2016

The Dobric-Ojeda Process with Applications to Option Pricing and the Stochastic Heat Equation

Mackenzie Wildman
Lehigh University

The Dobrić-Ojeda Process with Applications to Option Pricing and the Stochastic Heat Equation

by

Mackenzie Wildman

A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

Lehigh University
May 23, 2016

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Mackenzie Wildman
The Dobrić-Ojeda Process with Applications to Option Pricing and the Stochastic Heat Equation

Date

Daniel Conus, PhD, Dissertation Director, Chair

Accepted Date

Committee Members

Bennett Eisenberg, PhD

Don Davis, PhD

Thomas Gerrity, PhD

Acknowledgments

First and foremost, I am grateful I had the opportunity to work with Professor Vladimir Dobrić. He instilled in me a passion for tackling problems in finance from the perspective of a mathematician. I miss him very much and I hope to honor his memory by continuing the work we began together with the same contagious passion that originally drew me in.

I am extremely grateful to Professor Daniel Conus for his seamless transition into the role of my advior and for all of the time and effort he has spent guiding me through this journey and preparing me for what's next. Any success in my future as a mathematician will be a direct result of his investment in me. I also acknowledge the Mathematics Department at Lehigh, especially Mary Ann Dent, for all of their support, especially throughout unusual circumstances. It takes a village to raise a mathematician!

To my fellow graduate students - I have thoroughly enjoyed pursuing this feat together. To my mother, thank you for making me carry my own violin case and for teaching me to follow my dreams, especially when it's hard; because if it were easy, it would have already been done by now! Thank you, Dad, for all of your words of encouragement, from emergency pep talks to the notes you sent in the mail every week. Ian, I could not have done this without you. "You and me, together, we can do anything."

Finally, I would like to dedicate this dissertation to B. Thank you for constantly reminding me to take on each new challenge with vigor... and that there's always time to take a break.

Contents

List of Tables

List of Figures

Abstract

Replacing Black-Scholes' driving process, Brownian motion, with fractional Brownian motion allows for incorporation of a past dependency of stock prices but faces a few major downfalls, including the occurrence of arbitrage when implemented in the financial market. We present the development, testing, and implementation of a simplified alternative to using fractional Brownian motion for pricing derivatives. By relaxing the assumption of past independence of Brownian motion but retaining the Markovian property, we are developing a competing model that retains the mathematical simplicity of the standard Black-Scholes model but also has the improved accuracy of allowing for past dependence. This is achieved by replacing Black-Scholes' underlying process, Brownian motion, with the Dobrić-Ojeda process. In the second half of the dissertation, we introduce a Dobrić-Ojeda type stochastic noise. This noise is intended to serve as an approximation for fractional noise in a partial differential equation. We implement this Dobrić-Ojeda noise in the stochastic heat equation and compare the solution to the analogue with fractional noise. As in option pricing, we aim to provide a more mathematically tractable alternative to fractional noise with similar properties.

Chapter 1

Introduction

Under the Nobel prize-winning Black-Scholes model for pricing financial derivatives [3], we assume that the underlying stock price $(S_t)_{t\in[0,\infty)}$ behaves according to the stochastic differential equation (SDE)

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \tag{1.1}$$

with initial condition $S(0) = S_0 \in \mathbb{R}^+$ and where $(W_t)_{t\in[0,\infty)}$ is a standard Brownian motion process. The solution to this SDE is achieved using Itô calculus:

$$S_t = S_0 \exp\left\{\sigma W_t + \mu t - \frac{1}{2}\sigma^2 t\right\}.$$

Recall a few of the assumptions imposed by this model: the short-term interest rate r is known and constant, there are no transaction costs, stock prices have constant and known volatility σ and drift μ, changes in stock price are log normally distributed, and future stock prices are independent of past. The current study of Option Pricing Theory largely consists of relaxing one or more of the assumptions of the standard model and studying the result. Incorporating a stochastic volatility into the model relaxes the assumption that the underlying stock has constant volatility as in, for example, Hull [15] and Heston [10]. A Black-Scholes model that incorporates transaction costs was developed by Leland [20]. Incorporating a jump-diffusion process instead of Brownian motion is one way to relax the Gaussian property of log returns, as first considered by Merton [23]. Use of Brownian noise in the stock price process imposes the assumption that the log increments in stock price are independent over disjoint time intervals. One way to relax this assumption is by using fractional Brownian motion in the SDE (1.1) in place of Brownian motion.

Fractional Brownian motion, introduced by Mandelbrot and van Ness [22], is a Wiener process generalized to incorporate time dependence through an additional parameter, the *Hurst index* H, which measures the intensity of long-range dependence.

Definition 1.0.1. *Fractional Brownian motion is a real-valued Gaussian process* $(Z_H(t))_{t\in[0,\infty)}$*, where* $H \in (0,1)$*, such that* $Z_H(0) = 0$ *almost surely and*

$$\mathbb{E}[Z_H(t)Z_H(s)] = \tfrac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}.$$

Note that when $H = \frac{1}{2}$, this is equivalent to a standard Brownian motion process. For values of $H > \frac{1}{2}$, the increments of the process are positively correlated and the closer H is to 1, the stronger long-memory the process exhibits. Conversely, if $H < \frac{1}{2}$, the increments of fractional Brownian motion are negatively correlated. Hu and Øksendal [13] and Sottinen [28] have replaced Brownian motion with fractional Brownian motion in the Black-Scholes SDE:

$$dS_t = S_t(\mu\, dt + \sigma\, dZ_H(t)).$$

Hu and Øksendal [13] achieve a solution to this differential equation using Wick calculus:

$$S_t = S_0 \exp\left\{\sigma Z_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right\}.$$

One motivation for incorporating past dependency of stock prices is given by an empirical study of daily returns from 1962 to 1987 [25], which shows the Hurst index of the S&P 500 Index is approximately 0.61 with a 95% confidence interval of (0.57,0.69). If the index price showed no past dependency, we would expect the Hurst index to be 0.5. (Also see arguments that log returns have long-range dependence in [21] and [26].) A major disadvantage, however, to this model is that it results in a non-semi-martingale stock price process. This allows for arbitrage in the financial markets and it fails to admit an explicit hedging strategy through the use of Wick calculus instead of Itô calculus. See, for example [28] and its references.

With these issues surrounding the use of fractional Brownian motion in mind, we introduce and implement the "Dobrić-Ojeda process", as originally defined in [6]. The *Dobrić-Ojeda process* is a temporally dependent Gaussian Markov process with similar properties to those of fractional Brownian motion, and we propose this process as an alternative to fractional Brownian motion in the Black-Scholes stochastic differential equation (1.1). Following [6], we define the Dobrić-Ojeda process by first considering the fractional Gaussian field $Z = (Z_H(t))_{(t,H)\in[0,\infty)\times(0,1)}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by the covariance

$$\mathbb{E}\{Z_H(t)Z_{H'}(s)\} = \frac{a_{H,H'}}{2}\{|t|^{H+H'} + |s|^{H+H'} - |t-s|^{H+H'}\},$$

where

$$a_{H,H'} = \begin{cases} -\frac{2}{\pi}\sqrt{\Gamma(2H+1)\sin(\pi H)}\sqrt{\Gamma(2H'+1)\sin(\pi H')} & \\ \times\Gamma\left(-(H+H')\right)\cos\left((H'-H)\frac{\pi}{2}\right)\cos\left((H+H')\frac{\pi}{2}\right) & \text{for } H+H' \neq 1 \\ \sqrt{\Gamma(2H+1)\Gamma(3-2H)}\sin^2(\pi H) =: a_H =: a_{H'} & \text{for } H+H' = 1, \end{cases}$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the usual Gamma function. Existence of this field was established in [7]. Note that when $H = H'$, Z_H is a fractional Brownian motion process and when $H = H' = \frac{1}{2}$, Z_H is a standard Brownian motion process. On this field, for the case $H + H' = 1$, define the process

$$M_H = (M_H(t))_{t\in[0,\infty)} = (\mathbb{E}(Z_{H'}(t)|\mathcal{F}_t^H))_{t\in[0,\infty)} \tag{1.2}$$

where

$$\mathcal{F}_t^H = \sigma(Z_H(r) : 0 \leq r \leq t).$$

As proved in Proposition 2.1.1 below, the process $M_H(t)$ is a martingale with respect to $\mathcal{F}_t^H$. This fact is stated without proof in [6]. The second moment of $M_H(t)$ is given by

$$\mathbb{E}[M_H^2(t)] = c_M t^{2-2H}, \tag{1.3}$$

where $c_M = \frac{a_H^2 \Gamma(3/2-H)}{2H\Gamma(H+1/2)\Gamma(3-2H)}$ [6]. We will also show that $M_H(t)$ is Gaussian centered with independent increments and covariance $\mathbb{E}[M_H(t)M_H(s)] = c_M\,(s \wedge t)^{2-2H}$ (see Proposition 2.1.2).

We use this process $M_H(t)$ to capture some of the information of fractional Brownian motion by projecting a fractional Brownian motion onto the fractional Gaussian field Z.

We seek a process that approximates fractional Brownian motion and has the form $\Psi_H(t)M_H(t)$, where $\Psi_H(t)$ is some deterministic coefficient. We find such a coefficient for M_H to minimize the least-squares difference from Z_H, given by $\mathbb{E}(Z_H(t) - \Psi_H(t)M_H(t))^2$. Since this expectation is quadratic in Ψ_H, the minimizing Ψ_H is given by

$$\Psi_H(t) := \frac{\mathbb{E}(Z_H(t)M_H(t))}{\mathbb{E}M_H^2(t)}.$$

A closed form solution for $\Psi_H(t)$ is found in [6]:

$$\Psi_H(t) = \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{a_H\Gamma(3/2-H)} t^{2H-1} := c_\Psi t^{2H-1}.$$

We can finally define the Dobrić-Ojeda process $(V_H(t))_{t\in[0,\infty]}$ as

$$V_H(t) = \Psi_H(t)M_H(t) \tag{1.4}$$

where

$$\Psi_H(t) = c_\Psi t^{2H-1}$$

and

$$M_H(t) = \mathbb{E}[Z_{H'}(t)|\mathcal{F}_t^H],$$

where $H + H' = 1$. Note that when $H = \frac{1}{2}$, the process $V_H(t)$ is a Brownian motion.

To understand how closely the Dobrić-Ojeda process V_H approximates fractional Brownian motion Z_H, consider the difference process

$$Y_H(t) := Z_H(t) - V_H(t).$$

As proved in [6],

$$\mathbb{E}Y_H^2(t) = d_H^2 t^{2H} = d_H^2 \mathbb{E}Z_H^2(t),$$

for

$$d_H^2 = 1 - 2H\frac{\Gamma(1/2+H)\Gamma(3-2H)}{\Gamma(3/2-H)}.$$

Therefore, for $H > 1/4$, V_H approximates Z_H with a relative L^2 error of at most 32%. Moreover, for $H \in (0.4, 1)$, which we expect to be reasonable in most markets, V_H approximates Z_H with a relative L^2 error of at most 12%. We expect that H is approximately 0.6 in a typical market and rarely less than 0.4, as described and cited above.

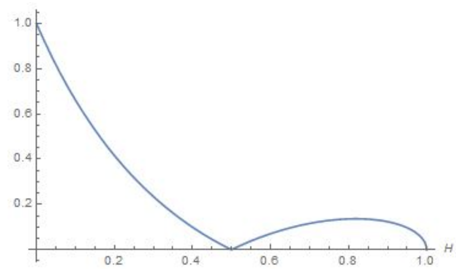

Figure 1.1: Graph of d_H.

One useful property of the Dobrić-Ojeda process is that it has an Itô diffusion representation and is a semi-martingale. See Proposition 2.2.1.

The major goal of the first half of this dissertation is to apply the Dobrić-Ojeda process as noise in the Black-Scholes SDE (1.1):

$$dS_t = S_t(\mu\, dt + \sigma\, dV_H(t)).$$

We emphasize that when $H = 1/2$ this is equivalent to the original Black-Scholes SDE. The main advantage to the Dobrić-Ojeda process, however, is its semi-martingale property that allows for use of Itô calculus.

In order to price options, the next natural step is to describe a risk-neutral measure for this model. This does not follow directly as in the Black-Scholes model due to the $1/t$ term in the drift, as we illustrate in Proposition 3.1.1. This causes explosion of the expectation of the process

$$\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 ds\right) \tag{1.5}$$

at 0. To remedy this issue, we define a modified Dobrić-Ojeda process in which the drift is 0 until time $t = \epsilon > 0$. Under the modified Dobrić-Ojeda process we achieve a risk-neutral measure using Novikov's condition [24]. In the case of a European call option, we find a price formula under this risk-neutral measure:

$$F_t = S_t^\epsilon \Phi\left(\sigma C\sqrt{\frac{T^{2H} - t^{2H}}{2H}} - d_1\right) - Ke^{-r(T-t)}\Phi(-d_1),$$

where C is a deterministic constant and as usual, T is the expiration, K is the strike price, Φ is the standard normal cumulative distribution function, and

$$d_1 = \frac{\ln\left(\frac{K}{S_t^\epsilon}\right) - r(T-t) + \frac{1}{2}\sigma^2 C^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)}{\sigma C\sqrt{\frac{T^{2H}-t^{2H}}{2H}}}.$$

Formal convergence of these ϵ-measures to a risk-neutral measure for S_t remains an open problem.

We conclude the first half of this dissertation by discussing techniques for estimating the Hurst index, H, and volatility, σ, using historical prices of the underlying asset, following with a comparison of historical option prices computed using Brownian motion, fractional Brownian motion, and the Dobrić-Ojeda process in the Black-Scholes SDE. We find that the model using the Dobrić-Ojeda process does, in fact, approximate the option price given using fractional Brownian motion when the parameter H is similar. When using a smaller value for the Hurst index H, however, the Dobrić-Ojeda process appears to outperform the competing models.

The study of stochastic differential equations is a leading topic in current mathematics with countless applications to fields as disparate as physics, engineering, biology, and finance. In Chapters 2-5, we study the Black-Scholes stochastic differential equation (1.1), which incorporates derivatives with respect to time. The Black-Scholes equation is an example of a stochastic ordinary differential equation. It is called "ordinary" because the solution S_t is a function of one parameter, time t and it is called "stochastic" because randomness is introduced through the Brownian motion process $(W_t)_{t\geq 0}$ so that the process S_t also depends on the outcome ω of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We have shown that, by changing the driving process $(W_t)_{t\geq 0}$ to the Dobrić-Ojeda process $(V_t)_{t\geq 0}$, the solution S_t is changed to allow for past dependence and correlated log increments.

In the study of partial differential equations, a space parameter is incorporated as well. For example, the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.6}$$

describes the dissipation of heat on a surface as a function of time t and position x. Other examples of differential equations are the Navier-Stokes equation, which models the motion of viscous fluids, Burgers' equation, which is used to study traffic flow and nonlinear acoustics, and the wave equation, which describes the movement of waves such as sound waves and water waves. See, for instance, [8]. A stochastic element can be introduced to these equations as well. For example, a stochastic element added to the heat equation describes the temperature of a surface when a random external heat source is applied. If this random external heat source is of a Gaussian type, and this force behaves independently over disjoint sets in x and t, then it is called *white noise* $\dot{W}$ and this heat equation is modeled as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)\dot{W}. \tag{1.7}$$

Recall that the Brownian field is nowhere differentiable; finding a solution $u(x, t)$ to this equation in a straight-forward manner is a hopeless task. Instead, we seek a "mild" solution of integral form, where, of the multiple methods studied, we consider the integral with respect to stochastic noise as a *Walsh integral*, as developed by John Walsh [29]. This method begins with defining a *martingale measure*, a process $(M_t(A))_{t\in[0,\infty),A\in\mathcal{B}}$ that for fixed t is a measure on Borel sets $\mathcal{B}$ and for fixed A is a martingale process. This martingale measure is then integrated against.

White noise is just one type of stochastic noise that can be applied to a differential equation. One area of current study is in implementing fractional noise, a stochastic noise that behaves like Brownian motion in x and fractional Brownian motion in t. Phenomena that are better suited to modeling by fractional noise than white noise include cyclic economic time series, fluctuations in solids, water levels of a river, and of course, the log returns of a stock [22]. Studying physical properties of solutions to stochastic partial differential equations driven by fractional noise is significantly more complicated than studying their white noise counterparts, since the same Walsh integral approach relies on the use of Itô calculus which is not compatible with fractional Brownian motion. This motivates our introduction of a Dobrić-Ojeda type noise to gain understanding of the physical properties of solutions to equations with fractional noise while using a more mathematically tractable approach.

We begin this study in Chapter 6 by defining a martingale measure that corresponds to the martingale process $(M_H(t))_{t\in[0,\infty)}$ defined in (1.2). This will allow us to apply techniques of Walsh integration to define an integral with respect to a Dobrić-Ojeda noise $\dot{V}$. Next we apply this noise to the stochastic heat equation.

Intuitively, the heat equation (1.7) describes the distribution of heat along an infinitely thin, infinitely long wire as it evolves over time. The wire has initial temperature $u_0(x)$ at every point $x \in \mathbb{R}$ along the wire. A random external heat source is applied to the wire with proportion $f(u(x,t))$ to the current temperature of the wire at position x. This equation has been thoroughly studied, both in the case of white noise and fractional noise. For white noise, see [29] and [18] for dimension $d = 1$ and [5] for dimension $d > 1$. For the stochastic heat equation with fractional noise, see for example, [1], [11] and [12]. In the case of fractional noise, a solution to the heat equation exists for $H > 1/2$. It is conjectured that a solution only exists for $H > 3/8$ (see [12]), but both existence for $3/8 < H < 1/2$ and non-existence when $H < 3/8$ appear to remain unproven to date. The Hölder continuity of the stochastic heat equation with this fractional noise is of order $1/4$ in time and $1/2$ in space, again only when $H > 1/2$ [14]. We explore the stochastic heat equation with Dobrić-Ojeda noise and compare our results with those of fractional noise.

We prove the existence and uniqueness of a mild solution to the stochastic heat equation with a Dobrić-Ojeda noise for $H > 1/4$. Note that the existence and uniqueness of a solution to the stochastic heat equation with fractional noise is only proven for $H > 1/2$. We also establish the Hölder continuity of the solution, which exhibits similar properties to the Hölder continuity of the solution to the stochastic heat equation driven by fractional Brownian noise. Table 1.1 shows the upper bound for the order of Hölder continuity of the solution to the stochastic heat equation with Dobrić-Ojeda noise. For example, the mild solution $u(x,t)$ is Hölder continuous of any order up to 1/4 in time, for $H \geq 1/2$. See Theorems 7.3.7 and 7.3.12 in Chapter 7.

H	$(1/4, 1/2)$	$[1/2, 1)$
Time	$H - 1/4$	$1/4$
Space	$2H - 1/2$	$1/2$

Table 1.1: Hölder continuity of the stochastic heat equation with Dobrić-Ojeda noise.

In the case $H > 1/2$ the orders of Hölder continuity are the same with both Dobrić-Ojeda and fractional noise. The solution to the stochastic heat equation with white noise also has Hölder continuity of order 1/4 in time and 1/2 in space. This shows that incorporating a "nicer" noise, or $H > 1/2$, does not improve the orders of continuity. For $H < 1/2$, we are not aware of an explicit result for the comparable noise that is white in space and fractional in time.

We point out that the stochastic heat equation is related to the fashionable Kardar-Parisi-Zhang (KPZ) equation [17]

$$\frac{\partial h}{\partial t} = \Delta h - |\nabla h|^2 + \lambda \dot{W}(t,x), \tag{1.8}$$

which models growth interfaces such as the movement of galaxies or the clustering of bacteria. The Hopf-Cole transformation, a type of logarithmic transformation, connects the KPZ equation to the stochastic heat equation. This connection was formalized by Martin Hairer [9], and this work was awarded a 2014 Fields medal.

Chapter 2

The Dobrić-Ojeda process

In this chapter we prove a few properties of the Dobrić-Ojeda process, as defined in Chapter 1.

2.1 Properties of $M_H(t)$

First note that the process $M_H(t)$ is Gaussian for all $t > 0$ because it is the conditional expectation of a Gaussian process, $Z_H(t)$. The process $M_H(t)$ also satisfies, by definition, $\mathbb{E}[M_H(t)] = 0$ and, by [6], $\mathbb{E}[M_H^2(t)] = c_M t^{2-2H}$. The following proposition is stated without proof in [6].

Proposition 2.1.1. *The process $M_H(t)$ is a martingale with respect to $\mathcal{F}_t^H$.*

Proof. First we will show for $t > 0$, $\mathbb{E}[|M_H(t)|] < \infty$. Since $M_H(t)$ is Gaussian centered, with variance $\sigma^2 = c_M t^{2-2H}$, we have

$$\begin{aligned}
&\mathbb{E}[|M_H(t)|] \\
=&\mathbb{E}[|M_H(t)||M_H(t) \geq 0]\mathbb{P}(M_H(t) \geq 0) + \mathbb{E}[|M_H(t)||M_H(t) < 0]\mathbb{P}(M_H(t) < 0) \\
=&\frac{1}{2}\left(\mathbb{E}[M_H(t)|M_H(t) \geq 0] + \mathbb{E}[-M_H(t)|M_H(t) < 0]\right) \\
=&\mathbb{E}[M_H(t)|M_H(t) \geq 0] \\
=&\frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2\sigma^2} dx \\
=&\frac{\sqrt{2}\sigma}{\sqrt{\pi}} \\
=&\frac{\sqrt{2c_M t^{2-2H}}}{\sqrt{\pi}} \\
=&\frac{\sqrt{2c_M} t^{1-H}}{\sqrt{\pi}} \\
<&\infty.
\end{aligned} \tag{2.1}$$

Next we will show for $0 \leq s < t$, $\mathbb{E}[M_H(t)|\mathcal{F}_s^H] = M_H(s)$. By the Tower Rule and by the definition of $(M_H(t))$ (1.2), we have

$$\begin{aligned}
\mathbb{E}[M_H(t)|\mathcal{F}_s^H] =&\mathbb{E}\left[\mathbb{E}(Z_{H'}(t)|\mathcal{F}_t^H)|\mathcal{F}_s^H\right] \\
=&\mathbb{E}\left[Z_{H'}(t)|\mathcal{F}_s^H\right] \\
=&\mathbb{E}\left[Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H\right] + \mathbb{E}\left[Z_{H'}(s)|\mathcal{F}_s^H\right] \\
=&\mathbb{E}\left[Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H\right] + M_H(s).
\end{aligned} \tag{2.2}$$

It remains to show that $\mathbb{E}\left[Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H\right] = 0$. Fix $V \in \mathcal{F}_s^H$. Without loss of generality, let $V = \mathbb{1}_{\{Z_H(u) \in B\}}$ for some $u \leq s$ and where B is a Borel set. Then

$$\begin{aligned}
&\mathbb{E}[V(Z_{H'}(t) - Z_{H'}(s))] \\
=&\mathbb{E}[\mathbb{1}_{\{Z_H(u)\in B\}}(Z_{H'}(t) - Z_{H'}(s))] \\
=&\mathbb{E}[\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(t)] - \mathbb{E}[\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(s)].
\end{aligned} \tag{2.3}$$

To simplify notation, let $X = Z_H(u)$ and $Y = Z_{H'}(t)$. Then

$$
\begin{aligned}
&\mathbb{E}[\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(t)] \\
=&\mathbb{E}[\mathbb{1}_{\{X\in B\}} Y] \\
=&\int_B \int_{\mathbb{R}} y f_{X,Y}(x,y)\, dy\, dx \\
=&\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_B \int_{\mathbb{R}} y \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right]\right) dy\, dx \qquad (2.4) \\
=&\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_B \exp\left(-\frac{1}{2(1-\rho^2)}\frac{x^2}{\sigma_X^2}\right) \\
&\times \int_{\mathbb{R}} y \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right]\right) dy\, dx.
\end{aligned}
$$

