arc of weight -2 but no positive weight arcs.

Lemma 2.2.3 rules out this possibility.

Case 3. Cycle C contains α arcs of weight q.

By Lemma 2.2.6, C must contain $\beta=\lceil\frac{q\alpha}{2}\rceil$ arcs of weight -2.

Case 3.1. Cycle C contains one arc with weight q (i.e., $\alpha=1$).
By Lemma 2.2.9, C corresponds to a $\left\lceil\frac{\mathbf{q+2p}}{\mathbf{p}}\right\rceil+\mathbf{1}$ where $p=2$, so a $\lceil\frac{\mathbf{q+4}}{\mathbf{2}}\rceil+\mathbf{1}$. Since q is odd this is $\frac{\mathbf{q+5}}{\mathbf{2}}+\mathbf{1}$ (structure 3.5b).

Case 3.2. Cycle C contains two arcs with weight q.
Thus, $\beta=\lceil\frac{2q}{2}\rceil=q=2s+1$.

By Proposition 3.1.2, the poset corresponding to C is isomorphic to one of the structures in Figure 3.1 where $a\in\{s+1,\ldots,2s\}$.

These are exactly the posets represented by the diagrams in Figure 3.5d.

Case 3.3. Cycle C contains three or more arcs of weight q (i.e., $\alpha\geq 3$).

This case is excluded by Lemma 2.2.10.

Thus, all possible minimal negative cycles have been considered and the resulting structures are in $\mathcal{F}_2^q$. Since each structure corresponds to a minimal negative cycle the structures are minimal forbidden substructures. To see minimality even more clearly, we reason that no structure in Figure 3.5 contains another. First, since each structure in family (d) contains more elements than (a), (b), and (c), and the same number of elements as every other structure in (d), no poset in family (d) can be contained in any other structure in the list. Similarly, (c) is not contained in (a) or (b), and (b) is not contained in (a). Now, (a) is obviously not contained in (b) or (c). There is also no $\mathbf{2}+\mathbf{2}$ in family (d) since the two chains are heavily connected. The chain in (b) has length $\frac{q+5}{2}$, and the longest chain incomparable to another element in (c) has length $\frac{q+4-2+1}{2}=\frac{q+3}{2}$. In a structure in (d) the longest chain incomparable to an element has length either $q+2-a\leq q+2-\lceil\frac{q}{2}\rceil=\lceil\frac{q}{2}\rceil+1=\frac{q+3}{2}$ or $\lceil\frac{q}{2}\rceil+1=\frac{q+3}{2}$. Thus, (b) is not contained in any of the other posets. Finally, the longest chain in (c) has length $q+2$, and the longest chain in any structure of (d) has length $a+2\leq q+1$. Therefore, the posets in Figure 3.5 are minimal and are appropriately defined as $\mathcal{F}_2^q$. □

How many posets are forbidden by $\mathcal{F}_2^q$? Proposition 3.2.3 answers this question for each q.

Proposition 3.2.3. The number of minimal forbidden subposets for interval lengths between 2 and odd q is $|\mathcal{F}_2^q|=3+\frac{(q+1)!}{24(q-2)!}$.

Proof. In Figure 3.5, (a), (b), and (c) contribute 3 posets. For part (d), there are $\frac{q+3}{2}-2=\frac{q-1}{2}$ structures without accounting for the dashed lines. Now, the first structure contains $q+6$ elements, 9 of which are part of the center structure. Each of the remaining $q-3$ elements have exactly one dashed line precedence. We can select at most one precedence from the top set of dashed lines and at most one precedence from the bottom set of dashed lines (recall: selecting a precedence close to the center implies all precedences farther from the center by transitivity). These choices can be made in $\left(\frac{q-3}{2}+1\right)\left(\frac{q-3}{2}+1\right)=\left(\frac{q-1}{2}\right)^2$ ways. The next structure in (d) has 11 elements in its center structure and so represents $\left(\frac{q-3}{2}\right)^2$ distinct posets. The third structure would have 13 elements in its center structure and so represents $\left(\frac{q-5}{2}\right)^2$ posets. Continuing to the last poset, the center structure contains all $q+6$ elements and so represents only one poset. Thus, $|\mathcal{F}_2^q|=3+\sum_{i=1}^{\frac{q-1}{2}} i^2=3+\frac{(q+1)(q)(q-1)}{24}=3+\frac{(q+1)!}{24(q-2)!}$. □

We note that this sequence appears starting at the third term of sequence A283195 on the Online Encyclopedia of Integer Sequences [16].

3.3 Small values of q

We provide the forbidden substructures for q=3,5, and 7 with some added discussion.

3.3.1 Lengths $[2,3]$

Figure 3.7 shows how to translate a poset into the associated digraph for interval orders with lengths between 2 and 3. Note: as mentioned in the introduction this is equivalent to a poset having an interval representation with lengths between 1 and 3/2.

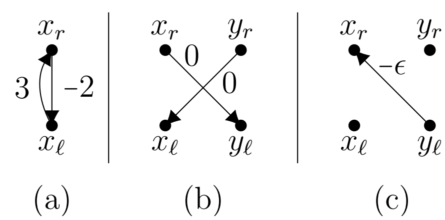
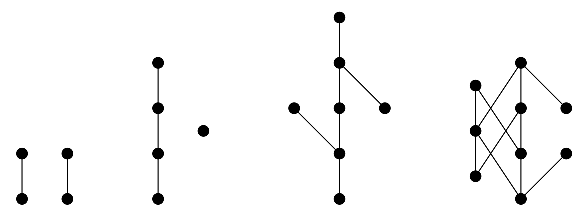


Figure 3.7: $\mathcal{P}[2,3]$ digraph representations of (a) x, (b) $x \cap y$, and (c) $x \prec y$

Figure 3.8 shows the four forbidden suborders which prevent a poset from having an interval representation with lengths between 2 and 3. Note that order (a) is forbidden even with no length restriction, order (b) is analogous to the added forbidden suborder for [1,q] interval orders (see Figure 1.3), and family (d) only contains one structure when q=3.



(a) (b) (c) (d)

Figure 3.8: Minimal induced suborders which prevent a poset from having an interval representation with lengths in $[2, 3]$

Now, since these are minimal forbidden substructures, there must be an interval representation when any single vertex is removed from the structure. We note that this is not how minimality was shown at the end of the proof of Theorem 3.2.2. It is included here to provide another view of minimality.

We leave structures (a) and (b) to the reader. For structure (c), see Figure 3.9 and for structure (d) see Figure 3.10. We provide an interval representation for one subposet of each type. For instance, since each structure is horizontally symmetric, removing the maximal element of a chain is analogous to removing the minimal element of that chain (the interval representations are vertical mirror images). These interval representations can be extended to larger values of q by adding length 2 intervals to the chains and adjusting the lengths of the other intervals. However, $q{=}3$ does not include the possibility of elements above or below the center structure. See figure 3.15 for an illustration of removing an element in this case.

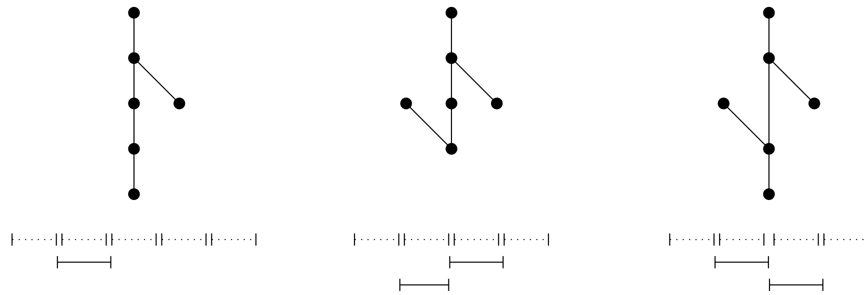


Figure 3.9: Interval representations with lengths in $[2,3]$ for subposets of minimal forbidden poset (c): Dotted intervals have length 2 and solid intervals have lengths between 2 and 3.

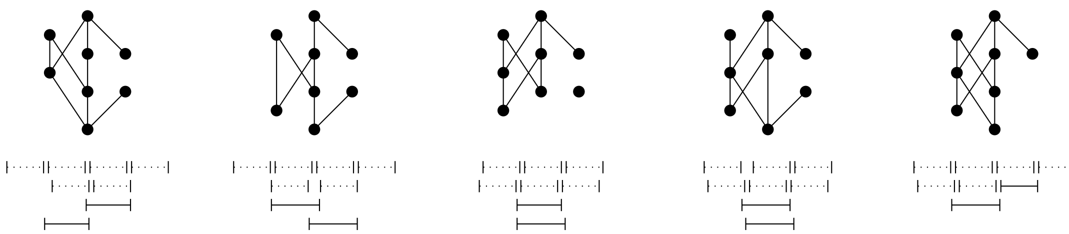
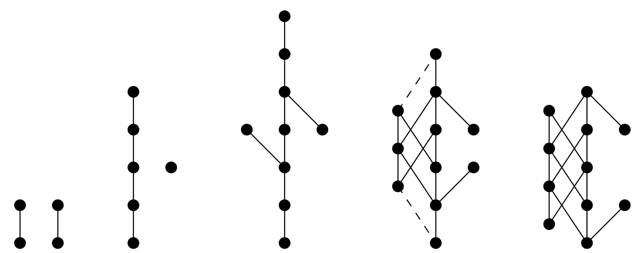


Figure 3.10: Interval representations with lengths in $[2,3]$ for subposets of minimal forbidden poset (d): Dotted intervals have length 2 and solid intervals have lengths between 2 and 3.

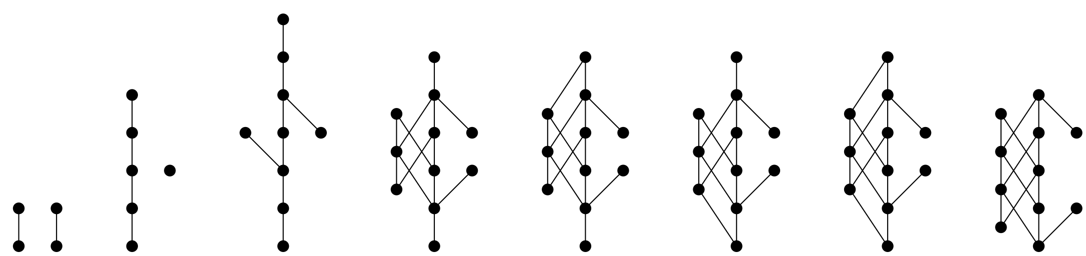
3.3.2 Lengths $[2,5]$

Figures 3.11 and 3.12 give the minimal forbidden induced subposets which prevent a poset from having an interval representation with interval lengths between 2 and 5. Figure 3.11 uses our compact dashed line notation while Figure 3.12 gives the standard Hasse diagrams.



(a) (b) (c) (d_1) (d_2)

Figure 3.11: Minimal forbidden induced subposets for lengths $[2, 5]$



(a) (b) (c) $(d_{1.1})$ $(d_{1.2})$ $(d_{1.3})$ $(d_{1.4})$ (d_2)

Figure 3.12: Minimal forbidden induced subposets for lengths $[2, 5]$ with the all eight posets given explicitly without the dashed lines

3.3.3 Lengths $[2, 7]$

Figure 3.13 shows the forbidden induced subposets for lengths between 2 and 7 using the dashed notation. Figure 3.14 gives the nine posets that part (d) of Figure 3.13 defines.

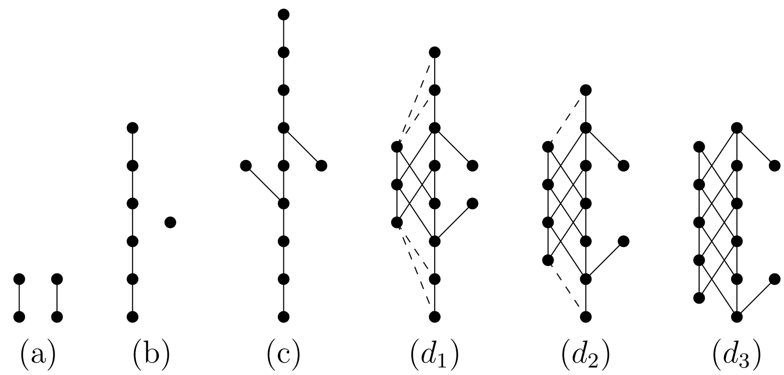


Figure 3.13: Minimal forbidden induced subposets for lengths $[2, 7]$

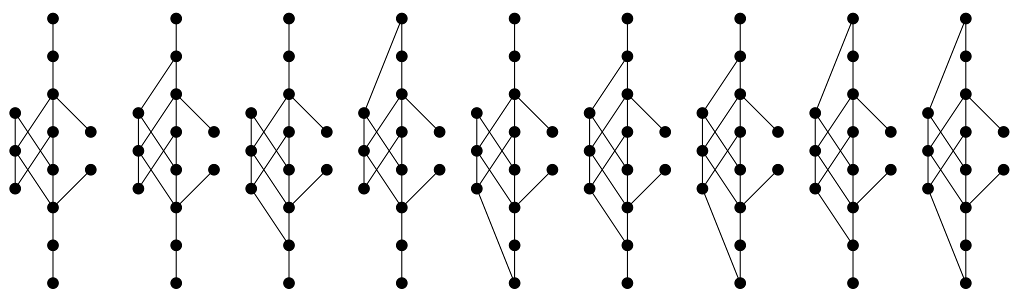


Figure 3.14: Minimal forbidden induced subposets (d_1) for lengths $[2, 7]$ with the all nine posets given explicitly without the dashed lines

As previously discussed, each poset in Figure 3.14 is minimal and so removing any vertex will create a $\mathcal{P}[2, 7]$ representable interval order. In Figure 3.10, we considered many cases of removing elements. Figure 3.15 illustrates the remaining case of removing an element above or below the central structure.

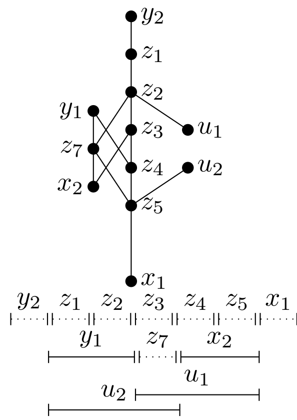


Figure 3.15: Interval representation with lengths in $[2, 7]$ for a subposet of a minimal forbidden poset in family (d): Dotted intervals have length 2 and solid intervals have lengths between 2 and 7.

This example assumes that all dashed lines become the incomparable relation. Selecting precedence instead would simply influence the lengths of intervals y_1 and/or x_2.

Chapter 4

Interval orders with lengths in $[3, q]$: $\mathcal{P}[3, q]$

In this chapter we focus on finding $\mathcal{F}_p^q$ for $p{=}3$ and integer q. For $q{=}3s$, the interval orders with interval representation with lengths in $[3, q]$ have representations with lengths in $[1, s]$. Thus, the result for q being a multiple of 3 is known. Therefore we restrict our investigation to $q{=}3s{+}1$ and $q{=}3s{+}2$. Theorem 4.2.2 gives the list of minimal forbidden posets for $\mathcal{P}[3, q]$, but first we prove a result which applies for all values of p.

4.1 Minimal negative cycles with three weight q arcs

Proposition 4.1.2 characterizes structures in $\mathcal{F}_p^q$ which correspond to minimal negative cycles in the digraph which contain exactly three weight q arcs. The smallest p value for which this proposition is useful is $p{=}3$. We first define $\mathcal{F}_p^q(3)$.

Definition 4.1.1. Let $\mathcal{F}_p^q(3)$ be the posets (and their horizontal reflections) labeled as in Figure 4.1 with the following relationships where $\beta{=}\left\lceil \frac{3q}{p} \right\rceil$, $s{=}\left\lfloor \frac{q}{p} \right\rfloor$, and $q(\text{mod } p){\leq}\frac{p}{3}$ or $\frac{p}{2}{<}q(\text{mod } p){\leq}\frac{2p}{3}$:

Subfigure (i) 1. u_1,

   (a) $\prec y_1$,
   (b) $\prec z_i$ for $i \in \left\{1,2,\dots \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$,
   (c) $\cap z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \dots, \beta\right\}$,
   (d) $\prec u_3$,
   (e) $\cap x_1$,
   (f) $\cap u_2$,

2. u_2,

   (a) $\prec y_1$,
   (b) $\prec z_i$ for $i \in \{1,2,\dots,s\}$,
   (c) $\cap z_i$ for $i \in \left\{s+1,\dots,\left\lceil \frac{2q}{p} \right\rceil\right\}$,
   (d) $\succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil + 1, \dots, \beta\right\}$,
   (e) $\succ x_1$,
   (f) $\cap u_3$,

3. u_3,

   (a) $\succ x_1$,
   (b) $\cap z_i$ for $i \in \{1,2,\dots s+1\}$,
   (c) $\succ z_i$ for $i \in \{s+2,\dots,\beta\}$,
   (d) $\cap y_1$,

4. $y_1 \succ z_1 \succ z_2 \succ \cdots \succ z_\beta \succ x_1$,

Subfigure (ii) 1. $u_1 \cap u_2$,

2. $\begin{cases} u_1 \cap u_3 & a < \left\lceil \frac{2q}{p} \right\rceil \\ u_1 \prec u_3 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

3. u_1,

   (a) $\prec y_2$,
   (b) $\prec z_i$ for $i \in \left\{1,2,\dots,a - \left\lceil \frac{q}{p} \right\rceil\right\}$,

(c) $\cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a\right\}$,

(d) $\cap z_j$ for $j \in \left\{a+1, a+2, \ldots, a + \left\lceil \frac{q}{p} \right\rceil\right\}$,

(e) $\succ z_j$ for $j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, \beta\right\}$,

(f) $\begin{cases} \succ x_2 & a < \left\lceil \frac{2q}{p} \right\rceil \\ \cap x_2 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(g) $\cap x_1$,

(h) $\cap y_1$,

4. u_2,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

(c) $\cap z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$,

(d) $\succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a\right\}$,

(e) $\begin{cases} \cap x_1 & a < \left\lceil \frac{2q}{p} \right\rceil \\ \succ x_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(f) $\begin{cases} \prec y_1 & a < \left\lceil \frac{2q}{p} \right\rceil \\ \cap y_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(g) $\prec z_j$ for $j \in \left\{a+1, a+2, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$,

(h) $\cap z_j$ for $j \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \ldots, \beta\right\}$,

(i) $\cap x_2$,

(j) $\cap u_3$,

5. u_3,

(a) $\cap z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil\right\}$,

(b) $\succ z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a\right\}$,

(c) $\succ x_1$,

(d) $\begin{cases} \cap y_2 & a \leq \left\lceil \frac{2q}{p} \right\rceil \\ \cap \succ y_2 & a > \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(e) $\cap z_j$ for $j \in \left\{a+1, a+2, \ldots, \left\lceil \frac{2q}{p} \right\rceil \right\}$,

(f) $\succ z_j$ for $j \in \left\{ \left\lceil \frac{2q}{p} \right\rceil +1, \ldots, \beta \right\}$,

(g) $\succ x_1$,

(h) $\cap y_2$,

6. y_1,

(a) $\begin{cases} \prec\cap y_2 & a < \left\lceil \frac{2q}{p} \right\rceil \\ \prec y_2 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(b) $\prec z_i$ for $i \in \left\{1, 2, \ldots, a - \left\lceil \frac{2q}{p} \right\rceil \right\}$,

(c) $\prec\cap z_i$ for $i \in \left\{a - \left\lceil \frac{2q}{p} \right\rceil +1, \ldots, a - \left\lceil \frac{q}{p} \right\rceil -1 \right\}$,

(d) $\cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil, \ldots, a - \left\lceil \frac{q}{p} \right\rceil +1 \right\}$,

(e) $\succ z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil +2, \ldots, a \right\}$,

(f) $\succ x_1$,

7. z_j for $j \in \{a+1, \ldots \beta\}$,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{1, 2, \ldots, j - \left\lceil \frac{q}{p} \right\rceil -1 \right\}$,

(c) $\cap z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil +1 \right\}$,

(d) $\succ z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil +2, \ldots, a \right\}$,

(e) $\begin{cases} \succ x_1 & j < a + \left\lceil \frac{q}{p} \right\rceil \\ \cap x_1 & a + \left\lceil \frac{q}{p} \right\rceil \leq j \leq a + \left\lceil \frac{q}{p} \right\rceil +1 \\ \prec\cap x_1 & j > a + \left\lceil \frac{q}{p} \right\rceil +1 \end{cases}$,

8. x_2,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{2q}{p} \right\rceil -1 \right\}$,

(c) $\cap z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil, \ldots, \left\lceil \frac{2q}{p} \right\rceil +1 \right\}$,

(d) $\cap\succ z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil +2, \ldots, a \right\}$,

(e) $\begin{cases} \prec\cap x_1 & a<\left\lceil\frac{2q}{p}\right\rceil-1 \\ \cap x_1 & \left\lceil\frac{2q}{p}\right\rceil-1\leq a\leq\left\lceil\frac{2q}{p}\right\rceil \\ \succ x_1 & a>\left\lceil\frac{2q}{p}\right\rceil \end{cases}$,

9. $y_2\succ z_1\succ z_2\succ\cdots\succ z_a\succ x_1$,

10. $y_1\succ z_{a+1}\succ z_{a+2}\succ\cdots\succ z_\beta\succ x_2$,

Subfigures (iii), (iv), and (v)

1. $u_1\cap u_2$,
2. $u_2\cap u_3$,
3. $\begin{cases} u_3\cap u_1 & a<\left\lceil\frac{2q}{p}\right\rceil \\ u_3\succ u_1 & a\geq\left\lceil\frac{2q}{p}\right\rceil \end{cases}$,
4. u_1,
   (a) $u_1\prec y_3$,
   (b) $u_1\prec z_i$ for $i\in\left\{1,2,\ldots,a-\left\lceil\frac{q}{p}\right\rceil\right\}$,
   (c) $u_1\cap z_i$ for $i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil+1,\ldots,a\right\}$,
   (d) $u_1\cap z_i$ for $i\in\left\{a+1,\ldots,a+\left\lceil\frac{q}{p}\right\rceil\right\}$,
   (e) $u_1\succ z_i$ for $i\in\left\{a+\left\lceil\frac{q}{p}\right\rceil+1,\ldots,a+b\right\}$,
   (f) $\begin{cases} u_1\cap x_2 & b<\left\lceil\frac{q}{p}\right\rceil \\ u_1\succ x_2 & b\geq\left\lceil\frac{q}{p}\right\rceil \end{cases}$,
   (g) $\begin{cases} u_1\prec y_2 & b<\left\lceil\frac{q}{p}\right\rceil \\ u_1\cap y_2 & b\geq\left\lceil\frac{q}{p}\right\rceil \end{cases}$,
   (h) $u_1\prec z_i$ for $i\in\left\{a+b+1,\ldots,a+\left\lceil\frac{q}{p}\right\rceil-1\right\}$,
   (i) $u_1\cap z_i$ for $i\in\left\{a+\left\lceil\frac{q}{p}\right\rceil,\ldots,\beta\right\}$,
   (j) $\begin{cases} u_1\prec\cap x_3 & a>\left\lceil\frac{2q}{p}\right\rceil \\ u_1\cap x_3 & a\leq\left\lceil\frac{2q}{p}\right\rceil \end{cases}$,
   (k) $u_1\cap x_1$,

(l) $u_1 \cap y_1$,

5. u_2,

(a) $u_2 \prec y_3$,

(b) $u_2 \prec z_i$ for $i \in \left\{1, 2, \ldots, a+b-\left\lceil \frac{2q}{p} \right\rceil\right\}$,

(c) $u_2 \cap z_i$ for $i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil+1\right\}$,

(d) $u_2 \succ z_i$ for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+2, \ldots, a\right\}$,

(e) $\begin{cases} u_2 \succ x_1 & b < \left\lceil \frac{q}{p} \right\rceil \\ u_2 \cap x_1 & b \geq \left\lceil \frac{q}{p} \right\rceil \end{cases}$

(f) $\begin{cases} , u_2 \cap y_1 & b < \left\lceil \frac{q}{p} \right\rceil \\ u_2 \prec y_1 & b \geq \left\lceil \frac{q}{p} \right\rceil \end{cases}$,

(g) $u_2 \prec z_i$ for $i \in \left\{a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil\right\}$,

(h) $u_2 \cap z_i$ for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+1, \ldots, a+b\right\}$,

(i) $u_2 \cap z_i$ for $i \in \{a+b+1, \ldots, \beta\}$,

(j) $u_2 \cap x_3$,

(k) $u_2 \cap x_2$,

(l) $u_2 \cap y_2$,

6. u_3,

(a) $u_3 \cap z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil\right\}$,

(b) $u_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil+1, \ldots, a\right\}$,

(c) $u_3 \succ x_1$,

(d) $\begin{cases} u_3 \cap \succ y_1 & a > \left\lceil \frac{2q}{p} \right\rceil \\ u_3 \cap y_1 & a \leq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(e) $u_3 \cap z_i$ for $i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$,

(f) $u_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil+1, \ldots, a+b\right\}$,

(g) $u_3 \succ x_2$,

(h) $u_3 \cap y_2$,

(i) $u_3 \cap z_i$ for $i \in \{a+b+1, \ldots, \beta\}$,

(j) $u_3 \cap x_3$,

(k) $u_3 \cap y_3$,

7. Chains one and two,

(a) y_1,

(I) $\begin{cases} y_1 \cap y_3 & a = \left\lceil \frac{q}{p} \right\rceil \\ y_1 \prec\cap y_3 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \\ y_1 \prec y_3 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(II) $y_1 \prec z_i$ for $i \in \left\{1, 2, \ldots, a - \left\lceil \frac{2q}{p} \right\rceil \right\}$,

(III) $y_1 \prec\cap z_i$ for $i \in \left\{a - \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(IV) $y_1 \cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil, a - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(V) $y_1 \succ z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$,

(VI) $y_1 \succ x_1$,

(b) z_j for $j \in \{a+1, a+2, \ldots, a+b\}$,

(I) $z_j \prec y_3$,

(II) $z_j \prec z_i$ for $i \in \left\{1, 2, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(III) $z_j \cap z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(IV) $z_j \succ z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$,

(V) $z_j \succ x_1$ for $j \in \left\{a+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(VI) $z_j \cap x_1$ for $j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil, \ldots, a + \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(VII) $z_j \prec\cap x_1$ for $j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b \right\}$,

(c) x_2,

(I) $x_2 \prec y_3$,

(II) $x_2 \prec z_i$ for $i \in \left\{1, 2, \ldots, a+b - \left\lceil \frac{q}{p} \right\rceil \right\}$,

(III) $x_2 \cap z_i$ for $i \in \left\{a+b - \left\lceil \frac{q}{p} \right\rceil + 1, a+b - \left\lceil \frac{q}{p} \right\rceil + 2 \right\}$,

(IV) $x_2 \cap\succ z_i$ for $i \in \left\{a+b-\left\lceil\frac{q}{p}\right\rceil+3,\dots,a\right\}$,

(V) $\begin{cases} x_2 \cap\succ x_1 & b<\left\lceil\frac{q}{p}\right\rceil-1 \\ x_2 \cap x_1 & b\in\left\{\left\lceil\frac{q}{p}\right\rceil-1,\left\lceil\frac{q}{p}\right\rceil\right\} \\ x_2 \prec\cap x_1 & b>\left\lceil\frac{q}{p}\right\rceil \end{cases}$,

8. Chains one and three,

   (a) y_2,

   (I) $\begin{cases} y_2 \prec\cap y_3 & a+b>\left\lceil\frac{2q}{p}\right\rceil \\ y_2 \cap y_3 & a+b=\left\lceil\frac{2q}{p}\right\rceil \end{cases}$,

   (II) $y_2 \prec\cap z_i$ for $i \in \left\{1,2,\dots,a+b-\left\lceil\frac{2q}{p}\right\rceil-1\right\}$,

   (III) $y_2 \cap z_i$ for $i \in \left\{a+b-\left\lceil\frac{2q}{p}\right\rceil,a+b-\left\lceil\frac{2q}{p}\right\rceil+1\right\}$,

   (IV) $y_2 \succ z_i$ for $i \in \left\{a+b-\left\lceil\frac{2q}{p}\right\rceil+2,\dots,a\right\}$,

   (V) $y_2 \succ x_1$,

   (b) z_j for $j \in \{a+b+1,\dots,\beta\}$,

   (I) $z_j \prec y_3$,

   (II) $z_j \prec z_i$ for $i \in \left\{1,2,\dots,j-\left\lceil\frac{2q}{p}\right\rceil-1\right\}$,

   (III) $z_j \cap z_i$ for $i \in \left\{j-\left\lceil\frac{2q}{p}\right\rceil,j-\left\lceil\frac{2q}{p}\right\rceil+1\right\}$,

   (IV) $z_j \succ z_i$ for $i \in \left\{j-\left\lceil\frac{2q}{p}\right\rceil+2,\dots,a\right\}$,

   (V) $z_j \succ x_1$,

   (c) x_3,

   (I) $x_3 \prec y_3$,

   (II) $x_3 \prec z_i$ for $i \in \left\{1,2,\dots,\left\lceil\frac{q}{p}\right\rceil-1\right\}$,

   (III) $x_3 \cap z_i$ for $i \in \left\{\left\lceil\frac{q}{p}\right\rceil,\left\lceil\frac{q}{p}\right\rceil+1\right\}$,

   (IV) $x_3 \cap\succ z_i$ for $i \in \left\{\left\lceil\frac{q}{p}\right\rceil+2,\dots,\left\lceil\frac{2q}{p}\right\rceil\right\}$,

   (V) $x_3 \succ z_i$ for $i \in \left\{\left\lceil\frac{2q}{p}\right\rceil+1,\dots,a\right\}$,

$$\text{(VI)} \begin{cases} x_3 \cap x_1 & a = \left\lceil \frac{q}{p} \right\rceil \\ x_3 \cap \succ x_1 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \succ x_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases},$$

9. Chains two and three,

(a) y_2,

$$\text{(I)} \begin{cases} y_2 \cap \succ y_1 & b < \left\lceil \frac{q}{p} \right\rceil - 1 \\ y_2 \cap y_1 & \left\lceil \frac{q}{p} \right\rceil - 1 \leq b \leq \left\lceil \frac{q}{p} \right\rceil \\ y_2 \prec \cap y_1 & \left\lceil \frac{q}{p} \right\rceil < b \end{cases},$$

(II) $y_2 \prec \cap z_i$ for $i \in \left\{a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

(III) $y_2 \cap z_i$ for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil, a+b-\left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

(IV) $y_2 \succ z_i$ for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b\right\}$,

(V) $y_2 \succ x_2$,

(b) z_j for $j \in \{a+b+1, \ldots, \beta\}$,

(I) $z_j \cap \succ y_1$ for $j \in \left\{a+b+1, \ldots, a+\left\lceil \frac{q}{p} \right\rceil - 2\right\}$,

(II) $z_j \cap y_1$ for $j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil - 1, a+\left\lceil \frac{q}{p} \right\rceil\right\}$,

(III) $z_j \prec y_1$ for $j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil + 1, \ldots, \beta\right\}$,

(IV) $z_j \prec z_i$ for $i \in \left\{a+1, \ldots, j-\left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

(V) $z_j \cap z_i$ for $i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil, j-\left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

(VI) $z_j \succ z_i$ for $i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b\right\}$,

(VII) $z_j \succ x_2$,

(c) x_3

$$\text{(I)} \begin{cases} x_3 \prec y_1 & a < \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \cap y_1 & \left\lceil \frac{2q}{p} \right\rceil \leq a \leq \left\lceil \frac{2q}{p} \right\rceil + 1 \\ x_3 \cap \succ y_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil + 2 \end{cases},$$

(II) $x_3 \prec z_i$ for $i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$,

(III) $x_3 \cap z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \left\lceil \frac{2q}{p} \right\rceil + 1\right\}$,

$$\text{(IV)}\quad x_3 \cap\!\succ z_i \text{ for } i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 2, \ldots, a+b \right\},$$

$$\text{(V)}\quad \begin{cases} x_3 \cap x_2 & a+b = \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \cap\!\succ x_2 & a+b > \left\lceil \frac{2q}{p} \right\rceil \end{cases},$$

10. $y_3 \succ z_1 \succ z_2 \succ \cdots \succ z_a \succ x_1$,

11. $y_1 \succ z_{a+1} \succ z_{a+2} \succ \cdots \succ z_{a+b} \succ x_2$,

12. $y_2 \succ z_{a+b+1} \succ z_{a+b+2} \succ \cdots \succ z_\beta \succ x_3$.

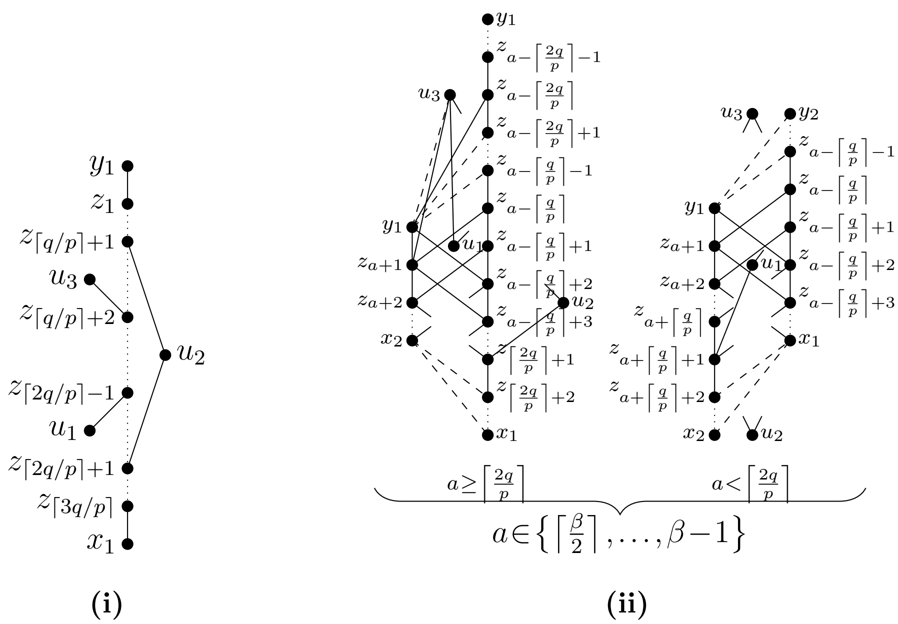
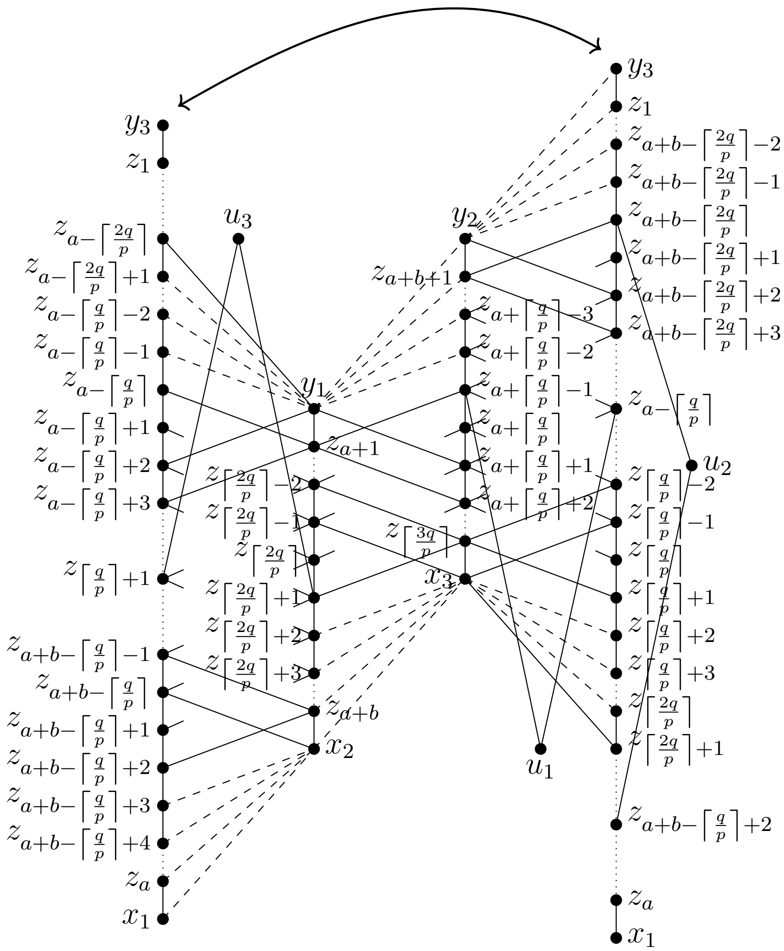


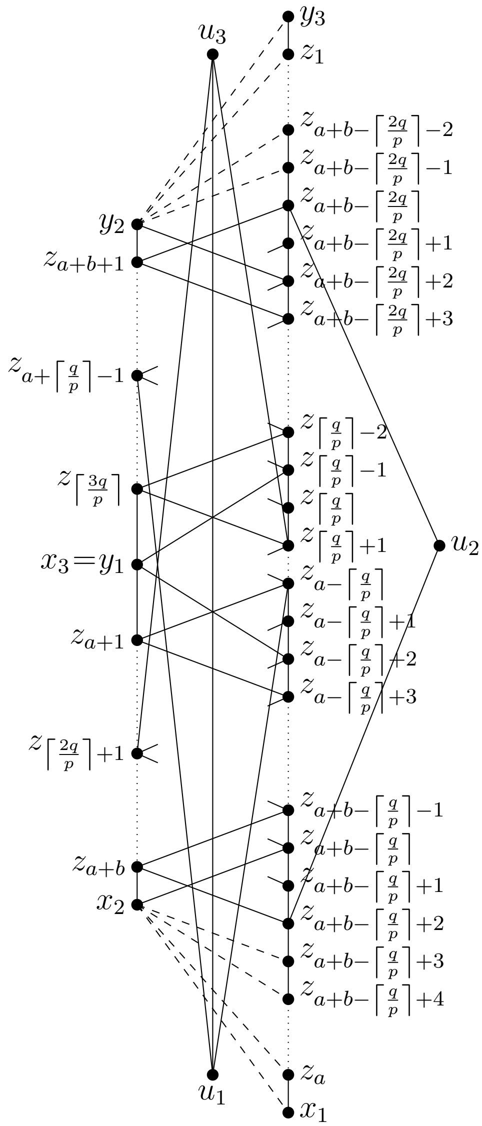
Figure 4.1: Minimal structures which correspond to a negative cycle with three weight q arcs that cannot appear in a $[p,q]$ representable interval order



(iii)

$b \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil - 1\right\}$

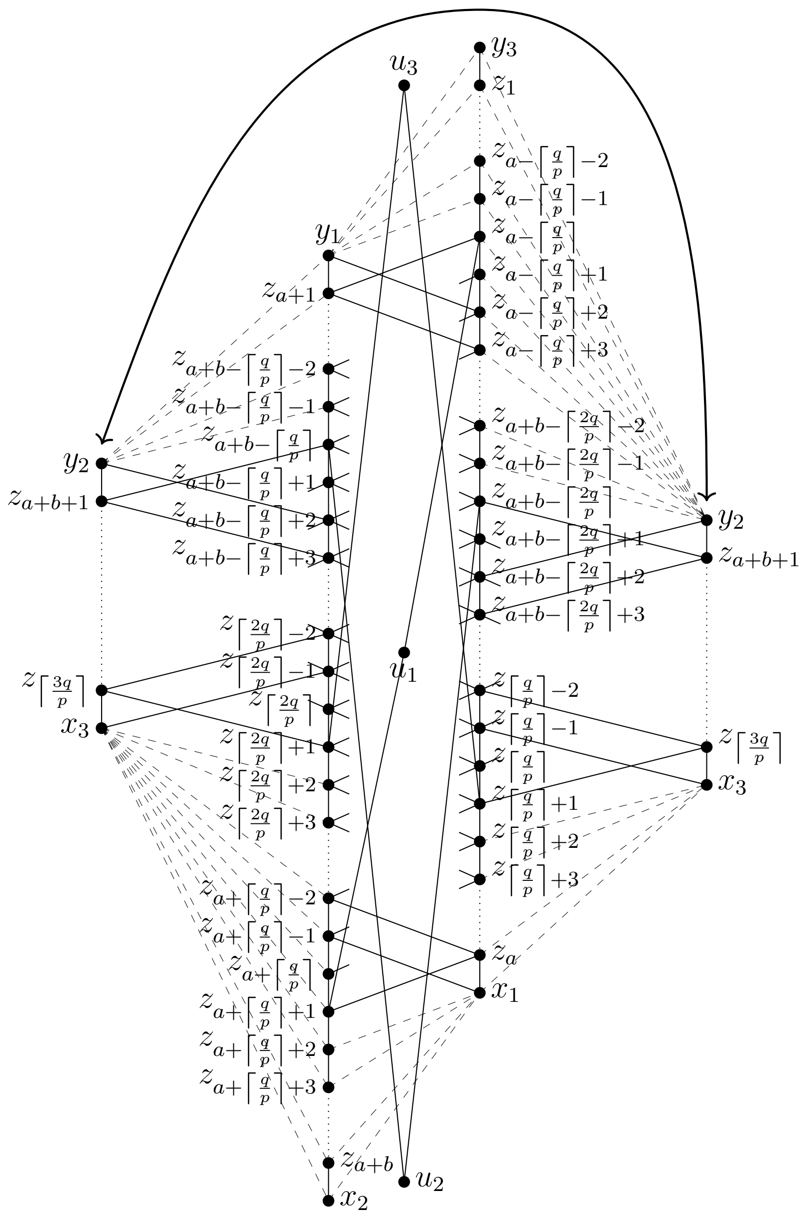
Figure 4.1 (cont): Minimal structures which correspond to a negative cycle with three weight q arcs that cannot appear in a $[p,q]$ representable interval order: The left and right chains contain the same elements. Note: $u_3 \succ u_1$ if $a \geq \left\lceil \frac{2q}{p} \right\rceil$, $y_2 \cap y_3$ if $a+b = \left\lceil \frac{2q}{p} \right\rceil$, $x_1 \cap x_2$ if $b = \left\lceil \frac{q}{p} \right\rceil - 1$.



