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The k-Fixed-Endpoint Path Partition Problem

by

Breeanne Alyse Baker

A Dissertation
Presented to the Graduate Committee
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The k-Fixed-Endpoint Path Partition Problem

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Abstract

The Hamiltonian path problem is to determine whether a graph has a Hamiltonian path. This problem is NP-complete in general. The path partition problem is to determine the minimum number of vertex-disjoint paths required to cover a graph. Since this problem is a generalization of the Hamiltonian path problem, it is also NP-complete in general. The k-fixed-endpoint path partition problem is to determine the minimum number of vertex-disjoint paths required to cover a graph G such that each vertex in a set T of k vertices is an endpoint of a path. Since this problem is a generalization of the Hamiltonian path problem and path partition problem, it is also NP-complete in general. For certain classes of graphs, there exist efficient algorithms for the k-fixed-endpoint path partition problem. We consider this problem restricted to trees, threshold graphs, block graphs, and unit interval graphs and show min-max theorems which characterize the k-fixed-endpoint path partition number.

Chapter 1

Introduction

The Hamiltonian path problem (HP) is to determine whether a graph has a path which contains all vertices in the graph, or a Hamiltonian path. In general, this problem is NP-complete. Efficient algorithms exist which determine whether a graph has a Hamiltonian path for cocomparability graphs [10], distance-hereditary graphs [18], interval graphs [9, 26], circular-arc graphs [9], and convex bipartite graphs [30]. It is shown that this problem is NP-complete on grid graphs [22], chordal bipartite graph [30], and strongly chordal split graphs [30].

The Hamiltonian path problem can be modified from a decision problem to the path partition problem (PP) which is to determine the minimum number of vertex disjoint paths required to cover the vertex set of a graph G. In general, this problem is NP-complete. Efficient algorithms exist which determine the size of a minimum path partition for trees [8, 12, 38], unicyclic graphs [12], cacti [29], block graphs [39, 40, 41], graphs with blocks which are complete graphs, cycles, or complete bipartite graphs [33], cographs [24, 31], P_4-sparse graphs [6, 13], P_4-extendible graphs [13], interval graphs [1, 20, 36], circular-arc graphs [17, 23], bipartite permutation graphs [39], bipartite distance hereditary graphs [42], and distance hereditary graphs [19]. Note that trees are block graphs and cacti; unicyclic graphs are cacti; interval graphs are circular-arc graphs; cographs are P_4-sparse graphs; and trees, block graphs, and cographs are distance-hereditary graphs.

The 1HP problem is to determine whether a graph has a Hamiltonian path with a specified vertex as an endpoint. The 2HP problem is to determine whether a graph has a Hamiltonian path with two specified vertices as endpoints. The 1HP, 2HP, and PP problems can be modified and extended to the k-fixed-endpoint path partition problem. A k-fixed-endpoint path partition with respect to a set T of size k is a path partition in which every vertex in T is an endpoint of a path. For a graph G and a given subset of the vertices T, the k-fixed-endpoint path partition problem is to determine the minimum size of a k-fixed-endpoint path partition. If $k = 0$, then the problem reduces to the path partition problem. Therefore, the k-fixed-endpoint path partition problem is NP-complete in general. Efficient algorithms exist which determine the size of a minimum path partition with respect to the given set T for trees [21], block graphs [16], cographs [2, 15], and proper (unit) interval graphs [5, 28]. An efficient algorithm exists for 2HP for grid graphs [22]. The complexity of the k-fixed-endpoint path partition problem is unknown for interval graphs except that when $k = 1$ an efficient algorithm exists [4]. Note that tree graphs are bipartite distance-hereditary graphs, bipartite distance-hereditary graphs are distance-hereditary graphs, and proper interval graphs are equivalent to unit interval graphs and are interval graphs. Definitions for the above graph classes can be found in [7].

While these efficient algorithms exist for the k-fixed-endpoint path partition problem, no characterization theorems exist. Our goal is to determine such characterization theorems which provide necessary and sufficient conditions for the k-fixed-endpoint path partition number for trees, threshold graphs, linear block graphs, block graphs, 2-connected unit interval graphs, and unit interval graphs. First, necessary notation and definitions will be discussed. Then lower bounds which apply to all graph classes will be established. The lower bounds yield a characterization for the k-fixed-endpoint path partition number for trees. In chapter 2, a min-max theorem which characterized the k-fixed-endpoint path partition number for threshold graphs is considered along with additional necessary definitions. In chapter 3, min-max theorems which characterize the k-fixed-endpoint path partition number for linear block graphs and block graphs are discussed along with additional necessary definitions. Linear block graphs are a specific case of block graphs and are considered since they are also unit interval graphs and provide insight into the unit interval graph case. In chapter 4, min-max theorems which characterize the k-fixed-endpoint path partition number for 2-connected and connected unit interval graphs are discussed along with additional necessary definitions. The characterization for 2-connected unit interval graphs is considered since the statement is concise and is necessary for the characterization of connected unit interval graphs.

1.1 Notation and Definitions

Definition 1. *A* path *is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A* Hamiltonian path *is a spanning path.*

Definition 2. *A* path partition *on a graph G is a set of vertex-disjoint paths which cover the vertices in G.*

Notation 1. *Let the k-fixed-endpoint path partition number be denoted $PP(G;T)$ for a graph G with a given set of vertices T. If $T = \emptyset$, then $PP(G)$ denotes the path partition number.*

Path partitions of G (with respect to T) of minimum size will be referred to as minimum path partitions of G (with respect to T).

Let G be a graph and $S \subset V(G)$. Throughout this dissertation, when the meaning is clear, notation such as $G - S$ will be used to represent the graph induced by $V(G) - S$.

Notation 2. *Let $c(G)$ be the number of components in G.*

Typically, this notation will be used to represent the number of components in a graph G when a subset of the vertices U has been removed; that is, $c(G - U)$.

Let T be a set of vertices in $V(G)$ and P be a subgraph of G. Notation such as $T \cap P$ or $T \cup P$ will be used to represent the vertices in both T and $V(P)$ or either T or $V(P)$, respectively. Let X and Y be sets of vertices. Then $X + y$ will be used

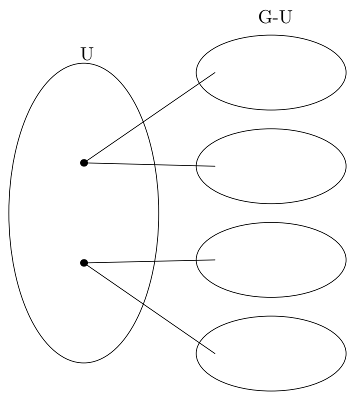

Figure 1.1: Each component of $G - U$ must contain at least one path and each vertex in U can connect at most two of those paths.

to represent $X \cup y$ where y is a vertex. $X - Y$ will represent the set of vertices in X but not Y. $X - y$ will represent the set X excluding the vertex y.

The following is a well-known lower bound for path partition number for general graphs.

Lemma 1. *For a graph G, $PP(G) \geq \max_{U \subseteq V}\{c(G - U) - |U|\}$.*

Consider Figure 1.1. Informally, each component of $G - U$ must contain at least one path and each vertex in U can connect at most two of those paths. Then G needs at least $c(G - U) - |U|$ paths for a minimum path partition.

1.2 Lower Bounds

Lemma 1 can be modified to determine an additional lower bound for the k-fixed-endpoint path partition number for all graphs.

Lemma 2 (k-fixed-endpoint path partition number lower bound)**.** *For any graph G,*

$$PP(G;T) \geq \max_{U \subseteq V}\{c(G - U) - |S|\}$$

where $S = U - T$.

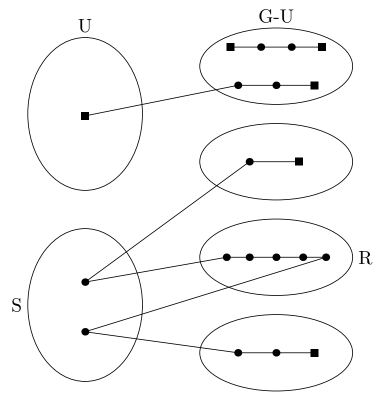

Figure 1.2: The square vertices are in T. Each component of $G - U$ needs at least $\left\lceil \frac{|C \cap T|}{2} \right\rceil$ or 1 path and each vertex in $S = U - T$ can connect at most two of those paths.

Proof. Consider a minimum path partition $\mathcal{P}$ on G with respect to T. Let $\mathcal{P}_U = \mathcal{P} - U$ for some subset of the vertices U. Each component of $G - U$ must be covered by at least one path in $\mathcal{P}_U$. Therefore, $|\mathcal{P}_U| \geq c(G - U)$. Each vertex in S can connect at most two paths in $\mathcal{P}_U$ to form $\mathcal{P}$. Vertices in $U \cap T$ cannot connect any of the paths. Therefore, $|\mathcal{P}| \geq |\mathcal{P}_U| - |S| \geq c(G - U) - |S|$. This holds for all subsets of the vertices U. Therefore, $PP(G;T) \geq \max_{U \subseteq V}\{c(G - U) - |S|\}$. □

This lower bound does not take into account the vertices in T when considering the components of $G - U$. First consider the following definition.

Definition 3. *Let C_i be the components of $G - U$. Let R be the number of C_i where $C_i \cap T = \emptyset$. Define*

$$c_T(G - U) = \sum_i \left\lceil \frac{|C_i \cap T|}{2} \right\rceil + R.$$

Now consider Figure 1.2. Informally, each component of $G - U$ now needs at least $\left\lceil \frac{|C \cap T|}{2} \right\rceil$ paths or 1 path if $C \cap T = \emptyset$. Each vertex in $S = U - T$ can connect at most two of these paths. Vertices in $U \cap T$ cannot connect any of the paths. Then G needs at least $\sum_i \left\lceil \frac{|C_i \cap T|}{2} \right\rceil + R - |S|$ paths in a minimum path partition with respect to T. The following lemma provides a second, tighter lower bound for the k-fixed-endpoint path partition number.

Lemma 3 (k-fixed-endpoint path partition number lower bound)**.** *For any graph G,*

$$PP(G;T) \geq \max_{U \subseteq V}\{c_T(G-U) - |S|\}$$

where $S = U - T$.

Proof. Consider a minimum path partition $\mathcal{P}$ on G with respect to T. Let $\mathcal{P}_U = \mathcal{P} - U$ for some subset of the vertices U. Each component of $G - U$ must be covered by at least $\left\lceil \frac{|C_i \cap T|}{2} \right\rceil$ paths in $\mathcal{P}_U$ or, if $C_i \cap T = \emptyset$, one path in $\mathcal{P}_U$. Therefore, $|\mathcal{P}_U| \geq c_T(G-U)$. Each vertex in S can connect at most two paths in $\mathcal{P}_U$ to form $\mathcal{P}$. Vertices in $U \cap T$ cannot connect any of the paths. Therefore, $|\mathcal{P}| \geq |\mathcal{P}_U| - |S| \geq c_T(G-U) - |S|$. This holds for all subsets of the vertices U. Therefore, $PP(G;T) \geq \max_{U \subseteq V}\{c_T(G-U) - |S|\}$.

□

Lemma 3 is used to determine a min-max theorem for trees and threshold graphs. It is also used within pieces of linear block and block graphs, as well as for threshold graphs.

If a pendant vertex is added to a graph G and the resulting graph $\hat{G}$ is in the same class as G, then the class is said to be closed under adding a pendant vertex. Trees and block graphs are two classes which are closed under adding a pendant vertex. This fact allows us to use the following lemma.

Lemma 4. *[11] If $\hat{G}$ is formed by adding a pendant vertex adjacent to every vertex in G which is in T and $\hat{G}$ is in the same class as G, then $PP(\hat{G}) = PP(G;T)$.*

Lemma 4 allows the k-fixed-endpoint path partition number for a tree G with respect to T to be determined using the path partition number for the tree $\hat{G}$ where $\hat{G}$ is formed by adding a pendant vertex adjacent to all vertices in T since trees have a nice characterization for the path partition number.

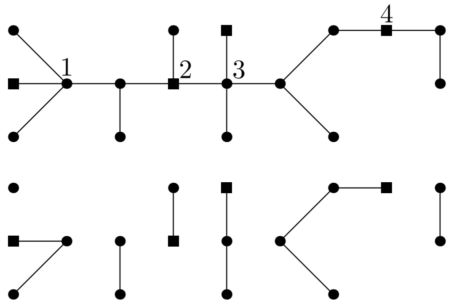

Figure 1.3: Square vertices are in T. When U includes the vertices labeled $1, 2, 3, 4$, $c_T(G - U) - |S| = 7$. The bottom figure shows that there exists a collection of 7 paths which cover G with respect to T.

1.3 Trees

Efficient algorithms exist for trees for the path partition problem [8, 12, 38] and thus the k-fixed-endpoint path partition problem by Lemma 4. In [21], the k-fixed-endpoint path partition problem for trees is solved directly. The following result is known for the path partition problem on trees.

Lemma 5. *Given a tree G, $PP(G) = \max_{U \subseteq V}\{c(G - U) - |U|\}$.*

Lemmas 4 and 5 together characterize the k-fixed-endpoint path partition number for trees when a pendant vertex is added adjacent to every vertex in T. However, the lower bound in Lemma 3 can be used to characterize the k-fixed-endpoint path partition number for trees without requiring G to be modified. Consider the example in Figure 1.3.

Theorem 1 (The k-Fixed-Endpoint Path Partition Problem for Trees)**.** *Given a tree G and a set of vertices T, $PP(G; T) = \max_{U \subseteq V}\{c(G - U) - |S|\}$ where $S = U - T$.*

Proof. Consider induction on the number of vertices.
Base If $n = 1$, then a minimum path partition is the trivial path and
$\max_{U \subseteq V}\{c(G - U) - |S|\} = c(G - \emptyset) - |\emptyset| = 1$.
Induction Let $n > 1$.

Case 1: Suppose there exists a vertex z which is adjacent to at least two leaves, x and y. By induction $PP(G-z;T-z) = \max_{U\subseteq V}\{c((G-z)-U)-|S-z|\}$. Let U' be optimal on $G-z$. Let G_i be the components of $G-z$ and $U_i = U'\cap G_i$ with $S_i = U_i - T$. Then $PP(G-z;T-z) = \sum_i PP(G_i;T) = \sum_i [c(G_i-U_i)-|S_i|]$. Let $U^* = U' + z$.

Case 1a: Suppose $z \notin T$. Then $S^* = U^* - T = \cup_i S_i + z$. In every minimum path partition on $G-z$, x and y are trivial paths. A path partition on G is a minimum path partition on $G-z$ with the paths x and y replaced with the path xzy. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G-z;T)-1\\
&= \sum_i [c(G_i-U_i)-|S_i|]-1\\
&= c(G-U^*)-(|S^*|-1)-1\\
&\leq \max_{U\subseteq V}\{c(G-U)-|S|\}.
\end{aligned}$$

Case 1b: Suppose $z \in T$. Then $S^* = U^* - T = \cup_i S_i$. In every minimum path partition on $G-z$, x and y are trivial paths. A path partition on G is a minimum path partition on $G-z$ with the path x replaced with the path xz. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G-z;T)\\
&= \sum_i [c(G_i-U_i)-|S_i|]\\
&= c(G-U^*)-|S^*|\\
&\leq \max_{U\subseteq V}\{c(G-U)-|S|\}.
\end{aligned}$$

Case 2: Suppose there does not exist a vertex which is adjacent to at least two leaves. Then there is a leaf y adjacent to a vertex w of degree 2.

Case 2a: Suppose $w \notin T$. By induction $PP(G-y;T-y) = \max_{U\subseteq V}\{c((G-y)-U)-|S|\}$. Let U' be a maximal optimal set on $G-y$ with $S' = U'-T$. In a minimum path partition on $G-y$, w is an endpoint of a path since it is a leaf in $G-y$. A path partition on G is a minimum path partition on $G-y$ with Pw

replaced by Pwy where P may be an empty path. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y) \\ &= c((G-y)-U') - |S'| \\ &\leq c(G-U') - |S'| \\ &\leq \max_{U \subseteq V}\{c(G-U) - |S|\}. \end{aligned}$$

Case 2b: Suppose $w \in T$. By induction,

$$PP(G-\{y,w\};T-\{y,w\}) = \max_{U \subseteq V}\{c((G-\{y,w\})-U) - |S-\{y,w\}|\}.$$

Let U' be a maximal optimal set in $G-\{y,w\}$ with $S' = U' - T$. Let $U^* = U' + w$. Then $S^* = U^* - T = S'$. A path partition on G is a minimum path partition on $G-\{y,w\}$ with the additional path yw. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-\{y,w\};T-\{y,w\}) + 1 \\ &= c((G-\{y,w\})-U') - |S'| + 1 \\ &\leq c(G-U^*) - 1 - |S^*| + 1 \\ &\leq \max_{U \subseteq V}\{c(G-U) - |S|\}. \end{aligned}$$

Therefore, the claim holds. □

Chapter 2

Threshold Graphs

Threshold graphs are contained within the class of cographs. Efficient algorithms exist for cographs for the path partition problem [24, 31], the 1HP and 2HP problems [3], and the k-fixed-endpoint path partition problem [2, 15]. The lower bound in Lemma 1 is tight for the path partition number for threshold graphs. The lower bound in Lemma 3 is not tight for the k-fixed-endpoint path partition number for threshold graphs; however, the k-fixed-endpoint path partition number for threshold graphs will be at most one greater than this lower bound. In this chapter, when this increase occurs will be discussed.