First, we compute the integral with respect to y:

$$
\begin{aligned}
&\int_{\mathbb{R}} y \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right]\right) dy \\
=&\int_{\mathbb{R}} y \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{\rho^2 x^2}{\sigma_X^2}\right] + \frac{\rho^2 x^2}{2(1-\rho^2)\sigma_X^2}\right) dy \\
=&\int_{\mathbb{R}} y \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{y}{\sigma_Y} - \frac{\rho x}{\sigma_X}\right]^2\right) \exp\left(\frac{\rho^2 x^2}{2(1-\rho^2)\sigma_X^2}\right) dy \\
=&\exp\left(\frac{\rho^2 x^2}{2(1-\rho^2)\sigma_X^2}\right) \frac{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \int_{\mathbb{R}} y \exp\left(-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left[y - \frac{\rho\sigma_Y x}{\sigma_X}\right]^2\right) dy \\
=&\exp\left(\frac{\rho^2 x^2}{2(1-\rho^2)\sigma_X^2}\right) \sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}\frac{\rho\sigma_Y x}{\sigma_X}. \qquad (2.5)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\int_B \exp\left(-\frac{1}{2(1-\rho^2)}\frac{x^2}{\sigma_X^2}\right)\\
&\quad\times\int_{\mathbb{R}} y\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right]\right)\,dy\,dx\\
=&\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\int_B \exp\left(-\frac{1}{2(1-\rho^2)}\frac{x^2}{\sigma_X^2}\right)\exp\left(\frac{\rho^2x^2}{2(1-\rho^2)\sigma_X^2}\right)\\
&\quad\times\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}\frac{\rho\sigma_Y x}{\sigma_X}\,dx\\
=&\frac{\rho\sigma_Y}{\sqrt{2\pi}\sigma_X^2}\int_B x\exp\left(-\frac{x^2}{2(1-\rho^2)\sigma_X^2}+\frac{\rho^2x^2}{2(1-\rho^2)\sigma_X^2}\right)\,dx\\
=&\frac{\rho\sigma_Y}{\sqrt{2\pi}\sigma_X^2}\int_B x\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\,dx\\
=&\frac{\mathbb{E}[XY]}{\sigma_X^2}\frac{1}{\sigma_X\sqrt{2\pi}}\int_{\mathbb{R}} x\mathbb{1}_{x\in B}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\,dx\\
=&\frac{\mathbb{E}[XY]}{\sigma_X^2}\mathbb{E}\left[\mathbb{1}_{X\in B}X\right]\\
=&\frac{\mathbb{E}[Z_H(u)Z_{H'}(t)]}{\mathbb{E}[Z_H^2(u)]}\mathbb{E}\left[\mathbb{1}_{Z_H(u)\in B}Z_H(u)\right]\\
=&\frac{a_Hu}{\mathbb{E}[Z_H^2(u)]}\mathbb{E}\left[\mathbb{1}_{Z_H(u)\in B}Z_H(u)\right].
\end{aligned}
\tag{2.6}
$$

Similarly, $\mathbb{E}[\mathbb{1}_{\{Z_H(u)\in B\}}Z_{H'}(s)] = \frac{a_Hu}{\mathbb{E}[Z_H^2(u)]}\mathbb{E}\left[\mathbb{1}_{Z_H(u)\in B}Z_H(u)\right]$. This shows $\mathbb{E}[V(Z_{H'}(t)-Z_{H'}(s))] = 0$ for all random variables $V \in \mathcal{F}_s^H$ and so $\mathbb{E}\left[Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H\right] = 0$. □

Proposition 2.1.2. *The martingale process $(M_H(t))_{t\in[0,\infty)}$ has independent increments and covariance $\mathbb{E}[M_H(t)M_H(s)] = c_M\,(s\wedge t)^{2-2H}$.*

Proof. Assume without loss of generality that $s < t$. Then by Proposition 2.1.1 and

(1.3) above,

$$\begin{aligned}\mathbb{E}[M_H(t)M_H(s)] =&\mathbb{E}\left[\left((M_H(t)-M_H(s))+M_H(s)\right)M_H(s)\right]\\ =&\mathbb{E}\left[(M_H(t)-M_H(s))M_H(s)\right]+\mathbb{E}\left[(M_H(s))^2\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[(M_H(t)-M_H(s))M_H(s)|\mathcal{F}_s^H\right]\right]+c_M s^{2-2H}\\ =&\mathbb{E}\left[M_H(s)\mathbb{E}\left[M_H(t)-M_H(s)|\mathcal{F}_s^H\right]\right]+c_M s^{2-2H}\\ =&\mathbb{E}\left[M_H(s)\mathbb{E}\left[M_H(t)|\mathcal{F}_s^H\right]-\mathbb{E}\left[M_H(s)|\mathcal{F}_s^H\right]\right]+c_M s^{2-2H}\\ =&\mathbb{E}\left[M_H(s)(M_H(s)-M_H(s))\right]+c_M s^{2-2H}\\ =&c_M s^{2-2H}.\end{aligned} \tag{2.7}$$

Therefore,

$$\mathbb{E}[M_H(t)M_H(s)]=c_M\left(s\wedge t\right)^{2-2H}. \tag{2.8}$$

Finally, to prove independence of increments, we again assume that $s<t$ and $h>0$ is small:

$$\begin{aligned}&\mathbb{E}\left[(M_H(t+h)-M_H(t))(M_H(s+h)-M_H(s))\right]\\ =&\mathbb{E}\left[M_H(t+h)M_H(s+h)\right]-\mathbb{E}\left[(M_H(t+h)M_H(s)\right]-\mathbb{E}\left[M_H(t)M_H(s+h)\right]\\ &+\mathbb{E}\left[M_H(t)M_H(s)\right]\\ =&c_M\left((s+h)^{2-2H}-s^{2-2H}-(s+h)^{2-2H}+s^{2-2H}\right)\\ =&0.\end{aligned} \tag{2.9}$$

Since $(M_H(t))$ is Gaussian, this suffices to show $(M_H(t))$ has independent increments. □

Next we will prove that the quadratic variation of the martingale process $(M_H(t))$ from 0 to t is given by $c_M t^{2-2H}$. First we will prove the following lemma, to be used in the proof of Proposition 2.1.4 and later in Theorem 4.2.2.

Lemma 2.1.3. *The following approximation holds for even moments of $M_t = M_H(t)$:*

$$\mathbb{E}[(\Delta M_{t_i})^{2k}] \le \begin{cases}(2k-1)!!\left(c_M(2-2H)t_i^{1-2H}\Delta t_i\right)^k & \text{if } H<1/2\\ (2k-1)!!\left(c_M(2-2H)t_{i-1}^{1-2H}\Delta t_i\right)^k & \text{if } H\ge 1/2,\end{cases} \tag{2.10}$$

where $k\ge 1$ and $\Delta M_{t_i} = M_{t_i} - M_{t_{i-1}}$.

Proof. Using (1.3) and the Mean Value Theorem,

$$\begin{aligned}\mathbb{E}[(\Delta M_{t_i})^2] =&\mathbb{E}[M_{t_i}^2] - 2\mathbb{E}[M_{t_i}M_{t_{i-1}}] + \mathbb{E}[M_{t_{i-1}}]\\ =&c_M t_i^{2-2H} - 2\mathbb{E}[(\Delta M_{t_i} + M_{t_{i-1}})M_{t_{i-1}}] + c_M t_{i-1}^{2-2H}\\ =&c_M t_i^{2-2H} - 2\mathbb{E}[\Delta M_{t_i} M_{t_{i-1}}] - 2\mathbb{E}[M_{t_{i-1}}^2] + c_M t_{i-1}^{2-2H}\\ =&c_M t_i^{2-2H} - 2c_M t_{i-1}^{2-2H} + c_M t_{i-1}^{2-2H} \qquad (2.11)\\ =&c_M(t_i^{2-2H} - t_{i-1}^{2-2H})\\ \le&\begin{cases} c_M(2-2H)t_i^{1-2H}\Delta t_i & \text{if } H < 1/2\\ c_M(2-2H)t_{i-1}^{1-2H}\Delta t_i & \text{if } H \ge 1/2.\end{cases}\end{aligned}$$

Since the process (M_t) is Gaussian, the result follows that for $k \ge 1$, as required. □

Proposition 2.1.4. *For $n > 0$, let $t_i = \frac{it}{n}$, $i = 0, ..., n$ be a partition sequence of $[0, t]$ and $M_t = M_H(t)$ as defined in* (1.2). *Then*

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} (\Delta M_{t_i})^2 - c_M t^{2-2H} \right\|_2 = 0$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\Delta M_{t_i})^2 = c_M t^{2-2H} \qquad a.s.$$

where $\Delta M_{t_i} = M_{t_i} - M_{t_{i-1}}$.

Proof. By the Triangle Inequality and the definition of a definite Riemann integral, it suffices to prove that the difference of the sample quadratic variation of M_t and $c_M t^{2-2H} = c_M(2-2H)\sum_{i=1}^{n} t_i^{1-2H}\Delta t$ converges to 0 as $n \to \infty$. Using the independent increments of M_t as proved in Proposition 2.1.2, we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} (\Delta M_{t_i})^2 - c_M(2-2H) \sum_{j=1}^{n} t_j^{1-2H} \Delta t \right\|_2 \\
=&\mathbb{E}\left[\left(\sum_{i=1}^{n} (\Delta M_{t_i})^2 - c_M(2-2H) \sum_{j=1}^{n} t_j^{1-2H} \Delta t \right)^2 \right] \\
=&\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[(\Delta M_{t_i})^2\right] \mathbb{E}\left[(\Delta M_{t_j})^2\right] - 2c_M(2-2H) \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[(\Delta M_{t_i})^2\right] t_j^{1-2H} \Delta t \\
&+ c_M^2(2-2H)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} t_i^{1-2H} t_j^{1-2H} (\Delta t)^2.
\end{aligned} \tag{2.12}$$

By Lemma 2.1.3, this is bounded above by 0 for either $H < 1/2$ or $H \geq 1/2$. □

2.2 Properties of $V_H(t)$

Next, we show that the Dobrić-Ojeda process has an Itô diffusion representation.

Proposition 2.2.1. *There exists a Brownian motion process $(W_t)_{t\in[0,\infty)}$ adapted to the filtration $(\mathcal{F}_t^H)_{t\in[0,\infty)}$ such that the Dobrić-Ojeda process $(V_H(t))_{t\in[0,\infty)}$ is an Itô diffusion process, satisfying the stochastic differential equation*

$$dV_H(t) = Ct^{H-1/2} dW_t + (2H-1)t^{-1} V_H(t) dt,$$

where $C = c_\Psi \sqrt{c_M(2-2H)}$.

Proof. By Proposition 2.1.4, the quadratic variation of $(M_H(t))$ is given by $[M_H, M_H]_t = c_M t^{2-2H}$. Therefore by the Representation Theorem for Martingales (see [16]), we have $dM_H(t) = \sqrt{c_M(2-2H)}t^{1/2-H} dW_t$, where W_t is a Brownian motion process adapted to the filtration $(\mathcal{F}_t^H)_{t\in[0,\infty)}$. Therefore,

$$\begin{aligned}
dV_H(t) &= d(\Psi_H(t)M_H(t)) \\
&= \Psi_H(t)dM_H(t) + M_H(t)d\Psi_H(t) \\
&= \Psi_H(t)\sqrt{c_M(2-2H)}t^{1/2-H}dW_t + (\Psi_H(t)^{-1}V_H(t))d(c_\Psi t^{2H-1}) \\
&= c_\Psi\sqrt{c_M(2-2H)}t^{2H-1}t^{1/2-H}dW_t + c_\Psi^{-1}t^{-2H+1}V_H(t)c_\Psi(2H-1)t^{2H-2}dt \\
&= c_\Psi\sqrt{c_M(2-2H)}t^{H-1/2}dW_t + (2H-1)t^{-1}V_H(t)dt.
\end{aligned}$$

Notice that this equation is well defined since $V_H(t)$ is of the order t^H. □

Note that the martingale part of this representation has a similar form to the Riemann-Liouville fractional integral $Z_H(t) = \frac{1}{\Gamma(H+1/2)}\int_0^t (t-s)^{H-1/2}dW_s$ (see [2]), but is non-anticipating and therefore Itô integrable while the fractional integral is not. We consider that the drift term of the diffusion compensates for this difference and works to imitate fractional Brownian motion while remaining a semi-martingale process.

A closed-form equation for the quadratic variation of the Dobrić-Ojeda process immediately follows:

Proposition 2.2.2. *The quadratic variation of* $(V_t)_{t\in[0,\infty)}$ *is given by*

$$[V,V]_t = \frac{C^2}{2H}t^{2H},$$

where $C = c_\Psi\sqrt{c_M(2-2H)}$*, as above.*

Proof. By Proposition 2.2.1, we have

$$[V,V]_t = \int_0^t C^2 s^{2H-1}ds = \frac{C^2}{2H}t^{2H}.$$

□

Chapter 3

Option pricing with the Dobrić-Ojeda process

We replace Brownian motion with the Dobrić-Ojeda process in the Black-Scholes stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dV_t). \tag{3.1}$$

To simplify notation, we drop the subscript H from $V_H(t)$. Note that when $H = 1/2$, we have a geometric Brownian motion process, so without loss of generality, we assume $H \neq 1/2$. Using Itô calculus, we can solve for S_t explicitly: Let $Y_t = \ln S_t$. Then we have

$$\begin{aligned} dY_t &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{(S_t)^2} \\ &= \mu dt + \sigma dV_t - \frac{1}{2}\sigma^2 d[V,V]_t, \end{aligned} \tag{3.2}$$

and thus by Proposition 2.2.2,

$$\begin{aligned} Y_t &= Y_0 + \mu t + \sigma V_t - \frac{1}{2}\sigma^2 [V,V]_t \\ &= Y_0 + \mu t + \sigma V_t - \frac{1}{2}\sigma^2 \frac{C^2}{2H} t^{2H}, \end{aligned} \tag{3.3}$$

which implies

$$S_t = S_0 \exp\left\{\mu t + \sigma V_t - \frac{1}{2}\sigma^2 \frac{C^2}{2H} t^{2H}\right\}. \tag{3.4}$$

3.1 Risk-neutral measure

The next natural step towards a comprehensive model for derivative pricing is to establish the existence of a risk-neutral measure. If we were to proceed as usual, we would consider the discounted stock price

$$dZ_t = Z_t(\sigma dV_t + (\mu - r)dt),$$

where r is a constant deterministic interest rate. By Proposition 2.2.1, we have

$$dZ_t = \sigma C t^{H-1/2} Z_t \left(dW_t + \gamma_t dt \right),$$

where

$$\gamma_t = \frac{\mu - r + \sigma(2H-1)t^{-1}V_t}{\sigma C t^{H-1/2}}. \tag{3.5}$$

We seek an equivalent probability measure $\mathbb{Q}$ so that Z_t is a $\mathbb{Q}$-martingale. The standard technique is to invoke Girsanov's Theorem by showing γ_t satisfies Novikov's Condition or Kazamaki's Condition (see [16] or [27]). To date, this remains an open problem as the usual techniques fail to work in this case. For example, we will show that Novikov's Condition fails to be satisfied in the following proposition.

Proposition 3.1.1. *For $0 \leq t \leq T$ and for γ_t as defined in* (3.5)*, we have*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 ds\right)\right] = \infty. \tag{3.6}$$

Proof. We have

$$\gamma_s^2 = A^2 s^{-1-2H} V_s^2 + 2ABs^{-2H} V_s + B^2 s^{1-2H},$$

where A and B are deterministic and constant. Therefore, by Jensen's Inequality and properties of V_t, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 ds\right)\right]\\
\geq &\exp\left(\mathbb{E}\left[\frac{1}{2}\int_0^t \gamma_s^2 ds\right]\right)\\
= &\exp\left(\mathbb{E}\left[\frac{1}{2}\int_0^t \left(A^2 s^{-1-2H} V_s^2 + 2ABs^{-2H}V_s + B^2 s^{1-2H}\right) ds\right]\right)\\
= &\exp\left(\mathbb{E}\left[\frac{1}{2}\int_0^t A^2 s^{-1-2H} V_s^2 ds\right] + \mathbb{E}\left[\frac{1}{2}\int_0^t 2ABs^{-2H}V_s ds\right]\right.\\
&\left. + \mathbb{E}\left[\frac{1}{2}\int_0^t B^2 s^{1-2H} ds\right]\right)\\
= &\exp\left(\mathbb{E}\left[\frac{1}{2}\int_0^t A^2 s^{-1-2H} V_s^2 ds\right]\right)\exp\left(\mathbb{E}\left[\frac{1}{2}\int_0^t 2ABs^{-2H}V_s ds\right]\right)\\
&\times\exp\left(\mathbb{E}\left[\frac{1}{2}\int_0^t B^2 s^{1-2H} ds\right]\right)\\
= &\exp\left(\frac{1}{2}\int_0^t A^2 s^{-1-2H}\mathbb{E}\left[V_s^2\right] ds\right)\exp\left(\frac{1}{2}\int_0^t 2ABs^{-2H}\mathbb{E}\left[V_s\right] ds\right)\\
&\times\exp\left(\frac{1}{2}\int_0^t B^2 s^{1-2H} ds\right)\\
= &\exp\left(\frac{1}{2}\int_0^t A^2 s^{-1-2H} c_M s^{2H} ds\right)\exp\left(\frac{B^2}{2(2-2H)}t^{2-2H}\right)\\
= &\exp\left(\frac{A^2 c_M}{2}\int_0^t s^{-1} ds\right)\exp\left(\frac{B^2}{2(2-2H)}t^{2-2H}\right)\\
= &\infty.
\end{aligned} \tag{3.7}$$

□

The determination of a risk-neutral probability measure without using Girsanov's Theorem remains an open problem. In the meantime, to resolve this issue and find a risk-neutral measure, we replace V_t with V_t^ϵ, defined to be slightly altered from the diffusion process given in 2.2.1. Since the issue lies in the $1/t$ term of the drift, we simply "turn off" the drift until some time $\epsilon > 0$. We can proceed with the standard techniques, as in [27], using the modified Dobrić-Ojeda process V_t^ϵ in the stock price SDE.

Definition 3.1.2. *Let $\epsilon > 0$. Define the Modified Dobrić-Ojeda process, $(V_t^\epsilon)_{t\in[0,\infty)}$, by*

$$dV_t^\epsilon = Ct^{H-1/2}dW_t + c_\Psi(2H-1)t^{2H-2}M_t\mathbb{1}_{[\epsilon,\infty)}(t)dt,$$

where $C = c_\Psi\sqrt{c_M(2-2H)}$ and from this point forward we call $M_H(t) = M_t$ for simplicity of notation.

The drift part of V_t which causes (3.6) to explode at time $t = 0$, is 0 until it "turns on" at time $t = \epsilon$ for any admissible $\epsilon > 0$, as we will see in Proposition 3.1.7. We will proceed towards derivative pricing using the model driven by V_t^ϵ and define an option price. We begin by proving a few properties about V_t^ϵ. First, we will prove the existence of a process (V_t^ϵ) that has this diffusion.

Proposition 3.1.3. *There is a unique solution to 3.1.2.*

Proof. By Definition 3.1.2, we have

$$V_t^\epsilon = C\int_0^t s^{H-1/2}\,dW_s + c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds. \tag{3.8}$$

It remains to show that both integrals are well-defined. First, using Itô Isometry,

$$\mathbb{E}\left[\left(C\int_0^t s^{H-1/2}\,dW_s\right)^2\right] = C^2\int_0^t s^{2H-1}\,ds = \frac{C^2}{2H}t^{2H} < \infty. \tag{3.9}$$

For $t \le \epsilon$, the second integral is 0. To show that the second integral is well-defined for $t > \epsilon$, first note that when $H = 1/2$, the second term is 0. Thus, without loss of generality, assume $H \neq 1/2$. Then we have by Proposition 2.1.2,

$$
\begin{aligned}
&\mathbb{E}\left[\left(c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right)^2\right]\\
=&c_\Psi^2(2H-1)^2\int_\epsilon^t\int_\epsilon^t s_1^{2H-2}s_2^{2H-2}\mathbb{E}\left[M_{s_1}M_{s_2}\right]\,ds_2\,ds_1\\
=&c_\Psi^2(2H-1)^2\int_\epsilon^t\int_\epsilon^t s_1^{2H-2}s_2^{2H-2}c_M(s_1\wedge s_2)^{2-2H}\,ds_2\,ds_1\\
=&2c_Mc_\Psi^2(2H-1)^2\int_\epsilon^t\int_\epsilon^{s_1} s_1^{2H-2}s_2^{2H-2}s_2^{2-2H}\,ds_2\,ds_1\\
=&2c_Mc_\Psi^2(2H-1)^2\int_\epsilon^t s_1^{2H-2}\int_\epsilon^{s_1}\,ds_2\,ds_1\\
=&2c_Mc_\Psi^2(2H-1)^2\int_\epsilon^t s_1^{2H-2}(s_1-\epsilon)\,ds_1\\
=&2c_Mc_\Psi^2(2H-1)^2\int_\epsilon^t (s_1^{2H-1}-\epsilon s_1^{2H-2})\,ds_1\\
=&2c_Mc_\Psi^2(2H-1)^2\left(\frac{1}{2H}(t^{2H}-\epsilon^{2H})-\frac{\epsilon}{2H-1}(t^{2H-1}-\epsilon^{2H-1})\right)\\
<&\infty.
\end{aligned}
\tag{3.10}
$$

This suffices to show that there is a unique solution to V_t^ϵ, as in Definition 3.1.2. □

Proposition 3.1.4. *The modified Dobrić-Ojeda process $(V_t^\epsilon)_{t\in[0,\infty)}$ satisfies, for all $t>0$,*

1. $\mathbb{E}[V_t^\epsilon]=0$ *for all* $\epsilon>0$ *and*
2. $\mathbb{E}[(V_t^\epsilon)^2]=\begin{cases}\frac{C^2t^{2H}}{2H} & \text{if } t\le\epsilon\\ \frac{C^2}{2H}t^{2H}+2C^2(2H-1)\frac{1}{2H}(t^{2H}-\epsilon^{2H}) & \\ +2c_Mc_\Psi^2(2H-1)^2\left(\frac{1}{2H}(t^{2H}-\epsilon^{2H})\right. & \\ \left.-\frac{\epsilon}{2H-1}(t^{2H-1}-\epsilon^{2H-1})\right) & \text{if } t>\epsilon.\end{cases}$

Proof. 1. For $t\le\epsilon$, by Definition 3.1.2, we have

$$\mathbb{E}[V_t^\epsilon]=\mathbb{E}\left[C\int_0^t s^{H-1/2}\,dW_s\right]=0 \tag{3.11}$$

since it's the expectation of a square-integrable Itô integral. For $t > \epsilon$, because the process (M_t) is a martingale and thus has zero expectation, we have

$$\begin{aligned}
\mathbb{E}[V_t^\epsilon] =& \mathbb{E}\left[C\int_0^t s^{H-1/2}\,dW_s + c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right] \\
=& \mathbb{E}\left[C\int_0^t s^{H-1/2}\,dW_s\right] + \mathbb{E}\left[c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right] \\
=& c_\Psi(2H-1)\int_0^t s^{2H-2}\mathbb{E}[M_s]\mathbb{1}_{[\epsilon,\infty)}(s)\,ds \\
=& 0.
\end{aligned} \tag{3.12}$$

2. For $t \leq \epsilon$, we have

$$\mathbb{E}\left[(V_t^\epsilon)^2\right] = \frac{C^2}{2H}t^{2H} \tag{3.13}$$

as in (3.9) above. For $t > \epsilon$, as in (3.10) above, we have

$$\begin{aligned}
&\mathbb{E}\left[(V_t^\epsilon)^2\right] \\
=& \mathbb{E}\left[\left(C\int_0^t s^{H-1/2}\,dW_s + c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right)^2\right] \\
=& \mathbb{E}\left[\left(C\int_0^t s^{H-1/2}\,dW_s\right)^2\right] \\
&+ 2Cc_\Psi(2H-1)\mathbb{E}\left[\int_0^t\int_0^t s_1^{H-1/2}s_2^{2H-2}M_{s_2}\mathbb{1}_{[\epsilon,\infty)}(s_2)\,dW_{s_1}\,ds_2\right] \\
&+ \mathbb{E}\left[\left(c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right)^2\right] \qquad (3.14) \\
=& \frac{C^2}{2H}t^{2H} + 2Cc_\Psi(2H-1)\mathbb{E}\left[\int_\epsilon^t\int_0^t s_1^{H-1/2}s_2^{2H-2}M_{s_2}\,dW_{s_1}\,ds_2\right] \\
&+ 2c_M c_\Psi^2(2H-1)^2\left(\frac{1}{2H}(t^{2H}-\epsilon^{2H}) - \frac{\epsilon}{2H-1}(t^{2H-1}-\epsilon^{2H-1})\right) \\
=& \frac{C^2}{2H}t^{2H} + 2C^2(2H-1)\frac{1}{2H}(t^{2H}-\epsilon^{2H}) \\
&+ 2c_M c_\Psi^2(2H-1)^2\left(\frac{1}{2H}(t^{2H}-\epsilon^{2H}) - \frac{\epsilon}{2H-1}(t^{2H-1}-\epsilon^{2H-1})\right).
\end{aligned}$$

Note that the middle term can be computed using the same Martingale representation as in the proof of Proposition 2.2.1:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_\epsilon^t s_1^{H-1/2} s_2^{2H-2} M_{s_2}\, dW_{s_1}\, ds_2\right] \\
=&\int_\epsilon^t s_2^{2H-2}\mathbb{E}\left[M_{s_2}\int_0^t s_1^{H-1/2}\, dW_{s_1}\right] ds_2 \\
=&\sqrt{c_M(2-2H)}\int_\epsilon^t s_2^{2H-2}\mathbb{E}\left[\int_0^{s_2} u^{1/2-H}\, dW_u \int_0^t s_1^{H-1/2}\, dW_{s_1}\right] ds_2 \\
=&\sqrt{c_M(2-2H)}\int_\epsilon^t s_2^{2H-2}\int_0^{s_2\wedge t} du\, ds_2 \\
=&\sqrt{c_M(2-2H)}\int_\epsilon^t s_2^{2H-1}\, ds_2 \\
=&\frac{\sqrt{c_M(2-2H)}}{2H}(t^{2H}-\epsilon^{2H}).
\end{aligned} \tag{3.15}$$

□

Proposition 3.1.5. *The quadratic variation of* $(V_t^\epsilon)_{t\in[0,\infty)}$ *is given by*

$$[V^\epsilon, V^\epsilon]_t = \frac{C^2}{2H}t^{2H},$$

where $C = c_\Psi\sqrt{c_M(2-2H)}$*, as above.*

Proof. By Definition 3.1.2, we have

$$[V^\epsilon, V^\epsilon]_t = \int_0^t C^2 s^{2H-1} ds = \frac{C^2}{2H}t^{2H}.$$

□

The modified Dobrić-Ojeda process has the same quadratic variation as the original Dobrić-Ojeda process because while the drift component has been modified, only the martingale part contributes to the quadratic variation.

