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Subdivisions with Distance Constraints in Large Graphs

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Abstract

In this dissertation we are concerned with sharp degree conditions that guarantee the existence of certain types of subdivisions in large graphs. Of particular interest are subdivisions with a certain number of arbitrarily specified vertices and with prescribed path lengths. Our non-standard approach makes heavy use of the Regularity Lemma (Szemerédi, 1978), the Blow-Up Lemma (Komlós, Sárközy, and Szemerédi, 1994), and the minimum degree panconnectivity criterion (Williamson, 1977).

Sharp minimum degree criteria for a graph G to be H-linked have recently been discovered. We define (H, w, d)-*linkage*, a condition stronger than H-linkage, by including a weighting function w consisting of required lengths for each edge-path of a desired H-subdivision. We establish sharp minimum degree criteria for a large graph G to be (H, w, d)-linked for all $d \geq 0$. We similarly define the weaker condition (H, S, w, d)-*semi-linkage*, where S denotes the set of vertices of H whose corresponding vertices in an H-subdivision are arbitrarily specified. We prove similar sharp minimum degree criteria for a large graph, i.e., a graph large enough to permit non-trivial use of the Regularity Lemma, to be (H, S, w, d)-semi-linked for all $d \geq 0$.

We also examine path coverings in large graphs, which are here viewed for the first time as a special case of (H, S, w)-semi-linkage. In 2000, Enomoto and Ota conjectured that a graph G of order n with degree sum $\sigma_2(G)$ satisfying

$$\sigma_2(G) \geq n + k - 1$$

may be partitioned into k paths, each of prescribed order and with a specified starting vertex. We prove the Enomoto-Ota Conjecture for graphs of sufficiently

large order.

Introduction

Extremal graph theory is the study of the ways in which the structure of a graph affects the existence or structure of certain subgraphs. This field of study began in 1941, when Turán proved [38] that a graph of order n not containing the complete graph K_{r+1} contains at most $\left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$ edges. Turán's Theorem gave rise to the question, "what is the upper bound for the number of edges for a graph of order n not containing a specific (non-complete) subgraph?" This question was answered by Erdös and Stone [11], who in 1946 generalized Turán's theorem for non-complete subgraphs of G. Since these two landmark results, much has been done to determine the necessary properties of a graph G so that G necessarily contains certain subgraphs.

In particular, a fundamental problem in extremal graph theory is a variant of an age-old problem from combinatorial optimization, the traveling salesman problem. There are many variants of this problem [6] but a simplified version goes as follows: given a list of cities and roads connecting the cites, find a route that allows the salesman to visit each city exactly once and return to the starting city. This problem can be modeled by a graph where the cities are the graph's vertices, roads between cities are the graph's edges, and the solution to the problem lies in finding a hamiltonian (spanning) cycle. After Dirac [8] and Ore [33] proved sharp minimum degree and degree-sum bounds for a graph to be hamiltonian, Bondy and Chvátal proved [2] in 1976 that a graph is hamiltonian if and only if its closure is hamiltonian. A modification of the traveling salesman problem was recently considered in [12] where the authors asked: how many vertices and edges must a graph G contain so that if we specify a certain number of vertices in G *and* the distances between them (assume that the distance between adjacent vertices is 1), then we may guarantee that G contains such a hamiltonian route? In [12] sharp minimum degree conditions on the host graph were proven to imply the existence of such subgraphs.

A natural extension of classifying hamiltonian graphs is to find the necessary and sufficient conditions for a graph to be partitioned into paths. Partitioning a graph into vertex-disjoint paths has been considered since Ore [33], who showed that a graph G of order n is hamiltonian if the minimum degree sum of G is at least n, which immediately implies that G may be partitioned into vertex-disjoint paths on any set of vertices, or that G may be partitioned into vertex-disjoint paths of arbitrary order. Prescribed path order on arbitrary endvertices is not something guaranteed simply by hamiltonicity, however. Enomoto and Ota conjectured [10] in 2000 that a graph of order n with minimum degree sum at least $n + k - 1$ can be partitioned into k paths, with each path having specified order and a specified endvertex. Progress on the Enomoto-Ota Conjecture has been made with weaker conditions [25], but no result includes the sharp bound $n+k-1$, the prescription of path lengths, *and* the prescription of the vertex locations. We prove the Enomoto-Ota Conjecture for graphs of sufficiently large order in Chapter 2.

An alternate extension of the traveling salesman problem is the question of the existence of subdivisions on specified vertices of prescribed size. The idea of ubiquitous H-subdivisions in a graph (i.e., of H*-linked* graphs) was first introduced by Jung in [24] and then developed in [18] and [39]. Further progress has been made in the last decade (see [17], [29], [16], and [5]). Each of these works provided sufficient (usually sharp) minimum degree conditions for a large graph to be H-linked. While the minimum degree requirements have always been high (slightly greater than $\frac{n}{2}$), this has been to assure high enough connectivity to permit the H-subdivision to travel between components of the host graph. We prove similar results in Chapter 3.

The Regularity Lemma [36], proved by Szemerédi in 1978, states that every sufficiently large graph can be partitioned into a bounded number of vertex sets called *clusters*, each of the same size, and one small "garbage" cluster such that all pairs of non-garbage clusters behave almost like random bipartite graphs. A weaker version of this surprising result was used to prove Szemerédi's Theorem [35] about arithmetic sequences in 1975. While the Regularity Lemma is true for all graphs, it is only meaningful for large graphs with many edges—this is the reason our results only concern large graphs. The Regularity Lemma concerns (ϵ, δ)-regularity, further discussed in Chapter 1, which measures the uniformity and density of the number of edges between a pair of clusters. A stronger version of (ϵ, d)-regularity is used in the statement of the powerful Blow-Up Lemma [27], proved by Komlós, Sárközy, and Szemerédi in 1994, which states that a partition of a sufficiently large graph as in the Regularity Lemma contains arbitrary subgraphs of bounded maximum degree between *every* highly dense pair of clusters. The Regularity and Blow-Up Lemmas combine to show that sufficiently large graphs may be partitioned into dense clusters that behave as complete bipartite graphs.

While the Regularity Lemma gives information about the entire structure of a large, dense graph, we require a method of forming paths of arbitrary length between any pair of vertices; i.e., we need our graph to be *panconnected*. Fortunately, Williamson and Alavi established a number of extremal results for graphs containing paths of arbitrary length between any pair of vertices, a property commonly called *panconnectivity*. In particular, they noted in [1] that both the square of a hamiltonian graph and the cube of any graph are panconnected. In [40], Willamson showed that graphs of order n and minimum degree at least $\frac{n+2}{2}$ are panconnected. We refer to this result as the Panconnectivity Criterion, as it is used heavily in this work. This wonderful result has been used in works that concern hamiltonian cycles with specified vertex locations (see [12], [13], [14]), cyclic decompositions of graphs (see [32]), and subdivisions (see [19]).

Chapter 1

Preliminaries

In this chapter, we detail the Regularity Lemma, the Blow-Up Lemma, and the Panconnectivity Criterion, the three main lemmas used heavily in this work. We start with an introduction to ϵ-regularity and then give a brief summary of the known results on panconnectivity.

For general definitions and notation of graph theory terminology, see [3].

1.1 Regularity

The Regularity Lemma, proved in [36] by Szemerédi in 1978, states that large graphs behave almost like random graphs. The Blow-Up Lemma, proved in [27] by Komlós, Sárközy, and Szemerédi in 1994, states that sufficiently large graphs with sufficiently many edges contain all subgraphs of certain fixed degree. This certain fixed degree can be a large number for our purposes, since we assume a minimum degree for G above $\frac{n}{2}$.

1.1.1 Density and ϵ-Regularity

The Regularity and Blow-Up Lemmas are based on the concept of ϵ-regularity, which gauges the edge-uniformity between two pairs of vertex sets. Let A and B be disjoint vertex sets. The *density* of the pair (A, B) is the value

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

Note that $0 \leq d(A, B) \leq 1$. Fix $\epsilon > 0$. A pair (A, B) is ϵ-*regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, we have $|d(X, Y) - d(A, B)| < \epsilon$. Some sources, such as [28], write $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$, but for our purposes this difference is insignificant. We say (A, B) is (ϵ, δ)-*regular* to mean (A, B) is ϵ-regular with density greater than δ. (Note that δ here is different from $\delta(G)$.)

Naturally, the smaller ϵ is, the more uniformly dense a pair is. Note that from [28], to show ϵ-regularity it suffices to only check all subsets X and Y of orders $\lfloor \epsilon|A| \rfloor + 1$ and $\lfloor \epsilon|B| \rfloor + 1$, respectively. Even with this sufficient condition, we still must check $\binom{|A|}{\lfloor \epsilon|A| \rfloor + 1}\binom{|B|}{\lfloor \epsilon|B| \rfloor + 1}$ subpairs of (A, B) to determine ϵ-regularity.

1.1.2 Examples

We want to keep two (albeit simple) examples in mind when considering ϵ-regular pairs. First, the pair (A, B) with no edges between A and B is trivially ϵ-regular for all $\epsilon > 0$. Second, the pair (A, B) with all possible edges between A and B (that is, the complete bipartite graph $K_{|A|,|B|}$ on (A, B)) is also ϵ-regular for all ϵ.

Example 1.1.1. A third, slightly more involved example is as follows. Fix $\epsilon > 0$, and consider a bipartite graph $G = A \cup B$ of sufficiently large order $n \geq n(\epsilon)$ that satisfies the following property: Assume $|A| = |B| = \frac{n}{2}$, and for each pair of vertices $a, b \in V(G)$, the probability that the edge $ab \in E(G)$ is $\frac{1}{2}$. Since n is a sufficiently large function of ϵ, we can say $|N(v)| \approx \frac{n}{2}$ for all but at most a tiny portion of vertices v in G with extremely high probability. In such instances, since almost all vertices have degree arbitrarily close to $\frac{n}{2}$, any large enough subsets $X \subset A$ and $Y \subset B$ form a pair whose density is well within ϵ of $d(A, B)$. Hence, the graph G forms an ϵ-regular pair with an extemely high probability.

1.1.3 Regularity Lemma

The following result was proved by Szemerédi in 1978.

Lemma 1.1.2 (Regularity Lemma - Szemerédi [36]). *For every $\epsilon > 0$, there is an $M = M(\epsilon)$ such that if G is any graph and $\delta \in (0,1)$ is any real number, then there is a partition of $V(G)$ into $r+1$ clusters $V_0, V_1, \ldots, V_r$, and there is a subgraph $G' \subseteq G$ with the following properties:*

(1) $r \leq M$,

(2) $|V_0| \leq \epsilon |V(G)|$,

(3) $|V_1| = \cdots = |V_r| = L \leq \epsilon |V(G)|$,

(4) $\deg_{G'}(v) > \deg_G(v) - (\delta + \epsilon)|V(G)|$ *for all* $v \in V(G)$,

(5) $e(G'[V_i]) = 0$ *for all* $i \geq 1$,

(6) for all $1 \leq i < j \leq r$ *the graph* $G'[V_i, V_j]$ *is* ϵ*-regular and has density either* 0 *or greater than* δ.

Although Lemma 1.1.2 holds for all positive values of ϵ, δ, and $|V(G)|$, for this work we assume the relations

$$0 < \frac{1}{|V(G)|} \ll \epsilon \ll \delta \ll 1, \tag{1.1}$$

with $|V(G)| \geq N(\epsilon, \delta)$ for some function N. The sets V_i in Lemma 1.1.2 are called *clusters*, with V_0 being the *garbage cluster*. Typically, we are concerned with graphs G with $|V(G)| \gg M$, since the result is trivially true for G of order M.

Lemma 1.1.2 states that for fixed values ϵ and δ, each graph G of large order has a spanning subgraph G' with almost as many edges as G (Item 4) and independent sets $V_1, \ldots, V_r$, all of which have the same number of vertices (Item 3). Furthermore, these clusters either have many edges between them in a highly uniform way, by ϵ-regularity, or no edges between them (Item 6). Since r is bounded above and below (Item 1), if n is large, then each cluster contains many, but not too many vertices.

Note that the garbage cluster V_0 disobeys these rules, but its order is bounded above (Item 2). We also do not know anything about the number of edges (if any) between V_0 and other clusters. This misbehavior is mainly problematic in Chapter 2, when we wish to find a set of disjoint paths of prescribed lengths that cover *all* vertices of G. Somewhat ironically in Chapter 3, the set V_0 often aids us in the proofs of Lemmas 3.4.1–3.4.4.

We consider a trivial application of the Regularity Lemma.

Example 1.1.3. For any $\epsilon > 0$, Lemma 1.1.2 is trivially true for a graph G of order $M = M(\epsilon)$ and any $\delta \in (0, 1)$. We can write $r = M$ and partition the graph into r clusters V_i for $1 \leq i \leq M$, each comprising a single vertex (i.e., $L = 1$), where $V_0 = \emptyset$. Then $G' = G$ trivially.

Unfortunately, the value M in Lemma 1.1.2 is so absurdly high that a more practial example cannot be illustrated. Gowers showed in [20] that the sharp lower bound for M, which we call m, is "given by a tower of 2s of height proportional to $\log(1/\epsilon)$," and that the upper bound for M is at least as large as a $\delta^{-1/16}$-level tower function of m. This means that an example involving clusters with more than one vertex each would require a graph with r *clusters*, where

$$m = \overbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}^{\log(1/\epsilon)} \leq M \leq \overbrace{m^{\cdot^{\cdot^{\cdot^{m}}}}}^{\delta^{-1/16}}.$$

1.1.4 Reduced Graph

We now define the reduced graph to easily encapsulate the general structure of G.

Definition 1.1.4. Given a graph G and appropriate choices of ϵ and δ, let G' be a spanning subgraph of G obtained from Lemma 1.1.2. The *reduced graph* $R = R(G, \epsilon, \delta)$ of G contains a vertex v_i for each cluster V_i in $G' \setminus V_0$ and has an edge between v_i and v_j if and only if $d(V_i, V_j) > \delta$. Hence, $V(R) = \{v_i \mid 1 \leq i \leq r\}$ and $E(R) = \{v_i v_j \mid 1 \leq i, j \leq r,\ d(V_i, V_j) > \delta\}$. See Figure 1.1 for an illustration of G, G', and $R(G, \epsilon, d)$.

Throughout this work, we let $r = |R|$.

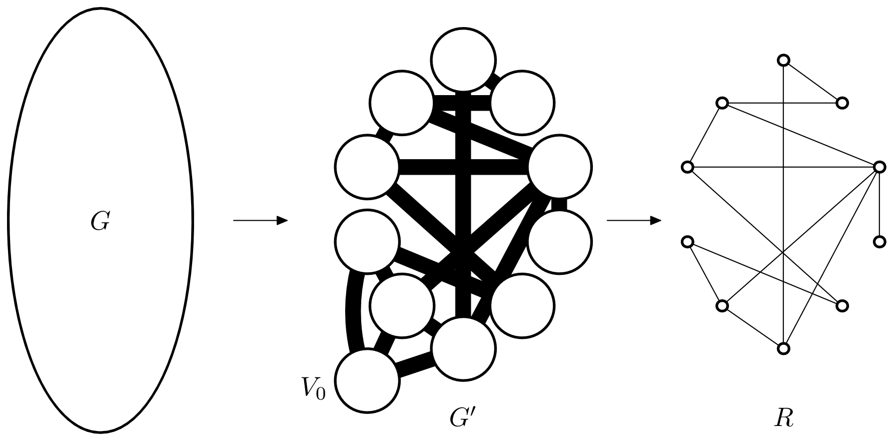

Figure 1.1: Applying Lemma 1.1.2 to obtain G' and $R(G, \epsilon, \delta)$.

1.1.5 Blow-Up Lemma

In order to state the Blow-Up Lemma, we introduce a stronger form of regularity. For fixed $\epsilon, \delta > 0$, a pair (A, B) is (ϵ, δ)-*super-regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, we have $d(X, Y) > \delta$, along with $\deg_B(a) > \delta|B|$ for all a in A and $\deg_A(b) > \delta|A|$ for all b in B.

In particular, if (A, B) is super-regular, then all vertices in A and B have degree at least 1. The primary example of an (ϵ, δ)-super-regular pair (A, B) is a complete bipartite graph, which is (ϵ, δ)-super-regular for all $\epsilon > 0$ and $\delta > 0$.

The following lemma says that we can remove a small number of vertices from an (ϵ, δ)-regular pair to form an $(\epsilon, \delta - \epsilon)$-super-regular pair.

Lemma 1.1.5 ([7] Lemma 7.5.1)**.** *Let (A, B) be an ϵ-regular pair of density d and let $Y \subseteq B$ have size $|Y| \geq \epsilon|B|$. Then all but at most $\epsilon|A|$ of the vertices in A have (each) at least $(d - \epsilon)|Y|$ neighbors in Y.*

We will use a simple corollary of this result.

Lemma 1.1.6. *Let (A, B) be an ϵ-regular pair of density d. Then there exist subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq (1-\epsilon)|A|$ and $|B| \geq (1-\epsilon)|B|$ such that the pair (A', B') is $(\epsilon, d-2\epsilon)$-super-regular.*

Frequently, when Lemma 1.1.5 is applied with $\delta = \delta_0$, it is followed immediately with an application of Lemma 1.1.6 so we may then use the value $\delta = \delta_0 - 2\epsilon$ in Lemma 1.1.7.

Lemma 1.1.7 (Blow-Up Lemma - Komlós, Sárközy, Szemerédi [27])**.** *Given a graph R of order r and positive parameters δ, Δ, there exists an $\epsilon_0 = \epsilon_0(\delta, \Delta, r) > 0$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers, and let us replace the vertices $v_1, v_2, \ldots, v_r$ of R with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ of orders $n_1, n_2, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The first graph $\mathbf{R}$ is obtained by replacing each edge $v_i v_j$ of R with the complete bipartite graph between the corresponding vertex-sets V_i and V_j. A sparser graph G is constructed by replacing each edge $v_i v_j$ with any (ϵ_0, δ)-super-regular pair between V_i and V_j. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $\mathbf{R}$, then it is already embeddable into G.*

Essentially, when finding subgraphs of bounded minimum degree Δ, we can treat super-regular pairs like complete bipartite graphs. The use of Lemmas 1.1.2, 1.1.5, and 1.1.7 ensures that a sufficiently large and dense graph G consisting of super-regular pairs of clusters contains *every* subgraph H of bounded maximum degree $\Delta(H)$. This will help us greatly when establishing (H, w)-linkage in a graph G. See Figure 1.2 for an application of Lemma 1.1.7 on a triangle T contained in a graph R.

Since we will apply Lemma 1.1.7 to a triangle, we extend the definitions of (ϵ, d)-regularity and (ϵ, d)-super-regularity to include triples. That is, (T_1, T_2, T_3) is an (ϵ, d)-*regular triple* if the pairs (T_1, T_2), (T_2, T_3), and (T_1, T_3) are all regular. Similarly, (T_1, T_2, T_3) is an (ϵ, d)-*super-regular triple* if the pairs (T_1, T_2), (T_2, T_3), and (T_1, T_3) are all super-regular.

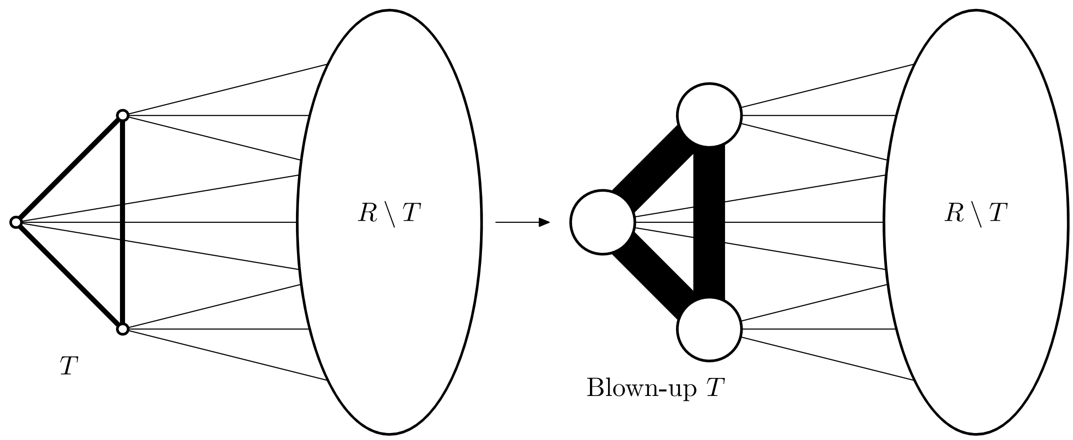

Figure 1.2: Lemma 1.1.7 is applied to a triangle within a graph R.

For the remainder of this work, when convenient, we will assume appropriate choices of $\epsilon > 0$ and n such that ϵn is an integer. We also adhere to (1.1) when choosing δ.

1.2 Panconnectivity

A graph G is *panconnected* if for every pair of vertices $u, v \in V(G)$ and all $2 \leq t \leq |G| - 1$, there exists a u, v-path of length t in G. Williamson proved the following sufficient condition for a graph to be panconnected.

Theorem 1.2.1 (Panconnectivity Criterion - Williamson [40]). *A graph G of order n is panconnected if $\delta(G) \geq \frac{n+2}{2}$.*

We often use Lemma 1.2.1 when we wish to construct a path of arbitrary length within a component of a graph with high minimum degree.

We define an analogous concept for bipartite graphs. A bipartite graph $U \cup V$ is *bipanconnected* if for every pair of vertices $x, y \in U \cup V$, there exist (x, y)-paths of all possible lengths at least 2 of appropriate parity in $U \cup V$. That is, for every pair of vertices $x \in U$ and $y \in V$, there exist (x, y)-paths of every possible odd length except 1, and for every pair of vertices $x, y \in U$ (and V), there exist (x, y)-paths of every even length. Note that we must exclude the value 1 from our definition in order to allow graphs $U \cup V$ that are not complete bipartite. Also observe that the partite sets of a bipanconnected graph must have order within one of each other.

The following lemma is a generalization of Claim 1 in [12] that establishes the odd bipanconnectivity of sufficiently dense balanced bipartite graphs. Recall that a bipartite graph $U \cup V$ is *balanced* if $|U| = |V|$.

Lemma 1.2.2. *If $U \cup V$ is a balanced bipartite graph of order $2m$ with $\delta(U \cup V) \geq \frac{3m}{4}$, then $U \cup V$ is bipanconnected.*

Proof. Consider a balanced bipartite graph $U \cup V$ of order $2m$ with $\delta(U \cup V) \geq \frac{3m}{4}$. First suppose $u \in U$ and $v \in V$. We prove this result using induction on the desired length of a (u, v)-path. The base case is straightforward since, under the assumed minimum degree of $U \cup V$, the vertices u and v must have adjacent neighbors. Now suppose there is a (u, v)-path P of length $2k - 1$ for some $3 \leq k < m$.

First assume $k \leq \frac{m}{2}$. For an edge $xy \in P$, there exists a vertex $w \in N(x)$ such that $N(w) \cap N(y) \setminus P \neq \emptyset$. Call a vertex in this set z. Replacing xy with $xwzy$ in P gives a path of length $2k + 1$. Next, suppose $k > \frac{m}{2}$ and further suppose there exists an edge wz with $w \in U \setminus P$ and $z \in V \setminus P$. Since $\delta(U \cup V) > \frac{3m}{4}$, the vertex z must be adjacent to more than half of $P \cap U$ and the vertex w must be adjacent to more than half of $P \cap V$. Then there must be some edge $xy \in P$ such that $xw, yz \in E(U \cup V)$. Replacing xy with $xwzy$ in P gives a path of length $2k + 1$.

Finally, suppose $k > \frac{m}{2}$ and there is no edge xy outside P. Since $\delta(U \cup V) > \frac{3m}{4}$, we must have $k > \frac{3m}{4}$. Consider vertices $x \in U \setminus P$ and $y \in V \setminus P$. Then x has $\delta(U \cup V)$ edges into $V \cap P$ and y has $\delta(U \cup V)$ edges into $U \cap P$. Call an vertex $z \in P$ *replaceable* if either x or y is adjacent to both neighbors of z in P (meaning this vertex can be replaced in P by x or y). Then at least $\frac{n}{2}$ vertices in $U \cap P$ and $\frac{m}{2}$ vertices in $V \cap P$ are replaceable. It follows then that there must be an edge $wz \in E(U \cap V)$ such that both w and z are replaceable. Replacing w with x and z with y gives a path of length $2k - 1$ with an edge outside of P. This reduces the problem to the previous case.

Now, suppose the two selected vertices are either both in U or both in V. Without loss of generality, suppose $u_1, u_2 \in U$. We again use induction on the desired length of a (u_1, u_2)-path. The base case is straightforward since, given the assumed minimum degree of $U \cup V$, the vertices u_1 and u_2 must share a neighbor. Now suppose there is a (u_1, u_2)-path P of length $2k$ for $2 \leq k < m$. By an argument identical to the previous case above, we get a (u_1, u_2)-path of length $2k + 2$.

