2016

A measure theoretic approach to the construction of scaling functions for wavelets

Sarah Dumnich
Lehigh University

A measure theoretic approach to the construction of scaling functions for wavelets

by

Sarah Raye Dumnich

A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

Lehigh University
May 23, 2016

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Sarah Raye Dumnich

A measure theoretic approach to the construction of scaling functions for wavelets

Date

Robert Neel, Dissertation Director, Chair

Accepted Date

Committee Members

Terrence Napier

Daniel Conus

Nathan Shank

Dedication

I thank my adviser, Rob Neel, for his excellent guidance, patience, and help. Without his knowledge, counsel and kindness, I would not have been able to write this dissertation. I would also like to acknowledge Vladimir Dobric. His unending positivity and passion for mathematics were truly inspirational qualities. The afternoons spent talking over chocolates or Turkish coffee in his office and living room laid the foundation for all of the work in this dissertation. His assistance, motivation, and unwavering confidence were essential to the completion of this work. I also wish to thank Daniel Conus, Terry Napier, and Nate Shank, for their valuable feedback and service as committee members.

I also must thank my fellow Lehigh graduate students for keeping me sane and providing me with countless memories and laughs during my time in graduate school. I would not be the person I am today without each of them. Finally, I thank my husband John for supporting me in every way, and for always having optimism greater than my own. And my parents, whose constant love, support, patience, and encouragement has made all of this possible.

Contents

List of Figures

Abstract

A multiresolution analysis is a tool used in the construction of orthogonal wavelets. The dilation equation is an equation that arises naturally when using an MRA to construct a wavelet basis. One way to understand the dilation equation, and its solution, the scaling function, is through a measure theoretic approach. By constructing a solution to the signed measure dilation equation, we give a new way of approximating the scaling function by dyadic step functions. We also give a method of controlling the support in the two-dimensional case.

Chapter 1

Introduction

In many applications, given a signal $f(t)$, one is interested in its frequency content locally in time. The standard Fourier transform gives a representation of the frequency content of f, but local features can get lost and if the signal is not stationary then this is not captured by the Fourier transform. Time-localization can be achieved by first windowing the signal to cut off only a well-localized slice of f, and then taking its Fourier transform. The way the function is windowed is by taking its inner-product with a time window function $g(t)$, which has unit norm and is centered at $t = 0$.

This is a standard technique for time-frequency localization, known as the windowed Fourier transform. The wavelet transform yields a similar time-frequency description, with a couple of important differences. One similarity between the wavelet and windowed Fourier transform is that they both take the inner products of f with a family of functions with two indices: $g^{\omega,t}$ for the windowed Fourier transform and $\psi^{a,b}$ for the wavelet transform. In each of these bases, one index represents frequency and the other represents time localization. The main difference between the wavelet and windowed Fourier transforms is in the shapes of the basis functions. The windowed Fourier transform basis functions all consist of the same function, translated to the proper time location, and "filled in" with higher frequency oscillations. Therefore, supports of the $g^{\omega,t}$ all have the same width. In contrast, the wavelet basis functions have widths adapted to their frequency: high frequency are narrow, while low frequency are broader. As a result, the wavelet transform is better able than the windowed Fourier transform to "zoom in" on very short lived high frequency information. The figure on page 4 of Daubechies' *Ten Lectures on Wavelets* [1] displays the differences in the shapes of the functions $g^{\omega,t}$ and $\psi^{a,b}$.

In the 80's, Mallat [2] and Meyer [3] formalized multiresolution analysis (MRA), which set the groundwork for the construction of orthogonal wavelets. The dilation equation is an equation that arises naturally from an MRA. A solution to the dilation equation, called a scaling function, canonically determines a corresponding wavelet basis [4].

Let $\Gamma \subset \mathbb{R}^d$ be a lattice and let M be an integer-valued *expanding* matrix. That is, all eigenvalues of M are greater than 1 in absolute value; so M preserves the lattice. A *dilation equation* is an equation of the form

$$\phi(x) = |\det M| \sum_{a_k \in \Gamma} p_k \phi(Mx - a_k). \tag{1.1}$$

If the sequence $(p) := (p_k)$ is in $l^2(\Gamma)$ then the dilation equation always has a solution in the distributional sense [1]. Functional solutions to dilation equations are useful in many applications such as subdivision schemes, interpolation methods, and the construction of wavelet bases of $L^2(\mathbb{R}^d)$ [1, 5, 6]. Depending on conditions placed upon the sequence (p), solutions to the dilation equation can be scaling functions or prescale functions. Integer translates of a *scaling function* form an orthonormal basis in the MRA, while integer translates of a *prescale function* form a Riesz basis in the MRA. Curry [7] has considered the class of dilation equations in multiple dimensions in which there are infinitely many coefficients and prescale functions are constructed. She did her work by looking at the dilation equation from the Fourier side. Gundy [8] considered the class of dilation equations in one dimension in which there are infinitely many coefficients and scaling functions are considered. He did his work by looking at the dilation equation from the Fourier side as well. Lawton et al. [9] have found general conditions which guarantee the existence and uniqueness of a scaling function; however the typical method used involves looking at the dilation equation from the Fourier side.

In the chapters that follow, we consider a special class of dilation equations: the class for which the sequence (p) is finitely supported (i.e. $p_k = 0$ for all but finitely many k), and p_k satisfy certain orthonormality conditions. In this case, the sequence (p) can be considered to be the set of weights of a signed measure which is defined to be the finite sum of weighted Dirac-δ measures. While measure-valued solutions are interesting in their own right, the absolute continuity of a signed measure can give us a solution to the functional dilation equation almost everywhere. We can state the problem as follows: Let μ be a signed measure defined on $\mathbb{R}^d$. Then μ is a solution to a *signed measure dilation equation* if μ satisfies

$$\mu(A) = \sum_k p_k \mu(MA - a_k), \tag{1.2}$$

for $A \in \mathcal{B}(\mathbb{R}^d)$, ie for Borel measurable sets A. If μ has a density, say f_μ, then f_μ solves (1.1) almost everywhere. Therefore, the questions we seek to answer are: Under certain orthonormality conditions, which we detail later, is there a measure valued solution? How can the solution to the signed measure dilation equation be used to find a scaling function?

A probabilistic approach to the construction of a scaling function has been considered by Dobric, Gundy, and Hitczenko (1-D case) [10] and by Belock and Dobric (2-D case) [11]. This is natural because the right-hand side of the dilation equation can be interpreted as the convolution of two probability measures (under the condition that all p_k are positive). This approach considers a random variable Z which satisfies a random variable dilation equation (which is explained in more detail on page 11). Assume that Z is absolutely continuous with respect to Lebesgue measure and denote its density by ϕ. Then ϕ satisfies the dilation equation almost everywhere [11]. However, by considering the dilation equation through a probabilistic approach, we limit ourselves to only constructing non-negative scaling functions. This is an unnatural constraint because several well-known scaling functions, including those of Daubechies' wavelets [4], aren't, in fact, non-negative. This is why we are now considering a signed measure approach.

In Chapter 3, we describe conditions and a method for constructing signed measure solutions to the dilation equation in one dimension and in Chapter 4 we extend these results to two-dimensions. In two-dimensions, this dilation equation involves a fractal object called the Twin Dragon, which creates a self-similar tiling of the plane. This tiling naturally makes use of a radix expansion of complex numbers helpful.

Our investigations of dilation equations are motivated by the application to multiresolution analysis and wavelet bases. Daubechies [1] has shown that if a scaling function satisfies certain conditions then it can be used to generate an MRA and therefore a wavelet basis. These conditions are detailed in the preliminary material in Chapter 2, but the main condition is that the lattice translates of the scaling function should form an orthonormal basis of its closed linear span.

Chapter 2

Preliminaries

2.1 Multiresolution Analysis

Burt and Adelson [12] introduced a multiresolution pyramid that can be used to process an image in low-resolution at first and then selectively increase the resolution locally wherever necessary. Mallat [2] and Meyer [3] built upon this idea by formalizing a multiresolution analysis (MRA), which set the groundwork for the construction of orthogonal wavelets. The approximation of a function f at a resolution 2^{-j} is given by averages of f over neighborhoods of size 2^{-j}. An MRA is composed of a nested sequence of subspaces V_j of $L^2(\mathbb{R}^2)$, which gives a finer approximation of a function at each subsequent space. Here we introduce the rigorous definition of an MRA. We will use some new notation; given an expanding linear transformation $M : \mathbb{R}^d \to \mathbb{R}^d$, for any function f, define the function $f_{j,k}(x) = f(M^j x - k)$.

Definition 1. *An MRA consists of an expanding linear transformation, $M : \mathbb{R}^d \to \mathbb{R}^d$, together with a sequence of closed subspaces V_j, which satisfy:*

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$

2. $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R}^d)$

3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

4. $f \in V_j \iff f_{1,0} \in V_{j+1}$

5. $f \in V_0 \iff f_{0,\gamma} \in V_0$ *for all* $\gamma \in \mathbb{Z}^d$

6. There exists a function $\phi \in V_0$ *called a scaling function, such that* $\{\phi_{0,\gamma} : \gamma \in \mathbb{Z}^d\}$ *forms an orthonormal basis for* V_0.

It is possible to generalize this by replacing $\mathbb{Z}^d$ with any discrete lattice $\Gamma \subset \mathbb{R}^d$. Since $f \in V_j \iff f_{1,0} \in V_{j+1}$ and for any $n \in \mathbb{Z}^d$, we have $f \in V_0 \iff f_{0,n} \in V_0$, we have that $\phi_{j,k} \in V_j$. In fact, $(\phi_{j,n})_n$ is an orthonormal basis in V_j.

Since $\phi \in V_0 \subset V_1$, and $(\phi_{1,n})_n$ forms an orthonormal basis in V_1, we have

$$\phi = \sum_n h_n \phi_{1,n}, \tag{2.1}$$

with $h_n = \langle \phi, \phi_{1,n} \rangle_{L^2}$.

We can write this in $\mathbb{R}^d$ as the *dilation equation*,

$$\phi(x) = |\det M| \sum_{k \in \Gamma} p_k \phi(Mx - a_k). \tag{2.2}$$

We only consider compactly supported scaling measures. In other words, only finitely many of the p_k are non-zero. For the sake of notation in the following conditions, we can assume that for $k \geq 2N$, $p_k = 0$. We work under the following conditions:

$$\begin{cases} \sum_{i=0}^{2N-1} p_i = 1 & (1) \\ \sum_{i=0}^{2N-1} p_i p_{i+2l} = \frac{1}{2}\delta_{0l}, & (2) \end{cases}$$

where δ_{ij} is the Kronecker delta function, i.e., $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. These conditions have been shown to be necessary for determining an MRA. Lawton [13] and Cohen [14] have independently established necessary and sufficient conditions under which the scaling function will be orthogonal to its integer translates[15]. Lawton's formulation is the following:

Theorem 2. *Define the operator* $G : l^2 \to l^2$ *by*

$$(Ga)_l = \frac{1}{2} \sum_{j,k} p_j p_k a_{2l+j-k}$$

for $a \in l^2$. Then the coefficients $\{p_k\}$ determine an MRA if and only if

1. *Conditions (1) and (2) are satisfied, and*
2. *δ_{0l} is the only eigenvector for G for the eigenvalue 1.*

Cohen's conditions, which have been shown to be equivalent to Lawton's are the following:

Theorem 3. *The coefficients $\{p_k\}$ determine an MRA if and only if*

1. *Conditions (1) and (2) are satisfied, and*
2. *there exists a $\gamma \in [-\pi/2, \pi/2]$ such that $\hat{f}(\gamma + 2k\pi) = 0$ for every $k \in \mathbb{Z}$,*

where f is the solution to the dilation equation.

The way that the scaling function relates to the wavelet function is as follows. Suppose you have MRA with scaling function ϕ which satisfies the dilation equation

$$\phi(x) = 2 \sum_{k=0}^{2N-1} p_k \phi(2x - k).$$

Then define the space $W_j := V_j - V_{j-1}$. So we have that $V_{j+1} = V_j \oplus W_{j+1}$ and $L^2(\mathbb{R})$ can be decomposed as a direct sum of the spaces W_j. The wavelet function is the function ψ where $\left\{\psi_{0,n} \middle| n \in \mathbb{Z}\right\}$ forms an orthonormal basis of W_0. Then, ψ can be written as [16],

$$\psi(x) = 2 \sum_{k=0}^{2N-1} (-1)^k \overline{p_{2N-1-k}} \phi(2x - k).$$

2.2 Twin Dragon

Given a dilation M, let $\mathcal{D}$ be a complete set of coset representatives for $\mathbb{Z}^n / M(\mathbb{Z}^n)$. We assume that $\mathcal{D}$, called the *digit set*, contains the zero vector. Let $\mathbb{P}$ denote the set of all $k \in \mathbb{Z}^n$ that can be written as a finite sum of the form

$$k = \sum_{j=0}^{N(k)} M^j d_j$$

with $d_j \in \mathcal{D}$. The pair $(M, \mathcal{D})$ is called a number system if $\mathbb{P} = \mathbb{Z}^n$. In this case M is said to be the *radix* of the system. If the digit set consists of all nonnegative multiples, $m = 0, 1, ..., (q-1)$, of a single coordinate unit vector, e_j, the system is called canonical [17]. Lagarias and Wang [18] have classified all expanding matrices in $\mathbb{R}^2$, up to integral similarity by a unimodular matrix $U \in M_2(\mathbb{Z})$. Their list is as follows: if $\det(M) = -2$,

$$M \sim C_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

is the canonical representative of the class. If $\det(M) = 2$ there are five classes, defined by the following canonical representatives:

$$C_2 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

$$\pm C_3 = \pm \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\pm C_4 = \pm \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}.$$

For each of these cases, a digit set $\mathcal{D}$ exists such that the set $T(M, \mathcal{D}) = \left\{ \sum_{j=1}^{\infty} M^{-j} d_j \right\}$ is a *tile* [17], where a tile T is a subset of the plane where translations of T by Gaussian integers γ are disjoint up to a set of Lebesgue measure 0 and $\cup_\gamma (T + \gamma) = \mathbb{R}^2$ covers the entire plane. The following theorem from Gundy and Jonsson [17] summarizes their results regarding these classes of dilation.