Proposition 3.1.6. *For* $H \in (0,1)$ *fixed, the process* $(V_t^\epsilon)_{t\in[0,\infty)}$ *as defined in Definition 3.1.2 converges both in* L^2 *and almost surely to the original Dobrić-Ojeda process* $(V_t)_{t\in[0,\infty)}$.

Proof. For $\epsilon > 0$, define the process $(N_t^\epsilon)_{t\in[0,\infty)}$ by

$$N_t^\epsilon = V_t - V_t^\epsilon \tag{3.16}$$

for all $t \geq 0$. Then by Proposition 2.2.1, Definition 3.1.2, and the original definition of the Dobrić-Ojeda process (1.4),

$$\begin{aligned} dN_t^\epsilon &= dV_t - dV_t^\epsilon \\ &= (2H-1)\left(t^{-1}V_t - c_\Psi t^{2H-2}M_t \mathbb{1}_{[\epsilon,\infty)}(t)\right) dt \\ &= (2H-1)t^{-1}\left(V_t - V_t \mathbb{1}_{[\epsilon,\infty)}(t)\right) dt \\ &= \begin{cases} (2H-1)t^{-1}V_t dt & \text{if } t < \epsilon \\ 0 & \text{if } t \geq \epsilon. \end{cases} \end{aligned} \tag{3.17}$$

When $t < \epsilon$,

$$N_t^\epsilon = (2H-1)\int_0^t s^{-1}V_s ds. \tag{3.18}$$

Therefore, by Proposition 2.1.2,

$$\begin{aligned} \mathbb{E}\left[(N_t^\epsilon)^2\right] =& \mathbb{E}\left[\left((2H-1)\int_0^t s^{-1}V_s ds.\right)^2\right] \\ =&(2H-1)^2\int_0^t\int_0^t s_1^{-1}s_2^{-1}\mathbb{E}\left[V_{s_1}V_{s_2}\right] ds_2\, ds_1 \\ =&c_\Psi^2(2H-1)^2\int_0^t\int_0^t s_1^{2H-2}s_2^{2H-2}\mathbb{E}\left[M_{s_1}M_{s_2}\right] ds_2\, ds_1 \\ =&c_M c_\Psi^2(2H-1)^2\int_0^t\int_0^t s_1^{2H-2}s_2^{2H-2}(s_1\wedge s_2)^{2-2H} ds_2\, ds_1 \\ =&2c_M c_\Psi^2(2H-1)^2\int_0^t s_1^{2H-2}\int_0^{s_1} ds_2\, ds_1 \\ =&2c_M c_\Psi^2(2H-1)^2\int_0^t s_1^{2H-1} ds_1 \\ =&\frac{2c_M c_\Psi^2(2H-1)^2}{2H}t^{2H}. \end{aligned} \tag{3.19}$$

When $t \geq \epsilon$, $dN_t^\epsilon = 0$ with initial condition $\mathbb{E}\left[(N_\epsilon^\epsilon)^2\right] = \frac{2c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H}$. Therefore, for $t \geq \epsilon$, $N_t^\epsilon = \frac{\sqrt{2c_M}c_\Psi(2H-1)}{\sqrt{2H}}\epsilon^H$ and thus

$$\mathbb{E}\left[(N_t^\epsilon)^2\right] = \mathbb{E}\left[\frac{2c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H}\right] = \frac{2c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H}. \tag{3.20}$$

Finally, to prove L^2 convergence, we have

$$\begin{aligned}
\sup_{0\le t<\infty} \mathbb{E}\left[(N_t^\epsilon)^2\right] &\le \sup_{0\le t<\epsilon} \mathbb{E}\left[(N_t^\epsilon)^2\right] + \sup_{\epsilon\le t<\infty} \mathbb{E}\left[(N_t^\epsilon)^2\right] \\
&= \sup_{0\le t<\epsilon}\left(\frac{2c_M c_\Psi^2(2H-1)^2}{2H} t^{2H}\right) + \sup_{\epsilon\le t<\infty}\left(\frac{2c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H}\right) \\
&= \frac{2c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H} + \frac{2c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H} \\
&= \frac{4c_M c_\Psi^2(2H-1)^2}{2H}\epsilon^{2H} \\
&\to 0
\end{aligned} \tag{3.21}$$

as $\epsilon \to 0$. Almost-sure convergence is straight-forward using the Dominated Convergence Theorem:

$$\begin{aligned}
\lim_{\epsilon\to 0} V_t^\epsilon &= \lim_{\epsilon\to 0}\left(C\int_0^t s^{H-1/2}\,dW_s + c_\Psi(2H-1)\int_0^t s^{2H-2}M_s \mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right) \\
&= C\int_0^t s^{H-1/2}\,dW_s + \lim_{\epsilon\to 0} c_\Psi(2H-1)\int_0^t s^{2H-2}M_s \mathbb{1}_{[\epsilon,\infty)}(s)\,ds \\
&= C\int_0^t s^{H-1/2}\,dW_s + c_\Psi(2H-1)\int_0^t s^{2H-2}M_s \lim_{\epsilon\to 0}\mathbb{1}_{[\epsilon,\infty)}(s)\,ds \\
&= C\int_0^t s^{H-1/2}\,dW_s + c_\Psi(2H-1)\int_0^t s^{2H-2}M_s\,ds \\
&= V_t.
\end{aligned} \tag{3.22}$$

□

Now we define S_t^ϵ:

$$dS_t^\epsilon = S_t^\epsilon(\sigma dV_t^\epsilon + \mu dt). \tag{3.23}$$

We will assume that the underlying stock price process follows $(S_t^\epsilon)_{t\in[0,\infty)}$, for some small $\epsilon > 0$. By Definition 3.1.2, we can use Itô Calculus to solve: Let $Y_t = \ln S_t^\epsilon$. Then we have

$$\begin{aligned}
dY_t &= \frac{dS_t^\epsilon}{S_t^\epsilon} - \frac{1}{2}\frac{(dS_t^\epsilon)^2}{(S_t^\epsilon)^2} \\
&= \mu dt + \sigma dV_t^\epsilon - \frac{1}{2}\sigma^2 d[V^\epsilon, V^\epsilon]_t,
\end{aligned} \tag{3.24}$$

and thus by Proposition 3.1.5,

$$\begin{aligned} Y_t &= Y_0 + \mu t + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 [V^\epsilon, V^\epsilon]_t \\ &= Y_0 + \mu t + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 \frac{C^2}{2H} t^{2H}, \end{aligned} \tag{3.25}$$

which implies

$$S_t^\epsilon = S_0 \exp\left\{ \mu t + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 \frac{C^2}{2H} t^{2H} \right\}. \tag{3.26}$$

Since $(V_t^\epsilon)_{t\in[0,\infty)}$ converges to $(V_t)_{t\in[0,\infty)}$ almost surely, convergence of $(S_t^\epsilon)_{t\in[0,\infty)}$ to $(S_t)_{t\in[0,\infty)}$, as in (3.4), immediately follows.

Define

$$\begin{aligned} Z_t^\epsilon &:= B_t^{-1} S_t^\epsilon \\ &= S_0 \exp\left\{ (\mu - r)t + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 \frac{C^2}{2H} t^{2H} \right\}, \end{aligned} \tag{3.27}$$

where $B_t = e^{rt}$ is the bond price process.

Then by Itô's Lemma and by Definition 3.1.2, we have

$$\begin{aligned} dZ_t^\epsilon =& Z_t^\epsilon \left(\sigma dV_t^\epsilon + (\mu - r)dt \right) \\ =& Z_t^\epsilon \left(\sigma (C t^{H-1/2} dW_t + c_\Psi (2H-1) t^{2H-2} M_t \mathbb{1}_{[\epsilon,\infty)}(t) dt) + (\mu - r) dt \right) \\ =& Z_t^\epsilon \left(\sigma C t^{H-1/2} dW_t + (\mu - r + \sigma c_\Psi (2H-1) t^{2H-2} M_t \mathbb{1}_{[\epsilon,\infty)}(t)) dt \right) \\ =& \sigma C t^{H-1/2} Z_t^\epsilon \left(dW_t + \frac{\mu - r + \sigma c_\Psi (2H-1) t^{2H-2} M_t \mathbb{1}_{[\epsilon,\infty)}(t)}{\sigma C t^{H-1/2}} dt \right) \\ =& \sigma C t^{H-1/2} Z_t^\epsilon \left(dW_t + \left(\frac{\mu - r}{\sigma C} t^{1/2-H} + \frac{c_\Psi (2H-1) M_t \mathbb{1}_{[\epsilon,\infty)}(t)}{C} t^{H-3/2} \right) dt \right). \end{aligned} \tag{3.28}$$

Let

$$\gamma_t = A t^{1/2-H} + B t^{H-3/2} M_t \mathbb{1}_{[\epsilon,\infty)}(t) \tag{3.29}$$

where

$$A = \frac{\mu - r}{\sigma C} \text{ and } B = \frac{c_\Psi (2H-1)}{C}. \tag{3.30}$$

In order to employ Girsanov's Theorem, we first verify Novikov's Condition (see [16]) for restricted values of ϵ. This restriction is discussed following the proof.

Proposition 3.1.7. *For γ_t as defined in* (3.29) *and for* $\epsilon > e^{\frac{-1}{2B^2 c_M}} T$,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2\, ds\right)\right] < \infty \tag{3.31}$$

for all $0 < t \leq T$.

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2\, ds\right)\right]\\
=&\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t (As^{1/2-H} + Bs^{H-3/2}M_s\mathbb{1}_{[\epsilon,\infty)}(s))^2\, ds\right)\right]\\
=&\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t (A^2 s^{1-2H} + 2ABs^{-1}M_s\mathbb{1}_{[\epsilon,\infty)}(s) + B^2 s^{2H-3}M_s^2\mathbb{1}_{[\epsilon,\infty)}(s))\, ds\right)\right]\\
=&\mathbb{E}\left[\exp\left(\frac{1}{2}A^2\int_0^t s^{1-2H}\, ds\right)\exp\left(AB\int_0^t s^{-1}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\, ds\right)\right.\\
&\left.\times\exp\left(\frac{1}{2}B^2\int_0^t s^{2H-3}M_s^2\mathbb{1}_{[\epsilon,\infty)}(s)\, ds\right)\right]\\
=&e^{\frac{A^2 t^{2-2H}}{2(2-2H)}}\mathbb{E}\left[\exp\left(AB\int_0^t s^{-1}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\, ds\right)\exp\left(\frac{1}{2}B^2\int_0^t s^{2H-3}M_s^2\mathbb{1}_{[\epsilon,\infty)}(s)\, ds\right)\right]\\
\leq&e^{\frac{A^2 t^{2-2H}}{2(2-2H)}}\left(\mathbb{E}\left[\exp\left(2AB\int_0^t s^{-1}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\, ds\right)\right]\right)^{1/2}\\
&\times\left(\mathbb{E}\left[\exp\left(B^2\int_0^t s^{2H-3}M_s^2\mathbb{1}_{[\epsilon,\infty)}(s)\, ds\right)\right]\right)^{1/2}.
\end{aligned} \tag{3.32}$$

Note that we can use the moment generating function of the Gaussian random variable $\int_0^t s^{-1}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\, ds$ to show that the first term is finite. To show that the last term is finite, we use the Taylor expansion of $f(x) = e^x$ and the Cauchy-Schwarz inequality:

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(B^2\int_0^t s^{2H-3}M_s^2\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right)\right]\\
=&\mathbb{E}\left[\exp\left(B^2c_M\int_0^t s^{2H-3}B_{s^{2-2H}}^2\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right)\right]\\
=&\mathbb{E}\left[\exp\left(\frac{B^2c_M}{2-2H}\int_0^{t^{2-2H}} r^{-2}B_r^2\mathbb{1}_{[\epsilon^{2-2H},\infty)}(r)\,dr\right)\right]\\
=&\mathbb{E}\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{B^2c_M}{2-2H}\int_0^{t^{2-2H}} r^{-2}B_r^2\mathbb{1}_{[\epsilon^{2-2H},\infty)}(r)\,dr\right)^k\right]\\
=&\sum_{k=0}^{\infty}\frac{(B^2c_M)^k}{(2-2H)^k k!}\,\mathbb{E}\left[\left(\int_{\epsilon^{2-2H}}^{t^{2-2H}} r^{-2}B_r^2\,dr\right)^k\right]\\
=&\sum_{k=0}^{\infty}\frac{(B^2c_M)^k}{(2-2H)^k k!}\int_{\epsilon^{2-2H}}^{t^{2-2H}}\dots\int_{\epsilon^{2-2H}}^{t^{2-2H}} r_1^{-2}\dots r_k^{-2}\mathbb{E}\left[B_{r_1}^2\dots B_{r_k}^2\right]\,dr_1\dots dr_k\\
\leq&1+\sum_{k=1}^{\infty}\frac{(B^2c_M)^k}{(2-2H)^k k!}\int_{\epsilon^{2-2H}}^{t^{2-2H}}\dots\int_{\epsilon^{2-2H}}^{t^{2-2H}} r_1^{-2}\dots r_k^{-2}\\
&\times\mathbb{E}\left[B_{r_1}^{2k}\right]^{1/k}\dots\mathbb{E}\left[B_{r_k}^{2k}\right]^{1/k}\,dr_1\dots dr_k \qquad (3.33)\\
=&1+\sum_{k=1}^{\infty}\frac{(B^2c_M)^k}{(2-2H)^k k!}\left(\int_{\epsilon^{2-2H}}^{t^{2-2H}} r^{-2}\mathbb{E}\left[B_r^{2k}\right]^{1/k}\,dr\right)^k\\
=&1+\sum_{k=1}^{\infty}\frac{(B^2c_M)^k}{(2-2H)^k k!}\left(\int_{\epsilon^{2-2H}}^{t^{2-2H}} r^{-2}\left(\frac{2^k\Gamma(k+1/2)}{\sqrt{\pi}}r^k\right)^{1/k}\,dr\right)^k\\
=&1+\sum_{k=1}^{\infty}\frac{(2B^2c_M)^k\Gamma(k+1/2)}{\sqrt{\pi}(2-2H)^k k!}\left(\int_{\epsilon^{2-2H}}^{t^{2-2H}} r^{-1}\,dr\right)^k\\
=&1+\sum_{k=1}^{\infty}\frac{(2B^2c_M)^k\Gamma(k+1/2)}{\sqrt{\pi}(2-2H)^k k!}\left(\ln\left(\frac{t^{2-2H}}{\epsilon^{2-2H}}\right)\right)^k\\
=&1+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(2B^2c_M)^k\Gamma(k+1/2)}{k!}\left(\ln\left(\frac{t}{\epsilon}\right)\right)^k\\
\leq&1+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(2B^2c_M)^k\Gamma(k+1)}{k!}\left(\ln\left(\frac{t}{\epsilon}\right)\right)^k\\
=&1+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\left(2B^2c_M\ln\left(\frac{t}{\epsilon}\right)\right)^k.
\end{aligned}
$$

This series converges when

$$\left|2B^2 c_M \ln\left(\frac{t}{\epsilon}\right)\right| < 1, \tag{3.34}$$

or when

$$te^{\frac{-1}{2B^2 c_M}} < \epsilon < te^{\frac{1}{2B^2 c_M}}. \tag{3.35}$$

□

The right-hand inequality is irrelevant since $te^{\frac{1}{2B^2 c_M}} > t$ and we intend for ϵ to be small. The left-hand inequality, $\epsilon > te^{\frac{-1}{2B^2 c_M}}$, has more important implications. This restriction on ϵ is significant for extreme H values but more reasonable for H values close to $1/2$, which corresponds to the H values we expect in a typical market. For example, for $H \in (.21, .68)$, we need not require ϵ to be any greater than 10% of t. To further consider this restriction on ϵ, set

$$\delta(H) = e^{\frac{-1}{2B^2 c_M}},$$

as shown in Figure 3.1.

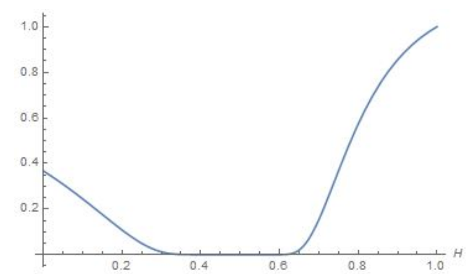

Figure 3.1: Graph of $\delta(H)$.

We do expect Theorem 3.1.7 to be satisfied for *any* $\epsilon > 0$ since intuitively, the Brownian motion process B_t behaves like $\sqrt{t}$ and the second term can be approximated (non-rigorously) by

$$\exp\left(B^2 c_M \int_\epsilon^t s^{-1}\, ds\right) < \infty, \tag{3.36}$$

however a rigorous proof of the theorem for any $\epsilon > 0$ remains a work in progress.

By Girsanov's Theorem (again, see [16]), there exists a measure $\mathbb{Q}^\epsilon$ equivalent to our original measure $\mathbb{P}$ such that

$$\begin{aligned} dW_t^\epsilon &= dW_t + \gamma_t dt \\ &= dW_t + \left(At^{1/2-H} + Bt^{H-3/2} M_t \mathbb{1}_{[\epsilon,\infty)}(t)\right) dt \end{aligned} \tag{3.37}$$

is a $\mathbb{Q}^\epsilon$-Brownian motion process. Therefore,

$$\begin{aligned} dZ_t^\epsilon &= \sigma C t^{H-1/2} Z_t^\epsilon \left(dW_t + \left(At^{1/2-H} + Bt^{H-3/2} M_t \mathbb{1}_{[\epsilon,\infty)}(t)\right) dt\right) \\ &= \sigma C t^{H-1/2} Z_t^\epsilon dW_t^\epsilon \end{aligned} \tag{3.38}$$

is a $\mathbb{Q}^\epsilon$-Martingale process. Note that under the measure $\mathbb{Q}^\epsilon$, we have

$$Z_t^\epsilon = S_0 \exp\left\{\sigma C \int_0^t s^{H-1/2} dW_s^\epsilon - \frac{\sigma^2 C^2}{2(2H)} t^{2H}\right\} \tag{3.39}$$

and similarly,

$$S_t^\epsilon = S_0 \exp\left\{rt + \sigma C \int_0^t s^{H-1/2} dW_s^\epsilon - \frac{\sigma^2 C^2}{2(2H)} t^{2H}\right\}. \tag{3.40}$$

Finally,

$$\mathbb{E}_{\mathbb{Q}^\epsilon}[S_t^\epsilon] = \mathbb{E}_{\mathbb{Q}^\epsilon}\left[S_0 \exp\left\{rt + \sigma C \int_0^t s^{H-1/2} dW_s^\epsilon - \frac{1}{2}\sigma^2 C^2 \frac{t^{2H}}{2H}\right\}\right] = S_0 e^{rt}, \tag{3.41}$$

using Itô Isometry and the moment generating function. Therefore $\mathbb{Q}^\epsilon$ is in fact a risk-neutral measure. Let $F(T)$ be the payoff of an option on an asset with price $(S_t^\epsilon)_{t\in[0,T]}$ for some $\epsilon > \delta(H)T$ at time $T > 0$. Note that we assume the underlying stock price follows (S_t^ϵ), NOT the original stock price process (S_t). Define

$$E_t = \mathbb{E}_{\mathbb{Q}^\epsilon}(B_T^{-1} F | \mathcal{F}_t^H). \tag{3.42}$$

Then by the Martingale Representation Theorem (see [27]), there exists an adapted process $(\phi_t)_{t\in[0,T]}$ such that

$$dE_t = \phi_t dZ_t^\epsilon. \tag{3.43}$$

For each $\epsilon > \delta(H)T$, we get a Δ-hedging portfolio given by $(\phi_t, \psi_t)_{t\in[0,T]}$, where ϕ_t is the number of shares of the risky asset and $\psi_t = E_t - \phi_t Z_t^\epsilon$ is the number of shares of the bond at time t. It can be easily verified that the portfolio is self-financing and replicating under the modified stock price process (S_t^ϵ). Then by the standard no-arbitrage argument (see, for instance, [27]), the value of the option is equal to the value of the portfolio at every time $t \in [0,T]$, given by

$$\begin{aligned} F_t &= \phi_t S_t^\epsilon + \psi_t B_t \\ &= B_t \mathbb{E}_{\mathbb{Q}^\epsilon}(B_T^{-1} F | \mathcal{F}_t). \end{aligned} \tag{3.44}$$

Furthermore, we can find the corresponding Black-Scholes partial differential equation:

Proposition 3.1.8. *Consider an option with underlying stock price* $(S_t^\epsilon)_{t\in[0,\infty)}$ *as defined in* (3.23) *that has payoff* F *at time* $T > 0$*. For simplicity of notation, denote* $x = S_t^\epsilon$*. The value of the option at time* $t \in [0,T]$ *is given by the solution* $f(x,t)$ *to the the partial differential equation*

$$rf(x,t) = rxf_x(x,t) + f_t + \frac{1}{2}\sigma^2 C^2 t^{2H-1} x^2 f_{xx}(x,t) \tag{3.45}$$

with terminal condition $f(x,T) = F$.