Hence, $U \cup V$ is bipanconnected. $\square$

We use Lemma 1.2.2 when the reduced graph of G is bipartite (and hence, the graph G is nearly bipartite) and we wish to construct paths of arbitrary length in G.

Chapter 2

The Enomoto-Ota Conjecture for Large Graphs

In this chapter, we prove the Enomoto-Ota Conjecture for graphs of sufficiently large order. We divide the proof into four lemmas and make heavy use of Lemmas 1.1.2 and 1.1.7, along with Theorem 1.2.1.

2.1 Introduction

The *degree sum* of a graph G, denoted $\sigma_2(G)$, is defined to be

$$\sigma_2(G) = \min_{uv \notin E(G)} \{d(u) + d(v)\}.$$

Assume that the term *disjoint* means vertex-disjoint when describing paths. In 2000, Enomoto and Ota conjectured the following.

Conjecture 2.1.1 (Enomoto, Ota [10]). *Given an integer $k \geq 3$, let G be a graph of order n and let $n_1, n_2, \ldots, n_k$ be a set of k positive integers with $\sum n_i = n$. If $\sigma_2(G) \geq n + k - 1$, then for any k distinct vertices $x_1, x_2, \ldots, x_k$ in G, there exists a set of disjoint paths $P_1, P_2, \ldots, P_k$ such that $|P_i| = n_i$ and P_i starts at x_i for all i with $1 \leq i \leq k$.*

Previously, in [23], Johansson showed that a graph of order $\sum_{i=1}^{k} n_i$ with minimum degree sum at least $\sum_{i=1}^{k} \lfloor n_i/2 \rfloor$ can be partitioned into k disjoint paths of order $n_1, \ldots, n_k$. This result was later improved by Chen in [4], who found a result for a lower minimum degree sum by incorporating the number of even integers in $\{n_1, \ldots, n_k\}$. In both cases, however, the endvertices of these paths are not specified.

Shortly after Enomoto and Ota made their conjecture in 2000, Kawarabayashi showed [25] in 2001 that a graph with the significantly larger minimum degree sum $\sum_{i=1}^{k} \max\left\{\left\lfloor \frac{4}{3} n_i \right\rfloor, n_i + 1\right\} - 1$ satisfies the conjecture. Magnant and Martin [31] later proved an asymptotic version of Conjecture 2.1.1 for large graphs with prescribed path orders that were fractions of n. Hall, Magnant, and Wang later used Szemerédi's Regularity Lemma to show in [21] that a large graph with minimum degree sum at least $n + k - 2$ contains a non-spanning collection of paths starting at specified vertices with prescribed lengths. We prove Theorem 2.2.1, which states that Conjecture 2.1.1 holds for graphs of sufficiently large order.

2.2 Theorem 2.2.1

When n is sufficiently large relative to k, we prove that Conjecture 2.1.1 is true for graphs of sufficiently large order.

Theorem 2.2.1. *Conjecture 2.1.1 holds for graphs of sufficiently large order.*

As in [21], the approach uses Regularity Lemma, along with the Blow-Up Lemma and the Panconnectivity Criterion. We use inductive arguments combined with the Panconnectivity Criterion to prove our results. Letting n_k be the largest prescribed path order, we show G satisfies our result when $\delta(G) \geq \frac{n_k}{8}$ for large $n = \sum_{i=1}^{k} n_i$. The subsequent results all focus on graphs with minimum degree greater than $\frac{n_k}{8}$. We then choose small, positive values of ϵ and d and a positive integer k in order to consider a graph G of order $n \geq n(\epsilon, d, k)$. After applying Lemma 1.1.2 to G, we examine the structure of the reduced graph R. We divide our proof into three extremal cases and one non-extremal case. The extremal cases are:

1. $\delta(G) \leq \frac{n_k}{8}$,

2. R has connectivity 0 or 1, and

3. R is bipartite save for a matching of at most $\frac{1}{145}$ edges.

The non-extremal case consists of all other possible structures of R with $\delta(G) > \frac{n_k}{8}$. To prove the non-extremal case, we also use the Blow-Up Lemma to ensure the existence of dense pairs of clusters in G. We also use Lemma 2.3.10, a stronger version of the Blow-Up Lemma, to prove that all pairs of vertices within these dense cluster pairs are connected by a short path.

2.2.1 Proof Outline

We use a sequence of lemmas to eliminate extremal cases of the proof. Without loss of generality, we assume $n_1 \leq n_2 \leq \cdots \leq n_k$.

Lemma 2.2.2. *Conjecture 2.1.1 holds when $\delta(G) \leq \frac{n_k}{8}$.*

Lemma 2.2.2 is proven in Subsection 2.3.1. By Lemma 2.2.2, we may assume $\delta(G) \geq \frac{n_k}{8}$.

Say that a graph G is λ-*almost-bipartite* if there exists a spanning edge-maximum bipartite subgraph B such that the largest matching in $G \setminus E(B)$ has at most $\lambda|G|$ edges. Call B a λ-*subgraph* of G.

Lemma 2.2.3. *Given small real numbers $\epsilon, \delta > 0$, an integer $k \geq 0$, and λ satisfying $0 < \lambda < \frac{1}{145k}$, let G be a graph of order $n \geq n(\epsilon, \delta, k)$ with $\delta(G) \geq \frac{n_k}{8}$. If the reduced graph of G is λ-almost-bipartite, then Conjecture 2.1.1 holds.*

Lemma 2.2.3 is proven in Subsection 2.3.2.

Lemma 2.2.4. *Given small real numbers $\epsilon, \delta > 0$ and a positive integer k, let G be a graph of order $n \geq n(\epsilon, \delta, k)$ with $\delta(G) \geq \frac{n_k}{8}$. If the reduced graph of G has connectivity at most 1, then Conjecture 2.1.1 holds.*

Lemma 2.2.4 is proven in Subsection 2.3.3.

Once all these (extremal) lemmas are in place, we use Ore's Theorem [33] to construct a long cycle in the reduced graph. Alternating edges of this cycle are made into super-regular pairs of the graph. This structure is then used to construct the desired paths.

Lemma 2.2.5. *Given small real numbers $\epsilon, \delta > 0$, a positive integer k, and λ satisfying $0 < \lambda < \frac{1}{145k}$, let G be a graph of order $n \geq n(\epsilon, \delta, k)$ with $\delta(G) \geq \frac{n_k}{8}$. If the reduced graph of G has connectivity at least 2 and is not λ-almost-bipartite, then Conjecture 2.1.1 holds.*

The complete proof of Theorem 2.2.1 comprises the previous four lemmas.

Proof of Theorem 2.2.1. Use Lemmas 2.2.2–2.2.5. □

2.3 Proof of Theorem 2.2.1

We present the proof of Lemma 2.2.2.

2.3.1 $\delta(G) \leq \frac{n_k}{8}$

Proof of Lemma 2.2.2. Let $a \in V(G)$, with $|N(a)| = \delta(G) \leq \frac{n_k}{8}$, and partition $V(G)$ as follows (see Figure 2.1);

$$\begin{aligned} B &= G \setminus (a \cup N(a)) \\ A &= \left\{ v \in a \cup N(a) \;\middle|\; |N(v) \cap V(B)| < \frac{1}{8}(n + k - \delta(G) - 1) \right\} \\ C &= \left\{ v \in a \cup N(a) \;\middle|\; |N(v) \cap V(B)| \geq \frac{1}{8}(n + k - \delta(G) - 1) \right\} \end{aligned}$$

Note that, since $\sigma_2(G) \geq n+k-1$, the set A induces a complete graph. Furthermore, the set B has order $n - 1 - \delta(G)$, and A is nonempty since $a \in A$. We also have k fixed vertices $X = \{x_1, \ldots, x_k\} \subseteq V(G)$. Since $\sigma_2(G) \geq n + k - 1$ and a has no edges to B, each vertex in B has degree at least $n + k - 1 - \delta(G)$ which means $\delta(G[B]) \geq n + k - 1 - 2\delta(G)$. Note that also G is at least $(k + 1)$-connected. First, we make a claim about subsets of B.

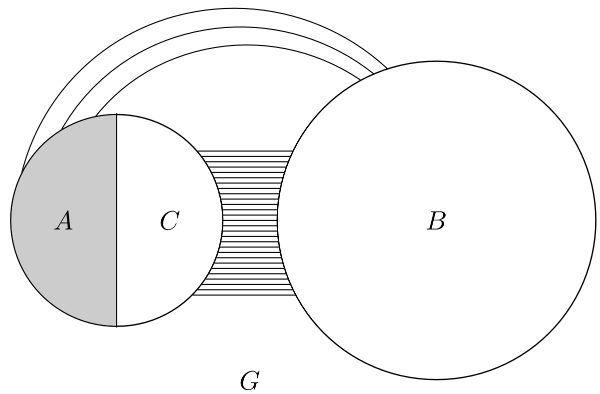

Figure 2.1: $G = (A \cup C) \cup B$, where A induces a complete graph.

Claim 2.3.1. *Every subset of B of order at least $\frac{3n_k}{8}$ is panconnected.*

Proof. With $|B| = n - \delta(G) - 1$ and $\delta(G[B]) \geq n + k - 1 - 2\delta(G)$, we see that $\delta(G[B]) \geq |B| - \delta(G) \geq |B| - \frac{n_k}{8}$. Therefore, for any subset $B' \subseteq B$ with $|B'| \geq \frac{3n_k}{8}$, we have $\delta(G[B']) \geq |B'| - \frac{n_k}{8} > \frac{|B'|+2}{2}$. By Theorem 1.2.1, we see that B' is panconnected. $\square$

Let X_A denote the (possibly empty) set $X \cap A$ and let X'_A denote $X_A \cup v$ where $v \in A \setminus X_A$ if such a vertex v exists. If no such vertex v exists, let $X'_A = X_A$. The vertices of X'_A will serve as start vertices for paths that will be used to cover all of A. By Menger's Theorem, since $\kappa(G) \geq k + 1$, there exists a set of disjoint paths $\mathscr{P}_A$ starting at the vertices of X'_A and ending in B and avoiding all other vertices of X. Choose such a collection so that each path is as short as possible, so each path contains only one vertex of B and, by construction, each path has order at most 4. If any of the paths in $\mathscr{P}_A$ begins at a selected vertex x_i and has order at least n_i, we call this desired path completed and remove the first n_i vertices of the path from the graph and continue the construction process. If $A \setminus V(\mathscr{P}_A) \neq \emptyset$, then let P_v be a path using all remaining vertices in A and ending at v. This path P_v together with the path of $\mathscr{P}_A$ corresponding to v provides a single path that cleans up the remaining vertices of A and ends in B. The ending vertices of these paths, the vertices of B, will serve as proxy vertices for the start vertices ($x_i \in X \cap A$ or v). Thusfar, we have constructed paths that cover all of A, start at vertices of $X \cap A$ (when such vertices exist) and end in B.

As vertices of B are selected and used on various paths, we continuously call the set of vertices in B that have not already been prescribed or otherwise mentioned the *remaining* vertices in B. For example, so far, $B \setminus (X \cup V(\mathscr{P}_A))$ are the remaining vertices of B. Our goal is to maintain at least $\frac{n_k}{2}$ remaining vertices to be able to apply Claim 2.3.1 as needed within the remaining vertices.

Since $|C| \leq \delta(G) \leq \frac{n_k}{8}$ and $d_B(u) \geq \frac{1}{8}(n+k-\delta(G)-1)$ for all $u \in C$, there exists a set of two distinct neighbors in $B \setminus (X \cup V(\mathscr{P}_A))$ for each vertex in C. For each vertex $x_i \in X \cap C$, select one such vertex to serve as a proxy for x_i and leave the other aforementioned neighbor in the remaining vertices of B. By Claim 2.3.1, there exists a path through the remaining vertices of B with at most one intermediate vertex from one neighbor of a vertex of C to a neighbor of another vertex of C. Since $|C| \leq \frac{n_k}{8}$, such paths can be built and strung together into a single path P_C starting and ending in B, containing all vertices of $C \setminus X$ with $|P_C| < 4|C| \leq \frac{n_k}{2}$.

We may now construct what is left of the desired paths within B. The paths $P_1, P_2, \ldots, P_{k-1}$ can be constructed in any order starting at corresponding proxy vertices and ending at arbitrary remaining vertices of B using the Claim 2.3.1 in the remaining vertices of B. Finally, there are at least

$$\begin{aligned} & |B| - |B \cap (\cup_{i=1}^{k-1} V(P_i))| - |B \cap V(\mathscr{P}_A)| - |B \cap V(P_C)| \\ \geq \; & (n-1-\delta(G)) - (k+1) - 3|C| \\ > \; & \frac{3n_k}{8} + 1 \end{aligned}$$

remaining vertices in B. With these and Claim 2.3.1, we construct a path with at most one internal vertex from an end of P_C to the proxy of v (if such a vertex exists) and a path containing all remaining vertices of B from x_k (or its proxy) to the other end of P_C. This completes the construction of the desired paths and thereby completes the proof of Lemma 2.2.2. $\square$

2.3.2 R is λ-Almost-Bipartite

We recall a result from [34] that provides a lower bound for a matching in a graph based solely on its order, minimum degree, and maximum degree.

Lemma 2.3.2 ([34]). *Let G be a graph of order n and let $S \subset V(G)$ with $|S| \leq \frac{n}{2}$. Then in G there is a matching of size at least*

$$\delta(G)\frac{n-|S|}{2(\delta(G)+\Delta(G))} \geq \delta(G)\frac{n}{8\Delta(G)}$$

such that for each matching edge at least one of the endpoints is from $V(G) \setminus S$.

Whenever we apply Lemma 2.3.2, we always let $S = \emptyset$. We apply Lemma 2.3.2 to a set $B \subset G$ that corresponds to an almost-partite set of R. The abundance of independent edges in B allows us to construct a path P_0 that uses vertices in B without using edges in R.

The following theorem from [30] details the similarity between the minimum degree sum of a graph and its reduced graph. In Theorem 2.3.3, assume Lemma 1.1.2 has been applied for small $\epsilon, \delta > 0$ on a graph G of order n with reduced graph R.

Theorem 2.3.3 ([30]). *Given a constant c, if $\sigma_2(G) \geq cn$, then $\sigma_2(R) \geq (c - 2\delta - 4\epsilon)r$.*

Using Theorem 2.3.3, we can determine upper and lower bounds on all vertices in $V(G) \setminus V_0$.

Fact 2.3.4. Given an integer $k \geq 0$ and small real numbers $\epsilon, \delta > 0$, let G be a sufficiently large graph of order $n = \sum_{i=1}^{k} n_i \geq n(\epsilon, \delta, k)$ with $\sigma_2(G) \geq n + k - 1$ and garbage set V_0. Suppose the reduced graph R of G is λ-almost-bipartite for some $0 < \lambda \leq \frac{1}{2}$ with λ-matching M_R, and let M be the vertex set in G corresponding to M_R. For all $v \in G$, we have

$$d(v) > \left(\frac{1}{2} - \delta - 2\epsilon - 2\lambda\right) n + k - 1, \tag{2.1}$$

and for all $v \in G \setminus (V_0 \cup M)$, we have

$$d(v) < \left(\frac{1}{2} + \delta + 2\epsilon + 2\lambda\right) n. \tag{2.2}$$

Proof. Recall that $r = |R|$. Let $U_R \cup V_R$ be the λ-subgraph of R. By definition, we have $V(R) = V(U_R \cup V_R)$ and $|M_R| \leq 2\lambda r$. By Theorem 2.3.3, we have $\sigma_2(R) \geq (1 - 2\delta - 4\epsilon)r$, and hence that

$$(1 - 2\delta - 4\epsilon - 4\lambda)r < |U_R|, |V_R| < (1 + 2\delta + 4\epsilon + 4\lambda)r.$$

Then for all $r \in R \setminus M_R$, we have $d(r) < \frac{1}{2}(1 + 2\delta + 4\epsilon + 4\lambda)r$. It follows from Items 3, 4, and 6 in Lemma 1.1.2, that for all $v \in G \setminus (V_0 \cup M)$, we have $d(v) < \left(\frac{1}{2} + \delta + 2\epsilon + 2\lambda\right) n$, which is precisely (2.2). This in turn implies (2.1). $\square$

There is no upper bound on the number of possible neighbors of vertices in $V_0 \cup M$, but this is not a problem. The established lower bound in (2.1) allows us to group vertices in $V_0 \cup M$ with other vertices in $V(G) \setminus (V_0 \cup M)$ depending on the location of the majority of their neighbors.

We present the proof of Lemma 2.2.3.

Proof of Lemma 2.2.3. Applying Lemma 1.1.2 to G, suppose we get a λ-almost-bipartite reduced graph R with λ-bipartite graph $A'_R \cup B'_R$ and λ-matching M_R. By definition, we see $V(M_R) \subset V(A'_R \cup B'_R)$, and by definition, we see $|M_R| \leq 2\lambda r$. Let $A_R = A'_R \setminus M_R$ and $B_R = B'_R \setminus M_R$. Let $V(G) = A \cup B \cup M \cup V_0$, where A, B, and M correspond to A_R, B_R, and M_R, respectively, and V_0 is the garbage cluster in G. Letting $M_0 = V_0 \cup M$, by Fact 2.3.4, our assumptions immediately imply

$$\begin{aligned} 0 &\leq |M_0| \leq (\epsilon + 2\lambda)\, n \\ &\ll \left(\frac{1}{2} - \delta - 2\epsilon - 2\lambda\right) n + k - 1 < |A|, |B| < \left(\frac{1}{2} + \delta + 2\epsilon + 2\lambda\right) n. \end{aligned}$$

Claim 2.3.5. *For $m \geq \frac{n}{9k} > 16\lambda n$, every balanced bipartite graph consisting of m vertices in A and m vertices in B is bipanconnected.*

Proof. By Fact 2.3.4, for all $v \in G$, we have $d(v) > \left(\frac{1}{2} - (\delta + 2\epsilon) - 2\lambda\right) n + k - 1$ and $|A| \leq |B| < \left(\frac{1}{2} + (\delta + 2\epsilon) + 2\lambda\right)$. Hence, if $v \in A$, then there are fewer than $(2\delta + 4\epsilon + 4\lambda)\, n < \frac{n}{36k}$ vertices in $B \setminus N(v)$. A symmetric result is true for all $v \in B$. Hence, for any sets $U \subseteq A$ and $V \subseteq B$ each containing $m \geq \frac{n}{9k} \geq 16\lambda n$ vertices, the graph $G[U \cup V]$ is a balanced bipartite graph satisfying $\delta(U \cup V) \geq \frac{3m}{4}$. By Lemma 1.2.2, the graph $G[U \cup V]$ is bipanconnected. $\square$

We cannot extend this result to include vertices in M_0, which may only be adjacent to roughly $\frac{n}{5}$ vertices in A or B. Hence, we construct a single path that contains all vertices in M_0. We accomplish this by using Claim 2.3.5 multiple times to string together neighbors of the vertices in M_0.

Let $X = \{x_1, \ldots, x_k\} \subset V(G)$, and let B' denote the set of all vertices with at most $9\lambda n < \frac{n}{16k}$ neighbors in B. Define A' symmetrically, and note that $A' \cap B' = \emptyset$. Let V_B be the set of vertices $v \in M_0$ satisfying $|N_{A'}(v)| \geq |N_{B'}(v)|$. Define V_A symmetrically; then $V_A \cup V_B \subseteq M_0$. Then

$$|B' \cup V_B| \leq |B| + |M_0| < \left(\frac{1}{2} + (\delta + 3\epsilon) + 4\lambda\right) n,$$

with a symmetric inequality for $|A' \cup V_A|$.

Given a graph G and a vertex x_i, let an *x_i-path* be a path in G that begins at x_i. For example, each path P_i we wish to create is an *x_i-path.* Let ι be the number of even (order) x_i-paths with $x_i \in B' \cup V_B$ minus the number of even x_i-paths with $x_i \in A' \cup V_A$. When creating our x_i-paths, we generally alternate between $A' \cup V_A$ and $B' \cup V_B$, and the value ι counts each extra vertex in $A' \cup V_A$ or $B' \cup V_B$ resulting from even paths. Lastly, let $d_{AB} = |B' \cup V_B| - |A' \cup V_A| + \iota$, and note that $d_{AB} < (2\delta + 8\epsilon + 4\lambda)\, n < \frac{n}{36k}$. Without loss of generality, assume $d_{AB} \geq 0$.

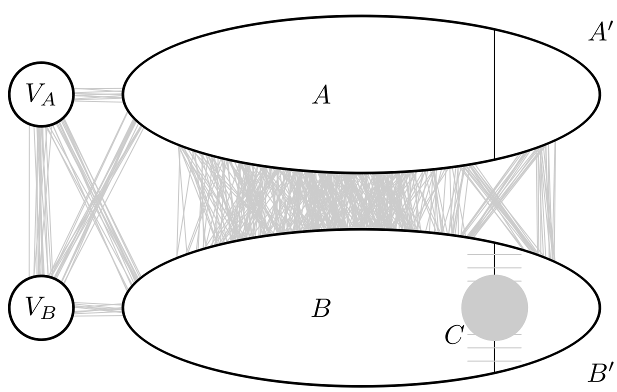

Figure 2.2: $G = (A' \cup V_A) \cup (B' \cup V_B)$.

Claim 2.3.6. *There exists a path P_0 in $G \setminus X$ with fewer than $5(d_{AB} + |M_0|) < 5(2\delta + 9\epsilon + 6\lambda)n < \frac{n}{4k}$ vertices that contains all vertices in $M_0 \setminus X$ and contains d_{AB} more vertices in $B' \cup V_B$ than in $A' \cup V_A$.*

Proof. Ignore all vertices in X for this proof. We create two paths, P_1 and P_2, where P_1 contains d_{AB} more vertices in B' than in A', and P_2 contains all remaining vertices in $M_0 \setminus P_1$. Linking these two paths together gives us P_0.

If $|V_B| \geq \frac{d_{AB}}{2}$, then each vertex in V_B has at least $\frac{n}{16k} > 2d_{AB}$ neighbors in B to string together all vertices in V_B into a single path P_1 in the following way. For $b_1 \in V_B$, reserve $v_1 \in B'$ and $a_1 \in A'$. For each $b_j \in V_B \setminus b_1$, reserve two distinct neighbors $u_j, v_j \in B'$ and a distinct vertex $a_j \in A'$. Since each vertex in B' has greater than $\frac{n}{3}$ neighbors in A', we may choose a_j so that $a_j \in N(v_j) \cap N(u_{j+1})$. Adjoin the paths $P_1 = \{b_1, v_1, a_1\}$ and each successive copy of $\{u_j, b_j, v_j, a_j\}$ to form the path P_1. Then P_1 consists of

- $\frac{d_{AB}}{2}$ vertices in V_B,
- d_{AB} vertices in B', and
- $\frac{d_{AB}}{2}$ vertices in A'.

Now suppose $|V_B| < \frac{d_{AB}}{2}$. Let C be the set of vertices in B' that have fewer than $\frac{1}{2}(n+k-1)$ neighbors in G, and note that $|C| < |M_0| + (\delta + 2\epsilon) < (\delta + 3\epsilon + 2\lambda)\, n < \frac{d_{B'}(b)}{5}$ for all $b \in V_B$. Note that C induces a complete graph; hence, if $|C| > d_{AB} - 2|V_B|$, then string together all vertices in V_B as above, and adjoin all vertices of C to create P_1. Here, the path P_1 consists of

- all vertices in V_B,
- $2|V_B|$ vertices in $B' \setminus C$,
- $d_{AB} - 2|V_B| + 1$ vertices in C, and
- $|V_B| + 1$ vertices in A'.

Lastly, if $|C| \leq d_{AB} - 2|V_B|$, then all vertices in $B' \setminus C$ have at least $d_{AB} - |V_B| > 0$ neighbors in B'. Create a new graph $\Gamma[B']$ by adjoining $\frac{d_{AB}}{2} - |V_B| - d(c)$ edges to each vertex $c \in C$. Then $\delta(\Gamma[B']) \geq \frac{d_{AB}}{2}$, and by Lemma 2.3.2, there are at least $\frac{d_{AB}/2 - |V_B|}{8(\delta + 2\epsilon + 9\lambda)}$ independent edges in $\Gamma[B']$. It then follows there are at least $\frac{d_{AB}/2 - |V_B|}{8(\delta + 2\epsilon + 9\lambda)} - |C|$ independent edges in $G[B']$. Since $\lambda < \frac{1}{145k}$ by assumption, there are

$$2|V_B| + \frac{d_{AB}/2 - |V_B|}{8(\delta + 2\epsilon + 9\lambda)} > d_{AB}$$

extra vertices in $B' \cup V_B$ than in A'; hence P_1 may always be constructed to contain exactly d_{AB} more vertices in $B' \cup V_B$ than in A' in the following way. Begin constructing P_1 by stringing together all vertices in V_B and adjoining all vertices in C as above. Given that each vertex in B' has greater than $\frac{n}{3}$ neighbors in A', for each pair of independent edges $u_1v_1, u_2v_2 \in G[B']$, we may choose a vertex $a \in N(v_1) \cap N(u_2) \subseteq A'$. String together $d_{AB} - 2|V_B| - |C|$ independent edges in B' with $d_{AB} - 2|V_B| - |C|$ vertices in A. In this final case, the path P_1 consists of

- all vertices in V_B,
- all vertices in C,
- $2|V_B| + 1 + 2(d_{AB} - 2|V_B| - |C|)$ vertices in $B' \setminus C$, and
- $|V_B| + 1 + (d_{AB} - 2|V_B| - |C|)$ vertices in A'.