Theorem 4. *For no choice of $\mathcal{D}$ is $(C_1, \mathcal{D})$ a number system. The matrices C_2, $-C_3$, $\pm C_4$ all generate number systems with the canonical digit set $\mathcal{D}_1 = \{0, \epsilon_1\}$, where $\epsilon_1 = (1, 0)'$. The pair $(+C_3, \mathcal{D}_1)$ generates a self-affine tile $T(+C_3, \mathcal{D}_1)$, but for no digit set $\mathcal{D}$ is $(+C_3, \mathcal{D})$ a number system.*

The pair $(+C_3, \mathcal{D}_1)$ is the exceptional case in the list in that it generates a self-affine tile but does not generate a number system. We find it easier to identify $\mathbb{R}^2$

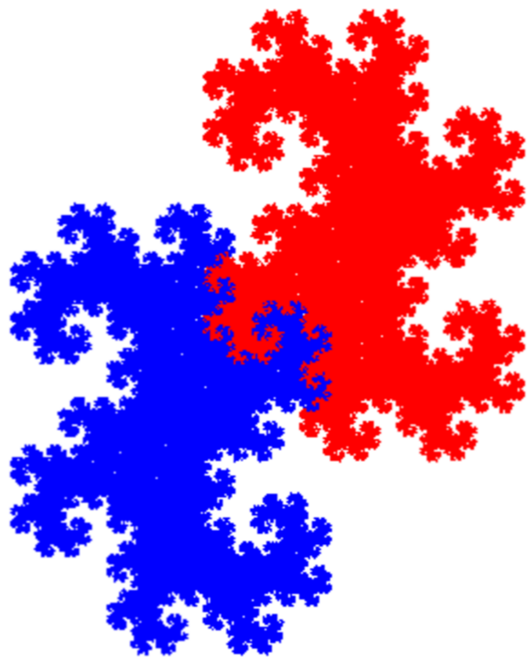

Figure 2.1: The Twin Dragon

with the complex plane $\mathbb{C}$ in order to simplify computations and notation. In this case, multiplication by the matrix $+C_3$ is equivalent to multiplication by $1+i$. The *Twin Dragon* is the tile which is generated by $(+C_3, \mathcal{D}_1)$ and can we written as the following:

$$T = \left\{ \sum_{k=1}^{\infty} \frac{\epsilon_k}{(1+i)^k} \middle| \epsilon_k \in \{0,1\} \quad \forall k \right\}.$$

2.3 Probabilistic Approach

One way to understand the dilation equation is through a probabilistic approach. Belock and Dobric [11] and Gundy and Zhang [19] examined this concept. This is natural because the right-hand side of the dilation equation can be interpreted as the convolution of two measures. Namely, the weighted sum of Dirac delta measures: $\sum p_k \delta_k$, and the measure μ, whose density ϕ satisfies the dilation equation. Since the measure $\sum p_k\delta_k$ does not have a density, we look at this from the measure side. We introduce here the Random Variable Dilation Equation. Consider a discrete random variable G with values in a subset Γ_1 of Γ and a random variable Z independent of G, with values in $\mathbb{R}^d$, both defined on a complete probability space, which satisfy

$$MZ \overset{d}{=} Z + G. \tag{2.3}$$

Here, $\overset{d}{=}$ denotes equality of the corresponding laws. Assume that Z is absolutely continuous with respect to Lebesgue measure and denote its density by ϕ. Equation (2.3) implies that ϕ satisfies the dilation equation almost everywhere. An approach to constructing candidates for prescale functions comes from understanding the structure of the solution of this random variable dilation equation [11].

In the one-dimensional case with $M = 2$, an unpublished result of Gundy and Zhang [19] proved that Z is absolutely continuous with respect to Lebesgue measure if and only if the fractional part of Z is uniform. They also gave a sufficient condition for the uniformity of the fractional part. The *fractional part* of a random variable Z can defined as $Z - \lfloor Z \rfloor$. In the higher dimensional case, Belock and Dobric [11] show that the statements of Gundy and Zhang hold true when a proper notion of the "fractional" part of a random variable is introduced.

However, by considering the dilation equation through a probabilistic approach, we limit ourselves to only constructing non-negative prescale functions. This is an unnatural constraint because several well known scaling functions, including those of Daubechies' wavelets, aren't, in fact, non-negative. Therefore, we are now considering a general measure theoretic approach.

2.4 Pseudo-Probability

We may define real random variables with *pseudo-probability* distributions, in other words, real valued Borel measures μ with $\mu(\mathbb{R}) = 1$. This allows consistent definition of independent random variables, even though, in general, the underlying pseudo-probability space may only support a finitely additive measure. Very much of probability theory may be transferred to this setting [20]. For example, Hochberg [21, 22] derived a generalization of Brownian motion governed by signed distributions which are the fundamental solutions of higher even-order parabolic partial differential equations. As a consequence of this research he proved some central limit theorems for equally distributed components. Our work does not explore generalizing probability results, but does take advantage of this notion of pseudo-probabilities.

Baez-Duarte [20] explored how signed measures “give the subject a decidedly different flavor.” The first instance of this is that it is not the case that $\mu_n \to \mu$ implies $(\mu_n)^+ \to (\mu)^+$. Moreover, the Portmanteau Theorem does not carry over. He goes on to state that the classical Lévy Convergence Theorem, which states that the weak convergence of a sequence of probability measures μ_n to a probability measure μ is equivalent to the pointwise convergence of the corresponding characteristic functions fails in the case of signed measures. Initially, we thought that we might be able to generalize the theorems in the paper by Belock and Dobric by applying similar techniques as what Belock and Dobric had used, but the work proved to be more complex than that. We eventually looked into the weak convergence of an approximating measure.

Chapter 3

Signed measure dilation equations

3.1 Existence

We begin by defining the signed measure dilation equation on $\mathcal{B}(\mathbb{R})$:

$$\mu(A) = \sum_{k=0}^{2N-1} p_k \mu(2A - k)$$

for any Borel set $A \subset \mathbb{R}$ and some integer N.

We work under the same orthogonality conditions on the p_k as in the functional dilation equation:

$$\begin{cases} \sum_{i=0}^{2N-1} p_i = 1 & (1) \\ \sum_{i=0}^{2N-1} p_i p_{i+2l} = \frac{1}{2}\delta_{0l}. & (2) \end{cases}$$

We form a solution to this dilation equation in an iterative manner. Let $\mu_0 = \sum_{i=0}^{2N-1} p_i \delta_{\frac{i}{2}}$. Then, we define the discrete measures μ_n. Let $D : \mathbb{R} \to \mathbb{R}$ be defined as $D(x) = \frac{x}{2}$. The *push forward* of the function D, denoted $D_\star$, is defined as $D_\star \nu(\cdot) = \nu(D^{-1}\cdot)$. The *convolution* of two measures ν and μ, denoted $\nu \star \mu$, is defined as $\nu \star \mu(\cdot) = \int \nu(\cdot - y) d\mu(y)$. Then, we define the sequence (μ_n) by

$$\mu_n = \mu_0 \star D_\star(\mu_{n-1}).$$

We claim that the limit of μ_n is a solution of the dilation equation for measures on $\mathbb{R}$. These discrete measures, μ_n, can be written as a linear combination of Dirac-delta measures: Let $S_n =: x \in \mathbb{R} | x = \frac{k}{2^{n+1}}$ for some $k \in \mathbb{Z}$. Note that $\text{supp}(\mu_n) \subset S_n$. So we can write $\mu_n = \sum_{x \in S_n} w_n(x)\delta_x$, where $w_n(x)$ are the pseudo probabilities associated with the points $x \in S_n$. Level 0 contains only the points $\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, ..., \frac{2N-1}{2}$ with pseudo probabilities p_0, p_1, p_2, ... p_{2N-1} respectively.

First, we show why this sequence of measures is worth looking at. That is, if the limit exists, it is indeed a solution to the dilation equation for measures.

Lemma 5. *If $(\mu_n)_n$ weakly converges to a limiting measure μ, then μ satisfies the dilation equation for measures.*

Proof. Assume that $\mu = \lim_{n\to\infty} \mu_n$. First, we can see that, by definition, $\mu_n = \bigstar_{k=0}^{n} (D_\star)^k(\mu_0)$. So, we have

$$\begin{aligned}
\mu = &\lim_{n\to\infty} \bigstar_{k=0}^{n} (D_\star)^k(\mu_0) \\
= &(\mu_0) \star \lim_{n\to\infty} \bigstar_{k=1}^{n} (D_\star)^k(\mu_0) \\
= &(\mu_0) \star \lim_{n\to\infty} (D_\star) \left(\bigstar_{k=0}^{n-1} (D_\star)^k(\mu_0)\right) \\
= &(\mu_0) \star (D_\star) \left(\lim_{n\to\infty} \bigstar_{k=0}^{n-1} (D_\star)^k(\mu_0)\right) \\
= &(\mu_0) \star (D_\star)(\mu).
\end{aligned}$$

Note that it is permissible to take the convolution of (μ_0) and $(D_\star)$ outside of the limit because they are both continuous operations. Therefore, it will not cause a problem to change the order in which we take the convolution and apply $D_\star$.

Therefore, this measure $\mu = \lim_{n\to\infty} \mu_n$ satisfies the dilation equation for measures. □

We've illustrated that if the limit of μ_n exists, then the limit will be a solution to the dilation equation for measures. It remains to prove that this limit does exist.

Lemma 6. *The coefficients at level n satisfy the following: $\sum_{x \in S_n} (w_n(x))^2 = \frac{1}{2^{n+1}}$.*

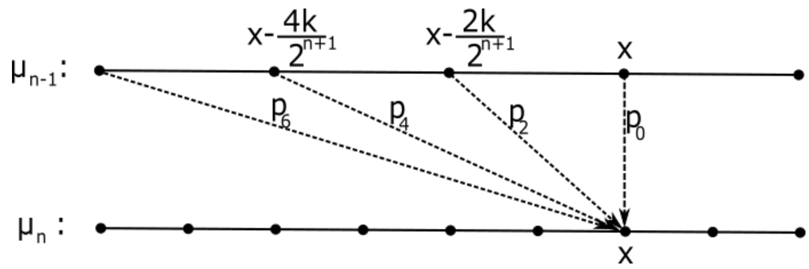

Figure 3.1: Finding $w_n(x)$ from $w_{n-1}(x)$

Proof. We proceed by induction on n. The base case is given as Condition (2) where $l = 0$. Assume the induction hypothesis that $\sum_{x \in S_{n-1}} (w_{n-1}(x))^2 = \frac{1}{2^n}$.

First, we use Figure 3.1 to see how the pseudo probabilities at level n can be written in terms of the pseudo-probabilities at level $n-1$.

So we have, for $x \in S_{n-1}$:

$$w_n(x) = \sum_k p_{2k} w_{n-1}\left(x - \frac{2k}{2^{n+1}}\right)$$
$$w_n(x + \frac{1}{2^{n+1}}) = \sum_k p_{2k+1} w_{n-1}\left(x - \frac{2k}{2^{n+1}}\right).$$

Squaring these, we have:

$$\begin{aligned}(w_n(x))^2 &= \sum_{k=0}\sum_{l=1} 2p_{2k}p_{2k+2l} w_{n-1}\left(x - \frac{k}{2^n}\right) w_{n-1}\left(x - \frac{k+l}{2^n}\right) \\ &+ \sum_k p_{2k}^2 w_{n-1}^2\left(x - \frac{k}{2^n}\right) \\ \left(w_n\left(x + \frac{1}{2^{n+1}}\right)\right)^2 &= \sum_{k=0}\sum_{l=1} 2p_{2k+1}p_{2k+1+2l} w_{n-1}\left(x - \frac{k}{2^n}\right) w_{n-1}\left(x - \frac{k+l}{2^n}\right) \\ &+ \sum_k p_{2k+1}^2 w_{n-1}^2\left(x - \frac{k}{2^n}\right).\end{aligned}$$

Note that in the following summations, we begin with summing over the indices k and j which cover $\mathbb{Z}^2$. We can re-index this by k and $m = j - k$. This is a bijection on $\mathbb{Z}^2$ and so it will still sum over the same points. Finally, by taking the sum of the squares, we obtain the following:

$$\begin{aligned}
\sum_{y\in S_n} (w_n(y))^2 &= \sum_{x\in S_{n-1}} (w_n(x))^2 + \left(w_n\left(x+\frac{1}{2^{n+1}}\right)\right)^2 \\
&= \sum_{j\in\mathbb{Z}} \left(\sum_{k=0} \sum_{l=1} 2\left(p_{2k}p_{2k+2l} + p_{2k+1}p_{2k+1+2l}\right) w_{n-1}\left(\frac{j-k}{2^n}\right) \right. \\
&\quad \left. \times\, w_{n-1}\left(\frac{j-k-l}{2^n}\right) + \sum_k \left(p_{2k}^2 + p_{2k+1}^2\right) w_{n-1}^2\left(\frac{j-k}{2^n}\right)\right) \\
&= \sum_{m\in\mathbb{Z}} \left(\sum_{l=1} \sum_{k=0} 2\left(p_{2k}p_{2k+2l} + p_{2k+1}p_{2k+1+2l}\right) w_{n-1}\left(\frac{m}{2^n}\right) w_{n-1}\left(\frac{m-l}{2^n}\right) \right. \\
&\quad \left. + \sum_k \left(p_{2k}^2 + p_{2k+1}^2\right)\left(w_{n-1}\left(\frac{m}{2^n}\right)\right)^2\right) \\
&= \sum_{m\in\mathbb{Z}} \sum_k \left(p_{2k}^2 + p_{2k+1}^2\right)\left(w_{n-1}\left(\frac{m}{2^n}\right)\right)^2 \qquad \text{by condition (2)} \\
&= \left(\frac{1}{2}\right) \sum_{m\in\mathbb{Z}} \left(w_{n-1}\left(\frac{m}{2^n}\right)\right)^2 \qquad \text{by condition (2)} \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{2^n}\right) \qquad \text{by the induction hypothesis} \\
&= \frac{1}{2^{n+1}}.
\end{aligned}$$

This gives us our desired equality. □

Using this information, we would like to prove that μ_n has uniformly bounded total variation.

Lemma 7. *For any continuous bounded function f, suppose $||f||_\infty \le B$, i.e., $f(x) \le B \quad \forall x$. Then $\forall n$,*

$$\left|\int f d\mu_n\right| \le B\sqrt{2N}.$$

Proof. It is helpful to first note that at any level n, an upper bound for the total number of points with nontrivial weight is $|\operatorname{supp}(\mu_n)| \le 2N \cdot 2^{n+1}$. Let f be a continuous bounded function with $\|f\|_\infty \le B$. Then, we have:

$$\begin{aligned}
\left|\int f d\mu_n\right| &= \left|\sum_{x\in S_n} w_n(x)f(x)\right| \\
&\leq \sqrt{\sum_{x\in S_n}|w_n(x)|^2 \cdot \sum_{x\in \mathrm{supp}(\mu_n)}|f(x)|^2} \quad \text{by Cauchy-Schwarz} \\
&= \sqrt{\frac{1}{2^{n+1}} \cdot \sum_{x\in \mathrm{supp}(\mu_n)}|f(x)|^2} \quad \text{by Lemma 4} \\
&\leq \sqrt{\frac{1}{2^{n+1}} \cdot \|f\|_\infty^2 \cdot \sum_{\mathrm{supp}(\mu_n)} 1} \\
&\leq B\sqrt{2N}.
\end{aligned}$$

Therefore, since the integral of any continuous bounded function against μ_n is bounded $\forall n$, then the sequence, $||\mu_n||_{TV}$, of the total variation of the measures μ_n must be bounded. □

Finally, we can show that μ_n converges weakly to a measure μ. Recall that a sequence of vectors x_n in a normed space E is called weakly convergent to a vector x if $l(x_n) \to l(x)$ for all $l \in E^*$, where E^* is the space of all continuous linear functions on E. This convergence can be described by means of the weak topology on E [23]. We show weak convergence by proving that $\int f d\mu_n$ is a Cauchy sequence for any continuous function f.