Proof. The underlying stock price process $(S_t^\epsilon)_{t\in[0,\infty)}$ satisfies, by (3.23) and Definition 3.1.2,

$$dS_t^\epsilon = \alpha(t) S_t^\epsilon dt + \sigma C t^{H-1/2} S_t^\epsilon dW_t, \tag{3.46}$$

where $\alpha(t) = \mu + \sigma c_\Psi (2H-1) t^{2H-2} M_t \mathbb{1}_{[\epsilon,\infty)}(t)$. Also note that the bond price process satisfies $dB_t = rB_t dt$. Then using Itô's formula, we have

$$\begin{aligned} df(x,t) =& f_x(x,t) dS_t^\epsilon + f_t(x,t) + \frac{1}{2} f_{xx}(x,t)(dS_t^\epsilon)^2 \\ =& f_x(x,t)\left(\alpha(t) S_t^\epsilon dt + \sigma C t^{H-1/2} S_t^\epsilon dW_t\right) + f_t(x,t) \\ &+ \frac{1}{2}\sigma^2 C^2 t^{2H-1} (S_t^\epsilon)^2 f_{xx}(x,t) dt \\ =& \alpha(t) S_t^\epsilon f_x(x,t) dt + \sigma C t^{H-1/2} S_t^\epsilon f_x(x,t) dW_t + f_t(x,t) \\ &+ \frac{1}{2}\sigma^2 C^2 t^{2H-1} (S_t^\epsilon)^2 f_{xx}(x,t) dt. \end{aligned} \tag{3.47}$$

Since the hedging portfolio (ϕ_t, ψ_t) is self-financing and replicates the value of the option at every time $t \in [0, T]$, we also have

$$\begin{aligned} df(x,t) =& \phi_t dS_t^\epsilon + \psi_t dB_t \\ =& \phi_t \left(\alpha(t) S_t^\epsilon dt + \sigma C t^{H-1/2} S_t^\epsilon dW_t\right) + \psi_t r B_t dt \\ =& \phi_t \alpha(t) S_t^\epsilon dt + \phi_t \sigma C t^{H-1/2} S_t^\epsilon dW_t + \psi_t r B_t dt \end{aligned} \tag{3.48}$$

Setting these equations (3.47) and (3.48) equal gives

$$\begin{aligned} &\left(\sigma C t^{H-1/2} S_t^\epsilon f_x(x,t) - \phi_t \sigma C t^{H-1/2} S_t^\epsilon\right) dW_t \\ =& \left(\phi_t \alpha(t) S_t^\epsilon + \psi_t r B_t - \alpha(t) S_t^\epsilon f_x(x,t) - f_t(x,t) - \frac{1}{2}\sigma^2 C^2 t^{2H-1} (S_t^\epsilon)^2 f_{xx}(x,t)\right) dt \end{aligned} \tag{3.49}$$

Since the left hand side of this equation is a martingale process and the right hand side is not, they must both be equal to zero almost surely. Therefore,

$$\phi_t = f_x(x,t) \tag{3.50}$$

and finally,

$$rf(x,t) = rxf_x(x,t) + f_t + \frac{1}{2}\sigma^2 C^2 t^{2H-1} x^2 f_{xx}(x,t) \tag{3.51}$$

as required. □

It follows that the number of shares of the underlying stock in the replicating portfolio $(\phi_t)_{t\in[0,T]}$ satisfies

$$\phi_t = \frac{\partial f(x,t)}{\partial x}. \tag{3.52}$$

3.2 Computation of a call option price

The payoff F of a call option on a risky asset with price $(S_t^\epsilon)_{t\in[0,T]}$ that has strike price K and expiration T is given by

$$F = (S_T^\epsilon - K)^+. \tag{3.53}$$

Suppose also that we have a risk-free interest rate r. Therefore by (3.44) and (3.40), we have

$$\begin{aligned} F_t &= B_t \mathbb{E}_{\mathbb{Q}^\epsilon}(B_T^{-1} F | \mathcal{F}_t) \\ &= B_t \mathbb{E}_{\mathbb{Q}^\epsilon}(B_T^{-1}(S_T^\epsilon - K)^+ | \mathcal{F}_t) \\ &= B_t \mathbb{E}_{\mathbb{Q}^\epsilon}\left(B_T^{-1} \left(S_t^\epsilon \frac{S_T^\epsilon}{S_t^\epsilon} - K \right)^+ \middle| \mathcal{F}_t \right) \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^\epsilon}\left(\left(S_t^\epsilon e^{r(T-t) + \sigma C \int_t^T s^{H-1/2} dW_s^\epsilon - \frac{1}{2}\sigma^2 C^2 \left(\frac{T^{2H} - t^{2H}}{2H} \right)} - K \right)^+ \middle| \mathcal{F}_t \right). \end{aligned} \tag{3.54}$$

Since S_t^ϵ is measurable with respect to $\mathcal{F}_t$, fix $x = S_t^\epsilon$. Then since $\int_t^T s^{H-1/2} dW_s^\epsilon$ is independent of $\mathcal{F}_t^H$, we have

$$F_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^\epsilon}\left(\left(x e^{r(T-t) + \sigma C \int_t^T s^{H-1/2} dW_s^\epsilon - \frac{1}{2}\sigma^2 C^2 \left(\frac{T^{2H} - t^{2H}}{2H} \right)} - K \right)^+ \middle| x = S_t^\epsilon \right) \tag{3.55}$$

Since $\int_t^T s^{H-1/2} dW_s^\epsilon$ is a centered Gaussian random variable with variance $\frac{T^{2H} - t^{2H}}{2H}$, we have

$$\begin{aligned} F_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^\epsilon}\left(\left(x e^{r(T-t) + \sigma C \sqrt{\frac{T^{2H} - t^{2H}}{2H}} Z - \frac{1}{2}\sigma^2 C^2 \left(\frac{T^{2H} - t^{2H}}{2H} \right)} - K \right)^+ \middle| x = S_t^\epsilon \right) \\ &= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_t^\epsilon e^{r(T-t) + \sigma C \sqrt{\frac{T^{2H} - t^{2H}}{2H}} z - \frac{1}{2}\sigma^2 C^2 \left(\frac{T^{2H} - t^{2H}}{2H} \right)} - K \right)^+ e^{-\frac{1}{2}z^2} dz, \end{aligned} \tag{3.56}$$

where Z is a standard normal random variable. We have

$$S_t^\epsilon e^{r(T-t) + \sigma C \sqrt{\frac{T^{2H} - t^{2H}}{2H}} z - \frac{1}{2}\sigma^2 C^2 \left(\frac{T^{2H} - t^{2H}}{2H} \right)} - K \geq 0 \tag{3.57}$$

when

$$z \geq d_1 := \frac{\ln\left(\frac{K}{S_t^\epsilon} \right) - r(T-t) + \frac{1}{2}\sigma^2 C^2 \left(\frac{T^{2H} - t^{2H}}{2H} \right)}{\sigma C \sqrt{\frac{T^{2H} - t^{2H}}{2H}}}, \tag{3.58}$$

and therefore

$$\begin{aligned}
F_t &= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} \left(S_t^{\epsilon} e^{r(T-t)+\sigma C\sqrt{\frac{T^{2H}-t^{2H}}{2H}}z-\frac{1}{2}\sigma^2 C^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)} - K \right) e^{-\frac{1}{2}z^2} dz \\
&= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} \left(S_t^{\epsilon} e^{r(T-t)+\sigma C\sqrt{\frac{T^{2H}-t^{2H}}{2H}}z-\frac{1}{2}\sigma^2 C^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)-\frac{1}{2}x^2} - K e^{-\frac{1}{2}z^2} \right) dz \\
&= S_t^{\epsilon} \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} e^{-\frac{1}{2}\left(z-\sigma C\sqrt{\frac{T^{2H}-t^{2H}}{2H}}\right)^2} dz - K e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} e^{-\frac{1}{2}z^2} dz \\
&= S_t^{\epsilon} \frac{1}{\sqrt{2\pi}} \int_{d_1-\sigma C\sqrt{\frac{T^{2H}-t^{2H}}{2H}}}^{\infty} e^{-\frac{1}{2}y^2} dy - K e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} e^{-\frac{1}{2}z^2} dz \\
&= S_t^{\epsilon} \Phi\left(\sigma C \sqrt{\frac{T^{2H}-t^{2H}}{2H}} - d_1 \right) - K e^{-r(T-t)} \Phi(-d_1).
\end{aligned} \tag{3.59}$$

We observe that when $H = 1/2$, this formula is consistent with the original Black-Scholes call option price.

Chapter 4

Parameter estimation techniques

In both the original Black-Scholes model, its analogue with fractional Brownian motion, and now the model with the Dobrić-Ojeda process as the driving noise for the stock price process, we assume that the stock price parameters μ, σ, and H (drift, volatility, and Hurst index, respectively) are constant for $t \in [0, T]$. In this chapter we discuss two methods for estimating these parameters based on historical stock price data.

4.1 Ratio method with Ergodic Theory

First, we examine a parameter estimation technique under the assumption that $H_{Z_H} = H_{V_H^\epsilon}$. We justify this assumption by noting that the processes $(Z_H(t))$ and $(V_H(t))$ behave similarly, with less than 12% relative error, as discussed in Chapter 1. Under this assumption, we can employ the stationary and ergodic properties of the increments of fractional Brownian motion in a ratio method for estimating H, as developed in [25].

Define the shift transformation τ on a stochastic process $\{Y(t)\}_{t\geq 0}$ by $(Y \circ \tau)(t) = Y(t + \Delta t) - Y(\Delta t)$ for some small fixed Δt. Next define the sequence of random variables $\{X_m\}_{m\in\mathbb{Z}^+}$ by $X_m = Z_H \circ \tau^m$, where $Z_H(t)$ is a fractional Brownian motion process. The shift τ is invariant on the process Z_H since $(Z_H \circ \tau^m)(t) = Z_H(t + m\Delta t) - Z_H(m\Delta t)$ and fractional Brownian motion has stationary increments. Thus the sequence $\{X_m\}$ is ergodic.

Therefore, by the ergodic theorem, the sum of increments of fractional Brownian motion converge to their mean, 0, and the sum of squared increments of fractional Brownian motion converge to their second moment. We will use this fact to estimate the parameters μ, σ, and H.

Suppose that s_i is the observed price of the underlying stock at time $t_i = \frac{iT}{n}$, for $i = 0, \ldots, n$. Note that the time between each observation, Δt, is fixed. For example, s_i may be daily closing prices. Without loss of generality, assume that the stock does not pay dividends during the interval $[0, T]$. Otherwise use the adjusted stock price. Define the log returns $y_i = \ln \frac{s_i}{s_{i-1}}$ for $i = 1, \ldots, n$. Then under the assumption that the stock price follows a geometric fractional Brownian motion process, set

$$y_i = \mu\Delta t + \sigma(Z_H(t_i) - Z_H(t_{i-1})) - \frac{1}{2}\sigma^2(t_i^{2H} - t_{i-1}^{2H}). \tag{4.1}$$

Then we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\sigma^2(t_i^{2H} - t_{i-1}^{2H}) \\
=&\frac{\sigma^2}{2n}\sum_{i=1}^{n}(\left(\frac{Ti}{n}\right)^{2H} - \left(\frac{T(i-1)}{n}\right)^{2H}) \\
=&\frac{\sigma^2}{2n}\left(\frac{T}{n}\right)^{2H}\sum_{i=1}^{n}(i^{2H} - (i-1)^{2H}) \\
\approx&\frac{2H\sigma^2}{2n}\left(\frac{T}{n}\right)^{2H}\sum_{i=1}^{n}i^{2H-1} \\
\approx&\frac{H\sigma^2}{n}\left(\frac{T}{n}\right)^{2H}\int_0^n x^{2H-1}\,dx \\
=&\frac{H\sigma^2}{n}\left(\frac{T}{n}\right)^{2H}\frac{1}{2H}n^{2H} \\
=&\frac{1}{n}\frac{\sigma^2T^{2H}}{2} \\
\to&0 \text{ as } n \to \infty.
\end{aligned} \tag{4.2}$$

By using the ergodic property of $(Z_H(t))$, we have

$$\frac{1}{n}\sum_{i=1}^{n}(Z_H(t_i) - Z_H(t_{i-1})) \to \mathbb{E}(Z_H(t_1) - Z_H(t_0)) = 0 \tag{4.3}$$

and so

$$\frac{1}{n}\sum_{i=1}^{n} y_i = \mu\Delta t + \frac{\sigma}{n}\sum_{i=1}^{n}(Z_H(t_i) - Z_H(t_{i-1})) - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\sigma^2(t_i^{2H} - t_{i-1}^{2H}) \to \mu\Delta t. \tag{4.4}$$

Therefore we will estimate the drift μ for n sufficiently large by

$$\mu \approx \hat{\mu} = \frac{1}{\Delta t}\frac{1}{n}\sum_{i=1}^{n} y_i. \tag{4.5}$$

Since it remains to estimate both the volatility σ and the Hurst index H, we will use a ratio of second moments to estimate H first, as in [25]. Let

$$SS_1 := \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{\mu}\Delta t)^2 \approx \frac{\sigma^2}{n}\sum_{i=1}^{n}(Z_H(t_i) - Z_H(t_{i-1}))^2 \to \sigma^2(\Delta t)^{2H} \tag{4.6}$$

and

$$\begin{aligned} SS_2 &:= \frac{1}{\lfloor n/2 \rfloor}\sum_{i=1}^{\lfloor n/2 \rfloor}\left(\ln\frac{s_{2i}}{s_{2i-1}} - \hat{\mu}(2\Delta t)\right)^2 \\ &\approx \frac{\sigma^2}{n}\sum_{i=2}^{n}(Z_H(t_i) - Z_H(t_{i-2}))^2 \\ &\to \sigma^2(2\Delta t)^{2H}, \end{aligned} \tag{4.7}$$

using the previously computed estimator $\hat{\mu}$. Then

$$\frac{SS_1}{SS_2} \to \left(\frac{1}{4}\right)^H \tag{4.8}$$

and so we will estimate the Hurst index H by

$$H \approx \hat{H} = \log_4\left(\frac{SS_1}{SS_2}\right). \tag{4.9}$$

Finally, we can use $\hat{\mu}$ and $\hat{H}$ to estimate the volatility σ:

$$\sigma^2 \approx \hat{\sigma}^2 = \frac{1}{(\Delta t)^{2\hat{H}}}\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{\mu}\Delta t)^2. \tag{4.10}$$

4.2 Parameter estimation using quadratic variation

Next we relax the assumption that the parameters of the Dobrić-Ojeda model are necessarily equal to the parameters of the fractional Brownian motion model, i.e. that $H_{Z_H} = H_{V_H^\epsilon}$. We aim to estimate H and σ using properties of the modified Dobrić-Ojeda process. (The drift μ plays no role in pricing an option so we omit its estimation.) Unlike fractional Brownian motion, the modified Dobrić-Ojeda process is not ergodic so we cannot use the technique described in 4.1. Therefore, we propose the use of quadratic variation to estimate parameters in this model.

4.2.1 Almost-sure convergence of the quadratic variation

First, recall the definition of quadratic variation:

Definition 4.2.1. *Let $f(t)$ be a function defined on the interval $[t_0, T]$. The quadratic variation of f from time t_0 to time T, ${}_{t_0}[f,f]_T$, is defined as*

$$ {}_{t_0}[f,f]_T = \lim_{||\Pi_n||\to 0} \sum_{j=1}^{n} (f(t_j) - f(t_{j-1}))^2 \tag{4.11}$$

where $\Pi_n = \{t_0, t_1, \ldots, t_n\}$, $t_0 < t_1 < \ldots < t_n = T$ and $||\Pi_n|| = \max_{j=1,\ldots,n}(t_j - t_{j-1})$.

As shown in Propositions 2.2.2 and 3.1.5, the quadratic variation of both the original Dobrić-Ojeda process ($V_H(t)$) and the modified Dobrić-Ojeda process ($V_H^\epsilon(t)$) is given by

$$I = \frac{C^2}{2H}(T^{2H} - t_0^{2H}).$$

We use the following theorem to construct a parameter estimation algorithm using the quadratic variation of ($V_H^\epsilon(t)$). We will prove convergence in L^2, where the L^2 norm, $||\cdot||_2$ is given by

$$||X||_2 = \sqrt{\mathbb{E}[X^2]}, \tag{4.12}$$

and also almost sure convergence, which will allow us to use another ratio method to estimate the Hurst index, H. We require a sampling rate strictly greater than n in order to ensure almost sure convergence.

Theorem 4.2.2. *Let* $t_i = \frac{iT}{\lfloor n^{1+\delta} \rfloor}$, $i = i_0, ..., \lfloor n^{1+\delta} \rfloor$, $i_0 = \frac{t_0 \lfloor n^{1+\delta} \rfloor}{T}$, *be a sequence of partitions of* $[t_0, T]$ *for some* $\delta > 0$ *and* $V_t = V_H(t)$ *as defined in* (1.4). *Then*

$$\lim_{n\to\infty} \left\| \sum_{i=i_0}^{\lfloor n^{1+\delta} \rfloor} (\Delta V_{t_i})^2 - I \right\|_2 = 0$$

and

$$\lim_{n\to\infty} \sum_{i=i_0}^{\lfloor n^{1+\delta} \rfloor} (\Delta V_{t_i})^2 = I \qquad a.s.$$

where $\Delta V_{t_i} = V_{t_i} - V_{t_{i-1}}$.

Corollary 4.2.3. *The sample quadratic variation of the modified Dobrić-Ojeda process* (V_t^ϵ) *converges in* L^2 *and almost surely to* $I = \frac{C^2}{2H}(T^{2H} - t_0^{2H})$.

Proof. As the only modification to the original Dobrić-Ojeda process is in the drift term and the drift term does not impact quadratic variation, the quadratic variation remains unchanged. □

Now we define the log of the stock price process, $X_t = \ln(S_t^\epsilon)$. Then we also have convergence of the quadratic variation of X_t:

Corollary 4.2.4. *The sample quadratic variation of the log stock price process* X_t *converges in* L^2 *and almost surely to* $\sigma^2 \frac{C^2}{2H}(T^{2H} - t_0^{2H})$.

Proof. As in (3.26), we can write X_t as

$$X_t = \ln(S_t^\epsilon) = \mu t + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 \frac{C^2}{2H} t^{2H} \tag{4.13}$$

and again, since the only difference between X_t and V_t^ϵ is in the drift, the quadratic variation is simply $\sigma^2 I$. □

We will utilize the following lemma in the proof of Theorem 4.2.2.

Lemma 4.2.5. *For*

$$I^* = C^2 \sum_{i=i_0}^{\lfloor n^{1+\delta} \rfloor} t_i^{2H-1} \Delta t, \tag{4.14}$$

where $\Delta t = t_i - t_{i-1}$, *we have*

$$\lim_{n\to\infty} I^* = I \qquad a.s.$$

Proof. By the definition of a definite Riemann Integral, we have

$$I = \int_0^t C^2 s^{2H-1} ds = \lim_{n\to\infty} \sum_{i=i_0}^{\lfloor n^{1+\delta} \rfloor} C^2 t_i^{2H-1} \Delta t. \tag{4.15}$$

□

4.2.2 Proof of Theorem 4.2.2

Proof. Let $m = \lfloor n^{1+\delta} \rfloor$. By the triangle inequality,

$$\left|\left| \sum_{i=i_0}^{m} (\Delta V_{t_i})^2 - I \right|\right|_2 \leq \left|\left| \sum_{i=i_0}^{m} (\Delta V_{t_i})^2 - I^* \right|\right|_2 + ||I^* - I||_2$$

and so by Lemma 4.2.5, it suffices to show that $\left|\left| \sum_{i=i_0}^{m} (\Delta V_{t_i})^2 - I^* \right|\right|_2 \longrightarrow 0$. We have, by (4.14),

$$I^{*2} = C^4 \sum_{i=i_0}^{m} \sum_{j=i_0}^{m} t_i^{2H-1} t_j^{2H-1} (\Delta t)^2.$$

We will need the approximations for M_t given in Lemma 2.1.3. Similarly, we can approximate $\Delta\Psi_{t_i}$ and $(\Delta\Psi_{t_i})^2$:

$$\Delta\Psi_{t_i} = c_\Psi (t_i^{2H-1} - t_{i-1}^{2H-1}) \approx c_\Psi (2H-1) t_i^{2H-2} \Delta t \tag{4.16}$$

and

$$(\Delta\Psi_{t_i})^2 \approx c_\Psi^2 (2H-1)^2 t_i^{4H-4} (\Delta t)^2. \tag{4.17}$$

Then we have

$$\begin{aligned}
&\left\|\sum_{i=i_0}^{m}(\Delta V_{t_i})^2 - I^*\right\|_2^2 \\
=&\mathbb{E}\left[\sum_{i=i_0}^{m}(\Delta V_{t_i})^2 - I^*\right]^2 \\
=&\mathbb{E}\left[\sum_{i=i_0}^{m}(\Delta V_{t_i})^2\right]^2 - 2I^*\mathbb{E}\left[\sum_{i=i_0}^{m}(\Delta V_{t_i})^2\right] + I^{*2} \\
=&\mathbb{E}\left[\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}(\Delta V_{t_i})^2(\Delta V_{t_j})^2\right] - 2I^*\mathbb{E}\left[\sum_{i=i_0}^{m}(\Delta V_{t_i})^2\right] + I^{*2}.
\end{aligned} \tag{4.18}$$

Note that we can write $(\Delta V_{t_i})^2$ as

$$(\Psi_{t_i}\Delta M_{t_i} + \Delta\Psi_{t_i}M_{t_{i-1}})^2 = (\Delta\Psi_{t_i})^2 M_{t_{i-1}}^2 + 2\Delta\Psi_{t_i}M_{t_{i-1}}\Psi_{t_i}\Delta M_{t_i} + \Psi_{t_i}^2(\Delta M_{t_i})^2, \tag{4.19}$$

so the last two terms give

$$\begin{aligned}
&-2I^*\mathbb{E}\left[\sum_{i=i_0}^{m}(\Delta V_{t_i})^2\right] + I^{*2} \\
=&-2C^2\sum_{j=i_0}^{m}t_j^{2H-1}\Delta t\sum_{i=i_0}^{m}\Big((\Delta\Psi_{t_i})^2\mathbb{E}[M_{t_{i-1}}^2] + 2\Delta\Psi_{t_i}\Psi_{t_i}\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i}] \\
&+ \Psi_{t_i}^2\mathbb{E}[(\Delta M_{t_i})^2]\Big) + C^4\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}t_i^{2H-1}t_j^{2H-1}\Delta t^2 \\
\approx&-2C^2\sum_{j=i_0}^{m}t_j^{2H-1}\Delta t\sum_{i=i_0}^{m}\big(c_M c_\Psi^2(2H-1)^2 t_i^{4H-4}(\Delta t)^2 t_i^{2-2H} \\
&+ c_M(2-2H)c_\Psi^2 t_i^{4H-2}t_i^{1-2H}\Delta t\big) + C^4\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}t_i^{2H-1}t_j^{2H-1}(\Delta t)^2 \\
=&\sum_{j=i_0}^{m}\sum_{i=i_0}^{m}\big(-2C^2c_Mc_\Psi^2(2H-1)^2t_i^{2H-2}t_j^{2H-1}(\Delta t)^3 - 2C^4t_i^{2H-1}t_j^{2H-1}(\Delta t)^2 \\
&+ C^4t_i^{2H-1}t_j^{2H-1}(\Delta t)^2\big) \\
=&\sum_{j=i_0}^{m}\sum_{i=i_0}^{m}\left(-2C^2c_Mc_\Psi^2(2H-1)^2t_i^{2H-2}t_j^{2H-1}(\Delta t)^3 - C^4t_i^{2H-1}t_j^{2H-1}(\Delta t)^2\right).
\end{aligned} \tag{4.20}$$

We will see that the first term of (4.20) converges and the second term,

$$-\sum\sum C^4 t_i^{2H-1} t_j^{2H-1} (\Delta t)^2, \tag{4.21}$$

is canceled by another term. The first term of (4.18) is slightly less enjoyable to compute:

$$\begin{aligned}
&\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}\mathbb{E}[(\Delta V_{t_i})^2(\Delta V_{t_j})^2] \\
=&\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}\mathbb{E}[((\Delta\Psi_{t_i})^2 M_{t_{i-1}}^2 + 2\Delta\Psi_{t_i} M_{t_{i-1}}\Psi_{t_i}\Delta M_{t_i} + \Psi_{t_i}^2(\Delta M_{t_i})^2) \\
&\cdot((\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 + 2\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} + \Psi_{t_j}^2(\Delta M_{t_j})^2)] \\
=&\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}\mathbb{E}[(\Delta\Psi_{t_i})^2 M_{t_{i-1}}^2\cdot((\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 + 2\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} + \Psi_{t_j}^2(\Delta M_{t_j})^2) \\
&+ 2\Delta\Psi_{t_i} M_{t_{i-1}}\Psi_{t_i}\Delta M_{t_i}\cdot((\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 + 2\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} + \Psi_{t_j}^2(\Delta M_{t_j})^2) \\
&+ \Psi_{t_i}^2(\Delta M_{t_i})^2\cdot((\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 + 2\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} + \Psi_{t_j}^2(\Delta M_{t_j})^2)] \\
=&\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}\mathbb{E}[(\Delta\Psi_{t_i})^2 M_{t_{i-1}}^2(\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 + 2(\Delta\Psi_{t_i})^2 M_{t_{i-1}}^2\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} \\
&+ (\Delta\Psi_{t_i})^2 M_{t_{i-1}}^2\Psi_{t_j}^2(\Delta M_{t_j})^2 + 2\Delta\Psi_{t_i} M_{t_{i-1}}\Psi_{t_i}\Delta M_{t_i}(\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 \\
&+ 4\Delta\Psi_{t_i} M_{t_{i-1}}\Psi_{t_i}\Delta M_{t_i}\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} + 2\Delta\Psi_{t_i} M_{t_{i-1}}\Psi_{t_i}\Delta M_{t_i}\Psi_{t_j}^2(\Delta M_{t_j})^2 \\
&+ \Psi_{t_i}^2(\Delta M_{t_i})^2(\Delta\Psi_{t_j})^2 M_{t_{j-1}}^2 + 2\Psi_{t_i}^2(\Delta M_{t_i})^2\Delta\Psi_{t_j} M_{t_{j-1}}\Psi_{t_j}\Delta M_{t_j} \\
&+ \Psi_{t_i}^2(\Delta M_{t_i})^2\Psi_{t_j}^2(\Delta M_{t_j})^2] \\
=&\sum_{i=i_0}^{m}\sum_{j=i_0}^{m}\Big[(\Delta\Psi_{t_i})^2(\Delta\Psi_{t_j})^2\mathbb{E}[M_{t_{i-1}}^2 M_{t_{j-1}}^2] + 2(\Delta\Psi_{t_i})^2\Delta\Psi_{t_j}\Psi_{t_j}\mathbb{E}[M_{t_{i-1}}^2 M_{t_{j-1}}\Delta M_{t_j}] \\
&+ (\Delta\Psi_{t_i})^2\Psi_{t_j}^2\mathbb{E}[M_{t_{i-1}}^2(\Delta M_{t_j})^2] + 2\Delta\Psi_{t_i}\Psi_{t_i}(\Delta\Psi_{t_j})^2\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i} M_{t_{j-1}}^2] \\
&+ 4\Delta\Psi_{t_i}\Psi_{t_i}\Delta\Psi_{t_j}\Psi_{t_j}\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i} M_{t_{j-1}}\Delta M_{t_j}] \\
&+ 2\Delta\Psi_{t_i}\Psi_{t_i}\Psi_{t_j}^2\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i}(\Delta M_{t_j})^2] \\
&+ \Psi_{t_i}^2(\Delta\Psi_{t_j})^2\mathbb{E}[(\Delta M_{t_i})^2 M_{t_{j-1}}^2] + 2\Psi_{t_i}^2\Delta\Psi_{t_j}\Psi_{t_j}\mathbb{E}[(\Delta M_{t_i})^2 M_{t_{j-1}}\Delta M_{t_j}] \\
&+ \Psi_{t_i}^2\Psi_{t_j}^2\mathbb{E}[(\Delta M_{t_i})^2(\Delta M_{t_j})^2]\Big].
\end{aligned} \tag{4.22}$$