In all cases, the path P_1 has fewer than $5d_{AB} < \frac{n}{7k}$ vertices and contains d_{AB} more vertices in B' than in A'.

We now create P_2 containing all vertices in $M_0 \setminus P_1$. We may choose distinct neighbors in A or B for each vertex in $M_0 \setminus P_1$. We repeatedly use Claim 2.3.5 between each pair of remaining vertices to create a path of length 2 or 3 between these neighbors, depending on whether or not both neighbors are in A (symmetrically B). It is clear that we may avoid P_1 when performing this process. Then there exists a path P_2 with fewer than $5|M_0| < \frac{6n}{29k}$ vertices that contains all vertices in M_0. We may construct P_2 to contain at most 1 more vertex in $A' \cup V_A$ than in $B' \cup V_B$, or vice versa.

Connect P_1 and P_2 using Claim 2.3.5 similarly and call the resulting path P_0. (If P_0 would no longer have exactly d_{AB} more vertices in $B' \cup V_B$ than in $A' \cup V_A$, then adjust P_1 appropriately.) Then P_0 contains fewer than $\left(\frac{1}{7k} + \frac{6}{29k}\right) n < \frac{n}{4k}$ vertices. Furthermore, the path P_0 contains all vertices in M_0 (save for vertices in X, of course) and contains precisely d_{AB} more vertices in $B' \cup V_B$ than in $A' \cup V_A$. $\square$

By the Pigeonhole Principle, there must exist some $n_i \geq \frac{n}{k} > 4|P_0|$. Without loss of generality, assume $n_1 \leq \cdots \leq n_k$, and hence $n_k \geq \frac{n}{k}$. We induct on i to construct all desired disjoint x_i-paths P_i. Let $X^i = X \setminus \{x_1, \ldots, x_i\} = \{x_{i+1}, \ldots, x_k\}$, let $m_i = \left\lceil \max\left\{n_i, \frac{n}{4k}\right\}\right\rceil$, and let $\mathscr{P}^i = \bigcup_{j=1}^{i} P_j$.

First consider the base case. By Fact 2.3.4, the vertex x_1 contains at least $\frac{n}{5}$ neighbors in either A or B. We may choose one neighbor of x_1 and include it in a balanced bipartite graph $U_1 \cup V_1 \subset (A \cup B) \setminus (X^1 \cup P_0)$ consisting of m_1 (or $m_1 + 1$ if m_1 is odd) vertices. By Claim 2.3.5, there exists an x_1-path $P_1 \subseteq x_1 \cup U_1 \cup V_1$ with n_1 vertices. Similarly, for all $i < k$, we have

$$\begin{aligned} n - |\mathscr{P}^{i-1} \cup P_0 \cup X^i| &\geq \frac{(k-i+1)n}{k} - (k-i+1) - \frac{n}{4k} \\ &> \frac{n}{k}. \end{aligned}$$

Hence, there are always enough vertices to create a balanced bipartite graph $U_i \cup V_i \subset (A \cup B) \setminus (\mathscr{P}^{i-1} \cup P_0 \cup X^i)$ that includes x_i with m_i vertices. By Claim 2.3.5, we may construct an x_i-path P_i in $U_i \cup V_i$ with n_i vertices.

Finally, for $i = k$, we wish for P_k to contain all vertices in $V(G) \setminus \mathscr{P}^{k-1}$. Note that this final path must include P_0. We have $X^k = \emptyset$, and hence

$$\begin{aligned} n - |\mathscr{P}^{k-1} \cup P_0 \cup X^k| &> n - \left(\frac{n}{4k} + \frac{k-1}{k}\right) n \\ &> \frac{3n}{4k}. \end{aligned}$$

By Claim 2.3.6 and the definition of d_{AB}, the graph $G \setminus \bigcup_{i=1}^{k-1} P_i$ is a balanced bipartite graph. Hence, we may construct a balanced bipartite graph of order $\frac{n}{4k}$ that contains x_k and an endpoint of P_0, and then use Claim 2.3.5 to create a path with 2 (or 3) vertices connecting x_k to P_0, so that what remains in G (including the other endpoint of P_0) is still a balanced bipartite graph. Since this path containing x_k and P_0 contains fewer than $\frac{n}{4k}$ vertices, we may create a path starting at the other endpoint of P_0 that consists of all remaining vertices in G. This resulting x_k-path is P_k, which necessarily contains n_k vertices.

We have therefore created disjoint x_i-paths on n_i vertices for all $1 \leq i \leq k$ that together cover $V(G)$. $\square$

2.3.3 $\kappa(R) \leq \epsilon r$

We begin with a lemma ensuring that low connectivity in the reduced graph R results in at most two components in the main graph G. As in previous sections, let $r = |R|$.

Lemma 2.3.7. *Let $\epsilon, \delta > 0$ be small reals and k be a positive integer. If G is a graph with $\sigma_2(G) \geq n + k - 1$ and reduced graph R with connectivity at most $\left(\frac{1}{10} - \frac{3}{5}(\delta + 2\epsilon)\right) r$, then R consists of only two components (and a cutset if $\kappa(R) > 0$).*

Proof. Applying Lemma 1.1.2 to G, let $G'' = G'[V(G) \setminus V_0]$. Since $d_{G''}(v) > d_G(v) - (\delta+2\epsilon)n$, it immediately follows that $\sigma_2(R) > (1-2(\delta+2\epsilon))r$. Let D be a cutset of R (if one exists). Suppose R (or $R \setminus D$) contains at least 3 components, three of which being A, B, and C. Let $a \in A$, $b \in B$ and $c \in C$. Then $d(a)+d(b) > (1-2(\delta+2\epsilon))r$, which implies $|A| + |B| > (1 - 2(\delta + 2\epsilon))r - 2|D|$. Similarly, the same is true for $|B| + |C|$ and $|A| + |C|$. So $2(|A| + |B| + |C|) > 3(1 - 2(\delta + 2\epsilon))r - 6|D|$, or $|D| > \left(\frac{1}{10} - \frac{3}{5}(\delta + 2\epsilon)\right) r$, a contradiction. $\square$

Note that the connectivity of R may be considerably larger than ϵr for us to be guaranteed two components in G. Also note that A, B, C, and D were vertex sets in R. In the following remark, the same symbols are used to denote vertex sets in G.

Remark 2.3.8. Given small real numbers $\epsilon, \delta > 0$ and a positive integer k, let G be a graph of order $n = \sum_{i=1}^{k} n_i \geq n(\epsilon, \delta, k)$ with $\sigma_2(G) \geq n + k - 1$ and $\delta(G) \geq \frac{n_k}{8}$. If the reduced graph of G has connectivity at most ϵr, then let $D \subset V(G)$ be the cluster corresponding to a cut vertex of R. (If R contains no cut vertices, then $D = \emptyset$.) Let V_0 be the garbage cluster of G resulting from Lemma 1.1.2, and let C be a minimum cutset of G. Then $C \subseteq D \cup V_0$. By Lemma 1.1.2, each vertex of R corresponds to a cluster in G of order $L = \xi n$. Hence, we have $k + 1 \leq |C| \leq |D| + |V_0| \leq \epsilon\xi rn + \epsilon n$. By Lemma 2.3.7, we may define A and B to be the components of $G \setminus C$ and write $G = A \cup C \cup B$. It immediately follows from $\sigma_2(G) \geq n + k - 1$ that

$$\begin{aligned}\delta(G[A]) &> |A| - |C| > |A| - (\epsilon\xi r + \epsilon)n,\\ \delta(G[B]) &> |B| - |C| > |B| - (\epsilon\xi r + \epsilon)n.\end{aligned} \tag{2.3}$$

From the condition $\delta(G) \geq \frac{n_k}{8} \geq \frac{n}{2k}$, we know $|A|, |B| \geq \frac{n_k}{8} - |C| \geq \left(\frac{1}{8k} - \epsilon\xi r - \epsilon\right) n > \frac{n}{8(k+1)}$.

Note that $\epsilon^2 r \ll 1$.

Lemma 2.3.9. *Let ϵ, δ, k, and $G = A \cup C \cup B$ be defined as in Remark 2.3.8. Then the induced graph on any subgraph of A or B of order at least $2(\epsilon\xi r + \epsilon)n$ is panconnected.*

Proof. We see from (2.3) that $\delta(G[A]) > |A| - |C| > |A| - (\epsilon\xi r + \epsilon)n$. Then for all $U \subset A$ of order at least $2(\epsilon\xi r + \epsilon)n$, we have

$$\begin{aligned}\delta(G[U]) &\geq |S| - (\epsilon\xi r + \epsilon)n + 1\\ &\geq \frac{|S| + 2}{2}.\end{aligned}$$

By Theorem 1.2.1, the graph $G[U]$ is panconnected. A symmetric argument shows that if $U \subset B$ has order at least $(\epsilon\xi r + \epsilon)n$, then $G[U]$ is panconnected. $\square$

While panconnected sets give paths of arbitrary length, only the endpoints are specified. Hence, to create disjoint paths of arbitrary length, we must create sets using vertices that are not part of an already existing desired path. Fortunately, even small subsets of A and B induce panconnected graphs.

Proof of Lemma 2.2.4. Suppose $\kappa(R) \leq \epsilon r$, and let $G = A \cup C \cup B$ as in Remark 2.3.8. As noted before (2.3), we know $k + 1 \leq |C| \leq (\epsilon\xi r + \epsilon)n$. As noted after (2.3), we know $|A|, |B| > \frac{n}{8(k+1)}$. For each $c \in C$, we may reserve 2 distinct neighbors $a_c \in A$ and $b_c \in B$. Call $A_C = \{a_c \in A \setminus X \mid c \in C\}$ (symmetrically $B_C = \{b_c \in B \setminus X \mid c \in C\}$) the set of *proxy* vertices in A (symmetrically B). In particular, note that every vertex in C has unique proxy vertices in both A_C and B_C, and hence

$$|C| = |A_C| = |B_C|.$$

Given a vertex x_i, let an x_i*-path* be a path containing x_i as an endpoint. Namely, each desired path P_i in G is an x_i-path. Recall that $X = \{x_i \mid n_i \leq n_{i+1}\}$. For some $i \leq k$, let

$$A^* = A \setminus \bigcup_{j=1}^{i-1} P_j,$$

and define B^* and C^* similarly. In particular, note that $A^* = A$ for $i = 1$. Let

$$A^v = (A^* \setminus (A_C \cup X)) \cup v,$$
$$B^v = (B^* \setminus (B_C \cup X)) \cup v;$$

i.e., $A^v \cap (X \cup A_C) = v$, and symmetrically for B^v and B_C. For the sake of notation, if $v = x_i$, then we write A^x and B^x. Both notations A^* and A^x will never cause an issue, as we never discuss A^* or x_i for different values of i at the same time.

We induct on i to prove our result. Consider the base case $i = 1$. If $x_1 \in A$ and $n_1 \leq |A^x| - 4(\epsilon\xi r + \epsilon)n$, then use Lemma 2.3.9 to construct an x_1-path $P_1 \subset A^x$ containing n_1 vertices. If $x_1 \in A$ and $n_1 > |A^x| - 4(\epsilon\xi r + \epsilon)n$, then let $c \in C$ with proxy vertices $a \in A_C$ and $b \in B_C$. Use Lemma 2.3.9 to create an x_1, a-path P_A consisting of all but $4(\epsilon\xi r + \epsilon)n$ vertices of $x_1 \cup A^a$. Also create a b-path $P_B \subseteq B^b$ with $n_1 - |P_A| - 1$ vertices. Then $P_1 = P_A \cup c \cup P_B$ is an x_1-path with n_1 vertices. If $x_1 \in B$, then a symmetric argument works. Lastly, if $x_1 \in C$, then suppose without loss of generality that $|A \setminus (A_C \cup X)| \geq |B \setminus (B_C \cup X)|$. Since $|C| + |A_C \cup X| \leq 2(\epsilon\xi r + \epsilon)n + k < 4(\epsilon\xi r + \epsilon)n$ and $n_1 \leq \frac{n}{k}$, this implies $n_1 < |A^x \cup a|$. Let $a \in A_C$ be the proxy vertex of x_1, and use Lemma 2.3.9 to create an a-path $P_A \subset A^a$ with $n_1 - 1$ vertices. Then $P_1 = x_1 \cup P_A$ is an x_1-path with n_1 vertices.

Now suppose $1 < i < k$ and that the disjoint x_j-paths $P_1, \ldots, P_{i-1}$ have been constructed in G. If $x_i \in A^* \cup C^*$ and $n_i \leq |A^x| - 4(\epsilon\xi r + \epsilon)n$, then use Lemma 2.3.9 to construct an x_i-path $P_i \subset A^x$ containing n_i vertices. If $x_i \in A^* \cup C^*$ and $n_i > |A^x| - 4(\epsilon\xi r + \epsilon)n$, then

$$\begin{aligned} |B^b| &= n - |A^*| - |C^*| - |\{x_i, \ldots, x_k\}| - \left|\bigcup_{j=1}^{i-1} P_j\right| \\ &\geq n - n_i - 4(\epsilon\xi r + \epsilon)n - (\epsilon\xi r + \epsilon)n - (k - i + 1) - \sum_{j=1}^{i-1} n_j \\ &\geq \sum_{j=i+1}^{k} n_j - 6(\epsilon\xi r + \epsilon)n. \end{aligned} \tag{2.4}$$

Hence, if $i < k - 1$, then $|B^b| > n_{i+1} \geq n_i$. If $i = k - 1$, then by the Pigeonhole Principle, we have

$$\begin{aligned} |B^b| &> n_k - 6(\epsilon\xi r + \epsilon)n \\ &> \frac{n}{k} - 6(\epsilon\xi r + \epsilon)n \end{aligned} \tag{2.5}$$

Furthemore, since $|C^* \setminus x_i| \geq k + 1 - (i - 1) - 1 \geq 2$, we know there exists $c \in C^*$ with proxy vertices $a \in A_C$ and $b \in B_C$. If $|A^x| > 4(\epsilon\xi r + \epsilon)n$, then use Lemma 2.3.9 to create an x_i, a-path P_A consisting of all but $4(\epsilon\xi r + \epsilon)n$ vertices of $a \cup A^x$. If $|A^x| \leq 4(\epsilon\xi r + \epsilon)n$ and $a \in A^*$, then use Lemma 2.3.9 to create an x_i, a-path P_A consisting of three vertices in $a \cup A^x$. If $|A^x| \leq 4(\epsilon\xi r + \epsilon)n$ and $a \in C^*$, then let $P_A = \emptyset$. Regardless of the initial size of A^x, we now have

$$3(\epsilon\xi r + \epsilon)n < |A^x \setminus P_A| \leq 4(\epsilon\xi r + \epsilon)n. \tag{2.6}$$

From (2.4) and (2.5), we may similarly use Lemma 2.3.9 to create a b-path $P_B \subset B^b$ with $n_i - |P_A| - 1$ vertices. Then $P_i = P_A \cup c \cup P_B$ is an x_i-path with n_i vertices. If $x_i \in B^*$, then a symmetric argument works.

Finally, suppose $i = k$ and that the disjoint x_j-paths $P_1, \ldots, P_{k-1}$ have been constructed in G. From (2.5) and (2.6), we know

$$\begin{aligned} |A^* \setminus (X \cup A_C)| &> 3(\epsilon\xi r + \epsilon)n, \\ |B^* \setminus (X \cup B_C)| &> 3(\epsilon\xi r + \epsilon)n. \end{aligned} \tag{2.7}$$

From the Pigeonhole Principle, we also know

$$n_k \geq \frac{n}{k} \gg |A_C| + |C^*| + |B_C|.$$

Without loss of generality, assume $x_k \in A^* \cup C^*$. Note that $|C^* \setminus x_k| \geq 1$, and hence, that we must have $n_k > |A^x|$. Define A^*_C and B^*_C similarly to the way A^* and B^* are defined. Use Lemma 2.3.9 $|C^*| \leq (\epsilon\xi r + \epsilon)n$ times within A^* and within B^* each to create a path P_C that strings together all vertices in C^* by using all proxy vertices in A^*_C and B^*_C. Since A^*_C, B^*_C, and C^* each have at most $(\epsilon\xi r + \epsilon)n$ vertices, and since $|C^* \setminus x_k| \geq 1$, we know $7 \leq |P_C| < 5(\epsilon\xi r + \epsilon)n$. Given the high value of $\delta(G)$, we may ensure that P_C starts with a proxy vertex in A^* and ends with a proxy vertex in B^* by including an additional vertex in A^* or B^* adjacent to some vertex in C^*. Let the endpoints of P_C be $a \in A^*$ and $b \in B^*$. Noting (2.7), we may use Lemma 2.3.9 in A^* and B^* to create an x_k, a-path P_A that contains all vertices in $A^* \setminus P_C$. Similarly, use (2.7) and Lemma 2.3.9 to create a b-path P_B that contains all vertices in $B^* \setminus P_C$. Then $P_k = P_A \cup P_C \cup P_B$ is an x_k-path that contains all remaining n_k vertices in G.

We have created k paths $P_1, \ldots, P_k$ in G, with each path P_i starting at x_i and having n_i vertices. □

2.3.4 Generalized Blow-Up Lemma

Before proving the non-extremal case of Theorem 2.2.1, weshow that every (ϵ, δ)-super-regular pair contains a short path between any two specified vertices. To prove this, we first state an expanded version of Lemma 1.1.7.

Lemma 2.3.10 (Generalized Blow-Up Lemma - Komlós, Sárközy, Szemerédi [27])**.** *Given a graph R of order r and positive parameters δ, Δ, there exists an $\epsilon_0 = \epsilon_0(\delta, \Delta, r) > 0$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers, and let us replace the vertices $v_1, v_2, \ldots, v_r$ of R with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ of orders $n_1, n_2, \ldots, n_r$ (blowing up). For a graph H with $\Delta(H) \leq \Delta$ and $r < \epsilon \min_i\{n_i\}$, let $\{u_1, \ldots, u_r\} \subseteq V(H)$, and let $U_1, \ldots, U_r$ be vertex sets with*

$U_i \subset V_j$ for some j and $|U_i| > \delta n_j$. If H with $\Delta(H) \leq \Delta$ is embeddable into $\mathbf{R}$*, then H is already embeddable into G with $u_i \in U_i$ for all i.*

See Theorem 1, Remark 13 of [27] for the proof. Lemma 2.3.10 is more general than Lemma 1.1.7 because it allows for a graph H to be embedded within specific areas of clusters within G. While these areas are not pinpoint accurate, the lower bound δn_j for $|U_i|$ is sufficiently small to establish the existence of a short path beginning and ending at specified vertices, as in Lemma 2.3.11.

Lemma 2.3.11. *Given an (ϵ, δ)-super-regular pair (A, B) and a pair of vertices $a \in A$ and $b \in B$, there exists a path of length at most 3 from a to b in (A, B).*

Proof. If $|A|$ or $|B|$ is small, then $A \cup B$ is a complete bipartite graph, and we are done. Assume $|A|$ and $|B|$ are large, and choose $a \in A$ and $b \in B$. We have $|N(a) \setminus a| \geq \frac{\delta |A|}{2}$ and $|N(b) \setminus b| \geq \frac{\delta |B|}{2}$. Since the pair $(A \setminus a, B \setminus b)$ is $\left(2\epsilon, \frac{\delta}{2}\right)$-super-regular, then by Lemma 2.3.10 there exists an edge uv with $u \in N(a)$ and $v \in N(b)$. Then $\{a, u, v, b\}$ is an a, b-path of length 3. □

The statement and proof of Lemma 2.3.11 are analogous to those in [22], where Hladky proves that every pair of endpoints within an (ϵ, δ)-super-regular pair is connected by a hamiltonian path.

We also use the following theorem of Ore.

Theorem 2.3.12 ([33])**.** *If G is 2-connected, then G contains a cycle of length at least $\sigma_2(G)$.*

2.3.5 Non-Extremal Case

Proof of Lemma 2.2.5. Given an integer $k \geq 3$ and desired path orders $n_1, \ldots, n_k$ as functions of the order n of a graph, we choose constants ϵ and d as follows:

$$\epsilon \ll d \ll \frac{1}{k}.$$

Let n be sufficiently large to apply Lemma 1.1.2 with constant ϵ to get large clusters and let R be the corresponding reduced graph. Note that, when applying Lemma 1.1.2, there are at least $\frac{n}{\epsilon}$ clusters so $|R| \geq \frac{n}{\epsilon}$.

By Lemma 2.2.4, we may assume R is 2-connected. By Theorem 2.3.3, we know that $\sigma_2(R) \geq (1 - 2\delta - 4\epsilon)|R|$. Thus, we may apply Theorem 2.3.12 to obtain a cycle C of length at least $(1 - 2\delta - 4\epsilon)|R|$ in R.

Color the edges of C with red and blue such that no two red edges are adjacent and, as few blue edges as possible are adjacent. Note that if C is even, the colors will alternate and if C is odd, there will be only one consecutive pair of blue edges while all others are alternating. Apply Lemma 1.1.6 on the pairs of clusters in G corresponding to the red edges of R to obtain super-regular pairs where the two sets of each super-regular pair have the same order. All vertices discarded in this process are added to the garbage set. Note that we have added at most ϵn vertices to the garbage set.

If C is odd, let c_0 be the vertex with two blue edges, let C_0 be the corresponding cluster and let C_0^+ and C_0^- be the neighboring clusters. Since the pairs (C_0^-, C_0) and (C_0, C_0^+) are both large and ϵ-regular, there exists a set of k vertices $T_0 \subseteq C_0$ with a matching to each of C_0^- and C_0^+. We will use these vertices as transportation and move all of $C_0 \setminus T_0$ to the garbage set.

Let G_C denote the set of vertices remaining in clusters associated with C that have not been moved to the garbage set and let D denote the garbage set. Note that $|D| \leq (2\delta + 6\epsilon)n$.

By Lemma 2.2.2, we may assume $\delta(G) \geq \frac{n_k}{8}$. In particular, the vertices in D each have at least $\frac{n_k}{8} - (|D| - 1) \gg \epsilon n$ edges to G_C.

A path is said to *balance* the super-regular pairs in G_C if for every super-regular pair the path visits, it uses an equal number of vertices from each set in the pair. Note that the removal of a balancing path preserves the fact that if a pair of clusters is super-regular, then the two clusters have the same order. Let (A, B) be a super-regular pair of clusters on C. A balancing path starting in A and ending in B which contains at least one vertex $v \in D$ is called *v-absorbing*.

Claim 2.3.13. *Avoiding any selected set of at most ϵr clusters and any set of at most $\frac{16(2\delta+6\epsilon)n}{\epsilon r}$ vertices in each of the remaining clusters, there exists a v-absorbing path of order at most* 16*. Otherwise, the desired path partition already exists.*

Proof of Claim 2.3.13. If for all selected vertices in the garbage cluster there exists an absorbing path then we are done. We create the most possible absorbing paths for selected vertices in X_0.

Absorbing paths are constructed iteratively, one for each vertex of D, in an arbitrary order. Suppose some number of such absorbing paths has been created. If we have created one for each vertex of D within the restrictions of the claim, the proof is complete so suppose we have constructed at most $|D|-1$ absorbing paths. Clusters that have lost at least $\frac{16(2\delta+6\epsilon)n}{\epsilon r}$ vertices removed from consideration in following iterations.

Fact 2.3.14. If we have created $|D|-1$ such paths, at most ϵr clusters would have order at most $L-\frac{16(2\delta+6\epsilon)n}{\epsilon r}$.

Proof of Fact 2.3.14. Since each absorbing path constructed in this claim has order at most 16, we lose at most 16 vertices from G_C for each vertex of D. The result follows. $\square$

Let $v \in X_0$ such that there is no absorbing path for v of order at most 16. Since $d(v) \geq \frac{n_k}{8}$, v must have edges to at least $\frac{r}{8k}$ clusters. Let A and B be two clusters which are not already ignored to which v has at least one edge to at least one vertex that is not already in a path or an absorbing path. For convenience, we call two clusters X and Y a *couple* or *spouses* if X and Y are consecutive on C and the pair is super-regular.