Theorem 8. *The sequence of integrals of any continuous function with respect to μ_n converges weakly.*

Proof. Let f be a continuous function with $\|f\|_\infty \leq B$. Let $\epsilon > 0$. Since $\mathrm{supp}(\mu)$ is compact, f is continuous and bounded on a compact set, so f is uniformly continuous on $\mathrm{supp}(\mu)$. So there exists δ such that for all x and y with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Choose M large enough so that $\delta > \frac{2N-1}{2^M}$. Then for $m > M$, we want to show that for any $k > 0$, $|\int f d\mu_{m+k} - \int f d\mu_m| < \epsilon$.

We define a set and make a couple of remarks first. Let

$$B_k = \{0, 1, 2, ..., 2N-1\}^k.$$

Note that

$$1 = 1^k = \left(\sum_{i=0}^{2N-1} p_i \right)^k = \sum_{(a)=(a_1,a_2,...,a_k)\in B_k} \left(\prod_{j=1}^{k} p_{a_j} \right).$$

Also, we have for $m > M$ and for any $k > 0$:

$$\sum_{j=1}^{k} \frac{a_j}{2^{m+j}} \leq \sum_{j=1}^{k} \frac{2N-1}{2^{m+j}} = \frac{2N-1}{2^m} \sum_{j=1}^{k} \frac{1}{2^j} < \frac{2N-1}{2^m} < \frac{2N-1}{2^M} < \delta.$$

It will be helpful to look at the integral $\int f d\mu_{m+k}$ in terms of the points in S_m. In order to do this, we will think about the definition of μ_{m+k}. Thus we have

$$\begin{aligned}
\mu_{m+k} &= \bigstar_{j=0}^{m+k} (D_\star)^j(\mu_0) \\
&= \left(\bigstar_{j=m+1}^{m+k} (D_\star)^j(\mu_0) \right) \star \mu_m \\
&= \left(\bigstar_{j=1}^{k} (D_\star)^{j+m}(\mu_0) \right) \star \left(\sum_{x\in S_m} w_m(x)\delta_x \right) \\
&= \sum_{x\in S_m} \left(\sum_{(a_j)\in B_k} \prod_{j=1}^{k} p_{a_j} \right) w_m(x) \delta_{x+\sum_{j=1}^{k} \frac{a_j}{2^{m+j}}}.
\end{aligned}$$

We will now use these observations to look at the difference of the integrals $\int f d\mu_{m+k}$ and $\int f d\mu_m$:

$$\begin{aligned}
&\left|\int f d\mu_{m+k} - \int f d\mu_m\right| \\
&\qquad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x) f\left(x+\sum_{j=1}^{k}\frac{a_j}{2^{m+j}}\right)\right)\right. \\
&\qquad \left. - \sum_{y\in S_m} w_m(y) f(y)\right| \\
&\qquad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x) f\left(x+\sum_{j=1}^{k}\frac{a_j}{2^{m+j}}\right)\right)\right. \\
&\qquad \left. - \sum_{y\in S_m}\left(\sum_{i=0}^{2N-1} p_i\right)^k w_m(y) f(y)\right| \\
&\qquad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x) f\left(x+\sum_{j=1}^{k}\frac{a_j}{2^{m+j}}\right)\right)\right. \\
&\qquad \left. - \sum_{y\in S_m}\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(y) f(y)\right| \\
&\qquad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x)\left(f\left(x+\sum_{j=1}^{k}\frac{a_j}{2^{m+j}}\right) - f(x)\right)\right)\right|.
\end{aligned}$$

In the above computations, we used the definition of the integral against the measure μ_m and the integral against the measure μ_{m+k} in terms of the points from S_m. From here, we used the fact that $(\sum_{i=0}^{2N-1} p_i)^k = 1$ in order to group terms together. So, we have

$$
\begin{aligned}
&\left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k}p_{a_j}\right)w_m(x)\left(f\left(x+\sum_{j=1}^{k}\frac{a_j}{2^{m+j}}\right)-f(x)\right)\right)\right|\\
&<\left|\sum_{x\in S_m}\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k}p_{a_j}\right)w_m(x)\epsilon\right|\\
&=\left|\sum_{x\in S_m}\left(\sum_{i=0}^{2N-1}p_i\right)^k w_m(x)\epsilon\right|\\
&=\left|\sum_{x\in S_m}w_m(x)\epsilon\right|.
\end{aligned}
$$

In the above approximation, we used the uniform continuity of f. The second and third line of this equation follow from the fact that $\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k}p_{a_j}\right)=\left(\sum_{i=0}^{2N-1}p_i\right)^k=1$. The final approximations follow from the Cauchy Schwarz inequality, Lemma 4, and the fact that we have the upper bound $|\operatorname{supp}(\mu_m)|\leq 2N\cdot 2^{m+1}$. We have

$$
\begin{aligned}
\Big|\sum_{x\in S_m}w_m(x)\epsilon\Big| &\leq \sqrt{\sum_{x\in S_m}|w_m(x)|^2\cdot\sum_{\operatorname{supp}(\mu_m)}|\epsilon|^2}\\
&\leq\sqrt{\frac{1}{2^{m+1}}\cdot\epsilon^2\cdot 2N\cdot 2^{m+1}}\\
&\leq \epsilon\sqrt{2N}.
\end{aligned}
$$

Therefore, our sequence $(\int f d\mu_n)_n$ is Cauchy, so (μ_n) converges weakly to a limiting measure, μ. $\square$

So we have found a solution for the dilation equation for signed measures! We would like to find whether or not this is the unique solution for the dilation equation for measures and whether or not this solution is absolutely continuous with respect to Lebesgue measure.

3.2 Uniqueness

Theorem 9. *The measure that we constructed, $\mu = \lim_{n\to\infty} \mu_n$ is the unique solution in the set of signed measures with compact support, up to scaling by a constant, for the signed measure dilation equation.*

Proof. Suppose ν is any signed measure with compact support. Recall the function $D(x) := \frac{x}{2}$.Then, we claim that $D_\star^{n+1}\nu \to \nu(\mathbb{R})\delta_0$ weakly as $n \to \infty$. We can show this by letting f be any bounded continuous function. Then, we have

$$\int f dD_\star^n \nu(x) = \int f(2^{-n}x)d\nu.$$

Now we take the limit, and have the following:

$$\begin{aligned}\lim_{n\to\infty} \int f(2^{-n}x)d\nu &= \int f(0)d\nu \\ &= f(0)\nu(\mathbb{R}).\end{aligned}$$

Therefore, we have the weak convergence, $\lim_{n\to\infty} D_\star^{n+1}\nu \to \nu(\mathbb{R})\delta_0$. So now, let $\tilde{\mu}$ be any solution to the signed measure dilation equation with compact support. We take the limit as $n \to \infty$ to get

$$\begin{aligned}\tilde{\mu} &= \lim_{n\to\infty} \left(\mu_0 \star D_\star\mu_0 \star D_\star^2\mu_0 \cdots \star D_\star^n\mu_0\right) \star D_\star^{n+1}\tilde{\mu} \\ &= (\bigstar_{n=0}^{\infty} D_\star^n\mu_0) \star \tilde{\mu}(\mathbb{R})\delta_0 \\ &= \tilde{\mu}(\mathbb{R})\left(\bigstar_{n=0}^{\infty} D_\star^n\mu_0\right).\end{aligned}$$

By definition of our solution, $\mu(x) = \bigstar_{n=0}^{\infty} D_\star^n\mu_0$. So it must be the unique solution up to multiplication by a constant, in particular, $\tilde{\mu}(\mathbb{R})$. Moreover, solutions are unique within the class of measures ν such that $\nu(2^{n+1}x) \to \nu(\mathbb{R})\delta_0$, which includes more than just measures with compact support. $\square$

We also showed that the standard uniqueness condition holds for the signed measure dilation equation. This condition is relevant to the Fourier transform of the resulting measure.

Proposition 10. *The Fourier Transform of μ is continuous at 0, ie:*

$$\lim_{\xi \to 0} \hat{\mu}(\xi) = \hat{\mu}(0) = \mu(\mathbb{R}).$$

Proof. We can show that the regular condition holds under the case of μ being a signed measure as well. So, we would like to show that

$$\lim_{\xi \to 0} \hat{\mu}(\xi) = \hat{\mu}(0).$$

We start with the following:

$$\begin{aligned}
\hat{\mu}(0) &= \mu(\mathbb{R}) \\
&= \widehat{\mu(\mathbb{R})\delta_0} \\
&= \lim_{n \to \infty} \widehat{D_\star^n}\mu(\xi) \\
&= \lim_{n \to \infty} \int_{\mathbb{R}} e^{-i\pi x \cdot \xi} dD_\star^n \mu(x) \\
&= \lim_{n \to \infty} \int_{\mathbb{R}} e^{-i\pi x \cdot \xi} d\mu(2^{n+1}x).
\end{aligned}$$

Now we can make a change of variable, where $y = 2^{n+1}x$. So, we have $x = \frac{y}{2^{n+1}}$. Therefore, we have:

$$\begin{aligned}
\lim_{n \to \infty} \int_{\mathbb{R}} e^{-i\pi x \cdot \xi} d\mu(2^{n+1}x) &= \lim_{n \to \infty} \int_{\mathbb{R}} e^{-i\pi \frac{y}{2^{n+1}} \cdot \xi} d\mu(y) \\
&= \lim_{n \to \infty} \int_{\mathbb{R}} e^{-i\pi y \cdot \frac{\xi}{2^{n+1}}} d\mu(y) \\
&= \lim_{\xi \to 0} \int_{\mathbb{R}} e^{-i\pi y \cdot \xi} d\mu(y) \\
&= \lim_{\xi \to 0} \hat{\mu}(\xi).
\end{aligned}$$

$\square$

3.3 Absolute Continuity

Now that we have the existence of the unique solution to the signed measure dilation equation, we must show that it is absolutely continuous with respect to Lebesgue measure. As long as this is the case, its Radon-Nikodym derivative will be a solution to the functional dilation equation almost everywhere.

Proposition 11. *The solution to the signed measure dilation equation, μ, is absolutely continuous with respect to Lebesgue measure.*

Proof. We approach this proof by contraction. Suppose that A is a Borel-measureable set with $\lambda(A) = 0$ but $|\mu|(A) = 2a > 0$. Then, since $|\mu|$ is a Borel regular measure, there is a compact subset $K \subset A$ with $|\mu|(K) > a$. Because K has Lebesgue measure 0, for any $\epsilon > 0$, we can cover K with open intervals whose areas have sum $< \epsilon$. And since K is compact, we have a finite subcover, E. Then, we see that the number of points in S_{m-1} contained in E is asymptotic to $|E| \cdot 2^m$. In addition, since the sum-of-squares of the $w_{m-1}(x)$ is $1/2^m$, this trivially bounds the sum-of-squares of the $w_{m-1}(x)$ in E. By Urysohn Lemma, we have a function f with $0 \leq f \leq 1$ that has $f = 1$ on K and $f = 0$ on $\mathbb{R} \setminus E$. Then we have the following:

$$\begin{aligned}
|\mu|(K) &\leq \int f d|\mu| \\
&\leq \limsup \int f d|\mu_{m-1}| \\
&= \limsup \sum f(x)|w_{m-1}(x)| \\
&\leq \limsup \sqrt{\sum_{x \in E \cap S_{m-1}} f^2(x) \sum_{x \in E \cap S_{m-1}} w_{m-1}^2(x)} \\
&\leq \limsup \sqrt{(|E| \cdot 2^m + o(2^m)) \left(\frac{1}{2^m}\right)} \\
&= \sqrt{|E|} \leq \sqrt{\epsilon}.
\end{aligned}$$

From here, it follows that the total variation of μ and $|\mu|$ over E is less than or equal to $\sqrt{\epsilon}$. So we can choose ϵ to be small enough that $\sqrt{\epsilon} < a$. However, since E

covers K, we assumed that $|\mu|(E) \geq |\mu|(K) > a$, which is a contraction. Therefore, we must have that $|\mu|(A) = 0$, so μ and $|\mu|$ are both absolutely continuous with respect to Lebesgue measure. $\square$

We also claim that the solution μ to the dilation equation for signed measures has density satisfying the functional dilation equation almost everywhere.

Corollary 12. *Let ϕ be the density of μ. We have that $\phi \in L^2(\mathbb{R})$.*

Proof. We begin by letting f be a continuous smooth function with compact support. Then, by definition of μ and μ_n, we have

$$\begin{aligned}\int f d\mu &= \lim_{n\to\infty} \int f d\mu_n \\ &= \lim_{n\to\infty} \sum_{x \in S_n} f(x) w_n(x).\end{aligned}$$

Now we can apply the Cauchy Schwarz inequality to the sum on the right-hand side, followed by Lemma 4. So we have,

$$\begin{aligned}\int f d\mu &\leq \lim_{n\to\infty} \sqrt{\sum_{x\in S_n} w_n^2(x) \sum_{x \in \operatorname{supp}(\mu_n)} f^2(x)} \\ &= \lim_{n\to\infty} \sqrt{\frac{1}{2^{n+1}} \sum_{x \in \operatorname{supp}(\mu_n)} f^2(x)} \\ &= \lim_{n\to\infty} \sqrt{N \cdot \frac{1}{N \cdot 2^{n+1}} \sum_{x \in \operatorname{supp}(\mu_n)} f^2(x)}.\end{aligned}$$

Note that $|\operatorname{supp}(\mu_n)| \leq N \cdot 2^{n+1}$. So, considering the refinement of the real line by dyadic intervals, by the definition of Lebesgue integral, we have the following:

$$\begin{aligned}\int f d\mu &\leq \sqrt{N} \sqrt{\int f^2(x) dx} \\ &= \sqrt{N} ||f||_2.\end{aligned}$$

Therefore integration of a smooth function f against μ is bounded by a constant multiple of $||f||_2$. Since any L^2 function can be approximated by smooth functions, we can find the integral of an L^2 function g against μ by taking the limit of integrals against approximating smooth functions. Therefore, μ is a bounded linear functional on L^2. So by Riesz Representation Theorem, $\phi \in L^2$.

$\square$

Corollary 13. *The density ϕ of the solution μ of the dilation equation for signed measures satisfies the functional dilation equation almost everywhere.*

Proof. Let ϕ be the density of the solution to the dilation equation for signed measures, μ. Let x_0 be any point in $\operatorname{supp}(\phi)$ and $r \in \mathbb{R}^+$. Let $B(x_0, r)$ denote the ball of radius r about the point x_0. Then we have the following, from the dilation equation for signed measures:

$$\mu(B(x_0, r)) = \sum_k p_k \mu(2(B(x_0, r)) - k).$$

We can re-write each side of this equation using ϕ, the density of μ. Thus we see

$$\begin{aligned} \int_{B(x_0,r)} \phi(x)dx &= \sum_k p_k \int_{2(B(x_0,r))-k} \phi(y)dy \\ &= 2\sum_k p_k \int_{B(x_0,r)} \phi(2x-k)dx, \end{aligned}$$

where the second equality is true by substituting in $x = \frac{1}{2}(y+k)$. Now we can take the limit as $r \to 0$.

$$\lim_{r\to 0} \int_{B(x_0,r)} \phi(x)dx = 2 \lim_{r\to 0} \sum_k p_k \int_{B(x_0,r)} \phi(2x-k)dx.$$

By the Lebesgue Differentiation Theorem, for almost every x_0, we have:

$$\phi(x_0) = 2\sum_k p_k \cdot \phi(2x_0 - k).$$

Therefore, the density of the solution of the dilation equation for signed measures satisfies the functional dilation equation almost everywhere.