By symmetry, this is equal to

$$
\begin{aligned}
&2\sum_{i=i_0}^{m}\sum_{i<j}\Big[(\Delta\Psi_{t_i})^2(\Delta\Psi_{t_j})^2\mathbb{E}[M_{t_{i-1}}^2M_{t_{j-1}}^2]+2(\Delta\Psi_{t_i})^2\Delta\Psi_{t_j}\Psi_{t_j}\mathbb{E}[M_{t_{i-1}}^2M_{t_{j-1}}\Delta M_{t_j}]\\
&+(\Delta\Psi_{t_i})^2\Psi_{t_j}^2\mathbb{E}[M_{t_{i-1}}^2(\Delta M_{t_j})^2]+2\Delta\Psi_{t_i}\Psi_{t_i}(\Delta\Psi_{t_j})^2\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i}M_{t_{j-1}}^2]\\
&+4\Delta\Psi_{t_i}\Psi_{t_i}\Delta\Psi_{t_j}\Psi_{t_j}\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i}M_{t_{j-1}}\Delta M_{t_j}]+2\Delta\Psi_{t_i}\Psi_{t_i}\Psi_{t_j}^2\mathbb{E}[M_{t_{i-1}}\Delta M_{t_i}(\Delta M_{t_j})^2]\\
&+\Psi_{t_i}^2(\Delta\Psi_{t_j})^2\mathbb{E}[(\Delta M_{t_i})^2M_{t_{j-1}}^2]+2\Psi_{t_i}^2\Delta\Psi_{t_j}\Psi_{t_j}\mathbb{E}[(\Delta M_{t_i})^2M_{t_{j-1}}\Delta M_{t_j}]\\
&+\Psi_{t_i}^2\Psi_{t_j}^2\mathbb{E}[(\Delta M_{t_i})^2(\Delta M_{t_j})^2]\Big]\\
&+\sum_{i=i_0}^{m}\Big[(\Delta\Psi_{t_i})^4\mathbb{E}[M_{t_{i-1}}^4]+4(\Delta\Psi_{t_i})^3\Psi_{t_i}\mathbb{E}[M_{t_{i-1}}^3\Delta M_{t_i}]\\
&+6(\Delta\Psi_{t_i})^2\Psi_{t_i}^2\mathbb{E}[M_{t_{i-1}}^2(\Delta M_{t_i})^2]+4\Delta\Psi_{t_i}\Psi_{t_i}^3\mathbb{E}[M_{t_{i-1}}(\Delta M_{t_i})^3]+\Psi_{t_i}^4\mathbb{E}[(\Delta M_{t_i})^4]\Big]
\end{aligned}
\tag{4.23}
$$

We generalize the cross terms as follows:

$$(\Delta\Psi_{t_i})^\beta\Psi_{t_i}^{4-\beta}\mathbb{E}[M_{t_{i-1}}^\beta(\Delta M_{t_i})^{4-\beta}], \tag{4.24}$$

for $\beta = 1, 2, 3, 4$. The only nonzero cross terms correspond to $\beta = 0, 2, 4$:

$$
\begin{aligned}
&(\Delta\Psi_{t_i})^4\mathbb{E}[M_{t_{i-1}}^4]\approx c_\psi^4(2H-1)^4t_i^{8H-8}(\Delta t)^4c_M^2t_i^{4-4H}=c_\Psi^4c_M^2(2H-1)^4t_i^{4H-4}(\Delta t)^4,\\
&6(\Delta\Psi_{t_i})^2\Psi_{t_i}^2\mathbb{E}[M_{t_{i-1}}^2(\Delta M_{t_i})^2]\approx 6c_\Psi^4c_M^2(2H-1)^2(2-2H)t_i^{4H-3}(\Delta t)^3,\text{ and}\\
&\Psi_{t_i}^4\mathbb{E}[(\Delta M_{t_i})^4]\approx c_\Psi^4c_M^2(2-2H)^2t_i^{4H-2}(\Delta t)^2.
\end{aligned}
\tag{4.25}
$$

To see that each of these terms converges, we compute in general, $\sum\limits_{i=i_0}^{m} t_i^{4H-K}(\Delta t_i)^K$ for $K \geq 2$. Setting $t_i = \frac{iT}{m}$, we have

$$\begin{aligned}
\sum_{i=i_0}^{m} t_i^{4H-K}(\Delta t)^K &= \left(\tfrac{T}{m}\right)^{4H} \sum_{i=i_0}^{m} i^{4H-K} \\
&= \left(\tfrac{T}{m}\right)^{4H} \left[i_0^{4H-K} + \sum_{i=i_0+1}^{m} i^{4H-K} \right] \\
&\leq \left(\tfrac{T}{m}\right)^{4H} \left[i_0^{4H-K} + \int_{i_0}^{m} x^{4H-K}\, \mathrm{d}x. \right] \\
&= \left(\tfrac{T}{m}\right)^{4H} \left[i_0^{4H-K} + \tfrac{1}{4H-K+1}(m^{4H-K+1} - i_0^{4H-K+1}) \right] \\
&= T^{4H} \left[\frac{\left(\frac{t_0 m}{T}\right)^{4H-K}}{m^{4H}} + \tfrac{1}{4H-K+1} \left(\frac{1}{m^{K-1}} - \frac{\left(\frac{t_0 m}{T}\right)^{4H-K+1}}{m^{4H}} \right) \right] \\
&= T^{4H} \left[\frac{\left(\frac{t_0}{T}\right)^{4H-K}}{m^{K}} + \tfrac{1}{4H-K+1} \left(\frac{1}{m^{K-1}} - \frac{\left(\frac{t_0}{T}\right)^{4H-K+1}}{m^{K-1}} \right) \right] \\
&= T^{4H} \left[\frac{\left(\frac{t_0}{T}\right)^{4H-K}}{\lfloor n^{1+\delta} \rfloor^{K}} + \tfrac{1}{4H-K+1} \left(\frac{1}{\lfloor n^{1+\delta} \rfloor^{K-1}} - \frac{\left(\frac{t_0}{T}\right)^{4H-K+1}}{\lfloor n^{1+\delta} \rfloor^{K-1}} \right) \right].
\end{aligned}$$

This converges strictly faster than $\frac{1}{n}$ for all $K \geq 2$. Note that if $K = 2$ and $\delta = 0$, it only converges at a rate of $\frac{1}{n}$. Thus we sample at a rate strictly faster than $\frac{1}{n}$. Next we generalize the $i < j$ terms:

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1} M_{t_{j-1}}^{\alpha_2} \Delta M_{t_i}^{\alpha_3} \Delta M_{t_j}^{\alpha_4}] \\
&= \mathbb{E}[M_{t_{i-1}}^{\alpha_1} ((M_{t_{j-1}} - M_{t_i}) + M_{t_i})^{\alpha_2} \Delta M_{t_i}^{\alpha_3} \Delta M_{t_j}^{\alpha_4}] \qquad (4.26) \\
&= \mathbb{E}[M_{t_{i-1}}^{\alpha_1} ((M_{t_{j-1}} - M_{t_i}) + \Delta M_{t_i} + M_{t_{i-1}})^{\alpha_2} \Delta M_{t_i}^{\alpha_3} \Delta M_{t_j}^{\alpha_4}]
\end{aligned}$$

Now we need cases:

1. If $\alpha_2 = 2$ then $\alpha_4 = 0$ and

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}((M_{t_{j-1}} - M_{t_i}) + \Delta M_{t_i} + M_{t_{i-1}})^2 \Delta M_{t_i}^{\alpha_3}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3}((M_{t_{j-1}} - M_{t_i})^2 + \Delta M_{t_i}^2 + M_{t_{i-1}}^2 \\
&+ 2(M_{t_{j-1}} - M_{t_i})\Delta M_{t_i} + 2(M_{t_{j-1}} - M_{t_i})M_{t_{i-1}} + 2\Delta M_{t_i} M_{t_{i-1}})] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3}(M_{t_{j-1}} - M_{t_i})^2 + M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3} \Delta M_{t_i}^2 \\
&+ M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3} M_{t_{i-1}}^2 + 2M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3}(M_{t_{j-1}} - M_{t_i})\Delta M_{t_i} \\
&+ 2M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3}(M_{t_{j-1}} - M_{t_i})M_{t_{i-1}} + 2M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3} \Delta M_{t_i} M_{t_{i-1}}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3}(M_{t_{j-1}} - M_{t_i})^2] + \mathbb{E}[M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3+2}] \\
&+ \mathbb{E}[M_{t_{i-1}}^{\alpha_1+2} \Delta M_{t_i}^{\alpha_3}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1} \Delta M_{t_i}^{\alpha_3+1}(M_{t_{j-1}} - M_{t_i})] \\
&+ 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1} \Delta M_{t_i}^{\alpha_3}(M_{t_{j-1}} - M_{t_i})] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1} \Delta M_{t_i}^{\alpha_3+1}].
\end{aligned} \tag{4.27}$$

Using the independence of disjoint increments of (M_t) and then that (M_t) is centered, this is

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]\mathbb{E}[(M_{t_{j-1}} - M_{t_i})^2] + \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+2}] \\
&+ \mathbb{E}[M_{t_{i-1}}^{\alpha_1+2}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}]\mathbb{E}[M_{t_{j-1}} - M_{t_i}] \\
&+ 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]\mathbb{E}[M_{t_{j-1}} - M_{t_i}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+2}] \\
&+ \mathbb{E}[M_{t_{i-1}}^{\alpha_1+2}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}].
\end{aligned} \tag{4.28}$$

If $\alpha_1 = \alpha_3 = 1$ then we have

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+2}] \\
&+ \mathbb{E}[M_{t_{i-1}}^{\alpha_1+2}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}] \\
=&\mathbb{E}[M_{t_{i-1}}]\mathbb{E}[\Delta M_{t_i}]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[M_{t_{i-1}}]\mathbb{E}[\Delta M_{t_i}^3] \\
&+ \mathbb{E}[M_{t_{i-1}}^3]\mathbb{E}[\Delta M_{t_i}] + 2\mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_i}^2] \\
=&2\mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_i}^2] \\
\approx&2c_M^2 t_i^{2-2H}(2-2H)t_i^{1-2H}\Delta t \\
=&2c_M^2(2-2H)t_i^{3-4H}\Delta t.
\end{aligned} \tag{4.29}$$

If $\alpha_1 = 0$ then $\alpha_3 = 2$ and then

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+2}] \\
&\quad + \mathbb{E}[M_{t_{i-1}}^{\alpha_1+2}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}] \\
=&\mathbb{E}[\Delta M_{t_i}^2]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[\Delta M_{t_i}^4] \\
&\quad + \mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_i}^2] + 2\mathbb{E}[M_{t_{i-1}}]\mathbb{E}[\Delta M_{t_i}^3] \\
\approx& c_M^2[(2-2H)t_i^{1-2H}\Delta t(t_j^{2-2H} - t_i^{2-2H}) + 3(2-2H)^2 t_i^{2-4H}\Delta t^2 \\
&\quad + (2-2H)t_i^{3-4H}\Delta t] \\
=& c_M^2[(2-2H)t_i^{1-2H}t_j^{2-2H}\Delta t + 3(2-2H)^2 t_i^{2-4H}\Delta t^2].
\end{aligned} \tag{4.30}$$

Finally, if $\alpha_1 = 2$ and $\alpha_3 = 0$ then

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+2}] \\
&\quad + \mathbb{E}[M_{t_{i-1}}^{\alpha_1+2}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}] + 2\mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}] \\
=&\mathbb{E}[M_{t_{i-1}}^2]c_M(t_{j-1}^{2-2H} - t_i^{2-2H}) + \mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_i}^2] \\
&\quad + \mathbb{E}[M_{t_{i-1}}^4] + 2\mathbb{E}[M_{t_{i-1}}^3]\mathbb{E}[\Delta M_{t_i}] \\
\approx& c_M^2[t_i^{2-2H}(t_j^{2-2H} - t_i^{2-2H}) + t_i^{2-2H}(2-2H)t_i^{1-2H}\Delta t + 3t_i^{4-4H}] \\
=& c_M^2[t_i^{2-2H}t_j^{2-2H} + (2-2H)t_i^{3-4H}\Delta t + 2t_i^{4-4H}].
\end{aligned} \tag{4.31}$$

2. If $\alpha_2 = 1$ then

$$\begin{aligned}
&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}((M_{t_{j-1}} - M_{t_i}) + \Delta M_{t_i} + M_{t_{i-1}})\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^{\alpha_4}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^{\alpha_4}(M_{t_{j-1}} - M_{t_i}) + M_{t_{i-1}}^{\alpha_1}\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^{\alpha_4}\Delta M_{t_i} \\
&\quad + M_{t_{i-1}}^{\alpha_1}\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^{\alpha_4}M_{t_{i-1}}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^{\alpha_4}(M_{t_{j-1}} - M_{t_i}) + M_{t_{i-1}}^{\alpha_1}\Delta M_{t_i}^{\alpha_3+1}\Delta M_{t_j}^{\alpha_4} \\
&\quad + M_{t_{i-1}}^{\alpha_1+1}\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^{\alpha_4}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]\mathbb{E}[M_{t_{j-1}} - M_{t_i}]\mathbb{E}[\Delta M_{t_j}^{\alpha_4}] \\
&\quad + \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}]\mathbb{E}[\Delta M_{t_j}^{\alpha_4}] + \mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]\mathbb{E}[\Delta M_{t_j}^{\alpha_4}] \\
=&\mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3+1}]\mathbb{E}[\Delta M_{t_j}^{\alpha_4}] + \mathbb{E}[M_{t_{i-1}}^{\alpha_1+1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]\mathbb{E}[\Delta M_{t_j}^{\alpha_4}].
\end{aligned} \tag{4.32}$$

If $\alpha_1 = 0$ then $\alpha_3 = 1$ and $\alpha_4 = 2$ and we have

$$\mathbb{E}[\Delta M_{t_i}^2]\mathbb{E}[\Delta M_{t_j}^2] + \mathbb{E}[M_{t_{i-1}}]\mathbb{E}[\Delta M_{t_i}]\mathbb{E}[\Delta M_{t_j}^2] \approx c_M^2(2-2H)^2 t_i^{1-2H}t_j^{1-2H}\Delta t^2. \tag{4.33}$$

If $\alpha_1 = 1$ then $\alpha_3 = \alpha_4 = 1$ and we have

$$\mathbb{E}[M_{t_{i-1}}]\mathbb{E}[\Delta M_{t_i}^2]\mathbb{E}[\Delta M_{t_j}] + \mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_i}]\mathbb{E}[\Delta M_{t_j}] = 0. \tag{4.34}$$

Finally, if $\alpha_1 = 2$ then $\alpha_3 = 0$ and $\alpha_4 = 1$ and we have

$$\mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_i}]\mathbb{E}[\Delta M_{t_j}] + \mathbb{E}[M_{t_{i-1}}^3]\mathbb{E}[\Delta M_{t_j}] = 0. \tag{4.35}$$

3. If $\alpha_2 = 0$ then $\alpha_4 = 2$ and we have

$$\mathbb{E}[M_{t_{i-1}}^{\alpha_1}\Delta M_{t_i}^{\alpha_3}\Delta M_{t_j}^2] = \mathbb{E}[M_{t_{i-1}}^{\alpha_1}]\mathbb{E}[\Delta M_{t_i}^{\alpha_3}]\mathbb{E}[\Delta M_{t_j}^2] \tag{4.36}$$

and so if $\alpha_1 = 0$ then $\alpha_3 = 2$ and we have

$$\mathbb{E}[\Delta M_{t_i}^2]\mathbb{E}[\Delta M_{t_j}^2] \approx c_M^2(2-2H)^2 t_i^{1-2H} t_j^{1-2H}\Delta t^2. \tag{4.37}$$

If $\alpha_1 = 1$ then $\alpha_3 = 1$ and we have

$$\mathbb{E}[M_{t_{i-1}}]\mathbb{E}[\Delta M_{t_i}]\mathbb{E}[\Delta M_{t_j}^2] = 0. \tag{4.38}$$

Finally, if $\alpha_1 = 2$ then $\alpha_3 = 0$ and

$$\mathbb{E}[M_{t_{i-1}}^2]\mathbb{E}[\Delta M_{t_j}^2] \approx c_M^2(2-2H) t_i^{2-2H} t_j^{1-2H}\Delta t. \tag{4.39}$$

After incorporating the Ψ_t terms, one term emerges to cancel with the term

$$C^4 \sum\sum t_i^{2H-1} t_j^{2H-1}(\Delta t)^2 \tag{4.40}$$

in (4.20). Otherwise, all remaining terms are of the form

$$\sum\sum t_i^{2H-M} t_j^{2H-N}(\Delta t)^{M+N},$$

for combinations of $N, M \in \{1,2\}$ except $M = N = 1$. For terms of this form, setting $t_i = \frac{iT}{m}$ and $t_j = \frac{jT}{m}$, we have

$$\begin{aligned}
&\sum_{i=i_0}^{m} \sum_{j=i+1}^{m} t_i^{2H-M} t_j^{2H-N} (\Delta t)^{M+N} \\
&= \left(\tfrac{T}{m}\right)^{4H} \sum_{i=i_0}^{m} i^{2H-M} \sum_{j=i+1}^{m} j^{2H-N} \\
&\leq \left(\tfrac{T}{m}\right)^{4H} \sum_{i=i_0}^{m} i^{2H-M} \int_i^m x^{2H-N} \, \mathrm{d}x. \\
&= \left(\tfrac{T}{m}\right)^{4H} \sum_{i=i_0}^{m} i^{2H-M} \tfrac{1}{2H-N+1} \left(m^{2H-N+1} - i^{2H-N+1}\right) \\
&= \tfrac{T^{4H}}{2H-N+1} \left[\tfrac{1}{m^{2H+N-1}} \sum_{i=i_0}^{m} i^{2H-M} - \tfrac{1}{m^{4H}} \sum_{i=i_0}^{m} i^{4H-N-M+1} \right] \\
&\approx \tfrac{T^{4H}}{2H-N+1} \left[\tfrac{1}{m^{2H+N-1}} \left[i_0^{2H-M} + \int_{i_0}^{m} x^{2H-M} \, \mathrm{d}x \right] \right. \\
&\quad \left. - \tfrac{1}{m^{4H}} \left[i_0^{4H-M-N+1} + \int_{i_0}^{m} x^{4H-N-M+1} \, \mathrm{d}x \right] \right]. \\
&= \tfrac{T^{4H}}{2H-N+1} \left[\frac{\left(\frac{t_0}{T}\right)^{2H-M}}{m^{M+N-1}} + \tfrac{1}{2H-M+1} \left(\frac{1}{m^{M+N-2}} - \frac{\left(\frac{t_0}{T}\right)^{2H-M+1}}{m^{M+N-2}} \right) \right. \\
&\quad \left. - \frac{\left(\frac{t_0}{T}\right)^{4H-M-N+1}}{m^{M+N-1}} - \tfrac{1}{4H-M-N+2} \left(\frac{1}{m^{M+N-2}} - \frac{\left(\frac{t_0}{T}\right)^{4H-M-N+2}}{m^{M+N-2}} \right) \right]
\end{aligned}$$

This converges strictly faster than $\frac{1}{n}$ for all $N, M \in \{1,2\}$, excluding $M = N = 1$, as required. Since the order is strictly faster than $\frac{1}{n}$ for all terms, Borel Cantelli implies almost sure convergence. □

4.2.3 Ratio method with quadratic variation

As in Section 4.1, suppose we have $m = \lfloor n^{1+\delta} \rfloor$ equally time-spaced observations of the stock price process (S_t^{ϵ}), called s_i, observed at time $t_i = \frac{iT}{n}$, $i = 0, \ldots, m$. Let $\Delta t = \frac{T}{m}$. Again, assume that the stock price does not pay dividends during this interval and define the log returns $y_i = \ln \frac{s_i}{s_{i-1}}$ for $i = 1, \ldots, m$. We assume the stock price process follows a geometric Dobrić-Ojeda process, as detailed in Chapter 3, where we have

$$S_t^\epsilon = S_0 \exp\left\{rt + \sigma C \int_0^t s^{H-1/2} dW_s^\epsilon - \frac{\sigma^2 C^2}{2(2H)} t^{2H}\right\}. \tag{4.41}$$

Assume

$$y_i = \mu \Delta t + \sigma(V_H^\epsilon(t_i) - V_H^\epsilon(t_{i-1})) - \frac{1}{2}\sigma^2 \frac{C^2}{2H}(t_i^{2H} - t_{i-1}^{2H}). \tag{4.42}$$

By Corollary 4.2.4, we have

$$\sum_{i=1}^m y_i^2 \to \sigma^2 \frac{C^2}{2H} T^{2H} \tag{4.43}$$

and similarly, the sample quadratic variation of half of the sample path converges to $\sigma^2 \frac{C^2}{2H}\left(\frac{T}{2}\right)^{2H}$:

$$\sum_{i=1}^{\lfloor m/2 \rfloor} y_i^2 \to \sigma^2 \frac{C^2}{2H}\left(\frac{T}{2}\right)^{2H}. \tag{4.44}$$

Therefore, since this convergence is almost sure, we can use a ratio of quadratic variations method to estimate the parameter H:

$$\frac{\sum_{i=1}^{\lfloor m/2 \rfloor} y_i^2}{\sum_{i=1}^m y_i^2} \to \frac{\sigma^2 \frac{C^2}{2H}\left(\frac{T}{2}\right)^{2H}}{\sigma^2 \frac{C^2}{2H} T^{2H}} = \left(\frac{1}{4}\right)^H. \tag{4.45}$$

Therefore for m sufficiently large, we will estimate the Hurst index H by

$$H \approx \hat{H} = \log_4\left(\frac{\sum_{i=1}^{\lfloor m/2 \rfloor} y_i^2}{\sum_{i=1}^m y_i^2}\right). \tag{4.46}$$

Finally, we can use the estimator $\hat{H}$ to obtain an estimate for the volatility σ:

$$\sigma^2 \approx \hat{\sigma}^2 = \frac{2H}{C(\hat{H})^2 T^{2\hat{H}}} \sum_{i=1}^m y_i^2. \tag{4.47}$$

Chapter 5

Simulation and case study

We conclude the development of this model with a brief mention of simulation and finally computation of the value of a European call option using historical stock price data.

5.1 Simulation

Using the Itô diffusion representation of the Dobrić-Ojeda process given in Proposition 2.2.1, we can use a sequence of i.i.d. standard normal random variables in order to simulate a discretized Dobrić-Ojeda sample path, assuming that $V_H(0) = 0$. More specifically, if $\{X_i\}_{i=1,\ldots,n}$ is a sequence of i.i.d. standard normal random variables, then we simulate increments of the martingale process $\Delta M_H(t_i)$ by

$$\Delta M_H(t_i) = \sqrt{c_M(2-2H)}\, t_i^{1/2-H} \sqrt{\Delta t} X_i. \tag{5.1}$$

We sum the increments $\Delta M_H(t_i)$ and multiply by the deterministic function $\Psi_H(t)$ to simulate a sample path of $V_H(t)$.

To describe implementation of the model, we price a historical European call option and compare this price with the actual trading price along with prices computed using the original Black-Scholes model and the model using fractional Brownian motion as its driving process, as developed by Hu and Oksendal [13] and Sottinen [28].

5.2 Case study

We consider a call option on American Airlines stock (AAL) with strike price $K = 38$ and expiration November 22, 2014. For each day beginning March 27, 2014 and ending October 15, 2014, we estimate H and σ using the previous 62 consecutive daily AAL closing prices. Figure 5.1 shows the daily closing price for the stock over this time period.

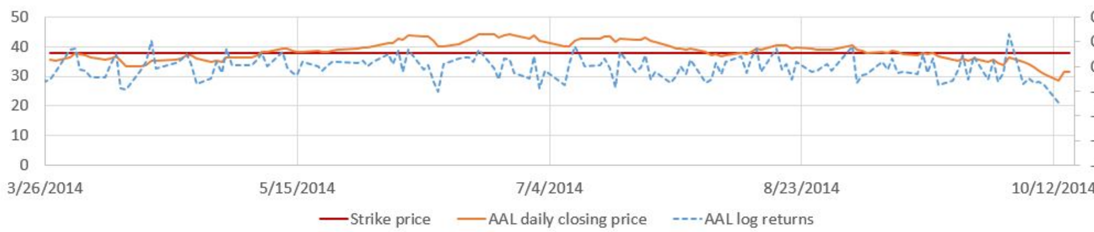

Figure 5.1: Graph of AAL daily closing prices.

For each day, we compute 3 estimations for the parameters: 1. assuming the stock price follows a geometric Brownian motion process and using standard Black-Scholes techniques; 2. assuming the stock price follows a geometric fractional Brownian motion process and using using a ratio of second moments technique as detailed in Section 4; 3. assuming the stock price follows a geometric Dobrić-Ojeda process and using a ratio of quadratic variations technique, also in Section 4. The latter two rolling H estimates are shown in Figure 5.2.

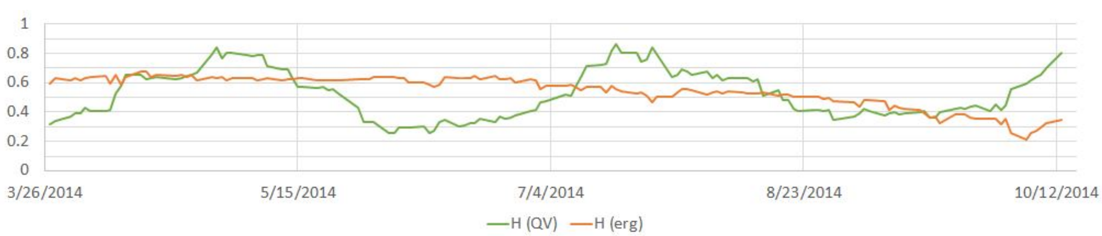

Figure 5.2: Graph of rolling H estimates.