The following facts are easily proven using using the structure we have provided and the lemmas proven before.

Fact 2.3.15. A and B are not a couple.

Otherwise, it would be trivial to produce a v-absorbing path.

We call the spouses of these two clusters A' and B', and define the following sets of clusters:

- $X_A := \{$all couples of clusters such that both clusters have an edge to A' in $R\}$,

- $X_B :=$ {all couples of clusters such that both clusters have an edge to B' in R}, and
- $X_C :=$ {all couples of clusters such that one spouse has an edge to both A' and B' in R}. In particular, let X'_C denote the clusters in X_C that are not the neighbors of A' and B'.

From the definitions, it is clear that these sets are disjoint since otherwise we could easily build a v-absorbing path of order at most 16. This fact follows from the fact that $\sigma_2(R) \geq (1 - 2\delta - 4\epsilon)|R|$.

Fact 2.3.16. There are at most $(2\delta - 4\epsilon)|R|$ clusters in C which are in none of X_A, X_B and X_C.

Since we are assuming there is no short v-absorbing path, we may also exclude several other edges from R.

Fact 2.3.17. There are no edges of the following form in R:

- from a cluster in X_A to a cluster in X_B,
- from a cluster in $X_A \cup X_B$ to cluster in X'_C, or
- between two clusters in X'_C.

If both $X_A \cup X_B$ and X_C are large, then the clusters in X'_C could not have enough edges to satisfy the degree sum condition. Thus, we get the following fact.

Fact 2.3.18. At least one of $X_A \cup X_B$ or X_C contains very few clusters.

By Fact 2.3.18, if X_C is large, then it must contain most of C. In this case, R contains an almost-spanning almost-bipartite subgraph with X'_C in one part and $X_C \setminus X'_C$ in the other. We may then apply Lemma 2.2.3 to obtain the desired result. Thus, we get this fact.

Fact 2.3.19. The set X_C contains very few clusters.

This means $X_A \cup X_B$ cover almost all of the clusters in C, and therefore almost all of G. If they are both large, then we may apply Lemma 2.2.4 since there are no edges between X_A and X_B. Thus, the following fact is immediate.

Fact 2.3.20. One of X_A or X_B must be very small.

Without loss of generality, suppose X_A is very small, meaning that X_B covers almost all of the clusters in C. This means that vertices in A' have almost no edges out, contradicting the minimum degree condition on R that follows from Lemma 2.2.2 and completing the proof of Claim 2.3.13. □

For each desired vertex x_i, if $x_i \notin G_C$, use Menger's Theorem to construct a shortest path to a vertex, say x_i', in G_C. Using an edge of a super-regular pair first, construct a balancing path from x_i' through every cluster of G_C. Note that, since the pairs are either ϵ-regular or (ϵ, δ)-super-regular, using Lemma 2.3.11, this path can be constructed to use at most 3 vertices from each cluster.

By Claim 2.3.13, since $|D| \leq (2\delta + 6\epsilon)n$, we can construct an absorbing path for each vertex $v \in D$ where these paths are all disjoint. Let P^v be an absorbing path for v with ends of P^v in clusters C_i and C_{i+1}. Suppose uw is the edge of P_k from C_i to C_{i+1}. Then using Lemma 2.3.11, we can replace the edge uw with the path P^v with the addition of at most 4 extra vertices at either end. By this process, all of D can be absorbed into P_k. This makes $|P_k|$ larger but since $|D| \leq (2\delta + 6\epsilon)n$ and each path P^v has order at most 16, we get $|P_k| \leq 3|C| + 16(2\delta + 6\epsilon)n < n_k$.

For each i with n_i small, absorb only a few vertices from each super-regular pair until P_i has the desired order. For each remaining index i, absorb entire super-regular pairs at a time (with possibly a few vertices from other super-regular pairs) until P_i has the desired order to complete the proof. □

Chapter 3

H-Linked Graphs

In this chapter, we prove that large graphs with high minimum degree contain H-subdivisions whose paths have prescribed length and have specified endpoints. To give a brief history of H-linked graphs and to state our results, we must first state some definitions.

3.1 Introduction

Unless otherwise noted, H refers to a multigraph with at least one edge, and G refers to a simple graph. Denote multiedges in H by (uv, i), where u and v are the endpoints, and t is the indexing integer. We may refer to multiedges in H simply as edges when the meaning is unambiguous. Let $|H| = |V(H)|$ and $e(H) = |E(H)|$. Let "$\hookrightarrow$" denote that a map is injective, and let $\mathcal{P}(G)$ be the set of all paths in G. By an *embedding* of H into G, we mean a pair of maps $(f : V(H) \hookrightarrow V(G), g : E(H) \to \mathcal{P}(G))$ that maps all edges $(uv, i) \in E(H)$ to edge-disjoint $f(u), f(v)$-paths in G. If the embedding (f, g) maps edges in $E(H)$ to internally (vertex-) disjoint paths in G, then we say the embedding, or corresponding subgraph of G, is called an *H-subdivision* in G (see Figure 3.1). We say image(f) is the set of *ground vertices* and image(g) is the set of *edge-paths* in G.

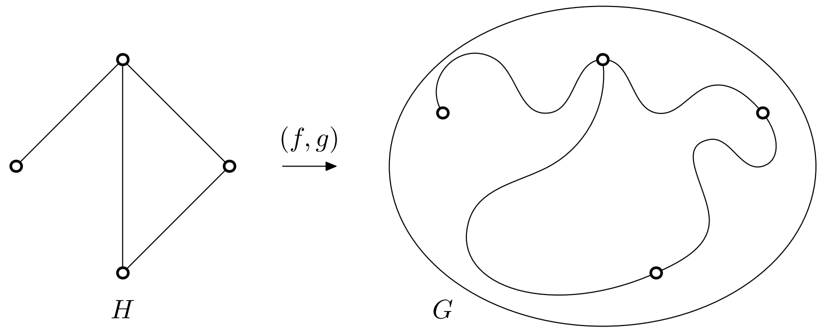

Figure 3.1: (f, g) is an H-subdivision in G.

Given a multigraph H, a graph G is H*-linked* if every map $f : V(H) \hookrightarrow V(G)$ can be extended into an H-subdivision in G (see Figure 3.2). Jung [24] first developed the concept of H-linked graphs, but it was not until Whalen [39] along with Ferrara, Gould, and Tansey [16] that specific criteria were established for a graph to be H-linked. At the same time, Kostochka and Yu proved [29] similar results about H-linked graphs for slightly fewer multigraphs H but significantly smaller graphs G. However, both sets of authors actually discovered the same sharp minimum degree condition for a large graph to be H-linked. Gould, Kostochka, and Yu combined their results in [17] to show that graphs of reasonably large order with minimum degree slightly higher than $\frac{|G|}{2}$ are H-linked. We specify the lower bounds on $|G|$ and $\delta(G)$ below.

Many of these same authors expanded upon the idea of H-linkage and considered the total number of edges of all H-subdivisions on specified vertices in G. In particular, letting h_0 and h_1 be the number of vertices in H with degree 0 and 1, respectively, Gould and Whalen showed in [18] that if a graph G of order n and minimum degree

$$\delta(G) \geq \frac{n + e(H) - |H| + h_1 + 2h_0}{2} \tag{3.1}$$

contains an H-subdivision $\mathcal{H}$, then G contains a spanning H-subdivision on the same set of ground vertices as $\mathcal{H}$. The bound on $\delta(G)$ is sharp. Ferrara, Magnant, and Powell in turn expanded upon this and proved a similar result for H-subdivisions in G of all sizes. In [15], Ferrara et al. showed that a sufficiently large graph G

satsifying (3.1) is *pan-H-linked*; i.e., contains H-subdivisions of all possible sizes on the same set of specified ground vertices for multigraphs H of average degree at least four. The bound in (3.1) is sharp for this result as well. In a sense, this contrasts the result in [17], which guaranteed the existence of a small H-subdivision in G. The use of (3.1) in both [18] and [15] gave rise to the question [5] of whether the sharp bound in [17] could be extended to show that sufficiently large graphs G are H-linked with varying H-subdivision sizes. Theorem 3.2.2 answers this question in the affirmative for large graphs G.

Given a multigraph H and an integer sequence $w = \{w_e \mid e \in E(H), w_e \geq 2\}$, a graph G is *(H, w, d)-linked* if every map $f : V(H) \hookrightarrow V(G)$ can be extended into an H-subdivision (f, g) in G such that each path $g(e)$ has length within d of w_e. We require $w_e \geq 2$ for all $w_e \in w$, for if even one $w_e = 1$, then G is (H, S, w)-semi-linked if and only if G is complete. If $d = 0$, then we say G is *(H, w)-linked*.

We assign weights to the *lengths* of edge-paths in our desired H-subdivions, and not the orders, as we did in Chapter 2 for Theorem 2.2.1. This is for two reasons. The first and most important reason is for convenience. Since we are specifying all endvertices for all desired edge-paths in an H-subdivision in G, it is easier to count edges between endvertices than multiply count the same endvertices of different edge-paths. Secondly, other similar terms (such as pan-H-linkage) also reference edge-path lengths instead of orders. No doubt they do this for convenience's sake as well.

For all sets w with each value $w_e \in w$ at least 14, we establish a sharp minimum degree condition for a large graph G to be $(H, w, 1)$-linked. The value 14 is used in the results solely for technical reasons (see Lemmas 3.3.7 and 3.3.8, along with the proof of Lemma 3.4.1) and most likely is not sharp. We hope to improve on this lower bound in the future.

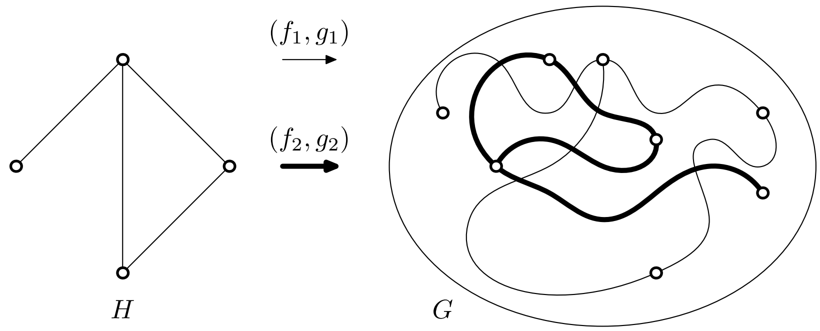

Figure 3.2: G is H-linked.

If G is (H, w, d)-linked for specific H and w, then we can choose the specific length of each edge-path for a given H-subdivision. While this is similar to the idea of pan-H-linked graphs, there are significant differences as well. Namely, pan-H-linkage does not specify lengths of individual edge-paths in G. On the other hand, pan-H-linkage implies an H-subdivision in G can be extended to span G. As a result, Theorems 3.2.1 and 3.2.2 are neither stronger nor weaker than Theorem 6 in [15].

In order to state our main results, we require a definition from [17]. Let $B(H)$ be the number of edges in a maximum edge-cut of H and let $c(H)$ be the number of components in H not containing an even cycle. Let

$$b(H) = \begin{cases} |H| - 1 & \text{H contains no even cycles,} \\ B(H) + c(H) & \text{otherwise.} \end{cases}$$

The authors of [17] showed that given a multigraph H, a graph G of order $n \geq 9.5(e(H)+c(H)+1)$ is H-linked if G satisfies the sharp condition $\delta(G) \geq \left\lceil \frac{n+b(H)}{2} \right\rceil - 1$. Note that this is the same as $\delta(G) \geq \frac{n+b(H)-2}{2}$, the same sharp bound we use in Theorem 3.2.1. Theorem 3.2.1 shows that given an integer sequence w with values all at least 14, a sufficiently large graph G with this same sharp minimum degree condition is also $(H, w, 1)$-linked. If we also consider $e(H)$ and the number of isolated vertices in H, we get a similar sharp lower bound for $\delta(G)$ when establishing (H, w)-linkage in G.

3.2 Minimum Degree Criteria for $(H, w, 1)$- and (H, w)-Linkage

We now state our main results.

Theorem 3.2.1. *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. If G is a graph of order $n \geq n(H, w)$ with $\delta(G) \geq \frac{n+b(H)-2}{2}$, then G is $(H, w, 1)$-linked. Furthermore, the lower bound for $\delta(G)$ is sharp.*

The sharpness of Theorem 3.2.1 is established in [17].

Theorem 3.2.2. *Let H be a multigraph with h_0 isolated vertices, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. If G is a graph of order $n \geq n(H, w)$ with $\delta(G) \geq \max\left\{\frac{n+b(H)-2}{2}, \frac{n+e(H)+h_0}{2}\right\}$, then G is (H, w)-linked. Furthermore, the lower bound for $\delta(G)$ is sharp.*

For Theorem 3.2.2, when $b(H)-2 \geq e(H)+h_0$ the example in [17] also establishes the sharpness of the bound for $\delta(G)$. If instead $b(H) - 2 < e(H) + h_0$, then the following example shows the sharpness of $\delta(G) \geq \frac{n+e(H)+h_0}{2}$.

Example 3.2.3. Let H be a multigraph with h_0 isolated vertices. Let H_0 denote the set of isolated vertices in H. Let G be a complete tripartite graph on n vertices with independent sets A, B, and A' satisfying

$$\begin{aligned} |A| &= \left\lceil \frac{n - (e(H) + h_0 - 1)}{2} \right\rceil \\ |B| &= \left\lfloor \frac{n - (e(H) + h_0 - 1)}{2} \right\rfloor \\ |A'| &= e(H) + h_0 - 1. \end{aligned}$$

As a result, we have $\delta(G) = \left\lceil \frac{n+e(H)+h_0-2}{2} \right\rceil$. Although G is complete tripartite, note that $G \setminus E(A', A)$ and $G \setminus E(A', B)$ are bipartite. In fact, the graph G is complete $\left(\frac{|B||A'|}{|A||A'|+|A||B|+|B||A'|}\right)$-almost-bipartite.

Consider a map $f : V(H) \hookrightarrow V(G)$ that is defined as follows: if $v \in H_0$, then let $f(v) \in A'$; otherwise, let $f(v) \in B$. Let w be a sequence of order $e(H)$ consisting of only odd integers. In order to create an H-subdivision (f, g) in G where each edge-path has odd length, we need to create paths of odd length between the ground vertices in B. It follows that each edge-path must use at least one vertex of $A' \setminus f(H_0)$. However, since $|A' \setminus f(H_0)| = e(H) - 1$, we cannot construct all of the edge-paths in our H-subdivision with the correct parity. Therefore, G is $(H, w, 1)$-linked but not (H, w)-linked. This shows that the condition $\delta(G) \geq \frac{n+e(H)+h_0}{2}$ in Theorem 3.2.2 is sharp. See Figure 3.3 for an example with $w = \{3, 3, 3, 3\}$.

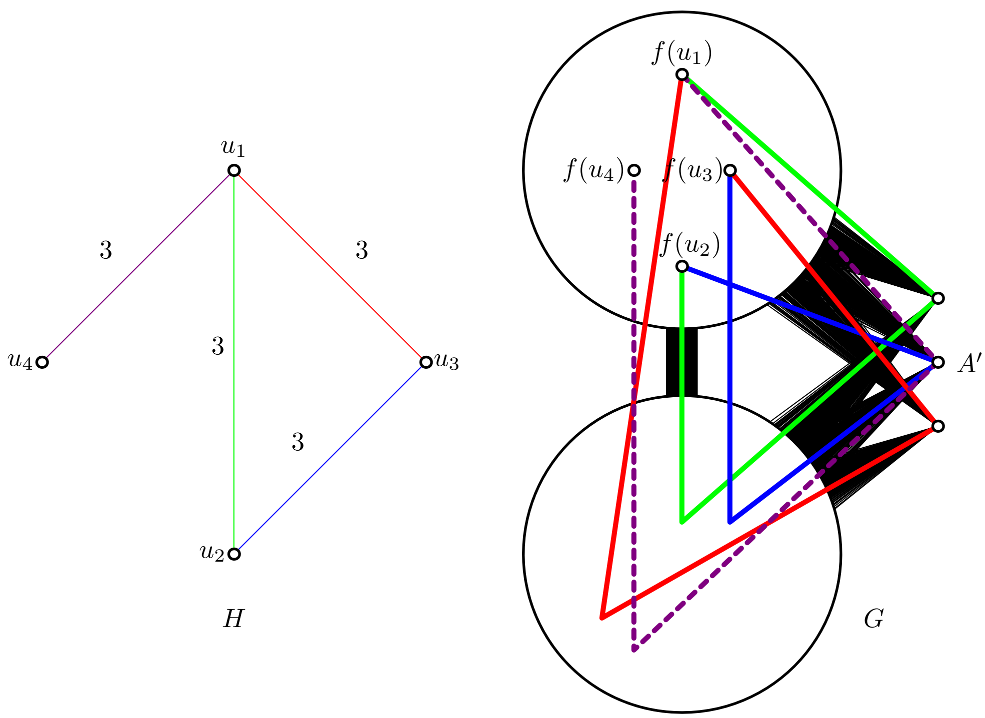

Figure 3.3: Given H and $w = \{3, 3, 3, 3\}$, the graph $G = A \cup B \cup A'$ is not (H, w)-linked.

In Section 3.3 we prove necessary results for Lemmas 3.4.1-3.4.4, which in turn constitute the proofs of Theorems 3.2.1 and 3.2.2.

3.3 Preliminaries

In this section, we frequently apply all definitions and lemmas from Section 1.1. Before we prove Theorems 3.2.1 and 3.2.2, we state and prove several lemmas. Throughout our proofs, we assume the relationships in (1.1) and that ϵn is an integer.

3.3.1 Reduced Graph with High Minimum Degree

The following lemma provides a minimum degree condition on the reduced graph.

Lemma 3.3.1. *If a graph G satisfies $\delta(G) \geq \frac{n}{2}$, then for fixed $\epsilon > 0$ and $\delta > 0$, the reduced graph $R = R(G, \epsilon, \delta)$ has minimum degree condition*

$$\delta(R) \geq \left(\frac{1}{2} - (\delta + 2\epsilon)\right) r.$$

Proof. The proof follows from a simple edge-counting argument. Fix ϵ and δ, and apply Lemma 1.1.2 on G to create the subgraph $G' = G'' \cup V_0$ and the reduced graph R. Since $\delta(G) \geq \frac{n}{2}$, we have $\delta(G') > \left(\frac{1}{2} - (\delta + \epsilon)\right) n$. From $|V_0| \leq \epsilon n$, it follows that

$$\delta(G') \geq \left(\frac{1}{2} - (\delta + 2\epsilon)\right) n.$$

Using Item 6 of Lemma 1.1.2, we see that if a vertex $v \in V_i$ is adjacent to vertices in another cluster V_j, then $d(V_i, V_j) > \delta$, and hence $v_i v_j \in E(R)$. Since each cluster in G'' has order L,

$$\delta(R) \geq \frac{\delta(G)}{L} \geq \left(\frac{1}{2} - (\delta + 2\epsilon)\right) r.$$

□

Our next lemma shows the existence of a triangle in the reduced graph.

Lemma 3.3.2. *Let G be a graph of order n with $\delta(G) \geq \frac{n}{2}$, and let $\epsilon, \delta > 0$ be small. If $R = R(G, \epsilon, \delta)$ is not bipartite, then R contains a triangle.*

Proof. The proof follows by considering a shortest odd cycle and using the degree assumption to force the cycle to be shorter than its assumed length. Since R is not bipartite, it contains an odd cycle. Consider a shortest odd cycle $C = \{c_1, \ldots, c_r, c_1\}$ in R, and suppose for a contradiction that $r \geq 5$. Furthermore, C is chosen to be a shortest odd cycle and hence contains no chords. We must have $N(c_i) \cap N(c_{i+1}) = \emptyset$ (where indices are taken modulo r), or else we have a triangle. Recall $r = |R|$. Letting $\gamma = (\delta + 2\epsilon)$, by Lemma 3.3.1, we have $|N(c_i)| \geq \left(\frac{1}{2} - \gamma\right) r$ for all i. There are at most

$$r - \left(\frac{1}{2} - \gamma\right) r = \left(\frac{1}{2} - \gamma\right) r + 2\gamma r$$

other vertices in R, with at least $\left(\frac{1}{2} - \gamma\right) r$ of them in $N(c_{i+1})$. It follows that we have at most $2\gamma r$ vertices in $R \setminus (N(c_i) \cup N(c_{i+1}))$. Hence, $|N(c_i) \cap N(c_{i+2})| \geq \left(\frac{1}{2} - 3\gamma\right) r$. In particular, we have $|N(c_1) \cap N(c_3)| \geq \left(\frac{1}{2} - 3\gamma\right) r$ and $|N(c_3) \cap N(c_5)| \geq \left(\frac{1}{2} - 3\gamma\right) r$.

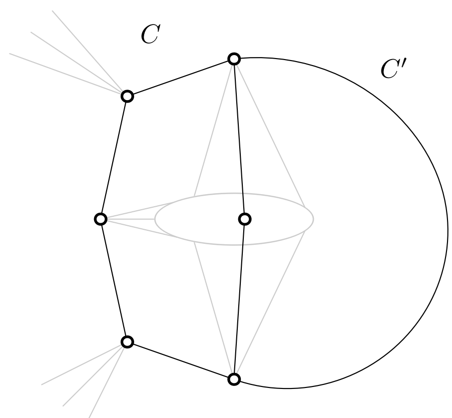

Figure 3.4: C' is a shorter cycle than C.

This implies $|N(c_1) \cap N(c_5)| \geq \left(\frac{1}{2} - 7\gamma\right) r$ for sufficiently small ϵ and d. Letting u be a vertex in $N(c_1) \cap N(c_5)$, we have the odd cycle $C' = \{c_1, u, c_5, \ldots, c_r, c_1\}$. If $r \geq 7$, then C' is a shorter odd cycle than C; if $r = 5$, then C' is the desired triangle (see Figure 3.4). Either way, we have a contradiction to the choice of C. Hence, R must contain a triangle. $\square$

3.3.2 Component Structure When $\delta(G) \geq \frac{n}{2}$ for Large n

The following fact provides upper and lower bounds on the sizes of components of a graph after a minimum cutset is removed.

Fact 3.3.3. Let G be a graph of order n with $\delta(G) \geq \frac{n}{2}$, and let S be a minimum cutset of G. It follows that $G = A \cup S \cup B$, where there are no edges between A and B, and we have

$$\frac{n}{2} - |S| + 1 \leq |A| \leq \frac{n}{2},$$

with the same upper and lower bounds for $|B|$.

This next lemma establishes a lower bound for $\kappa(G)$ when the reduced graph is connected.

Lemma 3.3.4. *Suppose G is a graph of sufficiently large order n with $\delta(G) \geq \frac{n}{2}$, and let $\epsilon, \delta > 0$ be small. If $R(G, \epsilon, \delta)$ has order r and is connected, then G is $\left(\frac{\epsilon(1-\epsilon)}{r}n\right)$-connected.*

Proof. The proof follows from an easy but technical contradiction argument. Apply Lemma 1.1.2 on G to obtain $G' = G'' \cup V_0$. We first prove G'' is $\left(\frac{\epsilon(1-\epsilon)}{r}n\right)$-connected and then extend this to G' and G.

Suppose $\kappa(G'') < \left(\frac{\epsilon(1-\epsilon)}{r}\right) n$ and let S be a minimum cutset of G''. Since $\delta(G) \geq \frac{n}{2}$, we know from Lemma 1.1.2 that $\delta(G'') > \left(\frac{1}{2} - (\delta + 2\epsilon)\right) n$. By Fact 3.3.3, $G'' = A \cup S \cup B$ with

$$\left(\frac{1}{2} - (\delta + 2\epsilon)\right) n - |S| + 1 \leq |A| \leq \left(\frac{1}{2} + (\delta + 2\epsilon)\right) n - 1, \text{ and}$$
$$\left(\frac{1}{2} - (\delta + 2\epsilon)\right) n - |S| + 1 \leq |B| \leq \left(\frac{1}{2} + (\delta + 2\epsilon)\right) n - 1.$$

From Lemma 1.1.2, we have $|V_0| \leq \epsilon n$ and $|V_i| = L \geq \left(\frac{(1-\epsilon)}{r}\right) n$ for all $1 \leq i \leq r$. Note, however, that a cluster in G'' could have vertices in A, B, and S.