Threshold graphs can be characterized in many ways. The following characterization will be most helpful for statements and proofs. Additional information regarding threshold graphs and their characterizations can be found in [25].

Definition 4. *A graph G is* threshold graph *if the vertex set of G can be partitioned into sets $R_0, R_1, \ldots, R_m, L_1, L_2, \ldots, L_m$ that satisfy:*

- *for each $v \in R_i$, $N(v) = \bigcup_{j=1}^{i} L_j$ for $1 \leq i \leq m$ and*
- *for each $v \in L_j$, $N(v) = \left(\bigcup_{i=j}^{m} R_i\right) \cup \left(\bigcup_{j=1}^{m} L_j\right)$ for $1 \leq j \leq m$.*

Note that only R_0 and R_m, may be empty. Figure 2.1 illustrates this characterization where the cliques on the left are adjacent to all other cliques and the

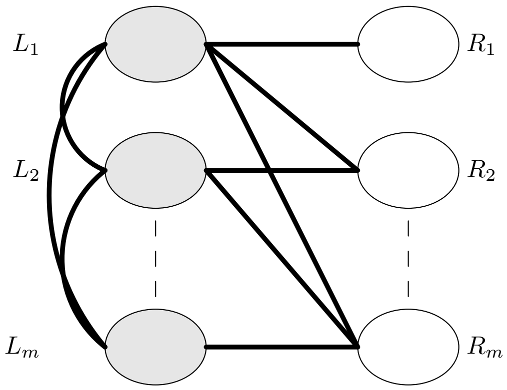

Figure 2.1: The shaded components are cliques and the white components are independent sets. Edges represent all possible edges between the vertex sets.

independent sets on the right at or below the level of the clique. Edges will be omitted in all other figures with threshold graphs. All threshold graphs considered in this dissertation will be connected. Therefore, $R_0 = \emptyset$ for all considered graphs.

Given the structure of a threshold graph, any set which maximizes the lower bound in Lemma 3 will be of the form $\bigcup_{j=1}^{a} L_j$, $0 \le a \le m$. If any vertex in L_j is not in U, then removing vertices from U which are in L_j will not decrease the number of components in $G-U$. If any vertex in R_i is in U, then removing that vertex from U will not decrease the number of components in $G-U$. These actions also will not increase the size of $S = U - T$. This means the only subsets of the vertices U which need to be considered are the empty set and a set of cliques on the left which are consecutive from the top down. The following lemma formalizes this notion.

Lemma 6. *If G is a connected threshold graph, then for some $a \ge 0$, $U = \bigcup_{j=1}^{a} L_j$ will maximize the lower bound in Lemma 3.*

Proof. Suppose $U = \left(\bigcup_{j=1}^{a} L_j\right) \cup X \cup Y$ maximizes the lower bound in Lemma 3 where $X \subseteq \bigcup_{j=a+1}^{m} L_j$ and $Y \subseteq \bigcup_{i=1}^{m} R_i$. Note that a is the smallest index such that there exists $v \in L_{a+1}$ and $v \notin U$. Let $U' = U - X - Y$. Then, $c_T(G-U) \leq c_T(G-U')$ since vertices in $U \cap T$ can be added to the component contained within $\bigcup_{j=a+1}^{m} (L_j \cup R_j)$ when U' is created from U and vertices in $U \cap \bigcup_{i=1}^{a} R_i$ can create new components in $c_T(G - U')$. Additionally, for $S = U - T$ and $S' = U' - T$, $|S| \geq |S'|$ since vertices are removed from U to create U'. Therefore, $c_T(G - U) - |S| \leq c_T(G - U') - |S'|$ and U' maximizes the lower bound in Lemma 3. □

Lemma 6 means that only subsets of the form $U = \bigcup_{j=1}^{a} L_j$ need to be considered for the lower bound in Lemma 3. For subsets U of this form, $G - U$ will have at most one nontrivial component C and a set of isolated vertices $\bigcup_{i=1}^{a} R_i$. Define the following function $\eta(a)$ to describe $c_T(G-U)-|S|$ where $S = U-T$ and $U = \bigcup_{j=1}^{a} L_j$.

Lemma 7. *Let $\eta_G(a)$, $0 \leq a \leq m$, denote the value for $c_T\left(G - \left(\bigcup_{j=1}^{a} L_j\right)\right) - \left|\left(\bigcup_{j=1}^{a} L_j\right) - T\right|$. Then*

$$\eta_G(a) = \begin{cases} \left\lceil \frac{|T|}{2} \right\rceil & \text{if } a = 0 \\ \left\lceil \dfrac{\left|\left(\bigcup_{j=a+1}^{m} (L_j \cup R_j)\right) \cap T\right|}{2} \right\rceil + \left|\bigcup_{i=1}^{a} R_i\right| - \left|\left(\bigcup_{j=1}^{a} L_j\right) - T\right| & \text{if } 1 \leq a \leq m. \end{cases}$$

When the graph G is clear from context, $\eta_G(a)$ will be denoted $\eta(a)$.

Figure 2.2 illustrates the values for $\eta(a)$, $0 \leq a \leq m$. Lemma 7 is easy to check.

The lower bound in Lemma 3 does not yield the k-fixed-endpoint path partition number for every threshold graph. For example, the graph in Figure 2.2 shows

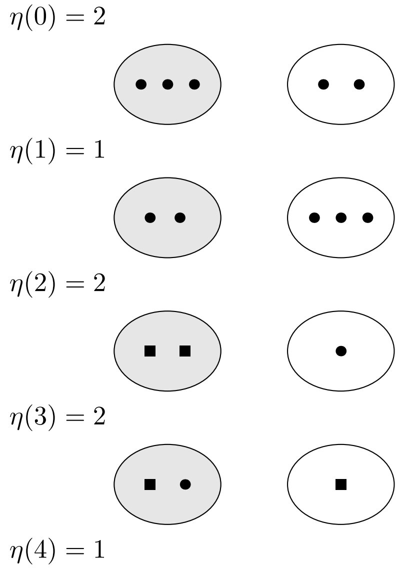

Figure 2.2: $\eta(a)$, $0 \leq a \leq m$, denotes $c_T(G - U) - |S|$ for subsets U which are described in Lemma 6. $PP(G;T) \geq 2$ for this graph.

$PP(G;T) \geq 2$ yet the k-fixed-endpoint path partition number is 3. A new lower bound is needed to account for this discrepancy.

When the maximum for the lower bound is attained for at least two different values of a, the lower bound may not yield the k-fixed-endpoint path partition number. When no vertices in T occur in the sets between where the maximums occur, then the number of these vertices on the left equals the number on the right. In addition, if the number of vertices in T below the lower level of the maximum is even, then the k-fixed-endpoint path partition number will be at least one greater than the lower bound in Lemma 3. The graph in Figure 2.3 satisfies these conditions. If the lower bounds above and below the lower level of the maximum are considered and added, the value will be one greater than the lower bound of the entire graph. This occurs since the lower bound above the lower level of the maximum will have its maximum occur at the higher level of the maximum. This lower bound will cause an increase in the overall lower bound since the subgraph now has a component below

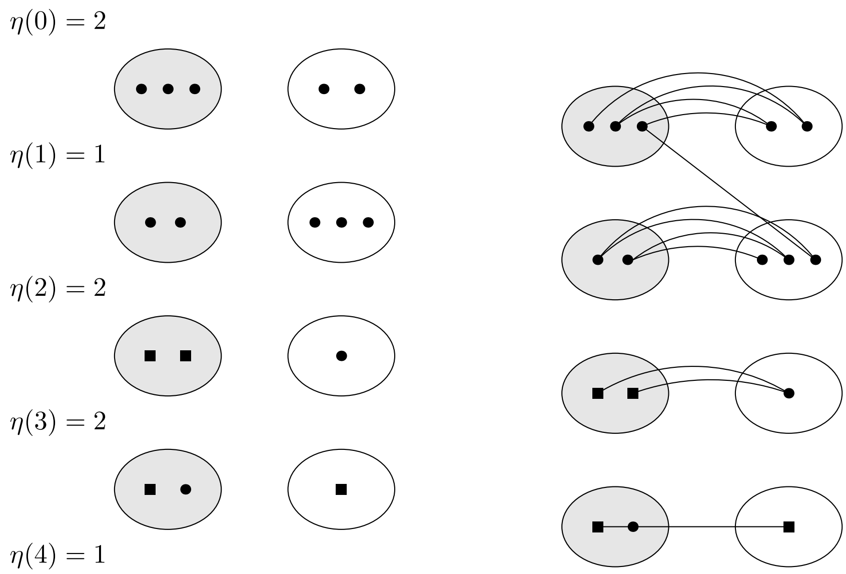

Figure 2.3: The graph attains the maximum $\eta(a) = 2$ twice, when $a = 0$ and $a = 2$. However, the k-fixed-endpoint path partition number is 3 as shown in the graph on the right.

the maximum which is not counted elsewhere. The conditions for the increase to occur are formally stated, and then the lemma formalizes these ideas.

Condition 1. *Let G be a connected threshold graph. We will say Condition 1 holds if there exist a_1 and a_2, $0 \leq a_1 < a_2 < m$, which maximize $\eta_G(a)$ such that the following hold:*

- $\left(\bigcup_{j=a_1+1}^{a_2} (L_j \cup R_j) \right) \cap T = \emptyset$ *and*
- $\left| \left(\bigcup_{j=a_2+1}^{m} (L_j \cup R_j) \right) \cap T \right|$ *is even.*

Note that $\left| \bigcup_{j=a_1+1}^{a_2} L_j \right| = \left| \bigcup_{i=a_1+1}^{a_2} R_i \right|$ when Condition 1 holds.

Lemma 8. *Given a connected threshold graph G,*

$$PP(G;T) \geq \begin{cases} \max\limits_{0\leq a\leq m} \{\eta(a)\} + 1 & \textit{if Condition 1 is satisfied} \\ \max\limits_{0\leq a\leq m} \{\eta(a)\} & \textit{otherwise} \end{cases}.$$

Proof. Since $\max\limits_{0\leq a\leq m} \{\eta(a)\} = \max\limits_{U\subseteq V}\{c_T(G-U) - |S|\}$ by Lemma 6 and Lemma 7, $PP(G;T) \geq \max\limits_{0\leq a\leq m} \{\eta(a)\}$ by Lemma 3. Suppose Condition 1 is satisfied by a_1 and a_2 with $a_1 < a_2$. Let $G_1 = \bigcup\limits_{j=1}^{a_1}(L_j \cup R_j)$, $G_2 = \bigcup\limits_{j=a_1+1}^{a_2}(L_j \cup R_j)$, and $G_3 = \bigcup\limits_{j=a_2+1}^{m}(L_j \cup R_j)$. Let $\alpha = \max\limits_{0\leq a\leq m} \{\eta_G(a)\}$. Then $\eta_{G_1}(a_1) = \alpha - \left\lceil \frac{|(G_2 \cap G_3) \cap T|}{2} \right\rceil = \alpha - \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil$ since $G_2 \cap T$ must be empty when Condition 1 is satisfied. Additionally, $\max\{\eta_{G_3}(a)\} \geq \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil$ and $\eta_{(G_1\cup G_2)}(a_1) = \eta_{G_1}(a_1) + 1 = \alpha - \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil + 1$.

Consider a minimum path partition $\mathcal{P}$ on G. Suppose no edges in $\mathcal{P}$ have an end in $G_1 \cup G_2$ and an end in G_3. In this case, $PP(G;T) = PP(G_1 \cup G_2; T \cap (G_1 \cup G_2)) + PP(G_3; T \cap G_3) \geq \alpha - \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil + 1 + \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil = \alpha + 1$.

Suppose b edges in $\mathcal{P}$ have one end in $G_1 \cup G_2$ and one end in G_3. Let B be the set of vertices in G_3 which are endpoints of these edges. Then $G_3' = G_3 - B$ will be covered by at least $\left\lceil \frac{|(G_3 - B) \cap T|}{2} \right\rceil \geq \left\lceil \frac{|G_3 \cap T| - |B|}{2} \right\rceil$ paths. Let $G' = G_1 \cup G_2 \cup B'$ where $|B'| = |B|$ and the vertices in B' are adjacent to $\bigcup\limits_{j=1}^{a_2} L_j$. Let $T^* = (G' \cap T) \cup B'$. Then $\eta_{G'}(a_2) = \alpha - \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil + |B|$. In this case, $PP(G;T) \geq PP(G';T^*) + PP(G_3'; T \cap G_3') \geq \left(\alpha - \left\lceil \frac{|G_3 \cap T|}{2} \right\rceil + |B|\right) + \left\lceil \frac{|G_3 \cap T| - |B|}{2} \right\rceil \geq \alpha + 1$.

Therefore,

$$PP(G;T) \geq \begin{cases} \max\limits_{0\leq a\leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\leq a\leq m} \{\eta(a)\} & \text{otherwise} \end{cases}.$$

□

The lower bound in Lemma 8 is tight for threshold graphs. To prove this fact, induction will be applied to a subgraph of a threshold graph G created by removing a dominating vertex y or an edge ry where $r \in R_1$. A path partition on G with respect to T will be created using a minimum path partition on $G - y$ with respect to $T - y$ or on $G - \{y, r\}$ with respect to $T - \{y, r\}$.

Theorem 2. *Given a connected threshold graph,*

$$PP(G;T) = \begin{cases} \max\limits_{0 \le a \le m} \{\eta(a)\} + 1 & \textit{if Condition 1 is satisfied} \\ \max\limits_{0 \le a \le m} \{\eta(a)\} & \textit{otherwise} \end{cases}.$$

Proof. Induction on the number of vertices n in G.

Base: Suppose $n = 1$. Then $PP(G;T) = 1$ and $\max\limits_{0 \le a \le 1} \{\eta(a)\} = 1$. Condition 1 cannot be satisfied. Therefore,

$$PP(G;T) = \begin{cases} \max\limits_{0 \le a \le m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \le a \le m} \{\eta(a)\} & \text{otherwise} \end{cases}.$$

Induction: Suppose $n \ge 2$. Let y be a vertex in L_1 of G. Let G' be the nontrivial connected threshold graph in $G - y$. Let the sets of vertices in G' be labeled $L'_1, L'_2, \ldots, L'_{m-1}, R'_1, R'_2, \ldots, R'_{m-1}$. Let a_1 be the smallest value for which $\eta_{G'}(a_1) = \max\limits_{0 \le a \le m} \{\eta_{G'}(a)\}$.