(iv)

$a\in\left\{\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,q-2\right\}$ $\left(\text{thus } b\in\left\{1,2,\ldots,\left\lceil\frac{q}{p}\right\rceil-3\right\}\right)$

Figure 4.1 (cont): Minimal structures which correspond to a negative cycle with three weight q arcs that cannot appear in a $[p, q]$ representable interval order



(v)

$b \in \left\{ \left\lceil \frac{q}{p} \right\rceil, \ldots \left\lfloor \frac{\beta-1}{2} \right\rfloor \right\}$

Figure 4.1 (cont): Minimal structures which correspond to a negative cycle with three weight q arcs that cannot appear in a $[p, q]$ representable interval order: The left and right chains contain the same elements.

Proposition 4.1.2. Let P be a poset. Let C be a minimal negative cycle in $D_p^q(P)$. If C has exactly three weight q arcs, then P contains an induced subposet isomorphic to one of the posets in $\mathcal{F}_p^q(3)$.

Proof. Let C be a minimal negative cycle in $D_p^q(P)$ with exactly three weight q arcs.

Let $q{=}ps{+}d$ with $d{\in}\{1,2,\dots,p{-}1\}$ and $gcd(p,d){=}1$. We have that $\beta{=}\left\lceil \frac{3q}{p} \right\rceil{=}$ $3s{+}\left\lceil \frac{3d}{p} \right\rceil{=}\begin{cases} 3s{+}1 & d{\le}\frac{p}{3} \\ 3s{+}2 & \frac{p}{3}{<}d{\le}\frac{2p}{3} \\ 3s{+}3 & \frac{2p}{3}{<}d \end{cases}$. There are three possible cycle structures to consider: (1) the three positive weight arcs are adjacent on the cycle, (2) two of the positive weight arcs are adjacent on the cycle and the third is not, and (3) no pair of positive weight arcs is adjacent on C. By Lemma 2.2.4, we can draw C as in Figure 4.2.

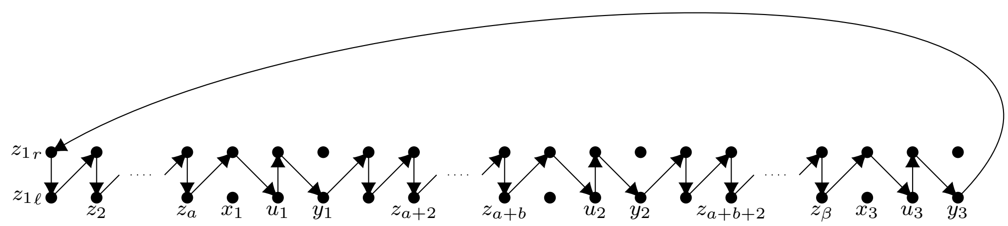


Figure 4.2: Cycle in $D_p^q(P)$ with three weight q arcs

To simplify the calculations, we will disregard the weight $-\epsilon$ arcs when finding cycle weights. Thus, if a cycle has weight 0 below, it will be considered a negative cycle because all of the cycles considered have at least one weight $-\epsilon$ arc.

Without loss of generality assume $a{\ge}b{\ge}3s{+}\left\lceil \frac{3d}{p} \right\rceil{-}a{-}b{\ge}0$. Thus, $a{\ge}\left\lceil \frac{1}{3}\left(3s{+}\left\lceil \frac{3d}{p} \right\rceil\right) \right\rceil{\ge}s{+}1$. Note: If $b{=}0$ and/or $3s{+}\left\lceil \frac{3d}{p} \right\rceil{-}a{-}b{=}0$, eliminate their corresponding x and y elements. Now, consider the relationship between $z_{a-(s+1)}$ and u_1 such that $z_{a-(s+1)}{=}y_3$ if $a{=}s{+}1$. By transitivity, $u_1{\not\succ}z_{a-(s+1)}$. If $z_{a-(s+1)}{\succ}$ u_1, then replacing $z_{a-(s+1)_\ell}{\to}u_{1r}$ in C with the arc $(z_{a-(s+1)_\ell},u_{1r})$ creates a cycle with weight $2q-p(3s+\lceil(3d)/p\rceil-(s+1))=2(ps+d)-p(2s+\lceil(3d)/p\rceil-1)=2d-$

$$p\lceil (3d)/p\rceil + p = \begin{cases} 2d & d \le \frac{p}{3} \\ 2d-p & \frac{p}{3} < d \le \frac{2p}{3} \\ 2d-2p & \frac{2p}{3} < d \end{cases}$$ which is non-positive when $d > \frac{2p}{3}$ or $\frac{p}{3} < d \le \frac{p}{2}$. If $z_{a-(s+1)} \cap u_1$, then replacing $u_{1r} \to z_{a-(s+1)_\ell}$ in C with the arc $(u_{1r}, z_{a-(s+1)_\ell})$ creates a cycle with weight $q - p(s+1) < 0$. Thus, when $d > \frac{2p}{3}$ or $\frac{p}{3} < d \le \frac{p}{2}$ all relationships between $z_{a-(s+1)}$ and u_1 yield shorter negative cycles. For the remainder of the proof, we will assume that $d \le \frac{p}{3}$ or $\frac{p}{2} < d \le \frac{2p}{3}$.

We also note that when $\frac{p}{2} < d \le \frac{2p}{3}$ or $d \le \frac{p}{3}$, $\left\lceil \frac{3q}{p} \right\rceil - \left\lceil \frac{2q}{p} \right\rceil = 3s + \left\lceil \frac{3d}{p} \right\rceil - 2s - \left\lceil \frac{2d}{p} \right\rceil = \begin{cases} s+1-1 & d \le \frac{p}{3} \\ s+2-2 & \frac{p}{2} < d \le \frac{2p}{3} \end{cases} = \left\lceil \frac{q}{p} \right\rceil - 1$, and so $\left\lceil \frac{3q}{p} \right\rceil - \left\lceil \frac{q}{p} \right\rceil = \left\lceil \frac{2q}{p} \right\rceil - 1$. We will use these facts in the following cases.

Case 1. All three positive weight arcs are adjacent on the cycle.

By Lemma 2.2.4, we can represent the cycle as

$$z_{1r}, z_{1\ell}, z_{2r}, z_{2\ell}, \ldots, z_{\beta_r}, z_{\beta_\ell}, x_{1r}, u_{1\ell}, u_{1r}, u_{2\ell}, u_{2r}, u_{3\ell}, u_{3r}, y_{1\ell}, z_{1r}.$$

By Corollary 2.2.8, x_1 and y_1 are unique. Cycle C contains both vertices corresponding to each of the other elements. Thus, all elements labeled are distinct.

We have $y_1 \succ z_1 \succ z_2 \succ \cdots \succ z_\beta \succ x_1 \cap u_1 \cap u_2 \cap u_3 \cap y_1$ (relationships i(1)e, i(1)f, i(2)f, i(3)d, and i4 of Definition 4.1.1). See Figure 4.3. The relationships between u_1 and u_1, y_1, and z_i for $i \in \{1, 2, \ldots, \beta\}$, between u_2 and x_1, y_1, and z_i for $i \in \{1, 2, \ldots, \beta\}$, and between u_3 and x_1 and z_i for $i \in \{1, 2, \ldots, \beta\}$ must be determined.

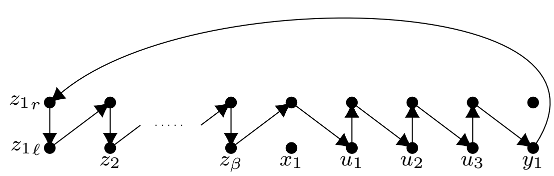


Figure 4.3: Cycle in $D_p^q(P)$ with three weight q arcs that are adjacent on C: Each arc directed vertically downward has weight $-p$, each arc directed vertically upward has weight q, arcs directed diagonally downward have weight 0, and arcs directed diagonally upward have weight $-\epsilon$.

We will start with u_1. By transitivity, $u_1 \not\succ y_1$. If $u_1 \prec y_1$, then replacing $y_{1\ell} \to u_{1r}$ in C with the arc $(y_{1\ell}, u_{1r})$ creates a cycle with weight $2q$. If $u_1 \cap y_1$, then replacing $u_{1r} \to y_{1\ell}$ in C with the arc $(u_{1r}, y_{1\ell})$ creates a cycle with weight $q - p(\beta) < 0$.

If $u_1 \succ z_i$, then replacing $u_{1\ell} \to z_{ir}$ in C with the arc $(u_{1\ell}, z_{ir})$ yields a shorter negative cycle for $i \in \{1, 2, \ldots, \beta\}$. If $u_1 \cap z_i$, then replacing $u_{1r} \to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $q - p(\beta - i)$ which is positive for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \ldots \beta\right\}$, and replacing $z_{ir} \to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $3q - p(i-1) > 0$. If $u_1 \prec z_i$, then replacing $z_{i\ell} \to u_{1r}$ in C with the arc $(z_{i\ell}, u_{1r})$ creates a cycle with weight $2q - p(i)$ which is positive for $i \in \left\{1, 2, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$.

If $u_1 \succ u_3$, then replacing $u_{1\ell} \to u_{3r}$ in C with the arc $(u_{1\ell}, u_{3r})$ creates a shorter negative cycle. If $u_1 \prec u_3$, then replacing $u_{3\ell} \to u_{1r}$ in C with the arc $(u_{3\ell}, u_{1r})$ creates a cycle with weight $q > 0$. If $u_1 \cap u_3$, then replacing $u_{1r} \to u_{3\ell}$ in C with the arc $(u_{1r}, u_{3\ell})$ creates a shorter negative cycle.

Thus, $\boldsymbol{u_1 \prec y_1}$, $\boldsymbol{u_1 \prec z_i}$ for $\boldsymbol{i \in \left\{1, 2, \ldots \left\lceil \frac{2q}{p} \right\rceil - 1\right\}}$, $\boldsymbol{u_1 \cap z_i}$ for $\boldsymbol{i \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \ldots, \beta\right\}}$, and $\boldsymbol{u_1 \prec u_3}$ (relationships i(1)a, i(1)b, i(1)c, and i(1)d of Definition 4.1.1).

Similarly, for u_3, if $u_3 \succ x_1$, then replacing $u_{3\ell} \to x_{1r}$ in C with the arc $(u_{3\ell}, x_{1r})$ creates a cycle with weight $2q > 0$. By transitivity, $u_3 \not\prec x_1$. If $u_3 \cap x_1$, then replacing $x_{1r} \to u_{3\ell}$ in C with the arc $(x_{1r}, u_{3\ell})$ creates a cycle with weight $q - p(\beta) < 0$.

If $u_3 \succ z_i$, then replacing $u_{3\ell} \to z_{ir}$ in C with the arc $(u_{3\ell}, z_{ir})$ creates a cycle with weight $2q - p(\beta - (i-1))$ which is positive for $i \in \{s+2, \ldots, \beta\}$. If $u_3 \cap z_i$, then replacing $u_{3r} \to z_{i\ell}$ in C with the arc $(u_{3r}, z_{i\ell})$ creates a cycle with weight $3q - p(\beta - i) > 0$, and replacing $z_{ir} \to u_{3\ell}$ in C with the arc $(z_{ir}, u_{3\ell})$ creates a cycle with weight $q - p(i-1)$ which is positive for $i \in \{1, 2, \ldots, s+1\}$. By transitivity, $u_1 \not\prec z_i$.

Thus, $\boldsymbol{u_3 \succ x_1}$, $\boldsymbol{u_3 \cap z_i}$ for $\boldsymbol{i \in \{1, 2, \ldots s+1\}}$, and $\boldsymbol{u_3 \succ z_i}$ for $\boldsymbol{i \in \{s+2, \ldots, \beta\}}$ (relationships i(3)a, i(3)b, and i(3)c of Definition 4.1.1).

For u_2, if $u_2 \succ y_1$, then replacing $u_{2\ell} \to y_{1\ell}$ in C with the path $u_{2\ell}, y_{1r}, y_{1\ell}$ creates a shorter negative cycle. If $u_2 \prec y_1$, then replacing $y_{1\ell} \to u_{2r}$ in C with the arc $(y_{1\ell}, u_{2r})$ creates a cycle with weight $q > 0$. If $u_2 \cap y_1$, then replacing $u_{2r} \to y_{1\ell}$ in C with the arc $(u_{2r}, y_{1\ell})$ creates a cycle with weight $2q - p(\beta) < 0$.

If $u_2 \succ z_i$ then replacing $u_{2\ell} \rightarrow z_{ir}$ in C with the arc $(u_{2\ell}, z_{ir})$ creates a cycle with weight $q - p(\beta - (i-1))$ which is positive for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta \right\}$. If $u_2 \prec z_i$, then replacing $z_{i\ell} \rightarrow u_{2r}$ in C with the arc $(z_{i\ell}, u_{2r})$ creates a cycle with weight $q - p(i)$ for $i \in \{1, 2, \ldots, s\}$. If $u_2 \cap z_i$, then replacing $u_{2r} \rightarrow z_{i\ell}$ in C with the arc $(u_{2r}, z_{i\ell})$ creates a cycle with weight $2q - p(\beta - i)$ which is positive for $i \in \{s+1, \ldots, \beta\}$ and replacing $z_{ir} \rightarrow u_{2\ell}$ in C with the arc $(z_{ir}, u_{2\ell})$ creates a cycle with weight $2q - p(i-1)$ which is positive for $i \in \left\{ 1, 2, \ldots, \left\lceil \frac{2q}{p} \right\rceil \right\}$.

If $u_2 \succ x_1$, then replacing $u_{2\ell} \rightarrow x_{1r}$ in C with the arc $(u_{2\ell}, x_{1r})$ creates a cycle with weight $q > 0$. If $u_2 \prec x_1$, then replacing $x_{1r} \rightarrow u_{2r}$ in C with the path $x_{1r}, x_{1\ell}, u_{2r}$ creates shorter negative cycle. If $u_2 \cap x_1$, then replacing $x_{1r} \rightarrow u_{2\ell}$ in C with the arc $(x_{1r}, u_{2\ell})$ creates a cycle with weight $2q - p(\beta) < 0$.

Thus, $\boldsymbol{u_2 \prec y_1}$, $\boldsymbol{u_2 \prec z_i}$ for $\boldsymbol{i \in \{1, 2, \ldots, s\}}$, $\boldsymbol{u_2 \cap z_i}$ for $\boldsymbol{i \in \left\{ s+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil \right\}}$, $\boldsymbol{u_2 \succ z_i}$ for $\boldsymbol{i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta \right\}}$, and $\boldsymbol{u_2 \succ x_1}$ (relationships i(2)a, i(2)b, i(2)c, i(2)d, and i(2)e of Definition 4.1.1).

The preceding analysis gives the following relationships which give poset (i) of Definition 4.1.1:

1. u_1,
   (a) $\prec y_1$,
   (b) $\prec z_i$ for $i \in \left\{ 1, 2, \ldots \left\lceil \frac{2q}{p} \right\rceil - 1 \right\}$,
   (c) $\cap z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil, \ldots, \beta \right\}$,
   (d) $\prec u_3$,
   (e) $\cap x_1$,
   (f) $\cap u_1$,
2. u_2
   (a) $\prec y_1$,
   (b) $\prec z_i$ for $i \in \{1, 2, \ldots, s\}$,
   (c) $\cap z_i$ for $i \in \left\{ s+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil \right\}$,

(d) $\succ z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta \right\}$,

(e) $\succ x_1$,

(f) $\cap u_3$,

3. u_3,

(a) $\succ x_1$,

(b) $\cap z_i$ for $i \in \{1, 2, \ldots s+1\}$,

(c) $\succ z_i$ for $i \in \{s+2, \ldots, \beta\}$,

(d) $\cap y_1$,

4. $y_1 \succ z_1 \succ z_2 \succ \cdots \succ z_\beta \succ x_1$.

Case 2. Two of the positive weight arcs are adjacent on C and the third is not. Here, C can be written as

$$z_{1r}, z_{1\ell}, z_{2r}, z_{2\ell}, \ldots, z_{ar}, z_{a\ell}, x_{1r}, u_{1\ell}, u_{1r}, y_{1\ell}, z_{a+1_r}, z_{a+1_\ell}, z_{a+2_r}, z_{a+2_\ell}, \cdots, z_{\beta_r}, z_{\beta_\ell}, x_{2r},$$

$$u_{2\ell}, u_{2r}, u_{3\ell}, u_{3r}, y_{2\ell}, z_{1r}.$$

This cycle is depicted in Figure 4.4. Each x, y, and z is distinct except possibly x_1, y_1, y_{b+2}, and x_1, but by Corollary 2.2.8 they are also distinct.

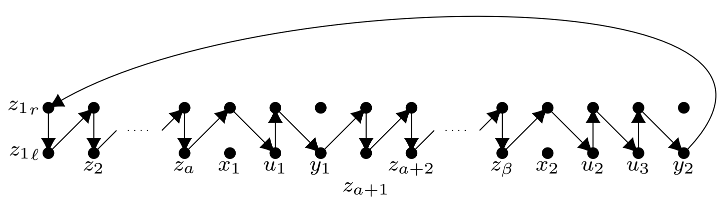


Figure 4.4: Cycle in $D_p^q(P)$ with two adjacent weight q arcs and one not adjacent

Cycle C produces different structures based on the value of a. We have $a \in \{1, 2, \beta-1\}$. Now, $a = \beta - a'$ gives the vertical reflection of $a = a'$, so we will only consider $a \in \{\lceil \beta/2 \rceil, \ldots, \beta-1\}$. We will first consider relationships in terms of a

and then analyze them based on the value of a. Let $b := \beta - a$. Cycle C gives $y_2 \succ z_1 \succ z_2 \succ \cdots \succ z_a \succ x_1 \cap u_1 \cap y_1 \succ z_{a+1} \succ \cdots \succ z_\beta \succ x_2 \cap u_2 \cap u_3 \cap y_2$ (relationships ii(3)g, ii(3)h, ii(4)i, ii(4)j, ii(5)h, ii9, and ii10 of Definition 4.1.1).

If $u_1 \succ u_2$, then replacing $u_{1\ell} \to u_{2r}$ in C with the arc $(u_{1\ell}, u_{2r})$ creates a cycle with weight $q - p(a) < 0$. If $u_1 \prec u_2$, then replacing $u_{2\ell} \to u_{1r}$ in C with the arc $(u_{2\ell}, u_{1r})$ creates a cycle with weight $-p(\beta - a) < 0$. If $u_1 \cap u_2$ then replacing $u_{1r} \to u_{2\ell}$ in C with the arc $(u_{1r}, u_{2\ell})$ creates a cycle with weight $3q - p(a) > 0$, and replacing $u_{2r} \to u_{1\ell}$ in C with the arc $(u_{2r}, u_{1\ell})$ creates a cycle with weight $2q - p(\beta - a) > 0$.

If $u_1 \succ u_3$, then replacing $u_{1\ell} \to u_{3r}$ in C with the arc $(u_{1\ell}, u_{3r})$ creates a cycle with weight $-p(a) < 0$. If $u_1 \prec u_3$, then replacing $u_{3\ell} \to u_{1r}$ in C with the arc $(u_{3\ell}, u_{1r})$ creates a cycle with weight $q - p(\beta - a)$ which is positive is $a \geq \left\lceil \frac{2q}{p} \right\rceil$. If $u_1 \cap u_3$ then replacing $u_{1r} \to u_{3\ell}$ in C with the arc $(u_{1r}, u_{3\ell})$ creates a cycle with weight $2q - p(a)$ which is positive when $a < \left\lceil \frac{2q}{p} \right\rceil$, and replacing $u_{3r} \to u_{1\ell}$ in C with the arc $(u_{3r}, u_{1\ell})$ creates a cycle with weight $3q - p(\beta - a) > 0$.

Thus, $\boldsymbol{u_1 \cap u_2}$, and $\boldsymbol{u_1 \prec u_3}$ if $\boldsymbol{a \geq \left\lceil \frac{2q}{p} \right\rceil}$ and $\boldsymbol{u_1 \cap u_3}$ if $\boldsymbol{a < \left\lceil \frac{2q}{p} \right\rceil}$ (relationships ii1 and ii2 of Definition 4.1.1).

Let $i \in \{1, 2, \ldots, a\}$. If $u_1 \succ z_i$, then replacing $u_{1\ell} \to z_{ir}$ in C with the arc $(u_{1\ell}, z_{ir})$ creates a cycle with weight $-p(a - (i-1)) < 0$. If $u_1 \succ y_2$, then replacing $u_{1\ell} \to y_{2\ell}$ in C with the path $u_{1\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $-p(a+1) < 0$. If $u_1 \prec z_i$, then replacing $z_{i\ell} \to u_{1r}$ in C with the arc $(z_{i\ell}, u_{1r})$ creates a cycle with weight $2q - p(b+i)$ which is positive for $i \in \left\{1, 2, \ldots a - \left\lceil \frac{q}{p} \right\rceil \right\}$. If $u_1 \prec y_2$, then replacing $y_{2\ell} \to u_{1r}$ in C with the arc $(z_{2\ell}, u_{1r})$ creates a cycle with weight $2q - p(b) > 0$.

If $u_1 \cap z_i$, then replacing $u_{1r} \to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $q - p(a - i)$ which is positive for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}$, and replacing $z_{ir} \to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $3q - p(b + (i-1)) > 0$. If $u_1 \cap y_2$, then replacing $u_{1r} \to y_{2\ell}$ in C with the arc $(u_{1r}, y_{2\ell})$ creates a cycle with weight $q - p(a) < 0$.

Thus, $\boldsymbol{u_1 \prec y_2}$, $\boldsymbol{u_1 \prec z_i}$ for $\boldsymbol{i \in \left\{1, 2, \ldots, a - \left\lceil \frac{q}{p} \right\rceil \right\}}$ and $\boldsymbol{u_1 \cap z_i}$ for $\boldsymbol{i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}}$ (relationships ii3(a)-(c) of Definition 4.1.1).

If $u_2 \succ z_i$, then replacing $u_{2\ell} \to z_{ir}$ in C with the arc $(u_{2\ell}, z_{ir})$ creates a cycle with weight $q-p(\beta-(i-1))$ which positive for $i\in\left\{\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,a\right\}$. If $u_2\succ y_2$, then replacing $u_{2\ell}\to y_{2\ell}$ in C with the path $u_{2\ell},y_{2r},y_{2\ell}$ creates a cycle with weight $q-p(\beta+1)<0$. If $u_2\succ x_1$, then replacing $u_{2\ell}\to x_{1r}$ in C with the arc $(u_{2\ell},x_{1r})$ creates a cycle with weight $q-p(b)$ which is positive when $a\geq\left\lceil\frac{2q}{p}\right\rceil$.

If $u_2\prec z_i$, then replacing $z_{i\ell}\to u_{2r}$ in C with the arc $(z_{i\ell},u_{2r})$ creates a cycle with weight $q-p(i)$ which is positive for $i\in\left\{1,2,\ldots\left\lceil\frac{q}{p}\right\rceil-1\right\}$. If $u_2\prec y_2$, then replacing $y_{2\ell}\to u_{2r}$ in C with the arc $(y_{2\ell},u_{2r})$ creates a cycle with weight $q>0$. If $u_2\prec x_1$, then replacing $x_{1r}\to u_{2r}$ in C with the path $x_{1r},x_{1\ell},u_{2r}$, creates a cycle with weight $q-p(a+1)<0$.

If $u_2\cap z_i$, then replacing $u_{2r}\to z_{i\ell}$ in C with the arc $(u_{2r},z_{i\ell})$ creates a cycle with weight $2q-p(\beta-i)$ which is positive for $i\in\left\{\left\lceil\frac{q}{p}\right\rceil,\ldots,a\right\}$, and replacing $z_{ir}\to u_{2\ell}$ in C with the arc $(z_{ir},u_{2\ell})$ creates a cycle with than $2q-p(i-1)$ which is positive for $i\in\left\{1,2,3,\ldots,\left\lceil\frac{2q}{p}\right\rceil\right\}$. If $u_2\cap y_2$, then replacing $u_{2r}\to y_{2\ell}$ in C with the arc $(u_{2r},y_{2\ell})$ creates a cycle with weight $2q-p(\beta)<0$. If $u_2\cap x_1$, then replacing $u_{2r}\to x_{1r}$ in C with the path $u_{2r},x_{1\ell},x_{1r}$ creates a cycle with weight $3q-p(b)>0$, and replacing $x_{1r}\to u_{2\ell}$ in C with the arc $(x_{1r},u_{2\ell})$ creates a cycle with than $2q-p(a)$ which is positive if $a<\left\lceil\frac{2q}{p}\right\rceil$.

Thus, $\boldsymbol{u_2\prec y_2}$, $\boldsymbol{u_2\prec z_i}$ for $\boldsymbol{i\in\left\{1,2,\ldots,\left\lceil\frac{q}{p}\right\rceil-1\right\}}$, $\boldsymbol{u_2\cap z_i}$ for $\boldsymbol{i\in\left\{\left\lceil\frac{q}{p}\right\rceil,\ldots,\left\lceil\frac{2q}{p}\right\rceil\right\}}$, $\boldsymbol{u_2\succ z_i}$ for $\boldsymbol{i\in\left\{\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,a\right\}}$ and $\boldsymbol{\begin{cases}u_2\cap x_1 & a<\left\lceil\frac{2q}{p}\right\rceil\\ u_2\succ x_1 & a\geq\left\lceil\frac{2q}{p}\right\rceil\end{cases}}$
(relationships ii4(a)-(e) of Definition 4.1.1).

If $u_3\succ z_i$ then replacing $u_{3\ell}\to z_{ir}$ in C with the arc $(u_{3\ell},z_{ir})$ creates a cycle with weight $2q-p(\beta-(i-1))$ which is positive for $i\in\{\lceil q/p\rceil+1,\ldots,a\}$. If $u_3\succ x_1$ then replacing $u_{3\ell}\to x_{1r}$ in C with the arc $(u_{3\ell},x_{1r})$ creates a cycle with weight $2q-p(b)>0$. By transitivity, $u_3\not\succ z_i$ and $u_3\not\succ x_1$.

If $u_3\cap z_i$, then replacing $z_{ir}\to u_{3\ell}$ in C with the arc $(z_{ir},u_{3\ell})$ creates a cycle with weight $q-p(i-1)$ which is positive for $i\in\{1,2,\ldots,\lceil q/p\rceil\}$, and replacing $u_{3r}\to z_{i\ell}$ in C with the arc $(u_{3r},z_{i\ell})$ creates a cycle with weight $3q-p(\beta-i)>0$. If $u_3\cap x_1$, then replacing $x_{1r}\to u_{3\ell}$ in C with the arc $(x_{1r},u_{3\ell})$ creates a cycle with weight $q-p(a)<0$.

Thus, $\boldsymbol{u_3 \cap z_i}$ for $\boldsymbol{i \in \left\{1, 2, \dots, \left\lceil \frac{q}{p} \right\rceil\right\}}$, $\boldsymbol{u_3 \succ z_i}$ for $\boldsymbol{i \in \left\{\left\lceil \frac{q}{p} \right\rceil + 1, \dots, a\right\}}$, and $\boldsymbol{u_3 \succ x_1}$ (relationships ii5(a)-(c) of Definition 4.1.1).

Let $j \in \{a+1, \dots, \beta\}$. If $u_1 \succ z_j$, then replacing $u_{1\ell} \to z_{j_r}$ in C with the arc $(u_{1\ell}, z_{j_r})$ creates a cycle with weight $2q - p(\beta - (j-1) + a)$ which is positive for $j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil + 1, \dots, \beta\right\}$. If $u_1 \succ x_2$, then replacing $u_{1\ell} \to x_{2r}$ in C with the arc $(u_{1\ell}, x_{2r})$ creates a cycle with weight $2q - p(a)$ which is positive if $a < \left\lceil \frac{2q}{p} \right\rceil$. By transitivity, $u_1 \not\prec z_j$ and $u_1 \not\prec x_2$.

If $u_1 \cap z_j$, then replacing $u_{1r} \to z_{j_\ell}$ in C with the arc (u_{1r}, z_{j_ℓ}) creates a cycle with weight $3q - p(\beta - j + a) > 0$, and replacing $z_{j_r} \to u_{1\ell}$ in C with the arc $(z_{j_r}, u_{1\ell})$ creates a cycle with weight $q - p(j - 1 - a)$ which is positive for $j \in \left\{a+1, a+2, \dots, a + \left\lceil \frac{q}{p} \right\rceil\right\}$. If $u_1 \cap x_2$, then replacing $u_{1r} \to x_{2r}$ in C with the path $u_{1r}, x_{2\ell}, x_{2r}$ creates a cycle with weight $4q - p(a) > 0$, and replacing $x_{2r} \to u_{1\ell}$ in C with the arc $(x_{2r}, u_{1\ell})$ creates a cycle with weight $q - p(b)$ which is positive for $a \geq \left\lceil \frac{2q}{p} \right\rceil$.

Thus, $\boldsymbol{u_1 \cap z_j}$ for $\boldsymbol{j \in \left\{a+1, a+2, \dots, a + \left\lceil \frac{q}{p} \right\rceil\right\}}$, $\boldsymbol{u_1 \succ z_j}$ for $\boldsymbol{j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil + 1, \dots, \beta\right\}}$, and $\boldsymbol{\begin{cases} u_1 \succ x_2 & a < \left\lceil \frac{2q}{p} \right\rceil \\ u_1 \cap x_2 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}}$ (relationships ii3(d)-(f) of Definition 4.1.1).

By transitivity $u_2 \not\succ z_j$ and $u_2 \not\succ y_1$. If $u_2 \prec z_j$, then replacing $z_{j_\ell} \to u_{2r}$ in C with the arc (z_{j_ℓ}, u_{2r}) creates a cycle with weight $2q - p(j)$ which positive for $j \in \left\{a+1, a+2, \dots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$. If $u_2 \prec y_1$, then replacing $y_{1\ell} \to u_{2r}$ in C with the arc $(y_{1\ell}, u_{2r})$ creates a cycle with weight $2q - p(a)$ which positive for $a < \left\lceil \frac{2q}{p} \right\rceil$.

If $u_2 \cap z_j$, then replacing $u_{2r} \to z_{j_\ell}$ in C with the arc (u_{2r}, z_{j_ℓ}) creates a cycle with weight $q - p(\beta - j)$ which is positive for $j \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \dots, \beta\right\}$, and replacing $z_{j_r} \to u_{2\ell}$ in C with the arc $(z_{j_r}, u_{2\ell})$ creates a cycle with weight $3q - p(\beta - j + a) > 0$. If $u_2 \cap y_1$, then replacing $u_{2r} \to y_{1\ell}$ in C with the arc $(u_{2r}, y_{1\ell})$ creates a cycle with weight $q - p(b)$ which is positive for $a \geq \left\lceil \frac{2q}{p} \right\rceil$, and replacing $y_{1\ell} \to u_{2\ell}$ in C with the path $y-1_\ell, y_{1r}, u_{2\ell}$ creates a cycle with weight $4q - p(a) > 0$.

Thus, $\boldsymbol{\begin{cases} u_2 \prec y_1 & a < \left\lceil \frac{2q}{p} \right\rceil \\ u_2 \cap y_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}}$, $\boldsymbol{u_2 \prec z_j}$ for $\boldsymbol{j \in \left\{a+1, a+2, \dots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}}$, and

$\boldsymbol{u_2 \cap z_j}$ **for** $\boldsymbol{j \in \left\{ \left\lceil \frac{2q}{p} \right\rceil, \ldots, \beta \right\}}$ (relationships ii4(f)-(h) of Definition 4.1.1).

If $u_3 \succ z_j$, then replacing $u_{3\ell} \to z_{j_r}$ in C with the arc $(u_{3\ell}, z_{j_r})$ creates a cycle with weight $q - p(\beta - (j-1))$ which is positive for $j \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta \right\}$. If $u_3 \succ y_1$, then replacing $u_{3\ell} \to y_{1\ell}$ in C with the path $u_{3\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $q - p(b+1)$ which is positive for $a > \left\lceil \frac{2q}{p} \right\rceil$. If $u_3 \succ x_2$ then replacing $u_{3\ell} \to x_{2r}$ in C with the arc $(u_{3\ell}, x_{2r})$ creates a cycle with weight $q > 0$.

If $u_3 \prec z_j$, then replacing $z_{j_\ell} \to u_{3r}$ in C with the arc (z_{j_ℓ}, u_{3r}) creates a cycle with weight $q - p(j) < 0$. If $u_3 \prec y_1$, then replacing $y_{1\ell} \to u_{3r}$ in C with the arc $(y_{1\ell}, u_{3r})$ creates a cycle with weight $q - p(a) < 0$. If $u_3 \prec x_2$, then replacing $x_{2r} \to u_{3r}$ in C with the path $x_{2r}, x_{2\ell}, u_{3r}$ creates a cycle with weight $q - p(\beta+1) < 0$.

If $u_3 \cap z_j$, then replacing $u_{3r} \to z_{j_\ell}$ in C with the arc (u_{3r}, z_{j_ℓ}) creates a cycle with weight $2q - p(\beta - j)$ which is positive for $j \in \left\{ \left\lceil \frac{q}{p} \right\rceil, \ldots, \beta \right\} = \{a+1, \ldots, \beta\}$, and replacing $z_{j_r} \to u_{3\ell}$ in C with the arc $(z_{j_r}, u_{3\ell})$ creates a cycle with weight $2q - p(j-1)$ which is positive for $j \in \left\{ a+1, a+2, \ldots \left\lceil \frac{2q}{p} \right\rceil \right\}$. If $u_3 \cap y_1$, then replacing $y_{1\ell} \to u_{3\ell}$ in C with the path $y_{1\ell}, y_{1r}, u_{3\ell}$ creates a cycle with weight $3q - p(a) > 0$, and replacing $u_{3r} \to y_{1\ell}$ in C with the arc $(u_{3r}, y_{1\ell})$ creates a cycle with weight $2q - p(b) > 0$. If $u_3 \cap x_2$, then replacing $x_{2r} \to u_{3\ell}$ in C with the arc $(x_{2r}, u_{3\ell})$ creates a cycle with weight $2q - p(\beta) \leq 0$.

Thus, $\boldsymbol{\begin{cases} u_3 \cap y_2 & a \leq \left\lceil \frac{2q}{p} \right\rceil \\ u_3 \cap \succ y_2 & a > \left\lceil \frac{2q}{p} \right\rceil \end{cases}}$**,** $\boldsymbol{u_3 \cap z_j}$ **for** $\boldsymbol{j \in \left\{ a+1, a+2, \ldots, \left\lceil \frac{2q}{p} \right\rceil \right\}}$**,** $\boldsymbol{u_3 \succ z_j}$ **for** $\boldsymbol{j \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta \right\}}$**, and** $\boldsymbol{u_3 \succ x_2}$ (relationships ii5(d)-(g) of Definition 4.1.1).

Lastly, we must consider the relationships between elements of the two chains. We start with the maximal element of the second chain.

Let $i \in \{1, 2, \ldots, a\}$. If $y_1 \succ z_i$, then replacing $y_{1\ell} \to z_{ir}$ in C with the arc $(y_{1\ell}, z_{ir})$ creates a cycle with weight $q - p(a - (i-1))$ which is positive for $i \in \left\{ a - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$. If $y_1 \succ y_2$, then replacing $y_{1\ell} \to y_{2\ell}$ in C with the path $y_{1\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q - p(a+1) < 0$. If $y_1 \succ x_1$, then replacing $y_{1\ell} \to x_{1r}$ in C with the arc $(y_{1\ell}, x_{1r})$ creates a cycle with weight $q > 0$.

If $y_1 \prec z_i$, then replacing $z_{i\ell} \to y_{1\ell}$ in C with the path $z_{i\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $2q - p(j+b+1)$ which is positive for $i \in \left\{ 1, 2, \ldots, a - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$. If $y_1 \prec y_2$

then replacing $y_{2\ell}\to y_{1\ell}$ in C with the path $y_{2\ell},y_{1r},y_{1\ell}$ creates a cycle with weight $2q-p(b+1)>0$. If $y_1\prec x_2$, then replacing $z_{a\ell}\to y_{1\ell}$ with the path $z_{a\ell},y_{1r},y_{1\ell}$ creates a cycle with weight $2q-p(\beta+1)<0$.

If $y_1\cap z_i$, then replacing $z_{ir}\to y_{1\ell}$ in C with the arc $(z_{ir},y_{1\ell})$ creates a cycle with weight $2q-p(i-1+b)$ which is positive for $i\in\left\{1,2,\ldots,a-\left\lceil\frac{q}{p}\right\rceil+1\right\}$, and replacing $y_{1\ell}\to z_{i\ell}$ in C with the path $y_{1\ell},y_{1r},z_{i\ell}$ creates a cycle with weight $2q-p(a-i)$ which is positive for $i\in\left\{a-\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,a\right\}$. If $y_1\cap y_2$, then replacing $y_{2\ell}\to y_{1\ell}$ in C with the path $y_{2\ell},y_{2r},y_{1\ell}$ creates a cycle with weight $3q-p(b)>0$, and replacing $y_{1\ell}\to y_{2\ell}$ in C with the path $y_{1\ell},y_{1r},y_{2\ell}$ creates a cycle with weight $2q-p(a)$ which is positive when $a<\left\lceil\frac{2q}{p}\right\rceil$. If $y_1\cap x_1$, then replacing $x_{1r}\to y_{1\ell}$ in C with the arc $(x_{1r},y_{1\ell})$ creates a cycle with weight $2q-p(\beta)<0$.

Thus, $\boldsymbol{\begin{cases} y_1\prec\cap y_2 & a<\left\lceil\frac{2q}{p}\right\rceil \\ y_1\prec y_2 & a\geq\left\lceil\frac{2q}{p}\right\rceil\end{cases}}$**,** $\boldsymbol{y_1\prec z_i}$ **for** $\boldsymbol{i\in\left\{1,2,\ldots,a-\left\lceil\frac{2q}{p}\right\rceil\right\}}$**,** $\boldsymbol{y_1\prec\cap z_i}$ **for** $\boldsymbol{i\in\left\{a-\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,a-\left\lceil\frac{q}{p}\right\rceil-1\right\}}$**,** $\boldsymbol{y_1\cap z_i}$ **for** $\boldsymbol{i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil,\ldots,a-\left\lceil\frac{q}{p}\right\rceil+1\right\}}$**,** $\boldsymbol{y_1\succ z_i}$ **for** $\boldsymbol{i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a\right\}}$**, and** $\boldsymbol{y_1\succ x_1}$ (relationships ii6(a)-(f) of Definition 4.1.1).

We will analyze the relationship of the remaining elements of the second chain to each element of the first, starting with the minimal element of the second chain, and then the middle elements.