$\square$

3.4 Example of computing a scaling function

We can consider the example of Daubechies' D4 wavelet. The dilation equation has pseudo-probabilities: $p_0 = \frac{1+\sqrt{3}}{8}$, $p_1 = \frac{3+\sqrt{3}}{8}$, $p_2 = \frac{3-\sqrt{3}}{8}$, and $p_3 = \frac{1-\sqrt{3}}{8}$. We know that $\text{supp}(\mu) \subset [0,3]$, so we apply the dilation equation the intervals of length 1: $[0,1]$, $[1,2]$, and $[2,3]$. By doing this, we obtain the following system of linear equations:

$$\begin{aligned}
\mu([0,1]) &= p_0\mu([0,2]-0) + p_1\mu([0,2]-1) + p_2\mu([0,2]-2) + p_3\mu([0,2]-3) \\
&= p_0\mu([0,1]) + p_0\mu([1,2]) + p_1\mu([-1,0]) + p_1\mu([0,1]) + p_2\mu([-2,-1]) \\
&+ p_2\mu([-1,0]) + p_3\mu([-3,-2]) + p_3\mu([-2,-1]) \\
&= (p_0+p_1)\mu([0,1]) + p_0\mu([1,2]),
\end{aligned}$$

$$\begin{aligned}
\mu([1,2]) &= p_0\mu([2,4]-0) + p_1\mu([2,4]-1) + p_2\mu([2,4]-2) + p_3\mu([2,4]-3) \\
&= p_0\mu([2,3]) + p_0\mu([3,4]) + p_1\mu([1,2]) + p_1\mu([2,3]) + p_2\mu([0,1]) \\
&+ p_2\mu([1,2]) + p_3\mu([-1,0]) + p_3\mu([0,1]) \\
&= (p_2+p_3)\mu([0,1]) + (p_1+p_2)\mu([1,2]) + (p_0+p_1)\mu([2,3]),
\end{aligned}$$

$$\begin{aligned}
\mu([2,3]) &= p_0\mu([4,6]-0) + p_1\mu([4,6]-1) + p_2\mu([4,6]-2) + p_3\mu([4,6]-3) \\
&= p_0\mu([4,5]) + p_0\mu([5,6]) + p_1\mu([3,4]) + p_1\mu([4,5]) + p_2\mu([2,3]) \\
&+ p_2\mu([3,4]) + p_3\mu([1,2]) + p_3\mu([2,3]) \\
&= p_3\mu([1,2]) + (p_2+p_3)\mu([2,3]).
\end{aligned}$$

Using this, we form the matrix A which has a right-1 eigenvector:

$$A = \begin{pmatrix} p_0 + p_1 & p_0 & 0 \\ p_2 + p_3 & p_1 + p_2 & p_0 + p_1 \\ 0 & p_3 & p_2 + p_3 \end{pmatrix} = \begin{pmatrix} \frac{4+2\sqrt{3}}{8} & \frac{1+\sqrt{3}}{8} & 0 \\ \frac{4-2\sqrt{3}}{8} & \frac{3}{4} & \frac{4+2\sqrt{3}}{8} \\ 0 & \frac{1-\sqrt{3}}{8} & \frac{4-2\sqrt{3}}{8} \end{pmatrix}.$$

We find that this has eigenvalue 1, so we look at $A - I$ to find the corresponding eigenspace:

$$A - I = \begin{pmatrix} \frac{\sqrt{3}-2}{4} & \frac{1+\sqrt{3}}{8} & 0 \\ \frac{2-\sqrt{3}}{4} & -\frac{1}{4} & \frac{2+\sqrt{3}}{4} \\ 0 & \frac{1-\sqrt{3}}{8} & \frac{-2-\sqrt{3}}{4} \end{pmatrix}.$$

After row reducing this, we have:

$$A - I = \begin{pmatrix} 1 & \frac{1+\sqrt{3}}{2\sqrt{3}-4} & 0 \\ 0 & 1 & \frac{2+2\sqrt{3}}{\sqrt{3}-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the eigenspace is one dimensional and is spanned by the vector V':

$$V' = \begin{pmatrix} \frac{-1-\sqrt{3}}{2\sqrt{3}-4} \\ 1 \\ \frac{\sqrt{3}-1}{-2-2\sqrt{3}} \end{pmatrix}.$$

We can normalize this vector so that its sum is 1, which would correspond with the scaling function having total mass 1, giving

$$V = \begin{pmatrix} \frac{-1-\sqrt{3}}{2-3\sqrt{3}} \\ \frac{2}{5+4\sqrt{3}} \\ \frac{\sqrt{3}-1}{-17-9\sqrt{3}} \end{pmatrix}.$$

This tells us, specifically, that $\mu(0,1) = \frac{-1-\sqrt{3}}{2-3\sqrt{3}}$, $\mu(1,2) = \frac{2}{5+4\sqrt{3}}$, and $\mu(2,3) = \frac{\sqrt{3}-1}{-17-9\sqrt{3}}$. From here, we use the signed measure dilation equation to find the measures of the intervals of length $\frac{1}{2}$, then $\frac{1}{4}$, and so on. We apply the dilation equation to the intervals of length $\frac{1}{2}$ (specifically $(0, \frac{1}{2})$, $\left(\frac{1}{2}, 1\right)$, $\left(1, \frac{3}{2}\right)$, etc.). This gives us the equations:

$$\begin{aligned}
\mu\left(0, \frac{1}{2}\right) &= p_0\mu(0,1)\\
\mu\left(\frac{1}{2}, 1\right) &= p_0\mu(1,2) + p_1\mu(0,1)\\
\mu\left(1, \frac{3}{2}\right) &= p_0\mu(2,3) + p_1\mu(1,2) + p_2\mu(0,1)\\
\mu\left(\frac{3}{2}, 2\right) &= p_1\mu(2,3) + p_2\mu(1,2) + p_3\mu(0,1)\\
\mu\left(2, \frac{5}{2}\right) &= p_2\mu(2,3) + p_3\mu(1,2)\\
\mu\left(\frac{5}{2}, 3\right) &= p_3\mu(2,3).
\end{aligned}$$

Using Matlab at this level, as well at the subsequent levels, we are able to find that

$$\begin{aligned}
\mu\left(0, \frac{1}{2}\right) &= 0.290170901\\
\mu\left(\frac{1}{2}, 1\right) &= 0.559508468\\
\mu\left(1, \frac{3}{2}\right) &= 0.227670901\\
\mu\left(\frac{3}{2}, 2\right) &= -0.061004234\\
\mu\left(2, \frac{5}{2}\right) &= -0.017841801\\
\mu\left(\frac{5}{2}, 3\right) &= 0.001495766.
\end{aligned}$$

Further,

$$\mu\left(0,\frac{1}{4}\right) = 1.866025404$$
$$\mu\left(\frac{1}{4},\frac{1}{2}\right) = 3.598076211$$
$$\mu\left(\frac{1}{2},\frac{3}{4}\right) = 4.696152423$$
$$\mu\left(\frac{3}{4},1\right) = 5.839745962$$
$$\mu\left(1,\frac{5}{4}\right) = 3.287187079$$
$$\mu\left(\frac{5}{4},\frac{3}{2}\right) = 1$$
$$\mu\left(\frac{3}{2},\frac{7}{4}\right) = -0.019237886$$
$$\mu\left(\frac{7}{4},2\right) = -1.129510429$$
$$\mu\left(2,\frac{9}{4}\right) = -0.445554338$$
$$\mu\left(\frac{9}{4},\frac{5}{2}\right) = 0.109581934$$
$$\mu\left(\frac{5}{2},\frac{11}{4}\right) = 0.030743609$$
$$\mu\left(\frac{11}{4},3\right) = -0.002577388.$$

This yields step function approximations of ϕ illustrated in Figure 3.2.

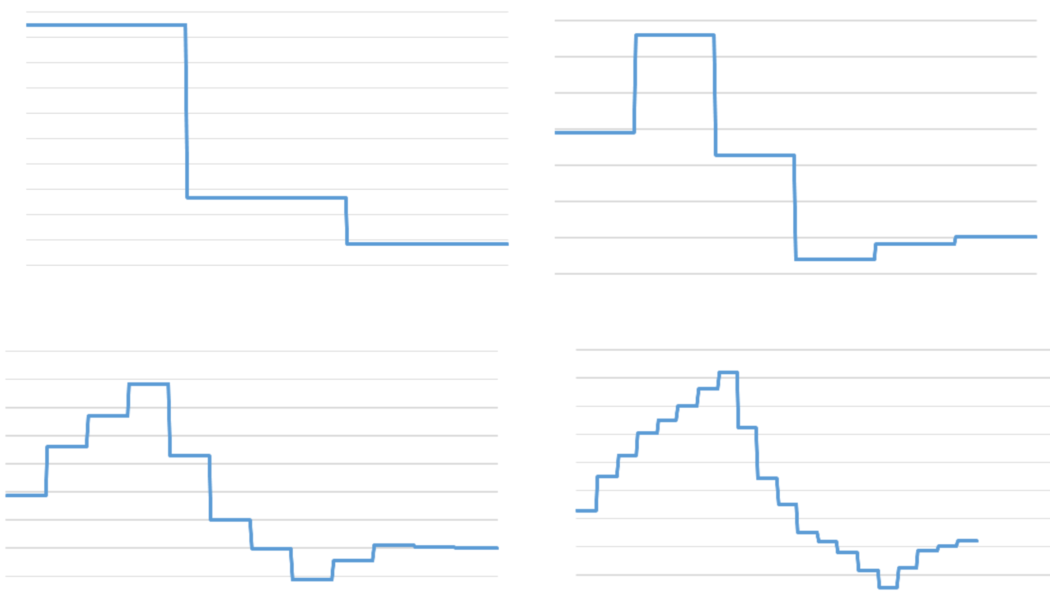

Figure 3.2: Dyadic step-function approximations of scaling function D4

Chapter 4

Two-dimensions

4.1 Scaling Functions in two-dimensions

It is helpful to compare the Twin Dragon tile, $T(+C_3, \mathcal{D}_1)$, with the unit interval $\{x : 0 \leq x \leq 1\}$, $T(2, \mathcal{D}_1)$, viewed as a tile with dilation 2 and a digit set $\mathcal{D}_1 = \{0, 1\}$. It is worth noting that the tiles $T(+C_3, \mathcal{D}_1)$ and $T(2, \mathcal{D}_1)$ both have the property that they contain exactly two lattice boundary points. The unit interval contains the points 0 and 1 while the twin dragon contains the points 0 and $-i$. These similarities are significant for a procedure that maps a space of binary sequences into the spaces $\mathbb{Z}$ and $\mathbb{Z}^2$. This coding is generated by the pair $(2, \mathcal{D}_1)$ on one hand, and by $(+C_3, \mathcal{D}_1)$ on the other.

In one dimension, the coding procedure is performed simply by writing the real number in its binary representation. In two dimensions, the coding procedure is performed in a similar manner, except that the base for this representation is $1 + i$. The first of these codings will map onto the non-negative half of the real line. Similarly, the latter will map onto half of the complex plane in some way [17].

We will detail two different methods of constructing scaling functions in two-dimensions. The first is inspired by techniques proposed by Gundy and Jonsson [17], while the second is analogous to our method used in one-dimension.

4.1.1 Gundy's method of pushing to two-dimensions

Gundy's principal result [17] is relevant to our work. It states that if there exists a scaling function in one-dimension with coefficients (p_k) in the dilation equation, then there exists a scaling function in two-dimensions with the same coefficients in the dilation equation with dilation by a factor of M, where M belongs to the class $+C_3$.

The way this is proven is by pushing a known scaling function (or measure) from one-dimension to two-dimensions. This is done by the following method. Suppose ϕ is a scaling function in one dimension. Then, each point in the positive real line can be written as its binary expansion: $x = \sum_k \frac{d_k}{2^k}$. So we can think of ϕ as a function on the sequences d_k. Now we will identify a point in the complex plane with each of these points from the real line in the following way: $x = \sum_k \frac{d_k}{2^k} \sim \sum_k \frac{d_k}{(1+i)^k} = x'$. Then we will have scaling function ϕ' defined by $\phi'(x') = \phi(x)$.

This transformation of the scaling function from one-dimension to two-dimensions, by identifying points with the same radix expansion, does not preserve continuity.

Consider for example, the Daubechies' D4 scaling function. The following figure illustrates the transformation for the D4 scaling function on the interval $[0, 1]$ to the Twin Dragon. The coloring in this figure represents the height of the function lying over the plane.