Case A: Suppose $L_1 = \{y\}$. Then $G' = G - y - R_1$. Note that $L'_j = L_{j+1}$, $1 \le j \le m-1$, and $R'_i = R_{i+1}$, $1 \le i \le m-1$. By induction,

$$PP(G-y;T-y) = |R_1| + \begin{cases} \max\limits_{0 \le a \le m-1} \{\eta_{G'}(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \le a \le m-1} \{\eta_{G'}(a)\} & \text{otherwise} \end{cases}.$$

Case A1: Suppose $\eta_{G'}(a_1) = \left\lceil \frac{|G' \cap T|}{2} \right\rceil$. Then $a_1 = 0$ and

$$\begin{aligned} PP(G-y;T-y) &= |R_1| + PP(G';T\cap G') \\ &= \begin{cases} |R_1| + \left\lceil \frac{|G'\cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied on } G' \\ |R_1| + \left\lceil \frac{|G'\cap T|}{2} \right\rceil & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \notin T$ and $|R_1| \geq 2$, then y can connect two vertices in R_1. Then a path partition on G with respect to T is a minimum path partition on $G-y$ with respect to $T-y$ with two trivial paths x, z in R_1 replaced by the path xyz. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y) - 1 \\ &= |R_1| + \left\lceil \frac{|G'\cap T|}{2} \right\rceil - 1 \\ &= |R_1| + \left\lceil \frac{|G'\cap T|}{2} \right\rceil - |L_1 - T| \\ &= \eta_G(1) \\ &\leq \begin{cases} \max\limits_{0\leq a\leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\leq a\leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \notin T$, $|R_1| = 1$, and $|G' \cap T|$ is odd, then a path partition on G with respect to T is a minimum path partition on $G-y$ with respect to $T-y$ with the vertex x in R_1 and a path P which has an end not in T replaced by xyP. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y) - 1 \\ &= |R_1| + \left\lceil \frac{|G'\cap T|}{2} \right\rceil - 1 \\ &= |R_1| + \left\lceil \frac{|G'\cap T|}{2} \right\rceil - |L_1 - T| \\ &= \eta_G(1) \\ &\leq \begin{cases} \max\limits_{0\leq a\leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\leq a\leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \notin T$, $|R_1| = 1$, $|G' \cap T|$ is even, and $R_1 \cap T \neq \emptyset$. Then $|T|$ is odd and $\frac{|G' \cap T|}{2} + |R_1| = \frac{|G' \cap T|}{2} + 1 = \left\lceil \frac{|T|}{2} \right\rceil$. Then a path partition on G with respect to T is a minimum path partition on $G - y$ with respect to $T - y$ with the vertex x in R_1 replaced by xy. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y) \\ &= |R_1| + \left\lceil \frac{|G' \cap T|}{2} \right\rceil \\ &= \left\lceil \frac{|T|}{2} \right\rceil \\ &= \eta_G(0) \\ &\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \notin T$, $|R_1| = 1$, $|G' \cap T|$ is even, $R_1 \cap T = \emptyset$, and the maximum of $\eta_{G'}(a)$ is achieved only when $a = 0$, then $\eta_G(0) = \eta_G(1)$, $(L_1 \cup R_1) \cap T = \emptyset$, and Condition 1 is satisfied. Then a path partition on G with respect to T is a minimum path partition on $G - y$ with respect to $T - y$ found inductively with the vertex x in R_1 replaced by xy. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y) \\ &= |R_1| + \left\lceil \frac{|G' \cap T|}{2} \right\rceil \\ &= \left\lceil \frac{|T|}{2} \right\rceil + 1 \\ &= \eta_G(0) + 1 \\ &\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \notin T$, $|R_1| = 1$, $|G' \cap T|$ is even, $R_1 \cap T = \emptyset$, the maximum is achieved at $a_2 \neq a_1$, and Condition 1 is satisfied on $G - y$, then by induction, $PP(G-y;T-y) =$

$\dfrac{|G' \cap T|}{2}+1$ and there exists a path P in every minimum path partition on $G-y$ with respect to $T-y$ which has an end which is not contained in T. Then a path partition on G with respect to T is this minimum path partition on $G-y$ with respect to $T-y$ with the vertex x in R_1 and the path P replaced by xyP. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y)-1 \\ &= |R_1| + \left\lceil \frac{|G' \cap T|}{2} \right\rceil + 1 - 1 \\ &= \left\lceil \frac{|T|}{2} \right\rceil + 1 \\ &= \eta_G(0) + 1 \\ &\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \notin T$, $|R_1| = 1$, $|G' \cap T|$ is even, $R_1 \cap T = \emptyset$, $a_2 \neq 0$, and Condition 1 is not satisfied on $G-y$, then Condition 1 is satisfied on G with $\eta_G(0)$ and $\eta_G(1)$. Then a path partition on G with respect to T is a minimum path partition on $G-y$ with respect to $T-y$ with the vertex x in R_1 replaced by xy. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G-y;T-y) \\ &= |R_1| + \left\lceil \frac{|G' \cap T|}{2} \right\rceil \\ &= \eta_G(0) + 1 \\ &\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y \in T$, Condition 1 is satisfied on G if and only if Condition 1 is satisfied on $G-y$ since the values $\eta_G(a) = \eta_{G'}(a-1)$ for $a \geq 1$ and y will not be part of $\bigcup\limits_{j=a_1+1}^{a_2} (L_j \cup R_j)$. Then a path partition on G with respect to T is a minimum path partition on $G-y$ with respect to $T-y$ with a vertex x in R_1 replaced by xy.

Therefore,

$$\begin{aligned}
PP(G;T) &\le PP(G-y;T-y)\\
&= \begin{cases} |R_1| + \left\lceil \dfrac{|G'\cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied}\\ |R_1| + \left\lceil \dfrac{|G'\cap T|}{2} \right\rceil & \text{otherwise} \end{cases}\\
&= \begin{cases} |R_1| + \left\lceil \dfrac{|G'\cap T|}{2} \right\rceil - |L_1 - T| + 1 & \text{if Condition 1 is satisfied}\\ |R_1| + \left\lceil \dfrac{|G'\cap T|}{2} \right\rceil - |L_1 - T| & \text{otherwise} \end{cases}\\
&= \begin{cases} \eta_G(1) + 1 & \text{if Condition 1 is satisfied}\\ \eta_G(1) & \text{otherwise} \end{cases}\\
&\le \begin{cases} \max\limits_{0\le a\le m}\{\eta(a)\} + 1 & \text{if Condition 1 is satisfied}\\ \max\limits_{0\le a\le m}\{\eta(a)\} & \text{otherwise} \end{cases}.
\end{aligned}$$

Case A2: Suppose $\eta_{G'}(a_1) > \left\lceil \dfrac{|G'\cap T|}{2} \right\rceil$. Then $a_1 > 0$. Let $C = \bigcup\limits_{j=a_1+1}^{m-1} (L'_j \cup R'_j)$. Then

$$PP(G-y;T-y) = \begin{cases} |R_1| + \eta_{G'}(a_1) + 1 & \text{if Condition 1 is satisfied on } G-y\\ |R_1| + \eta_{G'}(a_1) & \text{otherwise} \end{cases}$$

and

$$\begin{aligned}
|R_1| + \eta_{G'}(a_1) &= |R_1| + \left|\bigcup_{i=1}^{a_1} R'_i\right| + \left\lceil \frac{|C\cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=1}^{a_1} L'_j\right) - T\right|\\
&= \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C\cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=2}^{a_1+1} L_j\right) - T\right|.
\end{aligned}$$

Since $\max\limits_{0\le a\le m-1}\{\eta_{G'}(a)\} > \left\lceil \dfrac{|G'\cap (T-y)|}{2} \right\rceil$, there exists a path P in every minimum path partition on $G-y$ with respect to $T-y$ with an endpoint in $G'-T$. Condition 1 is satisfied on G if and only if Condition 1 is satisfied on $G-y$.

If $y \notin T$, then a path partition on G with respect to T is a minimum path partition on $G - y$ with respect to $T - y$ with a vertex x in R_1 and the path P replaced by xyP. Therefore,

$$
\begin{aligned}
PP(G;T) &\leq PP(G-y;T-y) - 1 \\
&= -1 + \begin{cases} \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C \cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=2}^{a_1+1} L_j\right) - T\right| + 1 & \text{if Cond 1} \\ \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C \cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=2}^{a_1+1} L_j\right) - T\right| & \text{otherwise} \end{cases} \\
&= \begin{cases} \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C \cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=1}^{a_1+1} L_j\right) - T\right| + 1 & \text{if Condition 1} \\ \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C \cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=1}^{a_1+1} L_j\right) - T\right| & \text{otherwise} \end{cases} \\
&= \begin{cases} \eta_G(a_1) + 1 & \text{if Condition 1 is satisfied} \\ \eta_G(a_1) & \text{otherwise} \end{cases} \\
&\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases} .
\end{aligned}
$$

If $y \in T$, then a path partition on G with respect to T is a minimum path partition on $G - y$ with respect to $T - y$ with a vertex x in R_1 replaced with xy.

Therefore,

$$
\begin{aligned}
PP(G;T) &\le PP(G-y;T-y)\\
&= \begin{cases} \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C\cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=2}^{a_1+1} L_j\right) - T\right| + 1 & \text{if Condition 1} \\ \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C\cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=2}^{a_1+1} L_j\right) - T\right| & \text{otherwise} \end{cases}\\
&= \begin{cases} \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C\cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=1}^{a_1+1} L_j\right) - T\right| + 1 & \text{if Condition 1} \\ \left|\bigcup_{i=1}^{a_1+1} R_i\right| + \left\lceil \frac{|C\cap T|}{2} \right\rceil - \left|\left(\bigcup_{j=1}^{a_1+1} L_j\right) - T\right| & \text{otherwise} \end{cases}\\
&= \begin{cases} \eta_G(a_1)+1 & \text{if Condition 1 is satisfied on G} \\ \eta_G(a_1) & \text{otherwise} \end{cases}\\
&\le \begin{cases} \max\limits_{0\le a\le m}\{\eta(a)\}+1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\le a\le m}\{\eta(a)\} & \text{otherwise} \end{cases}.
\end{aligned}
$$

Case B: Suppose $\{x,y\} \subseteq L_1$. Let $r \in R_1$ and $G^* = G - \{y,r\}$ with sets $L_1^*, L_2^*, \ldots, L_m^*, R_1^*, R_2^*, \ldots, R_m^*$. Note that $L_j^* = L_j$, $2 \le j \le m$ and $R_i^* = R_i$, $2 \le i \le m$. By induction,

$$PP(G^*;T\cap G^*) = \begin{cases} \eta_{G^*}(a_1)+1 & \text{if Condition 1 is satisfied} \\ \eta_{G^*}(a_1) & \text{otherwise} \end{cases}.$$

Case B1: Suppose $\eta_{G^*}(a_1) = \left\lceil \frac{|G^*\cap T|}{2} \right\rceil$. Then $a_1 = 0$ and

$$PP(G^*;T\cap G^*) = \begin{cases} \left\lceil \frac{|G^*\cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \frac{|G^*\cap T|}{2} \right\rceil & \text{otherwise} \end{cases}.$$

If $y \notin T$ and $r \notin T$, then $\eta_G(a) = \eta_{G^*}(a)$ for all a and Condition 1 is satisfied on G if and only if Condition 1 is satisfied on G^*. Since $|L_1| \ge 2$, let $x \in L_1$. Every minimum path partition on G^* contains a path $P = P_1 x P_2$ where P_1 or P_2 may be empty. Then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with path P replaced with $P_1 y r x P_2$. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G^*; T \cap G^*) \\
&= \begin{cases} \left\lceil \dfrac{|G^* \cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|G^* \cap T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \begin{cases} \left\lceil \dfrac{|T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \eta_G(0) \\
&\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases} .
\end{aligned}$$

If $y \notin T$, $R_1 \cap T = R_1$ and $|R_1|$ is even, then the above holds. If $R_1 \cap T = R_1$ and $|R_1|$ is odd, then $\left\lceil \dfrac{|G^* \cap T|}{2} \right\rceil = \left\lceil \dfrac{|T|}{2} \right\rceil - 1$ and a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with path the additional path yr. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G^*;T\cap G^*)+1 \\
&= 1+\begin{cases} \left\lceil \dfrac{|G^*\cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|G^*\cap T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \begin{cases} \left\lceil \dfrac{|T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \eta_G(0) \\
&\leq \begin{cases} \max\limits_{0\leq a\leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\leq a\leq m} \{\eta(a)\} & \text{otherwise} \end{cases} .
\end{aligned}$$

If $L_1 \cap T = L_1$ and $r \in T$ or $R_1 \cap T = \emptyset$ and $|T|$ is odd, then $\eta_G(a) = \eta_{G^*}(a)+1$ for all a and Condition 1 is satisfied on G if and only if Condition 1 is satisfied on G^*. Then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with the additional path yr. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G^*;T\cap G^*)+1 \\
&= 1+\begin{cases} \left\lceil \dfrac{|G^*\cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|G^*\cap T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \begin{cases} \left\lceil \dfrac{|T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \eta_G(0) \\
&\leq \begin{cases} \max\limits_{0\leq a\leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\leq a\leq m} \{\eta(a)\} & \text{otherwise} \end{cases} .
\end{aligned}$$

If $L_1 \cap T = L_1$, $R_1 \cap T = \emptyset$, and $|T|$ is even, then $\eta_G(0) = \eta_{G^*}(0)$, $\eta_G(a) = \eta_{G^*}(a) + 1$ for $1 \leq a \leq m$, and Condition 1 is satisfied on G if and only if Condition 1 is satisfied on G^*. If there exists a_2 such that $\eta_{G^*}(a_2) = \eta_{G^*}(a_1)$, then $\eta_G(a_2) = \eta_{G^*}(a_2) + 1 = \eta_{G^*}(a_1) + 1$. Then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with the additional path yr. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G^*; T \cap G^*) + 1 \\ &= 1 + \begin{cases} \eta_{G^*}(a_1) + 1 & \text{if Condition 1 is satisfied} \\ \eta_{G^*}(a_1) & \text{otherwise} \end{cases} \\ &= \begin{cases} \eta_G(a_2) + 1 & \text{if Condition 1 is satisfied} \\ \eta_G(a_2) & \text{otherwise} \end{cases} \\ &\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If no such a_2 exists, then $|R_1| \leq \left\lceil \frac{|L_1| - 1}{2} \right\rceil$ since $\eta_{G^*}(1) < \left\lceil \frac{|G^* \cap T|}{2} \right\rceil$. If $|L_1|$ is odd, then $|G^* \cap T| - |L_1|$ is odd and there exists a minimum path partition $\mathcal{P}$ on G^* with respect to $T \cap G^*$ such that $G^* - (L_1 \cup R_1)$ is covered by $\left\lceil \frac{|G^* \cap T| - |L - 1|}{2} \right\rceil$ paths. One of these paths will have one end not contained in T. Then a vertex v in L_1 can be made adjacent to that path. $(R_1 - r) + (L_1 - y - v)$ can be covered by $|R_1| - 1$ paths which have two ends in $L_1 - y - v$ and one interior vertex in $R_1 - r$. This leaves $(|L_1| - 2) - 2(|R_1| - 1) = |L_1| - 2 - (|L_1| - 1) - 2 = 1$ vertex x in L_1 to be covered. This vertex will be a trivial path in $\mathcal{P}$. Then a path partition on G with respect to T is this minimum path partition on G^* with respect to $T \cap G^*$ with the path x replaced by xry.

If $|L_1|$ is even, then $|G^* \cap T| - |L_1|$ is even and there exists a minimum path partition $\mathcal{P}$ on G^* with respect to $T \cap G^*$ such that $G^* - (L_1 \cup R_1)$ can be covered by $\frac{|G^* \cap T| - |L_1|}{2}$ paths. $(R_1 - r) + (L_1 - y)$ can be covered by $|R_1| - 1$ paths which have two ends in $L_1 - y$ and one interior vertex in $R_1 - r$. This leaves

$|L_1| - 1 - 2(|R| - 1) = |L_1| - 1 - |L_1| - 2 = 1$ vertex x in L_1 to be covered. This vertex will be a trivial path in $\mathcal{P}$. Then a path partition on G with respect to T is this minimum path partition on G^* with respect to $T \cap G^*$ with the path x replaced by xry. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G^*; T \cap G^*) \\
&= \begin{cases} \left\lceil \dfrac{|G^* \cap T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|G^* \cap T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \begin{cases} \left\lceil \dfrac{|T|}{2} \right\rceil + 1 & \text{if Condition 1 is satisfied} \\ \left\lceil \dfrac{|T|}{2} \right\rceil & \text{otherwise} \end{cases} \\
&= \eta_G(0) \\
&\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}.
\end{aligned}$$

Case B2: Suppose $\eta_{G^*}(a_1) > \left\lceil \dfrac{|G^* \cap T|}{2} \right\rceil$. Then $a_1 > 0$.

If $L_1 \cap T = \emptyset$ and $r \notin T$ or $R_1 \cap T = R_1$ and $|T|$ is even, then $\eta_G(a) = \eta_{G^*}(a)$ for all $0 \leq a \leq m$ and Condition 1 is satisfied on G if and only if Condition 1 is satisfied on G^*. Every minimum path partition on G^* with respect to $T \cap G^*$ contains a path $P = P_1 x P_2$ where $x \in L_1$. If $r \notin T$, then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with the path P replaced by $P_1 yrx P_2$. If $r \in T$ and $|T|$ is even, then $|G^* \cap T|$ is odd and there exists a path P with an end not contained in T. Then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with the path P replaced by Pyr. Therefore,

$$\begin{aligned} PP(G;T) &\le PP(G^*;T\cap G^*) \\ &= \begin{cases} \eta_{G^*}(a_1)+1 & \text{if Condition 1 is satisfied} \\ \eta_{G^*}(a_1) & \text{otherwise} \end{cases} \\ &= \begin{cases} \eta_{G}(a_1)+1 & \text{if Condition 1 is satisfied} \\ \eta_{G}(a_1) & \text{otherwise} \end{cases} \\ &\le \begin{cases} \max\limits_{0\le a\le m}\{\eta(a)\}+1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\le a\le m}\{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $L_1\cap T=\emptyset$, $R_1\cap T=R_1$ and $|T|$ is odd, then $\eta_G(0)=\eta_{G^*}(0)+1$ and $\eta_G(a)=\eta_{G^*}(a)$ for $1\le a\le m$. Then Condition 1 is satisfied on G if and only if Condition 1 is satisfied on G^*. Since $\eta_{G^*}(a_1)\ge\dfrac{|G^*\cap T|}{2}+1$, there exists a path P in every minimum path partition on G^* with respect to $T\cap G^*$ such that P has an end which is not contained in T. Then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T\cap G^*$ such that the path P is replaced by Pyr. Therefore,

$$\begin{aligned} PP(G;T) &\le PP(G^*;T\cap G^*) \\ &= \begin{cases} \eta_{G^*}(a_1)+1 & \text{if Condition 1 is satisfied} \\ \eta_{G^*}(a_1) & \text{otherwise} \end{cases} \\ &= \begin{cases} \eta_{G}(a_1)+1 & \text{if Condition 1 is satisfied} \\ \eta_{G}(a_1) & \text{otherwise} \end{cases} \\ &\le \begin{cases} \max\limits_{0\le a\le m}\{\eta(a)\}+1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0\le a\le m}\{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

If $y\in T$, then $\eta_G(a)=\eta_{G^*}(a)+1$ for $1\le a\le m$ except when $R_1\cap T=\emptyset$ and $|T|$ is odd, $\eta_G(0)=\eta_{G^*}(0)$. Then Condition 1 is satisfied on G if and only if Condition

1 is satisfied on G^*. Then a path partition on G with respect to T is a minimum path partition on G^* with respect to $T \cap G^*$ with the additional path yr. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G^*;T \cap G^*) + 1 \\ &= 1 + \begin{cases} \eta_{G^*}(a_1) + 1 & \text{if Condition 1 is satisfied} \\ \eta_{G^*}(a_1) & \text{otherwise} \end{cases} \\ &= \begin{cases} \eta_G(a_1) + 1 & \text{if Condition 1 is satisfied} \\ \eta_G(a_1) & \text{otherwise} \end{cases} \\ &\leq \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}. \end{aligned}$$

Therefore,

$$PP(G;T) = \begin{cases} \max\limits_{0 \leq a \leq m} \{\eta(a)\} + 1 & \text{if Condition 1 is satisfied} \\ \max\limits_{0 \leq a \leq m} \{\eta(a)\} & \text{otherwise} \end{cases}.$$

□

Chapter 3

Block Graphs

Efficient algorithms exist for block graphs for the path partition problem [39, 40, 41] and the k-fixed-endpoint path partition problem [16]. Lemma 4 applies to block graphs as well as trees. While efficient algorithms exist, there is no min-max theorem for the path partition problem on block graphs. Thus, Lemma 4 cannot be applied to yield a min-max theorem for the k-fixed-endpoint path partition number for block graphs. Therefore, a new method is required. In this chapter, min-max theorems for the k-fixed-endpoint path partition number for linear block graphs and block graphs are discussed. The standard definition for block graphs is below and followed by an additional definition and notation.

