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Some Uniqueness and Rigidity Results on Gradient Ricci Solitons

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Some Uniqueness And Rigidity Results On
Gradient Ricci Solitons

by

Qiang Chen

A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
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Some Uniqueness And Rigidity Results On Gradient Ricci Solitons

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Contents

Abstract	1
1 Preliminaries on Ricci Solitons	2
1.1 Definitions and Basic Identities	3
1.2 Examples of Ricci Solitons	7
1.3 Geometry of Gradient Ricci Solitons	10
1.3.1 Geometry of Gradient Shrinking Ricci Solitons	10
1.3.2 Geometry of Gradient Steady and Expanding Ricci Solitons . .	15
2 Bach Flat and Locally Conformally Flat Gradient Ricci Solitons	17
2.1 Major Results	17
2.1.1 Locally Conformally Flat Gradient Steady Ricci Solitons . . .	17
2.1.2 Bach Flat Shrinking Ricci Solitons	19
2.2 The covariant 3-tensor D_{ijk} and its Relation to Geometry	20
2.3 Proof of the Main Theorems	26
2.3.1 Proof of Theorem 2.1.1	26
2.3.2 Proof of Theorem 2.1.3 and Theorem 2.1.4	28
2.3.3 Proof of Theorem 2.1.2	35
3 Kähler-Ricci Solitons with Harmonic Bochner tensor	41
3.1 Basic Definitions and Identities	41
3.2 The Result and The Proof	43
4 Bach-flat Quasi-Einstein Manifolds	51

Abstract

This thesis contains my work during Ph.D. studies under the guidance of my advisor Huai-Dong Cao.

We initiated our research on Perelman's Conjecture stating that the three-dimensional steady gradient Ricci soliton is the Bryant soliton up to scaling, and we managed to prove this with the assumption that the metric is locally conformally flat.

Later, exploring the Bach tensor, we managed to show that a four-dimensional Bach flat shrinking Ricci soliton is either Einstein, the quotient of a Gaussian soliton \mathbb{R}^4 or the product $\mathbb{S}^3 \times \mathbb{R}$. For dimension $n \geq 5$, a Bach flat Ricci soliton is either Einstein, the quotient of Gaussian soliton \mathbb{R}^4 or the product of an Einstein manifold with a line, namely $\mathbb{N}^{n-1} \times \mathbb{R}$. A similar argument can be carried over to steady Ricci solitons with some additional assumptions.

In the proof we constructed a covariant 3-tensor called the D -tensor which is verified to be a key link for the geometry of Ricci solitons and the well-known Weyl curvature, Cotton tensor and Bach tensor.

As an extended study, joint with Meng Zhu, we establish the rigidity result for Kähler-Ricci solitons with harmonic Bochner tensor. Joint with Chenxu He, we also applied the Bach-flat argument to quasi-Einstein manifolds and prove the classification theorem.

Chapter 1

Preliminaries on Ricci Solitons

The concept of Ricci solitons was introduced by R. Hamilton [37] in the mid 1980's. The importance of Ricci solitons to the Ricci flow can be illustrated as follows:

- Ricci solitons are natural generalizations of Einstein metrics.
- Ricci solitons correspond to self-similar solutions to the Ricci flow.
- The Li-Yau-Hamilton inequality becomes an equality on expanding solitons.
- Ricci solitons often appear as singularity models, i.e., the dilation limits of singular solutions to the Ricci flow. For instance, type II and type III singularity models are steady and expanding solitons respectively; under certain conditions, type I singularity models are shrinking solitons.
- Ricci solitons are critical points of entropy functionals. For example, compact gradient steady solitons and shrinking solitons are the critical points of Perelman's λ and ν entropies, respectively.

In this chapter, we will give the definition and introduce some well-known results on Ricci solitons.

1.1 Definitions and Basic Identities

In differential geometry, the Ricci flow is an intrinsic geometric flow. It is a process that deforms the metric g_{ij} of a Riemannian manifold by its Ricci tensor R_{ij} ,

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \quad (1.1.1)$$

which is formally analogous to the diffusion of heat, smoothing out irregularities in the metric. It is the primary tool used in the Hamilton-Perelman solution to the Poincaré conjecture. A very important part in the study of the Ricci flow is to understand the geometry of Ricci solitons:

Definition 1.1.1. *A Ricci soliton is a Riemannian manifold whose metric satisfies*

$$R_{ij} + \mathcal{L}_V g_{ij} = \rho g_{ij}. \quad (1.1.2)$$

Here V is a smooth vector field, \mathcal{L} is the Lie derivative, and ρ is a real constant.

Ricci soliton metrics stay self-similar under the Ricci flow, and they are divided into three types called shrinking ($\rho > 0$), steady ($\rho = 0$) or expanding ($\rho < 0$).

Definition 1.1.2. *A gradient Ricci soliton is a special kind of Ricci soliton whose vector field V is the gradient of some potential function f , namely, $V = \frac{1}{2}\nabla f$, and hence for gradient Ricci solitons equation (1.1.2) reduces to:*

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}. \quad (1.1.3)$$

Definition 1.1.3. *An Einstein metric is a Riemannian metric whose Ricci curvature is constant, or namely*

$$R_{ij} = \rho g_{ij}.$$

It is easy to see that Einstein manifolds are necessarily Ricci solitons.

Remark 1.1.1. *Given a Ricci soliton (M, g_0, V) satisfying (1.1.2), it is easy to check the following self-similar solution to the Ricci flow with initial metric g_0 :*

$$g(t) = (1 - 2\rho t)\phi_t^* g_0,$$

where ϕ_t is the one-parameter family of diffeomorphisms generated by $\frac{1}{1-2\rho t}V$.

Moreover, by a result of Z.-H. Zhang [61], for a complete gradient steady or shrinking Ricci soliton, the family of diffeomorphisms $\{\phi_t\}$ exists on $(-\infty, T)$ for some T .

Before we start the computations, let us fix the notation and conventions used in this paper.

Let (M^n, g_{ij}) be a Riemannian manifold of dimension n , and denote by Γ_{ij}^k , Rm , Rc and R the Christoffel symbol, Riemannian curvature tensor, Ricci curvature tensor and scalar curvature respectively. In local coordinates $\{x^1, x^2, \dots, x^n\}$, we have the expression:

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \\ Rm(\partial_i, \partial_j, \partial_k, \partial_l) &= R_{ijkl} = g_{kp} \left(\frac{\partial \Gamma_{jl}^p}{\partial x^i} - \frac{\partial \Gamma_{il}^p}{\partial x^j} + \Gamma_{iq}^p \Gamma_{jl}^q - \Gamma_{jq}^p \Gamma_{il}^q \right) \\ Rc(\partial_i, \partial_k) &= R_{ik} = g^{jl} R_{ijkl} \\ R &= g^{ik} R_{ik}.\end{aligned}$$

Here, we use Einstein's convention which means that we take sum over repeated indices. For example $g^{ij}h_{jk} = \sum_{j=1}^n g^{ij}h_{jk}$. Also g^{ij} means the inverse matrix of the metric tensor g_{ij} , namely $g_{ij}g^{jk} = \delta_i^k$.

In some circumstances, we may also use \langle, \rangle to represent the metric tensor. Namely,

$$\langle X, Y \rangle = g_{ij}X^iY^j$$

The covariant derivative is given by

$$\nabla_i V_j = \frac{\partial V_j}{\partial x^i} - \Gamma_{ij}^k V_k.$$

We have the following Ricci identity:

$$\nabla_i \nabla_j V_k - \nabla_j \nabla_i V_k = R_{ijkl} V_m g^{lm}.$$

Furthermore, throughout this paper, we will always use normal coordinates near a given point where tensorial computation is performed. This means that for any

point p , we choose local coordinates $\{x^1, x^2, \dots, x^n\}$ near p such that $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$. Therefore, we may lower all of the indices and the Ricci identities above become

$$\nabla_i \nabla_j V_k - \nabla_j \nabla_i V_k = R_{ijkl} V_l.$$

Lemma 1.1.1. (Hamilton [39]) *Let (M^n, g_{ij}, f) be a complete gradient Ricci soliton (1.1.3). Then,*

$$R + \Delta f = n\rho \tag{1.1.4}$$

$$\nabla_j R_{ik} - \nabla_i R_{jk} = R_{ijkl} \nabla_l f \tag{1.1.5}$$

$$\nabla_i R = 2R_{ij} \nabla_j f, \tag{1.1.6}$$

and

$$R + |\nabla f|^2 - 2\rho f = C_0 \tag{1.1.7}$$

for some constant C_0 . Here R denotes the scalar curvature.

Proof. Taking the trace of equation (1.1.3) yields (1.1.4).

From

$$\nabla_j R_{ik} - \nabla_i R_{jk} = -\nabla_j \nabla_i \nabla_k f + \nabla_i \nabla_j \nabla_k f = R_{ijkl} \nabla_l f$$

we obtain (1.1.5)

Using the contracted second Bianchi identity, the first equality below, and the definition of gradient Ricci soliton (1.1.3), it follows that

$$\nabla_i R = 2\nabla_j R_{ij} = -2\nabla_j \nabla_i \nabla_j f = -2\nabla_i(\Delta f) - 2R_{il} \nabla_l f;$$

Applying the trace version of (1.1.3), namely $R + \Delta f = \rho n$, we derive

$$\nabla_i R = 2\nabla_i R - 2R_{il} \nabla_l f.$$

Hence (1.1.6) holds.

To prove (1.1.7), just verify that the covariant derivative of the left hand side equals zero:

$$\begin{aligned}
\nabla_i(R + |\nabla f|^2 - 2\rho f) &= \nabla_i R + 2\nabla_i \nabla_j f \nabla_j f - 2\rho \nabla_i f \\
&= 2R_{ij} \nabla_j f + 2\nabla_i \nabla_j f \nabla_j f - 2\rho \nabla_i f \\
&= 2(R_{ij} + \nabla_i \nabla_j f - \rho g_{ij}) \nabla_j f \\
&= 0.
\end{aligned}$$

□

Remark 1.1.2. *For shrinking Ricci solitons, it is always possible to rescale the metric and shift the function f by a constant, such that:*

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij} \text{ and } R + |\nabla f|^2 - f = 0. \quad (1.1.8)$$

Without ambiguity, when we refer to shrinking Ricci solitons later, we mean the shrinking Ricci solitons with this normalization.

Proposition 1.1.1. (Hamilton [39], Ivey [43]) *Any compact steady or expanding gradient Ricci soliton must be Einstein.*

Proof. We present the proof for the expanding case. The proof for the steady case is similar yet simpler. Let (M^n, g_{ij}) be a compact gradient expanding soliton satisfying (1.1.3) for some $\rho < 0$.

From Lemma 1.1.1, we have

$$R + \Delta f = n\rho$$

and

$$R + |\nabla f|^2 - 2\rho f = C_0$$

Taking the difference:

$$\Delta f - |\nabla f|^2 = -2\rho f + C$$

Thus using the maximum principle, we obtain:

$$\begin{aligned} -2\rho f|_{max} + C &\leq 0 \\ -2\rho f|_{min} + C &\geq 0 \end{aligned}$$

which forces $f|_{max} = f|_{min}$, or equivalently implies that f is constant, which in turn implies the soliton is Einstein. □

From the proposition above, in low dimensions ($n = 2$ or 3), there are no compact gradient steady or expanding Ricci solitons other than those of constant curvature. It turns out that this is also true for compact shrinking Ricci solitons.

Proposition 1.1.2. (Hamilton [40] for $n = 2$, Ivey [43] for $n = 3$) *In dimension $n \leq 3$, there are no compact gradient shrinking Ricci solitons other than those of constant positive curvature.*

Remark 1.1.3. *When $n \geq 4$, we can no longer expect such a proposition for compact shrinking solitons. Some non-Einstein compact shrinking Ricci soliton examples do exist.*

1.2 Examples of Ricci Solitons

In the first section, we saw that compact gradient steady and expanding solitons are Einstein. This is also true for compact shrinking Ricci solitons in low dimensions. However, as remarked, examples of nontrivial compact gradient shrinking Ricci solitons do exist when $n \geq 4$. Also there exist complete noncompact gradient steady, shrinking and expanding Ricci solitons which are not Einstein. In this section, we will present some of these examples.