In this case,

$$\begin{aligned} p_0 &= \frac{1+\sqrt{3}}{8} & p_1 &= \frac{3+\sqrt{3}}{8} \\ p_2 &= \frac{3-\sqrt{3}}{8} & p_3 &= \frac{1-\sqrt{3}}{8}. \end{aligned}$$

Recall the functional dilation equation:

$$\phi(x) = 2\sum_k p_k \phi(2x - k).$$

Then, by applying the dilation equation to the integers 0, 1, 2, and 3, we find the system of equations:

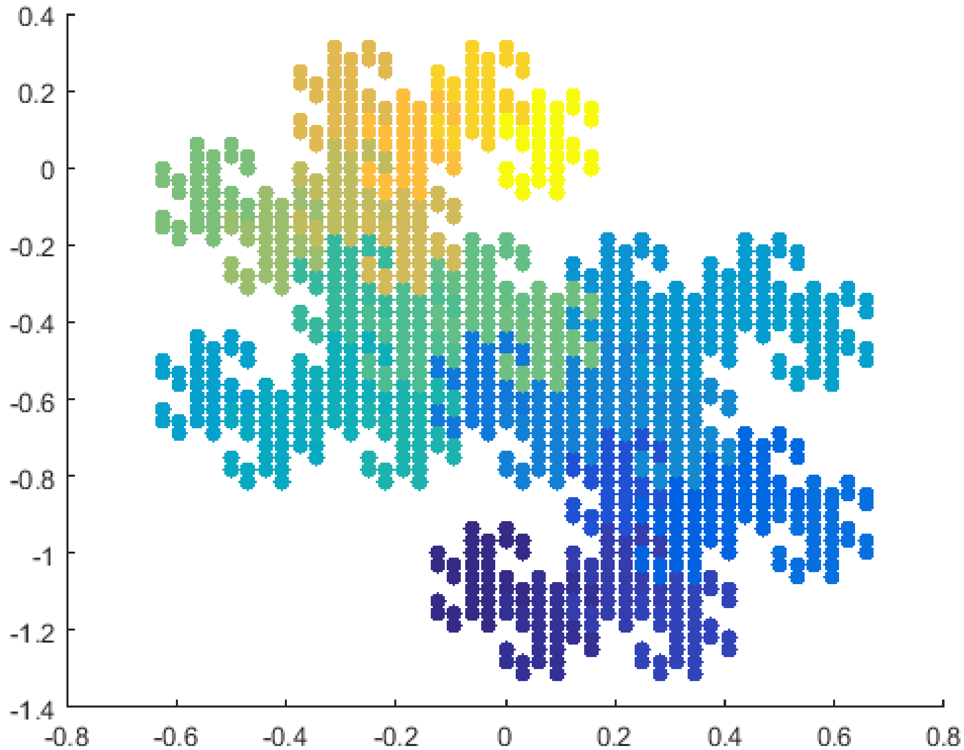

Figure 4.1: D4 scaling function restricted to $[0, 1]$, approximated to the refinement on intervals of length $\frac{1}{2^5}$, translated to the plane

$$\phi(0) = 2p_0\phi(0)$$
$$\phi(1) = 2p_0\phi(2) + 2p_1\phi(1) + 2p_2\phi(0)$$
$$\phi(2) = 2p_1\phi(3) + 2p_2\phi(2) + 2p_3\phi(1)$$
$$\phi(3) = 2p_3\phi(3).$$

This immediately gives $\phi(0) = \phi(3) = 0$. We are left with the following system of equations:

$$\phi(1) = 2p_0\phi(2) + 2p_1\phi(1)$$
$$\phi(2) = 2p_2\phi(2) + 2p_3\phi(1).$$

Solving these, we find the following possible solution:

$$\phi(1) = 1$$
$$\phi(2) = \frac{1-\sqrt{3}}{1+\sqrt{3}}.$$

Further, we can apply the dilation equation to the points which are multiples of $\frac{1}{2}$. This gives us the following:

$$\phi\left(\frac{1}{2}\right) = 2p_0\phi(1) = \frac{1+\sqrt{3}}{4}$$
$$\phi\left(\frac{3}{2}\right) = 2p_1\phi(2) + 2p_2\phi(1) = 0.$$

We can then apply the dilation equation to the points which are multiples of $\frac{1}{4}$. We find that

$$\phi\left(\frac{1}{4}\right) = p_0\phi\left(\frac{1}{2}\right) = \frac{2+\sqrt{3}}{16}$$
$$\phi\left(\frac{3}{4}\right) = p_0\phi\left(\frac{3}{2}\right) + p_1\phi\left(\frac{1}{2}\right) = \frac{3+2\sqrt{3}}{16}.$$

The dyadic expansion of the points $\frac{1}{4}$ and $\frac{3}{4}$ are, respectively, $.01$ and $.10\bar{1}$. Both of these expansions correspond with the complex number $\frac{-i}{2}$ when using the base $1+i$. Let $T_{.01}$ denote the tile which begins with the radix expansion $.01$ and $T_{.10}$ denote the tile which begins with the radix expansion $.10$. Then, we see that

$$\lim_{x \to \frac{-i}{2} \text{ via } T_{.01}} \phi(x) = \frac{2+\sqrt{3}}{16}$$

and we have

$$\lim_{x \to \frac{-i}{2} \text{ via } T.10} \phi(x) = \frac{3+2\sqrt{3}}{16}.$$

Therefore, this two-dimensional version of the Daubechies' D4 scaling function is not continuous. So the transformation does not preserve continuity.

4.1.2 A second method for constructing scaling functions on $\mathbb{R}^2$

We can create an ordering of a certain subset of the Gaussian integers, $\mathbb{Z}[i]$. Define a Gaussian integer to be *even* if and only if it can be written as the product of a Gaussian Integer with $1+i$. Additionally, if a Gaussian integer is not even, it is *odd*. This corresponds with the two different cosets of $\mathbb{Z}[i]/(1+i)\mathbb{Z}[i]$.We begin by choosing any odd Gaussian integer, s. Then define the sequence (a) by

$$a_{2k} = k(1+i)$$

and

$$a_{2k+1} = k(1+i) + s.$$

Note that this set of points is not the most general possible set to begin with, but we have found that it works. We will have $M = 1 + i$ as our dilation.

We begin by considering the dilation equation for measures on $\mathbb{R}^2$:

$$\mu(A) = \sum_k p_k \mu(MA - a_k).$$

Assume that only a finite number of p_k are non-zero. The following work is very similar to that in one-dimension. We form a solution to this dilation equation in an iterative manner. Let $\mu_0 = \sum_k p_k \delta_{M^{-1}a_k}$. Then, we define the discrete measures μ_n. Let $D : \mathbb{C} \to \mathbb{C}$ be defined as $D(x) = M^{-1}x$. Denote the push forward of this map as $D_\star$. Then, we have

$$\mu_n = \mu_0 \star D_\star(\mu_{n-1}).$$

We claim that the limit of μ_n is a solution of the signed measure dilation equation in two-dimensions. Define the set $S_n := M^{-(n+1)}\mathbb{Z}[i]$. Note that $\text{supp}(\mu_n) \subset S_n$. These discrete measures can be written as a linear combination of Dirac-delta measures: $\mu_n = \sum_{x \in S_n} w_n(x)\delta_x$, where $w_n(x)$ is the pseudo probability associated with the point x for the measure μ_n. We work under the same orthogonality conditions which were considered in the one-dimensional case.

$$\begin{cases} \sum_{i=0}^{2N-1} p_i = 1 & (1) \\ \sum_{i=0}^{2N-1} p_i p_{i+2l} = \frac{1}{2}\delta_{0l}. & (2) \end{cases}$$

We first show that these are necessary and almost sufficient conditions to form an orthonormal basis. Since there are a finite number of non-zero p's, say $\{p_n\}$ for all $n \in \Lambda$, where Λ is a finite subset of the Gaussian integers. (Our Λ has evens given by $0, 1+i, \ldots, N(1+i)$ for some N and odds given by all of these plus a fixed odd , for a total of $2N$ points.) Then by the coarse estimate on the support of μ, we know that the density $\phi(x) : \mathbb{R}^2 \to \mathbb{R}$ has support in some large ball around the origin of radius R (we can estimate R in terms of Λ, but this is only relevant in concrete cases). Thus, if we let

$$\alpha_z = \int_{\mathbb{R}^2} \phi(x)\phi(x-z)\,dx \quad \text{for } z \in \mathbb{Z}[i],$$

we see that $\alpha_z = 0$ whenever z is distance more than $2R$ from the origin, because the integrand is identically 0. Thus, we can restrict our attention to only those Gaussian integers inside B_{2R}, the ball of radius $2R$ centered at the origin. Note that ϕ and its Gaussian integer translates form an orthonormal basis for their span in $L^2(\mathbb{C})$ exactly when $\alpha_z = \delta_{0,z}$ (where δ here is Kronecker's delta function). This is then also the condition for ϕ to be a scaling function.

We want to give necessary and (almost) sufficient conditions for such a ϕ, given as the solution of a dilation equation, to be a scaling function. First, note that by applying the dilation equation, we have that

$$\alpha_z = \int_{\mathbb{R}^2} 4 \sum_{k,j \in \Lambda} p_k p_j \phi((1+i)x - k)\phi((1+i)x - (1+i)z - j)\,dx$$

and the change of indices $j = \ell - (1+i)z$ gives

$$\alpha_z = 4 \sum_{\substack{k \in \Lambda \\ \ell \in \Lambda + 2}} p_k p_{\ell - 2z} \int_{\mathbb{R}^2} \phi((1+i)x - k)\phi((1+i)x - \ell)\,dx.$$

Writing the integral on the right-hand side in terms of α, and gaining a factor of $1/2$ from the change of variables, we see that

$$\alpha_z = 2 \sum_{\substack{k\in\Lambda \\ \ell\in\Lambda+2z}} p_k p_{\ell-2z} \alpha_{\ell-k}.$$

If we adopt the convention that $p_k = 0$ for $k \notin \Lambda$, then we can neglect the range of summation for k and ℓ.

If we assume that ϕ is a scaling function, then $\alpha_z = \delta_{0,z}$, and we see that

$$\delta_{0,z} = 2 \sum_{k,\ell} p_k p_{\ell-2z} \delta_{\ell,k} = 2 \sum_k p_k p_{\ell-2z},$$

which are exactly the Lawton conditions mentioned in Chapter 3 [13]. Thus, these conditions are necessary for a solution to the doubling equation to be a scaling function.

On the other hand, suppose these conditions hold. Then making the change of indices $j = \ell - k$ and eliminating ℓ, we see that

$$\alpha_z = \sum_j \alpha_j \left(\sum_{k\in\Lambda\cap\Lambda+2z-j} 2 p_k p_{k+j-2z} \right). \tag{4.1}$$

Because we only need to consider $z \in B_{2R} \cap \mathbb{Z}^2$, this becomes a finite relationship among a finite number of α_z. To be more precise, order the points of $B_{2R} \cap \mathbb{Z}^2$ as $z_1, z_2, \ldots, z_L$, and assume that $z_1 = 0$. If we then consider the column L-vector $\alpha = [\alpha'_{z_j}]_{j=1}^{L}$ and let β be the column L-vector with first component 1 and all other components 0, then ϕ is a scaling function exactly when $\alpha = \beta$. Further, let A be the L-by-L matrix with entries

$$A_{nj} = \sum_k 2 p_k p_{k+j-2z_n}.$$

Then the system of equations given by (4.1) can be written as the matrix equation $\alpha = A\alpha$. In other words, α is a right 1-eigenvector of A. Next, note that β is always a right 1-eigenvector of A, since the nth component of $A\beta$ is

$$[A\beta]_n = \sum_k p_k p_{k-2z_n} = \delta_{1,n}.$$

So if A has a 1-dimensional right 1-eigenspace, α must be a multiple of β, and because ϕ is normalized to have L^2-norm 1, α must equal β. Thus, if the Lawton-type conditions hold *and* the associated matrix A has a 1-dimensional right 1-eigenspace, the solution ϕ to the corresponding dilation equation will be a scaling function. This is the "almost sufficiency" we referred to earlier.

Now that we have established the necessity and almost sufficiency of the orthogonality conditions in two dimensions, we begin with the following lemma, exemplifying why this sequence of measures is worth studying.

Lemma 14. *If* $\lim_{n\to\infty} \mu_n$ *exists, then it satisfies the dilation equation for measures.*

Note that, with only changing the definition of D to now be $D(x) := \frac{x}{1+i}$, the proof for the analogous lemma in one-dimension, Lemma 5 in Chapter 3, carries over for the two-dimensional case. Therefore, this measure $\mu = \lim_{n\to\infty} \mu_n$ satisfies the dilation equation for measures. It remains to prove that this limit does exist.

Lemma 15. *For the weights of measure* μ_n*, we have that* $\sum_x (w_n(x))^2 = \frac{1}{2^{n+1}}$.

Proof. We proceed by induction on n. The base case is given as Condition (2) with $l = 0$. We can see that going from level n to $n+1$ is just a zoomed in, rotated version of going from level -1 to 0. So, to keep notation simple, I will be relating w_0 back to w_{-1} Assume the induction hypothesis that $\sum_x (w_{-1}(x))^2 = 1$. For $x \in \operatorname{supp} w_{-1}$:

$$\begin{aligned} w_0(x) &= \sum_k p_{2k} w_{-1}(x - a_{2k}) \\ w_0(x+s) &= \sum_k p_{2k+1} w_{-1}(x - a_{2k}). \end{aligned}$$

Squaring each of these, we have:

$$
\begin{aligned}
(w_0(x))^2 &= \sum_k p_{2k}^2 \left(w_{-1}(x-a_{2k})\right)^2 \\
&\quad + 2\sum_k \sum_l p_{2k}p_{2k+2l}w_{-1}(x-a_{2k})w_{-1}(x-a_{2k+2l}) \\
\left(w_0^2(x+s)\right)^2 &= \sum_k p_{2k+1}^2 \left(w_{-1}(x-a_{2k})\right)^2 \\
&\quad + 2\sum_k \sum_l p_{2k+1}p_{2k+2l+1}w_{-1}(x-a_{2k})w_{-1}(x-a_{2k+2l}).
\end{aligned}
$$

Finally, by taking the sum of the squares, we obtain the following:

$$
\begin{aligned}
\sum_{y\in S_0} (w_n(y))^2 &= \sum_{x\in S_{-1}} (w_0(x))^2 + (w_0(x+s))^2 \\
&= \sum_{x\in S_{-1}} \Bigg(\sum_k \left(p_{2k}^2 + p_{2k+1}^2\right) \left(w_{-1}(x-a_{2k})\right)^2 \\
&+2\sum_k \sum_l \left(p_{2k}p_{2k+2l} + p_{2k+1}p_{2k+2l+1}\right) w_{-1}(x-a_{2k})w_{-1}(x-a_{2k+2l}) \Bigg) \\
&= \sum_{x\in S_{-1}} \Bigg(2\sum_{k=0} \sum_{l=1} p_k p_{k+2l} w_{-1}(x-a_k) w_{n-1}(x-a_{k+2l}) \\
&+ \sum_k p_k^2 \left(w_{-1}(x-a_k)\right)^2 \Bigg) \\
&= \sum_{x\in S_{n-1}} \sum_k p_k^2 \left(w_{-1}(x-a_k)\right)^2 \qquad \text{by condition (2)} \\
&= \left(\frac{1}{2}\right) \sum_{x\in S_{-1}} \left(w_{-1}(x-a_k)\right)^2 \qquad \text{by condition (2)} \\
&= \left(\frac{1}{2}\right) \qquad \text{by the induction hypothesis.}
\end{aligned}
$$

This gives us our desired equality. $\square$

Using this information, we would like to prove that μ_n has bounded total variation.

Lemma 16. *For any continuous bounded function f, suppose $\|f\|_\infty \leq B$, then $\forall n$,*

$$\left| \int f d\mu_n \right| \leq B\sqrt{2N}.$$

Note that the proof for this Lemma is identical to that of Lemma 7 from Chapter 3. Finally, we can show that μ_n converges weakly to a measure μ by proving that $\int f d\mu_n$ is a Cauchy sequence for any continuous function f.

Theorem 17. *The sequence μ_n weakly converges.*

Proof. Let f be a continuous function with $\|f\|_\infty \leq B$ and $\epsilon > 0$. Since $\text{supp}(\mu)$ is bounded and closed, f is continuous and bounded on a closed set. So f is uniformly continuous. So $\exists \delta$ such that for all x and y with $||x-y|| < \delta$, we have $|f(x)-f(y)| < \epsilon$. Let $A_{\max} = \max(||a_k||)$. Choose $\mathcal{M}$ large enough so that $\delta > \frac{A_{\max}}{\sqrt{2}^{\mathcal{M}}} \frac{1}{\sqrt{2}-1}$. Then for $m > \mathcal{M}$, we want to show that for any $k > 0$, $|\int f d\mu_{m+k} - \int f d\mu_m| < \epsilon$. We plan on writing both integrals, $\int f d\mu_{m+k}$ and $\int f d\mu_m$ in terms of the measure μ_m. We can do this by taking advantage of how μ_{m+k} can be derived from convolutions starting with μ_m.