If $x_2\succ z_i$, then replacing $x_{2r}\to z_{ir}$ in C with the path $x_{2r},x_{2\ell},z_{ir}$ creates a cycle with weight $q-p(\beta+1-(i-1))$ which is positive for $i\in\left\{\left\lceil\frac{2q}{p}\right\rceil+2,\ldots,a\right\}$. If $x_2\succ y_2$, then replacing $x_{2r}\to y_{2\ell}$ in C with the path $x_{2r},x_{2\ell},y_{2r},y_{2\ell}$ creates a cycle $q-p(\beta+2)<0$. If $x_2\succ x_1$, then replacing $x_{2r}\to x_{1r}$ in C with the path $x_{2r},x_{2\ell},x_{1r}$ creates a cycle with weight $q-p(b+1)$ which is positive for $a>\left\lceil\frac{2q}{p}\right\rceil$.

If $x_2\prec z_i$, then replacing $z_{i\ell}\to x_{2r}$ in C with the arc $(z_{i\ell},x_{2r})$ creates a cycle with weight $2q-p(i)$ which is positive for $i\in\left\{1,2,\ldots,\left\lceil\frac{2q}{p}\right\rceil-1\right\}$. If $x_2\prec y_2$, then replacing $y_{2\ell}\to x_{2r}$ in C with the arc $(y_{2\ell},x_{2r})$ creates a cycle with weight $2q>0$. If $x_2\prec x_1$, then replacing $x_{1r}\to x_{2r}$ in C with the path $x_{1r},x_{1\ell},x_{2r}$ creates a cycle with weight $2q-p(a+1)$ which is positive for $a<\left\lceil\frac{2q}{p}\right\rceil-1$.

If $x_2\cap z_i$, then replacing $z_{ir}\to x_{2r}$ in C with the path $z_{ir},x_{2\ell},x_{2r}$ creates a cycle with weight $3q-p(i-1)>0$, and replacing $x_{2r}\to z_{i\ell}$ in C with the arc $(x_{2r},z_{i\ell})$ creates a cycle with weight $q-p(\beta-i)$ which is positive for $i\in\left\{\left\lceil\frac{2q}{p}\right\rceil,\dots,a\right\}$. If $x_2\cap y_2$, then replacing $x_{2r}\to y_{2\ell}$ in C with the arc $(x_{2r},y_{2\ell})$ creates a cycle with weight $q-p(\beta)<0$. If $x_2\cap x_1$, then replacing $x_{2r}\to x_{1r}$ in C with the path $x_{2r},x_{1\ell},x_{1r}$ creates a cycle with weight $2q-p(b)>0$, and replacing $x_{1r}\to x_{2r}$ in C with the path $x_{1r},x_{2\ell},x_{2r}$ creates a cycle with weight $3q-p(a)>0$.

Thus, $\boldsymbol{x_2\prec y_2}$, $\boldsymbol{x_2\prec z_i}$ for $\boldsymbol{i\in\left\{1,2,\dots,\left\lceil\frac{2q}{p}\right\rceil-1\right\}}$, $\boldsymbol{x_2\cap z_i}$ for $\boldsymbol{i\in\left\{\left\lceil\frac{2q}{p}\right\rceil,\dots,\left\lceil\frac{2q}{p}\right\rceil+1\right\}}$, $\boldsymbol{x_2\cap\succ z_i}$ for $\boldsymbol{i\in\left\{\left\lceil\frac{2q}{p}\right\rceil+2,\dots,a\right\}}$, and

$$\begin{cases}\boldsymbol{x_2\prec\cap x_1} & \boldsymbol{a<\left\lceil\frac{2q}{p}\right\rceil-1}\\ \boldsymbol{x_2\cap x_1} & \boldsymbol{\left\lceil\frac{2q}{p}\right\rceil-1\le a\le\left\lceil\frac{2q}{p}\right\rceil}\\ \boldsymbol{x_2\succ x_1} & \boldsymbol{a>\left\lceil\frac{2q}{p}\right\rceil}\end{cases}$$ (relationships ii8(a)-(e) of Definition 4.1.1).

Let $j\in\{a+1,\dots\beta\}$. If $z_j\succ z_i$, then replacing $z_{j_\ell}\to z_{ir}$ in C with the arc (z_{j_ℓ},z_{ir}) creates a cycle with weight $q-p(j-(i-1))$ which is positive for $i\in\left\{j-\left\lceil\frac{q}{p}\right\rceil+2,\dots,a\right\}$. If $z_j\succ y_2$, then replacing $z_{j_\ell}\to y_{2\ell}$ in C with the path $z_{j_\ell},y_{2r},y_{2\ell}$ creates a cycle with weight $q-p(j+1)<0$. If $z_j\succ x_1$, then replacing $z_{j_\ell}\to x_{1r}$ in C with the arc (z_{j_ℓ},x_{1r}) creates a cycle with weight $q-p(j-a)$ which is positive for $j<a+\left\lceil\frac{q}{p}\right\rceil$.

If $z_j\prec z_i$, then replacing $z_{i\ell}\to z_{j_r}$ in C with the arc $(z_{i\ell},z_{j_r})$ creates a cycle with weight $2q-p(i+\beta-(j-1))$ which is positive for $i\in\left\{1,2,\dots,j-\left\lceil\frac{q}{p}\right\rceil-1\right\}$. If $z_j\prec y_2$, then replacing $y_{2\ell}\to z_{j_r}$ in C with the arc $(y_{2\ell},z_{j_r})$ creates a cycle with weight $2q-p(\beta-(j-1))$ which is positive for $j>\left\lceil\frac{q}{p}\right\rceil$ which is always true. If $z_j\prec x_1$, then replacing $x_{1r}\to z_{j_r}$ in C with the path $x_{1r},x_{1\ell},z_{j_r}$ creates a cycle with weight $2q-p(a+1+\beta-(j-1))$ which is positive when $j>a+\left\lceil\frac{q}{p}\right\rceil+1$.

If $z_j\cap z_i$, then replacing $z_{ir}\to z_{j_\ell}$ in C with the arc (z_{ir},z_{j_ℓ}) creates a cycle with weight $2q-p(i-1+\beta-j)$ which is positive for $i\in\left\{1,2,\dots,j-\left\lceil\frac{q}{p}\right\rceil+1\right\}$, and replacing $z_{j_r}\to z_{i\ell}$ in C with the arc $(z_{j_r},z_{i\ell})$ creates a cycle with weight $q-p(j-1-i)$ which is positive for $i\in\left\{j-\left\lceil\frac{q}{p}\right\rceil,\dots,a\right\}$. If $z_j\cap y_2$, then replacing $z_{j_r}\to y_{2\ell}$ in C with the arc $(z_{j_r},y_{2\ell})$ creates a cycle with weight $q-p(j-1)<0$. If $z_j\cap x_1$, then replacing $z_{j_r}\to x_{1r}$ in C with the path $z_{j_r},x_{1\ell},x_{1r}$ creates a cycle with weight $2q-p(j-1-a)>0$, and replacing $x_{1r}\to z_{j_\ell}$ in C with the arc (x_{1r},z_{j_ℓ}) creates a cycle with weight

$2q-p(a+\beta-j)$ which is positive for $j\geq a+\left\lceil\frac{q}{p}\right\rceil$.

Thus, for $\boldsymbol{j\in\{a+1,a+2,\ldots\beta\}}$**,** $\boldsymbol{z_j\prec y_2}$**,** $\boldsymbol{z_j\prec z_i}$ **for** $\boldsymbol{i\in\left\{1,2,\ldots,j-\left\lceil\frac{q}{p}\right\rceil-1\right\}}$**,** $\boldsymbol{z_j\cap z_i}$ **for** $\boldsymbol{i\in\left\{j-\left\lceil\frac{q}{p}\right\rceil,j-\left\lceil\frac{q}{p}\right\rceil+1\right\}}$**,** $\boldsymbol{z_j\succ z_i}$ **for** $\boldsymbol{i\in\left\{j-\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a\right\}}$**, and** $\boldsymbol{\begin{cases} z_j\succ x_1 & j<a+\left\lceil\frac{q}{p}\right\rceil \\ z_j\cap x_1 & a+\left\lceil\frac{q}{p}\right\rceil\leq j\leq a+\left\lceil\frac{q}{p}\right\rceil+1 \\ z_j\prec\cap x_1 & j>a+\left\lceil\frac{q}{p}\right\rceil+1 \end{cases}}$ (relationships ii7(a)-(e) of Definition 4.1.1).

Finally we will analyze the bold relationships based on the value of a. We first consider $a\geq\left\lceil\frac{2q}{p}\right\rceil$ and then $a<\left\lceil\frac{2q}{p}\right\rceil$.

Case 2.1. $a\geq\left\lceil\frac{2q}{p}\right\rceil$

We have the following relationships, which are marked with a (*) if they are impacted by the value of a:

1. $u_1\cap u_2$,

2. $u_1\prec u_3$ (*),

3. u_1,

   (a) $\prec y_2$,

   (b) $\prec z_i$ for $i\in\left\{1,2,\ldots,a-\left\lceil\frac{q}{p}\right\rceil\right\}$,

   (c) $\cap z_i$ for $i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil+1,\ldots,a\right\}$,

   (d) $\cap z_j$ for $j\in\{a+1,a+2,\ldots,\beta\}$ (*),

   (f) $\cap x_2$ (*),

4. u_2,

   (a) $\prec y_2$,

   (b) $\prec z_i$ for $i\in\left\{1,2,\ldots,\left\lceil\frac{q}{p}\right\rceil-1\right\}$,

   (c) $\cap z_i$ for $i\in\left\{\left\lceil\frac{q}{p}\right\rceil,\ldots,\left\lceil\frac{2q}{p}\right\rceil\right\}$,

(d) $\succ z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a \right\}$,

(e) $\succ x_1$ (*),

(f) $\cap y_1$ (*),

(h) $\cap z_j$ for $j \in \{a+1, a+2, \ldots, \beta\}$ (*),

5. u_3,

(a) $\cap z_i$ for $i \in \left\{ 1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil \right\}$,

(b) $\succ z_i$ for $i \in \left\{ \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}$,

(c) $\succ x_1$,

(d) $\begin{cases} \cap y_2 & a = \left\lceil \frac{2q}{p} \right\rceil \\ \cap \succ y_2 & a > \left\lceil \frac{2q}{p} \right\rceil \end{cases}$, (*)

(f) $\succ z_j$ for $j \in \{a+1, \ldots, \beta\}$ (*),

(g) $\succ x_2$,

6. y_1,

(a) $\prec y_2$ (*),

(b) $\prec z_i$ for $i \in \left\{ 1, 2, \ldots, a - \left\lceil \frac{2q}{p} \right\rceil \right\}$,

(c) $\prec \cap z_i$ for $i \in \left\{ a - \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(d) $\cap z_i$ for $i \in \left\{ a - \left\lceil \frac{q}{p} \right\rceil, \ldots, a - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(e) $\succ z_i$ for $i \in \left\{ a - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$,

(f) $\succ x_1$,

7. z_j for $j \in \{a+1, \ldots \beta\}$,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{ 1, 2, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(c) $\cap z_i$ for $i \in \left\{ j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(d) $\succ z_i$ for $i \in \left\{ j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$,

(e) $\succ x_1$ (*),

8. x_2,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{ 1, 2, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1 \right\}$,

(c) $\cap z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil, \ldots, \left\lceil \frac{2q}{p} \right\rceil + 1 \right\}$,

(d) $\cap \succ z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 2, \ldots, a \right\}$,

(e) $\begin{cases} \cap x_1 & a = \left\lceil \frac{2q}{p} \right\rceil \\ x_2 \succ x_1 & a > \left\lceil \frac{2q}{p} \right\rceil \end{cases}$ (*).

The relationships in the list above are the relationships of the left poset of Figure 4.1ii.

Case 2.2. $\left(\left\lceil \frac{3q}{2p} \right\rceil \leq \right) a < \left\lceil \frac{2q}{p} \right\rceil$

Again, relationships that are impacted by the value of a are marked with a (*). We have the following relationships:

1. $u_1 \cap u_2$,

2. $u_1 \cap u_3$ (*),

3. u_1,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{ 1, 2, \ldots, a - \left\lceil \frac{q}{p} \right\rceil \right\}$,

(c) $\cap z_i$ for $i \in \left\{ a - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}$,

(d) $\cap z_j$ for $j \in \left\{ a+1, a+2, \ldots, a + \left\lceil \frac{q}{p} \right\rceil \right\}$,

(e) $\succ z_j$ for $j \in \left\{ a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, \beta \right\}$

(f) $\succ x_2$ (*),

4. u_2,

   (a) $\prec y_2$,

   (b) $\prec z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

   (c) $\cap z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil, \ldots, a\right\}$ (*),

   (e) $\cap x_1$ (*),

   (f) $\prec y_1$ (*),

   (g) $\prec z_j$ for $j \in \left\{a+1, a+2, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$,

   (h) $\cap z_j$ for $j \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \ldots, \beta\right\}$,

5. u_3,

   (a) $\cap z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil\right\}$,

   (b) $\succ z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a\right\}$,

   (c) $\succ x_1$,

   (d) $\cap y_2$ (*),

   (e) $\cap z_j$ for $j \in \left\{a+1, a+2, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$,

   (f) $\succ z_j$ for $j \in \left\{\left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta\right\}$,

   (g) $\succ x_2$,

6. y_1,

   (a) $\prec \cap y_2$ (*),

   (c) $\prec \cap z_i$ for $i \in \left\{1, \ldots, a - \left\lceil \frac{q}{p} \right\rceil - 1\right\}$ (*),

   (d) $\cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil, \ldots, a - \left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

   (e) $\succ z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a\right\}$,

   (f) $\succ x_1$,

7. z_j for $j \in \{a+1, a+2, \ldots \beta\}$,

A. z_j for $j \in \left\{a+1, a+2, \dots a+\left\lceil \frac{q}{p} \right\rceil -1\right\}$,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \left\{1, 2, \dots, j-\left\lceil \frac{q}{p} \right\rceil -1\right\}$,

(c) $\cap z_i$ for $i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil, j-\left\lceil \frac{q}{p} \right\rceil +1\right\}$,

(d) $\succ z_i$ for $i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil +2, \dots, a\right\}$,

(e) $\succ x_1$ (*),

B. $z_{a+\left\lceil \frac{q}{p} \right\rceil}$,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \{1, 2, \dots, a-1\}$ (*),

(c) $\cap z_a$ (*),

(e) $\cap x_1$ (*),

C. $z_{a+\left\lceil \frac{q}{p} \right\rceil +1}$,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \{1, 2, \dots, a\}$, (*)

(e) $\cap x_1$ (*),

D. z_j for $j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil +2, \dots, \beta\right\}$,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \{1, 2, \dots, a\}$ (*),

(e) $\prec \cap x_1$ (*),

8. x_2,

(a) $\prec y_2$,

(b) $\prec z_i$ for $i \in \{1, 2, \dots, a\}$ (*),

(e) $\begin{cases} \prec \cap x_1 & a < \left\lceil \frac{2q}{p} \right\rceil -1 \\ \cap x_1 & a = \left\lceil \frac{2q}{p} \right\rceil -1 \end{cases}$ (*).

The relationship listed above are the relationships in the right poset of Figure 4.1ii.

Case 3. None of the three positive weight arcs are adjacent on C.

This cycle is depicted in Figure 4.5. We will again consider two cases based on the value of b. We will assume that $a \geq b \geq \beta - a - b$. Thus, $\left\lceil \frac{q}{p} \right\rceil \leq a \leq \beta - 2$, $1 \leq b \leq \left\lfloor \frac{\beta-1}{2} \right\rfloor$, and $1 \leq \beta - a - b \leq \left\lfloor \frac{q}{3} \right\rfloor$. The other combinations of a and b will produce vertical reflections of the structures produced using this convention. The cycle gives relationships ii(4)k, ii(4)l, ii(5)k, ii(5)l, ii(6)j, ii(6)k, ii10, ii11, and ii11 of Definition 4.1.1.

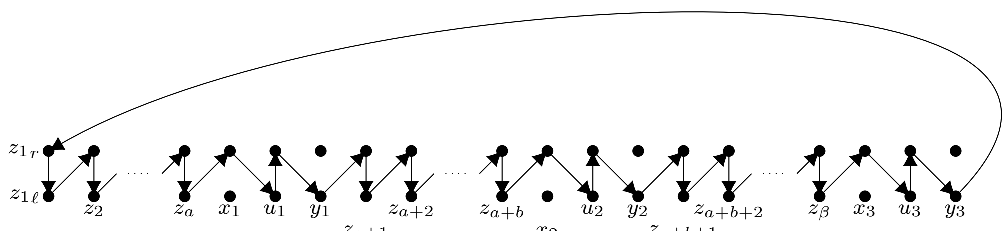


Figure 4.5: Cycle in $D_p^q(P)$ with three non-adjacent weight q arcs

All elements labeled in C are distinct except possibly x_1, y_1, x_2, y_2, x_3, and y_3. Since C uses x_{ir} for each x vertex and y_{j_ℓ} for each y vertex, the x vertices are distinct from each other and the y vertices are distinct from each other. By Corollary 2.2.8, x_1 is distinct from y_3 and y_1, x_2 is distinct from y_1 and y_2, and x_3 is distinct from y_2 and y_3. That leaves x_1 and y_2, x_2 and y_3, and x_3 and y_1. In the language of Lemma 2.2.7, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\beta_1 = a$, $\beta_2 = b$, and $\beta_3 = \beta - a - b$. Now, $1 + a + (\beta - a - b) = 1 + \beta_1 + \beta_3 \geq \left\lceil \frac{q}{p} \right\rceil$, so by Lemma 2.2.7(a)(ii), x_1 and y_2 are distinct. Also, $1 + a + b = 1 + \beta_1 + \beta_2 \geq \left\lceil \frac{q}{p} \right\rceil$, so x_2 and y_2 are distinct. Next, if $1 + b + (\beta - a - b) = 1 + \beta_2 + \beta_3 < \left\lceil \frac{q}{p} \right\rceil$ and $a = \beta_1 \geq \left\lceil \frac{2q}{p} \right\rceil$, then Lemma 2.2.7 does not exclude x_3 and y_1 from being the same element. Thus, if $a = \beta_1 \geq \left\lceil \frac{2q}{p} \right\rceil$, we must consider the case that $x_3 = y_1$ and the case that they are not the same element.

In the following analysis we will again disregard the weight $-\epsilon$ arcs when calculating cycle weights. Thus, a cycle with weight zero is actually a negative cycle once the $-\epsilon$ arcs are included. The following relationships are common to all minimal cycles with this structure.

First, we consider the relationships among the three u elements.

If $u_1 \prec u_2$, then replacing $u_{2\ell} \to u_{1r}$ in C with the arc $(u_{2\ell}, u_{1r})$ creates a cycle with no positive weight edges. If $u_1 \succ u_2$, then replacing $u_{1\ell} \to u_{2r}$ in C with the arc $(u_{1\ell}, u_{2r})$ creates a cycle with weight $q - p(a + \beta - a - b) = q - p(\beta - b) < 0$. If $u_1 \cap u_2$, then replacing $u_{1r} \to u_{2\ell}$ in C with the arc $(u_{1r}, u_{2\ell})$ creates a cycle with weight $3q - p(\beta - b) > 0$ and replacing $u_{2r} \to u_{1\ell}$ in C with the arc $(u_{2r}, u_{1\ell})$ creates a cycle with weight $2q - p(b) > 0$.

If $u_3 \prec u_2$, then replacing $u_{2\ell} \to u_{3r}$ in C with the arc $(u_{2\ell}, u_{3r})$ creates a cycle with weight $q - p(a + b) < 0$. If $u_3 \succ u_2$, then replacing $u_{3\ell} \to u_{2r}$ in C with the arc $(u_{3\ell}, u_{2r})$ creates a cycle with no positive weight edges. If $u_3 \cap u_2$, then replacing $u_{2r} \to u_{3\ell}$ in C with the arc $(u_{2r}, u_{3\ell})$ creates a cycle with weight $3q - p(a + b) > 0$ and replacing $u_{3r} \to u_{2\ell}$ in C with the arc $(u_{3r}, u_{2\ell})$ creates a cycle with weight $2q - p(\beta - a - b) > 0$.

Thus, $\boldsymbol{u_1 \cap u_2}$ and $\boldsymbol{u_2 \cap u_3}$ (relationships ii1 and ii2 of Definition 4.1.1).

If $u_3 \succ u_1$, then replacing $u_{3\ell} \to u_{1r}$ in C with the arc $(u_{3\ell}, u_{1r})$ creates a cycle with weight $q - p(\beta - a)$ which is positive if $a \geq \left\lceil \frac{2q}{p} \right\rceil$. If $u_3 \prec u_1$, then replacing $u_{1\ell} \to u_{3r}$ in C with the arc $(u_{1\ell}, u_{3r})$ creates a cycle with no positive weight edges. If $u_3 \cap u_1$, then replacing $u_{1r} \to u_{3\ell}$ in C with the arc $(u_{1r}, u_{3\ell})$ creates a cycle with weight $2q - p(a)$ which is positive if $a < \left\lceil \frac{2q}{p} \right\rceil$, and replacing $u_{3r} \to u_{1\ell}$ in C with the arc $(u_{3r}, u_{1\ell})$ creates a cycle with weight $3q - p(\beta - a) > 0$.

Thus, $\boldsymbol{u_3 \succ u_1}$ if $\boldsymbol{a \geq \left\lceil \frac{2q}{p} \right\rceil}$, and $\boldsymbol{u_3 \cap u_1}$ if $\boldsymbol{a < \left\lceil \frac{2q}{p} \right\rceil}$ (relationship ii3 of Definition 4.1.1).

Next, we consider the relationships between the u elements and the elements of the chains.

If $u_1 \succ y_3$, then there is a transitivity issue since $y_3 \succ x_1$ and $x_1 \cap u_1$. If $u_1 \cap y_3$ then replacing $u_{1r} \to y_{3\ell}$ in C with the arc $(u_{1r}, y_{3\ell})$ creates a cycle with weight $q - p(a) \leq q - p \left\lceil \frac{q}{p} \right\rceil < 0$. If $u_1 \prec y_3$, then replacing $y_{3\ell} \to u_{1r}$ in C with the arc $(y_{3\ell}, u_{1r})$ creates a cycle with weight $2q - p(\beta - a) > 2q - p \left\lfloor \frac{2q}{p} \right\rfloor > 0$.

Let $i \in \{1, 2, \ldots, a\}$. If $u_1 \succ z_i$, then replacing $u_{1\ell} \to z_{ir}$ in C with the arc $(u_{1\ell}, z_{ir})$ creates a cycle with weight -$p(a - (i - 1)) < 0$. If $u_1 \prec z_i$, then replacing $z_{i\ell} \to u_{1r}$

in C with the arc $(z_{i\ell}, u_{1r})$ creates a cycle with weight $2q-p(\beta-a+i)$ which is positive for $i\in\left\{1,2,\ldots,a-\left\lceil\frac{q}{p}\right\rceil\right\}$. If $u_1\cap z_i$, then replacing $u_{1r}\to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $q-p(a-i)$ which is positive for $i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil+1,\ldots,a\right\}$, and replacing $z_{ir}\to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $3q-p(\beta-a+i-1)>0$.

Thus, $\boldsymbol{u_1\prec y_3}$, $\boldsymbol{u_1\prec z_i}$ for $\boldsymbol{i\in\left\{1,2,\ldots,a-\left\lceil\frac{q}{p}\right\rceil\right\}}$, and $\boldsymbol{u_1\cap z_i}$ for $\boldsymbol{i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil+1,\ldots,a\right\}}$ (relationships ii4(a)-(c) of Definition 4.1.1).

If $u_2\succ y_3$, then replacing $u_{2\ell}\to y_{3\ell}$ in C with the path $u_{2\ell}, y_{3r}, y_{3\ell}$ creates a cycle with weight $q-p(a+b+1)<0$. If $u_2\cap y_3$, then replacing $u_{2r}\to y_{3\ell}$ in C with the arc $(u_{2r}, y_{3\ell})$ creates a cycle with weight $2q-p(a+b)<0$. If $u_2\prec y_3$, then replacing $y_{3\ell}\to u_{2r}$ in C with the arc $(y_{3\ell}, u_{2r})$ creates a cycle with weight $q-p(\beta-a-b)> q-p\left\lfloor\frac{q}{p}\right\rfloor>0$.

Let $i\in\{1,2,\ldots,a\}$. If $u_2\succ z_i$, then replacing $u_{2\ell}\to z_{ir}$ in C with the arc $(u_{2\ell}, z_{ir})$ creates a cycle with weight $q-p(a+b-(i-1))$ which is positive for $i\in\left\{a+b-\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a\right\}$. If $u_2\prec z_i$, then replacing $z_{i\ell}\to u_{2r}$ in C with the arc $(z_{i\ell}, u_{2r})$ creates a cycle with weight $q-p(i+\beta-a-b)$ which is positive for $i\in\left\{1,2,\ldots,a+b-\left\lceil\frac{2q}{p}\right\rceil\right\}$. If $u_2\cap z_i$, then replacing $u_{2r}\to z_{i\ell}$ in C with the arc $(u_{2r}, z_{i\ell})$ creates a cycle with weight $2q-p(a+b-i)$ which is positive for $i\in\left\{a+b-\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,a\right\}$, and replacing $z_{ir}\to u_{2\ell}$ in C with the arc $(z_{ir}, u_{2\ell})$ creates a cycle with weight $2q-p(i-1+\beta-a-b)$ which is positive for $i\in\left\{1,2,\ldots,a+b-\left\lceil\frac{q}{p}\right\rceil+1\right\}$.

If $u_2\succ x_1$, then replacing $u_{2\ell}\to x_{1r}$ in C with the arc $u_{2\ell}, x_{1r}$ creates a cycle with weight $q-p(b)$, which is positive if $b<\left\lceil\frac{q}{p}\right\rceil$. If $u_2\prec x_1$, then replacing $x_{1r}\to u_{2r}$ in C with the path $x_{1r}, x_{1\ell}, u_{2r}$ creates a cycle with weight $q-p(\beta-b+1)<0$. If $u_2\cap x_1$, then replacing $u_{2r}\to x_{1r}$ in C with the path $u_{2r}, x_{1\ell}, x_{1r}$ creates a cycle with weight $3q-p(b)>0$, and replacing $x_{1r}\to u_{2\ell}$ in C with the arc $(x_{1r}, u_{2\ell})$ creates a cycle with weight $2q-p(\beta-b)$ which is positive when $b\geq\left\lceil\frac{q}{p}\right\rceil$.

Thus, $\boldsymbol{u_2\prec y_3}$, $\boldsymbol{u_2\prec z_i}$ for $\boldsymbol{i\in\left\{1,2,\ldots,a+b-\left\lceil\frac{2q}{p}\right\rceil\right\}}$, $\boldsymbol{u_2\cap z_i}$ for $\boldsymbol{i\in\left\{a+b-\left\lceil\frac{2q}{p}\right\rceil+1,\ldots,a+b-\left\lceil\frac{q}{p}\right\rceil+1\right\}}$, $\boldsymbol{u_2\succ z_i}$ for $\boldsymbol{i\in\left\{a+b-\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a\right\}}$,

and $\begin{cases} \boldsymbol{u_2 \succ x_1} & \boldsymbol{b < \left\lceil \frac{q}{p} \right\rceil} \\ \boldsymbol{u_2 \cap x_1} & \boldsymbol{b \geq \left\lceil \frac{q}{p} \right\rceil} \end{cases}$ (relationships ii5(a)-(e) of Definition 4.1.1).

Let $i \in \{1, 2, \ldots, a\}$. If $u_3 \succ z_i$, then replacing $u_{3\ell} \to z_{ir}$ in C with the arc $(u_{3\ell}, z_{ir})$ creates a cycle with weight $2q - p(\beta - (i-1))$ which is positive for $i \in \left\{ \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}$. If $u_3 \prec z_i$, then replacing $z_{i\ell} \to u_{3r}$ in C with the arc $(z_{i\ell}, u_{3r})$ creates a cycle with weight -$p(i) < 0$. If $u_3 \cap z_i$, then replacing $u_{3r} \to z_{i\ell}$ in C with the arc $(u_{3r}, z_{i\ell})$ creates a cycle with weight $3q - p(\beta - i) > 0$, and replacing $z_{ir} \to u_{3\ell}$ in C with the arc $(z_{ir}, u_{3\ell})$ creates a cycle with weight $q - p(i-1)$ which is positive for $i \in \left\{ 1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil \right\}$.

If $u_3 \succ x_1$, then replacing $u_{3\ell} \to x_{1r}$ in C with the arc $u_{3\ell}, x_{1r}$ creates a cycle with weight $2q - p(\beta - a) > 0$. If $u_3 \prec x_1$, then replacing $x_{1r} \to u_{3r}$ in C with the path $x_{1r}, x_{1\ell}, u_{3r}$ creates a cycle with weight -$p(a+1) < 0$. If $u_3 \cap x_1$, then replacing $x_{1r} \to u_{3\ell}$ in C with the arc $(x_{1r}, u_{3\ell})$ creates a cycle with weight $q - p(a) < 0$.

Thus, $\boldsymbol{u_3 \cap z_i}$ **for** $\boldsymbol{i \in \left\{ 1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil \right\}}$**,** $\boldsymbol{u_3 \succ z_i}$ **for** $\boldsymbol{i \in \left\{ \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}}$**, and** $\boldsymbol{u_3 \succ x_1}$ (relationships ii6(a)-(c) of Definition 4.1.1).

Let $i \in \{a+1, \ldots, a+b\}$. If $u_1 \prec z_i$ or $u_1 \prec x_2$, then there is a transitivity issue since $u_1 \cap y_1 \succ z_i \succ x_2$. If $u_1 \succ z_i$, then replacing $u_{1\ell} \to z_{ir}$ in C with the arc $(u_{1\ell}, z_{ir})$ creates a cycle with weight $2q - p(\beta - (i-1) + a)$ which is positive for $i \in \left\{ a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a+b \right\}$. If $u_1 \cap z_i$, then replacing $u_{1r} \to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $3q - p(\beta - i + a) > 0$, and replacing $z_{ir} \to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $q - p(i - 1 - a) > 0$ which is positive for $i \in \left\{ 1, 2, \ldots, a + \left\lceil \frac{q}{p} \right\rceil \right\}$.

If $u_1 \succ x_2$, then replacing $u_{1\ell} \to x_{2r}$ in C with the arc $(u_{1\ell}, x_{2r})$ creates a cycle with weight $2q - p(\beta - b)$ which is positive when $b \geq \left\lceil \frac{q}{p} \right\rceil$. If $u_1 \cap x_2$, then replacing $u_{1r} \to x_{2r}$ in C with the path $u_{1r}, x_{2\ell}, x_{2r}$ creates a cycle with weight $4q - p(\beta - b) > 0$, and replacing $x_{2r} \to u_{1\ell}$ in C with the arc $(x_{2r}, u_{1\ell})$ creates a cycle with weight $q - p(b)$ which is positive when $b < \left\lceil \frac{q}{p} \right\rceil$.

Thus, $\boldsymbol{u_1 \cap z_i}$ **for** $\boldsymbol{i \in \left\{ a+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil \right\}}$**,** $\boldsymbol{u_1 \succ z_i}$ **for** $\boldsymbol{i \in \left\{ a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a+b \right\}}$**, and** $\begin{cases} \boldsymbol{u_1 \cap x_2} & \boldsymbol{b < \left\lceil \frac{q}{p} \right\rceil} \\ \boldsymbol{u_1 \succ x_2} & \boldsymbol{b \geq \left\lceil \frac{q}{p} \right\rceil} \end{cases}$ (relationships ii4(d)-(f) of Definition 4.1.1).

Let $i \in \{a+1, \ldots a+b\}$. If $u_2 \succ y_1$ or $u_2 \succ z_i$, then there is a transitivity issue since

$y_1 \succ z_i \succ x_2 \cap u_2$. If $u_2 \prec y_1$, then replacing $y_{1\ell} \to u_{2r}$ in C with the arc $(y_{1\ell}, u_{2r})$ creates a cycle with weight $2q-p(\beta-b)$, which is positive if $b \geq \left\lceil \frac{q}{p} \right\rceil$. If $u_2 \cap y_1$, then replacing $u_{2r} \to y_{1\ell}$ in C with the arc $(u_{2r}, y_{1\ell})$ creates a cycle with weight $q-p(b)$ which is positive if $b < \left\lceil \frac{q}{p} \right\rceil$, and replacing $y_{1\ell} \to u_{2\ell}$ in C with the path $y_{1\ell}, y_{1r}, u_{2\ell}$ creates a cycle with weight $4q-p(\beta-b)>0$.

If $u_2 \prec z_i$, then replacing $z_{i\ell} \to u_{2r}$ in C with the arc $(z_{i\ell}, u_{2r})$ creates a cycle with weight $2q-p(i+\beta-a-b)$ which is positive for $i \in \left\{a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil\right\}$. If $u_2 \cap z_i$, then replacing $u_{2r} \to z_{i\ell}$ in C with the arc $(u_{2r}, z_{i\ell})$ creates a cycle with weight $q-p(a+b-i)>0$ which is positive for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+1, \ldots, a+b\right\}$, and replacing $z_{ir} \to u_{2\ell}$ in C with the arc $(z_{ir}, u_{2\ell})$ creates a cycle with weight $3q-p(i-1+\beta-a-b)>0$.

Thus, $\boldsymbol{\begin{cases} u_2 \cap y_1 & b < \left\lceil \frac{q}{p} \right\rceil \\ u_2 \prec y_1 & b \geq \left\lceil \frac{q}{p} \right\rceil \end{cases}}$**,** $\boldsymbol{u_2 \prec z_i}$ **for** $\boldsymbol{i \in \left\{a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil\right\}}$**, and** $\boldsymbol{u_2 \cap z_i}$ **for** $\boldsymbol{i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+1, \ldots, a+b\right\}}$ (relationships ii5(f)-(h) of Definition 4.1.1).

If $u_3 \succ y_1$, then replacing $u_{3\ell} \to y_{1\ell}$ in C with the path $(u_{3\ell}, y_{1\ell}, y_{1r})$ creates a cycle of weight $q-p(\beta-a+1)$ which is positive if $a > \left\lceil \frac{2q}{p} \right\rceil$. If $u_3 \prec y_1$, then replacing $y_{1\ell} \to u_{3r}$ in C with the arc $(y_{1\ell}, u_{3r})$ creates a cycle with weight $q-p(a)<0$. If $y_1 \cap u_3$, then replacing $u_{3r} \to y_{1\ell}$ in C with the arc $(u_{3r}, y_{1\ell})$ creates a cycle with weight $2q-p(\beta-a)>0$, and replacing $y_{1\ell} \to u_{3\ell}$ in C with the path $y_{1\ell}, y_{1r}, u_{3\ell})$ creates a cycle with weight $3q-p(a)>0$.

Let $i \in \{a+1, \ldots, a+b\}$. If $u_3 \prec z_i$, then replacing $z_{i\ell} \to u_{3r}$ in C with the arc $(z_{i\ell}, u_{3r})$ creates a cycle with weight $q-p(i)$ which is positive if $i < \left\lceil \frac{q}{p} \right\rceil$, but $a \geq \left\lceil \frac{q}{p} \right\rceil$. If $u_3 \succ z_i$, then replacing $u_{3\ell} \to z_{ir}$ in C with the arc $(u_{3\ell}, z_{ir})$ creates a cycle with weight $q-p(\beta-(i-1))$ which is positive for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil+1, \ldots, a+b\right\}$. If $u_3 \cap z_i$, then replacing $z_{ir} \to u_{3\ell}$ in C with the arc $(z_{ir}, u_{3\ell})$ creates a cycle with weight $2q-p(i-1)$ which is positive for $i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$, and replacing $u_{3r} \to z_{i\ell}$ in C with the arc $(u_{3r}, z_{i\ell})$ creates a cycle with weight $2q-p(\beta-i)$ which is positive for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil, \ldots, a+b\right\} = \{a+1, \ldots, a+b\}$.

If $u_3 \prec x_2$, then replacing $x_{2r} \to u_{3r}$ in C with the path $x_{2r}, x_{2\ell}, u_{3r}$ creates a cycle with weight $q-p(a+b+1)<0$. If $u_3 \succ x_2$, then replacing $u_{3\ell} \to x_{2r}$ in C with the arc $(u_{3\ell}, x_{2r})$ creates a cycle with weight $q-p(\beta-a-b)>0$. If $u_3 \cap x_2$, then replacing $x_{2r} \to u_{3r}$ in C with the arc (x_{2r}, u_{3r}) creates a cycle with weight $q-p(a+b)<0$.

Thus, $\boldsymbol{\begin{cases} u_3 \cap\succ y_1 & a > \left\lceil \frac{2q}{p} \right\rceil \\ u_3 \cap y_1 & a \le \left\lceil \frac{2q}{p} \right\rceil \end{cases}}$**,** $\boldsymbol{u_3 \cap z_i}$ **for** $\boldsymbol{i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil \right\}}$**,** $\boldsymbol{u_3 \succ z_i}$ **for** $\boldsymbol{i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a+b \right\}}$**, and** $\boldsymbol{u_3 \succ x_2}$ (relationships ii6(d)-(g) of Definition 4.1.1).

If $u_1 \succ y_2$, then replacing $u_{1\ell} \to y_{2\ell}$ in C with the path $u_{1\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q-p(a)<0$. If $u_1 \prec y_2$, then replacing $y_{2\ell} \to u_{1r}$ in C with the arc $(y_{2\ell}, u_{1r})$ creates a cycle with weight $q-p(b)$ which is positive when $b < \left\lceil \frac{q}{p} \right\rceil$. If $u_1 \cap y_2$, then replacing $u_{1r} \to y_{2\ell}$ in C with the arc $(u_{1r}, y_{2\ell})$ creates a cycle with weight $2q-p(\beta-b)$ which is positive when $b \ge \left\lceil \frac{q}{p} \right\rceil$, and replacing $y_{2\ell} \to u_{1\ell}$ in C with the path $y_{2\ell}, y_{2r}, u_{1\ell}$ creates a cycle with weight $3q-p(b)>0$.

Let $i \in \{a+b+1, \ldots, \beta\}$. If $u_1 \succ z_i$, then replacing $u_{1\ell} \to z_{ir}$ in C with the arc $(u_{1\ell}, z_{ir})$ creates a cycle with weight $q-p(\beta-(i-1)+a)$ which is positive for $i \in \left\{a + \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, \beta \right\} = \emptyset$. If $u_1 \prec z_i$, then replacing $z_{i\ell} \to u_{1r}$ in C with the arc $(z_{i\ell}, u_{1r})$ creates a cycle with weight $q-p(i-a)$ which is positive for $i \in \left\{a+b+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$. If $u_1 \cap z_i$, then replacing $u_{1r} \to z_{i\ell}$ in C with the arc $(u_{1r}, z_{i\ell})$ creates a cycle with weight $2q-p(\beta-i+a)$ which is positive for $i \in \left\{a + \left\lceil \frac{q}{p} \right\rceil, \ldots, \beta \right\}$, and replacing $z_{ir} \to u_{1\ell}$ in C with the arc $(z_{ir}, u_{1\ell})$ creates a cycle with weight $2q-p(i-1-a)$ which is positive for $i \in \left\{a+b+1, \ldots, a + \left\lceil \frac{2q}{p} \right\rceil \right\} = \{a+b+1, \ldots, \beta\}$.

If $u_1 \prec x_3$, then replacing $x_{3r} \to u_{1r}$ in C with the path $x_{3r}, x_{3\ell}, u_{1r}$ creates a cycle with weight $q-p(\beta-a+1)$ which is positive for $a > \left\lceil \frac{2q}{p} \right\rceil$. If $u_1 \succ x_3$, then replacing $u_{1\ell} \to x_{3r}$ in C with the arc $u_{1\ell}, x_{3r}$ creates a cycle with weight $q-p(a)<0$. If $u_1 \cap x_3$, then replacing $u_{1r} \to x_{3r}$ in C with the path $u_{1r}, x_{3\ell}, x_{3r}$ creates a cycle with weight $3q-p(a)>0$, and replacing $x_{3r} \to u_{1\ell}$ in C with the arc $(x_{3r}, u_{1\ell})$ creates a cycle with weight $2q-p(\beta-a)>0$.