Definition 5. *A* block graph *is a graph in which all maximal 2-connected subgraphs are cliques.*

Definition 6. *A* linear block graph *is a block graph in which every block contains at most two cut vertices.*

Notation 3. *Let $\Lambda(G)$ denote the set of cut vertices in a block graph G. The shorthand Λ will be used when the graph G is clear from context.*

Definition 7. *The* interior *of a block B is the subgraph of G induced on the vertices in $B - \{u_i | i \in I\}$ where u_i, $i \in I$, are all the cut vertices in B.*

Note that the interior of a block may be empty.

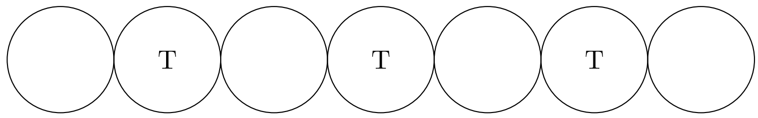

Figure 3.1: Circles represent blocks in the linear block graph. A T represents a vertex in the block which is also in T. Lemma 3 shows $PP(G;T) \geq 2$ yet a minimum path partition on G with respect to T requires 4 paths.

3.1 Linear Block Graphs

Linear block graphs are block graphs and any characterization for the k-fixed-endpoint path partition number for block graphs also applies to the k-fixed-endpoint path partition number for linear block graphs. Linear block graphs are considered separately since linear block graphs are also connected unit interval graphs. The characterization described for the k-fixed-endpoint path partition number for linear block graphs will require only slight modification to characterize the k-fixed-endpoint path partition number for unit interval graphs. The lower bound in Lemma 3 does not characterize the k-fixed-endpoint path partition number for linear block graphs. The difference between the value of the lower bound in Lemma 3 and the value of the k-fixed-endpoint path partition number of a linear block graph G can be arbitrarily large. Consider the graph in Figure 3.1. This graph can be extended to a linear graph with $2b+1$ blocks where b blocks contain exactly one vertex in T in the interior of the block and no adjacent blocks both contain a vertex in T. It can be shown that the lower bound in Lemma 3 would yield $\left\lceil \frac{b}{2} \right\rceil$ and the k-fixed-endpoint path partition number would be $PP(G;T) = b+1$ for such a linear block graph.

Let G be a linear block graph. Label the blocks B_i, $1 \leq i \leq \beta$, such that $B_i \cap B_{i+1} = c_i$, $1 \leq i \leq \beta - 1$, where $c_i \in \Lambda$. Note that a linear block graph G which has β blocks has $\beta - 1$ cut vertices.

Definition 8. *A block B_i is* left *of block B_j if $i < j$. A block B_i is* right *of block B_j if $i > j$. The* leftmost *block has smallest index while the* rightmost *block has largest index.*

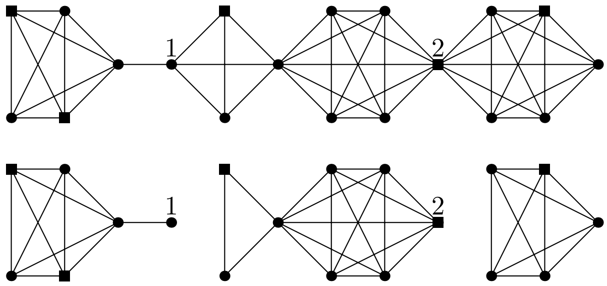

Figure 3.2: The graph on the bottom shows a partition formed by removing the vertices labeled 1 and 2 in the graph on the top and returning them to the components in which they are rightmost.

The linear block graph G will be partitioned into pieces to determine the k-fixed-endpoint path partition number. The parts of the partitions will be formed by removing some subset of the cut vertices and then returning the cut vertices to the component in which it is the rightmost vertex. See Figure 3.2.

Definition 9. *For a linear block graph G, let $\mathcal{P}(W)$, $W \subseteq \Lambda$, be the* partition of G *which is the set of $|W|+1$ induced subgraphs formed by removing the set of edges $\{c_i v | c_i \in W, v \in B_{i+1}\}$.*

Let i_j be the index of the jth leftmost vertex in W; that is, $c_{i_1}, c_{i_2}, \ldots, c_{i_{|W|}}$ where $i_1 < i_2 < \ldots < i_{|W|}$. Then each part P_j of $\mathcal{P}(W)$ can be defined as $P_1 = \bigcup_{l=1}^{i_1} B_l$, $P_j = \bigcup_{l=i_{j-1}+1}^{i_j} (B_l - c_{i_{j-1}})$ for $2 \leq j \leq |W|$ and $P_{|W|+1} = \bigcup_{l=i_{|W|}+1}^{\beta} (B_l - c_{i_{|W|}})$. Note that B_{i_j} is rightmost in part P_j and $\mathcal{P}(W)$ contains $|W|+1$ parts.

Observe that the interior of the leftmost block of each part of a partition must contain an end of a path since the end block contains exactly one cut vertex. If no vertex in T is in the interior of the leftmost block, then an arbitrary vertex can be chosen to be added to T in order to account for the end found in the leftmost block.

Definition 10. *For a partition $\mathcal{P}(W)$ on a linear block graph G, let $T'(W) = T \cup \{v_j | j \in J\}$ where $J \subseteq [|W|+1]$, $j \in J$ when the leftmost block B_1^j of P_j with* *cut vertex c satisfies $(B_1^j - c) \cap T = \emptyset$, and v_j is an arbitrary vertex in the interior of the leftmost block of P_j.*

Note that the shorthand T' will be used to represent $T'(W)$ when W is clear from context. Creating T' will not increase the size of a minimum path partition on G with respect to T. The vertices added to T to create T' were chosen since they can be made ends in a minimum path partition of G with respect to T restricted to a part P_i of $\mathcal{P}(W)$.

Lemma 9. *For a linear block graph G and a partition W of G, $PP(P_j; T \cap P_j) = PP(P_j; T'(W) \cap P_j)$, where T' is defined in Definition 10.*

Proof. Let $\mathcal{Q}$ be a minimum path partition on G with respect to T. Suppose $\mathcal{Q}$ restricted to P_j has no endpoint in the interior of the leftmost block B_1^j of part P_j of the partition. Then a path must enter B_1^j, traverse all vertices in the interior, and leave B_1^j. Paths can only enter and leave a block at cut vertices. Thus, B_1^j must have two cut vertices to satisfy the path condition. Then B_1^j cannot be leftmost in P_j since the leftmost block has exactly one cut vertex. Therefore, a contradiction exists, B_1^j, $1 \leq j \leq |W|+1$, must contain an end of a path of $\mathcal{Q}$ restricted to P_j, and $PP(P_j; T) = PP(P_j; T')$. □

Note that $|P_j \cap T'| = \begin{cases} |P_j \cap T| + 1 & \text{if } B_{i_j+1} \cap T = \emptyset \\ |P_j \cap T| & \text{otherwise} \end{cases}.$

Lemma 10. *Let G be a linear block graph G and T be a set of k vertices. Then $PP(G;T) \geq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$ where T' is defined in Definition 10.*

When $W = \emptyset$, we assume that

$$\max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|)}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\} = \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil.$$

Proof. Consider a partition $\mathcal{P}(W)$ and a path partition $\mathcal{Q}$ on G with respect to T. For each part $P_j \subseteq \mathcal{P}(W)$, count the number of paths in $\mathcal{Q}$ which have right endpoints in P_j. If a path in $\mathcal{Q}$ extends to the right of P_j, then P_j contains part of at least $\left\lceil \frac{|P_j \cap T'| + 1}{2} \right\rceil$ paths and thus $\left\lceil \frac{|P_j \cap T'| + 1}{2} \right\rceil - 1 = \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor$ right ends in this case. If no path in $\mathcal{Q}$ extends to the right of P_j, then P_j contains at least $\left\lfloor \frac{P_j \cap T'|}{2} \right\rfloor$ right ends. This holds for all P_j, $1 \leq j \leq |W|$. The rightmost part $P_{|W|+1}$ still needs to be considered. $P_{|W|+1}$ has $|P_{|W|+1} \cap T'|$ ends of paths. $\mathcal{Q}$ cannot have a path which extends to the right since $P_{|W|+1}$ is rightmost. Then $P_{|W|+1}$ has at least $\left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil$ right endpoints of paths in $\mathcal{Q}$. Therefore, $\mathcal{P}(W)$ contains at least $\left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lceil \frac{|P_j \cap T'|}{2} \right\rceil$ right endpoints in $\mathcal{Q}$. This holds for all W. Therefore, $PP(G;T) \geq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$. □

The lower bound in Lemma 10 is tight for linear block graphs. The "best" partition needs to be chosen to determine the k-fixed-endpoint path partition number of G with respect to T. The "best" partition W^* can be formed by working left to right. If the interior of the leftmost block contains no vertices in T, then the leftmost cut vertex c_i to be put into W^* will have the smallest index i for which $\left| T \cap \bigcup_{l=1}^{i} B_i \right|$ is odd. If the interior of the leftmost block contains at least one vertex in T, then the leftmost cut vertex c_i to be put into W^* will have the smallest index i for which $\left| T \cap \left(\bigcup_{l=1}^{i} B_i \right) \right|$ is even. Then repeat this process for $G - \bigcup_{l=1}^{i} B_i$. See Figure 3.3.

When $W = \emptyset$, we assume

$$\left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor = \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil .$$

Theorem 3. *Let G be a linear block graph G and T be a set of k vertices. Then*

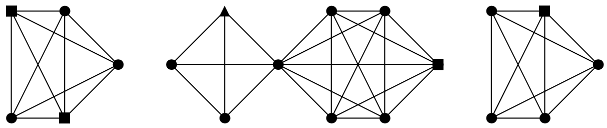

Figure 3.3: The square vertices are in T. The triangle vertex is in $T' - T$. This is an example of a "best" partition for G with respect to T.

$PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$ *where T' is defined in Definition 10.*

Proof. Induct on the number of blocks β. The lower bound follows from Lemma 10.
Base: Suppose $\beta = 1$. Then $\Lambda = \emptyset$ and

$$\max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_J \cap T'|}{2} \right\rfloor \right\} = \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil .$$

Since G is a clique in this case, $PP(G;T) = \left\lceil \frac{|T'|}{2} \right\rceil$. If $T = \emptyset$, then $|T'| = 1$ since G is a leftmost block without a vertex in T. Since $P_{|W|+1} = G$, $|P_{|W|+1} \cap T'| = |T'|$. Therefore, $PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$.

Induction: Suppose $\beta \geq 2$.

Case A: Suppose $|B_1 \cap T|$ is even and at least 2 or $|B_1 \cap T| = 1$ and $c_1 \in T$. By induction, $PP(G - B_1; T - B_1) = \left\lceil \frac{|P_{|W^*|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor$ where W^* is optimal on $G - B_1$ with respect to $T - B_1$. Let $W' = W^* + c_1$ and let $Q_1, Q_2, \ldots, Q_{|W'|+1}$ be the parts of $\mathcal{P}(W')$ on G. Then $P_i = Q_{i+1}$ for $1 \leq i \leq |W^*|+1$ and $Q_1 = B_1$. Since B_1 is a clique, B_1 can be covered by $\frac{|Q_1 \cap T'|}{2}$ paths. Note that $|Q_1 \cap T'|$ is always even since either $|B_1 \cap T|$ is even or $|(B_1 - c_1) \cap T| = \emptyset$ which means $|(B_1 - c_1) \cap T'| = 1$ and $|B_1 \cap T'| = 2$. Then a path partition on G with respect to T is a minimum path partition on $G - B_1$ with respect to $T - B_1$ found inductively with $\dfrac{|Q_1 \cap T'|}{2}$ additional paths. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G-B_1;T-B_1) + \frac{|Q_1 \cap T'|}{2} \\
&= \left\lceil \frac{|P_{|W^*|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor + \frac{|Q_1 \cap T'|}{2} \\
&= \left\lceil \frac{|Q_{|W'|+1} \cap T'|}{2} \right\rceil + \sum_{j=2}^{|W'|} \left\lfloor \frac{|Q_j \cap T'|}{2} \right\rfloor + \left\lfloor \frac{|Q_1 \cap T'|}{2} \right\rfloor \\
&= \left\lceil \frac{|Q_{|W'|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W'|} \left\lfloor \frac{|Q_j \cap T'|}{2} \right\rfloor \\
&\leq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}.
\end{aligned}$$

Case B: Suppose $|B_1 \cap T|$ is odd and at least three and $c_1 \in T$. By induction, $PP(G-(B_1-c_1);T-(B_1-c_1)) = \left\lceil \dfrac{|P_{W^*|+1} \cap T'|}{2} \right\rceil + \displaystyle\sum_{j=1}^{|W^*|} \left\lfloor \dfrac{|P_j \cap T'|}{2} \right\rfloor$ where W^* is optimal on $G-(B_1-c_1)$ with respect to $T-(B_1-c_1)$. Let $Q_1, Q_2, \ldots, Q_{|W^*|+1}$ be the parts of $\mathcal{P}(W^*)$ on G. Then $P_i = Q_i$ for $2 \leq i \leq |W^*|+1$ and $Q_1 = P_1 \cup (B_1 - c_1)$. Since $c_1 \in T$, $|(B_1-c_1) \cap T|$ is even, $\left\lfloor \dfrac{|Q_1 \cap T'|}{2} \right\rfloor = \left\lfloor \dfrac{|P_1 \cap T'|}{2} \right\rfloor + \dfrac{|(B_1-c_1) \cap T|}{2}$ and $B_1 - c_1$ can be covered by $\dfrac{|(B_1-c_1) \cap T|}{2}$ paths. Then a path partition on G with respect to T is a minimum path partition on $G-(B_1-c_1)$ with respect to

$T-(B_1-c_1)$ found inductively with $\dfrac{|(B_1-c_1)\cap T|}{2}$ additional paths. Therefore,

$$\begin{aligned} PP(G;T) &\le PP(G-(B_1-c_1);T-(B_1-c_1)) + \frac{|(B_1-c_1)\cap T|}{2} \\ &= \left\lceil \frac{|P_{|W^*|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_j\cap T'|}{2} \right\rfloor + \frac{|(B_1-c_1)\cap T'|}{2} \\ &= \left\lceil \frac{|Q_{|W^*|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|Q_j\cap T'|}{2} \right\rfloor \\ &\le \max_{W\subseteq\Lambda} \left\{ \left\lceil \frac{|P_{|W|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j\cap T'|}{2} \right\rfloor \right\}. \end{aligned}$$

Case C: Suppose $|B_1\cap T|$ is odd or 0 and $c_1\notin T$. By induction, $PP(G-(B_1-c_1);(T+c_1)-(B_1-c_1)) = \left\lceil \frac{|P_{|W^*|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_j\cap T'|}{2} \right\rfloor$ where W^* is optimal on $G-(B_1-c_1)$ with respect to $(T+c_1)-(B_1-c_1)$. Let $Q_1,Q_2,\ldots,Q_{|W^*|+1}$ be the parts of $\mathcal{P}(W^*)$ on G. Then $P_i=Q_i$ for $2\le i\le |W^*|+1$ and $Q_1=P_1\cup(B_1-c_1)$. Additionally, $|Q_1\cap T'| = |P_1\cap T'|-1+|(B_1-c_1)\cap T'|$. B_1-c_1 can be covered by $\left\lceil \frac{|(B_1-c_1)\cap T'|}{2} \right\rceil$ paths. Then a path partition on G with respect to T is a minimum path partition on $G-(B_1-c_1)$ with respect to $(T+c_1)-(B_1-c_1)$ found inductively with the path ending at c_1 joined with the path on B_1-c_1 which has exactly one endpoint in T and $\left\lfloor \frac{|(B_1-c_1)\cap T'|}{2} \right\rfloor$ additional paths. Therefore,

$$\begin{aligned} PP(G;T) &\le PP(G-(B_1-c_1);(T+c_1)-(B_1-c_1)) + \left\lfloor \frac{|(B_1-c_1)\cap T'|}{2} \right\rfloor \\ &= \left\lceil \frac{|P_{|W^*|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_J\cap T'|}{2} \right\rfloor + \left\lfloor \frac{|(B_1-c_1)\cap T'|}{2} \right\rfloor \\ &= \left\lceil \frac{|Q_{|W^*|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|Q_j\cap T'|}{2} \right\rfloor \\ &\le \max_{W\subseteq\Lambda} \left\{ \left\lceil \frac{|P_{|W|+1}\cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_J\cap T'|}{2} \right\rfloor \right\}. \end{aligned}$$

Therefore, $PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$ when G is a linear block graph.