One immediate observation is that the H estimate using quadratic variation is extremely sensitive to large changes in the log return of the underlying stock. We also notice that the estimates for H are in both cases often significantly lower than 0.6, our market-wide expected H estimate. These observations lead us to believe that H varies both over time and over stock selection. Next we compute the option price using the three competing models and their respective parameter estimation techniques and compare these prices to the actual trading price of the stock at market close each day. The results are shown in Figure 5.3.

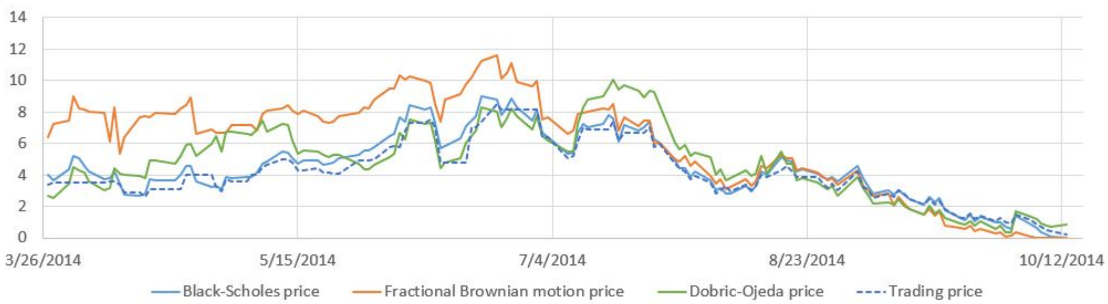

Figure 5.3: Graph of computed option prices.

We notice that when it appears the quadratic variation method overestimates H, the Dobrić-Ojeda model correspondingly overestimates the option price. However, when the H estimate using $V_H(t)$ is lower than expected, this model outperforms the others in approximating the actual trading price of the option. We also notice (less surprisingly) that the Black-Scholes price is fairly similar to the option's trading price. A more accurate method of testing the various models would be in building competing virtual historical portfolios and considering their performance.

Chapter 6

Dobrić-Ojeda stochastic noise

In the remaining two chapters, we develop and study a Dobrić-Ojeda type noise in stochastic partial differential equations, particularly in the stochastic heat equation. As with the Black-Scholes SDE, we propose this noise to be an alternative to fractional noise that gives similar results but allows for the use of Itô calculus because of its semi-martingale property.

We begin by defining a martingale measure $(M_t(A))_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^d)}$ inspired by the martingale process $(M_H(t))$. When a Borel set $A \in \mathcal{B}(\mathbb{R}^d)$ is fixed, the process $(M_t(A))_{t\geq 0}$ is a martingale and when $t \geq 0$ is fixed, $M_t(A)$ is a measure on $\mathcal{B}(\mathbb{R}^d)$. Next we use this martingale measure to define a stochastic integral with respect to a noise that is white in space and of a Dobrić-Ojeda type in time.

6.1 Martingale measure

In the Black-Scholes differential equation studied in the first half of this dissertation, random noise is incorporated in time. To consider the stochastic heat equation with a random external heat source, we incorporate a space-time random noise. Random noise that behaves like a Brownian motion in space and like another Brownian motion in time is called *space-time white noise*, $\dot{W}$. To understand an integral with respect to white noise, we follow the method of Walsh [29].

First, the definition of martingale measure:

Definition 6.1.1. *A process* $(X_t(A))_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^d)}$ *is a martingale measure with respect to a probability space* $(\Omega, \mathcal{F}_t, \mathbb{P})$ *if*

1. $X_0(A) = 0$ *a.s.,*

2. *If* $t > 0$ *then* X_t *is a sigma-finite* $L^2(\mathbb{P})$*-valued signed measure, and*

3. *For all Borel sets* $A \in \mathcal{B}(\mathbb{R}^d)$, $(X_t(A))_{t\geq 0}$ *is a mean-zero martingale with respect to the filtration* $\mathcal{F}_t = \sigma(X_t(A), t \geq 0)$.

With respect to any martingale measure, including white noise $\dot{W}$, we can define an integral:

Definition 6.1.2. *A Walsh integral is an integral with respect to a martingale measure* X*, first defined on elementary functions* $f(x,t,\omega) = Y(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_A(x)$ *as*

$$\iint_{B\times(0,t]} f(x,s)X(dx\,ds) = X(\omega)[X_{t\wedge b}(A\cap B) - X_{t\wedge a}(A\cap B)](\omega) \tag{6.1}$$

and extended in the usual fashion to adapted functions f.

Hence, Walsh defines an integral with respect to white noise. We wish to introduce a space dependence to white noise so we define the martingale measure below by replacing Brownian motion with the martingale process $(M_t)_{t\geq 0}$ as defined in Chapter 1, for $0 < H < 1$:

$$dM_t = \sqrt{c_M(2-2H)}t^{1/2-H}dW_t, \tag{6.2}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion process.

Proposition 6.1.3. *Then* $(M_t(A))_{t\geq 0, A\in\mathcal{B}(\mathbb{R})}$*, defined by*

$$\begin{aligned} M_t(A) &= \int_0^t \sqrt{c_M(2-2H)}s^{1/2-H}\,dW_s(A) \\ &= \iint\limits_{A\times(0,t]} \sqrt{c_M(2-2H)}s^{1/2-H}\,W(\,dx\,ds), \end{aligned} \tag{6.3}$$

is a martingale measure, where $\iint f(x,s)\,W(\,dx\,ds)$ *is the Walsh integral in Definition 6.1.2, as in [29].*

For similar reasons as in Part 1, $M_t(A)$ is well-defined: $\int_0^t s^{1-2H}\,ds < \infty$ for $0 < H < 1$. For $A \in \mathcal{B}(\mathbb{R})$, $M_t(A)$ is Gaussian as discussed in Part 1 as well. In order to show $(M_t(A))$ is a martingale measure, we will first prove the following lemma:

Lemma 6.1.4. *For $t, s > 0$ and $A, B \in \mathcal{B}(\mathbb{R})$ and $(M_t(A))_{t\geq 0, A\in\mathcal{B}(\mathbb{R})}$ as defined above, we have*

$$\mathbb{E}\left[M_t(A)M_s(B)\right] = c_M(t\wedge s)^{2-2H}\lambda(A\cap B), \tag{6.4}$$

where λ is the Lebesgue measure.

Proof of Lemma 6.1.4. For $(M_t(A))_{t\geq 0, A\in\mathcal{B}(\mathbb{R})}$ as defined above, we have

$$\begin{aligned}
&\mathbb{E}\left[M_t(A)M_s(B)\right] \\
=&\mathbb{E}\left[\iint\limits_{A\times(0,t]} \sqrt{c_M(2-2H)}u_1^{1/2-H}\,W(\,dx\,du_1) \iint\limits_{B\times(0,s]} \sqrt{c_M(2-2H)}u_2^{1/2-H}\,W(\,dy\,du_2)\right] \\
=&c_M(2-2H)\int_0^t\int_A\int_0^s\int_B u_1^{1/2-H}u_2^{1/2-H}\delta_0(u_1-u_2)\delta_0(x-y)\,dy\,du_2\,dx\,du_1 \\
=&c_M(2-2H)\int_0^\infty\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}} \mathbb{1}_{(0,t)}(u_1)\mathbb{1}_{(0,s)}(u_2)\mathbb{1}_A(x)\mathbb{1}_B(y)u_1^{1/2-H}u_2^{1/2-H} \\
&\times\delta_0(u_1-u_2)\delta_0(x-y)\,dy\,du_2\,dx\,du_1 \\
=&c_M(2-2H)\int_{(0,\infty)}\int_{\mathbb{R}}\mathbb{1}_{(0,t\wedge s)}(u)\mathbb{1}_{A\cap B}(x)u^{1-2H}\,dx\,du \\
=&c_M(2-2H)\frac{(t\wedge s)^{2-2H}}{2-2H}\lambda(A\cap B) \\
=&c_M(t\wedge s)^{2-2H}\lambda(A\cap B).
\end{aligned} \tag{6.5}$$

□

Proof of Proposition 6.1.3. 1. First,

$$M_0(A) = \iint\limits_{A\times(0,0]} \sqrt{c_M(2-2H)}s^{1/2-H}\,W(\,dx\,ds) = 0. \tag{6.6}$$

2. Fix $t > 0$. Then to show M_t is a sigma-finite $L^2(\mathbb{P})$-valued signed measure, it suffices to prove three things: (a) If $A, B \in \mathcal{B}(\mathbb{R}^d)$ are disjoint then $M_t(A)$ and $M_t(B)$ are independent random variables; (b) For all compact sets K, $\mathbb{E}\left[(M_t(K))^2\right] < \infty$; and (c) If $A_1 \supset A_2 \supset \ldots$ are all in $\mathcal{B}(\mathbb{R}^d)$ and $\cap A_n = \emptyset$, then $M_t(A_n) \to 0$ in $L^2(P)$ as $n \to \infty$. The proof of (a) is trivial by Lemma 6.1.4. To prove (b), we have

$$\mathbb{E}\left[(M_t(K))^2\right] = c_M t^{2-2H} \lambda^d(K) < \infty. \tag{6.7}$$

To prove (c),

$$\mathbb{E}\left[(M_t(A_n))^2\right] = c_M t^{2-2H} \lambda^d(A_n) \to 0 \tag{6.8}$$

as $n \to \infty$ since $\lambda^d(A_n) \to 0$.

3. Finally, for $A \in \mathcal{B}(\mathbb{R}^d)$, $(M_t(A))_{t\geq 0}$ is a mean-zero martingale with respect to $\mathcal{F}$ by Proposition 2.1.1.

□

Remark 1. *Alternatively, we could use Theorem 5.26 in [18], stated here without proof, to prove that* $(M_t(A))_{t\geq 0, a\in\mathcal{B}(\mathbb{R}^d)}$ *is a worthy martingale measure.*

Theorem 6.1.5. *Let X be a worthy martingale measure. Then for all predictable functions f with* $\mathbb{E}\left[\left(\iint_{K\times(0,t]} f\, dX\right)^2\right] < \infty$, *for all $K \in \mathbb{R}$ compact, we have*

$$\iint_{A\times(0,t]} f\, dX \tag{6.9}$$

is a worthy martingale measure.

Proof. See [18]. □

Alternate proof of Proposition 6.1.3. Let $f(x,t) = \sqrt{c_M(2-2H)}t^{1/2-H}$. Note that f does not depend on x. First, f is previsible because it's deterministic. Now let K

be any compact set in $\mathbb{R}$. Then

$$\mathbb{E}\left[\left(\iint\limits_{K\times(0,T]} f(x,t)W(\,dx\,dt)\right)^2\right] = c_M(2-2H)\int_0^T\int_K\int_K t^{1-2H}\delta_0(x-y)\,dx\,dy\,dt$$
$$= c_M T^{2-2H}\int_K dx$$
$$= c_M T^{2-2H}\lambda(K)$$
$$< \infty. \tag{6.10}$$

□

6.2 Dobrić-Ojeda stochastic noise

We define $\int_0^t\int_0^L F(y,s)\,V(\,dy\,ds)$ for any function F that satisfies

$$\int_0^t\int_0^L F^2(y,s)s^{2H-1}\,dy\,ds < \infty \tag{6.11}$$

and

$$\int_0^t s^{H-1}\left(\int_0^L F^2(y,s)\,dy\right)^{1/2}\,ds < \infty. \tag{6.12}$$

These conditions ensure that F is $V-$integrable.

Definition 6.2.1. *We define $\dot{V}$ as the Dobrić-Ojeda stochastic noise, for fixed $H \in (0,1)$ and for $F(y,s)$ that satisfies* (6.11) *and* (6.12)*,*

$$\int_0^t\int_0^L F(y,s)V(\,dy\,ds)$$
$$= \int_0^t\int_0^L F(y,s)s^{H-1/2}W(\,dy\,ds) + \int_0^t\left[\int_0^s\int_0^L F(y,s)r^{1/2-H}W(\,dy\,dr)\right]s^{2H-2}\,ds. \tag{6.13}$$

Note that we need $(M_t(A))$ to be a worthy martingale measure (as in Remark 1) to ensure that this stochastic noise is well-defined.

Remark 2. *With no space dependency, we have*

$$\begin{aligned}
&\int_0^t F(s)\,dV_s \\
&= \int_0^t F(s)s^{H-1/2}\,dW_s + \int_0^t F(s)\left[\int_0^s r^{1/2-H}\,dW_r\right] s^{2H-2}\,ds \qquad (6.14)\\
&= \int_0^t F(s)s^{H-1/2}\,dW_s + \int_0^t F(s)M_s s^{2H-2}\,ds,
\end{aligned}$$

where (M_s) *is the Martingale process defined in Part 1. When* $F(s) \equiv 1$*, we have*

$$V_t = \int_0^t dV_s = \int_0^t s^{H-1/2}\,dW_s + \int_0^t M_s s^{2H-2}\,ds, \qquad (6.15)$$

which corresponds with the Itô diffusion given in Proposition 2.2.1. This motivates our definition of $\dot{V}$.

Chapter 7

Stochastic heat equation

In this chapter we apply the Dobrić-Ojeda noise to the stochastic heat equation. We prove the existence and uniqueness of a solution for $H > 1/4$ and we establish the Hölder continuity of the solution.

7.1 Definition

Consider the following stochastic heat equation, for fixed $H \in (1/4, 1)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(u)\dot{V}, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{7.1}$$

where $\dot{V}$ is the Dobrić-Ojeda stochastic noise defined in (6.13), $u_0 : \mathbb{R} \to \mathbb{R}$ is nonrandom, measurable, and bounded; and $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz and bounded:

$$K := \sup_{0 \leq x \neq y} \frac{|f(x) - f(y)|}{|y - x|} + \sup_{0 \leq x \leq L} |f(x)| < \infty. \tag{7.2}$$

We begin by proving the existence of a unique and continuous solution to the stochastic heat equation (7.1) with a modified Dobrić-Ojeda noise. These results for the entire Dobrić-Ojeda noise, including the drift term, remain a work in progress. We expect that properties such as Hölder continuity and intermittency with respect to the time variable are invariant to the drift term because the drift term is differentiable in time and thus continuous. Motivated by this intuition, we redefine the Dobrić-Ojeda stochastic noise:

Definition 7.1.1. *The modified Dobrić-Ojeda stochastic noise, $\dot{V}$, is given by*

$$\int_0^t \int_{-\infty}^{\infty} F(y,s)V(\,dy\,ds) = \int_0^t \int_{-\infty}^{\infty} F(y,s)s^{H-1/2}W(\,dy\,ds), \tag{7.3}$$

for any function F satisfying the integrability condition

$$\int_0^t \int_{-\infty}^{\infty} F^2(y,s)s^{2H-1}\,dy\,ds < \infty. \tag{7.4}$$

Note that condition (6.11) is required so that the integral

$$\int_0^t \int_{-\infty}^{\infty} F(y,s)s^{H-1/2}W(\,dy\,ds)$$

is well-defined but without including the drift term, condition (6.12) is no longer necessary. From now on, we will refer to $\dot{V}$ as in Definition 7.1.1 as Dobrić-Ojeda noise.

7.2 Existence and uniqueness

As in [18], we know that the noise in (7.1) is not differentiable so we cannot find a strong solution $u(x,t)$ satisfying (7.1). Instead, we seek an integral solution, or "mild" solution, a function $u(x,t)$ that satisfies

$$\begin{aligned} &u(x,t) \\ &= \int_{-\infty}^{\infty} u_0(y)\Pi(t,x-y)\,dy + \int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s;x-y)V(\,dy\,ds) \\ &= \int_{-\infty}^{\infty} u_0(y)\Pi(t,x-y)\,dy + \int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s;x-y)s^{H-1/2}W(\,dy\,ds), \end{aligned} \tag{7.5}$$

where Π is the fundamental solution to the linear heat equation:

$$\Pi(t,a) = \frac{1}{(4\pi t)^{1/2}} e^{\frac{-a^2}{4t}}. \tag{7.6}$$

First, a few lemmas:

Lemma 7.2.1. *For $t > 0$, $x \in \mathbb{R}$, the function $\Pi(t, a)$ as defined in (7.6) satisfies*

$$\int_{-\infty}^{\infty} \Pi^2(t,a)\,da = \frac{1}{\sqrt{8\pi t}}. \tag{7.7}$$

Proof. By the definition of Π (7.6), we have

$$\begin{aligned}\int_{-\infty}^{\infty} \Pi^2(t,a)\,da &= \int_{-\infty}^{\infty} \left(\frac{1}{(4\pi t)^{1/2}} e^{\frac{-a^2}{4t}}\right)^2 da \\ &= \int_{-\infty}^{\infty} \frac{1}{4\pi t} e^{\frac{-a^2}{2t}}\,da \\ &= \frac{1}{\sqrt{8\pi t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{\frac{-a^2}{2t}}\,da \\ &= \frac{1}{\sqrt{8\pi t}}.\end{aligned} \tag{7.8}$$

□

Lemma 7.2.2. *When $\alpha > -1$ and $\gamma > -1$,*

$$\int_0^t s^\alpha (t-s)^\gamma\,ds = t^{\alpha+\gamma+1}\beta(\alpha+1,\gamma+1), \tag{7.9}$$

where $\beta(\alpha+1,\gamma+1) = \int_0^1 t^\alpha (1-t)^\beta\,dt$.

Proof. Let $u = \frac{s}{t}$. Then

$$\int_0^1 (tu)^\alpha (t-tu)^\gamma t\,du = t^{\alpha+\gamma+1}\int_0^1 u^\alpha (1-u)^\gamma\,du = t^{\alpha+\gamma+1}\beta(\alpha+1,\gamma+1). \tag{7.10}$$

□

Lemma 7.2.3. *For $H \in (1/4, 1)$ fixed, and $g(t)$ any bounded non-negative function, there exists a number $q > 2$ and a constant A such that for all $t \in [0, T]$,*

$$\int_0^t g(s) s^{2H-1} (t-s)^{-1/2}\,ds \le A\left(\int_0^t g^q(s)\,ds\right)^{1/q}. \tag{7.11}$$

Proof. Consider $p \in (1,2)$. Define $q > 2$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Then with Hölder's Inequality, we have

$$\begin{aligned}
&\int_0^t g(s) s^{2H-1} (t-s)^{-1/2} \, ds \\
&\leq \left(\int_0^t \left| s^{2H-1} (t-s)^{-1/2} \right|^p \, ds \right)^{1/p} \left(\int_0^t g^q(s) \, ds \right)^{1/q} \\
&= \left(\int_0^t s^{(2H-1)p} (t-s)^{-p/2} \, ds \right)^{1/p} \left(\int_0^t g^q(s) \, ds \right)^{1/q} .
\end{aligned} \tag{7.12}$$

If $H \geq 1/2$, then $(2H-1)p > 0$ so $s^{(2H-1)p}$ is increasing and if we choose, say, $p = 3/2$, we have

$$\begin{aligned}
&\left(\int_0^t s^{(2H-1)p} (t-s)^{-p/2} \, ds \right)^{1/p} \left(\int_0^t g^q(s) \, ds \right)^{1/q} \\
&= \left(\int_0^t s^{3/2(2H-1)} (t-s)^{-3/4} \, ds \right)^{2/3} \left(\int_0^t g^q(s) \, ds \right)^{1/q} \\
&\leq \left(T^{3/2(2H-1)} \int_0^t (t-s)^{-3/4} \, ds \right)^{2/3} \left(\int_0^t g^q(s) \, ds \right)^{1/q} \\
&= \left(T^{3/2(2H-1)} 4 t^{1/4} \right)^{2/3} \left(\int_0^t g^q(s) \, ds \right)^{1/q} \\
&\leq \left(4 T^{3/2(2H-1)+1/4} \right)^{2/3} \left(\int_0^t g^q(s) \, ds \right)^{1/q} .
\end{aligned} \tag{7.13}$$

Now consider $1/4 < H < 1/2$. Set $p = \frac{1}{3/2-2H}$. We verify $p \in (1,2)$:

$$\begin{aligned}
&1/4 < H < 1/2 \\
\Rightarrow &1/2 < 2H < 1 \\
\Rightarrow &-1 < -2H < -1/2 \\
\Rightarrow &3/2 - 1 < 3/2 - 2H < 3/2 - 1/2 \\
\Rightarrow &1/2 < 3/2 - 2H < 1 \\
\Rightarrow &1 < \frac{1}{3/2 - 2H} < 2.
\end{aligned} \tag{7.14}$$

We also have $(2H-1)p > -1$:

$$\begin{aligned}
&(2H-1)p > -1 \\
\iff &\frac{2H-1}{3/2-2H} > -1 \\
\iff &2H-1 > -(3/2-2H) \text{ (note that } 3/2-2H > 0 \text{ for } H < 1/2) \\
\iff &-1 > -3/2.
\end{aligned} \tag{7.15}$$

Then with Hölder's Inequality and Lemma 7.2.2, for q satisfying $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
&\left(\int_0^t s^{(2H-1)p}(t-s)^{-p/2}\,ds\right)^{1/p}\left(\int_0^t g^q(s)\,ds\right)^{1/q} \\
=&\left(t^0\beta((2H-1)p+1,-p/2+1)\right)^{1/p}\left(\int_0^t g^q(s)\,ds\right)^{1/q} \\
=&\beta((2H-1)p+1,-p/2+1)^{1/p}\left(\int_0^t g^q(s)\,ds\right)^{1/q}.
\end{aligned} \tag{7.16}$$

We require $p > 1$ so that $q > 0$ and we require $p < 2$ so that $-p/2 > -1$ and the first term is integrable. □

We will also need the following lemma, as stated in [18].

Lemma 7.2.4. *Suppose $\phi_1, \phi_2, \ldots : [0,T] \to \mathbb{R}_+$ are measurable and non-decreasing and there exists a constant A such that for all integers $n \geq 1$ and all $t \in [0,T]$,*

$$\phi_{n+1}(t) \leq A\int_0^t \phi_n(s)\,ds. \tag{7.17}$$

Then

$$\phi_n(t) \leq \phi_1(T)\frac{(At)^{n-1}}{(n-1)!} \tag{7.18}$$

for all $n \geq 1$ and $t \in [0,T]$ and therefore any positive power of $\phi_n(t)$ is summable in n. In the special case that ϕ_n does not depend on n, it follows that $\phi_n \equiv 0$.

Proof. For $n = 1$, the right hand side of (7.18) is simply $\phi_1(T)$ and since ϕ_1 is non-decreasing, $\phi_1(t) \leq \phi_1(T)$. Now suppose $\phi_n(t) \leq \phi_1(T)\frac{(At)^{n-1}}{(n-1)!}$. Then

$$\begin{aligned}\phi_{n+1}(t) \leq & A\int_0^t \phi_n(s)\,ds \\ \leq & A\int_0^t \phi_1(T)\frac{(As)^{n-1}}{(n-1)!}\,ds \\ = & A\phi_1(T)\frac{A^{n-1}}{(n-1)!}\int_0^t s^{n-1}\,ds \\ = & \phi_1(T)\frac{(At)^n}{n!}.\end{aligned} \tag{7.19}$$

□

Finally, we can prove the existence and uniqueness of a solution to (7.5). For simplicity, we assume the initial condition u_0 is constant.