Thus, $\boldsymbol{\begin{cases} u_1 \prec y_2 & b < \left\lceil \frac{q}{p} \right\rceil \\ u_1 \cap y_2 & b \ge \left\lceil \frac{q}{p} \right\rceil \end{cases}}$**,** $\boldsymbol{u_1 \prec z_i}$ **for** $\boldsymbol{i \in \left\{a+b+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil - 1 \right\}}$**,** $\boldsymbol{u_1 \cap z_i}$

for $\boldsymbol{i \in \left\{a + \left\lceil \frac{q}{p} \right\rceil, \ldots, \beta\right\}}$**, and** $\boldsymbol{\begin{cases} u_1 \prec\cap x_3 & a > \left\lceil \frac{2q}{p} \right\rceil \\ u_1 \cap x_3 & a \leq \left\lceil \frac{2q}{p} \right\rceil \end{cases}}$ (relationships ii4(g)-(j) of Definition 4.1.1).

Let $i \in \{a+b+1, \ldots, \beta\}$. If $u_2 \prec z_i$ or $u_2 \prec x_3$, then there is a transitivity issue since $y_2 \succ z_i \succ x_3$ and $y_2 \cap u_2$. If $u_2 \succ z_i$, then replacing $u_{2\ell} \to z_{ir}$ in C with the arc $(u_{2\ell}, z_{ir})$ creates a cycle with weight $2q - p(\beta - (i-1) + a + b) < 0$ since $a + b \geq \left\lceil \frac{2q}{p} \right\rceil$. If $u_2 \cap z_i$, then replacing $u_{2r} \to z_{i\ell}$ in C with the arc $(u_{2r}, z_{i\ell})$ creates a cycle with weight $3q - p(\beta - i + a + b) > 0$, and replacing $z_{ir} \to u_{2\ell}$ in C with the arc $(z_{ir}, u_{2\ell})$ creates a cycle with weight $q - p(i - 1 - a - b)$ which is positive for $i \in \left\{a+b+1, \ldots, a+b+\left\lceil \frac{q}{p} \right\rceil + 1\right\} = \{a+b+1, \ldots, \beta\}$.

If $u_2 \succ x_3$, then replacing $u_{2\ell} \to x_{3r}$ in C with the arc $(u_{2\ell}, x_{3r})$ creates a cycle with weight $2q - p(a+b) < 0$. If $u_2 \cap x_3$, then replacing $u_{2r} \to x_{3r}$ in C with the path $u_{2r}, x_{3\ell}, x_{3r}$ creates a cycle with weight $4q - p(a+b) > 0$, and replacing $x_{3r} \to u_{2\ell}$ in C with the arc $(x_{3r}, u_{2\ell})$ creates a cycle with weight $q - p(\beta - a - b) > 0$.

Thus, $\boldsymbol{u_2 \cap z_i}$ **for** $\boldsymbol{i \in \{a+b+1, \ldots, \beta\}}$ **and** $\boldsymbol{u_2 \cap x_3}$ (relationships ii5(i)-(j) of Definition 4.1.1).

Let $i \in \{a+b+1, \ldots \beta\}$. If $u_3 \succ y_2$ or $u_3 \succ z_i$, then there is a transitivity issue since $y_2 \succ z_i \succ x_3$ and $x_3 \cap u_3$. If $u_3 \prec y_2$, then replacing $y_{2\ell} \to u_{3r}$ in C with the arc $(y_{2\ell}, u_{3r})$ creates a cycle with weight $2q - p(a+b) < 0$. If $u_3 \cap y_2$, then replacing $u_{3r} \to y_{2\ell}$ in C with the arc $(u_{3r}, y_{2\ell})$ creates a cycle with weight $q - p(\beta - a - b) > 0$, and replacing $y_{2\ell} \to u_{3\ell}$ in C with the path $y_{2\ell}, y_{2r}, u_{3\ell}$ creates a cycle with weight $4q - p(a+b) > 0$.

If $u_3 \prec z_i$, then replacing $z_{i\ell} \to u_{3r}$ in C with the arc $(z_{i\ell}, u_{3r})$ creates a cycle with weight $2q - p(i) < 0$ since $a + b \geq \left\lceil \frac{2q}{p} \right\rceil$. If $u_3 \cap z_i$, then replacing $u_{3r} \to z_{i\ell}$ in C with the arc $(u_{3r}, z_{i\ell})$ creates a cycle with weight $q - p(\beta - i) > 0$, and replacing $z_{ir} \to u_{3\ell}$ in C with the arc $(z_{ir}, u_{3\ell})$ creates a cycle with weight $3q - p(i-1) > 0$.

Thus, $\boldsymbol{u_3 \cap y_2}$ **and** $\boldsymbol{u_3 \cap z_i}$ **for** $\boldsymbol{i \in \{a+b+1, \ldots, \beta\}}$ (relationships ii6(h)-(i) of Definition 4.1.1).

Finally, we consider the relationships between elements of the chains. We start with the first and third chains.

If $y_2 \succ y_3$, then replacing $y_{2\ell} \to y_{3\ell}$ in C with the path $y_{2\ell}, y_{3r}, y_{3\ell}$ creates a path with weight $2q-p(a+b+1)<0$. If $y_2 \prec y_3$, then replacing $y_{3\ell} \to y_{2\ell}$ in C with the path $y_{3\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q-p(\beta-a-b+1)$ which is positive if $a+b>\left\lceil \frac{2q}{p} \right\rceil$. If $y_2 \cap y_3$, then replacing $y_{2\ell} \to y_{3\ell}$ in C with the path $y_{2\ell}, y_{2r}, y_{3\ell}$ creates a cycle with weight $3q-p(a+b)>0$, and replacing $y_{3\ell} \to y_{2\ell}$ in C with the path $y_{3\ell}, y_{3r}, y_{2\ell}$ creates a cycle with weight $2q-p(\beta-a-b)>0$.

Let $i \in \{1,2,\ldots,a\}$. If $y_2 \succ z_i$, then replacing $y_{2\ell} \to z_{ir}$ in C with the arc $(y_{2\ell}, z_{ir})$ creates a cycle with weight $2q-p(a+b-(i-1))$ which is positive for $i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil+2,\ldots,a\right\}$. If $y_2 \prec z_i$, then replacing $z_{i\ell} \to y_{2\ell}$ in C with the path $z_{i\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q-p(i+\beta-a-b+1)$ which is positive for $i \in \left\{1,2,\ldots,a+b-\left\lceil \frac{2q}{p} \right\rceil-1\right\}$. If $y_2 \cap z_i$, then replacing $z_{ir} \to y_{2\ell}$ in C with the arc $(z_{ir}, y_{2\ell})$ creates a cycle with weight $q-p(i-1+\beta-a-b)$ for $i \in \left\{1,2,\ldots,a+b-\left\lceil \frac{2q}{p} \right\rceil+1\right\}$, and replacing $y_{2\ell} \to z_{i\ell}$ in C with the path $y_{2\ell}, y_{2r}, z_{i\ell}$ creates a cycle with weight $3q-p(a+b-i)>0$.

If $y_2 \succ x_1$, then replacing $y_{2\ell} \to x_{1r}$ in C with the arc $(y_{2\ell}, x_{1r})$ creates a cycle with weight $2q-p(b)>0$. If $y_2 \prec x_1$, then replacing $x_{1r} \to y_{2\ell}$ in C with the path $x_{1r}, x_{1\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $q-p(\beta-b+2)<0$. If $y_2 \cap x_1$, then replacing $x_{1r} \to y_{2\ell}$ in C with the arc $(x_{1r}, y_{2\ell})$ creates a cycle with weight $q-p(\beta-b)<0$.

Thus, $\boldsymbol{\begin{cases} y_2 \prec\cap y_3 & a+b>\left\lceil \frac{2q}{p} \right\rceil \\ y_2 \cap y_3 & a+b=\left\lceil \frac{2q}{p} \right\rceil \end{cases}}$**,** $\boldsymbol{y_2 \prec\cap z_i}$ **for** $\boldsymbol{i \in \left\{1,2,\ldots,a+b-\left\lceil \frac{2q}{p} \right\rceil-1\right\}}$**,** $\boldsymbol{y_2 \cap z_i}$ **for** $\boldsymbol{i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil, a+b-\left\lceil \frac{2q}{p} \right\rceil+1\right\}}$**,** $\boldsymbol{y_2 \succ z_i}$ **for** $\boldsymbol{i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil+2,\ldots,a\right\}}$**,** $\boldsymbol{y_2 \succ x_1}$ (relationships ii(8)a(I)-(V) of Definition 4.1.1).

If $x_3 \succ y_3$, then replacing $z_{q\ell} \to y_{3\ell}$ in C with the path $z_{q\ell}, y_{3r}, y_{3\ell}$ creates a cycle with weight $2q-p(\beta+1)<0$. This uses transitivity since $z_q \succ x_3 \succ y_3$. If $x_3 \prec y_3$, then replacing $y_{3\ell} \to x_{3r}$ in C with the arc $(y_{3\ell}, x_{3r})$ creates a cycle with weight $q>0$. If $x_3 \cap y_3$, then replacing $x_{3r} \to y_{3\ell}$ in C with the arc $(x_{3r}, y_{3\ell})$ creates a cycle with weight $2q-p(\beta)<0$.

If $x_3 \succ z_i$, then replacing $x_{3r} \to z_{ir}$ in C with the path $x_{3r}, x_{3\ell}, z_{ir}$ creates a cycle with weight $2q-p(\beta+1-(i-1))<0$ which is positive for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil+2,\ldots,a\right\}$. If

$x_3 \prec z_i$, then replacing $z_{i\ell} \to x_{3r}$ in C with the arc $(z_{i\ell}, x_{3r})$ creates a cycle with weight $q-p(i)$ which is positive for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil - 1\right\}$. If $x_3 \cap z_i$, then replacing $x_{3r} \to z_{i\ell}$ in C with the arc $(x_{3r}, z_{i\ell})$ creates a cycle with weight $2q-p(q)$ which is positive for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil, \ldots, a\right\}$, and replacing $z_{ir} \to x_{3r}$ in C with the path $z_{ir}, x_{3\ell}, x_{3r}$ creates a cycles with weight $2q-p(i-1)$ which is positive for $i \in \left\{1, 2, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$.

If $x_3 \prec x_1$, then replacing $x_{1r} \to x_{3r}$ in C with the path $x_{1r}, x_{1\ell}, x_{3r}$ creates a cycle with weight $q-p(a+1)<0$. If $x_1 \cap x_3$, then replacing $x_{1r} \to x_{3r}$ in C with the path $x_{1r}, x_{3\ell}, x_{3r}$ creates a cycle with weight $2q-p(a)$ which is positive if $a < \left\lceil \frac{2q}{p} \right\rceil$, and replacing $x_{3r} \to x_{1r}$ in C with the path $x_{3r}, x_{1\ell}, x_{1r}$ creates a cycle with weight $3q-p(\beta-a)>0$. If $x_3 \succ x_1$, then replacing $x_{3r} \to x_{1r}$ in C with the path $x_{3r}, x_{3\ell}, x_{1r}$ creates a cycle with weight $2q-p(\beta-a+1)<0$ which is positive when $a > \left\lceil \frac{q}{p} \right\rceil$.

Thus, $\boldsymbol{x_3 \prec y_3}$, $\boldsymbol{x_3 \prec z_i}$ for $\boldsymbol{i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil - 1\right\}}$, $\boldsymbol{x_3 \cap z_i}$ for $\boldsymbol{i \in \left\{\left\lceil \frac{q}{p} \right\rceil, \left\lceil \frac{q}{p} \right\rceil + 1\right\}}$, $\boldsymbol{x_3 \cap\!\succ z_i}$ for $\boldsymbol{i \in \left\{\left\lceil \frac{q}{p} \right\rceil + 2, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}}$, $\boldsymbol{x_3 \succ z_i}$ for $\boldsymbol{i \in \left\{\left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a\right\}}$, and $\boldsymbol{\begin{cases} x_3 \cap x_1 & a = \left\lceil \frac{q}{p} \right\rceil \\ x_3 \cap\!\succ x_1 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \succ x_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}}$ (relationships ii(8)c(I)-(VI) of Definition 4.1.1).

Let $j \in \{a+b+1, \ldots, \beta\}$. If $z_j \cap y_3$, then replacing $z_{j_r} \to y_{3\ell}$ in C with the arc $(z_{j_r}, y_{3\ell})$ creates a cycle with weight $2q-p(j-1)<0$. If $z_j \succ y_3$, then replacing $z_{j_\ell} \to y_{3\ell}$ in C with the path $z_{j_\ell}, y_{3r}, y_{3\ell}$ creates a cycle with weight $2q-p(j+1)<0$. If $z_j \prec y_3$, then replacing $y_{3\ell} \to z_{j_r}$ in C with the arc $(y_{3\ell}, z_{j_r})$ creates a cycle with weight $q-p(\beta-(i-1))>0$.

If $z_j \succ z_i$, then replacing $z_{j_\ell} \to z_{ir}$ in C with the arc (z_{j_ℓ}, z_{ir}) creates a cycle with weight $2q-p(j-(i-1))$ which is positive for $i \in \left\{j - \left\lceil \frac{2q}{p} \right\rceil + 2, \ldots, a\right\}$. If $z_j \prec z_i$, then replacing $z_{i\ell} \to z_{j_r}$ in C with the arc $(z_{i\ell}, z_{j_r})$ creates a cycle with weight $q-p(i+\beta-(j-1))$ which is positive for $i \in \left\{1, 2, \ldots, j - \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$. If $z_j \cap z_i$, then replacing $z_{j_r} \to z_{i\ell}$ in C with the arc $(z_{j_r}, z_{i\ell})$ creates a cycle with weight $2q-p(j-1-i)$ which is positive for $i \in \left\{j - \left\lceil \frac{2q}{p} \right\rceil, \ldots, a\right\}$, and replacing $z_{ir} \to z_{j_\ell}$ in C with the arc (z_{ir}, z_{j_ℓ}) creates a cycle with weight $q-p(i-1+\beta-j)$ which is positive for

$i \in \left\{1, 2, \ldots, j - \left\lceil \frac{2q}{p} \right\rceil + 1\right\}$.

If $z_j \prec x_1$, then replacing $x_{1r} \to z_{j_r}$ in C with the path $x_{1r}, x_{1\ell}, z_{j_r}$ creates a cycle with weight $q - p(a+1+\beta-(i-1)) < 0$. If $z_j \cap x_1$, then replacing $x_{1r} \to z_{j_\ell}$ in C with the arc (x_{1r}, z_{j_ℓ}) creates a cycle with weight $q - p(a+\beta-i) < 0$. If $z_j \succ x_1$, then replacing $z_{j_\ell} \to x_{1r}$ in C with the arc (z_{j_ℓ}, x_{1r}) creates a cycle with weight $2q - p(i-a) > 0$.

Thus, for $j \in \{a+b+1, \ldots, \beta\}$, $z_j \prec y_3$, $z_j \prec z_i$ for $i \in \left\{1, 2, \ldots, j - \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$, $z_j \cap z_i$ for $i \in \left\{j - \left\lceil \frac{2q}{p} \right\rceil, j - \left\lceil \frac{2q}{p} \right\rceil + 1\right\}$, $z_j \succ z_i$ for $i \in \left\{j - \left\lceil \frac{2q}{p} \right\rceil + 2, \ldots, a\right\}$, and $z_j \succ x_1$ (relationships ii(8)b(I)-(V) of Definition 4.1.1).

Next, we determine the relationships chains two and three starting with y_2 and each element of the second chain.

If $y_2 \succ y_1$, then replacing $y_{2\ell} \to y_{1\ell}$ in C with the path $y_{2\ell}, y_{3r}, y_{3\ell}$ creates a path with weight $q - p(b+1)$ which is positive if $b < \left\lceil \frac{q}{p} \right\rceil - 1$. If $y_2 \prec y_1$, then replacing $y_{1\ell} \to y_{2\ell}$ in C with the path $y_{1\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $2q - p(\beta - b + 1)$ which is positive if $b > \left\lceil \frac{q}{p} \right\rceil$. If $y_2 \cap y_1$, then replacing $y_{2\ell} \to y_{1\ell}$ in C with the path $y_{2\ell}, y_{2r}, y_{1\ell}$ creates a cycle with weight $2q - p(b) > 0$, and replacing $y_{1\ell} \to y_{2\ell}$ in C with the path $y_{1\ell}, y_{1r}, y_{2\ell}$ creates a cycle with weight $3q - p(\beta - b) > 0$.

Let $i \in \{a+1, \ldots, a+b\}$. If $y_2 \succ z_i$, then replacing $y_{2\ell} \to z_{ir}$ in C with the arc $(y_{2\ell}, z_{ir})$ creates a cycle with weight $q - p(a+b-(i-1))$ which is positive for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b\right\}$. If $y_2 \prec z_i$, then replacing $z_{i\ell} \to y_{2\ell}$ in C with the path $z_{i\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $2q - p(i + \beta - a - b + 1)$ for $i \in \left\{a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil - 1\right\}$. If $y_2 \cap z_i$, then replacing $z_{ir} \to y_{2\ell}$ in C with the arc $(z_{ir}, y_{2\ell})$ creates a cycle with weight $2q - p(i - 1 + \beta - a - b)$ which is positive for $i \in \left\{a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil + 1\right\}$ and replacing $y_{2\ell} \to z_{i\ell}$ in C with the path $y_{2\ell}, y_{2r}, z_{i\ell}$ creates a cycle with weight $2q - p(a+b-i)$ which is positive for $i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a+b\right\} = \{a+1, \ldots, a+b\}$.

If $y_2 \succ x_2$, then replacing $y_{2\ell} \to x_{2r}$ in C with the arc $(y_{2\ell}, x_{2r})$ creates a cycle with weight $q > 0$. If $y_2 \prec x_2$, then replacing $z_{a+b\ell} \to y_{2\ell}$ in C with the path $z_{a+b\ell}, y_{2r}, y_{2\ell}$ creates a cycle with weight $2q - p(\beta+1) < 0$. If $y_2 \cap x_2$, then replacing $x_{2r} \to y_{2\ell}$ in C

with the arc $(x_{2r},y_{2\ell})$ creates a cycle with weight $2q-p(q)<0$.

Thus, $\boldsymbol{\begin{cases} y_2 \cap\!\succ y_1 & b<\left\lceil \frac{q}{p}\right\rceil -1 \\ y_2 \cap y_1 & \left\lceil \frac{q}{p}\right\rceil -1 \le b \le \left\lceil \frac{q}{p}\right\rceil \\ y_2 \prec\!\cap y_1 & \left\lceil \frac{q}{p}\right\rceil < b \end{cases}}$**,** $\boldsymbol{y_2 \prec\!\cap z_i}$ **for** $\boldsymbol{i \in \left\{a+1,\dots,a+b-\left\lceil \frac{q}{p}\right\rceil -1\right\}}$**,** $\boldsymbol{y_2 \cap z_i}$ **for** $\boldsymbol{i \in \left\{a+b-\left\lceil \frac{q}{p}\right\rceil, a+b-\left\lceil \frac{q}{p}\right\rceil +1\right\}}$**, and** $\boldsymbol{y_2 \succ z_i}$ **for** $\boldsymbol{i \in \left\{a+b-\left\lceil \frac{q}{p}\right\rceil +2,\dots,a+b\right\}}$**, and** $\boldsymbol{y_2 \succ x_2}$ (relationships ii(9)a (I)-(V) of Definition 4.1.1).

Next, we determine the relationships between x_3 and the elements of chain two.

If $x_3 \succ y_1$, then replacing $x_{3r} \to y_{1\ell}$ in C with the path $x_{3r}, x_{3\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $q-p(\beta-a+2)$ which is positive if $a \ge \left\lceil \frac{2q}{p}\right\rceil +2$. If $x_3 \cap y_1$, then replacing $x_{3r} \to y_{1\ell}$ in C with the arc $(x_{3r}, y_{1\ell})$ creates a cycle with weight $q-p(\beta-a)$ which is positive when $a \ge \left\lceil \frac{2q}{p}\right\rceil$, and replacing $y_{1\ell} \to x_{3r}$ in C with the path $y_{1\ell}, y_{1r}, x_{3\ell}, x_{3r}$ creates a cycle with weight $4q-p(a)>0$. If $x_3 \prec y_1$, then replacing $y_{1\ell} \to x_{3r}$ in C with the arc $(y_{1\ell}, x_{3r})$ creates a cycle with weight $2q-p(a)$ which is positive when $a < \left\lceil \frac{2q}{p}\right\rceil$.

Let $i \in \{a+1,\dots,a+b\}$. If $x_3 \succ z_i$, then replacing $x_{3r} \to z_{ir}$ in C with the path $x_{3r}, x_{3\ell}, z_{ir}$ creates a cycle with weight $q-p(\beta+1-(i-1))$ which is positive for $i \in \left\{\left\lceil \frac{2q}{p}\right\rceil +2,\dots,a+b\right\}$. If $x_3 \prec z_i$, then replacing $z_{i\ell} \to x_{3r}$ in C with the arc $(z_{i\ell}, x_{3r})$ creates a cycle with weight $2q-p(i)$ which is positive for $i \in \left\{a+1,\dots,\left\lceil \frac{2q}{p}\right\rceil -1\right\}$. If $x_3 \cap z_i$, then replacing $x_{3r} \to z_{i\ell}$ in C with the arc $(x_{3r}, z_{i\ell})$ creates a cycle with weight $q-p(q-i)$ which is positive for $i \in \left\{\left\lceil \frac{2q}{p}\right\rceil,\dots,a+b\right\}$, and replacing $\to x_{3r}$ in C with the path $z_{ir}, x_{3\ell}, x_{3r}$ creates a cycle with weight $3q-p(i-1)>0$.

If $x_3 \prec x_2$, then replacing $x_{2r} \to x_{3r}$ in C with the path $x_{1r}, x_{1\ell}, x_{3r}$ creates a cycle with weight $2q-p(a+b+1)<0$. If $x_3 \succ x_2$, then replacing $x_{3r} \to x_{2r}$ in C with the path $x_{3r}, x_{3\ell}, x_{2r}$ creates a cycle with weight $q-p(\beta-a-b+1)$ which is positive for $a+b > \left\lceil \frac{2q}{p}\right\rceil$. If $x_3 \cap x_2$, then replacing $x_{3r} \to x_{2r}$ in C with the path $x_{3r}, x_{2\ell}, x_{2r}$ creates a cycle with weight $2q-p(\beta-a-b)>0$, and replacing $x_{2r} \to x_{3r}$ in C with the path $x_{2r}, x_{3\ell}, x_{3r}$ creates a cycle with weight $3q-p(a+b)>0$.

$$\textbf{Thus, } \begin{cases} \boldsymbol{x_3 \prec y_1} & \boldsymbol{a < \left\lceil \frac{2q}{p} \right\rceil} \\ \boldsymbol{x_3 \cap y_1} & \boldsymbol{\left\lceil \frac{2q}{p} \right\rceil \leq a \leq \left\lceil \frac{2q}{p} \right\rceil + 1} \\ \boldsymbol{x_3 \cap\!\succ y_1} & \boldsymbol{a \geq \left\lceil \frac{2q}{p} \right\rceil + 2} \end{cases} \boldsymbol{,\ x_3 \prec z_i} \textbf{ for } \boldsymbol{i \in \left\{ a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1 \right\},}$$

$\boldsymbol{x_3 \cap z_i}$ **for** $\boldsymbol{i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil, \left\lceil \frac{2q}{p} \right\rceil + 1 \right\}}$**,** $\boldsymbol{x_3 \cap\!\succ z_i}$ **for** $\boldsymbol{i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil + 2, \ldots, a+b \right\}}$**, and** $\begin{cases} \boldsymbol{x_3 \cap x_2} & \boldsymbol{a+b = \left\lceil \frac{2q}{p} \right\rceil} \\ \boldsymbol{x_3 \cap\!\succ x_2} & \boldsymbol{a+b > \left\lceil \frac{2q}{p} \right\rceil} \end{cases}$ (relationships ii(9)c(I)-(V) of Definition 4.1.1).

Now, we determine the relationships between z elements of the second and third chains.

Let $j \in \{a+b+1, \ldots, \beta\}$ and let $i \in \{a+1, \ldots, a+b\}$. If $z_j \succ z_i$, then replacing $z_{j_\ell} \rightarrow z_{ir}$ in C with the arc (z_{j_ℓ}, z_{ir}) creates a cycle of weight $q - p(j - (i-1))$ which is positive for $i \in \left\{ j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b \right\}$. If $z_j \prec z_i$, then replacing $z_{i\ell} \rightarrow z_{j_r}$ in C with the arc $(z_{i\ell}, z_{j_r})$ creates a cycle of weight $2q - p(i + \beta - (j-1))$ which is positive for $i \in \left\{ a+1, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$. If $z_j \cap z_i$, then replacing $z_{j_r} \rightarrow z_{i\ell}$ in C with the arc $(z_{j_r}, z_{i\ell})$ creates a cycle of weight $q - p(j - 1 - i)))$ which is positive for $i \in \left\{ j - \left\lceil \frac{q}{p} \right\rceil, \ldots, a+b \right\}$, and replacing $z_{ir} \rightarrow z_{j_\ell}$ in C with the arc (z_{ir}, z_{j_ℓ}) creates a cycle of weight $2q - p(i - 1 + \beta - j)$ which is positive for $i \in \left\{ a+1, \ldots, j - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$.

Thus, for $\boldsymbol{j \in \{a+b+1, \ldots, \beta\}}$**,** $\boldsymbol{z_j \prec z_i}$ **for** $\boldsymbol{i \in \left\{ a+1, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}}$**,** $\boldsymbol{z_j \cap z_i}$ **for** $\boldsymbol{i \in \left\{ j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}}$**,** $\boldsymbol{z_j \succ z_i}$ **for** $\boldsymbol{i \in \left\{ j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b \right\}}$ (relationships ii(9)b(IV)-(VI) of Definition 4.1.1).

Lastly, for chains two and three, we determine the relationships between y_1 and the z elements of chain three and between x_2 and the z elements of chain three.

Let $i \in \{a+b+1, \ldots, \beta\}$. If $y_1 \succ z_i$, then replacing $y_{1\ell} \rightarrow z_{ir}$ in C with the arc $(y_{1\ell}, z_{ir})$ creates a cycle with weight $2q - p(\beta - (i-1) + a)$ which is positive for $i \in \left\{ a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, \beta \right\}$. If $y_1 \prec z_i$, then replacing $z_{i\ell} \rightarrow y_{1\ell}$ in C with the path $z_{i\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $q - p(i - a + 1)$ which is positive for $i \in \left\{ a+b+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil - 2 \right\}$. If $y_1 \cap z_i$, then replacing $z_{ir} \rightarrow y_{1\ell}$ in C with the arc $(z_{ir}, y_{1\ell})$ creates a cycle with weight $q - p(i - 1 - a)$ which is positive for $i \in \left\{ a+b+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil \right\}$, and replacing $y_{1\ell} \rightarrow z_{i\ell}$ in C with the path $y_{1\ell}, y_{1r}, z_{i\ell}$ creates a cycle with weight $3q - p(\beta - i + a) > 0$.

Thus, $\boldsymbol{y_1 \prec\cap z_i}$ for $\boldsymbol{i\in\left\{a+b+1,\ldots,a+\left\lceil\frac{q}{p}\right\rceil-2\right\}}$, $\boldsymbol{y_1\cap z_i}$ for $\boldsymbol{i\in\left\{a+\left\lceil\frac{q}{p}\right\rceil-1, a+\left\lceil\frac{q}{p}\right\rceil\right\}}$, and $\boldsymbol{y_1\succ z_i}$ for $\boldsymbol{i\in\left\{a+\left\lceil\frac{q}{p}\right\rceil+1,\ldots,\beta\right\}}$ (relationships ii(9)b(I)-(III) of Definition 4.1.1).

Let $i\in\{a+b+1,\ldots,\beta\}$. If $x_2\succ z_i$, then replacing $x_{2r}\to z_{ir}$ in C with the path $x_{2r}, x_{2\ell}, z_{ir}$ creates a cycle with weight $2q-p(\beta-i+a+b+1)<0$. If $x_2\prec z_i$, then replacing $z_{i\ell}\to x_{2r}$ in C with the arc $(z_{i\ell}, x_{2r})$ creates a cycle with weight $q-p(i-a-b)>0$. If $x_2\cap z_i$, then replacing $x_{2r}\to z_{i\ell}$ in C with the arc $(x_{2r}, z_{i\ell})$ creates a cycles with weight $2q-p(a+b+\beta-i)<0$.

Thus, $\boldsymbol{x_2\prec z_i}$ for $\boldsymbol{i\in\{a+b+1,\ldots,\beta\}}$ (relationship ii(9)b(VII) of Definition 4.1.1).

We now analyze the relationships between the first and second chains. First, consider the relationships between y_1 and the elements of chain one.

If $y_1\succ y_3$, then replacing $y_{1\ell}\to y_{3\ell}$ in C with the path $y_{1\ell}, y_{3r}, y_{3\ell}$ creates a path with weight $q-p(a+1)<0$. If $y_1\prec y_3$, then replacing $y_{3\ell}\to y_{1\ell}$ in C with the path $y_{3\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $2q-p(\beta-a+1)$ which is positive if $a>\left\lceil\frac{q}{p}\right\rceil$. If $y_1\cap y_3$, then replacing $y_{3\ell}\to y_{1\ell}$ in C with the path $y_{3\ell}, y_{3r}, y_{1\ell}$ creates a cycle with weight $3q-p(\beta-a)>0$ and replacing $y_{1\ell}\to y_{3\ell}$ in C with the path $y_{1\ell}, y_{1r}, y_{3\ell}$ creates a cycle with weight $2q-p(a)$ which is positive if $a<\left\lceil\frac{2q}{p}\right\rceil$.

Let $i\in\{1,2,\ldots,a\}$. If $y_1\succ z_i$, then replacing $y_{1\ell}\to z_{ir}$ in C with the arc $(y_{1\ell}, z_{ir})$ creates a cycle with weight $q-p(a-(i-1))$ which is positive for $i\in\left\{a-\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a\right\}$. If $y_1\prec z_i$, then replacing $z_{i\ell}\to y_{1\ell}$ in C with the path $z_{i\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $2q-p(i-1+\beta-a)$ which is positive for $i\in\left\{1,2,\ldots,a-\left\lceil\frac{q}{p}\right\rceil-1\right\}$. If $y_1\cap z_i$, then replacing $z_{ir}\to y_{1\ell}$ in C with the arc $(z_{ir}, y_{1\ell})$ creates a cycle with weight $2q-p(i-1+\beta-a)$ which is positive for $i\in\left\{a+1,\ldots,a-\left\lceil\frac{q}{p}\right\rceil+1\right\}$ and replacing $y_{1\ell}\to z_{i\ell}$ in C with the path $y_{1\ell}, y_{1r}, z_{i\ell}$ creates a cycle with weight $2q-p(a-i)$ which is positive for $i\in\left\{a-\left\lceil\frac{2q}{p}\right\rceil+1,\ldots a\right\}$.

If $y_1\prec x_1$, then replacing $z_{a\ell}\to y_{1\ell}$ in C with the path $z_{a\ell}, y_{1r}, y_{1\ell}$ creates a cycle with weight $2q-p(\beta+1)<0$. This uses transitivity since $z_a\succ x_1\succ y_1$. If $y_1\succ x_1$, then replacing $y_{1\ell}\to x_{1r}$ in C with the arc $(y_{1\ell}, x_{1r})$ creates a cycle with weight $q>0$. If $y_1\cap x_1$, then replacing $x_{1r}\to y_{1\ell}$ in C with the arc $(x_{1r}, y_{1\ell})$ creates a cycle with weight $2q-p(\beta)<0$.

Thus, $\begin{cases} \boldsymbol{y_1 \cap y_3} & \boldsymbol{a=\left\lceil \frac{q}{p} \right\rceil} \\ \boldsymbol{y_1 \prec\cap y_3} & \boldsymbol{\left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil} \\ \boldsymbol{y_1 \prec y_3} & \boldsymbol{a \geq \left\lceil \frac{2q}{p} \right\rceil} \end{cases}$**,** $\boldsymbol{y_1 \prec z_i}$ **for** $\boldsymbol{i \in \left\{1,2,\dots,a-\left\lceil \frac{2q}{p} \right\rceil\right\}}$**,** $\boldsymbol{y_1 \prec\cap z_i}$ **for** $\boldsymbol{i \in \left\{a-\left\lceil \frac{2q}{p} \right\rceil+1,\dots,a-\left\lceil \frac{q}{p} \right\rceil-1\right\}}$**,** $\boldsymbol{y_1 \cap z_i}$ **for** $\boldsymbol{i \in \left\{a-\left\lceil \frac{q}{p} \right\rceil, a-\left\lceil \frac{q}{p} \right\rceil+1\right\}}$**,** $\boldsymbol{y_1 \succ z_i}$ **for** $\boldsymbol{i \in \left\{a-\left\lceil \frac{q}{p} \right\rceil+2,\dots,a\right\}}$**, and** $\boldsymbol{y_1 \succ x_1}$ (relationships ii(7)a(I)-(VI) of Definition 4.1.1).

Next, we determine the relationships between x_2 and the elements of chain one.

If $x_2 \succ y_3$, then replacing $x_{2r} \to y_{3\ell}$ in C with the path $x_{3r}, x_{3\ell}, y_{3r}, y_{3\ell}$ creates a cycle with weight $q-p(a+b+2)<0$. If $x_2 \prec y_3$, then replacing $y_{3\ell} \to x_{2r}$ in C with the arc $(y_{3\ell}, x_{2r})$ creates a cycle with weight $2q-p(\beta-a-b)>0$. If $x_2 \cap y_3$, then replacing $x_{2r} \to y_{3\ell}$ in C with the arc $(x_{2r}, y_{3\ell})$ creates a cycle with weight $q-p(a+b)<0$.

Let $i \in \{1,2,\dots,a\}$. If $x_2 \succ z_i$, then replacing $x_{2r} \to z_{ir}$ in C with the path $x_{2r}, x_{2\ell}, z_{ir}$ creates a cycle with weight $q-p(a+b+1-(i-1))$ which is positive for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+3,\dots,a\right\}$. If $x_2 \prec z_i$, then replacing $z_{i\ell} \to x_{3r}$ in C with the arc $(z_{i\ell}, x_{2r})$ creates a cycle with weight $2q-p(i+\beta-a-b)$ which is positive for $i \in \left\{1,2,\dots,a+b-\left\lceil \frac{q}{p} \right\rceil\right\}$. If $x_2 \cap z_i$, then replacing $x_{2r} \to z_{i\ell}$ in C with the arc $(x_{2r}, z_{i\ell})$ creates a cycle with weight $q-p(a+b-i)$ which is positive for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+1,\dots,a\right\}$, and replacing $z_{ir} \to x_{2r}$ in C with the path $z_{ir}, x_{2\ell}, x_{2r}$ creates a cycle with weight $3q-p(i-1+\beta-a-b)>0$.

If $x_2 \succ x_1$, then replacing $x_{2r} \to x_{1r}$ in C with the path $x_{2r}, x_{2\ell}, x_{1r}$ creates a cycle with weight $q-p(b+1)$ which is positive if $b<\left\lceil \frac{q}{p} \right\rceil-1$. If $x_2 \prec x_1$, then replacing $x_{1r} \to x_{2r}$ in C with the path $x_{1r}, x_{1\ell}, x_{2r}$ creates a cycle with weight $2q-p(\beta-b+1)$ which is positive if $b \geq \left\lceil \frac{q}{p} \right\rceil+1$. If $x_2 \cap x_1$, then replacing $x_{2r} \to x_{1r}$ in C with the path $x_{2r}, x_{1\ell}, x_{1r}$ creates a cycle with weight $2q-p(b)>0$ and replacing $x_{1r} \to x_{2r}$ in C with the path $x_{1r}, x_{2\ell}, x_{2r}$ creates a cycle with weight $3q-p(\beta-b)>0$.

Thus, $\boldsymbol{x_2 \prec y_3}$**,** $\boldsymbol{x_2 \prec z_i}$ **for** $\boldsymbol{i \in \left\{1,2,\dots,a+b-\left\lceil \frac{q}{p} \right\rceil\right\}}$**,** $\boldsymbol{x_2 \cap z_i}$ **for** $\boldsymbol{i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+1, a+b-\left\lceil \frac{q}{p} \right\rceil+2\right\}}$**,** $\boldsymbol{x_2 \cap\succ z_i}$ **for**

$\boldsymbol{i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+3,\dots,a\right\}}$, **and** $\begin{cases} \boldsymbol{x_2 \cap\succ x_1} & \boldsymbol{b < \left\lceil \frac{q}{p} \right\rceil -1} \\ \boldsymbol{x_2 \cap x_1} & \boldsymbol{b \in \left\{ \left\lceil \frac{q}{p} \right\rceil -1, \left\lceil \frac{q}{p} \right\rceil \right\}} \\ \boldsymbol{x_2 \prec\cap x_1} & \boldsymbol{b > \left\lceil \frac{q}{p} \right\rceil} \end{cases}$ (relationships ii(7)c(I)-(V) of Definition 4.1.1).

Lastly, we determine the relationships between the z elements of chain two and the elements of chain one.

Let $j \in \{a+1, a+2, \dots, a+b\}$. If $z_j \succ y_3$, then replacing $z_{j_\ell} \to y_{3\ell}$ in C with the path $z_{j_\ell}, y_{3r}, y_{3\ell}$ creates a cycle of weight $q-p(j+1)<0$. If $z_j \prec y_3$, then replacing $y_{3\ell} \to z_{j_r}$ in C with the arc $(y_{3\ell}, z_{j_r})$ creates a cycle of weight $2q-p(\beta-(j-1))>0$. If $z_j \cap y_3$, then replacing $z_{j_r} \to y_{3\ell}$ in C with the arc $(z_{j_r}, y_{3\ell})$ creates a cycle of weight $q-p(j-1)<0$.

Let $i \in \{1,2,\dots,a\}$. If $z_j \succ z_i$, then replacing $z_{j_\ell} \to z_{ir}$ in C with the arc (z_{j_ℓ}, z_{ir}) creates a cycle of weight $q-p(i-(i-1))$ which is positive for $i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil+2,\dots,a+b\right\}$. If $z_j \prec z_i$, then replacing $z_{i\ell} \to z_{j_r}$ in C with the arc $(z_{i\ell}, z_{j_r})$ creates a cycle of length $2q-p(i+\beta-(j-1))$ which is positive for $i \in \left\{1,2,\dots,j-\left\lceil \frac{q}{p} \right\rceil-1\right\}$. If $z_j \cap z_i$, then replacing $z_{j_r} \to z_{i\ell}$ in C with the arc $(z_{j_r}, z_{i\ell})$ creates a cycle of length $q-p(j-1-i)$ which is positive for $i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil,\dots,a\right\}$ and replacing $z_{ir} \to z_{j_\ell}$ in C with the arc (z_{ir}, z_{j_ℓ}) creates a cycle of length $2q-p(i-1+q-j)$ which is positive for $i \in \left\{1,2,\dots,j-\left\lceil \frac{q}{p} \right\rceil+1\right\}$.

If $z_j \succ x_1$, then replacing $z_{j_\ell} \to x_{1r}$ in C with the arc (z_{j_ℓ}, x_{1r}) creates a cycle with weight $q-p(j-a)$ which is positive for $j \in \left\{a+1,\dots,a+\left\lceil \frac{q}{p} \right\rceil-1\right\}$. If $z_j \prec x_1$, then replacing $x_{1r} \to z_{j_r}$ in C with the path $x_{1r}, x_{1\ell}, z_{j_r}$ creates a cycle with weight $2q-p(\beta-(i-1)+a+1)$ which is positive for $j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil+2,\dots,a+b\right\}$. If $z_j \cap x_1$, then replacing $x_{1r} \to z_{j_\ell}$ in C with the arc (x_{1r}, z_{j_ℓ}) creates a cycle with weight $2q-p(\beta-i+a)$ which is positive for $j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil,\dots,a+b\right\}$, and replacing $z_{j_r} \to x_{1r}$ in C with the path $z_{j_r}, x_{1\ell}, x_{1r}$ creates a cycle with weight $2q-p(i-1-a)>0$.

Thus, for $\boldsymbol{j \in \{a+1, a+2, \dots, a+b\}}$, $\boldsymbol{z_j \prec y_3}$, $\boldsymbol{z_j \prec z_i}$ **for** $\boldsymbol{i \in \left\{1,2,\dots,j-\left\lceil \frac{q}{p} \right\rceil-1\right\}}$, $\boldsymbol{z_j \cap z_i}$ **for** $\boldsymbol{i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil, j-\left\lceil \frac{q}{p} \right\rceil+1\right\}}$, $\boldsymbol{z_j \succ z_i}$ **for** $\boldsymbol{i \in \left\{j-\left\lceil \frac{q}{p} \right\rceil+2,\dots,a\right\}}$, $\boldsymbol{z_j \succ x_1}$ **for** $\boldsymbol{j \in \left\{a+1,\dots,a+\left\lceil \frac{q}{p} \right\rceil-1\right\}}$, $\boldsymbol{z_j \cap x_1}$ **for** $\boldsymbol{j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil,\dots,a+\left\lceil \frac{q}{p} \right\rceil+1\right\}}$, **and** $\boldsymbol{z_j \prec\cap x_1}$ **for** $\boldsymbol{j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil+2,\dots,a+b\right\}}$

(relationships ii(7)b(I)-(VII) of Definition 4.1.1).