□

3.2 Block Graphs

Let G be a block graph and let B be a block identified as a root block.

Definition 11. *Identify any vertex v in the interior of the root block B and call it the* root vertex.

Definition 12. *In a block graph G, every block B_i other than the root block B has a unique* parent vertex*, denoted $a(B_i)$, which is the cut vertex on every path from B to B_i. All other cut vertices in B_i are* children vertices *and the set of the vertices is denoted $c(B_i)$.*

Definition 13. *In a block graph G with root block B, every cut vertex v has a unique* parent block*, denoted $b(v)$, which is the unique block B_i containing v such that $a(B_i) \neq v$. All other blocks containing v are* children blocks *and the set of these blocks is denoted $c(v)$.*

Definition 14. *In a block graph G, a* leaf block *is a block which has no children vertices.*

Note that a leaf block contains exactly one cut vertex except when the block graph is a single block and the leaf block contains no cut vertices.

Definition 15. *A vertex v_i in block B_i is* below *another vertex v_j in block B_j if $i \neq j$ and every path from root block B to B_i contains $a(B_j)$. Then v_j is* above *v_i.*

A block graph G can be partitioned into pieces to determine the k-fixed-endpoint path partition number. The parts of the partition will be formed by removing some subset of the cut vertices and then returning these cut vertices to the component in which its children blocks reside. See Figure 3.4.

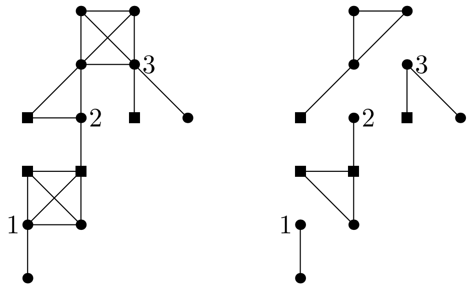

Figure 3.4: The graph on the right shows a partition formed by removing the vertices labeled 1, 2, and 3 in the graph on the left and returning the vertices to the components in which their children blocks reside.

Definition 16. *Let G be a block graph with root block B. Let $\mathcal{P}(W, B)$, $W \subseteq \Lambda$, denote the* partition of G with respect to B *which is formed by removing the edges* $\{vw|v \in W, w \in b(v)\}$.

Let P_0 be the part of $\mathcal{P}(W, B)$ which contains the vertices in $B - W$. Note that P_0 may be empty. Let P_v, $v \in W$, be the other parts of $\mathcal{P}(W, B)$ with root vertex v. Note that all parts of $\mathcal{P}(W, B)$ are block graphs.

Definition 17. *Let G be a block graph with root block B. Let $\mathcal{EP}(W, B)$ denote the* extended partition of G with respect to B *which is formed by adding a pendant edge and vertex adjacent to each root vertex v in P_v, $v \in W$.*

Let P'_v, $v \in W$, denote the parts of an extended partition on G with respect to B with root vertex v. Note that these parts are block graphs by Lemma 4. See Figure 3.5 which is the extended partition of the partition in Figure 3.4.

Lemma 11. *Let G be a block graph with root vertex B and an extended partition $\mathcal{EP}(W, B)$. Then*

$$PP(G; T) \geq PP(P_0; T_0) + \sum_{v \in W} [PP(P'_v; T_v) - 1]$$

where $T_0 = T \cap P_0$ and $T_v = T \cap P'_v$.

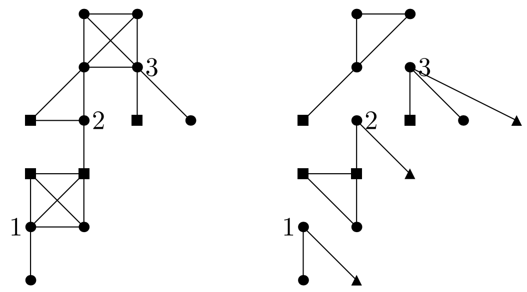

Figure 3.5: The graph on the right is the extended partition of the graph on the right with the vertices labeled 1, 2, and 3 in the set W. The triangle vertices are the vertices added to $\mathcal{P}(W, B)$ to form $\mathcal{EP}(W, B)$.

Proof. Consider a path partition $\mathcal{Q}$ on G with respect to T. Consider a part P'_v of $\mathcal{EP}(W, B)$. P'_v can be covered by $PP(P'_v; T_v)$ paths. Then P_v contains at least $PP(P'_v; T_v) - 1$ paths which do not have v as an end. P_0 can be covered by $PP(P_0; T_0)$ paths. Since P_0 does not have a root vertex, no path ends at the root vertex. Then a path partition on G with respect to T has at least $PP(P_0; T_0) + \sum_{v \in W} [PP(P'_v; T_v) - 1]$ paths. If a path is contained in multiple blocks, it will end at the root vertex in all but one part P_v where it gets counted. Therefore, $PP(G; T) \geq PP(P_0; T_0) + \sum_{v \in W} [PP(P'_v; T_v) - 1]$. $\square$

Observe that the interior of leaf blocks in parts P_v of $\mathcal{P}(W, B)$ must contain an end of a path since each leaf block contains exactly one cut vertex. If no vertex in T is in the interior of a leaf block, then an arbitrary vertex in the interior can be chosen to be added to T in order to account for the end found in the leaf block. Figure 3.6 shows the vertices added to T for the given partition.

Definition 18. *For a block graph G with root block B and partition $\mathcal{P}(W, B)$, let $T'(W, B) = T \cup \{v_i | i \in I\}$ where v_i is an arbitrary vertex in the interior of a leaf block B_i in P_v if $(B_i - c_i) \cap T = \emptyset$ where c_i is the cut vertex in B_i.*

Note that the shorthand T' will be used to represent $T'(W, B)$ when W and B are clear from context. Creating T' will not increase the size of a minimum path

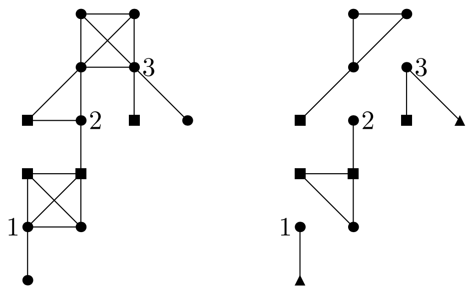

Figure 3.6: The vertices labeled 1,2 and 3 are cut vertices which form the partition. The square vertices are in T and the triangle vertices are added to T.

partition on G with respect to T. The vertices added to T to create T' were chosen since they can be ends in a minimum path partition on G respect to T restricted to the parts in $\mathcal{P}(W, B)$.

Lemma 12. *Let G be a block graph with root vertex $v \in B$ and $T'(W, B)$ as defined in Definition 18. Then $PP(P_v; T) = PP(P_v; T')$ for all $v \in W$.*

Proof. Suppose P_v is a single block. If $T = \emptyset$, then $T' \neq \emptyset$. Since G is a clique, $T = \emptyset$, and $|T'| = 1$, $PP(P_v; T) = 1$ and $PP(P_v; T') = 1$. If $T \neq \emptyset$, then $T = T'$. Then $PP(P_v; T) = PP(P_v; T')$.

Suppose P_v is not a single block. Then every leaf block in P_v must contain an end of a path in every minimum path partition on P_v with respect to T. If a leaf block does not contain an end of a path, then the path must enter the block, cover all vertices in the block, then leave the block. Since a leaf block has exactly one cut vertex, this is not possible without either using the cut vertex twice or having an end in the block. Then a minimum path partition on P_v with respect to T will have ends in each leaf block, including those which contain no vertices in T, and thus is also a minimum path partition on P_v with respect to T'.

Therefore, $PP(P_v; T) = PP(P_v; T')$. □

Lemma 13. *Let G' be a block graph G with root vertex v, a pendant edge and vertex adjacent to v, and $W = \emptyset$. Then $PP(G'; T) - 1 \geq \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v$ where C_i are*

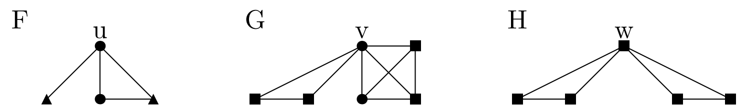

Figure 3.7: The block graph F on the left has root vertex u and $\delta_u = 1$ since at least one component of $F - u$ contains an odd number of vertices in T'. The block graph G in the middle has root vertex v and $\delta_v = 0$ since $G - v$ has no component which contains an odd number of vertices. The block graph H on the right has root vertex w and $\delta_w = 0$ since $w \in T$.

the components of $G - v$, $\delta_v = \begin{cases} 1 & \textit{if } v \notin T \textit{ and } |C_i \cap T'| \textit{ is odd for some } i \\ 0 & \textit{otherwise} \end{cases}$, *and* T' *is defined in Definition 18.*

Figure 3.7 illustrates when δ_v will be 0 or be 1.

Proof. **Case A**: Suppose $v \notin T$ and has exactly one child block. If $|T'|$ is odd, then $\delta_v = 1$ and by Lemma 3 with $U = v$,

$$\begin{aligned} PP(G';T) - 1 &\geq \left\lceil \frac{|T'|}{2} \right\rceil + 1 - |v| - 1 \\ &= \left\lceil \frac{|T'|}{2} \right\rceil - 1 \\ &= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - 1. \end{aligned}$$

If $|T'|$ is even, then $\delta_v = 0$ and by lemma 3 with $U = \emptyset$,

$$\begin{aligned} PP(G';T) - 1 &\geq \left\lceil \frac{|G' \cap T'|}{2} \right\rceil \\ &= \left\lceil \frac{|T'|}{2} \right\rceil \\ &= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \end{aligned}$$

since $|G' \cap T'| = |T'| + 1$ and $|T'|$ is odd.

Case B: Suppose $v \in T$ and has exactly one child block. Then $\delta_v = 0$ and by Lemma 3 with $U = v$,

$$\begin{aligned}
PP(G';T) - 1 &\geq \left\lceil \frac{|T'|}{2} \right\rceil + 1 - |v - T| - 1 \\
&= \left\lceil \frac{|T'|}{2} \right\rceil + 1 - 0 - 1 \\
&= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v.
\end{aligned}$$

Case C: Suppose $v \notin T$, has at least two children blocks, and $|C_i \cap T'|$ is even for all i. Then $\delta_v = 0$ and by Lemma 3 with $U = \emptyset$,

$$\begin{aligned}
PP(G';T) - 1 &\geq \left\lceil \frac{|G' \cap T'|}{2} \right\rceil - 1 \\
&= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil + 1 - 1 \\
&= \sum_i \left\lceil \frac{|C_i \cap T|}{2} \right\rceil - \delta_v
\end{aligned}$$

since $|G' \cap T'| = |T'| + 1$ given $|C_i \cap T'|$ is even for all i.

Case D: Suppose $v \notin T$, has at least two children blocks, and $|C_i \cap T'|$ is odd for some i. Then $\delta_v = 1$ and by Lemma 3 with $U = v$,

$$\begin{aligned}
PP(G';T) - 1 &\geq \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil + 1 - |v| - 1 \\
&= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - +1 - 1 - 1 \\
&= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v.
\end{aligned}$$

Case E: Suppose $v \in T$ and has at least two children blocks. Then $\delta_v = 0$ and by Lemma 3 with $U = v$,

$$\begin{aligned} PP(G';T) - 1 &\geq \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil + 1 - |v - T| - 1 \\ &= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil + 1 - 0 - 1 \\ &= \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v. \end{aligned}$$

Therefore, $PP(G';T) - 1 \geq \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v$. □

The lower bound in Lemma 13 is tight for block graphs. The "best" partition needs to be chosen to determine the k-fixed-endpoint path partition number of G with respect to T. The "best" partition W^* can be formed by working bottom to top. For a cut vertex c_i which only has leaf blocks as children blocks, the cut vertex will be in W^* in one of two situations:

- If $c_i \in T$ and the interior of exactly one child block contains either no vertices in T or an odd number of vertices in T, then $c_i \in W^*$.
- If $c_i \notin T$ and all children blocks contain an even, nonzero number of vertices in T or at least two children blocks contain either no vertices in T or an odd number of vertices in T, then $c_i \in W^*$.

If c_i is a cut vertex where all cut vertices below c_i have been considered and are not in W^*, then consider the components of $P_{c_i} - c_i$ where P_{c_i} is the graph below c_i. Such a c_i will be in W^* in one of two situations:

- If $c_i \in T$ and exactly one component of $P_{c_i} - c_i$ contains an odd number of vertices in T and leaves which contain no vertices in T, then $c_i \in W^*$.
- If $c_i \notin T$ and every component of $P_{c_i} - c_i$ contains an even number of vertices in T and leaves which contain no vertices in T or at least two components which contain an odd number of vertices in T and leaves which contain no vertices in T, then $c_i \in W^*$.

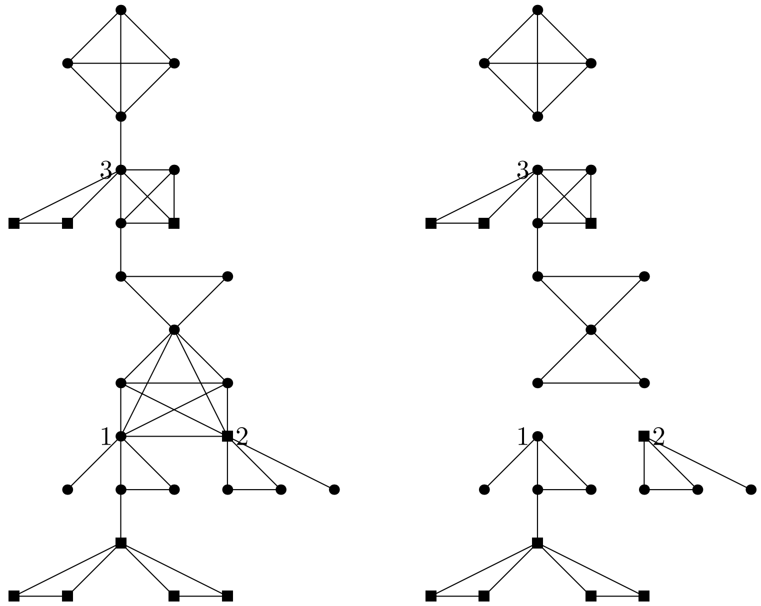

Figure 3.8: The graph on the right is the "best" partition W^* for the graph on the left. Vertices labeled 1, 2, and 3 are in W^*. Square vertices are in T.

Repeat this process for $G - P_{c_i}$. There exists a minimum path partition on G with respect to T such that all the paths will be contained within a part of such a "best" partition. See Figure 3.8.

Theorem 4. *Let G be a block graph with root block B. Then*

$$PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \right\}$$

where $T' = T'(W, B)$ as defined in Definition 18 and δ_v is as defined in Lemma 13.

When $W = \emptyset$, we assume that

$$\left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) = \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil .$$

Proof. Induct on the number of blocks β in G. The lower bound follows from Lemma 13.

Base: Suppose $\beta = 1$. Then G is a complete graph and $\Lambda = \emptyset$. G can be covered by $\left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil$ paths since G is a complete graph and

$$\begin{aligned} PP(G;T) &= \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil \\ &= \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \right\}. \end{aligned}$$

Induction: Suppose $\beta \geq 2$. Let $v' \in \Lambda$ be a cut vertex such that all of its children blocks are leaf blocks in G. Let $P_{v'}$ be the induced subgraph of G on v' and its children blocks.