Claim 3.3.5. *For all i satisfying $1 \leq i \leq r$, we have either*

$$|V_i \cap A| < \epsilon L, \text{ or}$$
$$|V_i \cap B| < \epsilon L.$$

Proof. Let $V_1^A = V_1 \cap A$ and $V_1^B = V_1 \cap B$. Suppose without loss of generality that $|V_1^A| \geq \epsilon L$ and $|V_1^B| \geq \epsilon L$. From the minimum degree condition on G'', there must be some cluster V_2 such that (V_1, V_2) forms an ϵ-regular pair with $d(V_1, V_2) > \delta$. Since $|S| < \left(\frac{\epsilon(1-\epsilon)}{r}\right) n < (1-\epsilon)\epsilon L$, we must have more than ϵL vertices of V_2 in either A or B. Without loss of generality, we will assume these vertices are in A. Call this set of vertices V_2^A. The pair (V_1^B, V_2^A) has no edges, contradicting the (ϵ, δ)-regularity of (V_1, V_2) (see Figure 3.5). Hence, either $|V_1 \cap A| < \epsilon L$ or $|V_1 \cap B| < \epsilon L$. $\square$

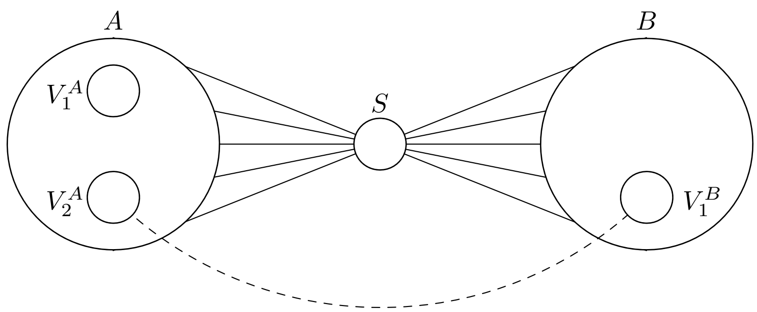

Figure 3.5: There are no edges between V_2^A and V_1^B.

The statement of Claim 3.3.5 is equivalent to saying that for all i satisfying $1 \leq i \leq r$, we have either

$$|V_i \cap A| > (1-\epsilon)L \tag{3.2}$$

or

$$|V_i \cap B| > (1-\epsilon)L. \tag{3.3}$$

Since R is connected and $r \geq 2$, there must be a cluster V_i satisfying (3.2) and a cluster V_j satisfying (3.3) such that (V_i, V_j) is an ϵ-regular pair. However, the pair (V_1^A, V_2^B) has no edges, contradicting the (ϵ, δ)-regularity of (V_1, V_2). This contradicts the bound on $|S|$, meaning that G'' is $\left(\frac{\epsilon(1-\epsilon)}{r}\right) n$-connected.

Now consider $G' = G'' \cup V_0$. Every vertex in V_0 has at least $\frac{n}{2} - \epsilon n$ edges into G''. Since the addition of a k, J-star to a k-connected graph J yields a k-connected graph, we can add each vertex of V_0 to G'' preserving the connectivity. It follows that G', and therefore G as well, is $\left(\frac{\epsilon(1-\epsilon)}{r}\right) n$-connected. $\square$

3.3.3 Proxy Vertices

Let $G = A \cup C \cup B$ be a graph with a minimum cutset C, and components A and B. Assume $|C| \leq \frac{1}{2}|A|, \frac{1}{2}|B|$. For each $c \in C$, let A_c denote a set of neighbors of c in A. Define B_c similarly. Call $A_c \cup B_c$ the set of *proxy* vertices of c. Let

$$A_p = \bigcup_{c \in C} A_c$$

be the set of all proxy vertices in A, and define B_p symmetrically.

The next fact establishes the existence of proxy vertices to represent the vertices in a minimum cutset.

Fact 3.3.6. Given a positive integer k and an injective map $f : V(H) \hookrightarrow V(G)$, if $G = A \cup C \cup B$ is a graph of sufficiently large order $n = n(k, f)$ with $\delta(G) \geq \frac{n}{2}$ and connectivity at most $\frac{n}{6}$, then for each $c \in C$, there exist sets $A_c \subset A$ and $B_c \subset B$ satisfying

1. $(A_{c_1} \cup B_{c_1}) \cap (A_{c_2} \cup B_{c_2}) = \emptyset$,
2. $(A_c \cup B_c) \cap f(V(H)) = \emptyset$,
3. $|(A_c \cup B_c)| \geq k$,
4. $|A_c|, |B_c| \geq 1$,
5. $|A_p|, |B_p| < \frac{n}{100}$.

We let $k = e(H)$ when citing Fact 3.3.6, and we let the map f be a vertex map of an H-subdivision in G.

The following lemma provides many short paths between every pair of vertices in G.

Lemma 3.3.7. *Given $\lambda > 0$, every graph G with $\kappa(G) \geq \lambda n$ and $\delta(G) \geq \frac{n}{2}$ has at least* $\min\left\{\frac{n}{24} - 2, \lambda n - 2\right\}$ *internally disjoint paths, each of length at most 6, between every pair of vertices.*

Proof. If $\kappa(G) \geq \frac{n}{3}$, then the average path length in G is at most 4. By Menger's Theorem, we know G contains at least $\frac{n}{3}$ internally disjoint paths between every pair of vertices in G. This means that at least $\frac{n}{9}$ of these paths have length at most 6.

Now suppose $\kappa(G) \leq \frac{n}{3}$, and let S be a minimum cutset of G. By the minimum degree condition on G, there exist at most 4 components of $G \setminus S$, each having at least $\frac{n}{6}$ vertices. Let A and B be two components of $G \setminus S$. Without loss of generality, for every $y, z \in G$, we have

1. $|N(y) \cap A|, |N(z) \cap A| \geq \frac{n}{24}$, and/or
2. $|N(y) \cap A|, |N(z) \cap B| \geq \frac{n}{24}$.

Case 1. $|N(y) \cap A|, |N(z) \cap A| \geq \frac{n}{24}$.

By assumption, we have $|N(y)|, |N(z)| \geq \frac{n}{2}$. Since $|A \cup S| \leq \frac{5n}{6}$, we have $|N(y) \cap N(z)| \geq \frac{n}{6}$. There are at least $\frac{n}{24}$ internally disjoint y, z-paths passing through $N(y) \cap N(z)$.

Case 2. $|N(y) \cap A| \geq \frac{n}{24}$ *and* $|N(z) \cap B| \geq \frac{n}{24}$

First suppose $|A| \geq |S|$ and $|B| \geq |S|$, and then define A_p and B_p as in Fact 3.3.6. For each proxy vertex $a \in A_p$, we have

$$|N(a) \cap N(y)| \geq \frac{n}{6}.$$

A similar argument shows $|N(b) \cap N(z)| \geq \frac{n}{6}$ for each $b \in B_p$ as well. For $\sigma \in S$, consider proxy vertices $a_\sigma \in A_p$ and $b_\sigma \in B_p$. Letting $a_{\sigma y}$ and $b_{\sigma z}$ denote vertices in $N(a_\sigma) \cap N(y)$ and $N(b_\sigma) \cap N(z)$, respectively, we have the path $\{y, a_{\sigma y}, a_\sigma, \sigma, b_\sigma, b_{\sigma z}, z\}$. It follows that G contains at least $\min\left\{\frac{n}{24}, \lambda n - 2\right\}$ internally disjoint y, z-paths of length 6 passing through $N(y) \cap N(A_p)$, A_p, S, B_p, and $N(z) \cap N(B_p)$.

Now suppose without loss of generality $|B| < |S|$. First assume $|A| \geq |S|$. Letting A_p be the set of proxy vertices in A, we have

$$\frac{n}{6} \leq |B| < |S| = |A_p| \leq \frac{n}{3} < |A| < \frac{n}{2}.$$

We also know

$$|N(y) \cap N(a)| \geq \frac{n}{6} \text{ for each } a \in A_p,$$
$$|N(z) \cap S| \geq \frac{n}{2} - |B| \geq \frac{n}{6}.$$

Then G contains at least $\frac{n}{24}$ internally disjoint y, z-paths of length 4 passing through $N(y) \cap N(A_p)$, A_p, and $N(z) \cap S$. Now assume $\frac{n}{6} \leq |A| < |S|$. We know

$$|N(y) \cap S| \geq \frac{n}{2} - |A| > \frac{n}{6},$$
$$|N(z) \cap S| \geq \frac{n}{2} - |B| > \frac{n}{6}.$$

Then we have at least $\lambda n - 2$ internally disjoint y, z-paths of length 2 passing though S. $\square$

The next lemma is similar to the well-known Fan Lemma, providing many short paths between a vertex and a set.

Lemma 3.3.8. *Let G be a graph with at least ξn internally disjoint paths of length at most 6 between each pair of vertices. Let $v \in V(G)$, and let W be a set of $\gamma n = \left\lfloor \frac{\xi n + 5}{6} \right\rfloor$ vertices in $G \setminus \{v\}$. There are γn paths, each of length at most 6, from v to W that are disjoint except for v.*

Proof. Let v be a vertex in G, and let $W = \{w_1, \dots, w_{\gamma n}\}$ be a set of γn vertices in $G \setminus \{v\}$. By assumption, we know there exist at least ξn paths, each of length at most 6, from v to each vertex in W. For each $w_i \in W$, let $\mathscr{P}_i = \{P \mid P \text{ is a } v, w_i\text{-path of length at most } 6\}$. We use an inductive construction to build the desired set of paths $\{P_1, P_2, \dots, P_{\gamma(n)}\}$. For P_1, choose any v, w_1-path in $\mathscr{P}_1$.

Now suppose we have selected $t < \gamma n$ paths from v to $\{w_1, \dots, w_t\}$. From Lemma 3.3.7, we know $|\mathscr{P}_{t+1}| \geq \xi n$. There are a total of at most $6t$ vertices (not counting v) used in the paths $P_1, \dots, P_t$. Each such vertex could be in at most one path in $\mathscr{P}_{t+1}$. Hence, there are at least $\xi n - 6t$ paths in $\mathscr{P}_{t+1}$, each of length at most 6, that do not intersect $\{P_1, P_2, \ldots, P_t\}$. Since we have

$$\begin{aligned}\xi n - 6t &\geq \xi n - 6\left(\left\lfloor \frac{\xi n + 5}{6} \right\rfloor - 1\right) \\ &\geq \xi n - (\xi n + 5 - 6) \\ &= 1,\end{aligned}$$

there is at least one path in $\mathscr{P}_{t+1}$ that is internally disjoint from $P_1 \cup P_2 \cup \cdots \cup P_t$. Setting P_{t+1} to be this path completes the induction step of the construction. Thus, there are at least γn internally disjoint paths, each of length at most 6, from v to W. $\square$

3.3.4 Structure of G when R is Disconnected

When using the following results in the proofs of Theorems 3.2.1 and 3.2.2, we will let $b = b(H) - 2$ or $b = \max\{b(H) - 2, e(H) + h_0\}$, depending on the situation.

When R is disconnected, the next lemma provides a bipartition of G into almost equal vertex sets with few edges in between.

Lemma 3.3.9. *Let b be a positive integer, let $\epsilon, \delta > 0$, and let G be a graph G order $n \geq n(\epsilon, \delta, b)$ with $\delta(G) \geq \frac{n+b}{2}$ and disconnected reduced graph $R = R(G, \epsilon, \delta)$ This implies there is a bipartition $G = A \cup B$ satisfying*

$$\frac{n+b}{2} - (\delta + 1.5\epsilon)n \leq |A|, |B| \leq \frac{n+b}{2} + (\delta + 1.5\epsilon)n, \tag{3.4}$$

and

$$e(A, B) < (d + 3\epsilon)n^2. \tag{3.5}$$

Proof. Let b, ϵ, δ, G and n be as given in the statement. Apply Lemma 1.1.2 to obtain the spanning subgraph $G' = G'' \cup V_0$. From Lemma 1.1.2, we have $d_{G'}(v) > \frac{n}{2} + b - (\delta + \epsilon)n$ for all n vertices $v \in G$. Since $|V_0| \leq \epsilon n$, it follows that

$$\delta(G'') > \frac{n+b}{2} - (\delta + 2\epsilon)n. \tag{3.6}$$

Since R is disconnected, we see that G'' must be disconnected as well. From (3.6), we see G'' has two components; call them C_1 and C_2. It follows that

$$\frac{n+b}{2} - (\delta + 2\epsilon)n + 1 < |C_i| < \frac{n+b}{2} + (\delta + 2\epsilon)n - 1$$

for $i = 1, 2$. Since G'' can have as many as n vertices, each having at most $(\delta + 2\epsilon)n$ edges in $G \setminus G''$, there must be fewer than $(\delta + 2\epsilon)n^2$ edges in G between C_1 and C_2.

From Lemma 1.1.2, we again note $|V_0| \leq \epsilon n$, which implies there are at most ϵn^2 edges in G incident to vertices of V_0. There at most $(\delta + 3\epsilon)n^2$ edges between the components C_1, C_2, and V_0. Now divide V_0 into two even sets, C_A and C_B. Create a bipartition of G by adjoining C_A with C_1 and C_B with C_2. Let

$$A = C_1 \cup C_A$$
$$B = C_2 \cup C_B.$$

Note $|C_A|, |C_B| \leq \frac{\epsilon}{2}n$. It follows that A and B have fewer than $(\delta + 3\epsilon)n^2$ edges between them, and that

$$\frac{n+b}{2} - (\delta + 1.5\epsilon)n \leq |A| \leq \frac{n}{2} + (\delta + 1.5\epsilon)n,$$

with the same bounds on $|B|$. $\square$

Although $(\delta + 3\epsilon)n^2$ may seem large, it is small compared to the minimum of $\frac{n^2}{4}$ edges in G. Also note that we may assume vertices in A have at least $\frac{n}{4}$ neighbors in A—otherwise, such a vertex should be put in B. A similar statement is true for vertices in B. This ensures that we have the smallest number of paths between A and B possible.

The next fact establishes an upper bound for the set of vertices in each of A and B that have many edges to the opposite set.

Lemma 3.3.10. *Let b be a positive integer, and let $\epsilon, \delta > 0$. Consider a graph G of sufficiently large order $n \geq n(\epsilon, \delta, b)$ with $\delta(G) \geq \frac{n+b}{2}$ and disconnected reduced graph $R(G, \epsilon, \delta)$. Bipartition G into $A \cup B$ such that A and B satisfy* (3.4) *and* (3.5). *Let D_A be the set of all vertices in A, each with more than $\frac{n}{100b^2}$ edges into B, and define D_B similarly. If $D = D_A \cup D_B$, then $|D| < 200b^2(\delta + 2\epsilon)n$.*

Proof. Let $b, \epsilon, \delta, G, n$ and D be as given in the statement. Consider a graph G of sufficiently large order $n \geq n(\epsilon, \delta, b)$ with $\delta(G) \geq \frac{n+b}{2}$ and disconnected reduced graph $R(G, \epsilon, \delta)$. Letting ξn be the number of vertices in D_A, there are at least $\xi n \cdot \frac{n}{100b^2}$ edges between D_A and B. However, from Lemma 3.3.9 we know

$$\frac{\xi}{100b^2} n^2 < (\delta + 3\epsilon) n^2.$$

Solving for ξ we get $\xi < 100b^2(\delta + 3\epsilon)$. Using the same logic for D_B, we have $|D| < 200b^2(\delta + 3\epsilon)n$. $\square$

3.3.5 Panconnected Induced Graphs in G

The following lemma, a generalization of Claim 5 in [16], shows that almost any reasonably-sized subset of A or B is panconnected.

Lemma 3.3.11. *Let b be a positive integer, and let $\epsilon, \delta > 0$. Consider a graph G of order $n \geq n(\epsilon, \delta, b)$ with $\delta(G) \geq \frac{n+b}{2}$ and disconnected reduced graph R. Bipartition G into $A \cup B$ such that A and B satisfy (3.4) and (3.5). Define $D = D_A \cup D_B$ as in Lemma 3.3.10. For each set $A' \subseteq A \setminus D_A$ and $B' \subseteq B \setminus D_B$ with $|A'|, |B'| \geq \frac{n}{10b^2}$, the graphs $G[A']$ and $G[B']$ are panconnected.*

Proof. This is immediate for $b \leq 2$. Consider $b > 2$. By Lemma 3.3.10 we know $\delta(A \setminus D_A) \geq \frac{n+b}{2} - (\delta + 1.5\epsilon)n - \frac{n}{100b^2} - 18b^2(\delta + 2\epsilon)n \geq \left(\frac{1}{2} - \frac{1}{50b^2}\right) n$. By symmetry, we know $\delta(B \setminus D_B) \geq \left(\frac{1}{2} - \frac{1}{50b^2}\right) n$ as well. It follows that we have

$$\begin{aligned}
\delta(A \setminus D_A) &\geq \left(\frac{1}{2} - \frac{1}{50b^2}\right) n \\
&= \frac{n}{2} - \frac{n}{50b^2} \\
&= \frac{|A| + |B|}{2} - \frac{n}{50b^2} \\
&\geq \frac{2|A| - \frac{n}{25b^2}}{2} - \frac{n}{50b^2} \\
&= |A| - \frac{n}{25b^2} \\
&\geq |A \setminus D_A| - \frac{n}{25b^2}.
\end{aligned}$$

Thus, given $A' \subseteq A \setminus D_A$ with $|A'| \geq \frac{n}{10b^2}$,

$$\begin{aligned}\delta(A') &\geq |A'| - \frac{n}{25b^2} \\ &\geq \frac{|A'|+2}{2} + \frac{n}{25b^2} - \frac{n}{25b^2} \\ &\geq \frac{|A'|+2}{2}.\end{aligned}$$

By Theorem 2.15, A' and likewise B' are panconnected. $\square$

We can also use the connectivity of such a graph G when $\kappa(G)$ is small. Consider a minimum cutset $S \subset G$ and the resulting components A and B that make up $G \setminus S$. The following lemma is analogous to Lemma 3.3.11 but deals with the partition $G = A \cup S \cup B$.

Lemma 3.3.12. *Let b be a positive integer, and let $\epsilon, \delta > 0$. Consider a graph G of order $n \geq n(\epsilon, \delta, b)$ with $\delta(G) \geq \frac{n+b}{2}$. Suppose in addition that $\kappa(G) \leq \frac{n}{6}$. Consider a minimum cutset $S \subset G$ and the resulting two components A and B that make up $G \setminus S$. For each set $A' \subseteq A$ (or $B' \subseteq B$) with $|A'| \geq \frac{3|A|}{4}$ (or $|B'| \geq \frac{3|B|}{4}$), the graphs $G[A']$ and $G[B']$ are panconnected.*

Proof. Letting $b, \epsilon, \delta, G, n, S, A$, and B be as given in the statement, we have $\frac{n}{3} \leq |A| \leq \frac{n}{2}$ and $\delta(G[A]) \geq \frac{2|A|}{3}$. It follows that for every set A' of order at least $\frac{3|A|}{4}$, we have

$$\begin{aligned}\delta(A') &\geq |A'| - \frac{n}{12} \\ &\geq \frac{|A'|+2}{2}.\end{aligned}$$

Similar logic is true for all $B' \subset B$ of order at least $\frac{3|B|}{4}$. By Theorem 2.15, A' and likewise B' are panconnected. $\square$

Note that this lemma does not require that $R(G, \epsilon, \delta)$ be connected. We use Lemma 3.3.12 to prove Lemma 3.4.2, where we assume a disconnected reduced graph for G.

3.3.6 Independent Edges when R is Bipartite

The next lemma provides either a large matching or several large stars.

Lemma 3.3.13. *If G is a graph of sufficiently large order n with $\delta(G) \geq k$, then either*

1. *there exist $2k$ independent edges in G, or*

2. *there exist k vertices of degree at least $\frac{n}{5k}$.*

Proof. Consider a graph G of sufficiently large order n with $\delta(G) \geq k$. If there exist $2k$ independent edges in G, then we are done, so suppose not. Consider the largest collection of $c < 2k$ independent edges, i.e., the largest perfect matching in G. Caling this set of edges M_c, we have $G = M_c \cup A$, where A must induce an independent set of $n - 2c$ vertices. Since $\delta(G) \geq k$, each vertex in A must be adjacent to k vertices in M_c. This means there are at least $k(n-2c)$ edges from A to M_c. Therefore, there are at least k vertices in M_c with degree at least $\frac{kn-2ck}{2c} = \left(\frac{k}{c}\right)\frac{n}{2} - k > \frac{n}{5k}$. $\square$

When G is close to being bipartite, it may be difficult to construct paths having length with the correct parity. For example, two vertices x and y in the same partite set of a bipartite graph can only have paths of even length between them, so all x, y-path lengths have even parity. All path lengths between a fixed pair of vertices in a bipartite graph must have the same parity. Since our main results involve specific path lengths between arbitrary pairs within any sufficiently large and dense graph (including potentially bipartite graphs), we sometimes will only be able to get within 1 of the desired length of a path.

Recall the definition of *bipanconnected* graphs and Lemma 1.2.2 from Chapter 2, as they will be used in the proof of Lemmas 3.4.3 and 3.4.4.

3.4 Proof Outline of Theorems 3.2.1 and 3.2.2

We prove Theorems 3.2.1 and 3.2.2 by applying Lemma 1.1.2 to G and then determining the structure of the reduced graph R. From there, we prove Lemmas 3.4.1–3.4.4, which combine to immediately imply Theorems 3.2.1 and 3.2.2.

The first lemma provides criteria for (H, w)-linkage in a sufficiently large graph with $\delta(G) \geq \frac{n}{2}$ and a connected, non-bipartite reduced graph, provided all integers in w are at least 14.

Lemma 3.4.1. *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. Let G be a graph of order $n \geq n(H, w)$ with $\delta(G) \geq \frac{n}{2}$ and reduced graph R. If R is connected and not bipartite, then G is (H, w)-linked.*

Our next lemma provides criteria for (H, w)-linkage in a sufficiently large graph G with $\delta(G) \geq \frac{n+b(H)-2}{2}$ whose reduced graph R is disconnected. Although we need a higher minimum degree here, we only need each prescribed path length to be at least 8.

Lemma 3.4.2. *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 8\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H, w)$ with $\delta(G) \geq \frac{n+b(H)-2}{2}$ and reduced graph R. If R is disconnected, then G is (H, w)-linked.*

Our next lemma provides criteria for $(H, w, 1)$-linkage in a sufficiently large graph with $\delta(G) \geq \frac{n}{2}$ and a bipartite reduced graph, provided all integers in w are at least 3.

Lemma 3.4.3. *Let H be a multigraph, and let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H, w)$ with $\delta(G) \geq \frac{n}{2}$ and reduced graph R. If R is bipartite, then G is $(H, w, 1)$-linked.*

Our final lemma provides criteria for (H, w)-linkage in a sufficiently large graph G with $\delta(G) \geq \frac{n}{2}$ and a bipartite reduced graph, provided all integers in w are at least 3.

Lemma 3.4.4. *Let H be a multigraph with h_0 isolated vertices, and let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H, w)$ with $\delta(G) \geq \frac{n+e(H)+h_0}{2}$ and reduced graph R. If R is bipartite, then G is (H, w)-linked.*

Lemmas 3.4.1–3.4.4 combine to prove Theorems 3.2.2 and 3.2.1.

Proof of Theorem 3.2.1. Use Lemmas 3.4.1, 3.4.2, and 3.4.3. □

Proof of Theorem 3.2.2. Use Lemmas 3.4.1, 3.4.2, and 3.4.4. □

It remains to be shown that a sufficiently large graph G is H-linked for H-subdivisions with prescribed distances each a fraction of $|G|$. Such a result could lead to a criterion for G to be H-linked for spanning H-subdivisions. We may use a proof technique similar to the ones above; in particular, we still would use the Regularity Lemma on G to determine its reduced graph. However, the Blow-Up Lemma only gives information about the existence of a small subgraph within blown-up clusters and would probably not be useful. Most likely, panconnectivity and bipanconnectivity would be the primary tools.

3.5 Proof of Theorems 3.2.1 and 3.2.2

In Lemmas 3.4.1-3.4.4, for a graph G, we assume appropriate choices of $\epsilon, \delta > 0$ to determine the reduced graph $R = R(G, \epsilon, \delta)$.

3.5.1 R is Connected and not Bipartite

Proof of Lemma 3.4.1. For a multigraph H, let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. Consider a sufficiently small $\delta_0 \in (0, 1)$. Choose parameters $\delta = \frac{\delta_0}{2}$ and $\Delta \gg \sum_{e \in E(H)} w_e$ as in Lemma 1.1.7. If T_1, T_2, and T_3 are independent sets, then let $\mathscr{B}(T_1, T_2, T_3)$ denote the complete tripartite graph on (T_1, T_2, T_3). By Lemma 1.1.7, there exists a value $\epsilon_0 = \epsilon_0\,(\delta, \Delta, 3)$ such that the following is true: if (T_1, T_2, T_3) is an (ϵ_0, δ)-super-regular triple on sufficiently many vertices, then (T_1, T_2, T_3) contains all subgraphs of maximum degree at most Δ that are contained in $\mathscr{B}(T_1, T_2, T_3)$. Since this result is true for all $\epsilon \leq \epsilon_0$, it suffices to choose $\epsilon \leq \epsilon_0$ that also satisfies $\epsilon \ll \delta_0$.