- **Examples of Compact Shrinking Solitons**

Example 1.2.1. The first example of a compact non-Einstein gradient shrinking Ricci soliton was found by H.-D. Cao [9] and N. Koiso [45] independently.

They proved the existence of a $U(n)$ symmetric gradient shrinking Kähler-Ricci soliton structure on the twisted projective line bundle $\mathbb{P}(L^k \oplus L^{-k})$ over $\mathbb{C}\mathbb{P}^{n-1}$ for $n \geq 2$, where L is the hyperplane line bundle over $\mathbb{C}\mathbb{P}^{n-1}$ and $1 \leq k \leq n - 1$. In particular, in real dimension 4, it implies that there is a shrinking Kähler-Ricci soliton structure on $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$.

Example 1.2.2. In [60], Wang-Zhu proved that there is a unique Kähler-Ricci soliton structure on any toric Kähler manifold with positive first Chern class and nonvanishing Futaki invariant. In particular, in complex dimension 2, this means that a Kähler-Ricci soliton exists on $\mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}$ with $U(1) \times U(1)$ symmetry.

- **Examples of Noncompact Shrinking Solitons**

Example 1.2.3. Feldman-Ilmanen-Knopf [32] discovered the first example of a complete noncompact non-Einstein gradient shrinking Ricci soliton. They found a family of shrinking Kähler-Ricci solitons with $U(n)$ symmetry and a cone-like end at infinity on the twisted line bundle over $\mathbb{C}\mathbb{P}^{n-1}$.

Example 1.2.4. In 2011, A. Dancer and M. Wang [30] constructed Ricci solitons on cohomogeneity-one manifolds.

Example 1.2.5. In the same year, A. Futaki and M. Wang [34] constructed gradient Kähler-Ricci solitons on Ricci-flat Kähler cone manifolds and on line bundles over toric Fano manifolds.

These examples above are constructed on Kähler manifolds, and we point out that so far, no example of a non-Kähler Riemannian shrinking soliton has been discovered.

- **Examples of Noncompact Steady Solitons**

Example 1.2.6. The first noncompact non-Einstein steady Ricci soliton was found by Hamilton [40] on \mathbb{R}^2 , called the cigar soliton. The metric and the

potential function are given by

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

and

$$f = -\log(1 + x^2 + y^2).$$

The cigar soliton has positive curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at infinity.

Example 1.2.7. R. Bryant [8] proved the existence and uniqueness of a complete noncompact rotationally symmetric gradient steady soliton with positive curvature on \mathbb{R}^n for $n \geq 3$.

Example 1.2.8. Examples of noncompact steady Ricci solitons on Kähler manifolds were first found by H.-D. Cao [9]. He constructed $U(n)$ symmetric gradient steady Kähler-Ricci solitons on both \mathbb{C}^n and the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin.

- **Examples of Noncompact Expanding Solitons**

Example 1.2.9. In addition to the steady solitons, R. Bryant [8] also proved the existence of noncompact rotationally symmetric gradient expanding Ricci solitons with positive curvature on \mathbb{R}^n .

Example 1.2.10. A one-parameter family of gradient Kähler-Ricci expanding solitons was discovered by H.-D. Cao [9] on \mathbb{C}^n . These solitons are $U(n)$ symmetric and have positive sectional curvature.

Example 1.2.11. More examples are found by Feldman-Ilmanen-Knopf [32] on the twisted line bundle L^{-k} on $\mathbb{C}\mathbb{P}^{n-1}$ for $k = n + 1, n + 2, \dots$, where L is the hyperplane bundle.

- **The Gaussian Solitons**

Example 1.2.12. The Euclidean space (R^n, δ_{ij}) with the flat metric can be considered as either a gradient shrinking, steady or expanding soliton, called the Gaussian shrinker, steady soliton or expander respectively.

i) The Gaussian shrinker has potential function $f = \frac{|x|^2}{4}$ satisfying

$$Rc + \nabla\nabla f = \frac{1}{2}g$$

ii) The Gaussian steady soliton has potential function $f = 0$ satisfying

$$Rc + \nabla\nabla f = 0$$

iii) The Gaussian expander has potential function $f = -\frac{|x|^2}{4}$ satisfying

$$Rc + \nabla\nabla f = -\frac{1}{2}g$$

For more examples, we refer the reader to the survey paper [11] of H.-D. Cao.

1.3 Geometry of Gradient Ricci Solitons

In this section, we are going to discuss some important geometric properties and classification results of gradient Ricci solitons.

1.3.1 Geometry of Gradient Shrinking Ricci Solitons

By an ancient solution, we mean a complete solution to the Ricci flow whose existing time is $(-\infty, T]$ for some T .

Lemma 1.3.1. *Let $(M^n, g_{ij}(t))$ be an ancient solution to the Ricci flow. Then it has nonnegative scalar curvature $R \geq 0$.*

Further, when $n = 3$, B.-L. Chen [25] showed more:

Lemma 1.3.2. *Any 3-dimensional ancient solution to the Ricci flow must have nonnegative sectional curvature.*

It follows from the completeness of the gradient vector field of the potential function f , that one can construct an ancient solution from a shrinking or steady Ricci soliton. Namely from the view of Ricci flow, shrinking or steady Ricci solitons are special cases of ancient solutions. Thus as a corollary:

Lemma 1.3.3. *Let (M^n, g_{ij}, f) be a complete gradient shrinking or steady soliton. Then it has nonnegative scalar curvature $R \geq 0$.*

Lemma 1.3.4. *Any 3-dimensional complete gradient shrinking or steady Ricci soliton must have nonnegative sectional curvature.*

Z.-H. Zhang [61] proved that a locally conformally flat shrinking or steady Ricci soliton has nonnegative curvature operator. Combined with the Lemma 1.3.4, we can conclude:

Lemma 1.3.5. *Let (M^n, g, f) be a complete gradient shrinking or steady Ricci soliton. Then the curvature operator $Rm \geq 0$ provided either*

- (i) $n = 3$ or
- (ii) $n \geq 4$ and g is locally conformally flat.

When a complete shrinking Ricci soliton has bounded nonnegative curvature operator, by a maximum principle of Hamilton, it either has positive curvature operator everywhere or its universal cover splits as $N \times \mathbb{R}^k$ with $k \geq 1$ and N a shrinking soliton with positive curvature operator. Moreover, if a shrinking soliton with positive curvature operator is compact, then it must be a finite quotient of the round sphere by the results of Hamilton [37, 38] (for $n = 3, 4$) and Böhm-Wilking [3] (for $n \geq 5$). Brendle and Schoen [7] and Brendle [6] got the same conclusion under some weaker suitable positive curvature conditions.

Perelman [54] showed that, in dimension 3, there is no noncompact gradient shrinking soliton with bounded positive curvature operator.

Lemma 1.3.6. (Perelman [54]) *Any complete 3-dimensional gradient shrinking Ricci soliton with bounded positive sectional curvature must be compact.*

Remark 1.3.1. *In the Kähler case, Ni [51] has shown the nonexistence of noncompact gradient shrinking Kähler-Ricci solitons with positive holomorphic bisectional curvature.*

Based on Lemma 1.3.6, Perelman obtained the following important classification result:

Theorem 1.3.1. (Perelman [54]) *Any complete 3-dimensional nonflat gradient shrinking Ricci soliton with bounded nonnegative sectional curvature must be either a quotient of \mathbb{S}^3 or a quotient of $\mathbb{S}^2 \times \mathbb{R}$.*

In the past decade, a lot of effort has been made to improve and generalize this result of Perelman. Ni-Wallach [52] and Naber [50] replaced the assumption of nonnegative sectional curvature by nonnegative Ricci curvature. In addition, instead of assuming bounded curvature, Ni-Wallach [52] allows the curvature to grow as fast as $e^{ar(x)}$, where $r(x)$ is the distance function to some arbitrarily fixed point and $a > 0$ is some constant. More specifically, they proved:

Proposition 1.3.1. (Ni-Wallach [52]) *Any 3-dimensional complete noncompact nonflat gradient shrinking Ricci soliton with $Rc \geq 0$ and $|Rm|(x) \leq e^{ar(x)}$ must be a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

Based on Lemma 1.3.4 and Proposition 1.3.1, Cao-Chen-Zhu [13] were able to remove all the assumptions on the curvature.

Theorem 1.3.2. (Cao-Chen-Zhu [13]) *Any 3-dimensional complete noncompact nonflat gradient shrinking Ricci soliton must be a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

For $n = 4$, Ni-Wallach [53] showed that any 4-dimensional gradient shrinking Ricci soliton with nonnegative curvature operator and positive isotropic curvature, satisfying certain additional assumptions, is a quotient of \mathbb{S}^4 or $\mathbb{S}^3 \times \mathbb{R}$. Using this result, Naber [50] proved

Theorem 1.3.3. (Naber [50]) *Any 4-dimensional complete noncompact shrinking Ricci soliton with bounded nonnegative curvature operator is isometric to \mathbb{R}^4 , or a finite quotient of $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$.*

For higher dimensions, the classification of gradient shrinking Ricci solitons was solved under the assumption that the Weyl tensor vanishes by the work of Eminenti-La Nave-Mantegazza [31], Ni-Wallach [52], Z.-H. Zhang [61], Petersen-Wylie [56] and Munteanu-Sesum [49].

Eminenti-La Nave-Mantegazza [31] showed that any compact shrinking Ricci soliton with vanishing Weyl tensor is a quotient of \mathbb{S}^n .

In the noncompact case, Ni-Wallach [52] proved

Proposition 1.3.2. (Ni-Wallach [52]) *Let (M^n, g, f) be a locally conformally flat gradient shrinking Ricci soliton with $Rc \geq 0$. Assume that*

$$|Rm|(x) \leq e^{a(r(x)+1)}$$

for some constant $a > 0$, where $r(x)$ is distance function to some fixed point. Then its universal cover is \mathbb{R}^n , \mathbb{S}^n or $\mathbb{S}^{n-1} \times \mathbb{R}$.

By showing that locally conformal flat gradient shrinking Ricci solitons have nonnegative curvature operator and utilizing the above result, Z.-H. Zhang [61] proved

Theorem 1.3.4. (Z.-H. Zhang [61]) *Any gradient shrinking soliton with vanishing Weyl tensor must be a finite quotient of \mathbb{R}^n , \mathbb{S}^n or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

The work of Petersen-Wylie [56], Cao-Wang-Zhang [20] and Munteanu-Sesum [49] gives another path to get the same classification result. Indeed, Petersen-Wylie first showed

Proposition 1.3.3. (Petersen-Wylie [56]) *Let (M^n, g, f) be a gradient shrinking Ricci soliton with potential function f . If the Weyl tensor vanishes and*

$$\int_M |Rc|^2 e^{-f} dV < \infty,$$

then (M^n, g, f) is a finite quotient of \mathbb{R}^n , \mathbb{S}^n or $\mathbb{S}^{n-1} \times \mathbb{R}$.

Munteanu-Sesum [49] later proved the L^2 integrability of the Ricci tensor based on the following Cao-Zhou's growth estimate of the potential function [18].

Lemma 1.3.7. (Cao-Zhou [18]) *Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton with normalization (1.1.8). Then,*

(i) *the potential function f satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2,$$

where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$, c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B(x_0, 1)$;

(ii) *there exists some constant $C > 0$ such that*

$$\text{Vol}(B(x_0, s)) \leq Cs^n$$

for $s > 0$ sufficiently large.