Case A: Suppose $v' \in T$ and at least one component of $P_{v'} - v'$ contains no vertices in T or an odd number of vertices in T. By induction,

$$PP(G - P_{v'}; T - P_{v'}) = \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right)$$

where W^* is an optimal subset of Λ. Let $W' = W^* + v'$ and let Q_0, Q_v, $v \in W'$ be the parts of the partition $\mathcal{P}(W', B)$ on G. Then $P_0 = Q_0$, $P_v = Q_v$ for all $v \in W^*$, and $Q_{v'} = P_{v'}$. Each component C_i of $P_{v'} - v'$ can be covered by $\left\lceil \frac{|C_i \cap T'|}{2} \right\rceil$ paths since each component is a clique. Then v' can be made adjacent to a path in $P_{v'} - v'$ which has exactly one end in T' which exists since at least one component contains an odd number of vertices in T'. Thus, $P_{v'}$ can be covered by $\sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil = \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'}$ since $\delta_{v'} = 0$. Then a path partition on G with respect to T is a minimum path partition on $G - P_{v'}$ with respect to $T - P_{v'}$ found inductively with $\displaystyle\sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'}$ additional paths. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G - P_{v'}; T - P_{v'}) + \sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'} \\
&= \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \\
&\quad + \sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'} \\
&= \left\lceil \frac{|Q_0 \cap T'|}{2} \right\rceil + \sum_{v \in W'} \left(\sum_{C_i \in Q_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \\
&\leq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \right\}.
\end{aligned}$$

Case B: Suppose $v' \notin T$ and every component of $P_{v'} - v'$ contains an even, nonzero number of vertices in T. By induction,

$$PP(G - P_{v'}; T - P_{v'}) = \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right)$$

where W^* is an optimal subset of Λ. Let $W' = W^* + v'$ and let Q_0, Q_v, $v \in W'$ be the parts of the partition $\mathcal{P}(W', B)$ on G. Then $P_0 = Q_0$, $P_v = Q_v$ for all $v \in W^*$, and $Q_{v'} = P_{v'}$. Each component of $P_{v'} - v'$ can be covered by $\left\lceil \frac{|C_i \cap T'|}{2} \right\rceil$ paths since each component is a clique. Then v' can be inserted into a path in $P_{v'} - v'$. Thus, $P_{v'}$ can be covered by $\displaystyle\sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil = \sum_i \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'}$ since $\delta_{v'} = 0$. Then a path partition on G with respect to T is a minimum path partition on $G - P_{v'}$ with respect to $T - P_{v'}$ found inductively with $\displaystyle\sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'}$ additional paths. Therefore,

$$
\begin{aligned}
PP(G;T) &\le PP(G-P_{v'};T-P_{v'}) + \sum_{C_i\in P_{v'}-v'} \left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_{v'} \\
&= \left\lceil \frac{|P_0\cap T'|}{2}\right\rceil + \sum_{v\in W^*}\left(\sum_{C_i\in P_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right) \\
&\quad + \sum_{C_i\in P_{v'}-v'} \left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_{v'} \\
&= \left\lceil \frac{|Q_0\cap T'|}{2}\right\rceil + \sum_{v\in W'}\left(\sum_{C_i\in Q_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right) \\
&\le \max_{W\subseteq\Lambda}\left\{\left\lceil \frac{|P_0\cap T'|}{2}\right\rceil + \sum_{v\in W}\left(\sum_{C_i\in P_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right)\right\}.
\end{aligned}
$$

Case C: Suppose $v' \notin T$ and at least two components of $P_{v'} - v'$ contain no vertices in T or an odd number of vertices in T. By induction,

$$
PP(G-P_{v'};T-P_{v'}) = \left\lceil \frac{|P_0\cap T'|}{2}\right\rceil + \sum_{v\in W^*}\left(\sum_{C_i\in P_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right)
$$

where W^* is an optimal subset of Λ. Let $W' = W^* + v'$ and let $Q_0, Q_v,\ v \in W'$ be the parts of the partition $\mathcal{P}(W', B)$ on G. Then $P_0 = Q_0,\ P_v = Q - v$, for all $v \in W^*$, and $Q_{v'} = P_{v'}$. Each component of $P_{v'} - v'$ can be covered by $\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil$ paths since each component is a clique. Then v' can connect two paths which have exactly one endpoint in T'. Thus, $P_{v'}$ can be covered by $\sum_i \left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - 1 = \sum_i \left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_{v'}$ since $\delta_{v'} = 1$. Then a path partition on G with respect to T is a minimum path partition on $G - P_{v'}$ with respect to $T - P_{v'}$ found inductively with $\displaystyle\sum_{C_i \in P_{v'}-v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'}$ additional paths. Therefore,

$$\begin{aligned}
PP(G;T) \leq PP(G-P_{v'};T-P_{v'}) + \sum_{C_i \in P_{v'}-v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'} \\
= \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \\
+ \sum_{C_i \in P_{v'}-v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_{v'} \\
= \left\lceil \frac{|Q_0 \cap T'|}{2} \right\rceil + \sum_{v \in W'} \left(\sum_{C_i \in Q_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \\
\leq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \right\}.
\end{aligned}$$

Case D: Suppose $v \in T$ and every component of $P_{v'} - v'$ contains an even, nonzero number of vertices in T. By induction,

$$PP(G-(P_{v'}-v');T-(P_{v'}-v')) = \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right).$$

Let $Q_0, Q_v,\ v \in W^*$ be the parts of the partition $\mathcal{P}(W^*, B)$ on G. Then $P_0 = Q_0,\ P_v = Q_v$ for $v \in W^* - v^*$, and $Q_{v^*} = P_{v^*} \cup P_{v'}$ where P_{v^*} contains v' in $G-(P_{v'}-v')$. Then $P_{v'} - v'$ can be covered by $\displaystyle\sum_{C_i \in P_{v'}-v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil$ paths and $\displaystyle\sum_{C_i \in P_{v^*}-v^*} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil + \sum_{C_i \in P_{v'}-v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil = \sum_{C_i \in Q_{v^*}-v^*} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil$ since each component of $P_{v'} - v'$ has an even number of vertices in T'. Then a path partition on G with respect to T is a minimum path partition on $G-(P_{v'}-v')$ with respect to $T-(P_{v'}-v')$ found inductively with $\displaystyle\sum_{C_i \in P_{v'}-v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil$ additional paths.

Therefore,

$$\begin{aligned}
PP(G;T) &\le PP(G-(P_{v'}-v');T-(P_{v'}-v')) + \sum_{C_i\in P_{v'}-v'} \left\lceil \frac{|C_i\cap T'|}{2}\right\rceil \\
&= \left\lceil \frac{|P_0\cap T'|}{2}\right\rceil + \sum_{v\in W^*}\left(\sum_{C_i\in P_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right) + \sum_{C_i\in P_{v'}-v'}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil \\
&= \left\lceil \frac{|Q_0\cap T'|}{2}\right\rceil + \sum_{v\in W^*}\left(\sum_{C_i\in Q_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right) \\
&\le \max_{W\subseteq\Lambda}\left\{\left\lceil \frac{|P_0\cap T'|}{2}\right\rceil + \sum_{v\in W}\left(\sum_{C_i\in P_v-v}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - \delta_v\right)\right\}.
\end{aligned}$$

Case E: Suppose $v\notin T$ and exactly one component of $P_{v'}-v'$ contains no vertices in T or an odd number of vertices in T. By induction, $PP(G-(P_{v'}-v');(T+v')-(P_{v'}-v')) = \left\lceil \frac{|P_0\cap(T'+v')|}{2}\right\rceil + \sum_{v\in W^*}\left(\sum_{C_i\in P_v-v}\left\lceil \frac{|C_i\cap(T'+v')|}{2}\right\rceil - \delta_v\right)$. Let $Q_0, Q_v, v\in W^*$ be the parts of the partition $\mathcal{P}(W',B)$ on G. Then $P_0=Q_0$, $P_v=Q_v$ for $v\in W^*-v^*$, and $Q_{v^*}=P_{v^*}\cup P_{v'}$ where P_{v^*} contains v' in $G-(P_{v'}-v')$. Then $P_{v'}-v'$ can be covered by $\sum_{C_i\in P_{v'}-v'}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil$ paths and the path in $G-(P_{v'}-v')$ which has endpoint v' can be combined with the path in $P_{v'}-v'$ which has exactly one endpoint in T'. Then $\sum_{C_i\in P_{v^*}-v^*}\left\lceil \frac{|C_i\cap(T'+v')|}{2}\right\rceil + \sum_{C_i\in P_{v'}-v'}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil - 1 = \sum_{C_i\in Q_{v^*}-v^*}\left\lceil \frac{|C_i\cap T'|}{2}\right\rceil$ since exactly one component of $P_{v'}-v'$ has an odd number of vertices in T'. Then a path partition on G with respect to T is a minimum path partition on $G-(P_{v'}-v')$ with respect to $(T+v')-(P_{v'}-v')$ found inductively with $\sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - 1$ additional paths. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G - (P_{v'} - v'); (T + v') - (P_{v'} - v')) + \sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - 1 \\
&= \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap (T' + v')|}{2} \right\rceil - \delta_v \right) \\
&\quad + \sum_{C_i \in P_{v'} - v'} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - 1 \\
&= \left\lceil \frac{|Q_0 \cap T'|}{2} \right\rceil + \sum_{v \in W^*} \left(\sum_{C_i \in Q_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \\
&\leq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \right\}.
\end{aligned}$$

Note that in this case, δ_{v^*} will be the same whether the graph $G - (P_{v'} - v')$ or G is considered.

Therefore,

$$PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_0 \cap T'|}{2} \right\rceil + \sum_{v \in W} \left(\sum_{C_i \in P_v - v} \left\lceil \frac{|C_i \cap T'|}{2} \right\rceil - \delta_v \right) \right\}.$$

□

Chapter 4

Unit Interval Graphs

Efficient algorithms exist for unit interval graphs for 1HP [5], 2HP [5], and the k-fixed-endpoint path partition problem [5, 28]. In this chapter, min-max theorems for the k-fixed-endpoint path partition number for 2-connected unit interval graphs and unit interval graphs with cut vertices are discussed. 2-connected unit interval graphs require $\left\lceil \frac{|T|}{2} \right\rceil$ paths except in a special case when one additional path is required. Consider the following definitions.

Definition 19. *An* interval representation *of a graph is a family of closed intervals assigned to the vertices so that vertices are adjacent if and only if the corresponding intervals intersect.*

Note that not all graphs have an interval representation.

Definition 20. *A* unit interval graph *has an interval representation where all intervals have unit length.*

It is well known that a unit interval representation can be drawn so that the intervals have distinct endpoints.

Definition 21. *A* proper interval graph *has an interval representation where no interval is properly contained within another.*

Theorem 5. *[35] A unit interval graph is a proper interval graph.*

The following two lemmas describe the path partition problem on 2-connected unit interval graphs and connected unit interval graphs, respectively.

Lemma 14. *[32] If G is a unit interval graph, then G is Hamiltonian if and only if G is 2-connected.*

This means every 2-connected unit interval graph contains a Hamiltonian cycle.

Lemma 15. *[34] Every connected unit interval graph contains a Hamiltonian path.*

4.1 2-Connected Unit Interval Graphs

Determining a characterization for the k-fixed-endpoint path partition number for a 2-connected unit interval graph will provide a value for the maximal 2-connected subgraphs of a connected unit interval graph. Consider an interval representation I of a unit interval graph G with distinct endpoints. Label the intervals of I, $1, 2, \ldots, n$, such that $l_1 < l_2 < \cdots < l_{n-1} < l_n$ where l_i is the left endpoint of interval i. Label the vertices of G $v_1, v_2, \ldots, v_n$ where v_i corresponds to interval i.

Definition 22. *A vertex v_i is* left *of vertex v_j in a unit interval graph G if $i < j$. A vertex v_i is* leftmost *if $i = 1$. Similarly, v_i is* right *of v_j if $i > j$ and v_i is* rightmost *if $i = n$.*

Label the vertices of T $t_1, t_2, \ldots, t_{|T|}$ such that t_i is the ith leftmost vertex in T. Note that every vertex in T has two labels, t_i and $v_{f(i)}$ where f is a function which maps the index of t to the index of v.

Definition 23. *An* available endpoint *in a path partition on G with respect to T is a vertex not in T which is an end of a path in the path partition or a vertex in T which is a trivial path in the path partition.*

Definition 24. *In a unit interval graph G, two cut sets, $X_1 = \{v_i, v_{i+1}\}$ and $X_2 = \{v_j, v_{j+1}\}$, $i < j$, are* distinct *if $j > i + 2$.*

The following lemma is a well known result.

Lemma 16. *A subgraph H of a 2-connected unit interval graph G which contains vertices $v_i, v_{i+1}, \ldots, v_{i+j}$ is a 2-connected unit interval graph.*

Lemma 17. *Let G be a 2-connected unit interval graph. Then*

$$PP(G;T) \geq \begin{cases} \frac{|T|}{2} + 1 & \textit{if } T \textit{ has } \frac{|T|}{2} \textit{ pairwise distinct cut sets} \\ \left\lceil \frac{|T|}{2} \right\rceil & \textit{otherwise} \end{cases}$$

Proof. By Lemma 3 with $U = \emptyset$, $PP(G;T) \geq \left\lceil \frac{|T|}{2} \right\rceil$.

Suppose T has $\frac{|T|}{2}$ pairwise distinct cut sets. Then $c(G - T) = \frac{|T|}{2} + 1$. By Lemma 3 with $U = T$,

$$\begin{aligned} PP(G;T) &\geq \max_{U \subseteq V}\{c_T(G-U) - |S|\} \\ &\geq c_T(G-T) - |\emptyset| \\ &= \frac{|T|}{2} + 1 - 0. \end{aligned}$$

Therefore,

$$PP(G;T) \geq \begin{cases} \frac{|T|}{2} + 1 & \text{if } T \text{ has } \frac{|T|}{2} \text{ pairwise distinct cut sets} \\ \left\lceil \frac{|T|}{2} \right\rceil & \text{otherwise.} \end{cases}$$

□

The following lemma is another well known result.

Lemma 18. *If G is a 2-connected unit interval graph, then v_i and v_{i+2} are adjacent for all $1 \leq i \leq n-2$.*

Proof. Suppose not. Since G is a unit interval graph and $v_i v_{i+2} \notin E(G)$, $N(v_i) \subseteq \{v_1, v_2, \ldots, v_{i-2}, v_{i-1}, v_{i+1}\}$ and $N(v_{i+2}) \subseteq \{v_{i+1}, v_{i+3}, v_{i+4}, \ldots, v_{n-1}, v_n\}$. Then $G - v_{i+1}$ has two components since $N(v_i) \cap N(v_{i+2}) = \emptyset$. This contradicts G being 2-connected. Therefore, $v_i v_{i+2} \in E(G)$ for $1 \leq i \leq n-2$. □

Mertzios and Unger [28] use Stair Normal Interval Representation (SNIR) form of a unit interval graph, initially described in [27], to characterize 2HP. Their theorems are below.

Theorem 6. *[28] Let G be a connected proper interval graph and u, v be two vertices of G, with $v \geq u + 2$. There is a Hamiltonian path in G with u, v as endpoints if and only if the submatrices $H_{1,u+1}$ and $H_{v-1,n}$ of H_G are two-way matrices.*

Theorem 7. *[28] Let G be a connected proper interval graph and u be a vertex of G. There is a Hamiltonian path in G with $u, u + 1$ as endpoints if and only if H_G is a two-way matrix and either $u \in \{1, n - 1\}$ or the vertices $u - 1$ and $u + 2$ are adjacent.*

These theorems can be restated without any knowledge of SNIR form. When H_G mentioned above is two-way, the unit interval graph G is 2-connected. It can be verified that the following lemma is equivalent to Theorems 6 and 7. The proofs that follow are shorter and included for completeness.

Lemma 19. *A connected unit interval graph G has a Hamiltonian path with endpoints v_i, v_j, $i < j$, if and only if at least one of the following two conditions is satisfied.*

- *In the case that $i + 1 < j - 1$, the subgraphs, H_1, H_2, of G which have disjoint vertex sets $\{v_1, v_2, \ldots, v_{i+1}\}$ and $\{v_{j-1}, v_j, \ldots, v_n\}$, respectively, are two-connected (restatement of Theorem 6).*
- *In the case that $i + 1 = j$, G is 2-connected and v_i, v_j do not form a cut set (restatement of Theorem 7).*

Lemma 19 can be restated in terms of when a Hamiltonian path will not exist rather than when a Hamiltonian path does exist as follows.

Lemma 20. *A connected unit interval graph G has a Hamiltonian path with endpoints v_i, v_j, $i < j$, except when there exists a cut vertex in $\{v_1, v_2, \ldots, v_i\}$ or $\{v_j, v_j, \ldots, v_n\}$ or when v_i and v_j form a cut set.*

Proof. ($\Rightarrow$) Suppose H_1 is connected but not 2-connected. Then there exists a with $a < i$ such that $v_{a-1}v_{a+1} \notin E(G)$. Then an endpoint of any Hamiltonian path on G must be contained within the set $\{v_1, v_2, \ldots, v_{a-1}\}$ which contains neither v_i nor v_j. Therefore, no desired Hamiltonian path exists. Similarly, if H_2 is connected but not 2-connected, any Hamiltonian path will have an endpoint in $\{v_{b+1}, v_{b+3}, \ldots, v_n\}$, $b > j$, which contains neither v_i nor v_j.