In the following cases we will list the bold relationships based on the value of b.

Case 3.1. $b < \left\lceil \frac{q}{p} \right\rceil$

In this case we must consider the possibility that $x_3 = y_1$. This condition does not affect the relationships of the chains with u_1, u_2, or u_3.

The following are the relationships from the preceding analysis when $b < \left\lceil \frac{q}{p} \right\rceil$, where the relationships from the general analysis that are impacted by the value of b are marked with a (*):

1. $u_1 \cap u_2$,
2. $u_2 \cap u_3$,
3. $\begin{cases} u_3 \cap u_1 & a < \left\lceil \frac{2q}{p} \right\rceil \\ u_3 \succ u_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,
4. u_1,
   (a) $u_1 \prec y_3$,
   (b) $u_1 \prec z_i$ for $i \in \left\{1, 2, \ldots, a - \left\lceil \frac{q}{p} \right\rceil \right\}$,
   (c) $u_1 \cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a \right\}$,
   (d) $u_1 \cap z_i$ for $i \in \{a+1, \ldots, a+b\}$ (*),
   (f) $u_1 \cap x_2$ (*),
   (g) $u_1 \prec y_2$ (*),
   (h) $u_1 \prec z_i$ for $i \in \left\{a+b+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,
   (i) $u_1 \cap z_i$ for $i \in \left\{a + \left\lceil \frac{q}{p} \right\rceil, \ldots, \beta \right\}$,
   (j) $\begin{cases} u_1 \prec \cap x_3 & a > \left\lceil \frac{2q}{p} \right\rceil \\ u_1 \cap x_3 & a \leq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,
5. u_2,

(a) $u_2 \prec y_3$,

(b) $u_2 \prec z_i$ for $i \in \left\{1, 2, \ldots, a+b-\left\lceil \frac{2q}{p} \right\rceil\right\}$,

(c) $u_2 \cap z_i$ for $i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil+1\right\}$,

(d) $u_2 \succ z_i$ for $i \in \left\{a+b-\left\lceil \frac{q}{p} \right\rceil+2, \ldots, a\right\}$,

(e) $u_2 \succ x_1$,

(f) $u_2 \cap y_1$,

(h) $u_2 \cap z_i$ for $i \in \{a+1, \ldots, a+b\}$ (*),

(i) $u_2 \cap z_i$ for $i \in \{a+b+1, \ldots, \beta\}$,

(j) $u_2 \cap x_3$,

6. u_3,

(a) $u_3 \cap z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil\right\}$,

(b) $u_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil+1, \ldots, a\right\}$,

(c) $u_3 \succ x_1$,

(d) $\begin{cases} u_3 \cap \succ y_1 & a > \left\lceil \frac{2q}{p} \right\rceil \\ u_3 \cap y_1 & a \leq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(e) $u_3 \cap z_i$ for $i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$,

(f) $u_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil+1, \ldots, a+b\right\}$,

(g) $u_3 \succ x_2$,

(h) $u_3 \cap y_2$,

(i) $u_3 \cap z_i$ for $i \in \{a+b+1, \ldots, \beta\}$,

7. Chains one and two,

(a) y_1,

(I) $\begin{cases} y_1 \prec \cap y_3 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \\ y_1 \prec y_3 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$ (*),

(II) $y_1 \prec z_i$ for $i \in \left\{1, 2, \ldots, a - \left\lceil \frac{2q}{p} \right\rceil \right\}$,

(III) $y_1 \prec\cap z_i$ for $i \in \left\{a - \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(IV) $y_1 \cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil, a - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(V) $y_1 \succ z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$,

(VI) $y_1 \succ x_1$,

(b) z_j for $j \in \{a+1, a+2, \ldots, a+b\}$,

(I) $z_j \prec y_3$,

(II) $z_j \prec z_i$ for $i \in \left\{1, 2, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1 \right\}$,

(III) $z_j \cap z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil + 1 \right\}$,

(IV) $z_j \succ z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a \right\}$,

(V) $z_j \succ x_1$ for $j \in \{a+1, \ldots, a+b\}$ (*),

(c) x_2,

(I) $x_2 \prec y_3$,

(II) $x_2 \prec z_i$ for $i \in \left\{1, 2, \ldots, a+b - \left\lceil \frac{q}{p} \right\rceil \right\}$,

(III) $x_2 \cap z_i$ for $i \in \left\{a+b - \left\lceil \frac{q}{p} \right\rceil + 1, a+b - \left\lceil \frac{q}{p} \right\rceil + 2 \right\}$,

(IV) $x_2 \cap\succ z_i$ for $i \in \left\{a+b - \left\lceil \frac{q}{p} \right\rceil + 3, \ldots, a \right\}$,

(V) $\begin{cases} x_2 \cap\succ x_1 & b < \left\lceil \frac{q}{p} \right\rceil - 1 \\ x_2 \cap x_1 & b = \left\lceil \frac{q}{p} \right\rceil - 1 \end{cases}$ (*),

8. Chains one and three,

(a) y_2,

(I) $\begin{cases} y_2 \prec\cap y_3 & a+b > \left\lceil \frac{2q}{p} \right\rceil \\ y_2 \cap y_3 & a+b = \left\lceil \frac{2q}{p} \right\rceil \end{cases}$,

(II) $y_2 \prec\cap z_i$ for $i \in \left\{1, 2, \ldots, a+b - \left\lceil \frac{2q}{p} \right\rceil - 1 \right\}$,

(III) $y_2 \cap z_i$ for $i \in \left\{a+b - \left\lceil \frac{2q}{p} \right\rceil, a+b - \left\lceil \frac{2q}{p} \right\rceil + 1 \right\}$,

(IV) $y_2 \succ z_i$ for $i \in \left\{a+b-\left\lceil \frac{2q}{p} \right\rceil+2,\ldots,a\right\}$,

(V) $y_2 \succ x_1$,

(b) z_j for $j \in \{a+b+1,\ldots,\beta\}$,

(I) $z_j \prec y_3$,

(II) $z_j \prec z_i$ for $i \in \left\{1,2,\ldots,j-\left\lceil \frac{2q}{p} \right\rceil-1\right\}$,

(III) $z_j \cap z_i$ for $i \in \left\{j-\left\lceil \frac{2q}{p} \right\rceil, j-\left\lceil \frac{2q}{p} \right\rceil+1\right\}$,

(IV) $z_j \succ z_i$ for $i \in \left\{j-\left\lceil \frac{2q}{p} \right\rceil+2,\ldots,a\right\}$,

(V) $z_j \succ x_1$,

(c) x_3,

(I) $x_3 \prec y_3$,

(II) $x_3 \prec z_i$ for $i \in \left\{1,2,\ldots,\left\lceil \frac{q}{p} \right\rceil-1\right\}$,

(III) $x_3 \cap z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil, \left\lceil \frac{q}{p} \right\rceil+1\right\}$,

(IV) $x_3 \cap\succ z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil+2,\ldots,\left\lceil \frac{2q}{p} \right\rceil\right\}$,

(V) $x_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil+1,\ldots,a\right\}$,

(VI) $\begin{cases} x_3 \cap\succ x_1 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \succ x_1 & a \geq \left\lceil \frac{2q}{p} \right\rceil \end{cases}$ (*),

9. Chains two and three,

(a) y_2,

(I) $\begin{cases} y_2 \cap\succ y_1 & b < \left\lceil \frac{q}{p} \right\rceil - 1 \\ y_2 \cap y_1 & b = \left\lceil \frac{q}{p} \right\rceil - 1 \end{cases}$ (*),

(IV) $y_2 \succ z_i$ for $i \in \{a+1,\ldots,a+b\}$ (*),

(V) $y_2 \succ x_2$,

(b) z_j for $j \in \{a+b+1,\ldots,\beta\}$,

(I) $z_j \cap\succ y_1$ for $j \in \left\{a+b+1,\ldots,a+\left\lceil \frac{q}{p} \right\rceil-2\right\}$,

(II) $z_j \cap y_1$ for $j \in \left\{a+\left\lceil \frac{q}{p} \right\rceil-1, a+\left\lceil \frac{q}{p} \right\rceil\right\}$,

(III) $z_j \prec y_1$ for $j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, \beta\right\}$,

(IV) $z_j \prec z_i$ for $i \in \left\{a+1, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

(V) $z_j \cap z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

(VI) $z_j \succ z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a+b\right\}$,

(VII) $z_j \succ x_2$,

(c) x_3,

(I) $\begin{cases} x_3 \prec y_1 & a < \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \cap y_1 & \left\lceil \frac{2q}{p} \right\rceil \le a \le \left\lceil \frac{2q}{p} \right\rceil + 1, \\ x_3 \cap\!\succ y_1 & a \ge \left\lceil \frac{2q}{p} \right\rceil + 2 \end{cases}$

(II) $x_3 \prec z_i$ for $i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil - 1\right\}$,

(III) $x_3 \cap z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil, \left\lceil \frac{2q}{p} \right\rceil + 1\right\}$,

(IV) $x_3 \cap\!\succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil + 2, \ldots, a+b\right\}$,

(V) $\begin{cases} x_3 \cap x_2 & a+b = \left\lceil \frac{2q}{p} \right\rceil \\ x_3 \cap\!\succ x_2 & a+b > \left\lceil \frac{2q}{p} \right\rceil \end{cases}.$

The relationships listed above match the poset family of Figure 4.1iii.

Note, if $y_1 = x_3$, some of the relationship possibilities between that element and the elements of chain one are eliminated. For example, if $y_1 \prec\!\cap z_i$ and $x_3 \prec z_i$, then $y_1 = x_3 \prec z_i$. After adjusting for the overlap, the listed relationships match the poset family of Figure 4.1iv.

Case 3.2. $b \ge \left\lceil \frac{q}{p} \right\rceil$

Since b is large, $a \not\ge \left\lceil \frac{2q}{p} \right\rceil$, and so $x_3 \ne y_1$. Therefore, all element labels in C represent unique elements of P.

The following are the relationships from the general analysis when $b < \left\lceil \frac{q}{p} \right\rceil$, where the relationships from the general analysis that are impacted by the value of b are marked with a (*):

1. $u_1 \cap u_2$,

2. $u_2 \cap u_3$,

3. $u_3 \cap u_1$ (*),

4. u_1,

   (a) $u_1 \prec y_3$,

   (b) $u_1 \prec z_i$ for $i \in \left\{1, 2, \ldots, a - \left\lceil \frac{q}{p} \right\rceil\right\}$,

   (c) $u_1 \cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a\right\}$,

   (d) $u_1 \cap z_i$ for $i \in \left\{a + 1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil\right\}$,

   (e) $u_1 \succ z_i$ for $i \in \left\{a + \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a + b\right\}$,

   (f) $u_1 \succ x_2$ (*),

   (g) $u_1 \cap y_2$ (*),

   (i) $u_1 \cap z_i$ for $i \in \{a + b, \ldots, \beta\}$ (*),

   (j) $u_1 \cap x_3$ (*),

5. u_2,

   (a) $u_2 \prec y_3$,

   (b) $u_2 \prec z_i$ for $i \in \left\{1, 2, \ldots, a + b - \left\lceil \frac{2q}{p} \right\rceil\right\}$,

   (c) $u_2 \cap z_i$ for $i \in \left\{a + b - \left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a\right\}$ (*),

   (e) $u_2 \cap x_1$ (*),

   (f) $u_2 \prec y_1$ (*),

   (g) $u_2 \prec z_i$ for $i \in \left\{a + 1, \ldots, a + b - \left\lceil \frac{q}{p} \right\rceil\right\}$,

   (h) $u_2 \cap z_i$ for $i \in \left\{a + b - \left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a + b\right\}$,

   (i) $u_2 \cap z_i$ for $i \in \{a + b + 1, \ldots, \beta\}$,

   (j) $u_2 \cap x_3$,

6. u_3,

(a) $u_3 \cap z_i$ for $i \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p} \right\rceil\right\}$,

(b) $u_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{q}{p} \right\rceil + 1, \ldots, a\right\}$,

(c) $u_3 \succ x_1$,

(d) $u_3 \cap y_1$ (*),

(e) $u_3 \cap z_i$ for $i \in \left\{a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil\right\}$,

(f) $u_3 \succ z_i$ for $i \in \left\{\left\lceil \frac{2q}{p} \right\rceil + 1, \ldots, a+b\right\}$,

(g) $u_3 \succ x_2$,

(h) $u_3 \cap y_2$,

(i) $u_3 \cap z_i$ for $i \in \{a+b+1, \ldots, \beta\}$,

7. Chains one and two,

(a) y_1,

(I) $\begin{cases} y_1 \cap y_3 & a = \left\lceil \frac{q}{p} \right\rceil \\ y_1 \prec\cap y_3 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \end{cases}$ (*),

(III) $y_1 \prec\cap z_i$ for $i \in \left\{1, \ldots, a - \left\lceil \frac{q}{p} \right\rceil - 1\right\}$ (*),

(IV) $y_1 \cap z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil, a - \left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

(V) $y_1 \succ z_i$ for $i \in \left\{a - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a\right\}$,

(VI) $y_1 \succ x_1$,

(b) z_j for $j \in \{a+1, a+2, \ldots, a+b\}$,

(I) $z_j \prec y_3$,

(II) $z_j \prec z_i$ for $i \in \left\{1, 2, \ldots, j - \left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

(III) $z_j \cap z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil, j - \left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

(IV) $z_j \succ z_i$ for $i \in \left\{j - \left\lceil \frac{q}{p} \right\rceil + 2, \ldots, a\right\}$,

(V) $z_j \succ x_1$ for $j \in \left\{a+1, \ldots, a + \left\lceil \frac{q}{p} \right\rceil - 1\right\}$,

(VI) $z_j \cap x_1$ for $j \in \left\{a + \left\lceil \frac{q}{p} \right\rceil, \ldots, a + \left\lceil \frac{q}{p} \right\rceil + 1\right\}$,

(VII) $z_j \prec\cap x_1$ for $j \in \left\{a+\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a+b\right\}$,

(c) x_2,

(I) $x_2 \prec y_3$,

(II) $x_2 \prec z_i$ for $i \in \{1,2,\ldots,a\}$ (*),

(V) $\begin{cases} x_2 \cap x_1 & b=\left\lceil\frac{q}{p}\right\rceil \\ x_2 \prec\cap x_1 & b>\left\lceil\frac{q}{p}\right\rceil \end{cases}$ (*),

8. Chains one and three,

(a) y_2,

(I) $y_2 \prec\cap y_3$ (*),

(II) $y_2 \prec\cap z_i$ for $i \in \left\{1,2,\ldots,a+b-\left\lceil\frac{2q}{p}\right\rceil-1\right\}$,

(III) $y_2 \cap z_i$ for $i \in \left\{a+b-\left\lceil\frac{2q}{p}\right\rceil, a+b-\left\lceil\frac{2q}{p}\right\rceil+1\right\}$,

(IV) $y_2 \succ z_i$ for $i \in \left\{a+b-\left\lceil\frac{2q}{p}\right\rceil+2,\ldots,a\right\}$,

(V) $y_2 \succ x_1$,

(b) z_j for $j \in \{a+b+1,\ldots,\beta\}$,

(I) $z_j \prec y_3$,

(IV) $z_j \prec z_i$ for $i \in \left\{1,2,\ldots,j-\left\lceil\frac{2q}{p}\right\rceil-1\right\}$,

(V) $z_j \cap z_i$ for $i \in \left\{j-\left\lceil\frac{2q}{p}\right\rceil, j-\left\lceil\frac{2q}{p}\right\rceil+1\right\}$,

(VI) $z_j \succ z_i$ for $i \in \left\{j-\left\lceil\frac{2q}{p}\right\rceil+2,\ldots,a\right\}$,

(VII) $z_j \succ x_1$,

(c) x_3,

(I) $x_3 \prec y_3$,

(II) $x_3 \prec z_i$ for $i \in \left\{1,2,\ldots,\left\lceil\frac{q}{p}\right\rceil-1\right\}$,

(III) $x_3 \cap z_i$ for $i \in \left\{\left\lceil\frac{q}{p}\right\rceil, \left\lceil\frac{q}{p}\right\rceil+1\right\}$,

(IV) $x_3 \cap\succ z_i$ for $i \in \left\{\left\lceil\frac{q}{p}\right\rceil+2,\ldots,a\right\}$ (*),

(VI) $\begin{cases} x_3 \cap x_1 & a = \left\lceil \frac{q}{p} \right\rceil \\ x_3 \cap \succ x_1 & \left\lceil \frac{q}{p} \right\rceil < a < \left\lceil \frac{2q}{p} \right\rceil \end{cases}$ (*),

9. Chains two and three,

   (a) y_2,

      (I) $\begin{cases} y_2 \cap y_1 & b = \left\lceil \frac{q}{p} \right\rceil \\ y_2 \prec \cap y_1 & \left\lceil \frac{q}{p} \right\rceil < b \end{cases}$ (*),

      (II) $y_2 \prec \cap z_i$ for $i \in \left\{ a+1, \ldots, a+b-\left\lceil \frac{q}{p} \right\rceil -1 \right\}$,

      (III) $y_2 \cap z_i$ for $i \in \left\{ a+b-\left\lceil \frac{q}{p} \right\rceil, a+b-\left\lceil \frac{q}{p} \right\rceil +1 \right\}$,

      (IV) $y_2 \succ z_i$ for $i \in \left\{ a+b-\left\lceil \frac{q}{p} \right\rceil +2, \ldots, a+b \right\}$,

      (V) $y_2 \succ x_2$,

   (b) z_j for for $j \in \{a+b+1, \ldots, \beta\}$,

      (III) $z_j \prec y_1$ for $j \in \{a+b+1, \ldots, \beta\}$ (*),

      (IV) $z_j \prec z_i$ for $i \in \left\{ a+1, \ldots, j-\left\lceil \frac{q}{p} \right\rceil -1 \right\}$,

      (V) $z_j \cap z_i$ for $i \in \left\{ j-\left\lceil \frac{q}{p} \right\rceil, j-\left\lceil \frac{q}{p} \right\rceil +1 \right\}$,

      (VI) $z_j \succ z_i$ for $i \in \left\{ j-\left\lceil \frac{q}{p} \right\rceil +2, \ldots, a+b \right\}$,

      (VII) $z_j \succ x_2$,

   (c) x_3,

      (I) $x_3 \prec y_1$ (*),

      (II) $x_3 \prec z_i$ for $i \in \left\{ a+1, \ldots, \left\lceil \frac{2q}{p} \right\rceil -1 \right\}$,

      (III) $x_3 \cap z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil, \left\lceil \frac{2q}{p} \right\rceil +1 \right\}$,

      (IV) $x_3 \cap \succ z_i$ for $i \in \left\{ \left\lceil \frac{2q}{p} \right\rceil +2, \ldots, a+b \right\}$,

      (V) $x_3 \cap \succ x_2$ (*).

The relationships listed above match the poset family of Figure 4.1v.

The preceding cases cover all possible minimal negative cycle structures. Thus, if a poset contains a minimal negative cycle with three weight q arcs, then the poset contains a subposet from Figure 4.1. □

The preceding proof shows that not every p, q pair will have a minimal forbidden substructure produced by a cycle with exactly three weight q arcs. This fact is captured in the following corollary.

Corollary 4.1.3. Let $q=ps+d$. If $\frac{p}{3}<d\leq\frac{p}{2}$ or $d>\frac{2p}{3}$, then a negative cycle with exactly three weight q arcs cannot be minimal.

Proof. We have $\beta=\left\lceil\frac{3q}{p}\right\rceil=\begin{cases}3s+1 & d\leq\frac{p}{3}\\ 3s+2 & \frac{p}{3}<d\leq\frac{2p}{3}\\ 3s+3 & \frac{2p}{3}<d\end{cases}$, and $\left\lceil\frac{2q}{p}\right\rceil=\begin{cases}2s+1 & d\leq\frac{p}{2}\\ 2s+2 & \frac{p}{2}<d<p\end{cases}$. As in the proof of Proposition 4.1.2, $d>\frac{2p}{3}$ and $\frac{p}{3}<d\leq\frac{p}{2}$ allowed a shorter negative cycle to be found. $\square$

For $p=3$, $d=1$ or $d=2$. Thus, $d\not>\frac{2p}{3}=2$ and there is not a d with $1=\frac{p}{3}<d\leq\frac{p}{2}=\frac{3}{2}$.

4.2 Structural result for lengths in [3,q], $q=2s+1$ or $q=2s+2$

Definition 4.2.1. Let $\mathcal{F}_3^q$ be the set of posets shown in Figure 4.6 and their horizontal reflections.

Theorem 4.2.2 will justify using the $\mathcal{F}_3^q$ notation for this collection of posets. The posets in $\mathcal{F}_3^q$ get quite complicated. Following the proof of this chapter's main theorem, $\mathcal{F}_3^q$ is shown for $q=4,5,7,8,10,11$, and 13.

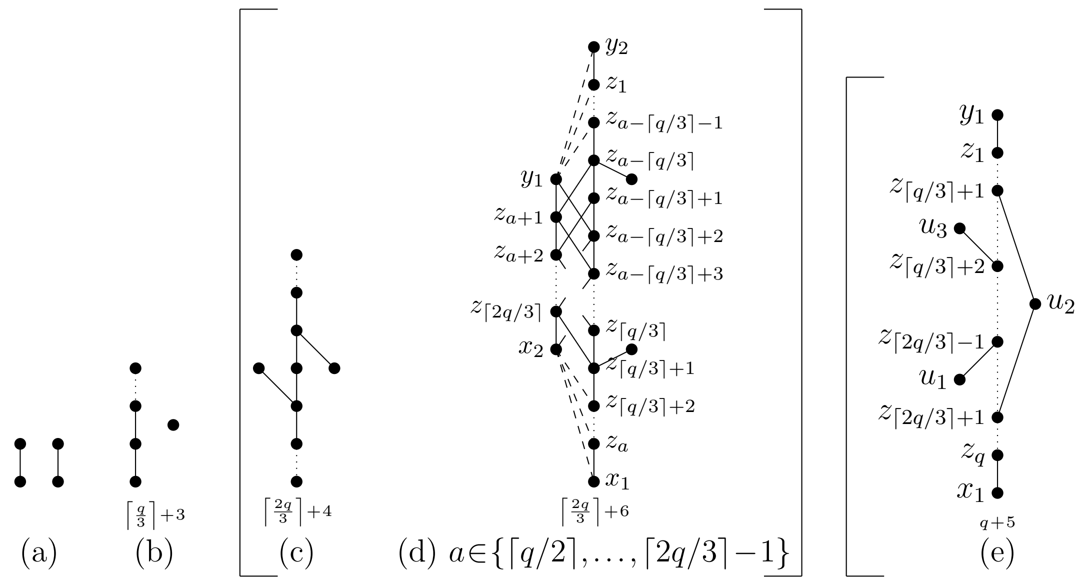
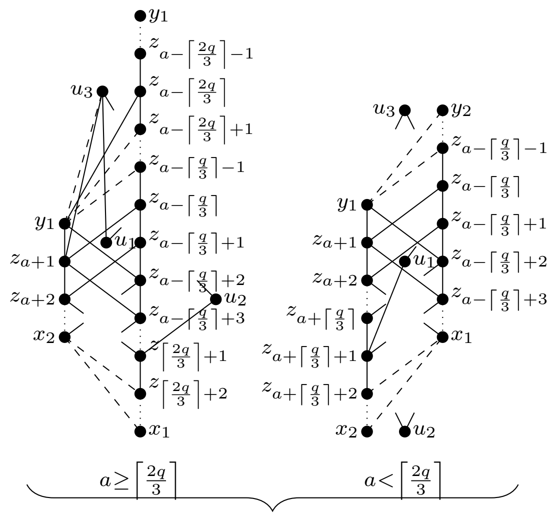


Figure 4.6: Minimal structures that cannot appear in a $[3, q]$ representable interval order: The numbers below the posets are the number of elements in the structure. Structure (c) and family (d) are minimal for $q=3s+1$ only. They are not minimal for $q=3s+2$. For $p=3$, (c) is structure (i) of Definition 3.1.1, (d) is family (ii) of Definition 3.1.1, and (e) is structure (i) of Definition 4.1.1.



(f) $a\in\left\{\left\lceil\frac{q}{2}\right\rceil,\ldots,q-1\right\}$

1. $\begin{cases} u_3\cap\succ y_1, u_3\succ z_{a+1} & \text{if } a>\left\lceil\frac{2q}{3}\right\rceil \\ u_3\succ z_{\left\lceil\frac{2q}{3}\right\rceil+1} & \text{if } a\leq\left\lceil\frac{2q}{3}\right\rceil \end{cases}$

2. $u_3\succ z_{\left\lceil\frac{q}{3}\right\rceil+1}$

3. $u_1\prec z_{a-\left\lceil\frac{q}{3}\right\rceil}$

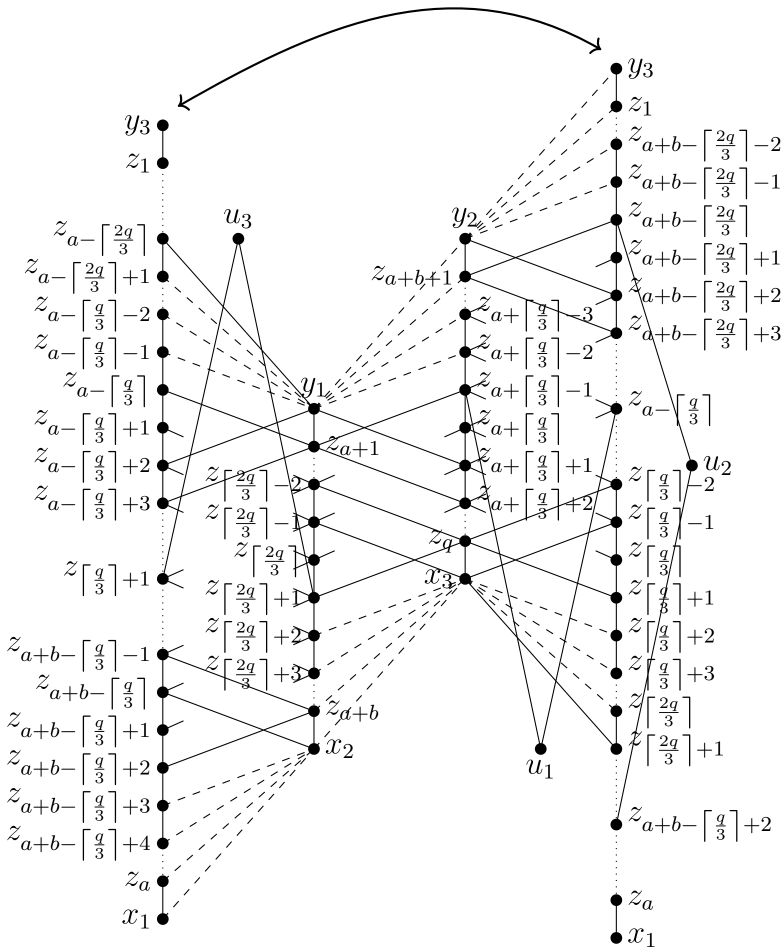
4. $u_3\succ u_1$ if $a\geq\left\lceil\frac{2q}{3}\right\rceil$

5. $u_1\succ z_{a+\left\lceil\frac{q}{3}\right\rceil+1}$ if $a<\left\lceil\frac{2q}{3}\right\rceil$

6. $u_2\prec z_{\left\lceil\frac{q}{3}\right\rceil-1}$

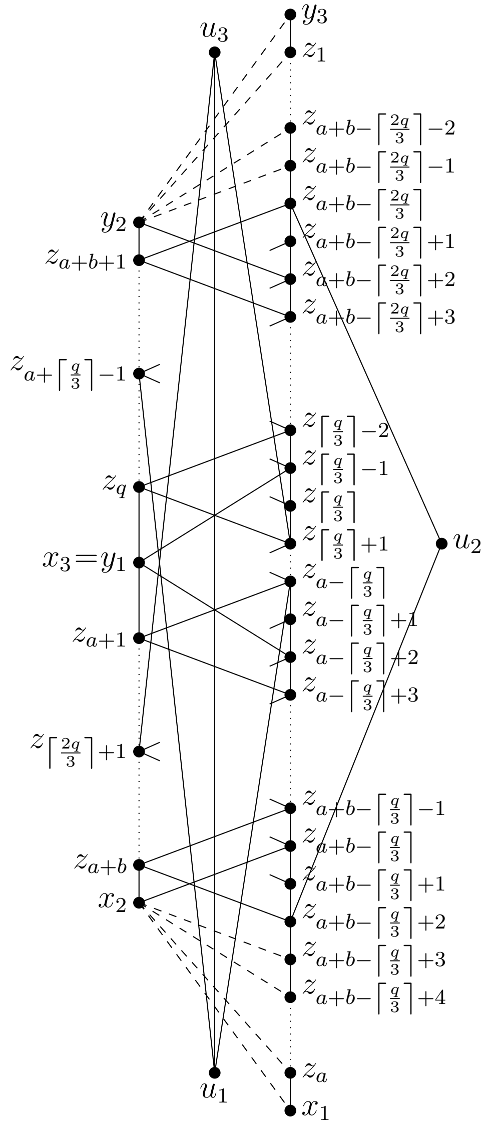
7. $\begin{cases} u_2\succ z_{\left\lceil\frac{2q}{3}\right\rceil+1} & \text{if } a\geq\left\lceil\frac{q}{3}\right\rceil \\ u_2\prec z_{\left\lceil\frac{2q}{3}\right\rceil-1} & \text{if } a<\left\lceil\frac{q}{3}\right\rceil \end{cases}$

Figure 4.6 (cont): Minimal structures that cannot appear in a $[3,q]$ representable interval order: Family (f) corresponds to family (ii) of Definition 4.1.1 when $p=3$.



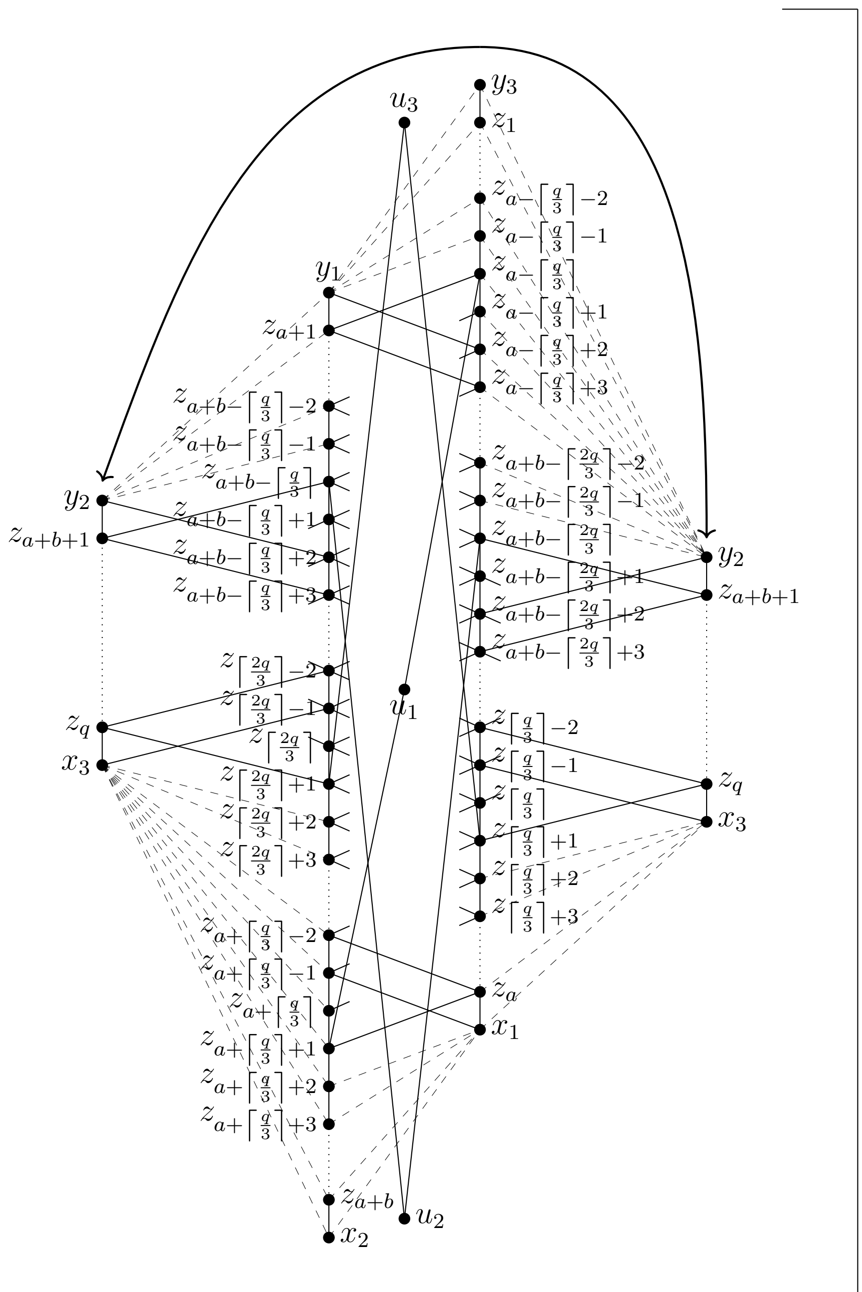
(g_1) $b \in \left\{1, 2, \ldots, \left\lceil \frac{q}{3} \right\rceil - 1\right\}$

Figure 4.6 (cont): Minimal structures that cannot appear in a $[3,q]$ representable interval order: Family (g_1) corresponds to family (iii) of Definition 4.1.1 when $p=3$. The left and right chains contain the same elements. Note: $u_3 \succ u_1$ if $a \geq \left\lceil \frac{2q}{3} \right\rceil$, $y_2 \cap y_3$ if $a+b = \left\lceil \frac{2q}{3} \right\rceil$, $x_1 \cap x_2$ if $b = \left\lceil \frac{q}{3} \right\rceil - 1$.



(g_2) $a\in\left\{\left\lceil\frac{2q}{3}\right\rceil+1,\ldots,q-2\right\}$ (thus $b\in\left\{1,2,\ldots,\left\lceil\frac{q}{3}\right\rceil-3\right\}$)

Figure 4.6 (cont): Minimal structures that cannot appear in a $[3,q]$ representable interval order: Family (g_2) corresponds to family (iv) of Definition 4.1.1 when $p=3$.



(g_3) $b \in \left\{ \left\lceil \frac{q}{3} \right\rceil, \ldots \left\lfloor \frac{q-1}{2} \right\rfloor \right\}$

Figure 4.6 (cont): Minimal structures that cannot appear in a $[3, q]$ representable interval order: Family (g_3) corresponds to family (v) of Definition 4.1.1 when $p{=}3$. The left and right chains contain the same elements.

Figure 4.7 illustrates poset (e) of $\mathcal{F}_3^q$. It gives the forbidden posets of family (e) for $q \leq 13$.

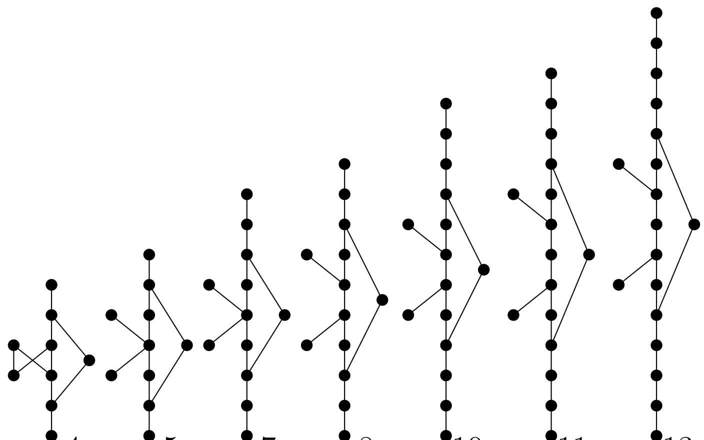


Figure 4.7: Minimal structures that cannot appear in a $[3,q]$ representable interval order with 3 weight q arcs and q weight -3 arcs where all q arcs are consecutive

We offer some notes on the structures in Figure 4.6.

For family (f), for the second case of relationship 1. and for relationship 4., if $\lceil \frac{q}{3} \rceil + 2 \leq \lceil \frac{2q}{3} \rceil - b$ which is $b \leq \begin{cases} \lceil \frac{q}{3} \rceil - 2 & q=3s+1 \\ \lceil \frac{q}{3} \rceil - 1 & q=3s+2 \end{cases} = \lceil \frac{q+2}{3} \rceil - 2$, then the relation is already present due to transitivity.

For posets in family (g), the thick double headed arrow indicates that the two chains it connects are actually the same chain. In diagrams (g_1) and (g_3) a chain was duplicated to simplify the drawing. Also, for structure (g_2), consider $z_{\lceil \frac{q}{3} \rceil - 1}$ and $z_{a - \lceil \frac{q}{3} \rceil + 2}$, the elements related to $x_3 = y_1$. We have $a - \lceil \frac{q}{3} \rceil + 2 > \lceil \frac{2q}{3} \rceil - \lceil \frac{q}{3} \rceil + 2 \geq \left(\lceil \frac{q}{3} \rceil - 1 \right) + 2$. This means that the elements precedent to/from $x_3 = y_2$ are at least 2 elements apart.

The following is the main theorem of this chapter.

Theorem 4.2.2. Let $P = (X; \prec)$ be a partial order and let $q = 3s+1$ or $q = 3s+2$, with $s \in \mathbb{Z}_{\geq 1}$. The following are equivalent:

1. Poset, P, has an interval representation with lengths between 3 and q.

2. The weighted digraph $D_3^q(P)$ contains no negative cycles.

3. Poset, P, contains no induced sub-poset from $\mathcal{F}_3^q$.

Proof. **(1) ⇔ (2)** This is a special case of Theorem 2.1.5.

(2) ⇒ (3) (by contrapositive) Recall that in $D_3^q(P)$, an edge $x_\ell \to x_r$ has weight q and the reverse edge has weight -3. All other edges have weight $-\epsilon$ or 0. If P contains an induced $\mathbf{2+2}$, say $(x \prec y) \cap (u \prec v)$, then $y_\ell, x_r, v_\ell, u_r, y_\ell$ is a cycle of weight -2ϵ. If P contains an induced $\lceil \frac{\mathbf{q+6}}{\mathbf{3}} \rceil + \mathbf{1}$, say $(x_1 \succ x_2 \succ \cdots \succ x_{\lceil (q+6)/3 \rceil}) \cap y$, then, the cycle $x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}, x_{3\ell}, x_{4r}, \cdots, x_{\lceil (q+6)/3 \rceil_r}, y_\ell, y_r, x_{1\ell}$ has weight $-3\left(\lceil \frac{q}{3} \rceil\right) + q - \epsilon\left(\lceil \frac{q+3}{3} \rceil\right) < 0$.

If P contains an induced subposet isomorphic to poset (c) of $\mathcal{F}_3^q$ from Figure 4.6 and $q = 3s+1$, then $x_1 \succ x_2 \succ \ldots, \succ x_{\lceil 2q/3 \rceil + 2} \cap y_1 \cap y_2$ with $y_1 \succ x_{(\lceil 2q/3 \rceil + 1)/2}$ and $y_2 \prec x_{(\lceil 2q/3 \rceil + 3)/2}$. Then, the cycle

$$x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}, x_{3\ell}, x_{4r}, \cdots, x_{\lceil 2q/3 \rceil + 2_r}, y_{1\ell}, y_{1r}, y_{2\ell}, y_{2r}, x_{1\ell}$$

has weight $-3\lceil \frac{2q}{3} \rceil + q(2) - \epsilon \lceil \frac{2q}{3} + 1 \rceil < 0$.

Next, consider the posets in family (d) labeled as in Figure 3.6.