We define a set and make a couple of remarks first. Let

$$B_k = \{0, 1, 2, ..., 2N-1\}^k.$$

Note that

$$1 = 1^k = \left(\sum_{i=0}^{2N-1} p_i \right)^k = \sum_{B_k} \left(\prod_{j=1}^{k} p_{a_j} \right).$$

Also, we have that for $m > \mathcal{M}$ and for any $k > 0$:

$$\begin{aligned} \left| \sum_{j=1}^{k} M^{-(m+j)} a_j \right| &\leq \left| \sum_{j=1}^{k} \frac{A_{\max}}{\sqrt{2}^{m+j}} \right| = \left| \frac{A_{\max}}{\sqrt{2}^{m}} \sum_{j=1}^{k} \frac{1}{\sqrt{2}^{j}} \right| \\ &< \left| \frac{A_{\max}}{\sqrt{2}^{m}} \frac{\sqrt{2}}{1-\sqrt{2}} \right| < \left| \frac{A_{\max}}{\sqrt{2}^{\mathcal{M}}} \frac{\sqrt{2}}{1-\sqrt{2}} \right| < \delta. \end{aligned}$$

So, by taking the difference of the integrals $\int f d\mu_{m+k}$, and $\int f d\mu_m$, we have

$$
\begin{aligned}
&\left|\int f d\ \mu_{m+k} - \int f d\mu_m\right| \\
&\quad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x) f\left(x+\sum_{j=1}^{k}\frac{a_j}{M^{m+j}}\right)\right) - \sum_{y\in S_m} w_m(y) f(y)\right| \\
&\quad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x) f\left(x+\sum_{j=1}^{k}\frac{a_j}{M^{m+j}}\right)\right)\right. \\
&\quad \left. - \sum_{y\in S_m}\left(\sum_{i=0}^{2N-1} p_i\right)^k w_m(y) f(y)\right| \\
&\quad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x) f\left(x+\sum_{j=1}^{k}\frac{a_j}{M^{m+j}}\right)\right)\right. \\
&\quad \left. - \sum_{y\in S_m}\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(y) f(y)\right| \\
&\quad = \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k} p_{a_j}\right) w_m(x)\left(f\left(x+\sum_{j=1}^{k}\frac{a_j}{M^{m+j}}\right) - f(x)\right)\right)\right|.
\end{aligned}
$$

In the above computations, we use the definition of the integral against measure μ_m and write the integral against measure μ_{m+k} in terms of the points from S_m. From here, we can use the fact that $(\sum_{i=0}^{2N-1} p_i) = 1$ to rearrange the terms to group them together:

$$\begin{aligned}
&\left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k}p_{a_j}\right)w_m(x)\;\left(f\left(x+\sum_{j=1}^{k}\frac{a_j}{M^{m+j}}\right)-f(x)\right)\right)\right|\\
&\qquad< \left|\sum_{x\in S_m}\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k}p_{a_j}\right)w_m(x)\epsilon\right)\right|\\
&\qquad= \left|\sum_{x\in S_m}\left(\sum_{i=0}^{2N-1}p_i\right)^k w_m(x)\epsilon\right|\\
&\qquad= \left|\sum_{x\in S_m}w_m(x)\epsilon\right|.
\end{aligned}$$

In the above approximation, we use the uniform continuity of f. The second and third line of this equation follow from the fact that $\left(\sum_{(a_j)\in B_k}\left(\prod_{j=1}^{k}p_{a_j}\right) = (\sum_{i=0}^{2N-1}p_i)^k = 1\right.$. The final approximations follow from Cauchy-Schwarz, Lemma 9, and the fact that we have the upper bound $|\operatorname{supp}(\mu_m)| \le 2N\cdot 2^{m+1}$.

$$\begin{aligned}
\left|\sum_{x\in S_m}w_m(x)\epsilon\right| &\le \sqrt{\sum_{x\in S_m}|w_m(x)|^2\cdot\sum_{\operatorname{supp}(\mu_m)}|\epsilon|^2}\\
&\le \sqrt{\frac{1}{2^{m+1}}\cdot\epsilon^2\cdot 2N\cdot 2^{m+1}}\\
&\le \epsilon\sqrt{2N}.
\end{aligned}$$

Therefore, the sequence $(\int f d\mu_n)_n$ is Cauchy and so it converges. Thus μ_n weakly converges. $\square$

Now we have shown that the sequence of the discrete measures μ_n converges weakly to some measure μ. Now we will show that this limiting measure μ is the unique solution, up to scaling by a constant for the signed measure dilation equation in two-dimensions.

Theorem 18. *This solution $\mu = \lim_{n\to\infty} \mu_n$ is the unique solution in the set of signed measures with compact support, up to scaling by a constant, for the signed measure dilation equation.*

Proof. Suppose ν is any signed measure with compact support. Recall the function $D(x) := \frac{x}{1+i}$.Then, we claim that $D_\star^{n+1}\nu \to \nu(\mathbb{R})\delta_0$ weakly as $n \to \infty$. We can show this by letting f be any bounded continuous function. Then, we have

$$\int f dD_\star^n \nu(x) = \int f(2^{-n}x) d\nu.$$

Now we take the limit, and have the following:

$$\begin{aligned} \lim_{n\to\infty} \int f(2^{-n}x) d\nu &= \int f(0) d\nu \\ &= f(0)\nu(\mathbb{C}). \end{aligned}$$

Therefore, we have the weak convergence, $\lim_{n\to\infty} D_\star^{n+1}\nu \to \nu(\mathbb{R})\delta_0$. So now, let $\tilde{\mu}$ be any solution to the signed measure dilation equation with compact support. We take the limit as $n \to \infty$ to obtain

$$\begin{aligned} \tilde{\mu} &= \lim_{n\to\infty} \left(\mu_0 \star D_\star \mu_0 \star D_\star^2 \mu_0 \cdots \star D_\star^n \mu_0\right) \star D_\star^{n+1} \tilde{\mu} \\ &= \left(\bigstar_{n=0}^{\infty} D_\star^n \mu_0\right) \star \tilde{\mu}(\mathbb{C})\delta_0 \\ &= \tilde{\mu}(\mathbb{C}) \left(\bigstar_{n=0}^{\infty} D_\star^n \mu_0\right). \end{aligned}$$

By definition of our solution, $\mu(x) = \bigstar_{n=0}^{\infty} D_\star^n \mu_0$. So it must be the unique solution up to multiplication by a constant. $\square$

Theorem 19. *The solution to the dilation equation for signed measures, μ, is absolutely continuous with respect to Lebesgue measure.*

Proof. We approach this proof by contradiction. Suppose that A is a Borel-measureable set with $\lambda(A) = 0$ but $|\mu|(A) = 2a > 0$. Then, since $|\mu|$ is a Borel regular measure, there is a compact subset $K \subset A$ with $|\mu|(K) > a$. Because K has Lebesgue measure 0, for any $\epsilon > 0$, we can cover K with open rectangles whose areas have sum $< \epsilon$. And since K is compact, we have a finite subcover, E. Then, we see that the number of points in S_{m-1} contained in E is asymptotic to $|E| \cdot 2^m$. In addition, since the sum-of-squares of the $w_{m-1}(x)$ is $1/2^m$, this trivially bounds the sum-of-squares of the $w_{m-1}(x)$ in E. By Urysohn Lemma, we have a function f with $0 \leq f \leq 1$ that has $f = 1$ on K and $f = 0$ on $\mathbb{C} \setminus E$. Then we have the following:

$$\begin{aligned}
|\mu|(K) &\leq \int f d|\mu| \\
&\leq \limsup \int f d|\mu_{m-1}| \\
&= \limsup \sum f(x)|w_{m-1}(x)| \\
&\leq \limsup \sqrt{\sum_{x \in E \cap S_{m-1}} f^2(x) \sum_{x \in E \cap S_{m-1}} w_{m-1}^2(x)} \\
&\leq \limsup \sqrt{(|E| \cdot 2^m + o(2^m)) \left(\frac{1}{2^m}\right)} \\
&= \sqrt{|E|} \leq \sqrt{\epsilon}.
\end{aligned}$$

From here, it follows that the total variation of μ and $|\mu|$ over E is less than or equal to $\sqrt{\epsilon}$. So we can choose ϵ to be small enough that $\sqrt{\epsilon} < a$. However, since E covers K, we assumed that $|\mu|(E) \geq |\mu|(K) > a$, which is a contraction. Therefore, we must have that $|\mu|(A) = 0$, so μ and $|\mu|$ are both absolutely continuous with respect to Lebesgue measure. □

We also claim that the solution μ to the dilation equation for signed measures has density satisfying the functional dilation equation almost everywhere.

Corollary 20. *For ϕ, the density of μ, we have that $\phi \in L^2(\mathbb{C})$.*

Note the proof in two-dimensions follows exactly the proof in one-dimension, found in Chapter 3, with the exception that we are now considering the refinement of the complex plane by sub-twin dragons, rather than the refinement of the real line by dyadic intervals.

Corollary 21. *The density ϕ of the solution μ of the dilation equation for signed measures satisfies the functional dilation equation almost everywhere.*

Proof. Let ϕ be the density of the solution to the dilation equation for signed measures, μ. Let x_0 be any point in $\operatorname{supp}(\mu)$ and $r \in \mathbb{R}^+$. Let $B(x_0, r)$ denote the ball of radius r about the point x. Then we have the following, from the dilation equation for signed measures:

$$\mu(B(x_0, r)) = \sum_k p_k \mu(M(B(x_0, r)) - a_k).$$

We can re-write each side of this equation using ϕ, the density of μ.

$$\begin{aligned} \int_{B(x_0,r)} \phi(x)dx &= \sum_k p_k \int_{M(B(x_0,r))-a_k} \phi(y)dy \\ &= \sum_k p_k |\det M| \cdot \int_{B(x_0,r)} \phi(Mx - a_k)dx. \end{aligned}$$

Where the second equality is true by substituting in $x = M^{-1}(y - a_k)$. Now we can take the limit as $r \to 0$:

$$\lim_{r\to 0} \int_{B(x_0,r)} \phi(x)dx = \lim_{r\to 0} \sum_k p_k |\det M| \int_{B(x_0,r)} \phi(Mx - a_k)dx.$$

By the Lebesgue Differentiation Theorem, for almost every x_0, we have:

$$\phi(x_0) = |\det M| \sum_k p_k \phi(Mx_0 - a_k).$$

Therefore, the density of the solution of the dilation equation for signed measures satisfies the functional dilation equation almost everywhere.

□

We summarize the results of this chapter in the following.

Theorem 22. *Under the orthogonality conditions, the sequence μ_n converges to the unique solution μ for the dilation equation for signed measures. Furthermore, this limiting measure is absolutely continuous with respect to Lebesgue measure and its density, f_μ is a scaling function which satisfies the functional dilation equation.*

4.2 Some examples

In this section, we will explore three different examples of finding the support of scaling and prescale measures in two dimensions. In the first example, we look at a simple, interesting case of using signed measures. This is the case which has $(a) = \{0, 1, 1+i, 2+i\}$. In the last two examples, we consider prescale measures which satisfy the probability case. A prescale function is one which generates a Riesz basis for an MRA rather than an orthonormal basis. A *Riesz basis* of V_0 is a sequence of functions $g_k \in V_0$ such that there exist constants $0 < c < C$ such that

$$c\left(\sum_k |a_k|^2\right) \leq \left|\left|\sum_k a_k g_k\right|\right|_{L^2}^2 \leq C\left(\sum_k |a_k|^2\right)$$

for all square summable sequences of scalars (a_k) and $\text{span}(g_k) = V_0$.

In addition, for the examples where we consider prescale functions, the coefficients satisfy the probability case. That is, they are positive, following the constructions considered by Belock and Dobric [11], who considered constructing prescale probability measures. The only restrictions placed upon their positive coefficients p_k were that

$$\sum_k p_k = 1,$$

$$\sum_{k\text{even}} p_k = \sum_{k\text{odd}} p_k = \frac{1}{2}$$

and

$$(p_k) \in l_2.$$

There is no orthogonality condition, and this is why these conditions will only guarantee prescale measures rather than scaling measures. For this reason, we will call them "prescale conditions." They have proven that under these conditions, the limiting measure will exist and be absolutely continuous [11]. Our work extended their results, in finding a method to solve for the support of the resulting prescale measure.

The main difficulty with finding the solution to the dilation equation in two-dimensions lies in identifying the support of measure μ. In one-dimension, the support is the sum of the set [0, 1] some number of times. This is easy to understand though, because $\sum_{k=0}^{2N-1}[0,1] = [0,2N]$. In two dimensions, it is much more complicated, because the support is the sum of the Twin Dragon tile, T some number of times. However, $\sum_{k=0}^{2N-1} T \neq 2N \cdot T$.

An important thing to mention is that in the following work, both Sage and Python were used to compute eigenvalues and corresponding eigenspaces. In all of these cases, unless otherwise noted, the resulting solutions are exact and not approximations.

4.2.1 A four coefficient case

We assume that μ is a signed measure on $\mathcal{B}(\mathbb{R}^2)$ (the Borel sets on $\mathbb{R}^2$) such that μ satisfies a dilation equation, which we now describe. For four real pseudo probabilities, p_0, p_1, p_2, and p_3, which satisfy the orthogonality conditions, we assume that μ satisfies the dilation equation with shifts 0, 1, $1+i$, and $2+i$. Obviously, the zero measure satisfies this equation for any choice of coefficients p_0, p_1, p_2, and p_3. Further, if μ is any solution, so is $c\mu$ for any $c \in \mathbb{R}$, and thus the natural form of uniqueness to consider is uniqueness up to scaling. We are interested in the existence, uniqueness, and also the computation of non-trivial μ satisfying this equation.

Support and "top level"

Recall the Twin Dragon, the subset of $\mathbb{C}$ given by

$$T = \left\{ \sum_{n=1}^{\infty} \frac{\gamma_n}{(1+i)^n} : \gamma_n \in \{0,1\} \text{ for all } n \right\}.$$

Note that T is a compact set, and the translates of T by the Gaussian integers tile the complex plane (see [17]). We're interested in the set of Gaussian integers $\tilde{S}$ such that the translates of the tile T by $\tilde{S}$ cover the support of μ, but no proper subset of

$\tilde{S}$ does. In other words, we want the set of translates of T that intersect the support of μ in a set of positive Lebesgue measure. This is interesting in its own right, as a way of controlling the support of μ, and also as the first step in determining μ on the "top level" of our dyadic decomposition scheme.