Apply Lemma 1.1.2 with parameters ϵ and δ_0 on the graph G of order $n \geq n(H, w, \epsilon, \delta_0)$ with $\delta(G) \geq \frac{n}{2}$ to obtain the reduced graph $R = R(G, \epsilon, \delta)$, which from Lemma 3.3.1 satisfies

$$\delta(R) \geq \left(\frac{1}{2} - (\delta + 2\epsilon)\right) r.$$

Suppose R is connected and not bipartite. From Lemma 3.3.2, we see that R must have a triangle, which implies the existence of a corresponding (ϵ, δ)-regular triple of clusters $(V_1, V_2, V_3) \subset G$. By Lemma 1.1.6, there exists an $(\epsilon, \delta - 2\epsilon)$-super-regular triple $(T_1, T_2, T_3) \subset (V_1, V_2, V_3)$. However, this implies that (T_1, T_2, T_3) is also an (ϵ, δ)-super-regular triple since $\delta < \delta_0 - 2\epsilon$.

By Lemma 1.1.7, there exists a complete tripartite graph $\mathscr{B}(X_1, X_2, X_3) \subset (T_1, T_2, T_3)$ with $|X_1| = |X_2| = |X_3| = \Delta$. Let $\gamma(n)$ be a sufficiently small fraction of n. Consider a map $f : V(H) \hookrightarrow V(G)$.

For each edge $e = (uv, s) \in E(H)$, do the following. Consider a set $\mathscr{X}_e \supset (X_1, X_2, X_3)$ of γn vertices. From Lemma 3.3.7, there exist both $f(u), X_e$- and $f(v), X_e$-fans, each consisting of γn internally disjoint paths of length at most 6. Choose a vertex in X_1 whose path in the $f(u), X_e$-fan has length b; denote this vertex by $x_{e,b+1}$. Similarly, choose a vertex in X_3 whose path in the $f(v), X_e$-fan has length c; denote this vertex by x_{e,w_e-c+1}. It follows that $w_e - b - c \geq 2$.

Case 1. $w_e - b - c \equiv 0 \mod 3$ *and is odd.*

Designate the vertex sets

- $\left\{x_{e,b+2m} \mid m = 1, \ldots, \frac{1}{2}(w_e - b - c - 1)\right\} \subset X_2$ and
- $\left\{x_{e,b+2m+1} \mid m = 1, \ldots, \frac{1}{2}(w_e - b - c - 1)\right\} \subset X_1$.

Case 2. $w_e - b - c \equiv 0 \mod 3$ *and is even.*

Designate vertices $x_{e,b+2} \in X_3$, $x_{e,b+3} \in X_2$, and $x_{e,b+4} \in X_1$, along with the vertex sets

- $\left\{x_{e,b+2m} \mid m = 2, \ldots, \frac{1}{2}(w_e - b - c - 1)\right\} \subset X_2$ and
- $\left\{x_{e,b+2m+1} \mid m = 2, \ldots, \frac{1}{2}(w_e - b - c - 1)\right\} \subset X_1$.

Case 3. $w_e - b - c \equiv 1 \mod 3$.

Designate vertex sets

- $\{x_{e,b+2+3m} \mid m = 0, \ldots, \frac{1}{3}(w_e - b - c - 4)\} \subset X_2$,
- $\{x_{e,b+3+3m} \mid m = 0, \ldots, \frac{1}{3}(w_e - b - c - 4)\} \subset X_3$, and
- $\{x_{e,b+4+3m} \mid m = 0, \ldots, \frac{1}{3}(w_e - b - c - 4\} \subset X_1$.

Case 4. $w_e - b - c \equiv 2 \mod 3$.

Designate the vertex sets

- $\{x_{e,b+2+3m} \mid m = 0, \ldots, \frac{1}{3}(w_e - b - c - 2)\} \subset X_3$,
- $\{x_{e,b+3+3m} \mid m = 0, \ldots, \frac{1}{3}(w_e - b - c - 5)\} \subset X_2$, and
- $\{x_{e,b+4+3m} \mid m = 0, \ldots, \frac{1}{3}(w_e - b - c - 5)\} \subset X_1$.

In each case, the path

$$\{f(u), \ldots, x_{e,b+1}, x_{e,b+2}, x_{e,b+3}, \ldots, x_{e,w_e-c-1}, x_{e,w_e-c}, x_{e,w_e-c+1}, \ldots, f(v)\}$$

has length w_e. See Figure 3.6 for an example of a length 5 edge-path construction in G. Also note that $\mathscr{B}(X_1, X_2, X_3)$ is large enough that we can choose each (w_i, w_j)-edge-path to be internally disjoint from all other such edge-paths. Hence, G is (H, w)-linked. $\square$

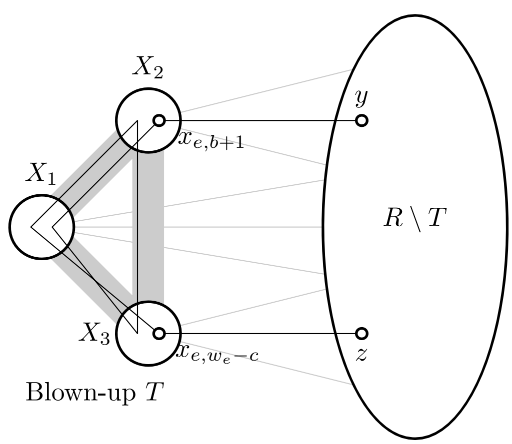

Figure 3.6: Construction of an $f(u), f(v)$-path.

There is little indication that the minimum degree assumption in the previous lemma is sharp.

3.5.2 R is Disconnected

Proof of Lemma 3.4.2. Let H be a multigraph, and let $\{w_e \mid e \in E(H), w_e \geq 8\}$ be a sequence of integers. Consider a vertex map $f : V(H) \hookrightarrow V(G)$. Consider a (simple) graph G of sufficiently large order n with $\delta(G) \geq \frac{n+b(H)-2}{2}$.

Apply Lemma 1.1.2 on G to obtain the reduced graph R of G. Since $\delta(G) \geq \frac{n+b(H)-2}{2}$, from Lemma 3.3.1 we have

$$\delta(R) \geq \left(\frac{1}{2} - (\delta + 2\epsilon)\right) r + \frac{b(H)-2}{2}.$$

We divide the remainder of the proof into two cases based on the connectivity of G. For each edge $e = (uv, s) \in E(H)$, let $y = f(u)$ and $z = f(v)$.

Case 1. $\kappa(G) \leq \frac{n}{6}$.

Let S be a minimum cutset of G, and let A and B be the components of $G \setminus S$. Recall the definition of the proxy vertex sets A_p and B_p from Fact 3.3.6. Given vertex sets $U, V \subset V(G)$, define a vertex set X in G to be (U, V)-*large* if

- $X \subset A$ or $X \subset B$,
- $|X| \geq \frac{n}{10b(H)^2}$,
- $X \cap f(V(H)) = U$, and
- $X \cap (A_p \setminus U) = V$ or $X \cap (B_p \setminus U) = V$.

By Lemma 3.3.12, the first two items guarantee that (U, V)-large sets are panconnected. The purpose of (U, V)-large sets is to construct paths of length 2 either between two proxy vertices or between a proxy vertex and y (or z). To construct all desired edge-paths in G, we create at most $2e(H)$ total (U, V)-large paths in G in the various cases below, using at most a total of $\frac{n}{5}$ vertices in G. Thus, G always contains enough vertices to create all desired (U, V)-large paths. We say a vertex

x is *unused* if, during the construction of the desired (H, w)-linkage, x is not in an edge-path.

Subcase 1.1. *$y, z \in A$ (or $y, z \in B$).*

Suppose without loss of generality that $y, z \in A$. Define a $(\{y, z\}, \emptyset)$-large set $A_e \subset A$. It follows that A_e contains a y, z-path of length $w_e \geq 2$. Similar logic works for $y, z \in B$.

Subcase 1.2. *$y \in A$ (or $y \in B$) and $z \in S$.*

Suppose without loss of generality that $y \in A$ and $z \in S$. First suppose that z has an unused proxy vertex in A_p. For some proxy vertex a_z of z, define a $(\{y\}, \{a_z\})$-large set $A_e \subset A$. Since there exists a y, a_z-path of length $w_e - 1$ in A_e, there exists a y, z-path of length w_e in G. Now suppose z does not have an unused proxy vertex in A_p; it follows that z has an unused proxy vertex $b_1 \in B_p$. Since $|S| \geq b(H) - 2$, there exists an unused vertex $\sigma_2 \in S$ with unused proxy vertices $b_2 \in B_p$ and $a_2 \in A_p$. If such an unused vertex did not exist, then G would not be H-linked, which would contradict Theorem 2 in [17]. Define a $(\emptyset, \{b_1, b_2\})$-large vertex set $B_e \subset B$. There exists a b_1, b_2-path of length 2 in B_e. Create a $(\{y\}, \{a_2\})$-large set $A_e \subset A$. There exists an a_2, y-path of length $w_e - 5$ in A_e. It follows that G contains a y, z-path of length $w_e \geq 7$.

Similar logic works for $y \in B$.

Subcase 1.3. *Without loss of generality, $y \in A$ and $z \in B$.*

Consider a vertex $\sigma \in S$ with proxy vertices $a \in A_p$ and $b \in B_p$. Create a $(\{y\}, \{a_2\})$-large set $A_e \subset A$ and a $(\{z\}, \{b_2\})$-large set $B_e \subset B$. There exists a y, a-path of length 2 within A_e and a z, b-path of length $w_e - 4$ within B_e. It follows that G contains a y, z-path of length $w_e \geq 6$.

Subcase 1.4. *$y, z \in S$*

This final case uses reasoning similar to that of Subcase 1.2. Recall that both y and z have sufficiently many proxy vertices to construct all necessary edge-paths, but also that y and z may not have multiple proxy vertices in both A_p and B_p.

Without loss of generality suppose that y has an unused proxy vertex $a_y \in A_p$. In addition, suppose first that z has an unused proxy vertex $a_z \in A_p$ as well. Create a $(\emptyset, \{a_y, a_z\})$-large set $A_e \subset A$. The set A_e contains an a_y, a_z-path of length $w_e - 2$. It follows that G contains a y, z-path of length w_e. Now suppose that z does not have any unused proxy vertices in A_p. The vertex z must have at least 1 unused proxy vertex $b_z \in B_p$. Since $|S| \geq b(H) - 2$, there exists an unused vertex $\sigma \in S$ with unused proxy vertices $a \in A_p$ and $b \in B_p$. If such an unused vertex did not exist, then G would not be H-linked, which would contradict Theorem 2 in [17]. Create a $(\emptyset, \{a_y, a\})$-large set $A_e \subset A$ and a $(\emptyset, \{b_z, b\})$-large set $B_e \subset B$. There exists an a_y, a-path of length 2 in A_e and a b_z, b-path of length $w_e - 6$ in B_e. It follows that G contains a y, z-path of length $w_e \geq 8$.

Note that in each subcase, we use at most one vertex in S (excluding y and z) when constructing the desired edge-path. From the sufficiently large order of G and the fact that $|f(V(H))| \leq |A_p|, |B_p|$, we can construct all necessary sets and paths to be disjoint where necessary. Hence, G is (H, w)-linked.

Case 2. $\kappa(G) \geq \frac{n}{6}$.

Consider two vertices $y, z \in f(V(H))$. By Lemma 3.3.9, we can bipartition G into sets A and B so that A and B satisfy (3.4) and (3.5) in Lemma 3.3.9. (Note that $G = A \cup B$ in this case, which is not to be confused with $G = A \cup S \cup B$ from Case 1.) Define

$$D_A = \left\{ x \in A \;\middle|\; |N(x) \cap B| > \frac{n}{100b(H)^2} \right\},$$

and symmetrically define D_B. By Lemma 3.3.10, each vertex in D_A has at least $\frac{n}{5}$ edges into $A \backslash D_A$. We consider several different scenarios depending on the locations of y and z. In each case, we construct a y, z-path in G that is internally disjoint from all other such paths. If $y, z \in A$, then since $\delta(G[A]) \geq \frac{n}{5}$, we can define sets $A_e \subset A$ such that

- $|A_e| \geq \frac{n}{10b(H)^2}$,
- $A_e \cap f(V(H)) = \{y, z\}$, and

- $A_e \cap D_A \subseteq \{y, z\}$.

Note that these sets can be chosen so that A_e is disjoint from all other such sets, except possibly for y and z. By Lemma 3.3.11, we see A_e is panconnected. Hence, we can construct a y, z-path of length w_e through A_e. An analogous argument works for the case when $y, z \in B$.

Next, suppose $y \in D_A$ and $z \in B$. By the definition of D_A and the fact that $\delta(G[A]) \geq \frac{n}{5}$, there exists a set $B_e \subset B \cup \{y\}$ such that

- $|B_e| \geq \frac{n}{10b(H)^2}$,
- $B_e \cap f(V(H)) = \{y, z\}$, and
- $(B_e \setminus \{z\}) \cap N(y) \neq \emptyset$.

Note that these sets can be chosen so that B_e is disjoint from all other such sets, except possibly for y and z. Again using Lemma 3.3.10 and Lemma 3.3.11, we see B_e is panconnected. Hence, we can construct a y, z-path of length w_e through B_e. A symmetric argument works for the case when $y \in D_B$ and $z \in A$. Note that in all of these cases, the orders of A_e and B_e are chosen to be small enough to be disjoint from all other such sets where necessary. It follows that the constructed y, z-paths are all internally disjoint.

The only remaining case is that in which $y \in A \setminus D_A$ and $z \in B \setminus D_B$. While this is by far the most difficult situation, we do use a similar technique as in the previous cases.

Since $\delta(G) \geq \frac{n+b(H)-2}{2}$, there exists a set M of exactly $b(H)$ disjoint paths from $A \setminus D_A$ to $B \setminus D_B$, each having length at most 2. Call these paths *transportation paths*. By Menger's Theorem, there are at least $\frac{n}{6} \gg e(H)$ transportation paths in G. Hence, it suffices to show that we use only one transportation path for each y, z-path with $y \in A \setminus D_A$ and $z \in B \setminus D_B$.

For each edge $e \in E(H)$, create the y, z-path in G as follows. Choose a vertex set $A_e \in A \setminus D_A$ such that

- $A_e \cap f(V(H)) = \{y\}$ and

- $A_e \cap M = \{a_e\}$ for a distinct $a_e \in P_e$.

Choose $B_e \in B \setminus D_B$ symmetrically. By Lemma 3.3.10 and Fact 3.3.6, we know

$$\begin{aligned} |D_A \cup (f(V(H) \setminus y))| &< 2(e(H))^2(\delta + 2\epsilon)n + e(H) \\ &< 3(e(H))^2(\delta + 2\epsilon)n. \end{aligned}$$

Similarly, $|D_B \cup B_p \cup (f(V(H) \setminus z))| < 3(e(H))^2(\delta + 2\epsilon)n$. It follows that a_e and B_e can be chosen to be disjoint from all other such sets where necessary. By Lemma 3.3.11, we know A_e and B_e are panconnected. Let P_e be a path in M corresponding to the edge $e \in E(H)$. Suppose P_e has length c (i.e., $c = 1$ or $c = 2$). Create a w_i, a_e-path $P_{A,e}$ of length $\left\lfloor \frac{w_e - c}{2} \right\rfloor$ in A_e and a b_e, w_j-path $P_{B,e}$ of length $\left\lceil \frac{w_e - c}{2} \right\rceil$ in B_e. It follows that $P_{A,e} \cup P_e \cup P_{B,e}$ is a y, z-path of length w_e in G. Hence, G is (H, w)-linked. $\square$

3.5.3 Bipartite R Implies G is $(H, w, 1)$-Linked

Proof of Lemma 3.4.3. Let H be a multigraph, and let $\{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a sufficiently large graph G of order n with $\delta(G) \geq \frac{n}{2}$ whose reduced graph R is bipartite. Let A_R and B_R be the independent sets composing R, and let A and B be the sets of clusters in G corresponding to A_R and B_R, respectively. For all $e = (uv, s) \in E(H)$, let $y = f(u)$ and $z = f(v)$. By Lemma 3.3.1 we have

$$\left(\frac{1}{2} - (\delta + 2\epsilon)\right) n + 1 \leq |A| \leq \left(\frac{1}{2} + (\delta + 2\epsilon)\right) n - 1,$$

and similarly for $|B|$. For each edge e, define sets $T_e \subset G$ satisfying

- $|T_e| = 5(\delta + 2\epsilon)n$,
- T_e induces a balanced bipartite graph in G (i.e., T_e has equally many vertices in A and in B), and
- $T_e \cap f(V(H)) = \{y, z\}$.

Note that these sets can be chosen so that T_e is disjoint from all other such sets, except possibly for y and z. By Lemma 1.2.2, T_e is bipanconnected. First suppose

$y, z \in A$ or $y, z \in B$. If w_e is even, then create a y, z-path in T_e of length w_e. If w_e is odd, then create a y, z-path in T_e of length $w_e - 1$. Now suppose $y \in A$ and $z \in B$ or vice versa. If w_e is odd, then create a y, z-path in T_e of length w_e. If w_e is even, then create a y, z-path in T_e of length $w_e - 1$.

It follows that G is $(H, w, 1)$-linked. □

We believe that the degree assumption used in the previous lemma is likely far from sharp under the assumption that the reduced graph is bipartite.

3.5.4 Bipartite R Implies G is (H, w)-Linked

Proof of Lemma 3.4.4. Let H be a multigraph. We can assume H has no isolated vertices since the images of these vertices can be removed from G, preserving the integrity of the minimum degree condition and the result. Let $\{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a sufficiently large graph G of order n with $\delta(G) \geq \frac{n+e(H)}{2}$ whose reduced graph R is bipartite. Let A_R and B_R be the independent sets composing R, and let A and B be the sets of clusters in G corresponding to A_R and B_R, respectively. By Lemma 3.3.1 we have

$$\left(\frac{1}{2} - (\delta + 2\epsilon)\right) n + 1 \leq |A| \leq \left(\frac{1}{2} + (\delta + 2\epsilon)\right) n - 1,$$

and similarly for $|B|$. By Lemma 3.3.13, A and B have either $2e(H)$ independent edges or $e(H)$ stars, each of size at least $\frac{n}{5e(H)}$. For each edge $e = (uv, s) \in E(H)$, let $y = f(u)$ and $z = f(v)$.

First suppose $G[A] \cup G[B]$ contains a set of $2e(H)$ independent edges. Call this set of independent edges I. For each edge $e \in E(H)$, consider a unique independent edge $i_e = jk \in I$. Also define sets $T_e \subset G$ satisfying

- $|T_e| = 5(\delta + 2\epsilon)n$,
- T_e induces a balanced bipartite graph in G (i.e., T_e has equally many vertices in A and in B),
- $T_e \cap f(V(H)) = \{y, z\}$, and

- $E(T_e) \cap I = \{i_e\}$.

Note that these sets can be chosen so that T_e is disjoint from all other such sets, except possibly for y and z. By Lemma 1.2.2, T_e is bipanconnected.

Suppose $y, z \in A$ or $y, z \in B$. If w_e is even, then there must exist a y, z-path in T_e of length w_e. If instead w_e is odd, then we must use the edge i_e. If i_e and y are both in A or both in B, then consider a y, j-path of length 2 (or length 0 if $y = j$) and an k, z-path of length $w_e - 3$ (or w_e if $y = j$), both within T_e. If instead $i_e \in A$ and $y \in B$ or vice versa, then there exists a y, j-path of length 3 and an k, z-path of length $w_e - 4$, both within T_e. In either case, we have a y, z-path in G of length w_e. A similar argument works for when $y \in A$ and $z \in B$. Regardless of the situation, since we have chosen sets T_e to be disjoint from one another (except possibly for y and z), the y, z-path is disjoint from all other edge-paths in G. It follows that G is (H, w)-linked.

Next, suppose $G[A] \cup G[B]$ have $e(H)$ stars of size at least $\frac{n}{5e(H)}$. Let Y denote the set of these stars. For each edge $e \in E(H)$, assign a star $\mathcal{S}_e$ to y and z and choose an edge $i_e = jk \in \mathcal{S}_e$. Create the set $T_e \subset G$ satisfying

- $|T_e| = 5(\delta + 2\epsilon)n$,
- T_e induces a balanced bipartite graph in G (i.e., T_e has equally many vertices in A and in B),
- $T_e \cap f(V(H)) = \{y, z\}$, and
- $E(T_e) \cap \mathcal{S}_e = \{i_e\}$ for some edge $i_e \in \mathcal{S}_e$.

Note that these sets can be chosen so that T_e is disjoint from all other such sets, except possibly for y and z. By Lemma 1.2.2, T_e is bipanconnected.

Let $y, z \in A$ or $y, z \in B$. If d_e is even, then there must exist a y, z-path in T_e of length w_e. If instead w_e is odd, then we must use the edge i_e. First suppose y and i_e are both in A (or B). Since T_e is bipanconnected, we can consider a y, j-path of length 2 (or length 0 if y is the center of $\mathcal{S}_e$) and a k, z-path of length $w_e - 3$ (or length $w_e - 1$ if y is the center of $\mathcal{S}_e$). If instead $y \in A$ and $i \in B$, then we can consider a y, j-path of length 3 and a k, z-path of length $w_e - 4$. A similar argument works with the edge yk in place of jk.

In all cases, we have a y, z-path in T_e of length w_e. Regardless of the situation, since we have chosen sets T_e to be disjoint from one another (except possibly for y and z), the y, z-path is disjoint from all other edge-paths in G. It follows that G is (H, w)-linked.

We can therefore conclude that G is (H, w)-linked. $\square$

Chapter 4

(H, S)-Semi-Linked Graphs

In this chapter we define (H, S)-*semi-linkage*, a weaker form of H-linkage, where we only specify the locations of some vertices in $V(H)$ and then map the rest into G in the way that uses the fewest number of vertices in a cutset of G. As with H-linkage, we show sharp minimum degree conditions for a graph to be (H, S)-semi-linked, both with and without prescribed path lengths. The proof structures for these results are similar (but not completely symmetric) to those in Chapter 3.

4.1 Introduction

Recall that a graph G is k-*connected* if for every pair of vertices u and v, there exist k disjoint u, v-paths in G. As noted in [3], we may remove the vertices u and v and say that a graph is k-connected if for every choice of $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exist k disjoint s_i, t_j-paths. A stronger form of k-connectivity is the concept of k-linkage. A graph G is k-*linked* if for every choice of $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exist k disjoint s_i, t_i-paths. It was shown in [37] that a graph G is k-linked if either G is $2k$-connected and has at least $5k|G|$ edges or if G is $10k$-connected, formally stated in Theorem 4.2.2. The authors of [26] modified the proof of the former criterion to show k-linkage for $2k$-connected graphs with average degree at least $12k$.

4.2 (H, S)-Semi-Linkage

Let G be a graph and let $\mathscr{P}(G)$ be the set of paths in G. Suppose we are given a multigraph H, possibly with loops and a subset $S \subseteq V(H)$. A graph G is (H, S)-*semi-linked* if, for every injective function $f : S \to V(G)$, there exists an injective function $g : V(H \setminus S) \to V(G \setminus f(S))$ and a set of $|E(H)|$ internally disjoint paths $\mathscr{P} \subseteq \mathscr{P}(G)$ connecting vertices of $f(S) \cup g(V(H \setminus S))$ for every edge between corresponding vertices of H. In particular, if $S = V(H)$ (with f_2 being the empty function), then a graph G is (H, S)-semi-linked if and only if G is H-linked. At the opposite extreme, if $S = \emptyset$, then a graph G is (H, S)-semi-linked if and only if G contains an H-subdivision. Given the function f, such a subgraph of G consisting of $[f(S) \cup g(V(H \setminus S))] \cup \mathscr{P}$ is called an (H, S)-*semi-linkage*. Call $f(S) \cup g(V(H \setminus S))$ the set of *ground vertices* and the paths in $\mathscr{P}$ *edge-paths*. In this chapter, when we refer to ground vertices and edge-paths, we are referring to those of an (H, S)-semi-linkage and not an H-linkage.

Figure 4.1: $(f_1 \cup f_2, g)$ is an (H, S)-semi-linkage in G.