Recently, Fernández-López and García-Riό [33] obtain the rigidity result under the harmonic Weyl assumption:

Proposition 1.3.4. (Fernández-López and García-Riό [33]) *Any complete gradient shrinking Ricci soliton (M^n, g, f) with harmonic Weyl tensor and*

$$\int_M |\text{div } Rm|^2 e^{-f} dV = \int_M |\nabla Rc|^2 e^{-f} dV \quad (1.3.1)$$

must be rigid, i.e. it is a quotient of $N^{n-k} \times \mathbb{R}^k$, where $0 \leq k \leq n$, N is an Einstein manifold and \mathbb{R}^k is the Gaussian shrinker.

Again, Munteanu-Sesum [49] used Cao-Zhou's potential function growth estimate to prove Equation (1.3.1). Therefore, they proved:

Theorem 1.3.5. (Munteanu-Sesum [49]) *Any complete gradient shrinking Ricci soliton with harmonic Weyl tensor must be rigid.*

1.3.2 Geometry of Gradient Steady and Expanding Ricci Solitons

Since any compact steady or expanding soliton is Einstein, our discussion here only concerns the noncompact cases.

Proposition 1.3.5. (Hamilton [39]) *Suppose that a noncompact gradient steady Ricci soliton (M^n, g, f) satisfies*

$$R_{ij} = \nabla_i \nabla_j f$$

for some function f . Assume that the Ricci curvature is positive and the scalar curvature attains its maximum R_{max} at some point x_0 . Then

$$|\nabla f|^2 + R = R_{max}$$

Moreover, the function is convex and attains its minimum at x_0 .

Remark 1.3.2. *Cao-Chen [14] also showed that in this case, the function f is an exhaustion function with linear growth. Hence we have*

Proposition 1.3.6. *A complete noncompact gradient steady soliton with positive Ricci curvature whose scalar curvature attains its maximum at some point must be diffeomorphic to \mathbb{R}^n .*

Similar results hold for expanding solitons:

Proposition 1.3.7. *If a complete noncompact expanding gradient Ricci soliton has nonnegative Ricci curvature, then its potential function f is a convex exhaustion function with quadratic growth and the manifold is diffeomorphic to \mathbb{R}^n .*

In the Kähler setting, Cao-Hamilton [16] first showed that any noncompact gradient steady Kähler-Ricci soliton with positive Ricci curvature whose scalar curvature attains its maximum at some point is Stein. Later, Chau-Tam [24] and Bryant [8] independently improved the result to the following

Theorem 1.3.6. (Chau-Tam [24] and Bryant [8]) *Any noncompact gradient steady Kähler-Ricci soliton with positive Ricci curvature whose scalar curvature attains its maximum at some point is biholomorphic to \mathbb{C}^n .*

Moreover, Chau-Tam [24] also showed

Theorem 1.3.7. (Chau-Tam [24]) *A complete noncompact gradient expanding soliton with nonnegative Ricci tensor must be biholomorphic to \mathbb{C}^n .*

The classification of steady Ricci solitons with positive curvature is one of the basic problems in the study of Ricci solitons. In dimension 2, Hamilton [40] proved the following important uniqueness Theorem:

Theorem 1.3.8. (Hamilton [40]) *The only complete steady Ricci soliton on a 2-dimensional manifold with bounded curvature R which assumes its maximum $R_{max} = 1$ at some point is the cigar soliton.*

In dimension 3, Perelman [54] claimed that any complete noncompact κ -noncollapsed (see [54] for definition) gradient steady Ricci soliton must be the Bryant soliton. However, he did not provide a proof. We initiated the study by showing the uniqueness under the assumption of locally conformal flatness, as will be described in the next chapters.

Chapter 2

Bach Flat and Locally Conformally Flat Gradient Ricci Solitons

2.1 Major Results

A fundamental question is the classification problem of Ricci solitons. Since compact steady and expanding Ricci solitons must be Einstein, our main focus will be on the noncompact cases as well as the shrinking Ricci solitons. This section is consist of the author's works, joint with H.-D. Cao, related to the classification of complete steady and shrinking Ricci solitons.

2.1.1 Locally Conformally Flat Gradient Steady Ricci Solitons

In dimension 2, Hamilton discovered the first example of a complete noncompact gradient steady Ricci soliton, called the cigar soliton. For dimension $n \geq 3$, Bryant proved that there exists, up to scaling, a unique complete rotationally symmetric gradient steady Ricci soliton on \mathbb{R}^n . A well-know conjecture by Perelman [55] is that when $n = 3$, the Bryant soliton should be the unique nonflat κ -noncollapsed

gradient steady Ricci soliton.

The first progress was made by the author and his advisor H.-D. Cao in which they classified the locally conformally flat steady Ricci solitons.

Locally conformal flatness of (M^n, g) means that at any point $p \in M$, there is a neighborhood $V \subset M$ and a real-valued function f on V , such that $(V, e^f g_{ij})$ is flat, namely its curvature vanishes. Fortunately such a property of a Riemannian manifold is well-understood by the Weyl-Schouten theorem, which says that a Riemannian metric is locally conformally flat if and only if its Weyl curvature tensor vanishes for $n \geq 4$ or the Cotton tensor vanishes for $n = 3$.

Here the Weyl tensor is given by

$$\begin{aligned} W_{ijkl} = & R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) \\ & + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \end{aligned} \quad (2.1.1)$$

and the Cotton tensor by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R). \quad (2.1.2)$$

Exploring these two tensors on the gradient steady Ricci solitons,

Theorem 2.1.1. (Cao— [14]) *Let (M^n, g, f) ($n \geq 3$) be a complete noncompact gradient steady Ricci soliton. If further we assume it is locally conformally flat, then it must be either flat or isometric to the Bryant soliton.*

Our proof was in part motivated by the works of physicists Israel (1967) and Robinson(1977) concerning the uniqueness of the Schwarzschild black hole among all static, asymptotically flat vacuum space-times. In the course of proving Theorem 2.1.1, we found a new covariant 3-tensor D_{ijk} defined on any gradient Ricci soliton, which turns out to be crucial and relates the classical Weyl tensor, the Cotton tensor and also the Bach tensor with the geometry of a Ricci soliton.

Later, exploring the D -tensor, X.-X. Chen and Y. Wang [28] extended our result replacing the condition by half-conformally flat when $n = 4$. Subsequently in [12] we further extended the result to the Bach-flat case, which is a weaker condition than half-conformal flatness.

Theorem 2.1.2. (Cao-Catino—Mantegazza-Mazzieri [12]) *Let (M^n, g, f) ($n \geq 4$) be a complete noncompact gradient steady Ricci soliton. If further we assume it has positive Ricci curvature that the scalar curvature attains its maximum, and is Bach flat, then it must be isometric to the Bryant soliton.*

Here we remark that with the help of Theorem 1.1 in [18], the Bach-flat condition works better for the shrinkers, which we are going to explore more in the next subsection.

2.1.2 Bach Flat Shrinking Ricci Solitons

The Bach tensor was introduced by R. Bach in the early 1920's to study conformal relativity. From the definition of the Bach tensor,

$$B_{ij} = \frac{1}{n-3} \nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}, \quad (2.1.3)$$

it is not hard to see that either Einstein or local conformal flatness will imply Bach-flatness. Moreover when $n = 4$, a Bach-flat metric is precisely a critical point of the following conformally invariant functional on the space of metrics,

$$\mathcal{W} = \int_M |W_g|^2 dV_g,$$

where W_g is the Weyl tensor of the metric g . Thus Bach-flatness is an invariant condition under conformal change. Furthermore it is well-known that when $n = 4$, half-conformal-flatness (either self-dual or anti-self-dual) also implies Bach-flatness.

Theorem 2.1.3. (Cao— [15]) *Let (M^4, g_{ij}, f) be a complete Bach-flat gradient shrinking Ricci soliton. Then, (M^4, g_{ij}, f) is either*

(i) *Einstein, or*

(ii) *Locally conformally flat, and hence a finite quotient of the Gaussian shrinking soliton \mathbb{R}^4 or $\mathbb{S}^3 \times \mathbb{R}$.*

Theorem 2.1.4. (Cao— [15]) *Let (M^n, g_{ij}, f) ($n \geq 5$) be a complete Bach-flat gradient shrinking Ricci soliton. Then, (M^n, g_{ij}, f) is either*

(i) Einstein, or

(ii) a finite quotient of the Gaussian shrinking soliton \mathbb{R}^n , or

(iii) a finite quotient of $N^{n-1} \times \mathbb{R}$, where N^{n-1} is an Einstein manifold of positive scalar curvature.

The main idea of the proof of the above two theorems is to explore the relation between the Weyl tensor, the Cotton tensor, the Bach tensor, and gradient Ricci soliton equations. The key link is the covariant 3-tensor D_{ijk} , defined by

$$\begin{aligned} D_{ijk} = & \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ & + \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f) \end{aligned} \quad (2.1.4)$$

which was first constructed by H.-D. Cao and the author, as mentioned previously. On the other hand, we proved the following key identity

$$|D|^2 = \frac{2|\nabla f|^4}{(n-2)^2}|h_{ab} - \frac{H}{n-1}g_{ab}|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_\Sigma R|^2, \quad (2.1.5)$$

where Σ is a level set of f at some regular value, h_{ab} is the second fundamental form of Σ , and $\nabla_\Sigma R$ is the projection of ∇R onto the tangential direction of Σ . This D tensor is closely related to the geometry of the f -level sets. In addition we can show the vanishing of D -tensor from Bach-flatness, and then following by some pointwise computation, equation (2.1.5) will yield the vanishing of the Cotton tensor, and therefore the work of Fernández-Lopéz and García-Río [33], and Munteanu-Sesum [49] will result in the rigidity.

2.2 The covariant 3-tensor D_{ijk} and its Relation to Geometry

In this section, we will recall some important tensors closely related to the geometry of gradient Ricci solitons, which is the starting point for the proof of Theorem 2.1.1, Theorem 2.1.3 and Theorem 2.1.4.

First of all, we recall that on any n -dimensional Riemannian manifold (M^n, g_{ij}) ($n \geq 3$), the Weyl curvature tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \quad (2.1.1)$$

and the Cotton tensor by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R). \quad (2.1.2)$$

It is well-known that, for $n = 3$, W_{ijkl} vanishes identically, while $C_{ijk} = 0$ if and only if (M^3, g_{ij}) is locally conformally flat; for $n \geq 4$, $W_{ijkl} = 0$ if and only if (M^n, g_{ij}) is locally conformally flat. Moreover, for $n \geq 4$, the Cotton tensor C_{ijk} is, up to a constant factor, the divergence of the Weyl tensor:

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}, \quad (2.2.1)$$

hence the vanishing of the Cotton tensor $C_{ijk} = 0$ (in dimension $n \geq 4$) is also referred as being harmonic Weyl.

Moreover, for $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2}R_{kl}W_{ikjl}. \quad (2.1.3)$$

By (2.2.1), we have

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl}W_{ikjl}). \quad (2.2.2)$$

Note that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices:

$$C_{ijk} = -C_{jik} \quad \text{and} \quad g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0. \quad (2.2.3)$$

Next, let us recall the covariant 3-tensor D_{ijk} on any gradient Ricci soliton introduced in our work [14] and its important properties.

For any gradient Ricci soliton satisfying the equation (1.1.3) the covariant 3-tensor D_{ijk} is defined as:

$$D_{ijk} = \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) - \frac{R}{(n-1)(n-2)}(g_{jk}\nabla_i f - g_{ik}\nabla_j f). \quad (2.1.4)$$

This 3-tensor D_{ijk} is closely tied to the Cotton tensor and played a significant role in our previous work [14] classifying locally conformally flat gradient steady solitons, as well as in the subsequent work of X. Chen and Y. Wang [28].

Most of the material in this section can be found in [14].