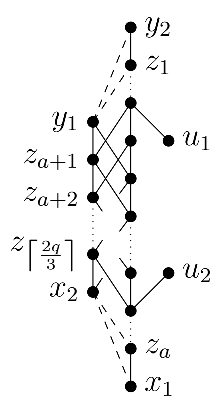


Figure 4.8: Labeling for the family (d) of posets from Figure 3.5: The $z_i's$ label the longer chain consecutively from top to bottom.

Now, the cycle

$$z_{1r}, z_{1\ell}, z_{2r}, z_{2\ell}, \ldots, z_{ar}, z_{a\ell}, x_{1r}, u_{1\ell}, u_{1r}, y_{1\ell}, z_{a+1r}, z_{a+1\ell}, z_{a+2r}, z_{a+2\ell}, \cdots, z_{\beta_r}, x_{\beta_\ell},$$

$$x_{2r}, u_{2\ell}, u_{2r}, y_{2\ell}, z_{1r}$$

has weight $2q-3(a)-3\left(\left\lceil\frac{2q}{3}\right\rceil-a\right)-\epsilon\left(\left\lceil\frac{2q}{3}\right\rceil+2\right)<0$.

Next, if P contains an induced poset isomorphic to (e) of Figure 4.6, say $x_1\succ x_2\succ\cdots\succ x_{q+2}\cap y_1\cap y_2\cap y_3$ with $y_1\prec x_{\lceil 2q/3\rceil}$, $x_{\lceil q/3\rceil}\succ y_2\succ x_{q+2-\lfloor q/3\rfloor}=x_{\lceil 2q/3\rceil+2}$, and $y_3\succ x_{q+2-\lfloor 2q/3\rfloor}=x_{\lceil q/3\rceil+2}$. Then, the cycle

$$x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}, x_{3\ell}, x_{4r}, \cdots, x_{q+2_r}, y_{1\ell}, y_{1r}, y_{2\ell}, y_{2r}, y_{3\ell}, y_{3r}, x_{1\ell}$$

has weight $-3(q)+q(3)-\epsilon(q+1)<0$.

Now, if P contains an induced subposet isomorphic to a poset in (f) labeled as in Figure 4.6, then the cycle

$$z_{1r}, z_{1\ell}, z_{2r}, z_{2\ell}, \ldots, z_{ar}, z_{a\ell}, x_{1r}, u_{1\ell}, u_{1r}, y_{1\ell}, z_{a+1r}, z_{a+1\ell}, z_{a+2r}, z_{a+2\ell}, \cdots, z_{q_r}, z_{q_\ell}, x_{2r},$$

$$u_{2\ell}, u_{2r}, u_{3\ell}, u_{3r}, y_{2\ell}, z_{1r}$$

has weight $-3(q)+q(3)-\epsilon(q+2)<0$.

Finally, if P contains an induced subset isomorphic to a poset in family (g) labeled as in Figure 4.6, then

$$z_{1r}, z_{1\ell}, z_{2r}, z_{2\ell}, z_{3r}, \cdots, z_{a\ell}, x_{1r}, u_{1\ell}, u_{1r}, y_{1\ell}, z_{a+1r}, z_{a+1\ell}, \cdots, z_{a+b\ell}, x_{2r}, u_{2\ell}, u_{2r}, y_{2\ell},$$

$$z_{a+b+1r}, z_{a+b+1\ell}\cdots, z_{q\ell}, x_{3r}, u_{3\ell}, u_{3r}, y_{3\ell}$$

is a cycle with weight $-3(q)+q(3)-\epsilon(q+3)<0$.

Thus, if a poset P contains an induced poset in $\mathcal{F}_3^q$, then $D_3^q(P)$ contains a negative cycle.

(3) $\Rightarrow$ (2) (By contrapositive) Assume $D_3^q(P)$ contains a negative cycle. We will show that P contains an element of $\mathcal{F}_3^q$ as an induced suborder.

Let C be a minimal negative cycle (shortest negative cycle with greatest (least negative) weight) in $D_3^q(P)$.

Case 1. All arcs of C have weight $-\epsilon$ or 0.

By Lemma 2.2.1, P contains an induced $\mathbf{2}+\mathbf{2}$ (structure (a) of Figure 4.6).

Case 2. Cycle C contains an arc of weight -3 but no positive weight arcs.

Lemma 2.2.3 rules out this possibility.

Case 3. Cycle C contains α arcs of weight q.

By Lemma 2.2.6, C must contain $\beta = \lceil \frac{q\alpha}{3} \rceil$ arcs of weight -3.

Case 3.1. $\alpha=1$

By Lemma 2.2.9, C corresponds to a $\left\lceil \frac{\mathbf{q}+\mathbf{2p}}{\mathbf{p}} \right\rceil + \mathbf{1}$ where $p=3$, so a $\lceil \frac{\mathbf{q}+\mathbf{6}}{\mathbf{3}} \rceil + \mathbf{1}$, which is $\left(\lceil \frac{\mathbf{q}}{\mathbf{3}} \rceil + \mathbf{2}\right) + \mathbf{1}$ (structure (b) of Figure 4.6).

Case 3.2. $\alpha=2$

If $q=3s+d$, then we have that $\beta = \lceil \frac{2q}{3} \rceil = s + \lceil \frac{2d}{3} \rceil = \begin{cases} 2s+1 & d=1 \\ 2s+2 & d=2 \end{cases}$. By Corollary 3.1.3, C is not minimal if $d=2$. By Proposition 3.1.2, C corresponds to one of the structures in $\mathcal{F}_3^q(2)$ which are the posets of families (c) and (d) of Figure 4.6.

Case 3.3. $\alpha=3$

We have that $\beta = \lceil \frac{3q}{3} \rceil = q$. By Proposition 4.1.2, C corresponds to one of the structures in $\mathcal{F}_3^q(3)$ which correspond to the posets in families (e), (f), and (g) of Figure 4.6.

Case 3.4. $\alpha>3$

By Lemma 2.2.10, this does not occur.

This covers all possibilities for α. Thus, if $D_3^q(P)$ contains a negative cycle, then P contains an induced sub-poset from $\mathcal{F}_3^q$. The proofs of Lemmas 2.2.1 and 2.2.9 and Propositions 3.1.2 and 4.1.2 also show that these structures are minimal since they correspond to minimal negative cycles.

Thus, $\mathcal{F}_3^q$ is a minimal list of minimal forbidden substructures for $\mathcal{P}[3,q]$. □

How many posets are forbidden by $\mathcal{F}_3^q$? Proposition 4.2.3 answers this question for each q. This result is analogous to Proposition 3.2.3.

Proposition 4.2.3. The number of minimal forbidden subposets for interval lengths between 3 and $q=2s+1$ is

$$
\begin{aligned}
|\mathcal{F}_3^q| = 4 + &\sum_{i=1}^{(\lceil 2q/3\rceil-1)/2} i^2 \\
&+\left\{2\sum_{a=\lceil q/2\rceil}^{\lceil 2q/3\rceil-1}\left[\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)\left(\left\lceil\frac{2q}{3}\right\rceil-a\right)\right]\right. \\
&+(1+(q \bmod 2))\left(\left\lceil\frac{q}{3}\right\rceil-1\right) \\
&\left.+2\sum_{a=\lceil 2q/3\rceil+1}^{q-1} 2\left(\left\lceil\frac{q}{3}\right\rceil-1\right)\left(a-\left\lceil\frac{2q}{3}\right\rceil+1\right)\right\} \\
&+\left\{\sum_{b=1}^{\lceil q/3\rceil-1}\left(\left\lceil\frac{q}{3}\right\rceil-1\right)^2\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^4\right. \\
&\qquad +2\sum_{b=1}^{\lceil q/3\rceil-1}\sum_{a=q-2b+1}^{q-b-1}\left(\left\lceil\frac{q}{3}\right\rceil-1\right)^2\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^2\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2 \\
&+\sum_{b=1}^{\lceil q/3\rceil-3}\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^2+2\sum_{b=1}^{\lceil q/3\rceil-3}\sum_{a=q-2b+1}^{q-b-1}\left(\left\lceil\frac{q}{3}\right\rceil-b\right)\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right) \\
&+\sum_{b=\lceil q/3\rceil}^{\lfloor (q-1)/2\rfloor}\left(b-\left\lceil\frac{q}{3}\right\rceil+1\right)^4\left(2b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2 \\
&\qquad \left.+2\sum_{b=\lceil q/3\rceil}^{\lfloor (q-1)/2\rfloor}\sum_{a=b+1}^{q-b-1}\left(b-\left\lceil\frac{q}{3}\right\rceil+1\right)^2\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)^2\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2\right\},
\end{aligned}
$$

and for lengths between 3 and $q{=}3s{+}2$ is

$$
\begin{aligned}
|\mathcal{F}_3^q| = 3 + \Bigg\{ & 2\sum_{a=\lceil q/2\rceil}^{\lceil 2q/3\rceil-1} \left[\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)\left(\left\lceil\frac{2q}{3}\right\rceil-a\right)\right] \\
& +(1+(q \bmod 2))\left(\left\lceil\frac{q}{3}\right\rceil\right) \\
& +2\sum_{a=\lceil 2q/3\rceil+1}^{q-1} 2\left(\left\lceil\frac{q}{3}\right\rceil\right)\left(a-\left\lceil\frac{2q}{3}\right\rceil+1\right)\Bigg\} \\
& +\Bigg\{\sum_{b=1}^{\lceil q/3\rceil-1}\left(\left\lceil\frac{q}{3}\right\rceil\right)^2\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^4 \\
& \qquad +2\sum_{b=1}^{\lceil q/3\rceil-1}\sum_{a=q-2b+1}^{q-b-1}\left(\left\lceil\frac{q}{3}\right\rceil\right)^2\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^2\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2 \\
& +\sum_{b=1}^{\lceil q/3\rceil-3}\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^2+2\sum_{b=1}^{\lceil q/3\rceil-3}\sum_{a=q-2b+1}^{q-b-1}\left(\left\lceil\frac{q}{3}\right\rceil-b\right)\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right) \\
& +\sum_{b=\lceil q/3\rceil}^{\lfloor (q-1)/2\rfloor}\left(b-\left\lceil\frac{q}{3}\right\rceil+1\right)^4\left(2b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2 \\
& \qquad +2\sum_{b=\lceil q/3\rceil}^{\lfloor (q-1)/2\rfloor}\sum_{a=b+1}^{q-b-1}\left(b-\left\lceil\frac{q}{3}\right\rceil+1\right)^2\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)^2\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2\Bigg\}.
\end{aligned}
$$

Proof. In Figure 4.6, (a), (b), (c), and (e) contribute 4 posets unless $q{=}3s{+}2$, then (c) is not included. The structures in family (d) are only counted when $q{=}3s{+}1$. There are $\left\lfloor \frac{\lceil \frac{2q}{3}\rceil}{2}\right\rfloor = s$ horizontally symmetric structures without accounting for the dashed lines, and each contains $\lceil\frac{2q}{3}\rceil+6=2s+7$ elements. In the first structure, 9 elements are part of the center structure. Each of the remaining $\lceil\frac{2q}{3}\rceil-3=2s-2$ elements have exactly one dashed line precedence. We can select at most one precedence from the top set of dashed lines and at most one precedence from the bottom set of dashed lines (selecting a precedence close to the center implies all precedences farther from the center by transitivity). These choices can be made in $\left(\frac{2s-2}{2}+1\right)\left(\frac{2s-2}{2}+1\right)=s^2$ ways. The next structure in (d) has 11 elements in its center structure and so represents $(s-1)^2$ distinct posets. The third structure would have 13 elements in its center structure and so represents $(s-2)^2$ posets. In the last

poset, the center structure contains all $2s+7$ elements and so represents only one poset. Thus, (d) contributes
$\sum_{i=1}^{s} i^2 = \sum_{i=1}^{(\lceil 2q/3\rceil-1)/2} i^2$ posets.

Next, we count the elements of family (f). Each structure contains at most two sets of dashed lines and if $a > \lceil \frac{2q}{3} \rceil$, a single dashed line from u_3. When $a < \lceil \frac{2q}{3} \rceil$, there are $1+a-\lceil \frac{q}{3} \rceil -1$ dashed lines in the top set and $q+1-\left(a+\lceil \frac{q}{3} \rceil+1\right)$ dashed lines in the bottom set. This creates $\left(1+a-\lceil \frac{q}{3} \rceil-1+1\right)\left(q+1-\left(a+\lceil \frac{q}{3} \rceil+1\right)+1\right) = \left(a-\lceil \frac{q}{3} \rceil+1\right)\left(\lceil \frac{2q}{3} \rceil-a\right)$ posets. We find a similar count for $a > \lceil \frac{2q}{3} \rceil$:

$$\begin{cases} 2\left(\lceil \frac{q}{3} \rceil-1\right)\left(a-\lceil \frac{2q}{3} \rceil+1\right) & q=3s+1 \\ 2\left(\lceil \frac{q}{3} \rceil\right)\left(a-\lceil \frac{2q}{3} \rceil+1\right) & q=3s+2 \end{cases}.$$

For $a = \lceil \frac{2q}{3} \rceil$, we get $\begin{cases} \left(\lceil \frac{q}{3} \rceil-1\right)\left(a-\lceil \frac{2q}{3} \rceil+1\right) & q=3s+1 \\ \left(\lceil \frac{q}{3} \rceil\right)\left(a-\lceil \frac{2q}{3} \rceil+1\right) & q=3s+2 \end{cases} = \begin{cases} \left(\lceil \frac{q}{3} \rceil-1\right) & q=3s+1 \\ \left(\lceil \frac{q}{3} \rceil\right) & q=3s+2 \end{cases}$
structures. Now, a structure, P is horizontally symmetric if its corresponding negative cycle in $D_3^q(P)$ is isomorphic to the cycle in reverse. Thus, a poset is horizontally symmetric if $a=b=\frac{q}{2}$ which only occurs when q is even. Thus, the count for family (f) is:

$$2\sum_{a=\lceil q/2\rceil}^{\lceil 2q/3\rceil-1}\left[\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)\left(\left\lceil\frac{2q}{3}\right\rceil-a\right)\right]+(1+(q \bmod 2))\left(\left\lceil\frac{q}{3}\right\rceil-1\right)$$
$$+2\sum_{a=\lceil 2q/3\rceil+1}^{q-1} 2\left(\left\lceil\frac{q}{3}\right\rceil-1\right)\left(a-\left\lceil\frac{2q}{3}\right\rceil+1\right)$$

when $q=3s+1$, and

$$2\sum_{a=\lceil q/2\rceil}^{\lceil 2q/3\rceil-1}\left[\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)\left(\left\lceil\frac{2q}{3}\right\rceil-a\right)\right]+(1+(q \bmod 2))\left(\left\lceil\frac{q}{3}\right\rceil\right)$$
$$+2\sum_{a=\lceil 2q/3\rceil+1}^{q-1} 2\left(\left\lceil\frac{q}{3}\right\rceil\right)\left(a-\left\lceil\frac{2q}{3}\right\rceil+1\right)$$

when $q=3s+2$.

The number of posets in family (g) can also be counted by determining the number of dashed lines in each set using the subscripts, adding one to each set, and multiplying them together. Again, we must determine which structures are horizontally symmetric and which are not and double the contribution of those that are not. The structure will be symmetric if two of the chains have the same length, meaning $a=b$ or $a=q-2b$, or all three chains have the same length. However, q is not divisible by three, so all three chains cannot have the same length.

For family (g_1), $b<\lceil \frac{q}{3}\rceil$, and $a\geq\lceil \frac{q}{3}\rceil$, so $a\neq b$, but $a=q-2b$ is possible. Family (g_1) contributes

$$\sum_{b=1}^{\lceil q/3\rceil-1}\left(\left\lceil\frac{q}{3}\right\rceil-\lambda\right)^2\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^4 +2\sum_{b=1}^{\lceil q/3\rceil-1}\sum_{a=q-2b+1}^{q-b-1}\left(\left\lceil\frac{q}{3}\right\rceil-\lambda\right)^2\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^2\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2$$

posets where the first sum counts the case that $a=q-2b$ and $\lambda=\begin{cases}1 & q=3s+1\\ 0 & q=3s+2\end{cases}$.

For family (g_2), $b<\lceil \frac{q}{3}\rceil-2$, and $a\geq\lceil \frac{q}{3}\rceil$, so again $a\neq b$, but $a=q-2b$ is possible. Thus, family (g_2) contributes

$$\sum_{b=1}^{\lceil q/3\rceil-3}\left(\left\lceil\frac{q}{3}\right\rceil-b\right)^2+2\sum_{b=1}^{\lceil q/3\rceil-3}\sum_{a=q-2b+1}^{q-b-1}\left(\left\lceil\frac{q}{3}\right\rceil-b\right)\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)$$

posets where the first sum counts the case that $a=q-2b$.

For family (g_3), $b\geq\lceil \frac{q}{3}\rceil$, and $a\geq\lceil \frac{q}{3}\rceil$, so $a\neq q-2b$, but $a=b$ is a possibility. Thus, family (g_3) contributes

$$\sum_{b=\lceil q/3\rceil}^{\lfloor (q-1)/2\rfloor}\left(b-\left\lceil\frac{q}{3}\right\rceil+1\right)^4\left(2b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2 +2\sum_{b=\lceil q/3\rceil}^{\lfloor (q-1)/2\rfloor}\sum_{a=b+1}^{q-b-1}\left(b-\left\lceil\frac{q}{3}\right\rceil+1\right)^2\left(a-\left\lceil\frac{q}{3}\right\rceil+1\right)^2\left(a+b-\left\lceil\frac{2q}{3}\right\rceil+1\right)^2$$

posets where the first sum counts the horizontally symmetric posets, and the double sum counts the non-symmetric posets twice. □

We illustrate Proposition 4.2.3 for $q=4$ and $q=5$. For $q=4$, we have

$$|\mathcal{F}_3^4|=4+1+2+1+0+1+0+0+0+0=9.$$

For $q=5$, we have

$$|\mathcal{F}_3^5|=3+2(2)+2(2)+0+4+0+0+0+1+0=16.$$

Notice that this result is more complicated than Proposition 3.2.3. It would be possible to find closed forms of the summations but they would not add clarity to the result. We also note that this sequence appears starting at the third term of sequence A153057 on the Online Encyclopedia of Integer Sequences [17].

Next, we illustrate $\mathcal{F}_3^q$ for small values of q.

4.3 Small values of q

The following sections contain the minimal forbidden substructures for $q=4,5,7,8,$ $10,11,$ and 13. In each set of structures a thick, double-headed arrow indicates that the chains it connects are the same chain, and a dashed line indicates optional precedence.

4.3.1 Lengths $[3,4]$

Figure 4.9 gives the minimal structures that are not representable by intervals with lengths in $[3,4]$. These structures together with their horizontal reflections comprise $\mathcal{F}_3^4$. Since all but one of the structures is horizontally symmetric (after possible transformations that do not change the relationships of the Hasse diagram), $\mathcal{P}[3,4]$ has eight minimal forbidden substructures.

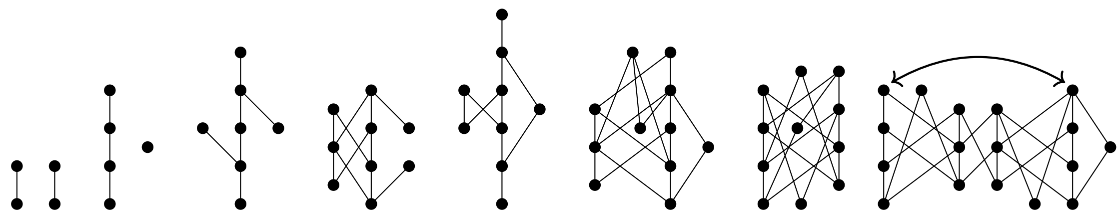


Figure 4.9: Structures that cannot appear in a [3,4] representable interval order: These and their horizontal reflections comprise $\mathcal{F}_3^4$.

There are eight structures in Figure 4.9. The sixth structure from the left is not horizontally symmetric, so $|\mathcal{F}_3^4|$=9. This confirms our count in Proposition 4.2.3.

4.3.2 Lengths $[3, 5]$

Figure 4.10 gives the minimal structures that are not representable by intervals with lengths in $[3, 5]$. These structures and their horizontal reflections make up $\mathcal{F}_3^5$.

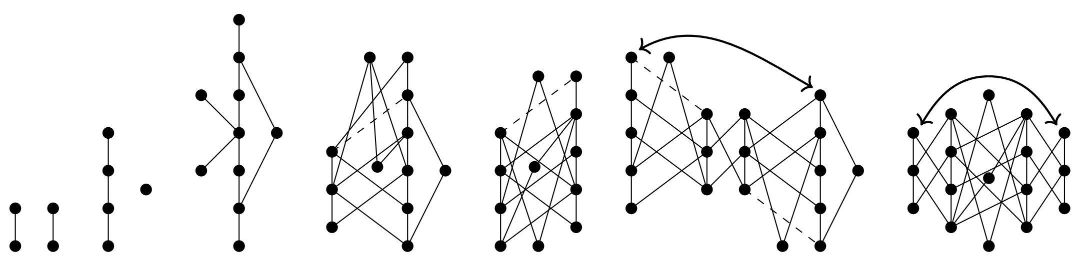


Figure 4.10: $\mathcal{F}_3^5$

There are seven structures in Figure 4.10 only two of which are not horizontally symmetric (in terms of the Hasse diagram relationships). The forth from the left has one dashed line and so represents two posets, but it is not horizontally symmetric, so it counts for four posets. Similarly, the fifth from the left adds four forbidden posets. The sixth from the left has two dashed lines between different pairs of chains and so represents four posets. Thus, $|\mathcal{F}_3^5|$ contains sixteen elements. This agrees with our calculation after Proposition 4.2.3.

4.3.3 Lengths $[3,7]$

Figure 4.11 gives the minimal structures that are not representable by intervals with lengths in $[3,7]$. These structures and their horizontal reflections make up $\mathcal{F}_3^7$.

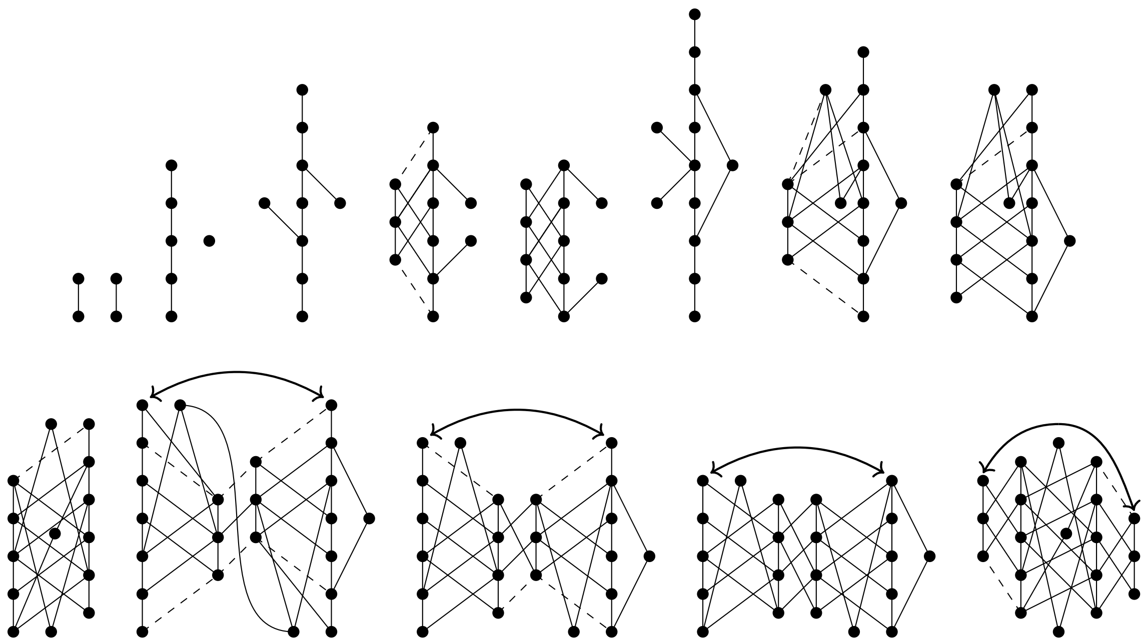


Figure 4.11: $\mathcal{F}_3^7$

4.3.4 Lengths $[3,8]$

Figure 4.12 gives the minimal structures that are not representable by intervals with lengths in $[3,8]$. These structures and their horizontal reflections make up $\mathcal{F}_3^8$.

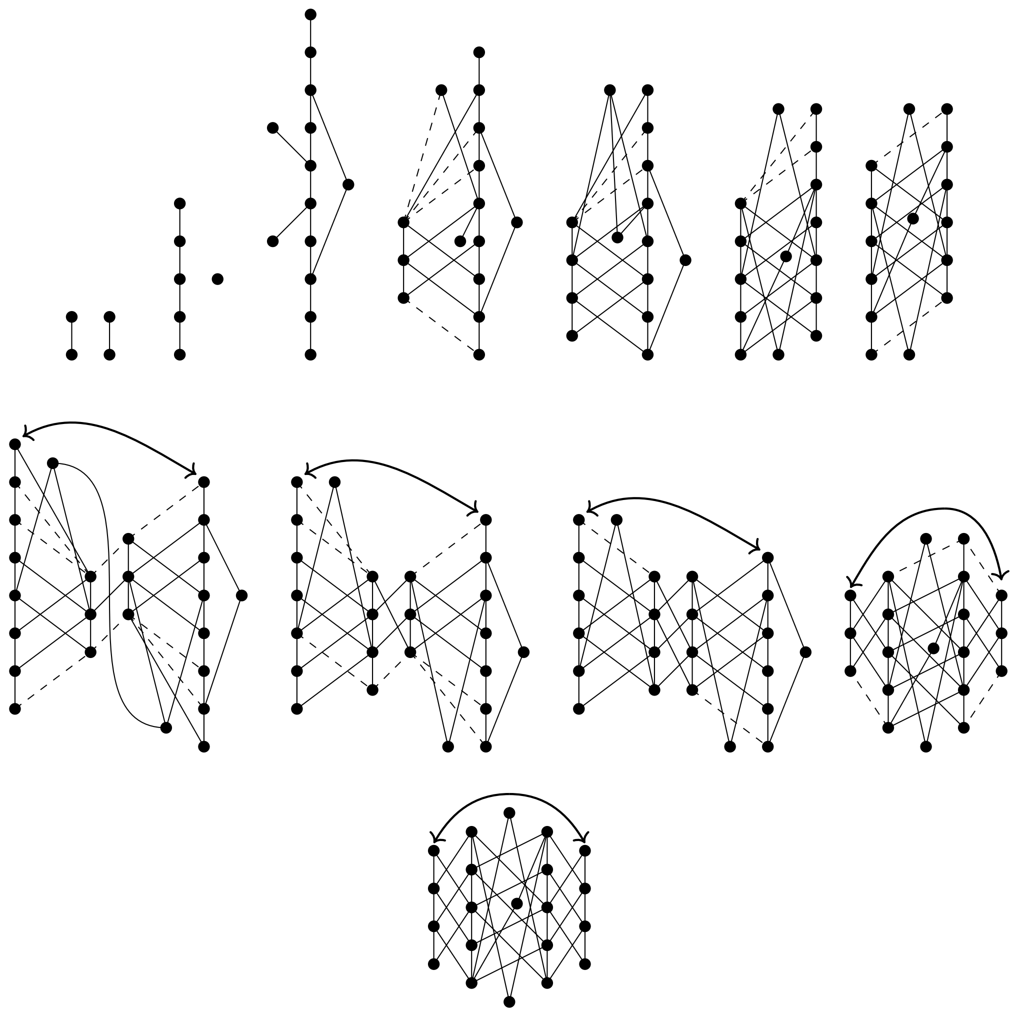


Figure 4.12: $\mathcal{F}_3^8$

4.3.5 Lengths $[3, 10]$

Figure 4.13 gives the minimal structures that are not representable by intervals with lengths in $[3, 10]$. These structures and their horizontal reflections make up $\mathcal{F}_3^{10}$.

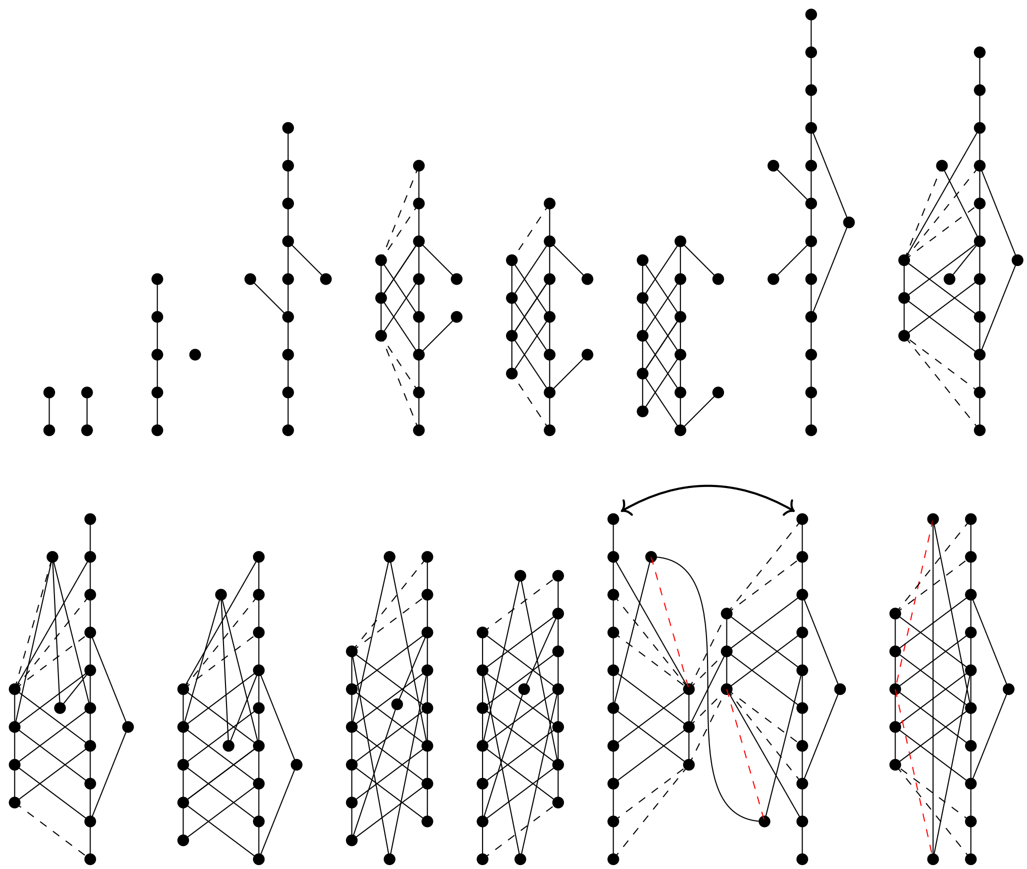


Figure 4.13: $\mathcal{F}_3^{10}$

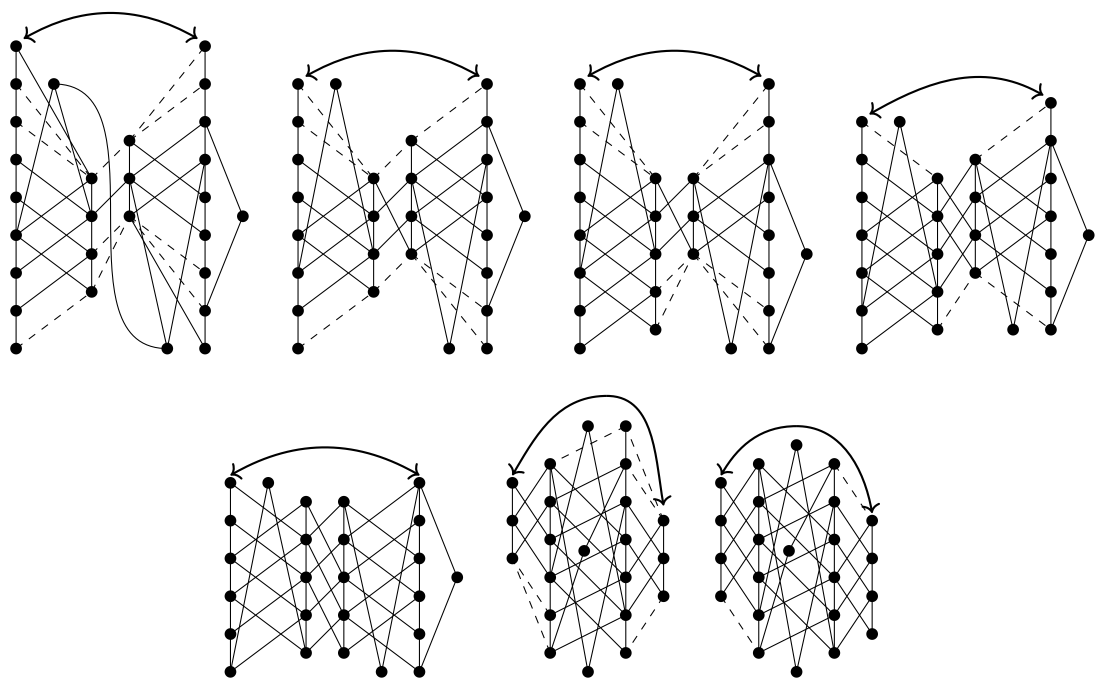


Figure 4.13 (cont): $\mathcal{F}_3^{10}$

4.3.6 Lengths $[3, 11]$

Figure 4.14 gives the minimal structures that are not representable by intervals with lengths in $[3, 11]$. These structures and their horizontal reflections comprise $\mathcal{F}_3^{11}$.

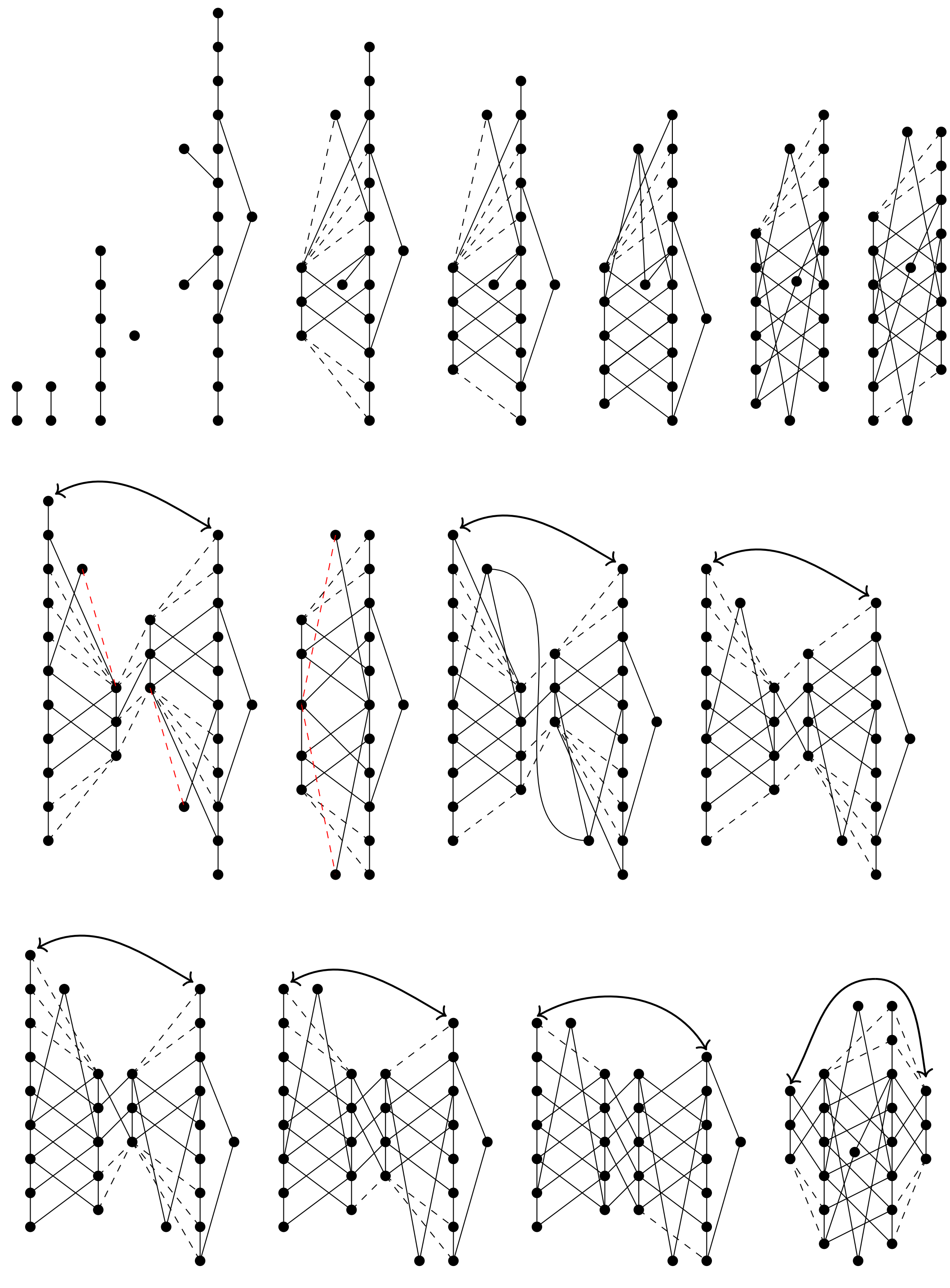


Figure 4.14: $\mathcal{F}_3^{11}$

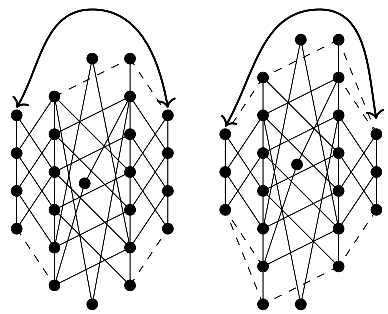


Figure 4.14 (cont): $\mathcal{F}_3^{11}$

4.3.7 Lengths $[3, 13]$

Figure 4.15 gives the minimal structures that are not representable by intervals with lengths in $[3, 13]$. These structures and their horizontal reflections comprise $\mathcal{F}_3^{13}$.