Suppose G is 2-connected and v_i, v_j form a cut set. Then $j = i+1$ and $v_{i-1}v_{i+2} \notin E(G)$. Therefore, an endpoint of any Hamiltonian path on G must be contained within the set $\{v_1, v_2, v_{i-1}\}$ which contains neither v_i nor v_j. Therefore, no desired Hamiltonian path exists.

($\Leftarrow$) Define the following paths.

$$P_1 = \begin{cases} v_i v_{i-2} \cdots v_4 v_2 v_1 v_3 \cdots v_{i-3} v_{i-1} & \text{if } i \text{ is even} \\ v_i v_{i-2} \cdots v_3 v_1 v_2 v_4 \cdots v_{i-3} v_{i-1} & \text{if } i \text{ is odd} \end{cases}$$

$$P_2 = \begin{cases} v_{i+1} v_{i+2} \cdots v_{j-2} v_{j-1} & \text{if } j \neq i+1 \\ \emptyset & \text{if } j = i+1 \end{cases}$$

$$P_3 = \begin{cases} v_{j+1} v_{j+3} \cdots v_{n-2} v_n v_{n-1} v_{n-3} \cdots v_{j+2} v_j & \text{if } j \text{ and } n \text{ have different parity} \\ v_{j+1} v_{j+3} \cdots v_{n-3} v_{n-1} v_n v_{n-2} \cdots v_{j+2} v_j & \text{if } j \text{ and } n \text{ have the same parity} \end{cases}$$

If $i = 1$, then $P_1 = v_1$. If $j = n$, then $P_3 = v_n$.

If H_1 and H_2 are 2-connected with disjoint vertex sets or if G is 2-connected and v_i, v_j do not form a cut set, then $P_1P_2P_3$ is a Hamiltonian path on G with endpoints v_i, v_j. Figure 4.1 illustrates some possibilities for such a path.

□

The following lemma will be used in the proof of Theorem 8.

Lemma 21. *If G is a 2-connected unit interval graph and $\{v_a, v_{a+1}\}$, forms a cut set on G, then there exists a path with endpoints v_a, v_{a+1} which contains the vertices $\{v_1, v_2, \ldots, v_{a-2}, v_{a-1}\}$ and there exists a path with endpoints v_a, v_{a+1} which contains the vertices $\{v_{a+2}, v_{a+3}, \ldots, v_{n-1}, v_n\}$.*

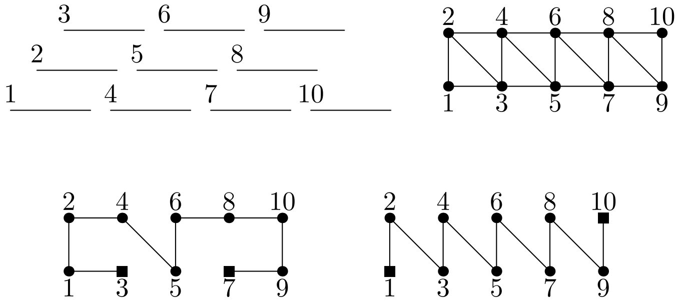

Figure 4.1: A interval representation of a 2-connected unit interval graph is shown in the upper left. The corresponding unit interval graph is shown in the upper right. The lower left graph illustrates a path when $a = 3$ and $b = 7$. The lower right graph illustrates a path when $a = 1$ and $b = 10$.

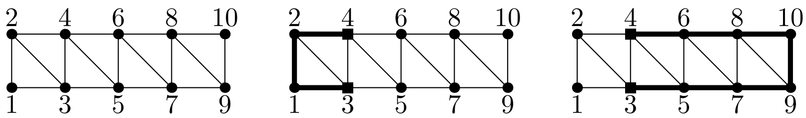

Figure 4.2: The graph on the left is a 2-connected unit interval graph. The graph in the center shows a path in bold with endpoints $a = 3$ and $a + 1 = 4$ which contains $\{1, 2, 3, 4\}$. The graph on the right shows a path in bold with endpoints $a = 3$ and $a + 1 = 4$ which contains $\{3, 4, 5, 6, 7, 8, 9, 10\}$.

Proof. Since $\{v_a, v_{a+1}\}$ form a cut set, $v_{a-1}v_{a+2} \notin E(G)$. By Lemma 19 applied to $G' = \{v_1, v_2, \ldots, v_a, v_{a+1}\}$, there exists a path with endpoints v_a, v_{a+1} which contains the vertices $\{v_1, v_2, \ldots, v_{a-2}, v_{a-1}\}$. By Lemma 19 applied to $G' = \{v_a, v_{a+1}, \ldots, v_n\}$, there exists a path with endpoints v_a, v_{a+1} which contains the vertices $\{v_{a+2}, v_{a+3}, \ldots, v_{n-1}, v_n\}$. Figure 4.2 illustrates these two paths. □

The lower bound in Lemma 17 yields the k-fixed-endpoint path partition number for 2-connected unit interval graphs. If $T = \emptyset$, then say that T forms 0 pairwise distinct cut sets. If all vertices in T form pairwise distinct cut sets, then paths can be formed all to the left or all to the right of the cut sets which leaves a set of vertices on the right end or left end, respectively, which need an additional path. If at least one vertex in T is not part of a pairwise distinct cut set, then paths can

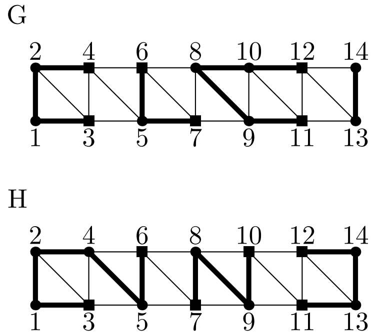

Figure 4.3: The square vertices are in T. In the graph G on the left, the vertices in T form 3 pairwise distinct cut sets. Therefore, $PP(G;T) = 4$. In the graph H on the right, the vertices in T do not form 3 pairwise distinct cut sets. Therefore, $PP(G;T) = 3$. A minimum path partition is shown in bold for each graph.

be created as in Lemma 19 so that all vertices in G are covered by $\left\lceil \frac{|T|}{2} \right\rceil$ paths. This can be proved using induction on the size of T. A leftmost portion of G can be removed and induction applied to the remaining rightmost portion. The vertices in the leftmost portion are determined by the two leftmost vertices in T, t_1 and t_2. Figure 4.3 illustrates the two possible values for $PP(G;T)$ when G is a 2-connected unit interval graph.

Theorem 8. *Let G be a 2-connected unit interval graph. Then*

$$PP(G;T) = \begin{cases} \frac{|T|}{2} + 1 & \textit{if } T \textit{ has } \frac{|T|}{2} \textit{ pairwise distinct cut sets} \\ \left\lceil \frac{|T|}{2} \right\rceil & \textit{otherwise} \end{cases}.$$

Proof. Induct on the size of T.

Base: Suppose $|T| = 0$. Then by Lemma 14, G has a Hamiltonian cycle and thus a Hamiltonian path. Since $T = \emptyset$, T forms 0 pairwise distinct cut sets and $PP(G;T) = \frac{|T|}{2} + 1 = 1$.

Induction: Suppose $|T| \geq 1$. Label the vertices in T as before.

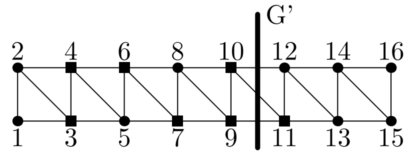

Figure 4.4: Square vertices are in T. $\{3,4\}$, $\{6,7\}$, and $\{9,10\}$ form distinct cut sets. $G' = \{11,12,13,14,15,16\}$ is used for induction.

Case 1: Suppose $\{t_1, t_2\}$ forms a cut set on G. For maximal i, suppose $\{t_1, t_2\}, \{t_3, t_4\}, \{t_5, t_6\}, \ldots, \{t_i, t_{i+1}\}$ form distinct cut sets on G. Let G' be the subgraph of G which contains the vertices $\{v_{j+1}, v_{j+2}, \ldots v_n\}$ where $t_{i+1} = v_j$. See Figure 4.4. G' is 2-connected by Lemma 16. Since i is the largest index which satisfies the above, $\{t_{i+2}, t_{i+3}\}$ either does not form a cut set in G or if $\{t_{i+2}, t_{i+3}\}$ does form a cut set in G, then $t_{i+2} = v_{j+1}$. Therefore, G' will not have $\dfrac{|G' \cap T|}{2}$ distinct cut sets unless $G' \cap T = \emptyset$. By induction,

$$PP(G'; T \cap G') = \begin{cases} 1 & \text{if } T \cap G' = \emptyset \\ \left\lceil \dfrac{|T \cap G'|}{2} \right\rceil & \text{if } T \cap G' \neq \emptyset \end{cases}$$
$$= \begin{cases} 1 & \text{if } T \cap G' = \emptyset \\ \left\lceil \dfrac{|T| - (i+1)}{2} \right\rceil & \text{if } T \cap G' \neq \emptyset \end{cases}.$$

Lemma 21 can be applied to the subgraphs $\{v_1, v_2, \ldots, v_{f(1)}, v_{f(2)}\}$, $\{v_{f(2)+1}, v_{f(2)+2}, \ldots, v_{f(3)}, v_{f(4)}\}$, …, $\{v_{f(i-1)+1}, v_{f(i-1)+2}, \ldots, v_{f(i)}, v_{f(i)+1}\}$. Then $G - G'$ can be covered with $\dfrac{i+1}{2}$ paths and a path partition on G is these paths

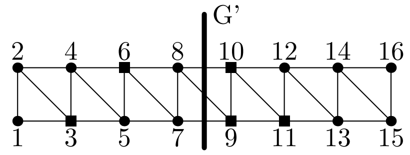

Figure 4.5: Square vertices are in T. $\{t_1, t_2\} = \{3, 6\}$ does not form a cut set. $G' = \{9, 10, 11, 12, 13, 14, 15, 16\}$ is used for induction.

along with the minimum path partition on G' found inductively. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G';T \cap G') + \frac{i+1}{2} \\ &= \frac{i+1}{2} + \begin{cases} 1 & \text{if } T \cap G' = \emptyset \\ \left\lceil \frac{|T| - (i+1)}{2} \right\rceil & \text{if } T \cap G' \neq \emptyset \end{cases} \\ &= \begin{cases} \frac{|T|}{2} + 1 & \text{if T forms } \frac{|T|}{2} \text{ pairwise distinct cut sets} \\ \left\lceil \frac{|T|}{2} \right\rceil & \text{otherwise} \end{cases} \end{aligned}$$

since $i + 1$ is even.

Case 2: Suppose $\{t_1, t_2\}$ does not form a cut set on G. Let G' be the subgraph of G which contains the vertices $\{t_3, v_{f(3)+1}, \ldots, v_n\}$. See Figure 4.5. By Lemma 16, G' is 2-connected. By induction, $PP(G'; T \cap G') = \left\lceil \frac{|T \cap G'|}{2} \right\rceil = \left\lceil \frac{|T| - 2}{2} \right\rceil = \left\lceil \frac{|T|}{2} \right\rceil - 1$ since G' cannot have $\frac{|T \cap G'|}{2}$ pairwise distinct cut sets since t_3 is leftmost in G'. Since $G - G'$ is 2-connected by Lemma 16, $G - G'$ has a Hamiltonian path with endpoints t_1 and t_2 by Lemma 19. Then a path partition on G is this path with

t_1, t_2 endpoints and the minimum path partition on G' found inductively. Therefore,

$$\begin{aligned} PP(G;T) &\leq PP(G';T \cap G') + 1 \\ &= \left\lceil \frac{|T}{2} \right\rceil - 1 + 1 \\ &= \left\lceil \frac{|T|}{2} \right\rceil \end{aligned}$$

since T does not form $\frac{|T|}{2}$ pairwise distinct cut sets on G.

Therefore,

$$PP(G;T) = \begin{cases} \frac{|T|}{2} + 1 & \text{if } T \text{ has } \frac{|T|}{2} \text{ pairwise distinct cut sets} \\ \left\lceil \frac{|T|}{2} \right\rceil & \text{otherwise} \end{cases}.$$

□

Note that when T forms $\frac{|T|}{2}$ pairwise distinct cut sets on G, the minimum path partition created in the proof of Theorem 8 contains $\frac{|T|}{2}$ paths with two endpoints in T and one path with no endpoints in T. Alternatively, a minimum path partition exists in this case which has two paths with exactly one endpoint in T and v_1 and v_n are endpoints of paths when $v_1, v_n \notin T$. The following corollaries will be useful when considering the k-fixed-endpoint path partition problem on unit interval graphs with cut vertices.

Corollary 1. *Let G be a 2-connected unit interval graph where T forms $\frac{|T|}{2}$ cut sets. There exists a minimum path partition on G with respect to T which contains two paths with exactly one endpoint in T and v_1, v_n are endpoints of paths.*

Proof. Apply Theorem 8 to G with the set $T' = T + \{v_1, v_n\}$. Then $PP(G;T') = \left\lceil \frac{|T'|}{2} \right\rceil = \frac{|T|}{2} + 1$ and v_1, v_n are ends of paths in the minimum path partitions on G with respect to T' and thus available endpoints in the minimum path partitions on G with respect to T. □

Corollary 2. *Let G be a 2-connected unit interval graph where T has odd size. There exists a minimum path partition on G with respect to T such that v_1 or v_n is an available endpoint.*

Proof. Apply Theorem 8 to G with the set $T' = T + v_1$. Then $PP(G;T) = \left\lceil \frac{|T'|}{2} \right\rceil = \left\lceil \frac{|T|}{2} \right\rceil$ and v_1 is an end of a path in the minimum path partition on G with respect to T' and thus an available endpoint in the minimum path partition on G with respect to T. Similarly, Theorem 8 applied to G with the set $T' = T + v_n$ shows there exists a minimum path partition on G with respect to T where v_n is an available endpoint. □

Corollary 3. *Let G be a 2-connected unit interval graph where T has even size and T does not form $\frac{|T|}{2}$ pairwise distinct cut sets. There exists a path partition on G with respect to T with size $\frac{|T|}{2} + 1$ such that v_1 and v_n are available endpoints except when $v_1 = t_1$,$v_n = t_{|T|}$, and $\{t_2, t_3, \dots t_{|T|-1}\}$ form $\frac{|T|}{2} - 1$ pairwise disjoint cut sets.*

Proof. Apply Theorem 8 to G with the set $T' = T + \{v_1, v_n\}$. Then $PP(G;T) = \left\lceil \frac{|T'|}{2} \right\rceil = \left\lceil \frac{|T|}{2} \right\rceil + 1$ and v_1 and v_n are ends of paths in the minimum path partition on G with respect to T' and thus available endpoints in the minimum path partition on G with respect to T. □

4.2 Connected Unit Interval Graphs

In this section, a characterization for the k-fixed-endpoint path partition number on connected unit interval graphs is considered. Connected unit interval graphs have a structure similar to linear blocks graphs. Each maximal 2-connected subgraph (or block) of a unit interval graph is a 2-connected unit interval graph, and these blocks can be ordered linearly from left to right. The characterization for the k-fixed-endpoint path partition number on unit interval graphs is the same as the characterization for linear block graphs except there is one additional situation which causes a vertex to be added to T'. First, definitions and notation will be recalled then lemmas which lead into the min-max theorem for the k-fixed-endpoint path partition number of unit interval graphs.

A connected unit interval graph has β blocks where each block is a 2-connected unit interval graph. Label each block B_i, $1 \leq i \leq \beta$, where the vertices in B_i have smaller indices than the vertices in B_j when $i < j$. Label each cut vertex c_i, $1 \leq i \leq \beta - 1$. Note that the cut vertices have two or three labels, c_i, $v_{g(i)}$, and potentially t_j where g maps the index of c to the index of v.

Recall Λ is the set of cut vertices in a unit interval graph G. A block B_i is *left* of block B_j if $i < j$. A block B_i is *right* of block B_j if $i > j$. The *leftmost* block has smallest index while the *rightmost* block has largest index. For a unit interval graph G, let $\mathcal{P}(W)$, $W \subseteq \Lambda$, be a *partition of* G which is a set of $|W| + 1$ induced subgraphs formed by removing the set of edges $\{v_iv_j \in E(G) | v_i \in W, i < j\}$ where $E(G)$ is the set of all edges in G.

Let i_j be the index of the jth leftmost vertex in W; that is, $c_{i_1}, c_{i_2}, \ldots, c_{i_{|W|}}$ where $i_1 < i_2 < \ldots < i_{|W|}$. Then each part P_j of $\mathcal{P}(W)$ can be defined as $P_1 = \bigcup_{l=1}^{i_1} B_1$, $P_j = \bigcup_{l=i_{j-1}+1}^{i_j} B_l - c_{i_{j-1}}$ for $2 \leq j \leq |W|$ and $P_{|W|+1} = \bigcup_{l=i_{|W|}+1}^{\beta} B_l - c_{i_{|W|}}$. Note that B_{i_j} is rightmost in part P_j and $\mathcal{P}(W)$ contains $|W| + 1$ parts.

In addition to the vertices which are added to T' for linear block graphs, when the leftmost block B_1^j of a part P_j has $\frac{|P_j \cap T|}{2}$ pairwise distinct cut sets in the interior of the block B_1^j, then a vertex to the right of all vertices in $P_j \cap T$ in block B_1^j will be added to T.