Theorem 7.2.5. *The stochastic heat equation* (7.5) *subject to* (7.2) *has an almost-sure unique solution* u *that satisfies*

$$\sup_{x\in\mathbb{R}}\sup_{0\leq t\leq T}\mathbb{E}\left(|u(x,t)|^2\right) < \infty \tag{7.20}$$

for all $T > 0$ *and for* $H \in (1/4, 1)$ *fixed.*

Proof. Existence: To show that a solution exists to (7.5) we use a Picard-type iteration scheme, as in [18]. First, let $u_0(x,t) = u_0$ and then define

$$\begin{aligned}u_{n+1}(x,t) = & \int_{-\infty}^{\infty} u_0\Pi(t, x-y)\,dy \\ & + \int_0^t\int_{-\infty}^{\infty} f(u_n(y,s))\Pi(t-s, x-y)s^{H-1/2}W(\,dy\,ds) \\ = & u_0 + \int_0^t\int_{-\infty}^{\infty} f(u_n(y,s))\Pi(t-s, x-y)s^{H-1/2}W(\,dy\,ds).\end{aligned} \tag{7.21}$$

for $n \geq 1$, where Π is defined as in (7.6). To see that the second term is well-defined, we show that its second moment is finite:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} f(u_n(y,s))\Pi(t-s,x-y)s^{H-1/2}W(\,dy\,ds)\right)^2\right]\\
=&\int_0^t \int_{-\infty}^{\infty} f^2(u_n(y,s))\Pi^2(t-s,x-y)s^{2H-1}\,dy\,ds\\
\leq&K^2\int_0^t \int_{-\infty}^{\infty} \Pi^2(t-s,x-y)s^{2H-1}\,dy\,ds\\
=&\frac{K^2}{\sqrt{8\pi}}\int_0^t s^{2H-1}(t-s)^{-1/2}\,ds\\
=&\frac{K^2}{\sqrt{8\pi}}t^{2H-1-1/2+1}\beta(2H-1+1,-1/2+1)\\
=&\frac{K^2}{\sqrt{8\pi}}t^{2H-1/2}\beta(2H,1/2)\\
<&\infty
\end{aligned} \tag{7.22}$$

by the condition (7.2) on f, Lemma 7.2.1, and Lemma 7.2.2. Let

$$\begin{aligned}
d_n(x,t) &= u_{n+1}(x,t) - u_n(x,t)\\
&= \int_0^t \int_{-\infty}^{\infty} \left(f(u_n(y,s)) - f(u_{n-1}(y,s))\right)\Pi(t-s,x-y)s^{H-1/2}W(\,dy\,ds).
\end{aligned} \tag{7.23}$$

Then by Burkholder's inequality [4] and the restriction (7.2) on f,

$$\begin{aligned}
&\mathbb{E}\left[(d_n(x,t))^2\right]\\
=&\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} \left(f(u_n(y,s)) - f(u_{n-1}(y,s))\right)\Pi(t-s,x-y)s^{H-1/2}W(\,dy\,ds)\right)^2\right]\\
\leq&\int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left[\left(f(u_n(y,s)) - f(u_{n-1}(y,s))\right)^2\right]\Pi^2(t-s,x-y)s^{2H-1}\,dy\,ds\\
\leq&K^2\int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left[\left(u_n(y,s) - u_{n-1}(y,s)\right)^2\right]\Pi^2(t-s,x-y)s^{2H-1}\,dy\,ds\\
=&K^2\int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left[(d_{n-1}(y,s))^2\right]\Pi^2(t-s,x-y)s^{2H-1}\,dy\,ds.
\end{aligned} \tag{7.24}$$

Note that in particular,

$$
\begin{aligned}
&\mathbb{E}\left[(d_0(x,t))^2\right] \\
=&\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} f(u_0)\Pi(t-s,x-y)s^{H-1/2}\,W(dy\,ds)\right)^2\right] \\
\leq&\frac{K^2}{\sqrt{8\pi}}t^{2H-1/2}\beta(2H,1/2) \\
\leq&\frac{K^2}{\sqrt{8\pi}}T^{2H-1/2}\beta(2H,1/2),
\end{aligned}
\tag{7.25}
$$

as in (7.22), since $H > 1/4$. Let $R_n^2(t) = \sup_{0\leq x\leq L}\sup_{0\leq s\leq t}\mathbb{E}\left[(d_n(x,t))^2\right]$. Then by Lemmas 7.2.1 and 7.2.3,

$$
\begin{aligned}
R_n^2(t) \leq& K^2\int_0^t \int_{-\infty}^{\infty} R_{n-1}^2(s)\Pi^2(t-s,x-y)s^{2H-1}\,dy\,ds \\
=&K^2\int_0^t R_{n-1}^2(s)s^{2H-1}\int_{-\infty}^{\infty}\Pi^2(t-s,x-y)\,dy\,ds \\
=&\frac{K^2}{\sqrt{8\pi}}\int_0^t R_{n-1}^2(s)s^{2H-1}(t-s)^{-1/2}\,ds \\
\leq&A\left(\int_0^t R_{n-1}^{2q}(s)\,ds\right)^{1/q}.
\end{aligned}
\tag{7.26}
$$

Raising both sides to the power q, we have

$$
R_n^{2q}(t) \leq A^q\int_0^t R_n^{2q}(s)\,ds. \tag{7.27}
$$

Then by Gronwall's lemma 7.2.4, we have

$$
\begin{aligned}
R_n^{2q}(t) \leq& R_0^{2q}(T)\frac{(At)^{n-1}}{(n-1)!} \\
\leq&\left(\frac{K^2}{\sqrt{8\pi}}T^{2H-1/2}\beta(2H,1/2)\right)^{2q}\frac{(At)^{n-1}}{(n-1)!},
\end{aligned}
\tag{7.28}
$$

which implies

$$
R_n(t) \leq \frac{K^2}{\sqrt{8\pi}}T^{2H-1/2}\beta(2H,1/2)\frac{(At)^{(n-1)/2q}}{((n-1)!)^{2q}}, \tag{7.29}
$$

and therefore

$$
\sum_{n=0}^{\infty} R_n(t) < \infty. \tag{7.30}
$$

Then we have a solution for (7.5) given by

$$u(x,t) = \sum_{n=1}^{\infty} d_n(x,t), \tag{7.31}$$

which converges as $n \to \infty$ in L^2 because

$$\left|\left|\sum_{n=1}^{\infty} d_n(x,t)\right|\right|_2 \le \sum_{n=1}^{\infty} ||d_n(x,t)||_2 = \sum_{n=1}^{\infty} R_n < \infty. \tag{7.32}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \int_0^t \int_{-\infty}^{\infty} & f(u_n(y,s))\Pi(t-s,x-y)s^{H-1/2}W(\,dy\,ds) \\ &= \int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s,x-y)s^{H-1/2}W(\,dy\,ds) \end{aligned} \tag{7.33}$$

in L^2 so $u(x,t)$ is a solution to (7.5). This also proves

$$\sup_{0\le x\le L}\sup_{0\le t\le T} \mathbb{E}\left(|u(x,t)|^2\right) = \left(\sum_{n=1}^{\infty} R_n(t)\right)^2 < \infty. \tag{7.34}$$

Uniqueness: Suppose u and v both solve (7.5) with the same initial condition and f satisfies the integrability condition (7.2). Let $d(x,t) = u(x,t) - v(x,t)$. Then using Burkholder's inequality [4], we have

$$\begin{aligned} &\mathbb{E}\left(|d(x,t)|^2\right) \\ =&\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} [f(u(y,s)) - f(v(y,s))]\,\Pi(t-s;x-y)s^{H-1/2}W(\,dy\,ds)\right)^2\right] \\ \le& \int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left[(f(u(y,s)) - f(v(y,s)))^2\right]\Pi^2(t-s;x-y)s^{2H-1}\,dy\,ds \\ \le& K^2 \int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left(|d(y,s)|^2\right)\Pi^2(t-s;x-y)s^{2H-1}\,dy\,ds. \end{aligned} \tag{7.35}$$

Let

$$R(t) := \sup_{0\le x\le L}\sup_{0\le s\le t} \mathbb{E}\left(|d(x,s)|^2\right). \tag{7.36}$$

Then using the definition of Π, we have

$$\begin{aligned}R(t) \leq & K^2 \int_0^t \int_{-\infty}^{\infty} R(s)\Pi^2(t-s;x-y)s^{2H-1}\,dy\,ds \\ =& K^2 \int_0^t R(s)s^{2H-1} \int_{-\infty}^{\infty} \Pi^2(t-s;x-y)\,dy\,ds \\ =& CK^2 \int_0^t R(s)s^{2H-1}(t-s)^{-1/2}\,ds.\end{aligned} \tag{7.37}$$

By Lemma 7.2.3, we have

$$R(t) \leq A \left(\int_0^t R^q(s)\,ds \right)^{1/q}, \tag{7.38}$$

which implies

$$R^q(t) \leq A^q \int_0^t R^q(s)\,ds, \tag{7.39}$$

for some constant A and $q > 2$, uniformly for all $t \in [0, T]$. Therefore by Gronwall's lemma 7.2.4, $R^q(t) \equiv 0$ and therefore $R(t) \equiv 0$. This concludes the uniqueness proof. □

In the case of fractional noise, the analogous functions $R_n(t)$ as in the above proof are finite for $H > 3/8$ but for $3/8 < H < 1/2$, summability remains unproven to date [12].

7.3 Continuity

Next, to prove that this unique solution $u(x,t)$ is continuous, we need some lemmas.

Lemma 7.3.1. *For $-1/2 < \alpha < 0$ and $x \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{x^{\alpha+n+1}}{n!(\alpha+n+1)} \leq 4e^x. \tag{7.40}$$

Proof. We prove the lemma in two cases:

1. If $x < 1$ then for all $n \geq 0$, $x^n \leq 1$. Also, $\alpha + 1 > 0$ so $\alpha + n + 1 > n$. Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{x^{\alpha+n+1}}{n!(\alpha+n+1)} \\
\leq& x^{\alpha+1} \sum_{n=0}^{\infty} \frac{1}{n!(\alpha+n+1)} \\
=& x^{\alpha+1} \left(\frac{1}{\alpha+1} + \sum_{n=1}^{\infty} \frac{1}{n!(\alpha+n+1)} \right) \\
\leq& x^{\alpha+1} \left(\frac{1}{\alpha+1} + \sum_{n=1}^{\infty} \frac{1}{n!n} \right) \\
\leq& x^{\alpha+1} \left(\frac{1}{\alpha+1} + \sum_{n=1}^{\infty} \frac{1}{n!} \right) \\
=& x^{\alpha+1} \left(\frac{1}{\alpha+1} + e - 1 \right) \\
\leq& x^{\alpha+1} \left(\frac{1}{\alpha+1} + 2 \right). \\
\leq& \left(\frac{1}{\alpha+1} + 2 \right) e^x \\
\leq& 4e^x.
\end{aligned} \tag{7.41}$$

2. Now suppose $x \geq 1$. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha+n+1}}{n!(\alpha+n+1)}$. We will show that $f(1) < e^1$ and for all $x > 1$, $f'(x) \leq e^x$, which suffices to show $f(x) \leq e^x \leq 4e^x$ for $x > 1$:

$$f(1) = \sum_{n=0}^{\infty} \frac{1}{n!(\alpha+n+1)} < e^1. \tag{7.42}$$

Next,

$$
\begin{aligned}
&f'(x)\\
=&\sum_{n=0}^{\infty}\frac{x^{n+\alpha}}{n!}\\
=&x^{\alpha}\sum_{n=0}^{\infty}\frac{x^{n}}{n!}\\
=&x^{\alpha}e^{x}\\
\leq&e^{x}\text{ (since }\alpha<0\text{ and }x\geq 1).
\end{aligned}
\tag{7.43}
$$

□

Lemma 7.3.2. *For $t \in [0,T]$ and $H \in (1/4, 1)$,*

$$
\int_0^t s^{2H-1}e^{-2(t-s)\xi^2}\,ds \leq \begin{cases} \frac{c_T}{1+\xi^2} & \text{if } H \geq 1/2 \\ \frac{2^{3-2H}}{\xi^{4H}} & \text{if } 1/4 < H < 1/2. \end{cases}
\tag{7.44}
$$

Proof. We prove this Lemma in two cases.

1. In the case $H \geq 1/2$, $2H-1>0$ so we have

$$
\begin{aligned}
&\int_0^t s^{2H-1}e^{-2(t-s)\xi^2}\,ds\\
&\leq t^{2H-1}\int_0^t e^{-2(t-s)\xi^2}\,ds\\
&\leq T^{2H-1}\int_0^t e^{-2(t-s)\xi^2}\,ds\\
&= T^{2H-1}e^{-2t\xi^2}\int_0^t e^{2s\xi^2}\,ds\\
&= T^{2H-1}e^{-2t\xi^2}\left(\frac{e^{2t\xi^2}-1}{2\xi^2}\right)\\
&= T^{2H-1}\left(\frac{1-e^{-2t\xi^2}}{2\xi^2}\right)\\
&\leq \frac{C_T}{1+\xi^2},
\end{aligned}
\tag{7.45}
$$

uniformly for all $0 \leq t \leq T$, where the last inequality is shown in two cases:

(a) When $|\xi| \leq 1$ and thus $\xi^2 \leq 1$, consider the Taylor expansion of $e^{-2t\xi^2}$:

$$\begin{aligned} e^{-2t\xi^2} &= 1 + -2t\xi^2 + \frac{(2t\xi^2)^2}{2} - \frac{(2t\xi^2)^3}{3!} + \ldots \\ \Rightarrow 1 - e^{-2t\xi^2} &= 2t\xi^2 - \frac{(2t\xi^2)^2}{2} + \frac{(2t\xi^2)^3}{3!} - \ldots \\ \Rightarrow \frac{1 - e^{-2t\xi^2}}{2\xi^2} &\leq t. \end{aligned} \tag{7.46}$$

Then since $|\xi| \leq 1$, we have $1 + \xi^2 \leq 2$ and so

$$\frac{1 - e^{-2t\xi^2}}{2\xi^2} \leq t = t\left(\frac{1+\xi^2}{1+\xi^2}\right) \leq \frac{2t}{1+\xi^2} \leq \frac{2T}{1+\xi^2}. \tag{7.47}$$

(b) When $|\xi| > 1$, $1 - e^{-2t\xi^2} < 1$ so we have

$$\frac{1 - e^{-2t\xi^2}}{2\xi^2} < \frac{1}{2\xi^2} = \frac{1}{2\xi^2}\left(\frac{1+\xi^2}{1+\xi^2}\right) = \frac{1}{1+\xi^2}\left(\frac{1}{2\xi^2} + \frac{1}{2}\right) \leq \frac{1}{1+\xi^2}. \tag{7.48}$$

2. In the case $1/4 < H < 1/2$, we use the Taylor expansion of $e^{2s\xi^2}$, Fubini's

theorem, and Lemma 7.3.1:

$$\begin{aligned}
&\int_0^t s^{2H-1} e^{-2(t-s)\xi^2}\, ds \\
&= e^{-2t\xi^2} \int_0^t s^{2H-1} e^{2s\xi^2}\, ds \\
&= e^{-2t\xi^2} \int_0^t s^{2H-1} \sum_{n=0}^{\infty} \frac{(2s\xi^2)^n}{n!}\, ds \\
&= e^{-2t\xi^2} \sum_{n=0}^{\infty} \frac{(2\xi^2)^n}{n!} \int_0^t s^{2H-1+n}\, ds \\
&= e^{-2t\xi^2} \sum_{n=0}^{\infty} \frac{(2\xi^2)^n}{n!(2H+n)} t^{2H+n} \\
&= t^{2H} e^{-2t\xi^2} \sum_{n=0}^{\infty} \frac{(2t\xi^2)^n}{n!(2H+n)} \\
&= \frac{t^{2H} e^{-2t\xi^2}}{(2t\xi^2)^{2H}} \sum_{n=0}^{\infty} \frac{(2t\xi^2)^{n+2H}}{n!(2H+n)} \\
&= \frac{e^{-2t\xi^2}}{(2\xi^2)^{2H}} \sum_{n=0}^{\infty} \frac{(2t\xi^2)^{n+2H}}{n!(2H+n)} \\
&\le \frac{4e^{-2t\xi^2}}{(2\xi^2)^{2H}} e^{2t\xi^2} \\
&= \frac{2^{2-2H}}{\xi^{4H}}, \text{ where } 1 < 4H < 2.
\end{aligned} \tag{7.49}$$

□

Lemma 7.3.3. *For $x \ge 0$,*

$$(1 - e^{-x})^2 \le \min(x^2, 1). \tag{7.50}$$

Proof. To see $(1-e^{-x})^2 \le 1$, we have

$$\begin{aligned} & x \ge 0 \\ \Rightarrow & 0 \le e^{-x} \le 1 \\ \Rightarrow & -1 \le -e^{-x} \le 0 \\ \Rightarrow & 0 \le 1-e^{-x} \le 1 \\ \Rightarrow & (1-e^{-x})^2 \le 1. \end{aligned} \tag{7.51}$$

To see $(1-e^{-x})^2 \le x^2$, we set $f(x) = x + e^{-x} - 1$. Then $f(0) = 0$ and $f'(x) = 1 - e^{-x} \ge 0$ as above and therefore $f(x) = x + e^{-x} - 1 \ge 0$ for $x \ge 0$. This proves $0 \le 1 - e^{-x} \le x$ and thus $(1-e^{-x})^2 \le x^2$, as required. □

Lemma 7.3.4. *For $1/4 < H < 1/2$ and $0 \le t \le t'$,*

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \int_0^t s^{2H-1} e^{-2(t-s)\xi^2}\, ds\, d\xi \\ &\qquad \le 2^{4-2H} \left(\frac{1}{5-4H} + \frac{1}{4H-1}\right)(t'-t)^{2H-1/2}. \end{aligned} \tag{7.52}$$

Proof. By Lemma 7.3.2 and Lemma 7.3.3,

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \int_0^t s^{2H-1} e^{-2(t-s)\xi^2}\, ds\, d\xi \\ \le& 2^{3-2H} \int_{-\infty}^{\infty} \frac{\left[1 - e^{-(t'-t)\xi^2}\right]^2}{\xi^{4H}}\, d\xi \\ =& 2^{4-2H} \int_0^{\infty} \frac{\left[1 - e^{-(t'-t)\xi^2}\right]^2}{\xi^{4H}}\, d\xi \\ \le& 2^{4-2H} \int_0^{\infty} \frac{\min((t'-t)^2\xi^4, 1)}{\xi^{4H}}\, d\xi \\ =& 2^{4-2H} \left((t'-t)^2 \int_0^{(t'-t)^{-1/2}} \xi^{4-4H}\, d\xi + \int_{(t'-t)^{-1/2}}^{\infty} \xi^{-4H}\, d\xi\right) \text{ (note } H > 1/4\text{)} \\ =& 2^{4-2H} \left(\frac{1}{5-4H} + \frac{1}{4H-1}\right)(t'-t)^{2H-1/2}. \end{aligned} \tag{7.53}$$

□

Lemma 7.3.5. *For* $1/2 \le H < 1$ *and* $0 \le t \le t' \le T$,

$$\int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \int_0^t s^{2H-1} e^{-2(t-s)\xi^2}\, ds\, d\xi \le \frac{4T^{2H-1}}{3}(t'-t)^{1/2}. \tag{7.54}$$

Proof. If $H \ge 1/2$ then for all $0 \le s \le t$, we have $s^{2H-1} \le t^{2H-1} \le T^{2H-1}$. So by Lemma 7.3.3,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \int_0^t s^{2H-1} e^{-2(t-s)\xi^2}\, ds\, d\xi \\
\le& T^{2H-1} \int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 e^{-2t\xi^2} \int_0^t e^{2s\xi^2}\, ds\, d\xi \\
=& T^{2H-1} \int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 e^{-2t\xi^2} \frac{\left(e^{2t\xi^2} - 1\right)}{2\xi^2}\, d\xi \\
=& T^{2H-1} \int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \frac{\left(1 - e^{-2t\xi^2}\right)}{2\xi^2}\, d\xi \\
\le& T^{2H-1} \int_{-\infty}^{\infty} \frac{\left[1 - e^{-(t'-t)\xi^2}\right]^2}{2\xi^2}\, d\xi \\
=& 2T^{2H-1} \int_0^{\infty} \frac{\left[1 - e^{-(t'-t)\xi^2}\right]^2}{2\xi^2}\, d\xi \qquad (7.55)\\
\le& 2T^{2H-1} \int_0^{\infty} \frac{\min((t'-t)^2\xi^4, 1)}{2\xi^2}\, d\xi \\
=& 2T^{2H-1} \left(\int_0^{(t'-t)^{-1/2}} \frac{(t'-t)^2\xi^4}{2\xi^2}\, d\xi + \int_{(t'-t)^{-1/2}}^{\infty} \frac{1}{2\xi^2}\, d\xi\right) \\
=& 2T^{2H-1} \left(\frac{(t'-t)^2}{2} \int_0^{(t'-t)^{-1/2}} \xi^2\, d\xi + \frac{1}{2} \int_{(t'-t)^{-1/2}}^{\infty} \xi^{-2}\, d\xi\right) \\
=& T^{2H-1} \left((t'-t)^2 \frac{1}{3}((t'-t)^{-1/2})^3 + ((t'-t)^{-1/2})^{-1}\right) \\
=& T^{2H-1} \left(\frac{1}{3}(t'-t)^{1/2} + (t'-t)^{1/2}\right) \\
=& \frac{4T^{\alpha}}{3}(t'-t)^{1/2}.
\end{aligned}$$

□

Lemma 7.3.6. *For $0 \le t \le t'$,*

$$\int_{-\infty}^{\infty} (\Pi(t'-s;x-y) - \Pi(t-s;x-y))^2\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2(t-s)\xi^2}\left(e^{-(t'-t)\xi^2}-1\right)^2\,d\xi. \tag{7.56}$$

Proof. The Fourier transform of $\Pi(t,x) = \frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}$ in x is

$$\begin{aligned}
\hat{\Pi}(t,\xi) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\Pi(t,x)\,dx \\
=& \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-ix\xi}e^{\frac{-x^2}{4t}}\,dx \\
=& \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{4t}-ix\xi}\,dx \\
=& \frac{1}{\sqrt{2\pi}}e^{-t\xi^2}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x+2ti\xi)^2}\,dx \\
=& \frac{1}{\sqrt{2\pi}}e^{-t\xi^2}.
\end{aligned} \tag{7.57}$$

Then by Plancherel's theorem and linearity of the Fourier transform,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\Pi(t'-s;x-y) - \Pi(t-s;x-y)\right)^2\,dy \\
=&\,||\Pi(t'-s;x-y) - \Pi(t-s;x-y)||_2^2 \\
=&\left|\left|\hat{\Pi}(t'-s;\xi) - \hat{\Pi}(t-s;\xi)\right|\right|_2^2 \\
=&\left|\left|\frac{1}{\sqrt{2\pi}}e^{-(t'-s)\xi^2} - \frac{1}{\sqrt{2\pi}}e^{-(t-s)\xi^2}\right|\right|_2^2 \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{-(t'-s)\xi^2} - e^{-(t-s)\xi^2}\right)^2\,d\xi \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2(t-s)\xi^2}\left(e^{-(t'-t)\xi^2}-1\right)^2\,d\xi.
\end{aligned} \tag{7.58}$$

□

We aim to show that there is a continuous solution for (7.1), i.e. a continuous modification of $u(x,t)$ where

$$u(x,t) = \int_{-\infty}^{\infty} u_0(y)\Pi(t, x-y)\,dy + \int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s, x-y)s^{H-1/2}W(\,dy\,ds), \tag{7.59}$$

as in (7.5). For simplicity, we will assume the initial condition u_0 is constant. Then the first term of (7.5) is

$$U_0(x,t) = C\int_{-\infty}^{\infty} \Pi(t, x-y)\,dy = C \tag{7.60}$$

since $\Pi(t,a)$ is a Gaussian density. Then the derivative of $U_0(x,t)$ with respect to both x and t is 0 and therefore bounded by, say 1. Then by the Mean Value Theorem,

$$|U_0(x,t) - U_0(x,t')|^k \leq |t-t'|^k \tag{7.61}$$

and

$$|U_0(x,t) - U_0(x',t)|^k \leq |x-x'|^k. \tag{7.62}$$

This is sufficient to show that the first term is continuous with Hölder continuity 1, in both space and time. Next we prove that there exists a continuous modification of the second term of (7.5), $U(x,t)$, where

$$U(x,t) := \int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s; x-y)s^{H-1/2}W(\,dy\,ds). \tag{7.63}$$

Theorem 7.3.7. *There exists a constant $C_k > 0$ such that uniformly for all (x,t), $(x,t') \in (-\infty,\infty)\times[0,T]$,*

$$\mathbb{E}\left(|U(x,t) - U(x,t')|^k\right) \leq C_k|t-t'|^{\gamma}, \tag{7.64}$$

where U is defined as in (7.63), and

$$\gamma = \begin{cases} (H-1/4)k & \text{if } 1/4 < H < 1/2 \text{ and} \\ k/4 & \text{if } 1/2 \leq H < 1. \end{cases} \tag{7.65}$$

Proof. Let $0 \le t \le t'$. Then

$$\begin{aligned} &U(x,t') - U(x,t) \\ =&\int_0^t \int_{-\infty}^{\infty} f(u(y,s)) \left[\Pi(t'-s; x-y) - \Pi(t-s; x-y)\right] s^{H-1/2} W(\,dy\,ds) \\ &+ \int_t^{t'} \int_{-\infty}^{\infty} f(u(y,s)) \Pi(t'-s; x-y) s^{H-1/2} W(\,dy\,ds). \end{aligned} \tag{7.66}$$

Define

$$\Lambda(s,t,t';x,y) := \left[\Pi(t'-s;x-y) - \Pi(t-s;x-y)\right]^2. \tag{7.67}$$

Then by Burkholder's inequality [4], the inequality $|a+b|^k \le 2^k|a|^k + 2^k|b|^k$, and (7.2),

$$\begin{aligned} &\mathbb{E}\left(|U(x,t) - U(x,t')|^k\right) \\ \le& 2^k c_k \mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} f^2(u(y,s)) \Lambda(s,t,t';x,y) s^{2H-1}\, dy\, ds\right)^{k/2}\right] \\ &+ 2^k c_k \mathbb{E}\left[\left(\int_t^{t'} \int_{-\infty}^{\infty} f^2(u(y,s)) \Pi^2(t'-s;x-y) s^{2H-1}\, dy\, ds\right)^{k/2}\right] \\ \le& 2^k c_k \left(\int_0^t \int_{-\infty}^{\infty} K^2 \Lambda(s,t,t';x,y) s^{2H-1}\, dy\, ds\right)^{k/2} \\ &+ 2^k c_k \left(\int_t^{t'} \int_{-\infty}^{\infty} K^2 \Pi^2(t'-s;x-y) s^{2H-1}\, dy\, ds\right)^{k/2} \\ =& (2K)^k c_k \left(\int_0^t \int_{-\infty}^{\infty} \Lambda(s,t,t';x,y) s^{2H-1}\, dy\, ds\right)^{k/2} \\ &+ (2K)^k c_k \left(\int_t^{t'} \int_{-\infty}^{\infty} \Pi^2(t'-s;x-y) s^{2H-1}\, dy\, ds\right)^{k/2}. \end{aligned} \tag{7.68}$$