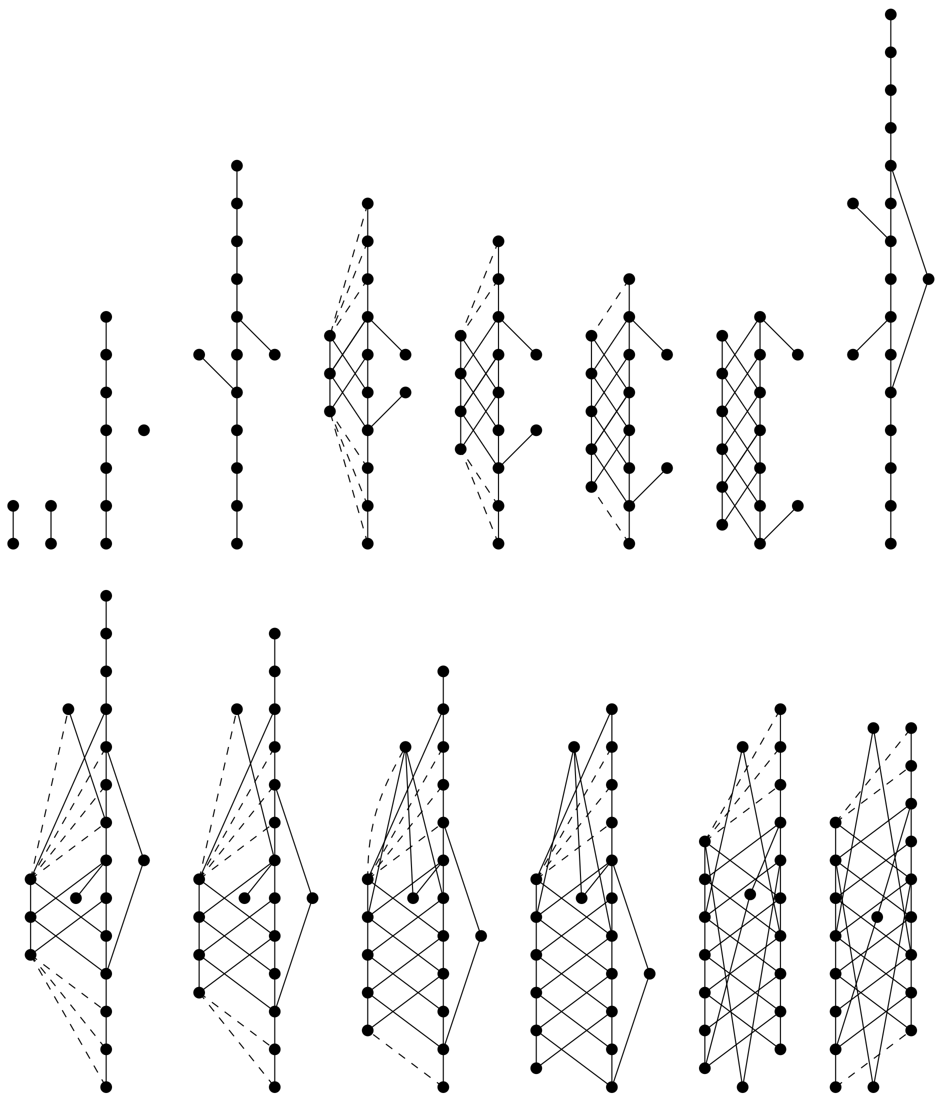


Figure 4.15: $\mathcal{F}_3^{13}$

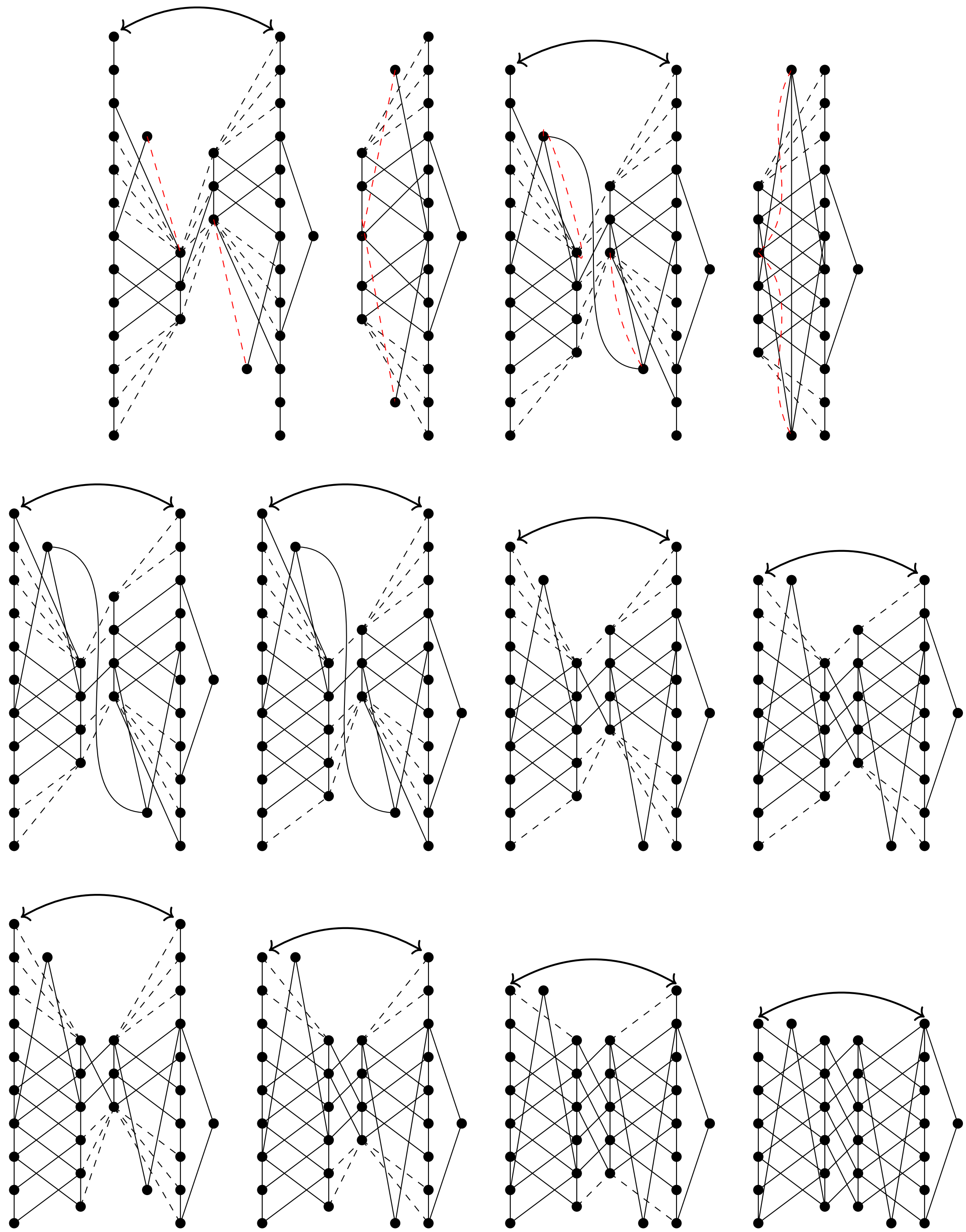


Figure 4.15 (cont): $\mathcal{F}_3^{13}$

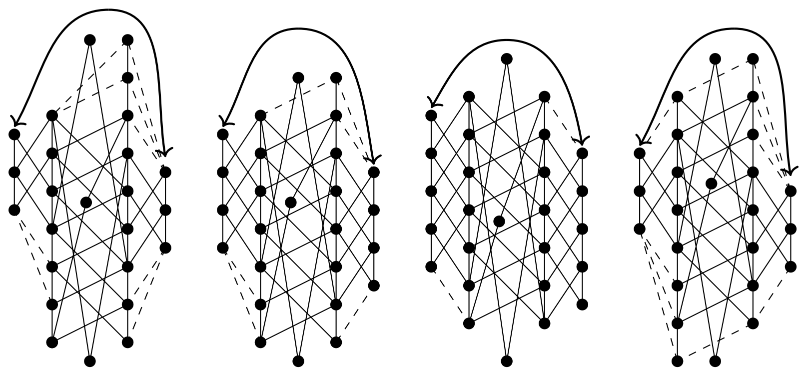


Figure 4.15 (cont): $\mathcal{F}_3^{13}$

Chapter 5

Interval orders with lengths $[p,q]$

Chapter 2 introduced the methods used in the subsequent chapters, Chapters 2 and 3 used these methods to prove results for interval lengths in $[2,q]$ and $[3,q]$ respectively. We conclude by providing incomplete results for larger values of p, and discussing why enumerating $\mathcal{F}_p^q$ becomes increasingly difficult as p increases. We will provide a general characterization of minimal forbidden structures in $\mathcal{F}_p^q$, but we are not yet able to count them.

5.1 $\mathcal{P}[p,kp{+}1], k{\in}\mathbb{Z}_+$

In this section, we state a result relating the minimal forbidden substructures for $\mathcal{P}[p,kp{+}1]$ and $\mathcal{P}[p{+}1,k(p{+}1){+}1]$. First, consider the example with $k{=}1$ of $\mathcal{P}[2,3]$ and $\mathcal{P}[3,4]$. Figure 3.8 gives $\mathcal{F}_2^3$, and Figure 4.9 illustrates $\mathcal{F}_3^4$. Notice that Figure 4.9 contains all of the elements of Figure 3.8. Thus, $\mathcal{F}_2^3 \subset \mathcal{F}_3^4$. Proposition 5.1.1 generalizes this result to all values of p and k.

Proposition 5.1.1. $\mathcal{F}_p^{kp+1} \subseteq \mathcal{F}_{p+1}^{k(p+1)+1}$, $k{\in}\mathbb{Z}_+$.

Proof. Let $P{\in}\mathcal{F}_p^{kp+1}$, and let C be a minimal negative cycle in $D_p^{kp+1}(P)$. Then, by Lemma 2.2.10, C contains $\alpha{\in}[p]$ weight $kp{+}1$ arcs, and by Lemma 2.2.6, $\beta{:=}\left\lceil \frac{\alpha(kp+1)}{p} \right\rceil$ weight $-p$ arcs. Since $(k\alpha{+}1)p \geq \alpha(kp{+}1) > k\alpha p$, we have $\beta{=}k\alpha{+}1$. By Lemma 2.2.4, C is a sequence of adjacent sets of adjacent weight $kp{+}1$ arcs and adjacent weight -p arcs.

Now, in $D_{p+1}^{k(p+1)+1}(P)$, C is still a negative cycle since it contains α weight $k(p+1)+1$ arcs, and $k\alpha+1$ weight -$(p+1)$ arcs for a total weight less than $\alpha(k(p+1)+1)-(k\alpha+1)(p+1)=\alpha-p-1\leq$-1. By Theorem 2.1.5, P is forbidden in $\mathcal{P}[p+1,k(p+1)+1]$.

Assume C is not a minimal negative cycle in $D_{p+1}^{k(p+1)+1}(P)$. Then, there exists a pair $x_i,x_j\in P$ represented in C such that an arc between two of their vertices in $D_{p+1}^{p+2}(P)$ creates a shorter negative cycle, C' in C. C' has $\alpha'\leq\alpha\leq p$ arcs of weight $k(p+1)+1$. For C' to have negative weight, it must contain $\beta'\geq\left\lceil\frac{\alpha'(k(p+1)+1)}{p+1}\right\rceil$ arcs of weight -$(p+1)$. Now, since $\alpha'\leq p<p+1$ and $k\alpha'$ is an integer, we have

$$\left\lceil k\alpha'+\frac{\alpha'}{p+1}\right\rceil=\left\lceil k\alpha'+\frac{\alpha'}{p}\right\rceil$$

and

$$\beta'\geq\left\lceil\frac{\alpha'(k(p+1)+1)}{p+1}\right\rceil=\left\lceil\frac{\alpha'(kp+1)}{p}\right\rceil.$$

Thus, C' also has negative weight in $D_p^{kp+1}(P)$ which contradicts our assumption that C is minimal in $D_p^{kp+1}(P)$. Thus, C is minimal in $D_{p+1}^{k(p+1)+1}(P)$, and $P\in\mathcal{F}_{p+1}^{k(p+1)+1}$. □

Corollary 5.1.2 motivated the investigation into this area, and so we give it some special attention.

Corollary 5.1.2. $\mathcal{F}_p^{p+1}\subseteq\mathcal{F}_{p+1}^{p+2}$.

Returning to our small example, we have $\mathcal{F}_2^3\subset\mathcal{F}_3^4$. Thus, if we remove any element from $P\in\mathcal{F}_2^3$, it is not only representable with lengths in $[2,3]$ it is also representable with lengths in $[3,4]$.

Recall $\mathcal{P}[p,q]=\mathcal{P}[1,q/p]$ [8]. That means we can restate the conclusion of the last paragraph as: if we remove any element from $P\in\mathcal{F}_2^3$, it is not only representable with lengths in $[1,3/2]$ it is also representable with lengths in $[1,4/3]$.

In general, if we remove any element from $P\in\mathcal{F}_p^{kp+1}$, it is not only representable with lengths in $\left[1,\frac{kp+1}{p}\right]$, but also representable with lengths in $\left[1,\frac{k(p+1)+1}{p+1}\right]$.

Now, as p increases $\frac{kp+1}{p}$ approaches k. Interestingly, $|\mathcal{F}_1^k|=2$ and $|\mathcal{F}_p^{kp+1}|$ increases as p increases (see Corollary 5.2.6).

5.2 Structures in $\mathcal{F}_p^q$ for all p

The following result holds for $p \geq 1$. It defines the element of $\mathcal{F}_p^q$ which corresponds to a negative cycle in the digraph with exactly one weight q arc.

Lemma 5.2.1. $P = \left\lceil \mathbf{\frac{q+2p}{p}} \right\rceil + \mathbf{1} \in \mathcal{F}_p^q$ for all p.

Proof. By Theorem 2.1.5, we must show that $\left\lceil \mathbf{\frac{q+2p}{p}} \right\rceil + \mathbf{1}$ corresponds to a minimal negative cycle in the digraph. Label the chain from top to bottom as $x_1, x_2, \ldots,$ $x_{\lceil (q+2p)/p \rceil}$ and the other element as y_1. Then, $D_p^q(P)$ contains the cycle

$$x_{1\ell}, x_{2r}, x_{2\ell}, x_{3r}, x_{3\ell}, \ldots, x_{\lceil (q+2p)/p \rceil - 1_r}, x_{\lceil (q+2p)/p \rceil - 1_\ell}, x_{\lceil (q+2p)/p \rceil_r}, y_{1\ell}, y_{1r}, x_{1\ell},$$

call it C. Now, C contains one weight q arc, $\left\lceil \frac{q+2p}{p} \right\rceil - 2$ arcs of weight $-p$, and $\left\lceil \frac{q+2p}{p} \right\rceil - 1$ arcs of weight $-\epsilon$; and has weight $q - p(\lceil (q+2p)/p \rceil - 2) - \epsilon(\lceil (q+2p)/p \rceil - 1) = q - p\lceil q/p \rceil - \epsilon(\lceil (q+2p)/p \rceil - 1)$. Clearly $\left\lceil \mathbf{\frac{q+2p}{p}} \right\rceil + \mathbf{1}$ does not contains a $\mathbf{2}+\mathbf{2}$ as a subposet, so by Lemma 2.2.1, $D_p^q(P)$ does not contain a negative cycle with only weight 0 and weight $-\epsilon$ arcs. By Corollary 2.2.2 and Lemma 2.2.3, a minimal negative cycle must contain a weight q arc. If a minimal negative cycle contains one weight q arc, by Lemma 2.2.6, it must contain $\left\lceil \frac{q}{p} \right\rceil$ weight $-p$ arcs. Cycle C contains these numbers of weight q and $-p$ arcs and the minimum number of weight 0 and $-\epsilon$ arcs as given by Lemma 2.2.4. Thus, C is minimal. $\square$

Lemma 5.2.2 defines the elements of $\mathcal{F}_p^q$ which correspond to negative cycles in the digraph with exactly two weight q arcs. Here, we must restrict the possible values of q as in Corollary 3.1.3 because their associated negative cycles are non-minimal and thus correspond to nonminimal forbidden structures. See the disscussion after Corollary 3.1.3 for an example with $p=4$ and $q=11$.

Lemma 5.2.2. Let P be one of the posets in $\mathcal{F}_p^q(2)$ (Definition 3.1.1). Then, $P \in \mathcal{F}_p^q$ for all p, q such that $q = ps + d$ with $d \leq \frac{p}{2}$.

Proof. Let P be as in the statement of the lemma. By Theorem 2.1.5, we must show that P corresponds to a minimal negative cycle in the digraph. First, $\left\lceil \frac{2q}{p} \right\rceil = \left\lceil \frac{2(ps+d)}{p} \right\rceil = 2s+1$.

For poset (i), label the chain from top to bottom as $y_1, z_1, z_2, \dots, z_{2s+1}, x_1$, label the element on the right u_1, and label the element on the left u_2. Then, $y_1 \succ z_1 \succ z_2 \succ \cdots \succ z_{2s+1} \succ x_1 \cap u_1 \cap u_2 \cap y_1$, and $D_p^q(P)$ contains the cycle

$$y_{1_\ell}, z_{1_r}, z_{1_\ell}, z_{2_r}, z_{2_\ell}, \ldots, z_{2s+1_r}, z_{2s+1_\ell}, x_{1_r}, u_{1_\ell}, u_{1_r}, u_{2_\ell}, u_{2_r}, y_{1_\ell}.$$

Call it C. Now, C contains two weight q arcs, $2s+1$ arcs of weight $-p$, and $2s+2$ arcs of weight $-\epsilon$; and has weight $2q-p(2s+1)-\epsilon(2s+2)=2(ps+d)-2ps-p-\epsilon(2s+2)=2d-p-\epsilon(2s+2)<0$ since $d \leq \frac{p}{2}$. Clearly P does not contain a $\mathbf{2+2}$ as a subposet, so by Lemma 2.2.1, $D_p^q(P)$ does not contain a negative cycle with only weight 0 and weight $-\epsilon$ arcs. Since $\left\lceil \frac{q}{p} \right\rceil = s+1$, P does not contain a $\left\lceil \frac{\mathbf{q+2p}}{\mathbf{p}} \right\rceil + \mathbf{1}$ which by Lemma 2.2.9 is the minimal cycle which corresponds to one weight q arc. If a minimal negative cycle contains two weight q arcs, by Lemma 2.2.6, it must contain $\left\lceil \frac{2q}{p} \right\rceil$ weight $-p$ arcs. Cycle C contains these numbers of weight q and $-p$ arcs and the minimum number of weight 0 and $-\epsilon$ arcs as given by Lemma 2.2.4. Thus, C is minimal.

For posets in (ii), label the right chain from top to bottom as $y_2, z_1, z_2, \dots, z_a, x_1$, label the left chain from top to bottom as $y_1, z_{a+1}, z_{a+2}, \dots, z_{2s+1}, x_2$, label the top right element as u_1 and the bottom right element as u_2. Then, $y_2 \succ z_1 \succ z_2 \succ \cdots \succ z_a \succ x_1 \cap u_1 \cap y_1 \succ z_{a+1} \succ z_{a+2} \succ z_{2s+1} \succ x_2 \cap u_2 \cap y_2$ and $D_p^q(P)$ contains the cycle

$$y_{2_\ell}, z_{1_r}, z_{1_\ell}, z_{2_r}, z_{2_\ell}, \ldots, z_{a_r}, z_{a_\ell}, x_{1_r}, u_{1_\ell}, u_{1_r}, y_{1_\ell}, z_{a+1_r}, z_{a+1_\ell}, z_{a+2_r}, z_{a+2_\ell}, \ldots, z_{2s+1_r},$$

$$z_{2s+1_\ell}, x_{2_r}, u_{2_\ell}, u_{2_r}, y_{2_\ell}.$$

Call it C'. Cycle C' contains two arcs of weight q, $2s+1$ arcs of weight $-p$, and $2s+3$ arcs of weight $-\epsilon$, and has weight $2q-p(2s+1)-\epsilon(2s+3)=2d-p-\epsilon(2s+3)<0$. Again P does not contain an induced $\mathbf{2+2}$ or an induced $\left(\left\lceil \frac{\mathbf{q}}{\mathbf{p}} \right\rceil + \mathbf{2}\right) + \mathbf{1}$ so C' cannot be shortened to a negative cycle with no positive weight arcs or only one positive weight arc. By Lemmas 2.2.6 and 2.2.4 C' is a minimal negative cycle which contains exactly two weight q arcs. Thus, P is a minimal forbidden substructure for $\mathcal{P}_p^q$ and so $P \in \mathcal{F}_p^q$. $\square$

Lemma 5.2.3 defines the elements of $\mathcal{F}_p^q$ which correspond to negative cycles in the digraph with exactly three weight q arcs. Again, we must restrict the possible values of q as in Corollary 4.1.3 if their associated negative cycles are non-minimal and thus correspond to nonminimal forbidden structures.

Lemma 5.2.3. Let P be one of the posets in $\mathcal{F}_p^q(3)$ (Definition 4.1.1). Then, $P \in \mathcal{F}_p^q$ for all p, q such that $q = ps + d$ with $d \leq \frac{p}{3}$ or $\frac{p}{2} < d \leq \frac{2p}{3}$.

Proof. Let P be as in the statement of the lemma. By Theorem 2.1.5, we must show that P corresponds to a minimal negative cycle in the digraph.

If P contains an induced poset isomorphic to Figure 4.1i, say $y_1 \succ z_1 \succ z_2 \succ \cdots \succ z_{\lceil 3q/p \rceil} \succ x_1 \cap u_1 \cap u_2 \cap u_3$ with $u_1 \prec z_{\lceil 2q/p \rceil - 1}$, $z_{\lceil q/p \rceil - 1} \succ u_2 \succ z_{\lceil 2q/p \rceil + 1}$, and $u_3 \succ z_{\lceil q/p \rceil + 1}$. Then, the cycle

$$z_{1_r}, z_{1_\ell}, z_{2_r}, z_{2_\ell}, \ldots, z_{\beta_r}, z_{\beta_\ell}, x_{1_r}, u_{1_\ell}, u_{1_r}, u_{2_\ell}, u_{2_r}, u_{3_\ell}, u_{3_r}, y_{1_\ell}, z_{1_r}$$

has weight $q(3) - p(\beta) - \epsilon(\beta + 1) < 0$.

Now, if P contains an induced subposet isomorphic to a poset in 4.1ii, then the cycle

$$z_{1_r}, z_{1_\ell}, z_{2_r}, z_{2_\ell}, \ldots, z_{a_r}, z_{a_\ell}, x_{1_r}, u_{1_\ell}, u_{1_r}, y_{1_\ell}, z_{a+1_r}, z_{a+1_\ell}, z_{a+2_r}, z_{a+2_\ell}, \cdots, z_{\beta_r}, z_{\beta_\ell}, x_{2_r},$$

$$u_{2_\ell}, u_{2_r}, u_{3_\ell}, u_{3_r}, y_{2_\ell}, z_{1_r}$$

has weight $q(3) - p(\beta) - \epsilon(\beta + 2) < 0$.

Finally, if P contains an induced subset isomorphic to a poset in family 4.1iii, 4.1iv, or 4.1v, then

$$z_{1_r}, z_{1_\ell}, z_{2_r}, z_{2_\ell}, z_{3_r}, \cdots, z_{a_\ell}, x_{1_r}, u_{1_\ell}, u_{1_r}, y_{1_\ell}, z_{a+1_r}, z_{a+1_\ell}, \cdots, z_{a+b_\ell}, x_{2_r}, u_{2_\ell}, u_{2_r}, y_{2_\ell},$$

$$z_{a+b+1_r}, z_{a+b+1_\ell} \cdots, z_{q_\ell}, x_{3_r}, u_{3_\ell}, u_{3_r}, y_{3_\ell}$$

is a cycle with weight $q(3) - p(\beta) - \epsilon(\beta + 3) < 0$.

From the proof of Proposition 4.1.2, the relationships of the structures in Figure 4.1 do not allow for shorter negative cycles. Thus, P is a minimal forbidden substructure for $\mathcal{F}_p^q$ and so $P \in \mathcal{F}_p^q$.

$\square$

The list of structures in $\mathcal{F}_p^3$ (Figure 4.6) is already quite long, and to draw them clearly, we chose to repeat one of the chains. As Lemma 2.2.10 shows the number of chains in the minimal structures of $\mathcal{F}_p^q$ could be as large as p. Drawing these structures would become cumbersome and potentially unhelpful. In what follows, we consider the minimal negative cycle structure and the posets they produce without attempting to draw the resulting posets.

5.2.1 Cycle structure

By Lemma 2.2.4, the minimal cycle structures and thus minimal cycles for larger values of p would look similar to the cycles already considered for $p{=}2$ and $p{=}3$.

In proposition, we consider the relationships not directly defined by the cycle. We also encounter a divisibility issue analogous to the one addressed in Corollary 4.1.3 in which there are no minimal negative cycles with $\alpha{=}2$ when $p{=}3$ and $d{=}q \bmod 3$ such that $\frac{p}{3}{<}d{\leq}\frac{p}{2}$ or $d{>}\frac{2p}{3}$. When this happens, the negative cycle is not minimal and thus corresponds to a forbidden structure that is not minimal. We will address but not settle these divisibility issues in Section 5.2.2.

Proposition 5.2.4. Let P be a finite poset. Let C be a minimal negative cycle in $D_p^q(P)$. If C contains exactly α weight q arcs and is labeled as in Figure 5.1, then P contains an induced subposet with the following relationships where $i,j\in\{1,2,\dots,\alpha\}$, $i{\leq}j$ and subscript arithmetic is considered modulo α with α as the additive identity:

1. $$\begin{cases} u_i \prec u_j & \displaystyle\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil \\ u_i \cap u_j & \displaystyle\left\lceil \frac{(j-i-1)q}{p} \right\rceil \leq \sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ u_i \succ u_j & \displaystyle\sum_{k=i+1}^{j} \beta_k \geq \left\lceil \frac{(j-i+1)q}{p} \right\rceil \end{cases},$$

2. $\begin{cases} u_i \prec y_j & \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i)q}{p} \right\rceil \\ u_i \cap y_j & \left\lceil \frac{(j-i)q}{p} \right\rceil \leq \sum\limits_{k=i+1}^{j} \beta_k \leq \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ u_i \cap\succ y_j & \left\lceil \frac{(j-i+1)q}{p} \right\rceil < \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i+2)q}{p} \right\rceil \\ u_i \succ y_j & \sum\limits_{k=i+1}^{j} \beta_k \geq \left\lceil \frac{(j-i+2)q}{p} \right\rceil \end{cases},$

3. for $n \in \left\{ \sum\limits_{k=1}^{j-1} \beta_k + 1, \ldots, \sum\limits_{k=1}^{j} \beta_k \right\}$ and $j' = n - \sum\limits_{k=1}^{j-1} \beta_k$,

$$\begin{cases} u_i \prec z_n & \sum\limits_{k=i+1}^{j-1} \beta_k + j' < \left\lceil \frac{(j-i-1)q}{p} \right\rceil \\ u_i \cap z_n & \left\lceil \frac{(j-i-1)q}{p} \right\rceil \leq \sum\limits_{k=i+1}^{j-1} \beta_k + j' \leq \left\lceil \frac{(j-i)q}{p} \right\rceil, \\ u_i \succ z_n & \sum\limits_{k=i+1}^{j-1} \beta_k + j' > \left\lceil \frac{(j-i)q}{p} \right\rceil \end{cases}$$

4. $\begin{cases} u_i \prec x_j & \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-2)q}{p} \right\rceil \\ u_1 \prec\cap x_j & \left\lceil \frac{(j-i-2)q}{p} \right\rceil \leq \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil - 1 \\ u_i \cap x_j & \left\lceil \frac{(j-i-1)q}{p} \right\rceil - 1 \leq \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i)q}{p} \right\rceil \\ u_i \succ x_j & \sum\limits_{k=i+1}^{j} \beta_k \geq \left\lceil \frac{(j-i)q}{p} \right\rceil \end{cases},$

5. $\begin{cases} y_i \succ y_j & \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil \\ y_1 \cap\succ y_j & \left\lceil \frac{(j-i-1)q}{p} \right\rceil \leq \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \\ y_i \cap y_j & \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \leq \sum\limits_{k=i+1}^{j} \beta_k \leq \left\lceil \frac{(j-i)q}{p} \right\rceil - 1, \\ y_1 \prec\cap y_j & \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 < \sum\limits_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ y_i \prec y_j & \left\lceil \frac{(j-i+1)q}{p} \right\rceil \leq \sum\limits_{k=i+1}^{j} \beta_k \end{cases}$

6. for $n \in \left\{\sum_{k=1}^{j-1}\beta_k+1,\ldots,\sum_{k=1}^{j}\beta_k\right\}$ and $j'=n-\sum_{k=1}^{j-1}\beta_k$,

$$\begin{cases} y_i \prec z_n & \sum_{k=i+1}^{j-1}\beta_k+j'<\left\lceil\frac{(j-i-2)q}{p}\right\rceil \\ y_i \prec\cap z_n & \left\lceil\frac{(j-i-2)q}{p}\right\rceil\le\sum_{k=i+1}^{j-1}\beta_k+j'<\left\lceil\frac{(j-i-1)q}{p}\right\rceil-1 \\ y_i \cap z_n & \left\lceil\frac{(j-i-1)q}{p}\right\rceil-1\le\sum_{k=i+1}^{j-1}\beta_k+j'\le\left\lceil\frac{(j-i-1)q}{p}\right\rceil \\ y_i \succ z_n & \sum_{k=i+1}^{j-1}\beta_k+j'>\left\lceil\frac{(j-i-1)q}{p}\right\rceil \end{cases},$$

7. $$\begin{cases} y_i \prec x_j & \sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i-3)q}{p}\right\rceil \\ y_i \prec\cap x_j & \left\lceil\frac{(j-i-3)q}{p}\right\rceil\le\sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i-1)q}{p}\right\rceil-2 \\ y_i \cap x_j & \left\lceil\frac{(j-i-1)q}{p}\right\rceil-2\le\sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i-1)q}{p}\right\rceil \\ y_i \succ x_j & \left\lceil\frac{(j-i-1)q}{p}\right\rceil\le\sum_{k=i+1}^{j}\beta_k \end{cases},$$

8. for $n \in \left\{\sum_{k=1}^{j-1}\beta_k+1,\ldots,\sum_{k=1}^{j}\beta_k\right\}$ and $j'=n-\sum_{k=1}^{j-1}\beta_k$,

$$\begin{cases} x_i \prec z_n & \sum_{k=i+1}^{j-1}\beta_k+j'<\left\lceil\frac{(j-i)q}{p}\right\rceil \\ x_i \cap z_n & \left\lceil\frac{(j-i)q}{p}\right\rceil\le\sum_{k=i+1}^{j-1}\beta_k+j'\le\left\lceil\frac{(j-i)q}{p}\right\rceil+1 \\ x_i \cap\succ z_n & \left\lceil\frac{(j-i)q}{p}\right\rceil+1<\sum_{k=i+1}^{j-1}\beta_k+j'\le\left\lceil\frac{(j-i+1)q}{p}\right\rceil \\ x_i \succ z_n & \left\lceil\frac{(j-i+1)q}{p}\right\rceil<\sum_{k=i+1}^{j-1}\beta_k+j' \end{cases},$$

$$9.\ \begin{cases} x_i \prec x_j & \sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil \\ x_i \prec\cap x_j & \left\lceil \frac{(j-i-1)q}{p} \right\rceil \leq \sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \\ x_i \cap x_j & \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \leq \sum_{k=i+1}^{j} \beta_k \leq \left\lceil \frac{(j-i)q}{p} \right\rceil \\ x_i \cap\succ x_j & \left\lceil \frac{(j-i)q}{p} \right\rceil < \sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ x_i \succ x_j & \left\lceil \frac{(j-i+1)q}{p} \right\rceil \leq \sum_{k=i+1}^{j} \beta_k \end{cases},$$

10. for $m \in \left\{ \sum_{k=1}^{i-1} \beta_k + 1, \ldots, \sum_{k=1}^{i} \beta_k \right\}$ with $i' = m - \sum_{k=1}^{i-1} \beta_k$ and $n \in \left\{ \sum_{k=1}^{j-1} \beta_k + 1, \ldots, \sum_{k=1}^{j} \beta_k \right\}$ with $j' = n - \sum_{k=1}^{j-1} \beta_k$,

$$\begin{cases} z_m \prec z_n & \sum_{k=i}^{j-1} \beta_k + j' - i' < \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \\ z_m \cap z_n & \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \leq \sum_{k=i}^{j-1} \beta_k + j' - i' \leq \left\lceil \frac{(j-i)q}{p} \right\rceil \\ z_m \succ z_n & \left\lceil \frac{(j-i)q}{p} \right\rceil < \sum_{k=i}^{j-1} \beta_k + j' - i' \end{cases}.$$

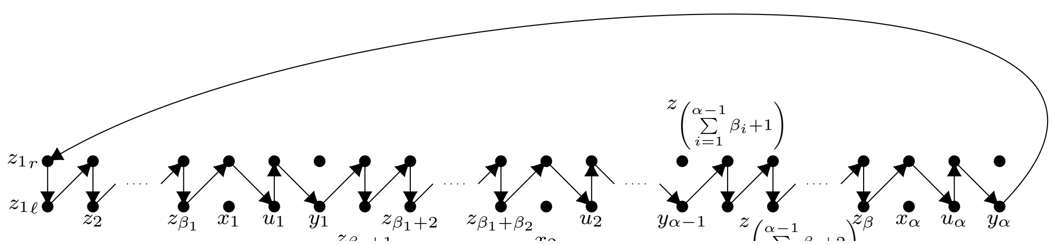


Figure 5.1: Cycle in $D_p^q(P)$ with α weight q arcs and β weight $-p$ arcs

Proof. Consider a minimal cycle, C, in $D_p^q(P)$ with α weight q arcs.

We will assume that $\left\lceil \frac{\alpha q}{p} \right\rceil - \left\lceil \frac{\alpha' q}{p} \right\rceil = \left\lceil \frac{(\alpha-\alpha')q}{p} \right\rceil - 1$ for all $\alpha' < \alpha$. In Remark 1 we will discuss why this is true when a cycle is minimal.

Let $q = ps + d$ with $d \in \{1, 2, \ldots, p-1\}$ and $gcd(p, d) = 1$.

We have $\beta = \left\lceil \frac{\alpha q}{p} \right\rceil = \left\lceil \frac{\alpha(ps+d)}{p} \right\rceil = \alpha s + \left\lceil \frac{\alpha d}{p} \right\rceil = \begin{cases} \alpha s+1 & d \leq \frac{p}{\alpha} \\ \alpha s+2 & \frac{p}{\alpha} < d \leq \frac{2p}{\alpha} \\ \alpha s+3 & \frac{2p}{\alpha} < d \leq \frac{3p}{\alpha} \\ \vdots & \\ \alpha s+\alpha & \frac{(\alpha-1)p}{\alpha} < d \end{cases}.$

By Lemma 2.2.4, we can draw C as in Figure 5.1.

To simplify the calculations, we will disregard the weight $-\epsilon$ arcs when finding cycle weights. Thus, if a cycle has weight 0 below, it will be considered a negative cycle because all of the cycles considered have at least one weight $-\epsilon$ arc.

In Figure 5.1, $\sum_{i=1}^{\alpha} \beta_i = \beta$. If $\beta_i = 0$, eliminate its corresponding x and y elements. Since $\beta \geq \alpha s+1$, $\exists i'$ such that $\beta_{i'} \geq s+1$. Now, consider the relationship between $z_{\beta_{i'}-(s+1)}$ and $u_{\beta_{i'}}$ such that $z_{\beta_{i'}-(s+1)} = y_{\beta_{i'}-1}$ if $\beta_{i'} = s+1$. By transitivity, $u_{\beta_{i'}} \not\succ z_{\beta_{i'}-(s+1)}$. If $z_{\beta_{i'}-(s+1)} \succ u_{\beta_{i'}}$, then replacing $z_{\beta_{i'}-(s+1)_\ell} \to u_{\beta_{i'}r}$ in C with the arc $(z_{\beta_{i'}-(s+1)_\ell}, u_{\beta_{i'}r})$ creates a cycle with weight

$$\begin{aligned} (\alpha-1)q - p(\alpha s + \lceil(\alpha d)/p\rceil - (s+1)) &= (\alpha-1)(ps+d) - p((\alpha-1)s + \lceil(\alpha d)/p\rceil - 1) \\ &= (\alpha-1)d - p\lceil(\alpha d)/p\rceil + p \\ &= \begin{cases} (\alpha-1)d & d \leq \frac{p}{\alpha} \\ (\alpha-1)d - p & \frac{p}{\alpha} < d \leq \frac{2p}{\alpha} \\ (\alpha-1)d - 2p & \frac{2p}{\alpha} < d \leq \frac{3p}{\alpha} \\ \vdots & \\ (\alpha-1)d - (\alpha-1)p & \frac{(\alpha-1)p}{\alpha} < d \end{cases} \end{aligned}$$

which is non-positive when $\frac{kp}{\alpha} < d \leq \frac{kp}{\alpha-1}$ $\forall k \in [\alpha-1]$. (Note $\frac{kp}{\alpha-1} \leq \frac{(k+1)p}{\alpha}$ $\forall k \in [\alpha-1]$.) If $z_{\beta_{i'}-(s+1)} \cap u_{\beta_{i'}}$, then replacing $u_{\beta_{i'}r} \to z_{\beta_{i'}-(s+1)_\ell}$ in C with the arc $(u_{\beta_{i'}r}, z_{a-(s+1)_\ell})$ creates a cycle with weight $q - p(s+1) < 0$. Thus, when $\frac{kp}{\alpha} < d \leq \frac{kp}{\alpha-1}$ for any $k \in [\alpha-1]$, all relationships between $z_{\beta_{i'}-(s+1)}$ and $u_{\beta_{i'}}$ yield shorter negative cycles. For the remainder of the proof, we will assume that $\frac{kp}{\alpha-1} < d \leq \frac{(k+1)p}{\alpha}$ for some $k \in [\alpha-1]$.

The preceding paragraph is an incomplete analysis for minimality. This only covers the case where C can be divided into two cycles: one with one weight q arc and one with $\alpha-1$ weight q arcs. This type of cycle division can also happen with different numbers of positive weight arcs. See Remark 1.

All elements labeled in C are distinct except we could have $x_i = y_j$ for some i, j pair. By Corollary 2.2.8, x_i is distinct from y_{i-1} and y_i. Lemma 2.2.7 gives the conditions when an element can repeat. Thus, each x_i, y_j pair which can represent the same element produces structures where they are different elements and ones where they are the same.

In what follows, we will use subscripts $i, j \in \{1, 2, \ldots, \alpha\}$. In each case, assume without loss of generality that $i \le j$ (rotate cycle if necessary). We consider subscript calculations modulo α with α as the additive identity. For example if $i = j$, $i - j = \alpha$ and $i - j - 2 = \alpha - 2$. Also, if $j < i+1$, we have $\sum_{k=i+1}^{j} \beta_k = \sum_{k=i+2}^{\alpha} \beta_k + \sum_{k=1}^{j-1} \beta_k = \beta - \sum_{k=j}^{i+1} \beta_j$.

First, we consider the relationships among the u elements.

Relationship 1:

If $u_i \prec u_j$, then replacing $u_{j_\ell} \to u_{ir}$ in C with the arc (u_{j_ℓ}, u_{ir}) creates a cycle with weight $(j-1-i)q - p\left(\sum_{k=i+1}^{j} \beta_k\right)$ which is positive when $\sum_{k=i+1}^{j} \beta_k < \frac{(j-i-1)q}{p}$. If $u_i \succ u_j$, then replacing $u_{i\ell} \to u_{j_r}$ in C with the arc $(u_{i\ell}, u_{j_r})$ creates a cycle with weight $(i-1+\alpha-j)q - p\left(\sum_{k=1}^{i} \beta_k + \sum_{k=j+1}^{\alpha} \beta_k\right) = (i-1+\alpha-j)q - p\left(\beta - \sum_{k=i+1}^{j} \beta_k\right)$ which is positive when $\sum_{k=i+1}^{j} \beta_k \ge \left\lceil \frac{(j-i+1)q}{p} \right\rceil$. If $u_i \cap u_j$, then replacing $u_{ir} \to u_{j_\ell}$ in C with the arc (u_{ir}, u_{j_ℓ}) creates a cycle with weight $(i+\alpha-(j-1))q - p\left(\sum_{k=1}^{i} \beta_k + \sum_{k=j+1}^{\alpha} \beta_k\right) = (i+\alpha-(j-1))q - p\left(\beta - \sum_{k=i+1}^{j} \beta_k\right)$ which is positive when $\sum_{k=i+1}^{j} \beta_k \ge \left\lceil \frac{(j-i-1)q}{p} \right\rceil$, and replacing $u_{2r} \to u_{1\ell}$ in C with the arc $(u_{2r}, u_{1\ell})$ creates a cycle with weight $(j-(i-1))q - p\left(\sum_{k=i+1}^{j} \beta_k\right)$ which is positive when $\sum_{k=i+1}^{j} \beta_k < \frac{(j-i+1))q}{p}$.

$$\textbf{Thus, } \begin{cases} \boldsymbol{u_i \prec u_j} & \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i-1)q}{p} \right\rceil \\ \boldsymbol{u_i \cap u_j} & \left\lceil \frac{(j-i-1)q}{p} \right\rceil \le \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ \boldsymbol{u_i \succ u_j} & \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} \ge \left\lceil \frac{(j-i+1)q}{p} \right\rceil \end{cases}.$$

Next, we consider the relationships between the u elements and the elements of the chains

Relationship 2:

If $u_i \prec y_j$, then replacing $y_{j_\ell} \to u_{ir}$ in C with the arc (y_{j_ℓ}, u_{ir}) creates a cycle with weight $(j-i)q - p\left(\sum_{k=i+1}^{j} \beta_k\right)$ which is positive when $\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i)q}{p} \right\rceil$. If $u_i \succ y_j$, then replacing $u_{i\ell} \to y_{j_\ell}$ in C with the path $u_{i\ell}, y_{j_r}, y_{j_\ell}$ creates a cycle with weight $(i-1+\alpha-j)q - p\left(1 + \sum_{k=1}^{i} \beta_k + \sum_{k=j+1}^{\alpha} \beta_k\right) = (\alpha+i-j-1)q - p\left(1+\beta - \sum_{k=i+1}^{j} \beta_k\right)$ which is positive for $\sum_{k=i+1}^{j} \beta_k > \left\lceil \frac{(j-i+1)q}{p} \right\rceil$. If $u_i \cap y_j$ then replacing $u_{ir} \to y_{j_\ell}$ in C with the arc (u_{ir}, y_{j_ℓ}) creates a cycle with weight $(i+\alpha-j)q - p\left(\sum_{k=1}^{i} \beta_k + \sum_{k=j+1}^{\alpha} \beta_k\right) = (\alpha+i-j)q - p\left(\beta - \sum_{k=i+1}^{j} \beta_k\right)$ which is positive for $\sum_{k=i+1}^{j} \beta_k \ge \left\lceil \frac{(j-i)q}{p} \right\rceil$, and replacing $y_{j_\ell} \to u_{i\ell}$ in C with the path $y_{j_\ell}, y_{j_r}, u_{i\ell}$ creates a cycle with weight $(j-(i-1)+1)q - p\left(\sum_{k=i+1}^{j} \beta_k\right)$ which is positive when $\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i+2)q}{p} \right\rceil$.

$$\textbf{Thus, } \begin{cases} \boldsymbol{u_i \prec y_j} & \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i)q}{p} \right\rceil \\ \boldsymbol{u_i \cap y_j} & \left\lceil \frac{(j-i)q}{p} \right\rceil \le \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} \le \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ \boldsymbol{u_i \cap\succ y_j} & \left\lceil \frac{(j-i+1)q}{p} \right\rceil < \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i+2)q}{p} \right\rceil \\ \boldsymbol{u_i \succ y_j} & \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} \ge \left\lceil \frac{(j-i+2)q}{p} \right\rceil \end{cases}.$$

Relationship 3:

Let $n\in\left\{\sum_{k=1}^{j-1}\beta_k+1,\dots,\sum_{k=1}^{j}\beta_k\right\}$, and let $j'=n-\sum_{k=1}^{j-1}\beta_k$. If $u_i\prec z_n$, then replacing $z_{n\ell}\to u_{ir}$ in C with the arc $(z_{n\ell},u_{ir})$ creates a cycle with weight $(j-1-i)q-p\left(\sum_{k=i+1}^{j-1}\beta_k+j'\right)$ which is positive when $\sum_{k=i+1}^{j-1}\beta_k+j'<\left\lceil\frac{(j-i-1)q}{p}\right\rceil$. If $u_i\succ z_n$, then replacing $u_{i\ell}\to z_{nr}$ in C with the arc $(u_{i\ell},z_{nr})$ creates a cycle with weight $(i-1+\alpha-(j-1))q-p\left(\sum_{k=1}^{i}\beta_k+\sum_{k=j}^{\alpha}\beta_k-(j'-1)\right)=(\alpha+i-j)q-p\left(\beta-\sum_{k=i+1}^{j-1}\beta_k-j'+1\right)$ which is positive for $\sum_{k=i+1}^{j-1}\beta_k+j'>\left\lceil\frac{(j-i)q}{p}\right\rceil$. If $u_i\cap z_n$, then replacing $u_{ir}\to z_{n\ell}$ in C with the arc $(u_{ir},z_{n\ell})$ creates a cycle with weight $(i+\alpha-(j-1))q-p\left(\sum_{k=1}^{i}\beta_k+\sum_{k=j}^{\alpha}\beta_k-j'\right)=(\alpha+i-j+1)q-p\left(\beta-\sum_{k=i+1}^{j-1}\beta_k-j'\right)$ which is positive for $\sum_{k=i+1}^{j-1}\beta_k+j'\geq\left\lceil\frac{(j-i-1)q}{p}\right\rceil$, and replacing $z_{nr}\to u_{i\ell}$ in C with the arc $(z_{nr},u_{i\ell})$ creates a cycle with weight $(j-1-(i-1))q-p\left(\sum_{k=i+1}^{j-1}\beta_k+j'-1\right)$ which is positive when $\sum_{k=i+1}^{j-1}\beta_k+j'\leq\left\lceil\frac{(j-i)q}{p}\right\rceil$.