Lemma 2.2.1. *Let (M^n, g_{ij}, f) ($n \geq 3$) be a complete gradient soliton. Then D_{ijk} is related to the Cotton tensor C_{ijk} and the Weyl tensor W_{ijkl} by*

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f.$$

Proof. From the soliton equation (1.1.3) and the Ricci identity, we have

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -\nabla_i \nabla_j \nabla_k f + \nabla_j \nabla_i \nabla_k f = -R_{ijkl}\nabla_l f.$$

Hence, using equation (2.1.1), (2.1.2) and (1.1.6), we obtain

$$\begin{aligned} C_{ijk} &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ &= -R_{ijkl}\nabla_l f - \frac{1}{(n-1)}(g_{jk}R_{il} - g_{ik}R_{jl})\nabla_l f \\ &= -W_{ijkl}\nabla_l f - \frac{1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) \\ &\quad + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) + \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f) \\ &= -W_{ijkl}\nabla_l f + D_{ijk}. \end{aligned}$$

□

Remark 2.2.1. *By Lemma 2.2.1, D_{ijk} is equal to the Cotton tensor C_{ijk} for three-dimensional gradient Ricci solitons. In addition, from $W_{ijkl} = -W_{ijlk}$, we have that*

$$D_{ijk}\nabla_k f = C_{ijk}\nabla_k f.$$

Also, clearly D_{ijk} vanishes if (M^n, g_{ij}, f) ($n \geq 3$) is either Einstein or locally conformally flat. Moreover, like the Cotton tensor C_{ijk} , D_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices:

$$D_{ijk} = -D_{jik} \quad \text{and} \quad g^{ij}D_{ijk} = g^{ik}D_{ijk} = 0. \quad (2.2.4)$$

What is so special about D_{ijk} is the following key identity, which links the norm of D_{ijk} to the geometry of the level surfaces of the potential function f .

Proposition 2.2.1. (Cao— [14]) *Let (M^n, g_{ij}, f) ($n \geq 3$) be an n -dimensional gradient Ricci soliton satisfying (1.1.3). Then, at any point $p \in M^n$ where $\nabla f(p) \neq 0$, we have*

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2} \left| h_{ab} - \frac{H}{n-1}g_{ab} \right|^2 + \frac{1}{2(n-1)(n-2)} |\nabla_a R|^2, \quad (2.1.5)$$

where h_{ab} and H are the second fundamental form and the mean curvature of the level surface $\Sigma = \{f = f(p)\}$, and g_{ab} is the induced metric on Σ .

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be any orthonormal frame, with $e_1 = \nabla f/|\nabla f|$ and e_2, \dots, e_n tangent to Σ . Then the second fundamental form h_{ab} and the mean curvature H are given respectively by

$$h_{ab} = g \left(\nabla_a \frac{\nabla f}{|\nabla f|}, e_b \right) = \frac{1}{|\nabla f|} \nabla_a \nabla_b f = \frac{\rho g_{ab} - R_{ab}}{|\nabla f|}, \quad a, b = 2, \dots, n \quad (2.2.5)$$

and

$$H = \frac{1}{|\nabla f|} [(n-1)\rho - (R - R_{11})].$$

Hence, it follows that

$$|h_{ab}|^2 = \frac{|\rho g_{ab} - R_{ab}|^2}{|\nabla f|^2} = \frac{1}{|\nabla f|^2} [(n-1)\rho^2 - 2\rho(R - R_{11}) + \sum_{a,b=2}^n |R_{ab}|^2],$$

and

$$H^2 = \frac{1}{|\nabla f|^2} [(n-1)^2\rho^2 - 2(n-1)\rho(R - R_{11}) + (R - R_{11})^2].$$

By (1.1.6),

$$R_{11} = \frac{1}{|\nabla f|^2} Rc(\nabla f, \nabla f) = \frac{1}{2|\nabla f|^2} g(\nabla f, \nabla R)$$

and

$$R_{1a} = \frac{1}{|\nabla f|} Rc(\nabla f, e_a) = \frac{1}{2|\nabla f|} \nabla_a R.$$

Moreover,

$$|\nabla_\Sigma R|^2 = \sum_{a=2}^n |\nabla_a R|^2 = |\nabla R|^2 - \frac{1}{|\nabla f|^2} \{g(\nabla R, \nabla f)\}^2.$$

Thus, by direct computation, we obtain

$$\begin{aligned} \left| h_{ab} - \frac{H}{n-1} g_{ab} \right|^2 &= |h_{ab}|^2 - \frac{H^2}{n-1} \\ &= \frac{1}{|\nabla f|^2} \sum_{a,b=2}^n |R_{ab}|^2 - \frac{1}{(n-1)|\nabla f|^2} (R - R_{11})^2 \\ &= \frac{1}{|\nabla f|^2} (|Rc|^2 - 2 \sum_{a=2}^n R_{1a}^2 - R_{11}^2) - \frac{(R - R_{11})^2}{(n-1)|\nabla f|^2} \\ &= \frac{1}{|\nabla f|^2} |Rc|^2 + \frac{R}{(n-1)|\nabla f|^4} \nabla f \cdot \nabla R - \frac{1}{2|\nabla f|^4} |\nabla R|^2 \\ &\quad + \frac{n-2}{4(n-1)|\nabla f|^6} |\nabla f \cdot \nabla R|^2 - \frac{R^2}{(n-1)|\nabla f|^2}, \end{aligned}$$

On the other hand, by (1.1.6) and (2.1.4), we obtain

$$\begin{aligned} |D_{ijk}|^2 &= \frac{1}{(n-2)^2} |R_{jk} \nabla_i f - R_{ik} \nabla_j f|^2 + \frac{1}{2(n-1)(n-2)^2} |\nabla R|^2 \\ &\quad + \frac{2R^2}{(n-1)(n-2)^2} |\nabla f|^2 + \frac{2}{(n-1)(n-2)^2} [R \nabla R \cdot \nabla f - Rc(\nabla f, \nabla R)] \\ &\quad - \frac{4R}{(n-1)(n-2)^2} [R |\nabla f|^2 - Rc(\nabla f, \nabla f)] - \frac{2R}{(n-1)(n-2)^2} \nabla R \cdot \nabla f \\ &= \frac{1}{(n-2)^2} (|R_{jk} \nabla_i f - R_{ik} \nabla_j f|^2 - \frac{2}{(n-1)} |R \nabla f - \frac{1}{2} \nabla R|^2) \\ &= \frac{2|\nabla f|^2}{(n-2)^2} |Rc|^2 - \frac{1}{2(n-2)^2} |\nabla R|^2 \\ &\quad - \frac{1}{2(n-1)(n-2)^2} (|\nabla R|^2 - 4R \nabla R \cdot \nabla f + 4R^2 |\nabla f|^2). \end{aligned}$$

Therefore, one can verify directly that

$$\frac{2|\nabla f|^4}{(n-2)^2} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 = |D_{ijk}|^2 - \frac{1}{2(n-1)(n-2)} |\nabla_{\Sigma} R|^2.$$

□

Finally, thanks to Proposition 2.2.1, the vanishing of D_{ijk} implies many nice properties about the geometry of the Ricci soliton (M^n, g_{ij}, f) and the level surfaces of the potential function f .

Proposition 2.2.2. (Cao— [14]) *Let (M^n, g_{ij}, f) ($n \geq 3$) be any complete gradient Ricci soliton with $D_{ijk} = 0$. Let γ be a regular value of f and $\Sigma_{\gamma} = \{f = \gamma\}$ be the level surface of f . Set $e_1 = \nabla f/|\nabla f|$ and pick any orthonormal frame e_2, \dots, e_n tangent to the level surface Σ_{γ} . Then:*

- (a) $|\nabla f|^2$ and the scalar curvature R of (M^n, g_{ij}, f) are constant on Σ_{γ} ;
- (b) $R_{1a} = 0$ for any $a \geq 2$ and $e_1 = \nabla f/|\nabla f|$ is an eigenvector of Rc ;
- (c) the second fundamental form h_{ab} of Σ_{γ} is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$;
- (d) the mean curvature H is constant on Σ_{γ} ;
- (e) on Σ_{γ} , the Ricci tensor of (M^n, g_{ij}, f) either has a unique eigenvalue λ , or has two distinct eigenvalues λ and μ of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \nabla f/|\nabla f|$ is an eigenvector of λ .

Proof. Clearly (a) and (c) follow immediately from $D_{ijk} = 0$, Proposition 2.2.1, and (1.1.7);

(b) follows from (a) and (1.1.6): $R_{1a} = \frac{1}{2|\nabla f|} \nabla_a R = 0$;

For (d), we consider the Codazzi equation

$$R_{1cab} = \nabla_a^{\Sigma_{\gamma}} h_{bc} - \nabla_b^{\Sigma_{\gamma}} h_{ac}, \quad a, b, c = 2, \dots, n. \quad (2.2.6)$$

Tracing over b and c in (2.2.6), we obtain

$$R_{1a} = \nabla_a^{\Sigma_{\gamma}} H - \nabla_b^{\Sigma_{\gamma}} h_{ab} = \left(1 - \frac{1}{n-1}\right) \nabla_a H.$$

Then (d) follows since $R_{1a} = 0$.

Finally, by (2.2.5) and (c), we know

$$R_{ab} = \rho g_{ab} - |\nabla f| h_{ab} = \left(\rho - \frac{H}{n-1} |\nabla f|\right) g_{ab}.$$

But both H and $|\nabla f|$ are constant on Σ_γ , so the Ricci tensor restricted to the tangent space of Σ_γ has only one eigenvalue μ :

$$\mu = R_{aa} = \rho - H|\nabla f|/(n-1), \quad a = 2, \dots, n,$$

which is constant along Σ_γ . On the other hand,

$$\lambda = R_{11} = R - \sum_{a=2}^n R_{aa} = R - (n-1)\rho + H|\nabla f|,$$

again a constant along Σ_γ . This proves (e). □

2.3 Proof of the Main Theorems

2.3.1 Proof of Theorem 2.1.1

From the well-known fact that the locally conformally flat condition is equivalent to the vanishing of the Weyl tensor (2.1.1) and the Cotton tensor (2.1.2), we can conclude the vanishing of the D tensor with the help of Lemma 2.2.1. And thus, we are free to use Proposition 2.2.2. Furthermore, under the assumption of locally conformal flatness, we can extend Proposition 2.2.2 by adding a roundness conclusion:

Proposition 2.3.1. (Cao— [14]) *Let (M^n, g_{ij}, f) ($n \geq 3$) be any complete locally conformally flat gradient Ricci soliton, and let c be a regular value of f with $\Sigma_\gamma = \{f = \gamma\}$ the corresponding level surface of f . Set $e_1 = \nabla f/|\nabla f|$ and pick any orthonormal frame e_2, \dots, e_n tangent to the level surface Σ_γ . Then:*

- (a) $|\nabla f|^2$ and the scalar curvature R of (M^n, g_{ij}, f) are constant on Σ_γ ;
- (b) $R_{1a} = 0$ for any $a \geq 2$ and $e_1 = \nabla f/|\nabla f|$ is an eigenvector of Rc ;
- (c) the second fundamental form h_{ab} of Σ_γ is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$;

(d) the mean curvature H is constant on Σ_γ ;

(e) on Σ_γ , the Ricci tensor of (M^n, g_{ij}, f) either has a unique eigenvalue λ , or has two distinct eigenvalues λ and μ of multiplicity 1 and $n - 1$ respectively. In either case, $e_1 = \nabla f / |\nabla f|$ is an eigenvector of λ

(f) Σ_γ with the induced metric has constant sectional curvature.

Proof. (f): By the Gauss equation, for $a \neq b$, and using (c):

$$R_{abab}^{\Sigma_\gamma} = R_{abab} + h_{aa}h_{bb} - h_{ab}^2 = R_{abab} + \frac{H^2}{(n-1)^2}. \quad (2.3.1)$$

Using (2.1.1):

$$\begin{aligned} R_{abab} &= \frac{1}{n-2}(R_{aa} + R_{bb}) - \frac{R}{(n-1)(n-2)} \\ &= \frac{2\mu}{n-2} - \frac{R}{(n-1)(n-2)}. \end{aligned}$$

Therefore, as is easy to see from (a), (d) and (e), that the sectional curvature of Σ_γ is a constant. □

Now we can complete the proof of Theorem 2.1.1:

Proof. Now (M^n, g, f) is a complete gradient steady Ricci soliton, and from lemma 1.3.5 we have $Rm \geq 0$. Together with equation (1.1.7), we have $0 \leq Rm \leq C$ for some constant C . By Hamilton's strong maximum principle, we know either $Rm > 0$ or the holonomy group is not transitive. The latter case splits into several cases:

(a) Riemannian Product: The only locally conformally flat Riemannian product with nonnegative curvature operator is a space form N^{n-1} with \mathbb{S}^1 or \mathbb{R} , and it is easy to see it must be flat to be a steady Ricci soliton.