Definition 25. *For a partition $\mathcal{P}(W)$ on a connected unit interval graph G, let $T'(W) = T \cup \{u_j | j \in J\}$ where $J \subseteq [|W| + 1]$, $j \in J$ when the leftmost block B_1^j of P_j with cut vertex c satisfies $(B_1^j - c) \cap T = \emptyset$ or when $(B_1^j - c)$ contains $\frac{|(B_1^j - c) \cap T|}{2}$ pairwise distinct cut sets, and u_j is an arbitrary vertex in the interior of B_1^j which is left of all vertices in $B_1^j \cap T$.*

When W is clear from context, T' will be used to represent $T'(W)$.

Lemma 22. *For a unit interval graph G with T and with W a subset of the cut vertices, $PP(P_j; T \cap P_j) = PP(P_j; T'(W) \cap P_j)$ where T' is as defined in Definition 25.*

Proof. Let $\mathcal{Q}$ be a minimum path partition on G with respect to T. Suppose $\mathcal{Q}$ restricted to P_j has no endpoint in the interior of the leftmost block B_1^j of part P_j of the partition. Then a path must enter B_1^j, traverse all vertices in the interior, and leave B_1^j. Paths can only enter and leave a block at cut vertices. Thus, B_1^j must have two cut vertices to satisfy the path condition. Then B_1^j cannot be leftmost in P_j since the leftmost block has exactly one cut vertex. Therefore, a contradiction exists, B_1^j, $1 \leq j \leq |W|+1$, must contain an end of a path of $\mathcal{Q}$ restricted to P_j.

Suppose B_1^j has $\dfrac{|B_1^j \cap T|}{2}$ pairwise distinct cut sets. Then by Theorem 8 applied to $B_1^j - c$ with $T \cap (B_1^j - c)$, B_1^j has $\dfrac{|B_1^j \cap T|}{2} + 1$ paths. By Lemma 21, there exists a path in $\mathcal{Q}$ restricted to P_j with an endpoint in B_1^j to the left of all vertices in $T \cap (B_1^j - c)$.

Therefore, $PP(P_j; T \cap P_j) = PP(P_j; T'(W) \cap P_j)$. $\square$

Note that

$$|P_j \cap T'| = \begin{cases} |P_j \cap T| + 1 & \text{if } (B_1^j - c) \cap T = \emptyset \text{ or} \\ & B_1^j \text{ contains } \dfrac{|(B_1^j - c) \cap T|}{2} \text{ pairwise distinct cut sets} \\ |P_j \cap T| & \text{otherwise} \end{cases} \cdot$$

Lemma 23. *Let G be a unit interval graph G and T be a set of k vertices. Then $PP(G;T) \geq \max\limits_{W \subseteq \Lambda} \left\{ \left\lceil \dfrac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \dfrac{|P_j \cap T|}{2} \right\rfloor \right\}$ where T' is defined in Definition 25.*

When $W = \emptyset$, we assume that

$$\max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\} = \left\lceil \frac{|G \cap T'|}{2} \right\rceil .$$

Proof. Consider a partition $\mathcal{P}(W)$ and a minimum path partition $\mathcal{Q}$ on G with respect to T. For each part $P_j \subseteq \mathcal{P}(W)$ count the number of paths in Q which have right endpoints in P_j. If a path in $\mathcal{Q}$ extends to the right of P_j, then P_j contains part of at least $\left\lceil \frac{|P_j \cap T'| + 1}{2} \right\rceil$ paths and thus $\left\lceil \frac{|P_j \cap T'| + 1}{2} \right\rceil - 1 = \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor$ right ends. If no path in $\mathcal{Q}$ extends to the right of P_j, then P_j contains at least $\left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor$ right ends. This holds for all P_j, $1 \leq j \leq |W|$. The rightmost part $P_{|W|+1}$ still needs to be considered. $P_{|W|+1}$ has $|P_{|W|+1} \cap T'|$ ends of paths. $\mathcal{Q}$ cannot have a path which extends to the right since $P_{|W|+1}$ is rightmost. Then $P_{|W|+1}$ has at least $\left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil$ right endpoints of paths in $\mathcal{Q}$. Therefore, $\mathcal{P}(W)$ contains at least $\left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor$ right endpoints in $\mathcal{Q}$. This holds for all $W \subseteq \Lambda$. Therefore, $PP(G;T) \geq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$. □

The lower bound in Lemma 23 is tight for unit interval graphs. The "best" partition needs to be chosen as before. The "best" partition W^* can be found by working left to right. If the interior of the leftmost block B contains no vertices in T or contains $\frac{|(B - c) \cap T|}{2}$ pairwise distinct cut sets, then the leftmost cut vertex c_i to be put into W^* will have the smallest index i for which $\left| T \cap \left(\bigcup_{l=1}^{i} B_l \right) \right|$ is odd. If the interior of the leftmost block contains vertices in T which do not form pairwise distinct cut sets, then the leftmost cut vertex c_i to be put into W^* will have the smallest index i for which $\left| T \cap \left(\bigcup_{l=1}^{i} B_l \right) \right|$ is even. Then repeat this process for $G - \bigcup_{l=1}^{i} B_l$. See Figure 4.6.

Theorem 9. *Let G be a unit interval graph G and T be a set of k vertices. Then* $PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$ *where T' is defined in Definition 25.*

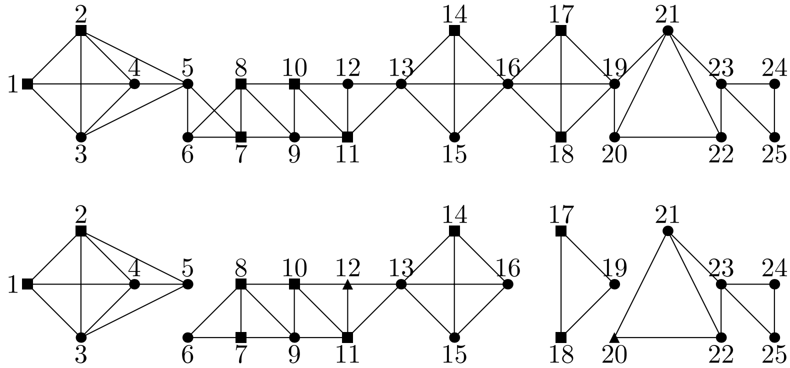

Figure 4.6: The square vertices are in T. The triangle vertices are in $T' - T$. The bottom graph is an example of a "best" partition for G with respect to T.

When $W = \emptyset$, we assume

$$\left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor = \left\lceil \frac{|G \cap T'|}{2} \right\rceil$$

Proof. Induct on the number of blocks β.

Base: Suppose $\beta = 1$. Then $\Lambda = \emptyset$ and $\max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$ $= \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil$. If the vertices in T form $\frac{|T|}{2}$ pairwise disjoint cut sets, then $|T'| = |T| + 1$ and $|T'|$ is odd. By Theorem 8,

$$\begin{aligned} PP(G;T) &= \begin{cases} \frac{|T|}{2} + 1 & \text{if } T \text{ has } \frac{|T|}{2} \text{ pairwise disjoint cut sets} \\ \left\lceil \frac{|T|}{2} \right\rceil & \text{otherwise} \end{cases} \\ &= \begin{cases} \left\lceil \frac{|T'|}{2} \right\rceil & \text{if } T \text{ has } \frac{|T|}{2} \text{ pairwise disjoint cut sets} \\ \left\lceil \frac{|T'|}{2} \right\rceil & \text{otherwise} \end{cases} \\ &= \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil \end{aligned}$$

Induction: Suppose $\beta \geq 2$. Let j' be the smallest index such that if the vertices in $(B_1 - c_1) \cap T$ form $\dfrac{|(B_1 - c_1) \cap T|}{2}$ pairwise distinct cut sets, then $\left|\left(\bigcup_{l=1}^{j'} B_l\right) \cap T\right|$ is odd and if the vertices in $(B_1 - c_1) \cap T$ do not form $\dfrac{|(B_1 - c_1) \cap T|}{2}$ pairwise distinct cut sets, then $\left|\left(\bigcup_{l=1}^{j'} B_1\right) \cap T\right|$ is even. Let $Q_1 = \bigcup_{l=1}^{j'} B_l$. Note that $G - Q_1$ is a unit interval graph. Then by induction,

$$PP(G - Q_1; T - Q_1) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$$
$$= \left\lceil \frac{|P_{|W^*|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor.$$

Let $W' = W^* + c_{j'}$ and $Q_1, Q_2, \ldots, Q_{|W'|+1}$ be the parts of $\mathcal{P}(W')$ on G. Then $P_i = Q_{i+1}$ for $1 \leq i \leq |W^*| + 1$.

If $j' = 1$, then by Theorem 8, Q_1 can be covered by $\dfrac{|Q_1 \cap T'|}{2}$ paths. If $j' > 1$, then for odd i, $1 \leq i \leq |Q_1 \cap T'|$, form the paths $H_i = H_{i1}H_{i2}H_{i3}$ with ends t_i and t_{i+1} where H_{i1}, H_{i2}, and H_{i3} are defined below with $\gamma_1 \neq \gamma_2$.

$$H_{i1} = \begin{cases} v_{f(i)}v_{f(i)-2} \cdots v_{g(\gamma_1)+1} \cdots v_{f(i)-3}v_{f(i)-1} & \text{if } t_i \in B_{\gamma_1}, t_{i-1} \in B_{\gamma_2} \\ v_{f(i)} & \text{if } t_i, t_{i-1} \in B_{\gamma_1} \end{cases}$$

$$H_{i2} = \begin{cases} v_{f(i)+1}v_{f(i)+2} \cdots v_{f(i+1)-1} & \text{if } t_i, t_{i+1} \in B_{\gamma_1} \\ v_{f(i)+1}v_{f(i)+2} \cdots v_{g(\gamma_1)} \cdots v_{g(\gamma_2)} \cdots v_{f(i+1)-1} & \text{if } t_i \in B_{\gamma_1}, t_{i+1} \in B_{\gamma_2} \end{cases}$$

$$H_{i3} = \begin{cases} v_{f(i+1)+1}v_{f(i+1)+3} \cdots v_{g(\gamma_1)} \cdots v_{f(i+1)+2}v_{f(i+1)} & \text{if } t_{i+1} \in B_{\gamma_1}, t_{i+2} \in B_{\gamma_2} \\ v_{f(i+1)+1}v_{f(i+1)+3} \cdots v_{f(i+2)-1} \cdots v_{f(i+1)+2}v_{f(i+1)} & \text{if } t_{i+1}, t_{i+2} \in B_{\gamma_1} \end{cases}$$

Figure 4.7 illustrate examples of these paths. This yields $\dfrac{|Q_1 \cap T'|}{2}$ paths which cover Q_1. Then a path partition on G with respect to T is a minimum path partition on $G - Q_1$ with respect to $T - Q_1$ found inductively with $\dfrac{|Q_1 \cap T'|}{2} = \left\lfloor \dfrac{|Q_1 \cap T'|}{2} \right\rfloor$

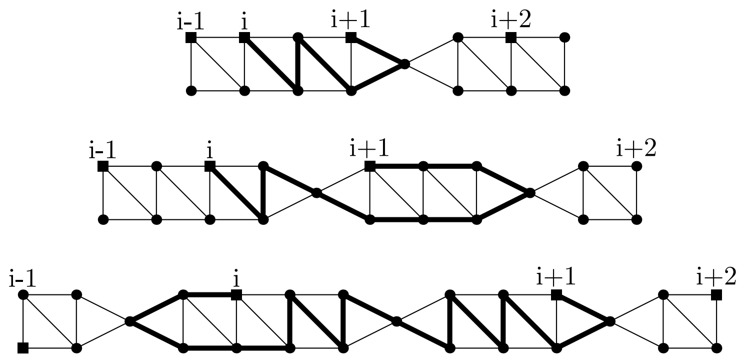

Figure 4.7: Square vertices are in T. The labels indicate $t_{i-1}, t_i, t_{i+1}, t_{i+2}$. The three graphs illustrate three possible paths in bold with endpoints t_i and t_{i+1} depending on the placement of $t_{i-1}, t_i, t_{i+1}, t_{i+2}$.

additional paths. Therefore,

$$\begin{aligned}
PP(G;T) &\leq PP(G-Q_1;T-Q_1) + \left\lfloor \frac{|Q_1 \cap T'|}{2} \right\rfloor \\
&= \left\lceil \frac{|P_{|W^*|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W^*|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor + \left\lfloor \frac{|Q_1 \cap T'|}{2} \right\rfloor \\
&= \left\lceil \frac{|Q_{|W'|+1} \cap T'|}{2} \right\rceil + \sum_{j=2}^{|W'|} \left\lfloor \frac{|Q_j \cap T'|}{2} \right\rfloor + \left\lfloor \frac{|Q_1 \cap T'|}{2} \right\rfloor \\
&= \left\lceil \frac{|Q_{|W'|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W'|} \left\lfloor \frac{|Q_j \cap T'|}{2} \right\rfloor \\
&\leq \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}.
\end{aligned}$$

Therefore, $PP(G;T) = \max_{W \subseteq \Lambda} \left\{ \left\lceil \frac{|P_{|W|+1} \cap T'|}{2} \right\rceil + \sum_{j=1}^{|W|} \left\lfloor \frac{|P_j \cap T'|}{2} \right\rfloor \right\}$ when G is a unit interval graph.

□

Chapter 5

Conclusion

Characterization theorems for the k-fixed-endpoint path partition problem were described in this dissertation. A direct characterization for trees was discussed in Chapter 1. A characterization for threshold graphs was discussed in Chapter 2. This characterization is the lower bound in Lemma 3 expect in a special case when an additional path is required. A characterization for block graphs is discussed in Chapter 3. Additionally, Chapter 3 contains a characterization for the k-fixed-endpoint path partition number for linear block graphs. This characterization is simpler than that for block graphs and provides insight into the characterization for unit interval graphs. The characterization for unit interval graphs is found in Chapter 4. A characterization for 2-connected unit interval graphs is also in Chapter 4. This characterization is $\left\lceil \frac{|T|}{2} \right\rceil$ expect in a special case when an additional path is required.

These characterizations could lead to simpler or more efficient algorithms for the k-fixed-endpoint path partition problem on these graph classes. They may also lead to certifying algorithms.

Since an efficient algorithm exists for the k-fixed-endpoint path partition problem for interval graphs when $k = 1$, this class would be a logical graph class to consider next to describe a characterization theorem for the k-fixed-endpoint path partition problem. Potentially the methods in Chapters 3 and 4 would extend to interval graphs when $k = 1$. An efficient algorithm for the k-fixed-endpoint path partition problem on cographs also exists. Cographs would be another logical graph class to consider to describe a characterization theorem for the k-fixed-endpoint path partition number.

Additionally, 2-trees would be a graph class to consider to determine a characterization theorem for the path partition number as well as the k-fixed-endpoint path partition number. No efficient algorithms have been published for these problems on K-trees, $K \geq 2$. 2-trees are an extension of trees and have many applications.

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PUBLICATIONS

- "The k-Fixed-Endpoint Path Partition Problem on Trees and Block Graphs." *in preparation*
- "Elliptic Curves and Cryptology." Meredith College Honors Thesis.
- "Thermal Imaging to Recover a Defect in Two- and Three-Dimensional Objects." Rose-Hulman Institute of Technology REU report.

MEMBERSHIPS

American Mathematical Society, August 2007-Present.

Graduate Student Intercollegiate Mathematics Seminar, August 2009-Present.

GRANTS

- Lehigh University Mathematics Department Summer Research Fellowship, 2012.

OTHER EXPERIENCE

Lehigh University Math Study/Help Center: Fall 2007-Present

- Bethlehem, PA.
- Tutored undergraduate and graduate students taking calculus, differential equations, linear algebra, probability, statistics, and analysis.

Meredith College Learning Center: Fall 2005-Spring 2007

- Raleigh, NC
- Tutored fellow students in math courses from algebra to calculus as well as introductory physics and introductory french.

Honors Thesis: Fall 2006-Spring 2007

- Meredith College, Raleigh, NC
- Performed research in cryptology with Dr Jennifer Hontz. Used Java to encrypt and decrypt messages.

Research Experiences for Undergraduates: Summer 2006

- Rose-Hulman Institue of Technology, Terre Haute, IN
- Performed research in inverse problems. Presented findings at conference for Indiana REUs.

AWARDS

- TA of the Year. Spring 2011. Lehigh University, Bethlehem, PA.
- Paschal Scholar. Fall 2005-Spring 2007. Meredith College, Raleigh, NC.
- Honors Scholar. Fall 2003-Spring 2007. Meredith College, Raleigh, NC.
- Kappa Nu Sigma. Inducted Spring 2006. Meredith College, Raleigh, NC.
- Pi Mu Epsilon. Inducted Spring 2005. Meredith College, Raleigh, NC.
- Vivian Kranes Mathematics Competition Winner. Spring 2004. Meredith College, NC.
- Alpha Lambda Delta. Inducted Spring 2004. Meredith College, Raleigh, NC.

COMPUTER SKILLS: MyMathLab, Course Site, Blackboard, LaTeX, Excel, PowerPoint