One way of approaching the support of μ (or more precisely, the minimal set of translated tiles that contain the support of μ) is to observe that supp(μ) has a representation analogous to that of T, namely

$$\operatorname{supp}(\mu) = \left\{ \sum_{n=1}^{\infty} \frac{\tilde{\gamma}_n}{(1+i)^n} : \tilde{\gamma}_n \in \{0, 1, 1+i, 2+i\} \text{ for all } n \right\}.$$

This is true by the definition of μ in terms of μ_n. Recall that

$$\begin{aligned} \mu &= \lim_{n\to\infty} \mu_n \\ &= \lim_{n\to\infty} \bigstar_{j=0}^{n} (D_\star)^j(\mu_0) \\ &= \lim_{n\to\infty} \bigstar_{j=0}^{n} (D_\star)^j \left(\sum_k p_k \delta_{\frac{p_k}{1+i}} \right), \end{aligned}$$

where $D(x) := \frac{x}{1+i}$. Two things follow from this. First, we find an estimate on the modulus of any element of supp(μ). Since any point, x_0, in the support satisfies

$$\begin{aligned} |x_0| &= \left| \sum_{n=1}^{\infty} \frac{\tilde{\gamma}_n}{(1+i)^n} \right| \\ &\le \left| \sum_{n=1}^{\infty} \frac{2+i}{(1+i)^n} \right| \\ &\le \sum_{n=1}^{\infty} \frac{\sqrt{5}}{\sqrt{2}^n} \\ &= \sqrt{5} \cdot \frac{\sqrt{2}}{\sqrt{2}-1}, \end{aligned}$$

no Gaussian integer with modulus greater than $\sqrt{5} \cdot \frac{\sqrt{2}}{\sqrt{2}-1} \approx 7.6344$ is in $\tilde{S}$. This leaves a finite set of candidates for $\tilde{S}$. Second, by taking all finite sums up to some level, we can determine a set of points that must belong to $\tilde{S}$.

If we compute all the points of the form

$$\left\{ \sum_{n=1}^{12} \frac{\tilde{\gamma}_n}{(1+i)^n} : \tilde{\gamma}_n \in \{0, 1, 1+i, 2+i\} \right\},$$

with Python, we find 14 Gaussian integers that must be in $\tilde{S}$, which, for future use, we give in order as follows

$$S = [0, -i, 1-i, 1, 1+i, i, -1+i, -1, -1-i, -2i, 1-2i, 2-2i, 2-i, 2] .$$

Now we will consider the other points which have modulus < 8, but are not included in S. The idea is as follows. Suppose we pick a Gaussian integer z and apply the dilation equation some number of times. This will express $\mu(z+T)$ as a linear combination of the measures of some other translated tiles, say $\mu(z_n+T)$ for $1 \leq n \leq N$. But if $|z_n| \geq 8$ for all n, then by our previous remarks about $\operatorname{supp}(\mu)$, we have $\mu(z_n+T) = 0$ for $1 \leq n \leq N$. Then $\mu(z+T) = 0$, and further, this reasoning applies to any Borel subset of $z+T$. Thus $z \notin \tilde{S}$. In other words once we know that the entire tile has measure 0, we know that it must not intersect the support of measure μ. This is because, the measure of any half-tile of this tile can be written in terms of a linear combination of measures of whole tiles, all of which have measure 0. Similarly, the measure of any quarter-tile of this tile can be written in terms of a linear combination of measures of half tiles, all of which have measure 0, and so on. We call a shifted tile with this quality of being a linear combination of measures of tiles with measure zero, a tile which is "pushed out." We now wish to carry out this procedure for every Gaussian integer with modulus less than or equal to 8 that is not in $\operatorname{set}(S)$, where $\operatorname{set}(S) = \{s | s \text{ is an entry in the vector } S\}$. However, there are an additional 14 Gaussian integers which don't get pushed out. We give these 14 points, in order, as

$$S' = [2i, -1+2i, -2+i, -2, -2-i, -1-2i, -3i,$$
$$1-3i, 2-3i, 3-2i, 3-i, 3, 2+i, 1+2i].$$

Now we need to consider the 28 points given in order by $S \oplus S'$, which is the vector whose first 14 components are given by S and whose last 14 components are given by S'.

So we have now determined the translates of T that contain the support of μ. However, it is possible that we have included more than necessary.

Let $V = v^S \oplus v^{S'}$ be the vector of real numbers, the kth entry of which is $\mu((S \oplus S')_k + T)$ for $1 \leq k \leq 28$. We are interested in computing the entries of V.

Remark 23. *Even for the four points* 0, 1, $1+i$, *and* $2+i$, *the size of the vectors and matrices under consideration is unwieldy. So even though* V *is more natural as a column vector, we write it as a row vector to save space and then transpose it as necessary.*

First note that $(1+i)T = T \cup (1+T)$. We have this because

$$\begin{aligned}
(1+i)T &= (1+i)\left\{\sum_{n=1}^{\infty} \frac{\gamma_n}{(1+i)^n} : \gamma_n \in \{0,1\}\right\} \\
&= (1+i)\left\{\left(\sum_{n=2}^{\infty} \frac{\gamma_n}{(1+i)^n}\right) \bigcup \left(\frac{1}{1+i} + \left(\sum_{n=2}^{\infty} \frac{\gamma_n}{(1+i)^n}\right)\right) : \gamma_n \in \{0,1\}\right\} \\
&= \left\{\left(\sum_{n=1}^{\infty} \frac{\gamma_n}{(1+i)^n}\right) \bigcup \left(1 + \left(\sum_{n=1}^{\infty} \frac{\gamma_n}{(1+i)^n}\right)\right) : \gamma_n \in \{0,1\}\right\} \\
&= T \cup (1+T).
\end{aligned}$$

Thus, from the dilation equation, we have

$$\begin{aligned}
\mu(T) = p_0\mu(T \cup (1+T)) + p_1\mu((-1+T) \cup (T)) + \\
p_2\mu((-i-1+T) \cup (-i+T) + p_3\mu((-i-2+T) \cup (-i-1+T)).
\end{aligned}$$

Because any two translates of T by distinct Gaussian integers are disjoint up to a set of Lebesgue measure zero (by the tiling property), and μ is absolutely continuous with respect to Lebesgue measure, we can split each of the four terms of the right-hand side of the above equation to get

$$\begin{aligned}\mu(T) = p_0\mu(T) + p_0\mu(1+T) + p_1\mu(-1+T) + p_1\mu(T) \\ + p_2\mu(-i-1+T) + \mu(-i+T) + p_3\mu(-i-2+T) + p_3\mu(-i-1+T).\end{aligned}$$

Further, all of the translates on the right-hand side belong to $\mathrm{set}(S) \cup \mathrm{set}(S')$, so none of these are zero a priori, so we have (in terms of V):

$$v_1 = (p_0 + p_1)\, v_1 + p_2v_2 + p_0v_4 + p_1v_8 + (p_2 + p_3)\, v_9 + p_3v_{19}.$$

A similar computation can be performed for the other 27 components of V, where any shifted tile involving a shift by a Gaussian integer outside of $\mathrm{set}(S) \cup \mathrm{set}(S')$ that appears is discarded, because we know that μ of such a shifted tile is necessarily zero. The result is a system of 28 linear equations, which we can write as

$$(V)^{\intercal} = \hat{A}(p_0, p_1, p_2, p_3)\, (V)^{\intercal},$$

where $\hat{A}(p_0, p_1, p_2, p_3)$ is a 28×28 matrix that we must now describe. We find that $\hat{A}$ has a block upper-triangular decomposition as

$$\hat{A} = \begin{bmatrix} A & * \\ 0 & A' \end{bmatrix},$$

where, if we let $P_{0,1} = p_0 + p_1$ and $P_{2,3} = p_2 + p_3$, A is

$$\begin{bmatrix}
P0,1 & p_2 & 0 & p_0 & 0 & 0 & 0 & p_1 & P_{2,3} & 0 & 0 & 0 & 0 & 0 \\
0 & p_1 & P_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & P_{2,3} & p_2 & 0 & p_0 & 0 \\
0 & p_3 & P_{2,3} & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & P_{0,1} \\
P_{2,3} & 0 & 0 & p_2 & P_{0,1} & p_1 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_2 & p_2 + p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_0 & P_{0,1} & p_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{0,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{0,1} & p_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & P_{0,1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & P_{2,3} & p_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 \\
0 & 0 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{2,3} \\
0 & 0 & 0 & 0 & P_{2,3} & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

A' is

$$\begin{bmatrix}
0 & p_0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_1 & p_0 + p_1 & p_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & p_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 \\
p_2 + p_3 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 \\
0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$*$ is a 14×14 block that we don't compute explicitly (for reasons that will be clear in a moment), and "0" is the 14×14 zero matrix.

Next, Sage computes that

$$\det(A' - I) = 1 - p_0^3 p_1 p_2 p_3^3$$

where I is the 14×14 identity matrix. Under our orthogonality conditions, the $p_\cdot$ all have absolute value strictly less than 1. Thus, under our orthogonality conditions, this determinant is always positive, and in particular 1 cannot be an eigenvalue of A'. This means that in order for $(V)^\intercal$ to be a right 1-eigenvector for $\hat{A}(p_0, p_1, p_2, p_3)$, we must have that $v^{S'}$ is the zero vector. One consequence of this (and the upper triangular block structure of $\hat{A}$) is that $\operatorname{set}(S')$ is not in $\tilde{S}$. In other words, we have now shown that $\tilde{S} = \operatorname{set}(S)$. (Note that this also explains what we saw by considering points which had "binary" expansions up to 12 places.) Further, we see that we only need to consider the system

$$\left(v^S\right)^\intercal = A(p_0, p_1, p_2, p_3)\left(v^S\right)^\intercal,$$

for the 14×14 matrix A given above. (This is the reason we don't bother to compute the upper right block "*", and the reason we chose our notation is this way.)

Since we are able to compute the scaling measure on full Twin Dragons, which are sets of Lebesgue measure 1, this uniquely determines the scaling measures on all measureable sets. This is because once we know the scaling measure on sets of Lebesgue measure 1, we can use the dilation equation to find the scaling measure on sets of Lebesgue measure $\frac{1}{2}$, and then $\frac{1}{4}$, and so on. Since μ is determined on all dyadic sets by the relationship dictated by the dilation equation, it is uniquely determined.

The right 1-eigenspace is always at least one-dimensional, since the vector of all 1's is a left 1-eigenvector. We can see this is true for this example since for the A above, we see that each column sums to $p_0 + p_1 + p_{1+i} + p_{2+i} = 1$.

Special cases

We specialize to the case when $p_0 = (1+\sqrt{3})/8$, $p_1 = (3+\sqrt{3})/8$, $p_2 = (3-\sqrt{3})/8$, and $p_3 = (1-\sqrt{3})/8$ to mimic the D4 case. We find that A has a 1-dimensional right 1-eigenspace, the right 1-eigenspace is spanned by

$$\begin{aligned}&[0.988473215486, 0.0991927845318, 0.0476287661136, 0.0956000986321,\\ &0.00421010706117, 0.0221507197936, 0.0104401680866,\\ &0.0305709339159, -0.00354125079226, -0.00205532069064, -0.00475426145644,\\ &0.000811194910171, -0.00886490283771, -0.00174490784237],\end{aligned}$$

where the notation was switched to a decimal approximation in order for this vector to fit on the page. So this gives $\left(v^S\right)^\intercal$ uniquely up to scaling.

4.2.2 The case of 0, 1, and i

We summarize the results in a parallel way to the previous case. We assume that μ is a signed measure on $\mathcal{B}(\mathbb{R}^2)$ such that μ is absolutely continuous with respect to Lebesgue measure and satisfies a dilation equation, which we now describe. For three constants p_0, p_1, and p_i which satisfy the prescale conditions, we assume that μ satisfies the dilation equation

$$\mu(A) = p_0\mu((1+i)A) + p_1\mu((1+i)A-1) + p_i\mu((1+i)A-i) \tag{4.2}$$

for all $A \in \mathcal{B}(\mathbb{C})$. Although this set of shifts do not satisfy our spacing conditions, they will still determine a prescale function [11].

4.2.3 The supporting tiles

We need the set $\tilde{S}$ of tiles that cover the support of μ. First, no Gaussian integer with modulus greater than $\frac{2}{\sqrt{2}-1} \approx 4.828$ is in $\tilde{S}$. This leaves a finite set of candidates for $\tilde{S}$. Second, by taking all finite sums up to some level, we can determine a set of points that must belong to $\tilde{S}$. In particular, by considering

$$\left\{\sum_{n=1}^{12} \frac{\tilde{\gamma}_n}{(1+i)^n} : \tilde{\gamma}_n \in \{0,1,i\}\right\},$$

which was done with a Python script, we see that there are at least 16 Gaussian integers in $\tilde{S}$. Namely, $\tilde{S}$ contains $\operatorname{set}(S)$ where S is the 16-dimensional vector

$$S = [0, i, 1+i, 1, 1-i, -i, -1, -1+i, 2i, 1+2i, 2+i, 2, 2-i, 1-2i, -2i, -1-i]\,.$$

We have written S as a vector, or equivalently, given these 16 points an order, because it will be useful for the linear algebra that follows. But for now, the question is whether these points are all of $\tilde{S}$, or whether we need more points.

We now try to show that no other Gaussian integers are in $\tilde{S}$ by showing that they "get pushed" outside of the ball of radius 5 (centered at the origin). If we use a simple Python script to apply the dilation equation, say 10 times, to each such Gaussian integer, we see that they all "get pushed" outside of the ball of radius 5 centered at the origin, so none of them are in $\tilde{S}$. In other words, we see that $\tilde{S} = \operatorname{set}(S)$.

The "top level" values of μ

Having determined the translates of T that contain the support of μ, we now turn our attention to the uniqueness and computation of μ. We consider the vector v^S, which gives μ of the corresponding shifted tiles. As before, the doubling equation gives relationships among the components of v^S. We have

$$\left(v^S\right)^\intercal = A(p_0, p_1, p_i)\left(v^S\right)^\intercal,$$

where $A(p_0, p_1, p_i)$ is the 16×16 matrix given by

$$
\begin{bmatrix}
P_{0,1} & 0 & 0 & p_0 & p_i & p_i & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_i & p_0 & 0 & 0 & 0 & 0 & p_i & P_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_i & p_i & 0 & 0 & 0 & 0 & 0 & p_{0,1} & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_1 & P_{0,1} & p_i & 0 & 0 & 0 & 0 & 0 & 0 & p_0 & p_i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{0,1} & p_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{0,1} & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & p_0 & p_i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_i & P_{0,1} \\
0 & 0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & p_i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_0 & P_{0,1} & 0
\end{bmatrix}
$$

where $P_{0,1} = p_0 + p_1$.