(b) Locally Symmetric Space: A locally symmetric steady Ricci soliton must be Ricci-flat, and hence flat since the Weyl tensor vanishes.

(c) **Irreducible and non-locally symmetric:** Referring to Berger's list, all possible cases with non-transitive holonomy are Ricci-flat and hence flat.

Now, it remains to consider the first case for which $Rm > 0$. From Gromoll-Meyer [35], M^n is diffeomorphic to \mathbb{R}^n . With $Rc = -Hessf > 0$, f can have at most one critical point. Letting $\Sigma_\gamma = \{x \in M | f(x) = \gamma\}$ and $\theta^2, \theta^3, \dots, \theta^n$ as the coordinates on Σ_γ for regular value γ , the metric on the regular set of f can be written as

$$g = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta^a d\theta^b. \quad (2.3.2)$$

With the condition $Rm > 0$, Proposition 2.3.1 shows (Σ_γ, g_{ab}) is a space form of positive curvature, and thus the round sphere, which in turn implies there exists a critical point O for f . Therefore (2.3.2) holds on $M - \{O\} = \mathbb{R}^n - \{O\}$, and we complete the proof that M is a rotationally symmetric steady Ricci soliton on \mathbb{R}^n , which must be the Bryant soliton. □

2.3.2 Proof of Theorem 2.1.3 and Theorem 2.1.4

Throughout this section, we assume that (M^n, g_{ij}, f) ($n \geq 4$) is a complete gradient shrinking soliton satisfying (1.1.8).

First of all, we relate the Bach tensor B_{ij} to the Cotton tensor C_{ijk} and the tensor D_{ijk} , and then show that Bach-flatness implies $D_{ijk} = 0$:

Lemma 2.3.1. *Let (M^n, g_{ij}, f) be a complete gradient shrinking soliton. If $B_{ij} = 0$, then $D_{ijk} = 0$.*

Proof. By direct computations, and using (2.2.2), (2.2.3) and lemma 2.2.1, we have

$$\begin{aligned}
B_{ij} &= -\frac{1}{n-2}\nabla_k C_{ikj} + \frac{1}{n-2}R_{kl}W_{ikjl} \\
&= -\frac{1}{n-2}\nabla_k(D_{ikj} - W_{ikjl}\nabla_l f) + \frac{1}{n-2}R_{kl}W_{ikjl} \\
&= -\frac{1}{n-2}(\nabla_k D_{ikj} - \nabla_k W_{jlik}\nabla_l f) + \frac{1}{n-2}(R_{kl} + \nabla_k\nabla_l f)W_{ijkl}.
\end{aligned}$$

Hence, by (1.1.8) and (2.2.1)

$$B_{ij} = -\frac{1}{n-2}(\nabla_k D_{ikj} + \frac{n-3}{n-2}C_{jli}\nabla_l f). \quad (2.3.3)$$

Next, we use (2.3.3) to show that Bach flatness implies vanishing of the tensor D_{ijk} . By Lemma 1.3.7, sublevel sets $\Omega_r = \{x \in M | f(x) \leq r\}$ of f are compact. Now by the definition of D_{ijk} , the above identity (2.3.3), as well as properties (2.2.3) and (2.2.4), we have

$$\begin{aligned}
\int_{\Omega_r} B_{ij}\nabla_i f\nabla_j f dV &= -\frac{1}{(n-2)}\int_{\Omega_r} \nabla_k D_{ikj}\nabla_i f\nabla_j f dV \\
&= \frac{1}{(n-2)}\left(\int_{\Omega_r} D_{ikj}\nabla_i f\nabla_k\nabla_j f dV - \int_{\Omega_r} \nabla_k(D_{ikj}\nabla_i f\nabla_j f)dV\right) \\
&= -\frac{1}{(n-2)}\left(\int_{\Omega_r} D_{ikj}\nabla_i f R_{jk} dV + \int_{\partial\Omega_r} D_{ikj}\nabla_i f\nabla_j f\nu_k dS\right) \\
&= -\frac{1}{2(n-2)}\int_{\Omega_r} D_{ikj}(\nabla_i f R_{jk} - \nabla_k f R_{ij})dV \\
&= -\frac{1}{2}\int_{\Omega_r} |D_{ikj}|^2 dV.
\end{aligned}$$

Here we have used (2.2.4) and the following equation concerning the boundary term:

$$\int_{\partial\Omega_r} D_{ikj}\nabla_i f\nabla_j f\nu_k dS = \int_{\partial\Omega_r} D_{ikj}\nabla_i f\nabla_j f\nabla_k f \frac{1}{|\nabla f|} dS = 0.$$

By taking $r \rightarrow \infty$, we immediately obtain

$$\int_M B_{ij}\nabla_i f\nabla_j f dV = -\frac{1}{2}\int_M |D_{ikj}|^2 dV.$$

This completes the proof of Lemma 2.3.1. □

Lemma 2.3.2. *Let (M^n, g_{ij}, f) ($n \geq 4$) be a complete gradient shrinking Ricci soliton with vanishing D_{ijk} . Then the Cotton tensor $C_{ijk} = 0$ at all points where $\nabla f \neq 0$.*

Proof. First of all, $D_{ijk} = 0$ and Lemma 2.2.1 also imply

$$C_{ijk} = -W_{ijkl}\nabla_l f, \quad (2.3.4)$$

hence

$$C_{ijk}\nabla_k f = -W_{ijkl}\nabla_k f\nabla_l f = 0. \quad (2.3.5)$$

Then, for any point $p \in M$ with $\nabla f(p) \neq 0$, we choose a local coordinates system $(\theta^2, \dots, \theta^n)$ on the level surface $\Sigma = \{f = f(p)\}$. Then, in an open neighborhood U of Σ in M , we use the local coordinate system

$$(x^1, x^2, \dots, x^n) = (f, \theta^2, \dots, \theta^n)$$

adapted to level surfaces. In the following, we use a, b, c to represent indices on the level sets which range from 2 to n , while i, j, k from 1 to n . Under the above chosen local coordinate system, the metric g can be expressed as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta^a d\theta^b.$$

Next, we set $\nu = -\frac{\nabla f}{|\nabla f|}$. It is then easy to see that

$$\nu = -|\nabla f|\partial_f, \quad \text{or} \quad \partial_f = \frac{1}{|\nabla f|^2}\nabla f.$$

Also ∂_1 and ∂_f shall be interchangeable below. And we have

$$\nabla_1 f = 1, \quad \text{and} \quad \nabla_a f = 0 \text{ for } a \geq 2.$$

Then, in this coordinate, (2.3.5) implies that

$$C_{ij1} = 0.$$

Claim 1: $D_{ijk} = 0$ implies $C_{abc} = 0$.

To show $C_{abc} = 0$, we make use of Proposition 2.2.2 as follows: from the Codazzi equation (2.2.6) and $h_{ab} = Hg_{ab}/(n-1)$, we obtain

$$R_{1cab} = \nabla_a^\Sigma h_{bc} - \nabla_b^\Sigma h_{ac} = \frac{1}{n-1}(g_{bc}\partial_a(H) - g_{ac}\partial_b(H)).$$

But we also know that the mean curvature H is constant on the level surface Σ of f , so

$$R_{1abc} = 0.$$

Moreover, since $R_{1a} = 0$, we easily obtain

$$W_{1abc} = R_{1abc} = 0.$$

By (2.3.4), we have

$$C_{abc} = -W_{abci}\nabla_j f g^{ij} = W_{1cab}\nabla_1 f g^{11} = 0.$$

This finishes the proof of Claim 1.

Claim 2: $D_{ijk} = 0$ implies $C_{1ab} = C_{a1b} = 0$. To see this, let us compute the second fundamental form in the preferred local coordinate system $(f, \theta^2, \dots, \theta^n)$:

$$h_{ab} = -\langle \nu, \nabla_a \partial_b \rangle = -\langle \nu, \Gamma_{ab}^1 \partial_f \rangle = \frac{\Gamma_{ab}^1}{|\nabla f|}.$$

But the Christoffel symbol Γ_{ab}^1 is given by

$$\Gamma_{ab}^1 = \frac{1}{2}g^{11}\left(-\frac{\partial g_{ab}}{\partial f}\right) = \frac{1}{2}|\nabla f|\nu(g_{ab}).$$

Hence, we obtain

$$h_{ab} = \frac{1}{2}\nu(g_{ab}).$$

On the other hand, since $|\nabla f|$ is constant along level surfaces, we have

$$[\partial_a, \nu] = -[\partial_a, |\nabla f|\partial_f] = 0.$$

Then using the fact that $\langle \nu, \nu \rangle = 1$ and $\langle \nu, \partial_a \rangle = 0$, it is easy to see that

$$\nabla_\nu \nu = 0.$$

By direct computations and using Proposition 2.2.2, we can compute the following component of the Riemannian curvature tensor:

$$\begin{aligned} Rm(\nu, \partial_a, \nu, \partial_b) &= \langle \nabla_\nu \nabla_a \partial_b - \nabla_a \nabla_\nu \partial_b, \nu \rangle \\ &= \langle \nabla_\nu (\nabla_a^\Sigma \partial_b + \nabla_a^\perp \partial_b), \nu \rangle - \langle \nabla_a \nabla_\nu \partial_b, \nu \rangle \\ &= \langle \nabla_a^\Sigma \partial_b, -\nabla_\nu \nu \rangle + \langle \nabla_\nu (-h_{ab} \nu), \nu \rangle + \langle \nabla_b \nu, \nabla_a \nu \rangle \\ &= -\nu(h_{ab}) + h_{ac} h_{cb} \\ &= -\frac{\nu(H)}{n-1} g_{ab} + \frac{H^2}{(n-1)^2} g_{ab}. \end{aligned}$$

Taking the trace over a, b yields

$$Rc(\nu, \nu) = -\nu(H) + \frac{H^2}{n-1}$$

Thus

$$\begin{aligned} Rm(\nu, \partial_a, \nu, \partial_b) &= -\frac{\nu(H)}{n-1} g_{ab} + \frac{H^2}{(n-1)^2} g_{ab} \\ &= \frac{Rc(\nu, \nu)}{n-1} g_{ab}. \end{aligned}$$

Finally, we are ready to compute C_{1ab} :

$$C_{1ab} = -W_{1abi} \nabla_j f g^{ij} = W_{1a1b} |\nabla f|^2 = W(\nu, \partial_a, \nu, \partial_b),$$

but using proposition 2.2.2 (e), we have:

$$\begin{aligned}
W(\nu, \partial_a, \nu, \partial_b) &= Rm(\nu, \partial_a, \nu, \partial_b) + \frac{Rg_{ab}}{(n-1)(n-2)} - \frac{1}{n-2}(Rc(\nu, \nu)g_{ab} + R_{ab}) \\
&= \frac{Rc(\nu, \nu)}{n-1}g_{ab} + \frac{Rg_{ab}}{(n-1)(n-2)} - \frac{1}{n-2}(Rc(\nu, \nu)g_{ab} + R_{ab}) \\
&= \frac{\lambda}{n-1}g_{ab} + \frac{(\lambda + (n-1)\mu)g_{ab}}{(n-1)(n-2)} - \frac{1}{n-2}(\lambda g_{ab} + \mu g_{ab}) \\
&= 0.
\end{aligned}$$

Therefore,

$$C_{1ab} = W_{1a1b} = 0.$$

This finishes the proof of Claim 2.