To bound the first term, we use Lemma 7.3.6, Fubini's theorem, and finally Lemmas

7.3.4 and 7.3.5:

$$
\begin{aligned}
&(2K)^k c_k \left(\int_0^t \int_{-\infty}^{\infty} \Lambda(s,t,t';x,y) s^{2H-1}\, dy\, ds \right)^{k/2} \\
=&\frac{(2K)^k c_k}{2\pi} \left(\int_0^t s^{2H-1} \int_{-\infty}^{\infty} e^{-2(t-s)\xi^2} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \, d\xi\, ds \right)^{k/2} \\
=&\frac{(2K)^k c_k}{2\pi} \left(\int_{-\infty}^{\infty} \left[1 - e^{-(t'-t)\xi^2}\right]^2 \left(\int_0^t s^{2H-1} e^{-2(t-s)\xi^2}\, ds \right) d\xi \right)^{k/2} \\
\leq& \begin{cases} \frac{(2K)^k c_k}{2\pi} \left(2^{3-\alpha}\left(\frac{1}{3-2\alpha} + \frac{1}{2\alpha+1}\right)(t'-t)^{\alpha+1/2}\right)^{k/2} & \text{if } 1/4 < H < 1/2 \\ \frac{(2K)^k c_k}{2\pi} \left(\frac{4T^\alpha}{3}(t'-t)^{1/2}\right)^{k/2} & \text{if } 1/2 \leq H < 1 \end{cases} \qquad (7.69) \\
=& \begin{cases} \frac{(2K)^k c_k}{2\pi} \left(2^{4-2H}\left(\frac{1}{5-4H} + \frac{1}{4H-1}\right)(t'-t)^{2H-1/2}\right)^{k/2} & \text{if } 1/4 < H < 1/2 \\ \frac{(2K)^k c_k}{2\pi} \left(\frac{4T^\alpha}{3}(t'-t)^{1/2}\right)^{k/2} & \text{if } 1/2 \leq H < 1 \end{cases} \\
=& \begin{cases} D_k (t'-t)^{(H-1/4)k} & \text{if } 1/4 < H < 1/2 \\ D_k' (t'-t)^{k/4} & \text{if } 1/2 \leq H < 1 \end{cases} .
\end{aligned}
$$

A bound for the second term in (7.68) uses the definition of Π and the Gaussian probability density function:

$$
\begin{aligned}
&(2K)^k c_k \left(\int_t^{t'} \int_{-\infty}^{\infty} \Pi^2(t'-s;x-y) s^{2H-1}\, dy\, ds \right)^{k/2} \\
=&(2K)^k c_k \left(\int_t^{t'} \int_{-\infty}^{\infty} \left(\frac{1}{(4\pi(t'-s))^{1/2}} e^{\frac{-(x-y)^2}{4(t'-s)}} \right)^2 s^{2H-1}\, dy\, ds \right)^{k/2} \\
=&(2K)^k c_k \left(\int_t^{t'} s^{2H-1} \int_{-\infty}^{\infty} \left(\frac{1}{4\pi(t'-s)} e^{\frac{-(x-y)^2}{2(t'-s)}} \right) dy\, ds \right)^{k/2} \\
=&(2K)^k c_k \left(\int_t^{t'} s^{2H-1} \frac{2}{4(2\pi(t'-s))^{1/2}} \int_{-\infty}^{\infty} \left(\frac{1}{(2\pi(t'-s))^{1/2}} e^{\frac{-(x-y)^2}{2(t'-s)}} \right) dy\, ds \right)^{k/2} \\
=&\frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(\int_t^{t'} s^{2H-1} (t'-s)^{-1/2}\, ds \right)^{k/2} .
\end{aligned} \qquad (7.70)
$$

If $H \geq 1/2$, then $2H - 1 \geq 0$ so

$$\begin{aligned} &\frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(\int_t^{t'} s^{2H-1}(t'-s)^{-1/2}\, ds \right)^{k/2} \\ \leq & T^{2H-1} \frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(\int_t^{t'} (t'-s)^{-1/2}\, ds \right)^{k/2} \\ = & T^{2H-1} \frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(2(t'-t)^{1/2} \right)^{k/2} \\ = & C_k (t'-t)^{k/4}. \end{aligned} \tag{7.71}$$

If $1/4 < H < 1/2$, then the function $f(x) = x^{2H-1}$ is decreasing so by Lemma 7.2.2, we have

$$\begin{aligned} &\frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(\int_t^{t'} s^{2H-1}(t'-s)^{-1/2}\, ds \right)^{k/2} \\ = & \frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(\int_0^{t'-t} (u+t)^{2H-1}(t'-t-u)^{-1/2}\, du \right)^{k/2} \quad \text{(where } u = s-t) \\ \leq & \frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left(\int_0^{t'-t} u^{2H-1}(t'-t-u)^{-1/2}\, du \right)^{k/2} \\ = & \frac{(2K)^k c_k}{2(2\pi)^{1/2}} \left((t'-t)^{2H-1/2} \beta(2H, 1/2) \right)^{k/2} \\ = & C'_k (t'-t)^{(H-1/4)k}. \end{aligned} \tag{7.72}$$

□

Before we prove continuity in x, a few more lemmas:

Lemma 7.3.8. *For $x, x' \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} |\Pi(t-s; x-y) - \Pi(t-s; x'-y)|^2 \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2(t-s)\xi^2} \left| 1 - e^{i\xi(x'-x)} \right|^2 d\xi. \tag{7.73}$$

Proof. As shown in the proof of Lemma 7.3.6, the Fourier transform of $\Pi(t, x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$ in x is

$$\hat{\Pi}(t, \xi) = \frac{1}{\sqrt{2\pi}} e^{-t\xi^2}. \tag{7.74}$$

Let $u = x - y$. Then by Plancherel's theorem and properties of the Fourier transform,

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\Pi(t-s;x-y) - \Pi(t-s;x'-y)|^2 \, dy \\
=&\int_{-\infty}^{\infty} |\Pi(t-s;u) - \Pi(t-s;x'-x+u)|^2 \, du \\
=&\,||\Pi(t-s;u) - \Pi(t-s;x'-x+u)||_2^2 \\
=&\left|\left|\hat{\Pi}(t-s,\xi) - e^{-i\xi(x'-x)}\hat{\Pi}(t-s,\xi)\right|\right|_2^2 \\
=&\left|\left|\hat{\Pi}(t-s,\xi)\left(1 - e^{-i\xi(x'-x)}\right)\right|\right|_2^2 \\
=&\left|\left|\frac{1}{\sqrt{2\pi}}e^{-(t-s)\xi^2}\left(1 - e^{-i\xi(x'-x)}\right)\right|\right|_2^2 \\
=&\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{2\pi}}e^{-(t-s)\xi^2}\left(1 - e^{-i\xi(x'-x)}\right)\right|^2 dy \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2(t-s)\xi^2}\left|1 - e^{-i\xi(x'-x)}\right|^2 d\xi.
\end{aligned} \tag{7.75}$$

□

Lemma 7.3.9. *For $x, x' \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty}\left|1 - e^{i\xi(x-x')}\right|^2 \frac{1}{\xi^{4H}}\, d\xi = 4\int_0^{\infty} |1 - \cos(\xi(x'-x))| \frac{1}{\xi^{4H}}\, d\xi. \tag{7.76}$$

Proof. By elementary arithmetic, we have

$$\begin{aligned}
\left|1 - e^{i\xi(x-x')}\right|^2 =& \,|1 - \cos(\xi(x-x')) - i\sin(\xi(x-x'))|^2 \\
=&\,(1 - \cos(\xi(x-x')))^2 + \sin^2(\xi(x-x')) \\
=&1 - 2\cos(\xi(x-x')) + \cos^2(\xi(x-x')) + \sin^2(\xi(x-x')) \\
=&2 - 2\cos(\xi(x-x'))
\end{aligned} \tag{7.77}$$

and therefore since $(1 - \cos(\xi(x-x')))\frac{1}{\xi^{4H}}$ is even,

$$\begin{aligned}
\int_{-\infty}^{\infty}\left|1 - e^{i\xi(x-x')}\right|^2 \frac{1}{\xi^{4H}}\, d\xi =& 2\int_{-\infty}^{\infty} (1 - \cos(\xi(x-x')))\frac{1}{\xi^{4H}}\, d\xi \\
=&4\int_0^{\infty} (1 - \cos(\xi(x-x')))\frac{1}{\xi^{4H}}\, d\xi.
\end{aligned} \tag{7.78}$$

□

Lemma 7.3.10. *For all $\theta \in \mathbb{R}$,*

$$2 - 2\cos(\theta) \leq \min(4, \theta^2) \tag{7.79}$$

Proof. Since $-1 \leq \cos(\theta)$ for all $\theta \in \mathbb{R}$, $2 - 2\cos(\theta) \leq 4$. To see $1 - \cos(\theta) \leq \theta^2$, let $f(\theta) = 2\cos(\theta) + \theta^2 - 2$. Then $f(0) = 0$ and for $x \geq 0$, $f'(\theta) = -2\sin(\theta) + 2\theta \geq 2\theta + 2\theta = 0$. Thus for $\theta \geq 0$, $f(\theta) \geq 0$. Since f is even, we have $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$, as required. $\square$

Lemma 7.3.11. *For $x, x' \in \mathbb{R}$,*

$$\int_0^t \int_{-\infty}^{\infty} |\Pi(t-s, x-y) - \Pi(t-s, x'-y)|^2 \, dy \, ds \leq \frac{5|x'-x|}{2\pi}. \tag{7.80}$$

Proof. By Lemma 7.3.8, Fubini's theorem, and Lemma 7.3.10,

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} |\Pi(t-s, x-y) - \Pi(t-s, x'-y)|^2 \, dy \, ds \\
=&\int_0^t \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2(t-s)\xi^2} \left|1 - e^{i\xi(x'-x)}\right|^2 \, d\xi \, ds \\
=&\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - e^{i\xi(x'-x)}\right|^2 \int_0^t e^{-2(t-s)\xi^2} \, ds \, d\xi \\
=&\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - e^{i\xi(x'-x)}\right|^2 e^{-2t\xi^2} \int_0^t e^{2s\xi^2} \, ds \, d\xi \\
=&\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - e^{i\xi(x'-x)}\right|^2 e^{-2t\xi^2} \frac{1}{2\xi^2} \left(e^{2t\xi^2} - 1\right) \, d\xi \\
=&\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - e^{i\xi(x'-x)}\right|^2 \frac{1 - e^{-2t\xi^2}}{2\xi^2} \, d\xi \\
=&\frac{1}{2\pi} \int_{-\infty}^{\infty} (2 - 2\cos(\xi(x'-x))) \frac{1 - e^{-2t\xi^2}}{2\xi^2} \, d\xi \qquad (7.81) \\
=&\frac{1}{\pi} \int_0^{\infty} (2 - 2\cos(\xi(x'-x))) \frac{1 - e^{-2t\xi^2}}{2\xi^2} \, d\xi \\
\leq&\frac{1}{\pi} \int_0^{\infty} \min(4, \xi^2(x'-x)^2) \frac{1 - e^{-2t\xi^2}}{2\xi^2} \, d\xi \\
\leq&\frac{1}{\pi} \int_0^{\infty} \frac{\min(4, \xi^2(x'-x)^2)}{2\xi^2} \, d\xi \\
=&\frac{1}{2\pi} \left((x'-x)^2 \int_0^{|x'-x|^{-1}} 1 \, d\xi + 4 \int_{|x'-x|^{-1}}^{\infty} \xi^{-2} \, d\xi \right) \\
=&\frac{1}{2\pi} \left((x'-x)^2 |x'-x|^{-1} + 4(|x'-x|^{-1})^{-1} \right) \\
=&\frac{5|x'-x|}{2\pi}.
\end{aligned}$$

□

Theorem 7.3.12. *There exists a constant $D_k > 0$ such that uniformly for all $(x,t), (x',t) \in (-\infty, \infty) \times [0,T]$,*

$$\mathbb{E}\left(|U(x,t) - U(x',t)|^k\right) \leq D_k |x - x'|^{\gamma}, \qquad (7.82)$$

where U is defined as in (7.63), *and*

$$\gamma = \begin{cases} (2H-1/2)k & \text{if } 1/4 < H < 1/2 \text{ and} \\ k/2 & \text{if } 1/2 \le H < 1. \end{cases} \tag{7.83}$$

Proof. Let $x, x' \in [0, L]$. Then by Burkholder's inequality [4] and condition (7.2) of f,

$$\begin{aligned}
&\mathbb{E}\left[|U(x,t) - U(x',t)|^k\right] \\
=&\mathbb{E}\left[\left|\int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s; x-y)s^{H-1/2}W(\,dy\,ds)\right.\right. \\
&\left.\left. - \int_0^t \int_{-\infty}^{\infty} f(u(y,s))\Pi(t-s; x'-y)s^{H-1/2}W(\,dy\,ds)\right|^k\right] \\
=&\mathbb{E}\left[\left|\int_0^t \int_{-\infty}^{\infty} f(u(y,s))(\Pi(t-s; x-y) - \Pi(t-s; x'-y))s^{H-1/2}W(\,dy\,ds)\right|^k\right] \\
\le&\mathbb{E}\left[c_k\left(\int_0^t \int_{-\infty}^{\infty} f^2(u(y,s))(\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 s^{2H-1}\,dy\,ds\right)^{k/2}\right] \\
=&c_k\left(\int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left[f^2(u(y,s))\right](\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 s^{2H-1}\,dy\,ds\right)^{k/2} \\
\le&c_k K^k\left(\int_0^t \int_{-\infty}^{\infty}(\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 s^{2H-1}\,dy\,ds\right)^{k/2} \\
=&c_k K^k\left(\int_0^t s^{2H-1}\int_{-\infty}^{\infty}(\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2\,dy\,ds\right)^{k/2}.
\end{aligned} \tag{7.84}$$

If $H \geq 1/2$ then we have, by Lemma 7.3.11,

$$\begin{aligned}
&c_k K^k \left(\int_0^t s^{2H-1} \int_{-\infty}^{\infty} (\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 \, dy \, ds \right)^{k/2} \\
\leq &c_k K^k \left(T^{2H-1} \int_0^t \int_{-\infty}^{\infty} (\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 \, dy \, ds \right)^{k/2} \\
= &c_k K^k T^{(2H-1)k/2} \left(\int_0^t \int_{-\infty}^{\infty} (\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 \, dy \, ds \right)^{k/2} \qquad (7.85) \\
\leq &c_k K^k T^{(2H-1)k/2} \left(\frac{5|x-x'|}{2\pi} \right)^{k/2} \text{ by Lemma 7.3.11} \\
= &c_k K^k T^{(2H-1)k/2} \left(\frac{5}{2\pi} \right)^{k/2} |x-x'|^{k/2} \\
= &J_k |x-x'|^{k/2}.
\end{aligned}$$

If $1/4 < H < 1/2$, then by Lemmas 7.3.8, 7.3.2, 7.3.9, and 7.3.10, we have

$$
\begin{aligned}
&c_k K^k \left(\int_0^t s^{2H-1} \int_{-\infty}^{\infty} (\Pi(t-s; x-y) - \Pi(t-s; x'-y))^2 \, dy \, ds \right)^{k/2} \\
=&c_k K^k \left(\int_0^t s^{2H-1} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2(t-s)\xi^2} \left| 1 - e^{i\xi(x-x')} \right|^2 d\xi \right) ds \right)^{k/2} \\
=&\frac{c_k K^k}{(2\pi)^{k/2}} \left(\int_{-\infty}^{\infty} \left| 1 - e^{i\xi(x-x')} \right|^2 \int_0^t s^{2H-1} e^{-2(t-s)\xi^2} \, ds \, d\xi \right)^{k/2} \\
\leq&\frac{c_k K^k}{(2\pi)^{k/2}} \left(\int_{-\infty}^{\infty} \left| 1 - e^{i\xi(x-x')} \right|^2 \frac{2^{3-2H}}{\xi^{4H}} \, d\xi \right)^{k/2} \\
=&\frac{c_k K^k 2^{(3-2H)k/2}}{(2\pi)^{k/2}} \left(\int_{-\infty}^{\infty} \left| 1 - e^{i\xi(x-x')} \right|^2 \frac{1}{\xi^{4H}} \, d\xi \right)^{k/2} \\
=&\frac{c_k K^k 2^{(2-2H)k/2}}{\pi^{k/2}} \left(\int_0^{\infty} \left| 1 - \cos(\xi(x'-x)) \right| \frac{1}{\xi^{4H}} \, d\xi \right)^{k/2} \\
\leq&\frac{c_k K^k 2^{(1-H)k}}{(2\pi)^{k/2}} \left(\int_0^{\infty} \min(4, \xi^2 (x'-x)^2) \frac{1}{\xi^{4H}} \, d\xi \right)^{k/2} \\
=&\frac{c_k K^k 2^{(1-H)k}}{(2\pi)^{k/2}} \left((x'-x)^2 \int_0^{|x-x'|^{-1}} \xi^{2-4H} \, d\xi + 4 \int_{|x-x'|^{-1}}^{\infty} \xi^{-4H} \, d\xi \right)^{k/2} \\
=&\frac{c_k K^k 2^{(1-H)k}}{(2\pi)^{k/2}} \left(\frac{1}{3-4H} (x'-x)^2 (|x-x'|^{-1})^{3-4H} + \frac{4}{4H-1} (|x-x'|^{-1})^{1-4H} \right)^{k/2} \\
=&\frac{c_k K^k 2^{(1-H)k}}{(2\pi)^{k/2}} \left(\frac{4}{3-4H} |x-x'|^{4H-1} + \frac{1}{4H-1} |x-x'|^{4H-1} \right)^{k/2} \\
=&\frac{c_k K^k 2^{(1-H)k}}{(2\pi)^{k/2}} \left(\left(\frac{1}{3-4H} + \frac{4}{4H-1} \right) |x-x'|^{4H-1} \right)^{k/2} \\
=&J'_k |x-x'|^{(2H-1/2)k}.
\end{aligned}
\tag{7.86}
$$

□

The Hölder continuity in space and time is summarized in Table 1.1. Finally, since we have continuity in both x and t, we can show that $u(x,t)$ has a Hölder continuous modification, using Kolmogorov's continuity theorem [19]: For fixed H, define the norm $||(x,t)||_H$ on $\mathbb{R} \times \mathbb{R}$ by

$$||(x,t)||_H = \begin{cases} |x|^{2H-1/2} + |t|^{H-1/4} & \text{if } 1/4 < H < 1/2, \text{ and} \\ |x|^{1/2} + |t|^{1/4} & \text{if } 1/2 \leq H < 1. \end{cases} \tag{7.87}$$

Note that the H-norm $||.||_H$ is topologically equivalent to the standard Euclidean norm. Now we can combine our continuity results, Theorems 7.3.7 and 7.3.12:

Theorem 7.3.13. *There exists a constant $A_k > 0$ such that uniformly for all $(x,t), (x',t') \in [0,T] \times (-\infty, \infty)$,*

$$\mathbb{E}\left(|U(x,t) - U(x',t')|^k\right) \leq A_k ||(x,t) - (x',t')||_H^k. \tag{7.88}$$

Moreover, by Kolmogorov's continuity theorem [19], it follows that $U(x,t)$ has a continuous modification.

Bibliography

[1] R. Balan and C. Tudor. "The Stochastic Heat Equation with Fractional-Colored Noise: Existence of the Solution". In: *Latin American Journal of Probability and Mathematical Statistics* 4 (2007).

[2] F. Biagini et al. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Red. by J. Gani et al. Probability and Its Applications. London: Springer London, 2008.

[3] F. Black and M. Scholes. "The Pricing of Options and Corporate Liabilities". In: *The Journal of Political Economy* 81.3 (1973), pp. 637–654.

[4] D. L. Burkholder. "Martingale Inequalities". In: *Martingales*. Ed. by Hermann Dinges. Lecture Notes in Mathematics 190. Springer Berlin Heidelberg, 1971, pp. 1–8.

[5] R. C. Dalang. "Extending martingale measure stochastic integral with applications to spatially homogeneous S. P. D. E's." In: *Electronic Journal of Probability [electronic only]* 4 (1999), Paper No. 6, p.1–29.

[6] V. Dobrić and F. M. Ojeda. "Conditional expectations and martingales in the fractional Brownian field". In: *Institute of Mathematical Statistics Collections* (2009), pp. 224–238.

[7] V. Dobrić and F. M. Ojeda. "Fractional Brownian fields, duality, and martingales". In: *Institute of Mathematical Statistics Lecture Notes - Monograph Series*. Beachwood, Ohio, USA: Institute of Mathematical Statistics, 2006, pp. 77–95.

[8] L. C. Evans. *Partial Differential Equations: Second Edition*. 2nd ed. Providence, R.I: American Mathematical Society, 2010. 749 pp.

[9] M. Hairer. "Solving the KPZ equation". In: *Annals of Mathematics* 178.2 (2013), pp. 559–664.

[10] S. L. Heston. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options". In: *Review of Financial Studies* 6.2 (1993), pp. 327–343.

[11] Y. Hu. "Heat Equations with Fractional White Noise Potentials". In: *Applied Mathematics and Optimization* 43.3 (2014), pp. 221–243.

[12] Y. Hu and D. Nualart. "Stochastic heat equation driven by fractional noise and local time". In: *Probability Theory and Related Fields* 143.1 (2008), pp. 285–328.

[13] Y. Hu and B. Øksendal. "Fractional white noise calculus and applications to finance". In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 06.1 (2003), pp. 1–32.

[14] Y. Hu et al. "Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency". In: *Electronic Journal of Probability* 20.0 (2015).

[15] J. Hull and A. White. "The Pricing of Options on Assets with Stochastic Volatilities". In: *The Journal of Finance* 42.2 (1987), pp. 281–300.

[16] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Vol. 113. Graduate Texts in Mathematics. New York, NY: Springer New York, 1998.

[17] M. Kardar, G. Parisi, and Y. Zhang. "Dynamic Scaling of Growing Interfaces". In: *Physical Review Letters* 56.9 (1986), pp. 889–892.

[18] D. Khoshnevisan and F. Rassoul-Agha, eds. *A Minicourse on Stochastic Partial Differential Equations*. Vol. 1962. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.

[19] H. Kunita. “Stochastic differential equations and stochastic flows of diffeomorphisms”. In: *École d’Été de Probabilités de Saint-Flour XII - 1982*. Ed. by P. L. Hennequin. Lecture Notes in Mathematics 1097. Springer Berlin Heidelberg, 1984, pp. 143–303.

[20] H. E. Leland. “Option Pricing and Replication with Transactions Costs”. In: *The Journal of Finance* 40.5 (1985), pp. 1283–1301.

[21] B. Mandelbrot. *Fractals and Scaling in Finance*. New York, NY: Springer New York, 1997.

[22] B. Mandelbrot and J.W. Van Ness. “Fractional Brownian motions, fractional noises and applications.pdf”. In: *SIAM Review* 10 (1968), pp. 422–437.

[23] R. C. Merton. “Option pricing when underlying stock returns are discontinuous”. In: *Journal of Financial Economics* 3.1 (1976), pp. 125–144.

[24] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Red. by S. S. Chern et al. Vol. 293. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999.

[25] D. J. Scansaroli. “Stochastic Modeling with Temporally Dependent Gaussian Processes: Applications to Financial Engineering, Pricing and Risk Management”. PhD thesis. 2012.

[26] A. N. Shiryaev. *Essentials of Stochastic Finance: Facts, Models, Theory*. Vol. 3. Advanced Series on Statistical Science and Applied Probability. World Scientific, 1999.

[27] S. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. New York, NY: Springer, 2004.

[28] T. Sottinen. “Fractional Brownian motion, random walks and binary market models”. In: *Finance and Stochastics* 5.3 (2001), pp. 343–355.

[29] J. B. Walsh. "An introduction to stochastic partial differential equations". In: *École d'Été de Probabilités de Saint Flour XIV - 1984*. Ed. by P. L. Hennequin. Lecture Notes in Mathematics 1180. Springer Berlin Heidelberg, 1986, pp. 265–439.

Vita

Mackenzie Wildman was born Mackenzie Elizabeth Bergstrom to Bradley and Kristin Bergstrom in December 1987. She grew up in Colorado before attending her first two years of college at Azusa Pacific University in Azusa, CA. She then transferred to Drexel University in Philadelphia, PA where she continued her studies of Mathematics. In 2010, Mackenzie entered into the graduate program at Lehigh University as a University Fellow. She has since worked as a Teaching Assistant for the Department of Mathematics while completing her degree. Mackenzie began her dissertation work under the guidance of Professor Vladimir Dobrić, who passed away in April 2015. She has continued her research under Professor Daniel Conus and was awarded the degree of Doctor of Philosophy in Mathematics in May 2016.