Remark 24. *The order of the components in S is chosen so that they spiral out clockwise as points in the plane. This is related to the action multiplication by $1+i$ on the plane, and, maybe more importantly, gives A the structure seen above of one roughly diagonal band and one roughly above-diagonal band.*

So v^S must be a right 1-eigenvector of A. If the right 1-eigenspace is 1-dimensional, then any solution must be unique up to scaling. This is because once we have v^S, μ is completely determined. To see this, note that the translations of T by elements of set(S),

$$\bigcup_{k=1}^{16} S_k + T$$

decomposes into 32 half-tiles, given by $T/(1+i)$ translated by the 32 elements of the rescaled lattice $\mathbb{Z}^2/(1+i)$ given by

$$\operatorname{set}(S) \cup \left\{ z_k + \frac{1}{1+i} : z_k \in \operatorname{set}(S) \right\}.$$

Said more simply, we just cut each of the original 16 tiles in half in the standard way coming from M. Then applying the doubling equation to each of these 32 half-tiles, μ of each of them is given as a linear combination of components of v^S. Then we can similarly compute μ of each of the quarter-tiles, and so on. This shows that v^S determines μ on arbitrary dyadically-subdivided tiles, and since such tiles generate $\mathcal{B}(\mathbb{R}^2)$, v^S uniquely determines μ. Viewed differently, this gives an iterative procedure for determining μ on dyadically-subdivided tiles as many "levels down" as we wish to go.

Special cases

From the earlier paper of Dobric and Belock [11], we know that there will be an absolutely continuous probability measure μ satisfying the dilation equation if $p_0 = \frac{1}{2}$ and p_1 and p_i are strictly between 0 and 1 and sum to 1/2. Restricting our attention to this case, we have one degree of freedom; namely, set $p_1 = (1/2) - p_i$ and let $p_i \in (0, 1/2)$. Making these substitutions explicitly in A makes the resulting matrix too big to include here. However, for any value of p_i, the resulting matrix has a 1-dimensional right 1-eigenspace.

Specializing even further, we take the concrete example with $p_0 = 1/2$ and

$p_1 = p_i = 1/4$. Then we get that $A(1/2, 1/4, 1/4)$ is

$$\begin{bmatrix}
\frac{3}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{3}{4} & 0
\end{bmatrix}.$$

The (right) 1-eigenspace is one-dimensional and is spanned by the vector

$$\left[1, \frac{767262}{1370695}, \frac{795934}{4112085}, \frac{104324}{274139}, \frac{130917}{1370695}, \frac{131013}{1370695}, \frac{13105}{274139}, \frac{32768}{1370695}, \right.$$
$$\left. \frac{8192}{1370695}, \frac{2048}{1370695}, \frac{512}{1370695}, \frac{128}{274139}, \frac{128}{1370695}, \frac{32}{1370695}, \frac{8}{1370695}, \frac{22}{1370695}\right].$$

This is normalized so that the first coordinate is 1, rather than being normalized to give a probability. Nonetheless, this determines v^S up to a scaling factor, and thus the corresponding μ is unique up to scaling.

4.2.4 The case of -1, 0, and 1

We now give the parallel computations in the case where our points are -1, 0, and 1, with corresponding weights p_{-1}, p_0, and p_1 satisfying the prescale conditions. Because it is so similar to the previous, we'll just summarize many things.

Here, the dilation equation is

$$\mu\left(A\right) = p_0\mu\left((1+i)A\right) + p_1\mu\left((1+i)A-1\right) + p_{-1}\mu\left((1+i)A+1\right).$$

Support and "top level"

First, we need the set $\tilde{S}$ of tiles that cover the support of μ. If we compute all the points of the form

$$\left\{\sum_{n=1}^{12} \frac{\tilde{\gamma}_n}{(1+i)^n} : \tilde{\gamma}_n \in \{0, 1, -1\}\right\},$$

with Python, we find 10 Gaussian integers that must be in $\tilde{S}$, which, for future use, we give in order as follows

$$S = [i, 0, 1, 1+i, 2i, -1+2i, -1+i, -1, -i, 1-i].$$

However, if we now try to show that no other Gaussian integers are in $\tilde{S}$ by showing that they "get pushed" outside of the ball of radius 5 (centered at the origin), we don't succeed. In particular, there are an additional 12 Gaussian integers which don't get pushed out. We give these 12 points as

$$S' = [3i, -1+3i, -2+2i, -2+i, -2, -1-i, -2i, 1-2i, 2-i, 2, 2+i, 1+2i].$$

Now we need to consider the 22 points given in order by $S \oplus S'$, which is the vector whose first 10 components are given by S and whose last 12 components are given by S'.

Next, we consider the vector $v^S \oplus v^{S'}$, which gives μ of the corresponding shifted tiles. As before, the dilation equation gives relationships among the components of $v^S \oplus v^{S'}$. We have

$$\left(v^S \oplus v^{S'}\right)^\intercal = \hat{A}(p_{-1}, p_0, p_1)\left(v^S \oplus v^{S'}\right)^\intercal,$$

where $\hat{A}(p_{-1}, p_0, p_1)$ is a 22×22 matrix that we must now describe. We find that $\hat{A}$ has a block upper-triangular decomposition as

$$\hat{A} = \begin{bmatrix} A & * \\ 0 & A' \end{bmatrix}$$

where A is

$$\begin{bmatrix}
P_{0,-1} & 0 & 0 & p_{-1} & 0 & 0 & P_{0,-1} & 0 & 0 & 0 \\
0 & P_{0,1} & P_{0,-1} & 0 & 0 & 0 & 0 & p_1 & 0 & 0 \\
p_1 & 0 & 0 & P_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{0,1} & p_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{-1} & P_{0,-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_{-1} & 0 & 0 & 0 \\
0 & p_{-1} & 0 & 0 & 0 & 0 & 0 & P_{0,-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{0,-1} & p_{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & P_{0,1} \\
0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

where $P_{0,-1} = p_0 + p_{-1}$, $P_{0,1} = p_0 + p_1$ and A' is

$$\begin{bmatrix}
0 & p_{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P_{0,1} & P_{0,-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 \\
p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{0,-1} & P_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$*$ is a 10×12 block that we don't compute explicitly, and "0" is the 12×10 zero matrix.

We first compute that A' does not have a 1-eigenvector for any real values of p_{-1}, p_0, and p_1. The determinant of the "bad block" minus the identity, $\det(A' - I)$, is $1 - p_{-1}^4 p_1^4$. Since we're in the probability case, where p_{-1}, p_0, and p_1 are all non-negative and sum to 1, for any admissible coefficient values, this determinant is positive, and thus A' has no 1-eigenvectors. This means that in order for $\left(v^S \oplus v^{S'}\right)^\intercal$ to be a 1-eigenvector for $\hat{A}(p_{-1}, p_0, p_1)$, we must have that $v^{S'}$ is the zero vector. One consequence of this and the upper triangular block structure of $\hat{A}$ is that set(S') is not in $\tilde{S}$. In other words, we have now shown that $\tilde{S} = \mathrm{set}(S)$. Further, we see that we only need to consider the system

$$\left(v^S\right)^\intercal = A(p_{-1}, p_0, p_1) \left(v^S\right)^\intercal ,$$

for the 10×10 matrix A given above. Again, studying the 1-eigenspace of A is the key to both the uniqueness of μ and to computing v^S, which then allows us to compute μ on successive levels of dyadic decomposition.

Special cases

Similar to before, we consider the case where $p_0 = 1/2$ and $p_1 = (1/2) - p_{-1}$, because then we know that an absolutely continuous μ solving the doubling equation exists. Analogously to the above, Sage claims that the resulting matrix $A(p_{-1}, 1/2, (1/2) - p_{-1})$ has a 1-dimensional right 1-eigenspace for any value of p_{-1}. If we specialize further to the case when $p_0 = 1/2$ and $p_1 = p_{-1} = 1/4$, then we get that the matrix $A(1/2, 1/4, 1/4)$ is

$$\begin{bmatrix}
\frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\
0 & \frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
\frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

This matrix has a 1-dimensional 1-eigenspace, spanned by

$$\left[1,\,1,\,\frac{4}{13},\,\frac{1}{13},\,\frac{1}{13},\,\frac{1}{13},\,\frac{4}{13},\,\frac{1}{13},\,\frac{1}{13},\,\frac{1}{13}\right].$$

Bibliography

[1] Daubechies, I. *Ten lectures on wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics* (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992). URL http://dx.doi.org/10.1137/1.9781611970104.

[2] Mallat, S. G. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. Amer. Math. Soc.* **315**, 69–87 (1989). URL http://dx.doi.org/10.2307/2001373.

[3] Meyer, Y. *Wavelets and operators*, vol. 37 of *Cambridge Studies in Advanced Mathematics* (Cambridge University Press, Cambridge, 1992). Translated from the 1990 French original by D. H. Salinger.

[4] Daubechies, I. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41**, 909–996 (1988). URL http://dx.doi.org/10.1002/cpa.3160410705.

[5] Strang, G. Convolution, reconstruction and wavelets. In *Advances in computational mathematics (New Delhi, 1993)*, vol. 4 of *Ser. Approx. Decompos.*, 213–223 (World Sci. Publ., River Edge, NJ, 1994).

[6] Heil, C. & Colella, D. Dilation equations and the smoothness of compactly supported wavelets. In *Wavelets: mathematics and applications*, Stud. Adv. Math., 163–201 (CRC, Boca Raton, FL, 1994).

[7] Curry, E. Low-pass filters and scaling functions for multivariable wavelets. *Canad. J. Math.* **60**, 334–347 (2008). URL `http://dx.doi.org/10.4153/CJM-2008-016-1`.

[8] Gundy, R. F. Probability, ergodic theory, and low-pass filters. In *Topics in harmonic analysis and ergodic theory*, vol. 444 of *Contemp. Math.*, 53–87 (Amer. Math. Soc., Providence, RI, 2007). URL `http://dx.doi.org/10.1090/conm/444/08575`.

[9] Lawton, W., Lee, S. L. & Shen, Z. Stability and orthonormality of multivariate refinable functions. *SIAM J. Math. Anal.* **28**, 999–1014 (1997). URL `http://dx.doi.org/10.1137/S003614109528815X`.

[10] Dobrić, V., Gundy, R. & Hitczenko, P. Characterizations of orthonormal scale functions: a probabilistic approach. *J. Geom. Anal.* **10**, 417–434 (2000). URL `http://dx.doi.org/10.1007/BF02921943`.

[11] Belock, J. & Dobric, V. Random variable dilation equation and multidimensional prescale functions. *Trans. Amer. Math. Soc.* **353**, 4779–4800 (electronic) (2001). URL `http://dx.doi.org/10.1090/S0002-9947-01-02833-1`.

[12] Burt, P. J., Edward & Adelson, E. H. The laplacian pyramid as a compact image code. *IEEE Transactions on Communications* **31**, 532–540 (1983).

[13] Lawton, W. M. Necessary and sufficient conditions for constructing orthonormal wavelet bases. *J. Math. Phys.* **32**, 57–61 (1991). URL `http://dx.doi.org/10.1063/1.529093`.

[14] Cohen, A. Ondelettes, analyses multirésolutions et filtres miroirs en quadrature. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7**, 439–459 (1990).

[15] Heil, C. Methods of solving dilation equations. In *Probabilistic and stochastic methods in analysis, with applications (Il Ciocco, 1991)*, vol. 372 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 15–45 (Kluwer Acad. Publ., Dordrecht, 1992).

[16] Hernandez, E. & Weiss, G. *A First Course on Wavelets*. Studies in Advanced Mathematics (CRC Press, 1996). URL `https://books.google.com/books?id=Dq95ngIhy0oC`.

[17] Gundy, R. F. & Jonsson, A. L. Scaling functions on $\mathbb{R}^2$ for dilations of determinant ± 2. *Appl. Comput. Harmon. Anal.* **29**, 49–62 (2010). URL `http://dx.doi.org/10.1016/j.acha.2009.08.004`.

[18] Lagarias, J. C. & Wang, Y. Haar type orthonormal wavelet bases in $\mathbf{R}^2$. *J. Fourier Anal. Appl.* **2**, 1–14 (1995). URL `http://dx.doi.org/10.1007/s00041-001-4019-2`.

[19] Gundy, R. & Zhang, C. Dilation equations. Lehigh University Probability Seminar (1994).

[20] Báez-Duarte, L. Central limit theorem for complex measures. *J. Theoret. Probab.* **6**, 33–56 (1993). URL `http://dx.doi.org/10.1007/BF01046767`.

[21] Hochberg, K. J. A signed measure on path space related to wiener measure. *The Annals of Probability* **6**, 433–458 (1978). URL `http://www.jstor.org/stable/2243147`.

[22] Hochberg, K. J. Central limit theorem for signed distributions. *Proc. Amer. Math. Soc.* **79**, 298–302 (1980). URL `http://dx.doi.org/10.2307/2043254`.

[23] Bogachev, V. I. *Measure theory. Vol. I, II* (Springer-Verlag, Berlin, 2007). URL `http://dx.doi.org/10.1007/978-3-540-34514-5`.

Vita

Contact Information

Sarah Dumnich

Lehigh University, Department of Mathematics, 14 E. Packer Ave, Bethlehem, PA 18015

E-mail: sarah.dumnich@gmail.com

Cell: 724-875-9896

Education

Lehigh University, Bethlehem, PA

Ph.D., Mathematics, May 2016, Area: Analysis, Advisor: Robert Neel

M.S., Mathematics, May 2012

Washington & Jefferson College, Washington, PA

B.A. in Mathematics, May 2010

Teaching Experience

Frostburg State University - expected August 2016

Assistant Professor of Mathematics

Lehigh University

Instructor

– Introduction to Mathematical Thought - Spring 2016

– Calculus I Part A - Fall 2014

– Calculus with Business Applications - Spring 2014

Participant in Teacher Development Program - Fall 2014 - Spring 2015

Teaching Assistant

– Calculus I - Fall 2015

– Calculus with Business Applications - Fall 2013

– Calculus III - Spring 2013, Fall 2012

– Calculus II - Spring 2012, Fall 2011

Northampton Community College - Summer 2013

Adjunct Faculty Instructor

– Intermediate Algebra - Summer Sessions 1 & 2

Publications

Dumnich, S., Neel, R. *A measure theoretic construction of scaling functions in one and two dimensions* (in progress to be submitted)

Grants & Honors

Graduate Student Senate travel grant, 2015 and 2016.

Honorable Mention for Teaching Assistant of the Year, Spring 2015.

Lehigh University Summer Research Grant, 2014 and 2015.

Recent Talks

Lafayette College Math Seminar, Easton, PA, Feb 2016.

Joint Mathematics Meetings, Seattle, WA, Jan 2016.

Mini-Symposium in Memory of Vladimir Dobric, Bethlehem, PA, Dec 2015.

AMS Sectional Meeting, New Brunswick, NJ, Nov 2015.

The Moravian College Epsilon Series, Bethlehem, PA, Oct 2015.

University of Scranton Mathematics Seminar, Scranton, PA, Sept 2015.

Lehigh University GSIMS, Bethlehem, PA, Sept 2015.

Joint Mathematics Meetings, San Antonio, TX, Jan 2015.

Recent Service Activities

College of Arts and Sciences Dean's Advisory Council, Fall 2014-present

Graduate Student Liaison Committee, Spring 2012-present

Virtual Graduate Fair Representative, Fall 2015

Graduate Study in the STEM Disciplines Panel Speaker, Fall 2014, Fall 2015

Graduate Student Intercollegiate Mathematics Seminar

– President, Fall 2013-Spring 2015

– Vice President, Fall 2011-Spring 2013