Therefore we have shown that $C_{ij1} = 0$, $C_{abc} = 0$ and $C_{1ab} = 0$. This proves Lemma 2.3.2. □

For dimension $n = 4$, it turns out that we can prove a stronger result:

Lemma 2.3.3. *Let (M^4, g_{ij}, f) be a complete gradient shrinking Ricci soliton with vanishing D_{ijk} . Then the Weyl tensor $W_{ijkl} = 0$ at all points where $\nabla f \neq 0$.*

Proof. From Lemma 2.3.2 we know that $D_{ijk} = 0$, implies $C_{ijk} = 0$. Hence it follows from Lemma 2.2.1 that

$$W_{ijkl}\nabla_l f = 0$$

for all $1 \leq i, j, k, l \leq 4$. For any p where $|\nabla f| \neq 0$, we can attach an orthonormal frame at p with $e_1 = \frac{\nabla f}{|\nabla f|}$, and then we have

$$W_{1ijk}(p) = 0, \quad \text{for } 1 \leq i, j, k \leq 4. \quad (2.3.6)$$

Thus it remains to show

$$W_{abcd}(p) = 0$$

for all $2 \leq a, b, c, d \leq 4$. However, this reduces to showing the Weyl tensor is zero in three-dimensional case (cf. [37], p.276–277): observing that the Weyl tensor W_{ijkl} has all the symmetry of the R_{ijkl} and is trace free in any two indices. Thus,

$$W_{2121} + W_{2222} + W_{2323} + W_{2424} = 0,$$

and so, by (2.3.6),

$$W_{2323} = -W_{2424}.$$

Similarly, we have

$$W_{2424} = -W_{3434} = W_{2323},$$

which implies $W_{2323} = 0$. On the other hand,

$$W_{1314} + W_{2324} + W_{3334} + W_{4344} = 0,$$

so $W_{2324} = 0$. This shows that $W_{abcd} = 0$ unless a, b, c, d are all distinct. But, there are only three choices for the indices a, b, c, d as they range from 2 to 4 so we can conclude that $W_{abcd} = 0$ for all $2 \leq a, b, c, d \leq 4$. \square

Now we are ready to finish the proof of our main theorems:

Conclusion of the proof of Theorem 2.1.3: Let (M^4, g_{ij}, f) be a complete Bach-flat gradient shrinking Ricci soliton. Then, by Lemma 2.3.1, $D_{ijk} = 0$. We divide the arguments into two cases:

- Case 1: the set $\Omega = \{p \in M \mid \nabla f(p) \neq 0\}$ is dense.

By Lemma 2.3.3, we know that $W_{ijkl} = 0$ on Ω . By continuity, we know that $W_{ijkl} = 0$ on M^4 . Therefore we conclude that (M^4, g_{ij}, f) is locally conformally flat. Furthermore, according to the classification result for locally conformally flat gradient shrinking Ricci solitons mentioned in the introduction, (M^4, g_{ij}, f) is a finite quotient of either \mathbb{S}^4 , or \mathbb{R}^4 , or $\mathbb{S}^3 \times \mathbb{R}$.

- Case 2: $|\nabla f|^2 = 0$ on some nonempty open set. In this case, since any gradient shrinking Ricci soliton is analytic in harmonic coordinates, it follows that $|\nabla f|^2 = 0$ on M , i.e., (M^4, g_{ij}) is Einstein.

This completes the proof of Theorem 2.1.3. □

Conclusion of the proof of Theorem 2.1.4: Let (M^n, g_{ij}, f) , $n \geq 5$, be a Bach-flat gradient shrinking Ricci soliton. Then, by Lemma 2.3.1, Lemma 2.3.2 and the same argument as in the proof of Theorem 2.1.3 above, we know that (M^n, g_{ij}, f) either is Einstein, or has harmonic Weyl tensor. In the latter case, by the rigidity theorem of Fernández-López and García-Río [33] and Munteanu-Sesum [49], (M^n, g_{ij}, f) is either Einstein or isometric to a finite quotient of $N^{n-k} \times \mathbb{R}^k$ ($k > 0$) the product of an Einstein manifold N^{n-k} with the Gaussian shrinking soliton \mathbb{R}^k . However, Proposition 2.2.2 (e) says that the Ricci tensor either has one unique eigenvalue or two distinct eigenvalues with multiplicity of 1 and $n - 1$ respectively. Therefore, only $k = 1$ and $k = n$ can occur in $N^{n-k} \times \mathbb{R}^k$. □

2.3.3 Proof of Theorem 2.1.2

The major difficulty in applying the Bach-flat condition to the steady Ricci soliton is that we do not have a nice property for the potential function f , as we have Lemma 1.3.7 for shrinking solitons. A consequence of this is that we cannot do integration by parts on steady Ricci solitons, and thus we do not have an analogue of Lemma 2.3.1. If we assume the potential function f is an exhaustion function, then the computation can be carried over. Hence in the paper [12], we gave a sufficient condition to guarantee that f is an exhaustion function.

Lemma 2.3.4. *Let (M^n, g, f) be a complete noncompact gradient steady soliton with positive Ricci curvature, and further assume that the scalar curvature R attains its maximum at some origin x_0 . Then there exist some constants $0 < c_1 < c_0$ and $c_2 > 0$ such that*

$$c_1 r(x) - c_2 \leq -f(x) \leq c_0 r(x) + |f(x_0)|$$

where $r(x) = d(x, x_0)$.

Proof. From

$$R + |\nabla f|^2 = C_0 \tag{1.1.7}$$

and the condition $R = \text{tr}(Rc) > 0$, we can take $c_0 = \sqrt{C_0}$ to get the upper bound.

To get the lower bound, we consider any minimizing unit-speed geodesic $\gamma(s)$, $0 \leq s \leq s_0$ for large $s_0 > 0$, starting from the origin $x_0 = \gamma(0)$. Denote $X(s) = \dot{\gamma}(s)$, the unit tangent vector along γ , and $\dot{f} = \nabla_X f(\gamma(s))$. By (1.1.3), we have

$$\nabla_X \dot{f} = \nabla_X \nabla_X f = -Rc(X, X). \tag{2.3.7}$$

Integrating it along γ , and noting that x_0 is the critical point of f , we get, for $s \geq 1$,

$$-\dot{f}(\gamma(s)) = \int_0^s Rc(X, X)ds \geq \int_0^1 Rc(X, X)ds \geq c_1,$$

where $c_1 > 0$ is taken to be the least eigenvalue of Rc on the unit geodesic ball $B_{x_0}(1)$. Thus,

$$-f(\gamma(s_0)) = -\int_1^{s_0} \dot{f}(\gamma(s))ds - f(\gamma(1)) \geq c_1 s_0 - c_1 - f(\gamma(1)).$$

Define $c_2 = c_1 + f(\gamma(1))$, and we finish the proof. \square

Then we are able to work on the Bach flat steady Ricci solitons:

Lemma 2.3.5. *Let (M^n, g, f) be a complete steady gradient Ricci soliton with positive Ricci curvature and with scalare curvature R attaining its maximum at some point. If we assume $B_{ij} = 0$, then on (M^n, g, f) , $D_{ijk} = 0$.*

Proof. The proof is similar to that of Lemma 2.3.1, replacing the use of Lemma 1.3.7 with Lemma 2.3.4. \square

Conclusion of the proof of Theorem 2.1.2: Let (M^n, g, f) , be a complete Bach-flat gradient steady Ricci soliton with positive Ricci curvature such that the

scalar curvature R attains its maximum at some interior point $O \in M$. Then, by Lemma 2.3.4 we know that f is proper, strictly concave, has a unique critical point at O , and that M^n is diffeomorphic to \mathbb{R}^n . On the other hand, by Lemma 2.3.5, we have $D_{ijk} = 0$.

First of all, on $M \setminus \{O\}$, the soliton metric g_{ij} can be expressed as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta^a d\theta^b,$$

where $(\theta^2, \dots, \theta^n)$ is any local coordinates system on the level surface $\Sigma = \{f = f(p)\}$ at $p \in M \setminus \{O\}$. Note that, since $D_{ijk} = 0$, $|\nabla f|^2$ depends only on f by Proposition 2.2.2 (a). Hence, by a suitable change of variable, we can further express g_{ij} as

$$ds^2 = dr^2 + g_{ab}(r, \theta) d\theta^a d\theta^b, \quad 0 < r < \infty.$$

Here $r(x)$ is the distance function from O .

Claim 1: For $r > 0$, the induced metric $\bar{g}_{\Sigma_r} = g_{ab}(r, \theta) d\theta^a d\theta^b$ on each level surface Σ_r is Einstein.

This is a result of Proposition 2.2.2 (e) and the Gauss equation.

Claim 2: On $M \setminus \{O\}$, the metric g takes the form of a warped product metric:

$$ds^2 = dr^2 + w(r)^2 \bar{g}_E, \quad r \in (0, +\infty), \quad (2.3.8)$$

where w is some nonnegative smooth function on M^n vanishing only at O , and $\bar{g}_E = \bar{g}_{\Sigma_1}$ is the Einstein metric defined on the level surface Σ_1 .

Indeed, by the definition of the second fundamental form (2.2.5) and Proposition 2.2.2, we have

$$\frac{\partial}{\partial r} g_{ab} = -2h_{ab} = \phi(r) g_{ab},$$

where $\phi(r) = -2H(r)/(n-1)$. Thus, it follows easily that

$$g_{ab}(r, \theta) = e^{\Phi(r)} g_{ab}(1, \theta),$$

where

$$\Phi(r) = \int_1^r \phi(r) dr.$$

This proves Claim 2.

By scaling, we can assume that

$$Ric_{\bar{g}_E} = (n - 2) \bar{g}_E \tag{2.3.9}$$

Claim 3: We have

$$\lim_{r \rightarrow 0^+} \frac{w(r)}{r} = 1.$$

Clearly, $w(r) \rightarrow 0$ as $r \rightarrow 0^+$. On the other hand, on $M \setminus \{O\}$, the Ricci tensor and the scalar curvature of the metric g in (2.3.8) take the form (see [1, Proposition 9.106])

$$Ric_g = -(n - 1) \frac{w''}{w} dr \otimes dr + ((n - 2)(1 - (w')^2) - w w'') \bar{g}_E,$$

and

$$R_g = -2(n - 1) \frac{w''}{w} + \frac{(n - 1)(n - 2)}{w^2} (1 - (w')^2)$$

respectively. Here we have used the Claim 1 and the normalization (2.3.9).

On the other hand, Rc is bounded, because it is assumed to be positive and the scalar curvature $R = 1 - |\nabla f|^2 \leq 1$ is bounded. Thus from the expression of the Ricci tensor above and the boundedness of Rc on M^n , it is easy to see that w''/w must be bounded as $r \rightarrow 0^+$. Hence, from the above scalar curvature expression, it is easy to deduce the claim.

Claim 4: \bar{g}_E is equal to the standard round metric $\bar{g}_{\mathbb{S}^{n-1}}$ on the unit sphere \mathbb{S}^{n-1} .