4.2.1 Minimum Degree Criterion for (H, S)-Semi-Linkage

Suppose we are given a multigraph H and a subset $S \subseteq V(H)$. Let c_S and $c_{V(H\setminus S)}$ be colorings of S and $V(H \setminus S)$, respectively, using the color set {**red**, **blue**, **green**}. Given a coloring c_S, let $m(c_S, H)$ be the minimum, over all colorings $c_{V(H\setminus S)}$, of the number of green vertices in H plus the number of edges between red and blue vertices in H. If G is a graph with minimum cutset C, then C must be large enough to contain all green ground vertices and allow the images of all red-to-blue edges to pass through C. I.e., we must have

$$|C| \geq \max_{c_S}\{m(c_S, H)\}$$

since this lower bound for $|C|$ assumes the highest-connectivity mapping of S into G and the resulting lowest-connectivity mapping of $V(H \setminus S)$ into G. Letting

$$s(H, S) = \max_{c_S}\{m(c_S, H)\} - 2,$$

the condition $\delta(G) \geq \frac{n+s(H,S)}{2}$ guarantees $\kappa(G) \geq \max\{m(c_S, H)\}$. We now state our first main result, which gives a sharp minimum degree condition for a graph to be (H, S)-semi-linked.

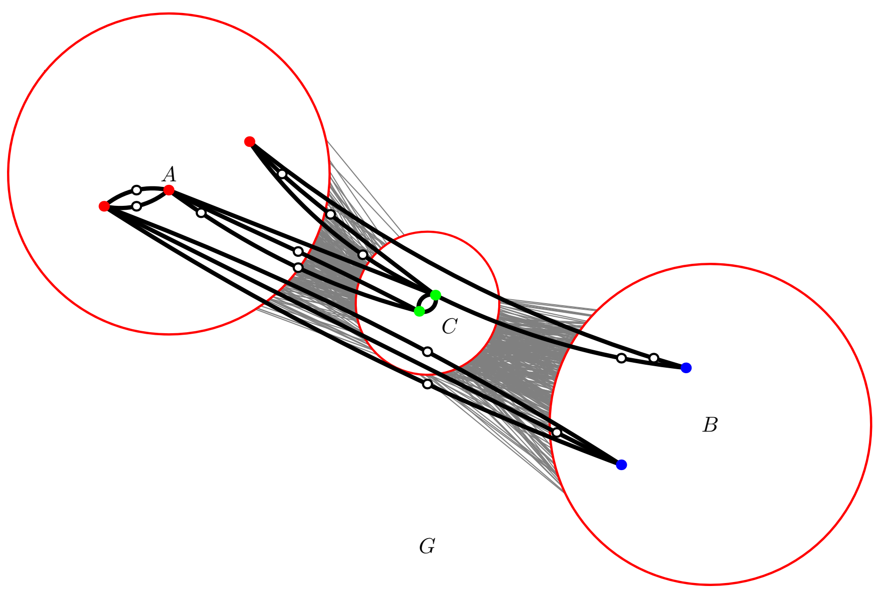

Figure 4.2: G must have connectivity at least $s(H, S) + 2$ to be (H, S)-semi-linked.

Theorem 4.2.1. *Given a multigraph H and a subset $S \subseteq V(H)$, if a graph G of order $n \geq 13(10e(H)+|H|)^3$ satisfies $\delta(G) \geq \frac{n+s(H,S)}{2}$, then G is (H,S)-semi-linked.*

By the definition of $s(H,S)$, such a result is certainly sharp. The concept behind $s(H,S)$ is similar to those behind $\eta(H)$ in [16] and $b(H)$ in [17]. As in Chapters 2 and 3, we make no attempt to optimize the bound on n.

4.2.2 Component Degree and Panconnectivity in G

In order to prove our main theorem, we use the following result from [37].

Theorem 4.2.2 ([37]). *If a graph G is $10k$-connected, then G is k-linked.*

Recall that a graph G of order n is *panconnected* if for every pair of vertices $u, v \in G$ and all t satisfying $2 \leq t \leq n$, there exists a u,v-path of length t in G, and that Theorem 1.2.1 guarantees that a graph of order n with $\delta(G) \geq \frac{n+2}{2}$ is panconnected. The following corollary of Theorem 1.2.1 states that every large induced subgraph of a sufficiently large graph with very high minimum degree is panconnected.

Corollary 4.2.3. *Let h be an integer and G be a graph of order $n \geq 13h^3$ with $\delta(G) \geq n - 6h^2$. For every set $A \subseteq V(G)$ satisfying $|A| \geq \frac{n}{h}$, the graph $G[A]$ is panconnected.*

Proof. Given $A \subseteq V(G)$ with $|A| \geq \frac{n}{h}$, we have

$$\begin{aligned} \delta(G[A]) &\geq |A| - 6h^2 \\ &\geq \frac{|A|+2}{2}. \end{aligned}$$

By Theorem 1.2.1, the graph $G[A]$ is panconnected. $\square$

The next lemma will be useful when G has low connectivity. First note that if G as defined in Theorem 4.2.1 has a minimum cutset C with $|C| < 10e(H)+|H|$, then $G \setminus C$ consists of exactly two components A and B. We simply write $G = A \cup C \cup B$. Combined with Corollary 4.2.3, Lemma 4.2.4 shows that if $G = A \cup B \cup C$ has connectivity less than $10e(H) + |H|$, then sufficiently large induced subgraphs of $G[A]$ and $G[B]$ are panconnected.

Lemma 4.2.4. *Given a multigraph H, let $G = A \cup C \cup B$ be a graph of order $n \geq 13(10e(H) + |H|)^3$ with $\delta(G) \geq \frac{n+s(H,S)}{2}$ and $\kappa(G) < 10e(H) + |H|$. The graph $G[A]$ (respectively $G[B]$) satisfies $\delta(G[A]) > |A| - (10e(H) + |H|)$ (respectively $\delta(G[B]) > |B| - (10e(H) + |H|)$).*

Proof. Let H be a multigraph with $S \subseteq V(H)$, and let G, A, B, and C be as defined above. Since $s(H,S) < 10e(H) + |H|$, the condition $\delta(G) \geq \frac{n+s(H,S)}{2}$ ensures that

$$\frac{n+s(H,S)}{2} - (10e(H)+|H|) < \frac{n+s(H,S)}{2} - |C| \leq |A|, |B| < \frac{n-s(H,S)}{2}. \quad (4.1)$$

Each vertex in A can only be adjacent to vertices in A and C, which gives

$$\begin{aligned}\delta(G[A]) &\geq \frac{n+s(H,S)}{2} - 2(10e(H) + |H|) \\ &> |A| - (10e(H) + |H|).\end{aligned}$$

Similarly, we have $\delta(G[B]) > |B| - (10e(H) + |H|)$. $\square$

Note the similarity between Lemma 3.3.11 and Lemma 4.2.4.

4.2.3 Blocked Vertices

Given a multigraph H, let $G = A \cup C \cup B$ be a graph of order $n \geq 13(10e(H) + |H|)^3$ with $\kappa(G) < 10e(H) + |H|$. For all $c \in C$ satisfying $|N(c) \cap A| \geq 5|C|(e(H) + |H|)$, consider a set $A_c \subset N(c) \cap A$ of $5(e(H) + |H|)$ vertices disjoint from all other such sets. For all other $c \in C$, define $A_c = \emptyset$. Call A_c the set of *proxy* vertices of c in A. If $A_c = \emptyset$, then c is *blocked* to A. Let C_A denote the set of all vertices in C that are blocked to A. Define B_c and C_B symmetrically.

Fact 4.2.5. Given a multigraph H with $S \subseteq V(H)$, if $G = A \cup C \cup B$ is a graph of order $n \geq 13(10e(H) + |H|)^3$ with $\delta(G) \geq \frac{n+s(H,S)}{2}$ and $\kappa(G) < 10e(H) + |H|$, then $C_A \cap C_B = \emptyset$.

Proof. From the high minimum degree condition on G, we see that $s(H, S) + 2 \leq |C| < 10e(H) + |H|$ and since n is large, all $c \in C$ satisfy either $|N(c) \cap A| \geq 5|C|(e(H) + |H|)$ or $|N(c) \cap B| \geq 5|C|(e(H) + |H|)$. □

4.2.4 Proof of Theorem 4.2.1

Proof of Theorem 4.2.1. Let H be a multigraph with $S \subseteq V(H)$. Let G be a graph of order $n \geq 13(10e(H) + |H|)^3$ with $\delta(G) \geq \frac{n+s(H,S)}{2}$, and let $f(S) \subseteq G$ be the image of S under f.

First suppose $\kappa(G) \geq 10e(H) + |H|$. In this case, we observe that G is H-linked and therefore (H, S)-semi-linked. For each vertex $v \in H$ and incident edge $e \in E(H)$, let v'_e be a neighbor of $f(v)$ in G. Since $\delta(G) \geq \frac{n}{2}$ and n is sufficiently large, the vertices $\{v'_e \mid e \in E(H)\}$ can be chosen to be distinct. By Theorem 4.2.2, we know that $G \setminus f(S)$ is $e(H)$-linked. In particular, for each edge $e = (uv, k) \in E(H)$, we can link the pairs (u'_e, v'_e). This implies that G is H-linked, which in turn implies that G is (H, S)-semi-linked. Furthermore, every edge-path in this (H, S)-semi-linkage has length 3. Thus, we may assume $\kappa(G) < 10e(H) + |H|$.

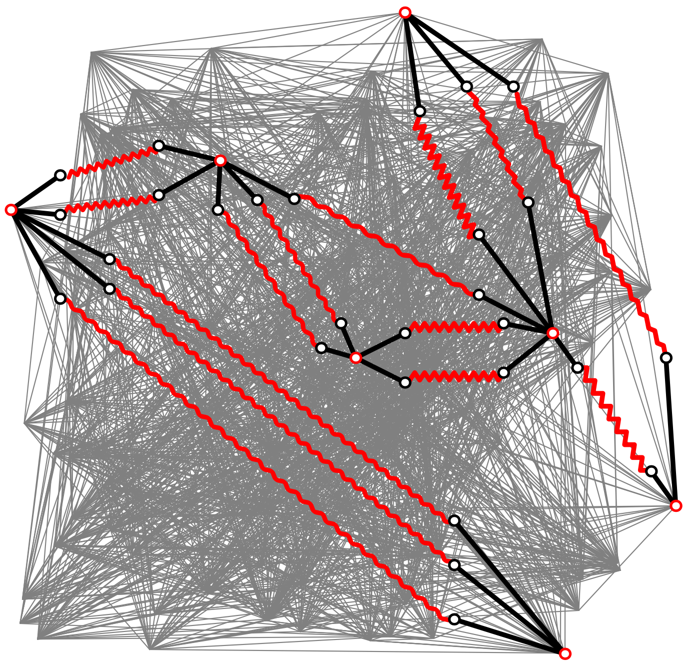

Figure 4.3: $\kappa(G) \geq 10e(H) + |H|$ implies G is k-linked.

If $e(H) = 0$, then G is clearly H-linked and hence (H, S)-semi-linked for all $S \subseteq V(H)$. As a result, we may assume $e(H) \geq 1$. The high minimum degree and low connectivity of G imply that $G = A \cup C \cup B$ for some minimum cutset C with $s(H, S) + 2 \leq |C| = \kappa(G) < 10e(H) + |H|$. Both components A and B must have order approximately $\frac{n}{2}$ and be very dense. More specifically, we have (4.1), and by Lemma 4.2.4 and Corollary 4.2.3, any reasonably large induced subgraph of $G[A]$ or $G[B]$ is panconnected.

After choosing $S \subseteq V(H)$ and $f : S \to V(G)$, determine the function $g : V(H \setminus S) \to V(G \setminus f(S))$ by mapping all vertices in such a way that, when creating all edge-paths between all pairs of vertices in $f(S) \cup g(V(H \setminus S))$, the minimum number of vertices in C must be used (note that this is at most $s(H, S) + 2$). Such an optimal mapping must exist since G contains a finite number of vertices.

We now prove that a sufficiently large graph $G' = A' \cup C' \cup B'$ with low connectivity, dense induced subgraphs $G'[A']$ and $G'[B']$, and no vertices blocked to A' or B' is (H, S)-semi-linked.

Fact 4.2.6. Let H be a multigraph with $S \subseteq V(H)$. If $G' = A' \cup C' \cup B'$ has order $n \geq 13(10e(H) + |H|)^3$ and satisfies

1. $s(H, S) + 2 \leq \kappa(G') < 10e(H) + |H|$,
2. $\delta(G'[A']) \geq |A'| - 6(10e(H) + |H|)^2$ and $\delta(G[B']) \geq |B'| - 6(10e(H) + |H|)^2$,
3. $C'_{A'} \cup C'_{B'} = \emptyset$,

then G' is (H, S)-semi-linked. In particular, for every set of $|S|$ vertices in G, there exists an (H, S)-semi-linkage in G whose edge-paths have length at most 6.

Proof. We show the existence of all necessary internally disjoint edge-paths between ground vertices. Note that $C'_{A'} \cup C'_{B'} = \emptyset$ is equivalent to saying $|A_c|, |B_c| = 5(e(H) + |H|)$ for all $c \in C$. Let A'_p and B'_p be the set of all proxy vertices in A' and B', respectively. Since we assume H contains at least one edge, we have

$$\begin{aligned} |A'_p \cup B'_p| &= 10(e(H) + |H|)(10e(H) + |H|) \\ &< \frac{n}{10}, \end{aligned}$$

which implies $|A' \setminus A'_p|, |B' \setminus B'_p| > \frac{4n}{5}$. By Corollary 4.2.3, there exist internally disjoint paths of length 2 between all pairs of ground vertices in A' (respectively B'). Denote multiedges between vertices $u, v \in H$ by (uv, t). Suppose we wish to construct an edge-path between ground vertices $a \in A'$ and $b \in B'$ to correspond to the multiedge $(v_a v_b, t) \in E(H)$. By Item 1 and the definition of $s(H, S)$, there exists a distinct vertex $c \in C'$ corresponding to $(v_a v_b, k)$. By Item 3, the sets A'_c and B'_c each contain many more vertices than there are edges in H. Let $a_c \in A'_c$ and $b_c \in B'_c$ be distinct proxy vertices of $c \in C'$. By Corollary 4.2.3, there exists an a, a_c-path of length 2 in A' and a b, b_c-path of length 2 in B'. It follows that G' contains an a, b-edge-path of length 6. For ground vertices $a \in A'$ and $c \in C'$, we again use Corollary 4.2.3 and a proxy vertex in A' to show that G' contains an a, c-edge-path of length 3. A symmetric approach works for ground vertices $b \in B'$ and $c \in C'$. Lastly, to create an edge-path between ground vertices $c_1, c_2 \in C'$, simply designate a proxy vertex $a_1 \in A'_{c_1}$ for c_1 and $a_2 \in A'_{c_2}$ for c_2. By Corollary 4.2.3, there exists an a_1, a_2-path of length 2 internally disjoint from all other paths created, and hence, G' contains a c_1, c_2-edge-path in G' of length 4. Since all necessary edge-paths exist in G' and are internally disjoint, we have that G' is (H, S)-semi-linked. Furthermore, each edge-path has length at most 6 if desired. $\square$

Absent from the assumptions on G' in Fact 4.2.6 is the condition $\delta(G') \geq \frac{n+s(H,S)}{2}$, which, by Lemma 4.2.4, implies the density of $G'[A']$ and $G'[B']$. Indeed, Fact 4.2.6 only requires $\delta(G') > 10(e(H) + |H|)$. We start with this weaker assumption on G' so that Fact 4.2.6 can be used to prove a certain spanning subgraph of G is (H, S)-semi-linked when G does contain vertices blocked to A or B.

Corollary 4.2.7. *If $C_A \cup C_B = \emptyset$, then G is (H, S)-semi-linked. In particular, for every set of $|S|$ vertices in G, there exists an (H, S)-semi-linkage in G whose edge-paths have length at most 6.*

Proof. By Lemma 4.2.4, we have $\delta(G[A]) \geq |A| - (10e(H) + |H|)$ and $\delta(G[B]) \geq |B| - (10e(H) + |H|)$. It follows that G is (H, S)-semi-linked with each edge-path having length at most 6 as desired by Fact 4.2.6. $\square$

In the proof of Fact 4.2.6, the only edge-paths whose interiors contained one vertex in the minimum cutset were those from one component to another. However, if G contains vertices blocked to A or B, then this may not be possible. If a vertex $c \in C$ blocked to A requires more than $|A_c|$ edge-paths into A, then we must use an extra vertex in C for each additional edge-path. A similar issue arises when constructing an edge-path between a vertex blocked from A and a vertex blocked from B. We must show that C is large enough to contain these "extra" required vertices. We start by giving an upper bound on $|C_A \cup C_B|$, the number of blocked vertices.

Fact 4.2.8. $|C_A \cup C_B| \leq |C| - (s(H, S) + 2)$.

Proof. We prove this result by first showing $|C_A| \le |C| - \left(\frac{n+s(H,S)}{2} - |A| + 1\right)$ and $|C_B| \le |C| - \left(\frac{n+s(H,S)}{2} - |B| + 1\right)$. Suppose $|C_A| > |C| - \left(\frac{n+s(H,S)}{2} - |A| + 1\right)$. It follows that

$$\begin{aligned} e(A, C_A) &< 5|C|(e(H) + |H|) \cdot |C| \\ &< 5(e(H) + |H|)(10e(H) + |H|)^2 \\ &< 5(10e(H) + |H|)^3 \end{aligned}$$

and

$$e(A, C \setminus C_A) \le |A| \left(\frac{n + s(H,S)}{2} - |A|\right),$$

and hence

$$e(A, C) < 5(10e(H) + |H|)^3 + |A| \left(\frac{n + s(H,S)}{2} - |A|\right). \tag{4.2}$$

However, since $\delta(G) \ge \frac{n+s(H,S)}{2}$, each vertex in A is adjacent to at least $\frac{n+s(H,S)}{2} - |A| + 1$ vertices in C, which gives

$$e(A, C) \ge |A| \left(\frac{n + s(H,S)}{2} - |A| + 1\right). \tag{4.3}$$

Combining (4.2) and (4.3) gives

$$\begin{aligned} |A| \left(\frac{n + s(H,S)}{2} - |A| + 1\right) &\le e(A, C) \\ &< 5(10e(H) + |H|)^3 + |A| \left(\frac{n + s(H,S)}{2} - |A|\right), \end{aligned}$$

which reduces to $|A| \le 5(10e(H) + |H|)^3$. By (4.1), this is a contradiction. Hence, $C_A \le |C| - \left(\frac{n+s(H,S)}{2} - |A| + 1\right)$, and symmetrically, we see that $C_B \le |C| - \left(\frac{n+s(H,S)}{2} - |B| + 1\right)$. It follows that

$$\begin{aligned} |C_A \cup C_B| &\le 2|C| - \left(\frac{n + s(H,S)}{2} - |A| + 1 + \frac{n + s(H,S)}{2} - |B| + 1\right) \\ &= 2|C| - (|C| + s(H,S) + 2) \\ &= |C| - (s(H,S) + 2). \end{aligned}$$

□

Consider the spanning subgraph $G' \subset G$ that results from removing all edges between C_A and A, between C_B and B, and between C_A and C_B.

Fact 4.2.9. G' is (H, S)-semi-linked. In particular, for every set of ground vertices in G, there exists an (H, S)-semi-linkage in G whose edge-paths have length at most 6.

Proof. Let

$$\begin{aligned} A' &= A \cup C_B \\ B' &= B \cup C_A \\ C' &= C \setminus (C_A \cup C_B). \end{aligned}$$

By definition, we see $G' = A' \cup C' \cup B'$ and has order $n \geq 13(10e(H) + |H|)^3$ with $C'_{A'} \cup C'_{B'} = \emptyset$. The inequality $s(H, S) + 2 \leq \kappa(G') < 10e(H) + |H|$ follows immediately from Fact 4.2.8. Also note that

$$\begin{aligned} \delta(G'[A']) &> |A'| - (|C| - |C_A|) - 5|C|(e(H) + |H|) \\ &> |A'| - 6(10e(H) + |H|)^2 \end{aligned}$$

which by Corollary 4.2.3 implies that all sets $A^* \subseteq A'$ of order at least $\frac{n}{10e(H)+|H|}$ the graph $G'[A^*]$ is panconnected. Then G' satisfies all the criteria for Fact 4.2.6 and hence is (H, S)-semi-linked with each edge-path having length at most 6 if desired. $\square$

Since $G' \subset G$, Fact 4.2.9 implies that G is (H, S)-semi-linked. This completes the proof of Theorem 4.2.1. $\square$

4.2.5 Conclusion

Corollary 4.2.3 and Lemma 4.2.4 could have been combined into a single lemma. However, the proof of Fact 4.2.6 in Section 4.2.4 involves a spanning subgraph $G' \subseteq G$ with $\delta(G') \leq \delta(G)$, which means we cannot necessarily apply Lemma 4.2.4 to G'. Instead, we apply Corollary 4.2.3 to the components of G' and let $h = 10e(H)+|H|$.

4.3 $(H, S, w, 1)$- and (H, S, w)-Semi-Linkage

Now that we have determined a minimum degree criterion for a graph to be (H, S)-semi-linked, we establish minimum degree criteria for a graph to be (H, S, w)-linked and $(H, S, w, 1)$-linked. Our approach to (H, S)-semi-linked graphs here mimics that for H-linked graphs in Chapter 3. In Chapter 3, we cited various works that showed certain conditions for a graph to be H-linked and then extended their results by showing $(H, w, 1)$- and (H, w)-linkage in large graphs using the Regularity Lemma. We proceed similarly here.

Let H be a multigraph and $S \subseteq V(H)$. Considering a set $w = \{w_e \mid e \in E(H), w_e \geq 2\}$ and a tolerance value d, a graph G is (H, S, w, d)-*semi-linked* if for every $f_1 : S \hookrightarrow V(G)$, there exist the maps $f_2 : V(H \setminus S) \to V(G)$ and $g : E(H) \to$ paths(G) such that $(f_1 \cup f_2, g)$ is an H-subdivision in G with each edge-path $g(e)$ having length w_e. If $d = 0$, then we omit d and say G is (H, S, w)-*semi-linked*. In particular, if $S = V(H)$ (with f_2 being the empty function), then a graph G is (H, S, w)-semi-linked if and only if G is (H, w)-linked. An analogous relation holds between $(H, S, w, 1)$-semi-linked and $(H, w, 1)$-linked graphs for $S = V(H)$.

To give an example of this notion, we restate Conjecture 2.1.1 in terms of (H, S, w)-semi-linkage.

Conjecture 2.1.1 (Enomoto, Ota [10]). *Let H be a matching on k edges, let S contain exactly one end of each edge of the matching, and let w be an integer sequence $w = \{w_e \mid e \in E(H), w_e \geq 2, \sum_{e \in E(H)} w_e = n - k\}$. If $\sigma_2(G) \geq n + k - 1$, then G is (H, S, w)-semi-linked.*

4.3.1 Minimum Degree Criteria for $(H, S, w, 1)$- and (H, S, w)-Semi-Linkage

We now state our main results, which give sharp minimum degree conditions for a graph to be $(H, S, w, 1)$- and (H, S, w)-semi-linked.

Theorem 4.3.1. *Let H be a multigraph with $S \subseteq V(H)$, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. If G is a graph of order $n \geq n(H, S, w)$*

with $\delta(G) \geq \frac{n+s(H,S)}{2}$, then G is $(H,S,w,1)$-semi-linked. Furthermore, the lower bound for $\delta(G)$ is sharp.

The sharpness of Theorem 4.3.1 is established in 4.2.1. Note the parallel between the statements of Theorems 4.3.1 and3.2.1. In fact, the proof of Theorem 4.3.1 largely parallels that of Theorem 3.2.1 as well. Lemmas 3.4.1 and 3.4.3 are used to prove Theorem 4.3.1 when R is connected. Furthemore, Lemma 4.3.4 is symmetric to Lemma 3.4.2, and Lemma 4.3.4 is symmetric to Lemma 3.4.2.

To state Theorem 4.3.2, we must first define the value $t(H,S)$. We combine the sharp example of G (a large, complete $\left(\frac{|B||A'|}{|A||A'|+|A||B|+|B||A'|}\right)$-almost-bipartite graph) with the coloring technique used to create $s(H,S)$ in Section 4.2.1. Suppose we are given a multigraph H and a subset $S \subseteq V(H)$. Let c_S and $c_{V(H\setminus S)}$ be colorings of S and $V(H \setminus S)$, respectively, using the color set $\{$**red**, **blue**, **green**$\}$. Also let the *assigned value* of an edge e be the value in w. Given a coloring c_S, let

- **g** be the number of green vertices in H,
- **rr** denote the number of edges with odd assigned values between red vertices,
- **bb** denote the number of edges with odd assigned values between blue vertices,
- **rb** denote the number of edges with even assigned values between red and blue vertices.

Let $p(c_S, H)$ be the minimum, over all colorings $c_{V(H\setminus S)}$, of **g** + **rr** + **bb** + **rb**, and let

$$t(H,S) = \max_{c_S}\{p(c_S,H)\} - 2.$$

The condition $\delta(G) \geq \frac{n+t(H,S)}{2}$ guarantees $\kappa(G) \geq \max\{p(c_S,H)\}$. We now state our main result, an extension of Theorem 4.2.1 for large graphs.