Thus, for $\boldsymbol{n\in\left\{\sum_{k=1}^{j-1}\beta_k+1,\dots,\sum_{k=1}^{j}\beta_k\right\}}$ and $\boldsymbol{j'=n-\sum_{k=1}^{j-1}\beta_k}$, we have

$$\boldsymbol{\begin{cases} u_i\prec z_n & \sum_{k=i+1}^{j-1}\beta_k+j'<\left\lceil\frac{(j-i-1)q}{p}\right\rceil\\ u_i\cap z_n & \left\lceil\frac{(j-i-1)q}{p}\right\rceil\leq\sum_{k=i+1}^{j-1}\beta_k+j'\leq\left\lceil\frac{(j-i)q}{p}\right\rceil\\ u_i\succ z_n & \sum_{k=i+1}^{j-1}\beta_k+j'>\left\lceil\frac{(j-i)q}{p}\right\rceil\end{cases}}.$$

Relationship 4:

If $u_i\prec x_j$, then replacing $x_{j_r}\to u_{ir}$ in C with the path $x_{j_r},x_{j_\ell},u_{ir}$ creates a cycle with weight $(j-1-i)q-p\left(1+\sum_{k=i+1}^{j}\beta_k\right)$ which is positive for $\sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i-1)q}{p}\right\rceil-1$. If $u_i\succ x_j$, then replacing $u_{i\ell}\to x_{j_r}$ in C with the arc $u_{i\ell},x_{j_r}$ creates a cycle with weight $(i-1+\alpha-(j-1))q-p\left(\sum_{k=1}^{i}\beta_k+\sum_{k=j+1}^{\alpha}\beta_k\right)=(\alpha+i-j)q-p\left(\beta-\sum_{k=i+1}^{j}\beta_k\right)$ which is positive for $\sum_{k=i+1}^{j}\beta_k\geq\left\lceil\frac{(j-i)q}{p}\right\rceil$. If $u_i\cap x_j$, then replacing $u_{ir}\to x_{j_r}$ in C with the path

$u_{ir}, x_{j\ell}, x_{j_r}$ creates a cycle with weight $(i+\alpha-(j-1)+1)q-p\left(\sum_{k=1}^{i}\beta_k+\sum_{k=j+1}^{\alpha}\beta_k\right)=$ $(\alpha+i-j+2)q-p\left(\beta-\sum_{k=i+1}^{j}\beta_k\right)$ which is positive for $\sum_{k=i+1}^{j}\beta_k\geq\left\lceil\frac{(j-i-2)q}{p}\right\rceil$, and replacing $x_{j_r}\to u_{i\ell}$ in C with the arc $(x_{j_r}, u_{i\ell})$ creates a cycle with weight $(j-1-(i-1))q-p\left(\sum_{k=i+1}^{j}\beta_k\right)$ which is positive for $\sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i)q}{p}\right\rceil$.

$$\textbf{Thus, }\begin{cases} \boldsymbol{u_i \prec x_j} & \boldsymbol{\sum_{k=i+1}^{j}\beta_k < \left\lceil\frac{(j-i-2)q}{p}\right\rceil} \\ \boldsymbol{u_1 \prec\cap x_j} & \boldsymbol{\left\lceil\frac{(j-i-2)q}{p}\right\rceil \leq \sum_{k=i+1}^{j}\beta_k < \left\lceil\frac{(j-i-1)q}{p}\right\rceil - 1} \\ \boldsymbol{u_i \cap x_j} & \boldsymbol{\left\lceil\frac{(j-i-1)q}{p}\right\rceil - 1 \leq \sum_{k=i+1}^{j}\beta_k < \left\lceil\frac{(j-i)q}{p}\right\rceil} \\ \boldsymbol{u_i \succ x_j} & \boldsymbol{\sum_{k=i+1}^{j}\beta_k \geq \left\lceil\frac{(j-i)q}{p}\right\rceil} \end{cases}.$$

Next, we consider the relationships between elements of the chains. We start with the maximal elements of each chain.

Relationship 5:

If $y_i\succ y_j$, then replacing $y_{j\ell}\to y_{i\ell}$ in C with the path $y_{j\ell}, y_{ir}, y_{i\ell}$ creates a path with weight $(j-i)q-p\left(\sum_{k=i+1}^{j}\beta_k+1\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i)q}{p}\right\rceil-1$. If $y_i\prec y_j$, then replacing $y_{i\ell}\to y_{j\ell}$ in C with the path $y_{i\ell}, y_{j_r}, y_{j\ell}$ creates a cycle with weight $(i+\alpha-j)q-p\left(1+\beta-\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k>\left\lceil\frac{(j-i)q}{p}\right\rceil-1$. If $y_i\cap y_j$, then replacing $y_{j\ell}\to y_{i\ell}$ in C with the path $y_{j\ell}, y_{j_r}, y_{i\ell}$ creates a cycle with weight $(j-i+1)q-p\left(\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k<\left\lceil\frac{(j-i+1)q}{p}\right\rceil$, and replacing $y_{i\ell}\to y_{j\ell}$ in C with the path $y_{i\ell}, y_{ir}, y_{j\ell}$ creates a cycle with weight $(i+1+\alpha-j)q-p\left(\beta-\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k\geq\left\lceil\frac{(j-i-1)q}{p}\right\rceil$.

$$\textbf{Thus, } \begin{cases} \boldsymbol{y_i \succ y_j} & \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i-1)q}{p} \right\rceil \\ \boldsymbol{y_1 \cap\!\succ y_j} & \left\lceil \frac{(j-i-1)q}{p} \right\rceil \leq \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \\ \boldsymbol{y_i \cap y_j} & \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 \leq \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} \leq \left\lceil \frac{(j-i)q}{p} \right\rceil - 1. \\ \boldsymbol{y_1 \prec\!\cap y_j} & \left\lceil \frac{(j-i)q}{p} \right\rceil - 1 < \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} < \left\lceil \frac{(j-i+1)q}{p} \right\rceil \\ \boldsymbol{y_i \prec y_j} & \left\lceil \frac{(j-i+1)q}{p} \right\rceil \leq \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k} \end{cases}$$

Next, we consider the maximal element of one chain a middle element of another.

Relationship 6:

Let $n \in \left\{ \sum_{k=1}^{j-1} \beta_k + 1, \ldots, \sum_{k=1}^{j} \beta_k \right\}$, and let $j' = n - \sum_{k=1}^{j-1} \beta_k$. If $y_i \prec z_n$, then replacing $z_{n\ell} \to y_{i\ell}$ in C with the path $z_{n\ell}, y_{ir}, y_{i\ell}$ creates a cycle with weight $(j-1-i)q - p\left(\sum_{k=i+1}^{j-1} \beta_k + j' + 1\right)$ which is positive when $\sum_{k=i+1}^{j-1} \beta_k + j' < \left\lceil \frac{(j-i-1)q}{p} \right\rceil - 1$. If $y_i \succ z_n$, then replacing $y_{i\ell} \to z_{nr}$ in C with the arc $(y_{i\ell}, z_{nr})$ creates a cycle with weight $(i + \alpha - (j-1))q - p\left(\sum_{k=1}^{i} \beta_k + \sum_{k=j}^{\alpha} \beta_k - (j'-1)\right) = (\alpha + i - j + 1)q - p\left(\beta - \sum_{k=i+1}^{j-1} \beta_k - j' + 1\right)$ which is positive for $\sum_{k=i+1}^{j-1} \beta_k + j' > \left\lceil \frac{(j-i-1)q}{p} \right\rceil$. If $y_i \cap z_n$, then replacing $y_{i\ell} \to z_{n\ell}$ in C with the path $y_{i\ell}, y_{ir}, z_{n\ell}$ creates a cycle with weight $(i + 1 + \alpha - (j-1))q - p\left(\sum_{k=1}^{i} \beta_k + \sum_{k=j}^{\alpha} \beta_k - j'\right) = (\alpha + i - j + 2)q - p\left(\beta - \sum_{k=i+1}^{j-1} \beta_k - j'\right)$ which is positive for $\sum_{k=i+1}^{j-1} \beta_k + j' \geq \left\lceil \frac{(j-i-2)q}{p} \right\rceil$, and replacing $z_{nr} \to y_{i\ell}$ in C with the arc $(z_{nr}, y_{i\ell})$ creates a cycle with weight $(j-1-i)q - p\left(\sum_{k=i+1}^{j-1} \beta_k + j' - 1\right)$ which is positive when $\sum_{k=i+1}^{j-1} \beta_k + j' \leq \left\lceil \frac{(j-i-1)q}{p} \right\rceil$.

Thus, for $\boldsymbol{n \in \left\{ \sum_{k=1}^{j-1} \beta_k + 1, \ldots, \sum_{k=1}^{j} \beta_k \right\}}$ and $\boldsymbol{j' = n - \sum_{k=1}^{j-1} \beta_k}$, we have

$$\begin{cases} \boldsymbol{y_i \prec z_n} & \displaystyle\sum_{k=i+1}^{j-1} \boldsymbol{\beta_k + j' < \left\lceil \frac{(j-i-2)q}{p} \right\rceil} \\ \boldsymbol{y_i \prec\cap z_n} & \boldsymbol{\left\lceil \frac{(j-i-2)q}{p} \right\rceil \le \displaystyle\sum_{k=i+1}^{j-1} \beta_k + j' < \left\lceil \frac{(j-i-1)q}{p} \right\rceil - 1} \\ \boldsymbol{y_i \cap z_n} & \boldsymbol{\left\lceil \frac{(j-i-1)q}{p} \right\rceil - 1 \le \displaystyle\sum_{k=i+1}^{j-1} \beta_k + j' \le \left\lceil \frac{(j-i-1)q}{p} \right\rceil} \\ \boldsymbol{y_i \succ z_n} & \displaystyle\sum_{k=i+1}^{j-1} \boldsymbol{\beta_k + j' > \left\lceil \frac{(j-i-1)q}{p} \right\rceil} \end{cases}.$$

Now, we consider the maximal element of one chain and the minimal element of another.

Relationship 7:

If $y_i \succ x_j$, then replacing $y_{i\ell} \to x_{j_r}$ in C with the arc $(y_{i\ell}, x_{j_r})$ creates a cycle with weight $(i+\alpha-(j-1))q - p\left(\beta - \sum_{k=i+1}^{j} \beta_k\right)$ which is positive for $\sum_{k=i+1}^{j} \beta_k \ge \left\lceil \frac{(j-i-1)q}{p} \right\rceil$. If $y_i \prec x_j$, then replacing $x_{j_r} \to y_{i\ell}$ in C with the path $x_{j_r}, x_{j\ell}, y_{ir}, y_{i\ell}$ creates a cycle with weight $(j-1-i)q - p\left(\sum_{k=i+1}^{j} \beta_k + 2\right)$ which is positive for $\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil - 2$. If $y_i \cap x_j$, then replacing $x_{j_r} \to y_{i\ell}$ in C with the arc $(x_{j_r}, y_{i\ell})$ creates a cycle with weight $(j-1-i)q - p\left(\sum_{k=i+1}^{j} \beta_k\right)$ which is positive for $\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil$, and replacing $y_{i\ell} \to x_{j_r}$ in C with the path $y_{i\ell}, y_{ir}, x_{j\ell}, x_{j_r}$ creates a cycle with weight $(i+1+\alpha-(j-1)+1)q - p\left(\beta - \sum_{k=i+1}^{j} \beta_k\right)$ which is positive for $\sum_{k=i+1}^{j} \beta_k \ge \left\lceil \frac{(j-i-3)q}{p} \right\rceil$.

$$\textbf{Thus, } \begin{cases} \boldsymbol{y_i \prec x_j} & \displaystyle\sum_{k=i+1}^{j} \boldsymbol{\beta_k < \left\lceil \frac{(j-i-3)q}{p} \right\rceil} \\ \boldsymbol{y_i \prec\cap x_j} & \boldsymbol{\left\lceil \frac{(j-i-3)q}{p} \right\rceil \le \displaystyle\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil - 2} \\ \boldsymbol{y_i \cap x_j} & \boldsymbol{\left\lceil \frac{(j-i-1)q}{p} \right\rceil - 2 \le \displaystyle\sum_{k=i+1}^{j} \beta_k < \left\lceil \frac{(j-i-1)q}{p} \right\rceil} \\ \boldsymbol{y_i \succ x_j} & \boldsymbol{\left\lceil \frac{(j-i-1)q}{p} \right\rceil \le \displaystyle\sum_{k=i+1}^{j} \beta_k} \end{cases}.$$

Next, we consider the minimal element of one chain and a middle element of another.

Relationship 8:

Let $n \in \left\{\sum_{k=1}^{j-1}\beta_k+1,\ldots,\sum_{k=1}^{j}\beta_k\right\}$, and let $j'=n-\sum_{k=1}^{j-1}\beta_k$. If $x_i \succ z_n$, then replacing $x_{ir} \to z_{nr}$ in C with the path $x_{ir}, x_{i\ell}, z_{nr}$ creates a cycle with weight $(i-1+\alpha-(j-1))q-p\left(\beta-\sum_{k=i+1}^{j-1}\beta_k-(j'-1)+1\right)$ which is positive for $\sum_{k=i+1}^{j-1}\beta_k+j' > \left\lceil\frac{(j-i)q}{p}\right\rceil+1$. If $x_i \prec z_n$, then replacing $z_{n\ell} \to x_{ir}$ in C with the arc $(z_{n\ell}, x_{ir})$ creates a cycle with weight $(j-1-(i-1))q-p\left(\sum_{k=i+1}^{j-1}\beta_k+j'\right)$ which is positive for $\sum_{k=i+1}^{j-1}\beta_k+j' < \left\lceil\frac{(j-i)q}{p}\right\rceil$. If $x_i \cap z_n$, then replacing $x_{ir} \to z_{n\ell}$ in C with the arc $(x_{ir}, z_{n\ell})$ creates a cycle with weight $(i-1+\alpha-(j-1))q-p\left(\beta-\sum_{k=i+1}^{j-1}\beta_k-j'\right)$ which is positive for $\sum_{k=i+1}^{j-1}\beta_k+j' \geq \left\lceil\frac{(j-i)q}{p}\right\rceil$, and replacing $z_{nr} \to x_{ir}$ in C with the path $z_{nr}, x_{i\ell}, x_{ir}$ creates a cycles with weight $(j-1-(i-1)+1)q-p\left(\sum_{k=i+1}^{j-1}\beta_k+j'-1\right)$ which is positive for $\sum_{k=i+1}^{j-1}\beta_k+j' \leq \left\lceil\frac{(j-i+1)q}{p}\right\rceil$.

Thus, for $\boldsymbol{n \in \left\{\sum_{k=1}^{j-1}\beta_k+1,\ldots,\sum_{k=1}^{j}\beta_k\right\}}$ and $\boldsymbol{j'=n-\sum_{k=1}^{j-1}\beta_k}$, we have

$$\boldsymbol{\begin{cases} x_i \prec z_n & \sum_{k=i+1}^{j-1}\beta_k+j' < \left\lceil\frac{(j-i)q}{p}\right\rceil \\ x_i \cap z_n & \left\lceil\frac{(j-i)q}{p}\right\rceil \leq \sum_{k=i+1}^{j-1}\beta_k+j' \leq \left\lceil\frac{(j-i)q}{p}\right\rceil+1 \\ x_i \cap\!\succ z_n & \left\lceil\frac{(j-i)q}{p}\right\rceil+1 < \sum_{k=i+1}^{j-1}\beta_k+j' \leq \left\lceil\frac{(j-i+1)q}{p}\right\rceil \\ x_i \succ z_n & \left\lceil\frac{(j-i+1)q}{p}\right\rceil < \sum_{k=i+1}^{j-1}\beta_k+j' \end{cases}}.$$

Now, we consider the minimal elements of two chains.

Relationship 9:

If $x_i \succ x_j$, then replacing $x_{ir} \to x_{j_r}$ in C with the path $x_{ir}, x_{i\ell}, x_{ir}$ creates a cycle with weight $(i+\alpha-j)q-p\left(1+\beta-\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k > \left\lceil\frac{(j-i)q}{p}\right\rceil$. If $x_i \prec x_j$, then replacing $x_{j_r} \to x_{ir}$ in C with the path $x_{j_r}, x_{j_\ell}, x_{ir}$ creates a cycle with weight $(j-i)q-p\left(1+\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k <$

$\left\lceil\frac{(j-i)q}{p}\right\rceil-1$. If $x_i \cap x_j$, then replacing $x_{ir} \to x_{j_r}$ in C with the path $x_{j_r}, x_{i\ell}, x_{ir}$ creates a cycle with weight $(j-i+1)q-p\left(\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k < \left\lceil\frac{(j-i+1)q}{p}\right\rceil$, and replacing $x_{ir} \to x_{j_r}$ in C with the path $x_{ir}, x_{j_\ell}, x_{j_r}$ creates a cycle with weight $(i+\alpha-j+1)q-p\left(\beta-\sum_{k=i+1}^{j}\beta_k\right)$ which is positive when $\sum_{k=i+1}^{j}\beta_k \geq \left\lceil\frac{(j-i-1)q}{p}\right\rceil$.

$$\textbf{Thus, }\begin{cases} \boldsymbol{x_i \prec x_j} & \sum_{k=i+1}^{j}\boldsymbol{\beta_k} < \left\lceil\frac{(j-i-1)q}{p}\right\rceil \\ \boldsymbol{x_i \prec\cap x_j} & \left\lceil\frac{(j-i-1)q}{p}\right\rceil \leq \sum_{k=i+1}^{j}\boldsymbol{\beta_k} < \left\lceil\frac{(j-i)q}{p}\right\rceil - 1 \\ \boldsymbol{x_i \cap x_j} & \left\lceil\frac{(j-i)q}{p}\right\rceil - 1 \leq \sum_{k=i+1}^{j}\boldsymbol{\beta_k} \leq \left\lceil\frac{(j-i)q}{p}\right\rceil \\ \boldsymbol{x_i \cap\succ x_j} & \left\lceil\frac{(j-i)q}{p}\right\rceil < \sum_{k=i+1}^{j}\boldsymbol{\beta_k} < \left\lceil\frac{(j-i+1)q}{p}\right\rceil \\ \boldsymbol{x_i \succ x_j} & \left\lceil\frac{(j-i+1)q}{p}\right\rceil \leq \sum_{k=i+1}^{j}\boldsymbol{\beta_k} \end{cases}.$$

Finally, we consider middle elements of two different chains.

Relationship 10:

Let $m \in \left\{\sum_{k=1}^{i-1}\beta_k+1, \ldots, \sum_{k=1}^{i}\beta_k\right\}$, and let $i'=m-\sum_{k=1}^{i-1}\beta_k$. Let $n \in \left\{\sum_{k=1}^{j-1}\beta_k+1, \ldots, \sum_{k=1}^{j}\beta_k\right\}$, and let $j'=n-\sum_{k=1}^{j-1}\beta_k$. Note, since $i<j$, $m<n$. If $z_m \succ z_n$, then replacing $z_{m\ell} \to z_{nr}$ in C with the arc $(z_{m\ell}, z_{nr})$ creates a cycle with weight $(i-1+\alpha-(j-1))q-p\left(\beta-\sum_{k=i}^{j-1}\beta_k+i'-j'+1\right)$ which is positive when $\sum_{k=i}^{j-1}\beta_k+j'-i' > \left\lceil\frac{(j-i)q}{p}\right\rceil$. If $z_m \prec z_n$, then replacing $z_{n\ell} \to z_{mr}$ in C with the arc $(z_{n\ell}, z_{mr})$ creates a cycle with weight $(j-1-(i-1))q-p\left(\sum_{k=i}^{j-1}\beta_k+j'-(i'-1)\right)$ which is positive when $\sum_{k=i}^{j-1}\beta_k+j'-i' < \left\lceil\frac{(j-i)q}{p}\right\rceil-1$. If $z_m \cap z_n$, then replacing $z_{mr} \to z_{n\ell}$ in C with the arc $(z_{mr}, z_{n\ell})$ creates a cycle with weight $(i-1+\alpha-(j-1))q-p\left(\beta-\sum_{k=i}^{j-1}\beta_k+i'-1-j'\right)$ which is positive when $\sum_{k=i}^{j-1}\beta_k+j'-i' \geq \left\lceil\frac{(j-i)q}{p}\right\rceil-1$, and replacing $z_{nr} \to z_{m\ell}$ in C with the arc $(z_{nr}, z_{m\ell})$ creates a cycle with weight $(j-1-(i-1))q-p\left(\sum_{k=i}^{j-1}\beta_k-i'+j'-1\right)$ which is positive when $\sum_{k=i}^{j-1}\beta_k+j'-i'\leq\left\lceil\frac{(j-i)q}{p}\right\rceil$.

Thus, for $\boldsymbol{m\in\left\{\sum_{k=1}^{i-1}\beta_k+1,\ldots,\sum_{k=1}^{i}\beta_k\right\}}$ **with** $\boldsymbol{i'=m-\sum_{k=1}^{i-1}\beta_k}$ **and** $\boldsymbol{n\in\left\{\sum_{k=1}^{j-1}\beta_k+1,\ldots,\sum_{k=1}^{j}\beta_k\right\}}$ **with** $\boldsymbol{j'=n-\sum_{k=1}^{j-1}\beta_k}$**, we have**

$$\boldsymbol{\begin{cases} z_m\prec z_n & \sum_{k=i}^{j-1}\beta_k+j'-i'<\left\lceil\frac{(j-i)q}{p}\right\rceil-1 \\ z_m\cap z_n & \left\lceil\frac{(j-i)q}{p}\right\rceil-1\leq\sum_{k=i}^{j-1}\beta_k+j'-i'\leq\left\lceil\frac{(j-i)q}{p}\right\rceil \\ z_m\succ z_n & \left\lceil\frac{(j-i)q}{p}\right\rceil<\sum_{k=i}^{j-1}\beta_k+j'-i' \end{cases}}.$$

The preceding analysis and the relationships in bold give minimal forbidden substructures for $\mathcal{P}[p,q]$. □

Given p, q, and α, the bold relationships would give the minimal forbidden structures associated with a minimal negative cycle in $D_p^q(P)$ with α weight q arcs. However, if the bold relationships do not provide a relationship for a pair of elements, then there are no minimal forbidden structures which correspond to that set of p, q, α values.

The proof of the Proposition 5.2.4 is much shorter than that of Propositions 3.1.2 and 4.1.2. However, the former results provide the specific structures for $p=2$ and $p=3$. To ascertain the specific structures for higher values of p, we would need for analyze the relationships in following proposition for each pair of elements. A structure in $\mathcal{F}_p^q$ will have between 4 and $q+3p$ elements, so there are as many as $\binom{q+3p}{2}=\frac{(q+3p)(q+3p-1)}{2}$ pairs.

The relationships found in the Proposition 5.2.4 are similar to those found in Chapters 3 and 4, but we are lacking the divisibility conditions to determine which values of α will produce minimal structures for a given p, q pair.

5.2.2 Divisibility

Let $q{=}ps{+}d$. Consider the values of β for small values of α and then for general α.

$$\left\lceil\frac{q}{p}\right\rceil=\left\lceil\frac{ps+d}{p}\right\rceil=s+\left\lceil\frac{d}{p}\right\rceil=s+1$$

$$\left\lceil\frac{2q}{p}\right\rceil=\left\lceil\frac{2(ps+d)}{p}\right\rceil=2s+\left\lceil\frac{2d}{p}\right\rceil=\begin{cases}2s+1 & d\leq\frac{p}{2}\\ 2s+2 & \frac{p}{2}<d\end{cases}$$

$$\left\lceil\frac{3q}{p}\right\rceil=\left\lceil\frac{3(ps+d)}{p}\right\rceil=3s+\left\lceil\frac{3d}{p}\right\rceil=\begin{cases}3s+1 & d\leq\frac{p}{3}\\ 3s+2 & \frac{p}{3}<d\leq\frac{2p}{3}\\ 3s+3 & \frac{2p}{3}<d\end{cases}$$

$$\left\lceil\frac{4q}{p}\right\rceil=\left\lceil\frac{4(ps+d)}{p}\right\rceil=4s+\left\lceil\frac{4d}{p}\right\rceil=\begin{cases}4s+1 & d\leq\frac{p}{4}\\ 4s+2 & \frac{p}{4}<d\leq\frac{p}{2}\\ 4s+3 & \frac{p}{2}<d\leq\frac{3p}{4}\\ 4s+4 & \frac{3p}{4}<d\end{cases}$$

$$\vdots$$

$$\left\lceil\frac{\alpha q}{p}\right\rceil=\left\lceil\frac{\alpha(ps+d)}{p}\right\rceil=\alpha s+\left\lceil\frac{\alpha d}{p}\right\rceil=\begin{cases}\alpha s+1 & d\leq\frac{p}{\alpha}\\ \alpha s+2 & \frac{p}{\alpha}<d\leq\frac{2p}{\alpha}\\ \alpha s+3 & \frac{2p}{\alpha}<d\leq\frac{3p}{\alpha}\\ \vdots & \\ \alpha s+\alpha & \frac{(\alpha-1)p}{\alpha}<d\end{cases}$$

To have a minimal negative cycle with exactly three weight q arcs, we cannot have $\frac{p}{3}<d\leq\frac{p}{2}$ or $\frac{2p}{3}<d$ because any relationship between $z_{a-(s+1)}$ and u_1 would give a shorter negative cycle (see the proof of Proposition 4.1.2).

To have a minimal cycle with exactly four weight q arcs, we cannot have $\frac{p}{4}<d\leq\frac{p}{3}$, $\frac{p}{3}<d\leq\frac{p}{2}$ (2 and 2), $\frac{p}{2}<d\leq\frac{2p}{3}$ (3 and 1), or $d>\frac{3p}{4}$ (2 and 2 or 1 and 3). For $\frac{p}{4}<d\leq\frac{p}{3}$, the cycle could be divided into either one side with three weight q arcs and the other with one or one side with two q arcs and the other side also with two, $\frac{p}{3}<d\leq\frac{p}{2}$ could just be two and two, $\frac{p}{2}<d\leq\frac{2p}{3}$ is just three and one, and $d>\frac{3p}{4}$ is three and one or two and two. For the three and one splits, the shortcut could again be $u_1\succ z_{a-(s+1)}$ if the cycle is labeled as in the proof of Proposition 4.1.2. The two, two split is not as straight forward. The sizes of the $-p$ arc sets are important. Let β_1,β_2,β_3, and

β_4 be the sizes of the $-p$ weight sets. We have $\beta_1+\beta_2+\beta_3+\beta_4 \geq 4s+1$, so there will be an adjacent pair with $\beta_i+\beta_{i+1} \geq \lceil 2d/p \rceil$ (otherwise $2(\beta_1+\beta_2+\beta_3+\beta_4) < 2\lceil 2d/p \rceil$). If $\beta_{i+1} < \lceil 2d/p \rceil$, then u_{i+1} and $z_{\sum_{k=1}^{\beta_{i+1}} \beta_k - \lceil 2d/p \rceil}$ will result in a shorter negative cycle regardless of their relationship. If $\beta_{i+1} \geq \lceil 2d/p \rceil$, then u_{i+2} and $z_{\sum_{k=1}^{\beta_{i+2}} \beta_k - \lceil 2d/p \rceil}$ will create the shortcut.

Proposition 5.2.5. $\mathcal{F}_p^q$ contains the minimal structures described in Proposition 5.2.4 for $\alpha = p$.

Proof. Let C be a negative cycle in $D_p^q(P)$ with p weight q arcs and q weight $-p$ arcs with the structure of Figure 5.1. For C to contain a shorter negative cycle, we must be able to divide the cycle into two parts such that one side has α' weight q arcs and the other has $\alpha'' \geq \alpha'$ weight q arcs, with $q = \left\lceil \frac{\alpha' q}{p} \right\rceil + \left\lceil \frac{\alpha'' q}{p} \right\rceil$. Since $\alpha' + \alpha'' = p$, $q = \frac{\alpha' q}{p} + \frac{\alpha'' q}{p}$. Now, since $gcd(p,q) = 1$, and $\alpha', \alpha'' < p$, both $\left\lceil \frac{\alpha' q}{p} \right\rceil$ and $\left\lceil \frac{\alpha'' q}{p} \right\rceil$ will round up. Thus, $q \neq \left\lceil \frac{\alpha' q}{p} \right\rceil + \left\lceil \frac{\alpha'' q}{p} \right\rceil$ for any values of α' and α''. Therefore, C is minimal. □

Corollary 5.2.6. $\mathcal{F}_p^{kp+1}$ contains minimal structures corresponding to negative cycles in $D_p^{kp+1}(P)$, $\forall \alpha \in \{1, 2, \ldots, p\}$.

Proof. By Proposition 5.1.1, $\mathcal{F}_{p-1}^{k(p-1)+1} \subseteq \mathcal{F}_p^{kp+1}$. By induction, $\mathcal{F}_p^{kp+1}$ contains minimal structures corresponding to $\alpha \in \{1, 2, \ldots, p-1\}$. By Proposition 5.2.5, $\mathcal{F}_p^{kp+1}$ contains minimal structures corresponding to $\alpha = p$. □

This supports the statement after the proof of Proposition 5.1.1 that $|\mathcal{F}_p^{kp+1}|$ increases as p increases.

Corollary 5.2.7. $\mathcal{F}_p^{kp+1} \subset \mathcal{F}_{p+1}^{k(p+1)+1}$.

Although we do not see shortcut behavior when $\alpha = p$, in the following remark, we develop the conditions on d ($q = ps+d$) that could produce shortcut behavior.

Remark 1. Let C be a negative cycle in $D_p^q(P)$ with the structure of Lemma 2.2.4 which contains $\alpha < p$ weight $q = ps+d$ arcs. In Proposition 5.2.5, we addressed $\alpha = p$.

Question 1. In general, can we divide C into two parts such that one side has α' weight q arcs and the other has $\alpha'' \geq \alpha'$ weight q arcs, so that $\left\lceil \frac{\alpha q}{p} \right\rceil = \left\lceil \frac{\alpha' q}{p} \right\rceil + \left\lceil \frac{\alpha'' q}{p} \right\rceil$?

We have $\left\lceil\frac{\alpha q}{p}\right\rceil = \alpha s + \left\lceil\frac{\alpha d}{p}\right\rceil$, $\left\lceil\frac{\alpha' q}{p}\right\rceil = \alpha' s + \left\lceil\frac{\alpha' d}{p}\right\rceil$, and $\left\lceil\frac{\alpha'' q}{p}\right\rceil = \alpha'' s + \left\lceil\frac{\alpha'' d}{p}\right\rceil$.

The question becomes:

Question 2. When is $\left\lceil\frac{\alpha d}{p}\right\rceil = \left\lceil\frac{\alpha' d}{p}\right\rceil + \left\lceil\frac{\alpha'' d}{p}\right\rceil$?

Let $\alpha d = ps' + d'$ and let $\alpha' d = ps'' + d''$. Since $gcd(p,d) = 0$ and $\alpha, \alpha' < p$, $d', d'' \neq 0$. Then,

$$\begin{aligned}\alpha'' d &= \alpha d - \alpha' d\\ &= p(s' - s'') + d' - d''\\ &= \begin{cases} p(s'-s'') + d' - d'' & d' \geq d''\\ p(s'-s''-1) + d'' - d' & d' < d'' \end{cases}\end{aligned}$$

.

Question 3. When is $\left\lceil\frac{d'}{p}\right\rceil = \left\lceil\frac{d''}{p}\right\rceil + \left\lceil\frac{d'-d''}{p}\right\rceil$?

Since $0 \leq d', d'' < p$, we have $-p < d' - d'' < p$. If $d' > d''$, then $\left\lceil\frac{d'-d''}{p}\right\rceil = 1$ and $1 \neq 1+1$. If $d' \leq d''$, then $\left\lceil\frac{d'-d''}{p}\right\rceil = 0$ and $1 = 1+0$. Thus, $d' \leq d''$, i.e., $\alpha d (\text{mod } p) \leq \alpha' d (\text{mod } p)$.

Question 4. When is $\alpha d (\text{mod } p) \leq \alpha' d (\text{mod } p)$?

We have,

$$\alpha d = s'p + d' = \begin{cases} d' & d < \frac{p}{\alpha}\\ p + d' & \frac{p}{\alpha} \leq d < \frac{2p}{\alpha}\\ 2p + d' & \frac{2p}{\alpha} \leq d < \frac{3p}{\alpha}\\ \vdots & \\ (\alpha-1)p + d' & \frac{(\alpha-1)p}{\alpha} \leq d < p \end{cases},$$

and

$$\alpha' d = s''p + d'' = \begin{cases} d'' & d < \frac{p}{\alpha'}\\ p + d'' & \frac{p}{\alpha'} \leq d < \frac{2p}{\alpha'}\\ \vdots & \\ (\alpha'-1)p + d'' & \frac{(\alpha'-1)p}{\alpha'} \leq d < p \end{cases}.$$

Thus, for $\frac{kp}{\alpha} \leq d < \frac{(k+1)p}{\alpha}$, we have $\alpha d = kp + d'$, and for $\frac{k'p}{\alpha'} \leq d < \frac{(k'+1)p}{\alpha'}$, we have $\alpha' d = k'p + d''$.

Next,

$$\left\lceil \frac{\alpha d}{p} \right\rceil = s' + \left\lceil \frac{d'}{p} \right\rceil = s' + 1 = \begin{cases} 1 & d \leq \frac{p}{\alpha} \\ 1+1 & \frac{p}{\alpha} < d \leq \frac{2p}{\alpha} \\ 2+1 & \frac{2p}{\alpha} < d \leq \frac{3p}{\alpha} \\ \vdots & \\ (\alpha-1)+1 & \frac{(\alpha-1)p}{\alpha} < d < p \end{cases},$$

and

$$\left\lceil \frac{\alpha' d}{p} \right\rceil = s'' + \left\lceil \frac{d''}{p} \right\rceil = \begin{cases} 1 & d \leq \frac{p}{\alpha'} \\ 1+1 & \frac{p}{\alpha'} < d \leq \frac{2p}{\alpha'} \\ \vdots & \\ (\alpha'-1)+1 & \frac{(\alpha'-1)p}{\alpha'} < d < p \end{cases}.$$

Thus, for $\frac{kp}{\alpha} < d \leq \frac{(k+1)p}{\alpha}$, we have $\left\lceil \frac{\alpha d}{p} \right\rceil = k+1$, and for $\frac{k'p}{\alpha'} < d \leq \frac{(k'+1)p}{\alpha'}$, we have $\left\lceil \frac{\alpha' d}{p} \right\rceil = k'+1$.

If $\alpha d (\text{mod } p) \leq \alpha' d (\text{mod } p)$, we have $\alpha d - s'p \leq \alpha' d - s''p \Longrightarrow d \leq \frac{(s'-s'')p}{\alpha - \alpha'}$.

Answer. $\left\lceil \frac{\alpha q}{p} \right\rceil = \left\lceil \frac{\alpha' q}{p} \right\rceil + \left\lceil \frac{\alpha'' d}{p} \right\rceil$ when if $\frac{kp}{\alpha} < d \leq \frac{(k+1)p}{\alpha}$ and $\frac{k'p}{\alpha'} < d \leq \frac{(k'+1)p}{\alpha'}$, then $d \leq \frac{(s'-s'')p}{\alpha-\alpha'} = \frac{(k-k')p}{\alpha-\alpha'}$.

This ends Remark 1.

These conditions on d could be those needed to determine minimality, but more work needs to be done.

5.3 Future work

Proposition 5.2.4 and Remark 1 from the previous section inspire Conjecture 5.3.1.

Conjecture 5.3.1. Let P be a finite poset. Let C be a minimal negative cycle in $D_p^q(P)$ which contains α weight q arcs. Let $q = ps + d$, $\alpha' \in \{1, 2, \ldots, \lceil \alpha/2 \rceil\}$, $k \in \{0, 1, \ldots, \alpha - 1\}$, $k' \in \{0, 1, \ldots, \alpha' - 1\}$, $\frac{kp}{\alpha} \leq d < \frac{(k+1)p}{\alpha}$, and $\frac{k'p}{\alpha'} \leq d < \frac{(k'+1)p}{\alpha'}$. Then, $d > \frac{(k-k')p}{\alpha-\alpha'}$.

Proving this conjecture would require showing that when $d \leq \frac{(k-k')p}{\alpha-\alpha'}$ there is a pair of elements which create a shorter negative cycle regardless of their relationship. We would also need to show that there is no such pair of elements when $d > \frac{(k-k')p}{\alpha-\alpha'}$.

The goal of the previous sections and Conjecture 5.3.1 would be a complete result for $\mathcal{F}_p^q$. Proposition 5.2.4 provides a complete description of the relationships in the posets of $\mathcal{F}_p^q$ based on p, q, and α (the number of q weight arcs in a corresponding negative cycle) when the corresponding negative cycle is minimal. Resolving 5.3.1 would determine the values of p, q, and α for which such a minimal negative cycle exists.

Beyond posets, there are also other relation sets as discussed in the introduction to which it could be interesting to consider adding length constraints.

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and

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