This essentially follows from the previous claims and the elementary fact that infinitesimally the metric g is approximately Euclidean near O . In fact, the standard expansion of the metric g around O , written in any normal coordinates (x^1, \dots, x^n) , gives

$$\begin{aligned}
g &= (\delta_{ij} + \sigma_{ij}(x)) dx^i \otimes dx^j \\
&= g_{\mathbb{R}^n} + \sigma_{ij} dx^i \otimes dx^j,
\end{aligned}$$

where $\sigma_{ij} = \mathcal{O}(|x|^2)$. To pass to polar coordinates, we write $x^i = r\phi^i(\theta^1, \dots, \theta^{n-1})$, with $r \in (0, +\infty)$ and $(\theta^1, \dots, \theta^{n-1})$ being local coordinates on \mathbb{S}^{n-1} . Notice that $|\phi^1|^2 + \dots + |\phi^n|^2 = 1$ and $|x| = r$. Thus, one has

$$\begin{aligned}
g &= (1 + \sigma_{ij}\phi^i\phi^j)dr \otimes dr + r\sigma_{ij}\frac{\partial\phi^i}{\partial\theta^\alpha}\phi^j dr \otimes d\theta^\alpha + r\sigma_{ij}\frac{\partial\phi^j}{\partial\theta^\alpha}\phi^i d\theta^\alpha \otimes dr + \\
&\quad + (r^2\bar{g}_{\alpha\beta}^{\mathbb{S}^{n-1}} + r^2\sigma_{ij}\frac{\partial\phi^i}{\partial\theta^\alpha}\frac{\partial\phi^j}{\partial\theta^\beta})d\theta^\alpha \otimes d\theta^\beta,
\end{aligned}$$

with $\sigma_{ij} = \mathcal{O}(r^2)$. Comparing with (2.3.8), we see that $\sigma_{ij}\phi^j = 0$ and

$$w^2(r)\bar{g}_E = r^2\bar{g}_{\mathbb{S}^{n-1}} + r^2\sigma_{ij}\frac{\partial\phi^i}{\partial\theta^\alpha}\frac{\partial\phi^j}{\partial\theta^\beta}d\theta^\alpha \otimes d\theta^\beta, \quad r \in (0, +\infty).$$

Now using the fact that $\sigma_{ij} = \mathcal{O}(r^2)$ and Claim 3, and taking the limit as $r \rightarrow 0$, we obtain

$$\bar{g}_E = \bar{g}_{\mathbb{S}^{n-1}}.$$

Therefore, on $M \setminus \{O\}$, we have

$$ds^2 = dr^2 + w(r)^2\bar{g}_{\mathbb{S}^{n-1}}, \quad r \in (0, +\infty),$$

proving that the soliton metric g is rotationally symmetric. Therefore, it follows that (M^n, g, f) is the Bryant soliton, because we know that M^n is diffeomorphic to \mathbb{R}^n and the Bryant soliton is the only non-flat rotationally symmetric gradient steady soliton on \mathbb{R}^n up to scaling. This completes the proof of Theorem 2.1.2. \square

Remark 2.3.1. *Very recently, Perelman's conjecture stating that a κ -noncollapsed 3-dimensional steady gradient Ricci soliton must be the Bryant soliton was verified by S. Brendle [4] using a Killing vector argument:*

Theorem 2.3.1. (Brendle [4]) *Any complete 3-dimensional nonflat κ -noncollapsed gradient steady Ricci soliton must be isometric to the Bryant soliton up to scaling.*

Brendle's argument also works in higher dimensions under the assumption of some asymptotic behavior.

Theorem 2.3.2. (Brendle [5]) *Let (M^n, g, f) be a gradient steady Ricci soliton of dimension $n \geq 4$. Assume that M has positive sectional curvature and is asymptotically cylindrical. Then (M^n, g, f) is rotationally symmetric.*

Chapter 3

Kähler-Ricci Solitons with Harmonic Bochner tensor

Motivated by the work on Ricci solitons with vanishing Weyl tensor and Cotton tensor described above, and as extended exploration, similar considerations lead to the classification of Kähler-Ricci Solitons if we turn to the Bochner tensor.

3.1 Basic Definitions and Identities

First, given holomorphic coordinate $\{z^1, z^2, \dots, z^m\}$ of a complex manifold M in complex dimension m , suppose the complex manifold M has a Hermitian metric $g_{i\bar{j}}$, and this thesis will only focus on the case when the Hermitian metric is Kähler:

Definition 3.1.1. $(M^m, g_{i\bar{j}})$ is Kähler iff

$$\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

is a closed $(1, 1)$ -form.

The complexified tangent space of M^m is a $2m$ complex-dimensional space, which in local coordinate is spanned by

$$\left\{ \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \dots, \frac{\partial}{\partial z^m}, \frac{\partial}{\partial \bar{z}^1}, \frac{\partial}{\partial \bar{z}^2}, \dots, \frac{\partial}{\partial \bar{z}^m} \right\}$$

Similar to the real case, on a Kähler manifold we also have the Christoffel symbols, the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature respectively:

$$\begin{aligned}\Gamma_{ij}^k &= g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j} \\ R_{i\bar{j}k\bar{l}} &= \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l} \\ R_{i\bar{j}} &= g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log(\det(g_{i\bar{j}})) \\ R &= g^{i\bar{j}} R_{i\bar{j}}.\end{aligned}$$

The covariant derivative is given by

$$\begin{aligned}\nabla_i V_j &= \frac{\partial V_j}{\partial z^i} - \Gamma_{ij}^k V_k \\ \nabla_i V_{\bar{j}} &= \frac{\partial V_{\bar{j}}}{\partial z^i} \\ \nabla_{\bar{i}} V_{\bar{j}} &= \frac{\partial V_{\bar{j}}}{\partial \bar{z}^i} - \overline{\Gamma_{ij}^k} V_{\bar{k}} \\ \nabla_{\bar{i}} V_j &= \frac{\partial V_j}{\partial \bar{z}^i},\end{aligned}$$

we have the Ricci identities:

$$\begin{aligned}\nabla_i \nabla_j - \nabla_j \nabla_i &= 0 \\ \nabla_i \nabla_{\bar{j}} V_k - \nabla_{\bar{j}} \nabla_i V_k &= -R_{i\bar{j}k\bar{l}} V_l \\ \nabla_i \nabla_{\bar{j}} V_{\bar{l}} - \nabla_{\bar{j}} \nabla_i V_{\bar{l}} &= -R_{i\bar{j}k\bar{l}} V_{\bar{k}}.\end{aligned}$$

Similarly, if a gradient Ricci soliton is also a Kähler manifold, we call it as a Kähler-Ricci soliton.

Definition 3.1.2. *An m -complex dimensional Kähler manifold $(M^m, g_{i\bar{j}})$ is called a gradient Kähler-Ricci soliton if there is a real-valued smooth function f satisfying the soliton equation*

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \lambda g_{i\bar{j}} \quad (3.1.1)$$

for some constant $\lambda \in \mathbb{R}$ and such that ∇f is a holomorphic vector field, i.e. $\nabla_i \nabla_j f = 0$.

As in Lemma 1.1.1, we have the basic properties for the Kähler-Ricci solitons:

Lemma 3.1.1. *On a gradient Kähler-Ricci soliton (3.1.1), we have*

$$R + |\nabla f|^2 - \lambda f = C_0; \quad (3.1.2)$$

$$R + \Delta f = n\lambda; \quad (3.1.3)$$

$$\nabla_i R_{k\bar{j}} = R_{i\bar{j}k\bar{l}} \nabla_l f; \quad (3.1.4)$$

and

$$\nabla_i R = R_{i\bar{j}} \nabla_j f. \quad (3.1.5)$$

On Kähler manifolds, there is a tensor similar to the Weyl tensor, called the Bochner tensor, which is defined as

$$\begin{aligned} W_{i\bar{j}k\bar{l}} = & R_{i\bar{j}k\bar{l}} + \frac{R}{(n+1)(n+2)} (g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}) \\ & - \frac{1}{n+2} (R_{i\bar{j}} g_{k\bar{l}} + R_{k\bar{l}} g_{i\bar{j}} + R_{i\bar{l}} g_{k\bar{j}} + R_{k\bar{j}} g_{i\bar{l}}). \end{aligned} \quad (3.1.6)$$

We also define its divergence as the tensor $C_{i\bar{j}k}$, which is a parallel notion of the Cotton tensor:

$$\begin{aligned} C_{i\bar{j}k} = & \nabla_q W_{i\bar{j}k\bar{q}} \\ = & \frac{n}{n+2} \nabla_i R_{k\bar{j}} - \frac{n}{(n+1)(n+2)} (g_{k\bar{j}} \nabla_i R + g_{i\bar{j}} \nabla_k R). \end{aligned} \quad (3.1.7)$$

3.2 The Result and The Proof

By using a similar argument to that of [14], Y. Su and K. Zhang [59] first proved a rigidity result for the Kähler-Ricci soliton: assuming the vanishing of the Bochner tensor, a Kähler Ricci soliton must be Kähler-Einstein, and hence a quotient of the corresponding space-form. Later, joint with Meng Zhu, we improved the result by only assuming the harmonic Bochner tensor, namely the vanishing of the tensor $C_{i\bar{j}k}$.

Theorem 3.2.1. (**—Zhu [27]**) *Any complete gradient Kähler-Ricci soliton with harmonic Bochner tensor must be isometric to $N^k \times \mathbb{C}^{n-k}$, where N^k is Kähler-Einstein and \mathbb{C}^{n-k} has a flat metric.*

The proof, which is a pointwise argument, is different from that of Fernández-Lopéz and García-Río [33] and Munteanu-Sesum [49] for the harmonic Weyl case, since the $C_{i\bar{j}k}$ tensor does not have such a nice identity as equation (2.1.5) for the D_{ijk} tensor.

From now on, we assume that $(M^n, g_{i\bar{j}}, f)$ is a gradient Kähler-Ricci soliton with harmonic Bochner tensor so that

$$\nabla_i R_{k\bar{j}} = \frac{1}{n+1} (\nabla_i R g_{k\bar{j}} + \nabla_k R g_{i\bar{j}}). \quad (3.2.1)$$

Lemma 3.2.1. *We have*

$$\begin{aligned} & \lambda R_{i\bar{j}} - R_{i\bar{j}k\bar{l}} R_{\bar{k}l} \\ = & \frac{1}{n+1} \left[\frac{1}{n+1} \nabla_k R \nabla_{\bar{k}} f g_{i\bar{j}} + (\lambda R - |Rc|^2) g_{i\bar{j}} - \frac{n}{n+1} \nabla_i R \nabla_{\bar{j}} f \right. \\ & \left. + \lambda R_{i\bar{j}} - R_{i\bar{k}} R_{k\bar{j}} \right], \end{aligned} \quad (3.2.2)$$

and

$$\begin{aligned} & 2(n+1)\lambda \nabla_i R - 2R \nabla_i R - 2R_{i\bar{j}} \nabla_j R \\ = & -\frac{1}{n+1} \nabla_i R |\nabla f|^2 - \frac{1}{n+1} \nabla_k R \nabla_{\bar{k}} f \nabla_i f. \end{aligned} \quad (3.2.3)$$

Proof. On the one hand, by differentiating (3.1.5), we obtain

$$\Delta R = \nabla_k \nabla_{\bar{k}} R = \nabla_k R \nabla_{\bar{k}} f + R_{k\bar{l}} \nabla_{\bar{k}} \nabla_l f.$$

From (3.2.1), we obtain

$$\begin{aligned} \nabla_k \nabla_{\bar{k}} R_{i\bar{j}} &= \frac{1}{n+1} (\Delta R g_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} R) \\ &= \frac{1}{n+1} (\nabla_k R \nabla_{\bar{k}} f g_{i\bar{j}} + R_{k\bar{l}} \nabla_{\bar{k}} \nabla_l f g_{i\bar{j}} + \nabla_i R_{k\bar{j}} \nabla_{\bar{k}} f + R_{k\bar{j}} \nabla_i \nabla_{\bar{k}} f) \\ &= \frac{1}{n+1} \left[\nabla_k R \nabla_{\bar{k}} f g_{i\bar{j}} + (\lambda R - |Rc|^2) g_{i\bar{j}} + \frac{1}{n+1} \nabla_i R \nabla_{\bar{j}} f \right. \\ & \quad \left. + \frac{1}{n+1} \nabla_k R \nabla_{\bar{k}} f g_{i\bar{j}} + \lambda R_{i\bar{j}} - R_{i\bar{k}} R_{k\bar{j}} \right]. \end{aligned} \quad (3.2.4)$$

On the other hand, by differentiating (3.1.4), we have

$$\begin{aligned}
\nabla_k \nabla_{\bar{k}} R_{i\bar{j}} &= \nabla_i R_{\bar{j}l} \nabla_{\bar{l}} f + R_{i\bar{j}k\bar{l}} \nabla_{\bar{k}} \nabla_l f \\
&= \nabla_k R_{i\bar{j}} \nabla_{\bar{k}} f + R_{i\bar{j}k\bar{l}} \nabla_{\bar{k}} \nabla_l f \\
&= \nabla_k R_{i\bar{j}} \nabla_{\bar{k}} f + \lambda R_{i\bar{j}} - R_{i\bar{j}k\bar{l}} R_{\bar{k}l}.
\end{aligned}$$

Now, by plugging in formula (3.2.4), we obtain (3.2.2).