Theorem 4.3.2. *Let H be a multigraph with $S \subseteq V(H)$, and let $w = \{w_e \mid e \in E(H), w_e \geq 14\}$ be a sequence of integers. If G is a graph of order $n \geq n(H,S,w)$ with $\delta(G) \geq \frac{n+t(H,S)}{2}$, then G is (H,S,w)-semi-linked. Furthermore, the lower bound for $\delta(G)$ is sharp.*

To see that $\delta(G) \geq \frac{n+t(H,S)}{2}$ is sharp, consider the following example.

Example 4.3.3. Given a multigraph H, a set $S \subseteq V(H)$, and an integer sequence $w = \{w_e \mid e \in E(H), w_e \geq 14\}$, let G be a complete tripartite graph on n vertices with independent sets A, B, and A' satisfying

$$|A| = \left\lceil \frac{n-(t(H,S)-1)}{2} \right\rceil$$
$$|B| = \left\lfloor \frac{n-(t(H,S)-1)}{2} \right\rfloor$$
$$|A'| = t(H,S) - 1.$$

As a result, we have $\delta(G) = \left\lceil \frac{n+t(H,S)-1}{2} \right\rceil$. Although G is tripartite, note that $G \setminus E(A', A)$ and $G \setminus E(A', B)$ are bipartite. We can think of G as being "almost" bipartite.

Let c_S and $c_{V(H \setminus S)}$ be the colorings such that $\mathbf{g} + \mathbf{rr} + \mathbf{bb} + \mathbf{rb} = t(H,S)$, and let $c = c_S \cup c_{V(H\setminus S)}$. Let $c(v)$ represent the coloring of a vertex under c. Define f_1 such that

$$f_1(v) \begin{cases} \in A & \text{if } v \text{ is red} \\ \in B & \text{if } v \text{ is blue} \\ \in A' & \text{if } v \text{ is green.} \end{cases}$$

By the definition of $t(H,S)$, it follows that defining f_2 such that

$$f_2(v) \begin{cases} \in A & \text{if } v \text{ is red} \\ \in B & \text{if } v \text{ is blue} \\ \in A' & \text{if } v \text{ is green} \end{cases}$$

results in the smallest necessary size of A', which is $t(H,S)$. However, we see $|A'| = t(H,S) - 1$, which means A' is not large enough to contain all necessary paths in our desired (H,S)-semi-linkage in G. This shows that $\delta(G) \geq \frac{n+t(H,S)}{2}$ is sharp.

4.3.2 Proof of Theorems 4.3.1 and 4.3.2

As with Theorems 3.2.1 and 3.2.2, we split the proof into four lemmas. Two of those lemmas have already been proven: If G is a sufficiently large graph of order n with $\delta(G) \geq \frac{n}{2}$, then Lemma 3.4.1 establishes (H, w)-linkage when R is connected and not bipartite, and 3.4.3 establishes $(H, w, 1)$-linkage when R is connected.

We prove a sharp minimum degree condition for a graph G with disconnected reduced graph to be (H, S, w)-semi-linked.

Lemma 4.3.4. *Let H be a multigraph with $S \subseteq V(H)$, and let $w = \{w_e \mid e \in E(H), w_e \geq 8\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H, S, w)$ with $\delta(G) \geq \frac{n+s(H,S)}{2}$ and reduced graph R. If R is disconnected, then G is (H, S, w)-semi-linked.*

Proof. The proof of Lemma 4.3.4 is similar to that of Lemma 4.3.4.

Let H be a multigraph with $S \subseteq V(H)$, and let $\{w_e \mid e \in E(H), w_e \geq 8\}$ be a sequence of integers. Consider a vertex map $f : V(H) \hookrightarrow V(G)$. Consider a (simple) graph G of sufficiently large order n with $\delta(G) \geq \frac{n+s(H,S)}{2}$.

Apply Lemma 1.1.2 on G to obtain the reduced graph R of G. Since $\delta(G) \geq \frac{n+s(H,S)}{2}$, from Lemma 3.3.1 we have

$$\delta(R) \geq \left(\frac{1}{2} - (\delta + 2\epsilon)\right) r + \frac{s(H, S)}{2}.$$

We divide the remainder of the proof into two cases based on the connectivity of G. Let $f = f_1 \cup f_2$, and for each edge $e = (uv, i) \in E(H)$, let $y = f(u)$ and $z = f(v)$.

Case 1. $\kappa(G) \leq \frac{n}{6}$.

Let C be a minimum cutset of G, and let A and B be the components of $G \setminus C$. Recall the definition of the proxy vertex sets A_p and B_p from Fact 3.3.6. Given vertex sets $U, V \subset V(G)$, define a vertex set X in G to be *(U, V)-large* if

- $X \subset A$ or $X \subset B$,

- $|X| \geq \frac{n}{10s(H,S)^2}$,
- $X \cap f(V(H)) = U$, and
- $X \cap (A_p \setminus U) = V$ or $X \cap (B_p \setminus U) = V$.

By Lemma 3.3.12, the first two items guarantee that (U,V)-large sets are panconnected. The purpose of (U,V)-large sets is to construct paths of length 2 either between two proxy vertices or between a proxy vertex and y (or z). To construct all desired edge-paths in G, we create at most $2e(H)$ total (U,V)-large paths in G in the various cases below, using at most a total of $\frac{n}{5}$ vertices in G. Thus, G always contains enough vertices to create all desired (U,V)-large paths. We say a vertex x is *unused* if, during the construction of the desired (H,S,w)-semi-linkage, x is not in an edge-path.

Subcase 1.1. *$y, z \in A$ (or $y, z \in B$).*

Suppose without loss of generality that $y, z \in A$. Define a $(\{y,z\}, \emptyset)$-large set $A_e \subset A$. It follows that A_e contains a y, z-path of length $w_e \geq 2$. Similar logic works for $y, z \in B$.

Subcase 1.2. *$y \in A$ (or $y \in B$) and $z \in C$.*

Suppose without loss of generality that $y \in A$ and $z \in C$. First suppose that z has an unused proxy vertex in A_p. For some proxy vertex a_z of z, define a $(\{y\}, \{a_z\})$-large set $A_e \subset A$. Since there exists a y, a_z-path of length $w_e - 1$ in A_e, there exists a y, z-path of length w_e in G. Now suppose z does not have an unused proxy vertex in A_p; it follows that z has an unused proxy vertex $b_1 \in B_p$. Since $|C| \geq s(H,S)$, there exists an unused vertex $\sigma_2 \in C$ with unused proxy vertices $b_2 \in B_p$ and $a_2 \in A_p$. If such an unused vertex did not exist, then G would not be (H,S)-semi-linked, which would contradict Theorem 4.2.1. Define a $(\emptyset, \{b_1, b_2\})$-large vertex set $B_e \subset B$. There exists a b_1, b_2-path of length 2 in B_e. Create a $(\{y\}, \{a_2\})$-large set $A_e \subset A$. There exists an a_2, y-path of length $w_e - 5$ in A_e. It follows that G contains a y, z-path of length $w_e \geq 7$.

Similar logic works for $y \in B$.

Subcase 1.3. *Without loss of generality, $y \in A$ and $z \in B$.*

Consider a vertex $\sigma \in C$ with proxy vertices $a \in A_p$ and $b \in B_p$. Create a $(\{y\}, \{a_2\})$-large set $A_e \subset A$ and a $(\{z\}, \{b_2\})$-large set $B_e \subset B$. There exists a y, a-path of length 2 within A_e and a z, b-path of length $w_e - 4$ within B_e. It follows that G contains a y, z-path of length $w_e \geq 6$.

Subcase 1.4. $y, z \in C$

This final case uses reasoning similar to that of Subcase 1.2. Recall that both y and z have sufficiently many proxy vertices to construct all necessary edge-paths, but also that y and z may not have multiple proxy vertices in both A_p and B_p. Without loss of generality suppose that y has an unused proxy vertex $a_y \in A_p$. In addition, suppose first that z has an unused proxy vertex $a_z \in A_p$ as well. Create a $(\emptyset, \{a_y, a_z\})$-large set $A_e \subset A$. The set A_e contains an a_y, a_z-path of length $w_e - 2$. It follows that G contains a y, z-path of length w_e. Now suppose that z does not have any unused proxy vertices in A_p. It follows that z must have at least 1 unused proxy vertex $b_z \in B_p$. Since $|C| \geq s(H, S)$, there exists an unused vertex $\sigma \in C$ with unused proxy vertices $a \in A_p$ and $b \in B_p$. If such an unused vertex did not exist, then G would not be (H, S)-semi-linked, which would contradict Theorem 4.2.1. Create a $(\emptyset, \{a_y, a\})$-large set $A_e \subset A$ and a $(\emptyset, \{b_z, b\})$-large set $B_e \subset B$. There exists an a_y, a-path of length 2 in A_e and a b_z, b-path of length $w_e - 6$ in B_e. It follows that G contains a y, z-path of length $w_e \geq 8$.

Note that in each subcase, we use at most one vertex in C (excluding y and z) when constructing the desired edge-path. From the sufficiently large order of G and the fact that $|f(V(H))| \leq |A_p|, |B_p|$, we can construct all necessary sets and paths to be disjoint where necessary. Hence, G is (H, S, w)-semi-linked.

Case 2. $\kappa(G) \geq \frac{n}{6}$.

By Lemma 3.3.9, we can bipartition G into sets A and B so that A and B satisfy (3.4) and (3.5) in Lemma 3.3.9. (Note that $G = A \cup B$ in this case, which is not to be confused with $G = A \cup C \cup B$ from Case 1.) Define

$$D_A = \left\{ x \in A \;\middle|\; |N(x) \cap B| > \frac{n}{100s(H,S)^2} \right\},$$

and symmetrically define D_B. By Lemma 3.3.10, each vertex in D_A has at least $\frac{n}{5}$ edges into $A \backslash D_A$. We consider several different scenarios depending on the locations of y and z. In each case, we construct a y,z-path in G that is internally disjoint from all other such paths. If $y, z \in A$, then since $\delta(G[A]) \geq \frac{n}{5}$, we can define sets $A_e \subset A$ such that

- $|A_e| \geq \frac{n}{10s(H,S)^2}$,
- $A_e \cap f(V(H)) = \{y, z\}$, and
- $A_e \cap D_A \subseteq \{y, z\}$.

Note that these sets can be chosen so that A_e is disjoint from all other such sets, except possibly for y and z. By Lemma 3.3.11, we see A_e is panconnected. Hence, we can construct a y,z-path of length w_e through A_e. An analogous argument works for the case when $y, z \in B$.

Next, suppose $y \in D_A$ and $z \in B$. By the definition of D_A and the fact that $\delta(G[A]) \geq \frac{n}{5}$, there exists a set $B_e \subset B \cup \{y\}$ such that

- $|B_e| \geq \frac{n}{10s(H,S)^2}$,
- $B_e \cap f(V(H)) = \{y, z\}$, and
- $(B_e \setminus \{z\}) \cap N(y) \neq \emptyset$.

Note that these sets can be chosen so that B_e is disjoint from all other such sets, except possibly for y and z. Again using Lemma 3.3.10 and Lemma 3.3.11, we see B_e is panconnected. Hence, we can construct a y,z-path of length w_e through B_e. A symmetric argument works for the case when $y \in D_B$ and $z \in A$. Note that in all of these cases, the orders of A_e and B_e are chosen to be small enough to be disjoint from all other such sets where necessary. It follows that the constructed y,z-paths are all internally disjoint.

The only remaining case is that in which $y \in A \setminus D_A$ and $z \in B \setminus D_B$. While this is by far the most difficult situation, we do use a similar technique as in the previous cases.

Since $\delta(G) \geq \frac{n+s(H,S)}{2}$, there exists a set M of exactly $s(H,S)+2$ disjoint paths from $A \setminus D_A$ to $B \setminus D_B$, each having length at most 2. Call these paths *transportation paths*. By Menger's Theorem, there are at least $\frac{n}{6} \gg e(H)$ transportation paths in G. Hence, it suffices to show that we use only one transportation path for each y,z-path with $y \in A \setminus D_A$ and $z \in B \setminus D_B$.

For each edge $e \in E(H)$, create the y,z-path in G as follows. Choose a vertex set $A_e \in A \setminus D_A$ such that

- $A_e \cap f(V(H)) = \{y\}$ and
- $A_e \cap M = \{a_e\}$ for a distinct $a_e \in P_e$.

Choose $B_e \in B \setminus D_B$ symmetrically. By Lemma 3.3.10 and Fact 3.3.6, we know

$$\begin{aligned}|D_A \cup (f(V(H) \setminus y))| &< 2(e(H))^2(\delta + 2\epsilon)n + e(H)\\ &< 3(e(H))^2(\delta + 2\epsilon)n.\end{aligned}$$

Similarly, $|D_B \cup B_p \cup (f(V(H) \setminus z))| < 3(e(H))^2(\delta+2\epsilon)n$. It follows that a_e and B_e can be chosen to be disjoint from all other such sets where necessary. By Lemma 3.3.11, we know A_e and B_e are panconnected. Let P_e be a path in M corresponding to the edge $e \in E(H)$. Suppose P_e has length c (i.e., $c = 1$ or $c = 2$). Create a w_i, a_e-path $P_{A,e}$ of length $\left\lfloor \frac{w_e - c}{2} \right\rfloor$ in A_e and a b_e, w_j-path $P_{B,e}$ of length $\left\lceil \frac{w_e - c}{2} \right\rceil$ in B_e. It follows that $P_{A,e} \cup P_e \cup P_{B,e}$ is a y,z-path of length w_e in G. Hence, G is (H,S,w)-semi-linked. $\square$

We now prove a sharp minimum degree condition for a graph G with bipartite reduced graph to be (H,S,w)-semi-linked.

Lemma 4.3.5. *Let H be a multigraph with $S \subseteq V(H)$, and let $w = \{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a graph G of order $n \geq n(H,S,w)$ with $\delta(G) \geq \frac{n+t(H,S)}{2}$ and reduced graph R. If R is bipartite, then G is (H,S,w)-semi-linked.*

Proof. Let H be a multigraph with $S \subseteq V(H)$. Let $\{w_e \mid e \in E(H), w_e \geq 3\}$ be a sequence of integers. Consider a sufficiently large graph G of order n with $\delta(G) \geq \frac{n+t(H)}{2}$ whose reduced graph R is bipartite. Let A_R and B_R be the independent sets composing R, and let A and B be the sets of clusters in G corresponding to A_R and B_R, respectively. By Lemma 3.3.1 we have

$$\left(\frac{1}{2} - (\delta + 2\epsilon)\right) n + 1 \leq |A| \leq \left(\frac{1}{2} + (\delta + 2\epsilon)\right) n - 1,$$

and similarly for $|B|$. By Lemma 3.3.13, A and B have either $2t(H,S)$ independent edges or $t(H,S)$ stars, each of size at least $\frac{n}{5t(H,S)}$. For each edge $e = (uv, s) \in E(H)$, let $y = f(u)$ and $z = f(v)$.

First suppose $G[A] \cup G[B]$ contains a set of $2t(H,S)$ independent edges. Call this set of independent edges I. For each edge $e \in E(H)$, consider a unique independent edge $i_e = jk \in I$. Also define sets $T_e \subset G$ satisfying

- $|T_e| = 5(\delta + 2\epsilon)n$,
- T_e induces a balanced bipartite graph in G (i.e., T_e has equally many vertices in A and in B),
- $T_e \cap f(V(H)) = \{y, z\}$, and
- $E(T_e) \cap I = \{i_e\}$.

Note that these sets can be chosen so that T_e is disjoint from all other such sets, except possibly for y and z. By Lemma 1.2.2, T_e is bipanconnected.

Suppose $y, z \in A$ or $y, z \in B$. If w_e is even, then there must exist a y,z-path in T_e of length w_e. If instead w_e is odd, then we must use the edge i_e. If i_e and y are both in A or both in B, then consider a y,j-path of length 2 (or length 0 if $y = j$) and an k,z-path of length $w_e - 3$ (or w_e if $y = j$), both within T_e. If instead $i_e \in A$ and $y \in B$ or vice versa, then there exists a y,j-path of length 3 and an k,z-path of length $w_e - 4$, both within T_e. In either case, we have a y,z-path in G of length w_e. A similar argument works for when $y \in A$ and $z \in B$. Regardless of the situation, since we have chosen sets T_e to be disjoint from one another (except possibly for y

and z), the y, z-path is disjoint from all other edge-paths in G. It follows that G is (H, S, w)-semi-linked.

Next, suppose $G[A] \cup G[B]$ have $t(H, S)$ stars of size at least $\frac{n}{5t(H,S)}$. Let Y denote the set of these stars. For each edge $e \in E(H)$, assign a star $\mathcal{S}_e$ to y and z and choose an edge $i_e = jk \in \mathcal{S}_e$. Create the set $T_e \subset G$ satisfying

- $|T_e| = 5(\delta + 2\epsilon)n$,
- T_e induces a balanced bipartite graph in G (i.e., T_e has equally many vertices in A and in B),
- $T_e \cap f(V(H)) = \{y, z\}$, and
- $E(T_e) \cap \mathcal{S}_e = \{i_e\}$ for some edge $i_e \in \mathcal{S}_e$.

Note that these sets can be chosen so that T_e is disjoint from all other such sets, except possibly for y and z. By Lemma 1.2.2, T_e is bipanconnected.

Let $y, z \in A$ or $y, z \in B$. If d_e is even, then there must exist a y, z-path in T_e of length w_e. If instead w_e is odd, then we must use the edge i_e. First suppose y and i_e are both in A (or B). Since T_e is bipanconnected, we can consider a y, j-path of length 2 (or length 0 if y is the center of $\mathcal{S}_e$) and a k, z-path of length $w_e - 3$ (or length $w_e - 1$ if y is the center of $\mathcal{S}_e$). If instead $y \in A$ and $i \in B$, then we can consider a y, j-path of length 3 and a k, z-path of length $w_e - 4$. A similar argument works with the edge yk in place of jk.

In all cases, we have a y, z-path in T_e of length w_e. Regardless of the situation, since we have chosen sets T_e to be disjoint from one another (except possibly for y and z), the y, z-path is disjoint from all other edge-paths in G. It follows that G is (H, S, w)-semi-linked.

We can therefore conclude that G is (H, S, w)-semi-linked. □

Lemmas 3.4.1, 3.4.3, 4.3.4, and 4.3.5 combine to prove Theorems 4.3.2 and 4.3.1.

Proof of Theorem 3.2.1. Use Lemmas 3.4.1, 4.3.4, and 3.4.3. □

Proof of Theorem 3.2.2. Use Lemmas 3.4.1, 3.4.2, and 4.3.5. □

Chapter 5

Conclusion

We have proved several degree conditions for a large graph to contain certain types of subdivisions. We would like to extend our results in a number of ways.

In particular, our large-graph approach using the Regularity Lemma might be applied to make progress on a longstanding conjecture by El-Zahar [9].

Conjecture 5.0.6 ([9])**.** *Suppose $n = \sum_{i=1}^{k} n_i$ with $n_i \geq 3$ for all i and $\delta(G) \geq \sum_{i=1}^{k} \left\lceil \frac{n_i}{2} \right\rceil$. Then G can be partitioned into cycles of length $n_1, \ldots, n_k$.*

Note that $\frac{n}{2} \leq \sum_{i=1}^{k} \left\lceil \frac{n_i}{2} \right\rceil \leq \frac{n+k}{2}$. Although Conjecture 5.0.6 uses $\delta(G)$, it may be possible to use techniques similar to those in Subsection 2.3.5 to create long paths such with adjacent endpoints, resulting in cycles. Another route may be to prove a result similar to Conjecture 5.0.6 for large graphs using $\sigma_2(G)$ instead of $\delta(G)$.

With both (H, w, d)-linkage and (H, S, w, d)-linkage in a large graph G of order n, we would like to increase the order of the resulting H-subdivisions to be approximately $(1 - \epsilon)n$, with the discrepancy of ϵn coming from the garbage set V_0 obtained from Lemma 1.1.2. Naturally, the sharp minimum degree bounds from Theorems 3.2.1, 3.2.2, 4.3.1, and 4.3.2 should all remain the same. We believe that the v-absorbing paths technique from Subsection 2.3.5 may be used to extend the lengths of each edge-path in an H-subdivision to be fractions of n.

We would also like to define the values $s(H, S)$ and $t(H, S)$ without using colorings. Such a description would reveal more about the effects the structure of H has on the connectivity of G and would make $s(H, S)$ and $t(H, S)$ easier to determine (although finding $s(H, S)$ and $t(H, S)$ for a large multigraph H still might be difficult). An ideal result would be a description similar to those of $b(H)$ in [17]. We imagine that, as with $b(H)$, both $s(H, S)$ and $t(H, S)$ must relate to the size of a maximum edge-cut of H, as the conditions for $\delta(G)$ in Theorems 4.3.1 and 4.3.2 are sharp because of the resulting connectivity of G.

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Vita

Education

Lehigh University — Bethlehem, PA

Doctor of Philosophy, Mathematics — May 2014

Dissertation: *Subdivisions with Distance Constraints in Large Graphs*

Advisors: Vincent Coll and Colton Magnant

Master of Sciences, Mathematics — May 2011

University of Rochester — Rochester, NY

Honors Bachelor of Arts, Mathematics — May 2008

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Research Interests

Extremal Graph Theory, Rainbow Ramsey Theory

Teaching Experience

Salisbury University — Salisbury, MD

Visiting Assistant Professor — August 2014–June 2015

Lehigh University — Bethlehem, PA

Graduate Student/Instructor — August 2008–May 2014

Courses taught:

- Math 76. Calculus I, Part B — Summer 2011, Spring 2014
- Math 43. Survey of Linear Algebra — Fall 2013

Teaching Assistant for the Engineering Calculus series — Fall 2008–Spring 2013

Drown Learning Center Tutor — Fall 2009, Fall 2012

Northampton Community College — Bethlehem, PA

Adjunct Faculty Instructor — Summer 2011, Fall 2012

Courses taught:

- Math 150. Introductory Statistics (2)

Talks

Invited Talks

Department of Mathematics Colloquium, Salisbury University	October 2013
Graduate Student Intercollegiate Mathematics Seminar (GSIMS), Lehigh University	October 2013
Graduate Student Seminar, Wesleyan University	February 2013
GSIMS, Lehigh University	February 2013
GSIMS, Lehigh University	April 2011
GSIMS, Lehigh University	September 2009
Society of Undergraduate Mathematics Students, University of Rochester	April 2008

Contributed Talks

Joint Mathematics Meetings, Baltimore	January 2014
Graph Theory Day Sixty-Six, Purchase College, SUNY	November 2013
AMS Fall Eastern Sectional Meeting, Temple University	October 2013

MAA MD-DC-VA Section Spring 2013 Meeting, Salisbury University April 2013
Joint Mathmatics Meetings, San Diego January 2008

Publications

(In Progress) *Placing vertices on a hamiltonian cycle*,
with S. Fujita, R. J. Gould, and C. Magnant.

(In Progress) *A proof of the Enomoto-Ota Conjecture for large graphs*,
with V. Coll, C. Magnant, and P. Salehi.

(Submitted) *Semi-linkage with distance constraints*,
with V. Coll and C. Magnant.

(Submitted) *On semi-linkage in large graphs*,
with C. Magnant and H. Wang, Discrete Mathematics.

(Submitted) *H-linked graphs with prescribed lengths*,
with V. Coll and C. Magnant, Journal of Graph Theory.

3: *A decomposition of Gallai multigraphs*,
with C. Magnant and K. Pula, Discussiones Mathematicae Graph Theory Vol. 34, Issue 2 (2014) Pages 331–352.

2: *On distance between graphs*,
with C. Magnant and D.M. Martin, Graphs and and Combinatorics Vol. 29, Issue 5 (2013) Pages 1391–1402.

1: *Long path lemma concerning connectivity and independence number*,
with S. Fujita and C. Magnant, Electronic Journal of Combinatorics Vol. 18(1), (2011) P149.

Grants and Honors

Academic Dean's List, University of Rochester	August 2004–May 2008
Research Experience for Undergraduates, University of Akron	June–July 2007
Meliora Grant, University of Rochester	August 2004
Rush Rhees Scholarship, University of Rochester	August 2004
NY Lottery Leaders of Tomorrow Scholarship, University of Rochester	August 2004

Memberships

American Mathematical Society	2008–Present
Society for Industrial and Applied Mathematics	2009–2010

Service Activities

Member, Lehigh University Committee on Discipline	2013–2014
Session Moderator, MAA MD-DC-VA Section Spring 2013 Meeting	2013
President, Graduate Student Intercollegiate Mathematics Seminar	2011–2013
Tutor, Lehigh University Math Help & Study Center	2008–present
Treasurer, Society of Undergraduate Mathematics Students	2007–2008

Computer Skills

Languages & Software

LaTeX, C++, Maple, WebAssign, WeBWorK, Course Site

Operating Systems

Linux, Windows