Next, by taking the divergence on both sides of (3.2.2),

$$\begin{aligned}
&\lambda \nabla_i R - (\nabla_i R_{k\bar{l}}) R_{\bar{k}l} - R_{i\bar{j}k\bar{l}} \nabla_j R_{\bar{k}l} \\
= &\frac{1}{n+1} \left[\frac{1}{n+1} \nabla_i \nabla_k R \nabla_{\bar{k}} f + \frac{1}{n+1} \nabla_k R \nabla_i \nabla_{\bar{k}} f + \lambda \nabla_i R - \nabla_i |Rc|^2 \right. \\
&\quad \left. - \frac{n}{n+1} \nabla_j \nabla_i R \nabla_{\bar{j}} f - \frac{n}{n+1} \nabla_i R \Delta f + \lambda \nabla_i R - (\nabla_j R_{i\bar{k}}) R_{k\bar{j}} - R_{i\bar{k}} \nabla_k R \right] \\
= &\frac{1}{n+1} \left[\frac{1}{n+1} \nabla_i R_{k\bar{l}} \nabla_l f \nabla_{\bar{k}} f + \frac{\lambda}{n+1} \nabla_i R - \frac{1}{n+1} R_{i\bar{k}} \nabla_k R + \lambda \nabla_i R - 2R_{k\bar{l}} \nabla_i R_{\bar{k}l} \right. \\
&\quad \left. - \frac{n}{n+1} \nabla_i R_{j\bar{k}} \nabla_k f \nabla_{\bar{j}} f - \frac{\lambda n^2}{n+1} \nabla_i R + \frac{n}{n+1} R \nabla_i R + \lambda \nabla_i R \right. \\
&\quad \left. - R_{k\bar{j}} \nabla_i R_{j\bar{k}} - R_{i\bar{k}} \nabla_k R \right].
\end{aligned}$$

It follows that,

$$\begin{aligned}
&\lambda \nabla_i R - (\nabla_i R_{k\bar{l}}) R_{\bar{k}l} - R_{i\bar{j}k\bar{l}} \nabla_j R_{\bar{k}l} \\
= &\frac{1}{n+1} \left[-\frac{n-1}{(n+1)^2} \nabla_i R |\nabla f|^2 - \frac{n-1}{(n+1)^2} \nabla_k R \nabla_{\bar{k}} f \nabla_i f \right. \\
&\quad \left. + (3-n) \lambda \nabla_i R - \left(1 + \frac{1}{n+1}\right) R_{i\bar{k}} \nabla_k R - 3R_{k\bar{l}} \nabla_i R_{\bar{k}l} + \frac{n}{n+1} R \nabla_i R \right].
\end{aligned}$$

We note,

$$\begin{aligned}
R_{l\bar{k}} \nabla_i R_{k\bar{l}} &= \frac{1}{n+1} R_{l\bar{k}} (\nabla_i R g_{k\bar{l}} + \nabla_k R g_{i\bar{l}}) \\
&= \frac{1}{n+1} R \nabla_i R + \frac{1}{n+1} R_{i\bar{j}} \nabla_j R,
\end{aligned}$$

and

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}\nabla_j R_{l\bar{k}} &= \frac{1}{n+1}R_{i\bar{j}k\bar{l}}(\nabla_j Rg_{l\bar{k}} + \nabla_l Rg_{j\bar{k}}) \\
&= \frac{1}{n+1}R_{i\bar{j}}\nabla_j R + \frac{1}{n+1}R_{i\bar{l}}\nabla_l R \\
&= \frac{2}{n+1}R_{i\bar{j}}\nabla_j R.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\lambda\nabla_i R - \frac{1}{n+1}R\nabla_i R - \frac{3}{n+1}R_{i\bar{j}}\nabla_j R \\
&= \lambda\nabla_i R - (\nabla_i R_{k\bar{l}})R_{k\bar{l}} - R_{i\bar{j}k\bar{l}}\nabla_j R_{k\bar{l}} \\
&= \frac{1}{n+1}\left[-\frac{n-1}{(n+1)^2}\nabla_i R|\nabla f|^2 - \frac{n-1}{(n+1)^2}\nabla_k R\nabla_{\bar{k}}f\nabla_i f\right. \\
&\quad \left.+(3-n)\lambda\nabla_i R - \left(1 + \frac{1}{n+1}\right)R_{i\bar{k}}\nabla_k R - 3R_{k\bar{l}}\nabla_i R_{k\bar{l}} + \frac{n}{n+1}R\nabla_i R\right].
\end{aligned}$$

Therefore, formula (3.2.3) follows easily. \square

Now, suppose that $\nabla f \neq 0$ at some point p . Then we may choose an orthonormal frame $\{e_1, e_2, \dots, e_n\}$ of holomorphic vector fields at p such that e_1 is parallel to ∇f . Therefore, we have $|\nabla_1 f| = |\nabla f|$ and $\nabla_k f = 0$ for $k = 2, \dots, n$.

Lemma 3.2.2. *Suppose $\nabla f \neq 0$ at p . Then, under the frame $\{e_1, e_2, \dots, e_n\}$ chosen above, we have*

$$R_{k\bar{1}} = R_{1\bar{k}} = 0 \quad \text{for } k \geq 2.$$

Proof. From (3.1.4) and (3.2.1), we have at p ,

$$R_{i\bar{j}k\bar{1}}\nabla_1 f = \frac{1}{n+1}(\nabla_i Rg_{k\bar{j}} + \nabla_k Rg_{i\bar{j}}) = \frac{1}{n+1}(R_{i\bar{1}}g_{k\bar{j}} + R_{k\bar{1}}g_{i\bar{j}})\nabla_1 f.$$

It follows that

$$R_{i\bar{j}k\bar{1}} = \frac{1}{n+1}(R_{i\bar{1}}g_{k\bar{j}} + R_{k\bar{1}}g_{i\bar{j}}).$$

In particular, for $k \geq 2$, we have

$$R_{1\bar{1}k\bar{1}} = \frac{1}{n+1}R_{k\bar{1}} \quad \text{and} \quad R_{1\bar{k}1\bar{1}} = 0.$$

However, on the other hand, it is easy to see that

$$R_{1\bar{1}k\bar{1}} = \overline{R_{\bar{1}1\bar{k}1}} = \overline{R_{1\bar{k}1\bar{1}}} = 0.$$

Therefore, $R_{k\bar{1}} = R_{1\bar{k}} = 0$ for $k \geq 2$.

□

Lemma 3.2.2 tells us that ∇f is an eigenvector of the Ricci curvature tensor. Thus we may choose another orthonormal frame $\{w_1 = e_1, w_2, \dots, w_n\}$ at p such that $|\nabla_1 f| = |\nabla f|$ and the Ricci curvature is diagonalized at p , i.e.

$$R_{i\bar{j}} = R_{i\bar{i}}\delta_{ij}.$$

Proposition 3.2.1. *Suppose that $\nabla f \neq 0$ at p . Then under the orthonormal frame $\{w_1, w_2, \dots, w_n\}$ chosen above, we have the following identities at p :*

$$n\lambda R_{1\bar{1}} - RR_{1\bar{1}} = \lambda R - |Rc|^2 - \frac{n-1}{n+1}R_{1\bar{1}}|\nabla f|^2, \quad (3.2.5)$$

and

$$(n+1)\lambda R_{1\bar{1}} - RR_{1\bar{1}} - R_{1\bar{1}}^2 = -\frac{1}{n+1}R_{1\bar{1}}|\nabla f|^2. \quad (3.2.6)$$

Proof. In (3.2.2), setting $i = j = 1$, we have

$$\begin{aligned}
& \lambda R_{1\bar{1}} - \frac{1}{n+1} R_{1\bar{1}}^2 - \frac{1}{n+1} R R_{1\bar{1}} \\
= & \lambda R_{1\bar{1}} - \frac{2}{n+1} R_{1\bar{1}}^2 - \frac{1}{n+1} R_{1\bar{1}} (R - R_{1\bar{1}}) \\
= & \lambda R_{1\bar{1}} - \frac{2}{n+1} R_{1\bar{1}}^2 - \frac{1}{n+1} R_{1\bar{1}} \sum_{k=2}^n R_{k\bar{k}} \\
= & \lambda R_{1\bar{1}} - R_{1\bar{1}1\bar{1}} R_{1\bar{1}} - \sum_{k=2}^n R_{1\bar{1}k\bar{k}} R_{k\bar{k}} \\
= & \lambda R_{1\bar{1}} - \sum_{k=1}^n R_{1\bar{1}k\bar{k}} R_{k\bar{k}} \\
= & \frac{1}{n+1} \left[\frac{1}{n+1} R_{1\bar{1}} |\nabla f|^2 + \lambda R - |Rc|^2 - \frac{n}{n+1} R_{1\bar{1}} |\nabla f|^2 + \lambda R_{1\bar{1}} - R_{1\bar{1}}^2 \right].
\end{aligned}$$

Thus, formula (3.2.5) follows immediately.

Next, by setting $i = 1$ in (3.2.3) and dividing both sides of the equation by $\nabla_1 f$, (3.2.6) follows. □

Proposition 3.2.2. *At a point p where $\nabla f \neq 0$, we have either*

$$Rc(\nabla f, \nabla f) = 0,$$

or

$$Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4} |\nabla f|^2.$$

Proof. Since at point p , $\nabla f \neq 0$, formula (3.2.6) implies that in a neighborhood of p we have

$$\left[(n+1)\lambda - R - \frac{R_{j\bar{i}} \nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} + \frac{1}{n+1} |\nabla f|^2 \right] \frac{R_{j\bar{i}} \nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} = 0. \quad (3.2.7)$$

Therefore, there are two possibilities

$R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f = 0$ at p , or $R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f \neq 0$ at p .

The former case is one possible conclusion of the proposition.

In the latter case, near p we have

$$-(n+1)\lambda + R + \frac{R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} - \frac{1}{n+1}|\nabla f|^2 = 0.$$

Taking the covariant derivative on both sides gives us

$$\begin{aligned} 0 &= \nabla_k R + \frac{1}{|\nabla f|^2}(\nabla_i f \nabla_{\bar{j}} f \nabla_k R_{j\bar{i}} + R_{j\bar{i}} \nabla_i f \nabla_k \nabla_{\bar{j}} f) - \frac{\nabla_j f \nabla_k \nabla_{\bar{j}} f}{|\nabla f|^4} R_{i\bar{i}} \nabla_i f \nabla_{\bar{i}} f \\ &\quad - \frac{1}{n+1}(\nabla_j f \nabla_k \nabla_{\bar{j}} f) \\ &= \nabla_k R + \frac{1}{(n+1)|\nabla f|^2} \nabla_i f \nabla_{\bar{j}} f (\nabla_k R g_{j\bar{i}} + \nabla_j R g_{k\bar{i}}) + \frac{1}{|\nabla f|^2} (\lambda \nabla_k R - R_{k\bar{j}} \nabla_j R) \\ &\quad - \frac{\lambda \nabla_k f - \nabla_k R}{|\nabla f|^4} \nabla_i R \nabla_{\bar{i}} f - \frac{1}{n+1} (\lambda \nabla_k f - \nabla_k R). \end{aligned}$$

Evaluating the identity above at p under the orthonormal frame $\{w_1, w_2, \dots, w_n\}$ yields

$$\begin{aligned} 0 &= R_{1\bar{1}} + \frac{2}{(n+1)|\nabla f|^2} R_{1\bar{1}} |\nabla f|^2 + \frac{1}{|\nabla f|^2} (\lambda R_{1\bar{1}} - R_{1\bar{1}}^2) \\ &\quad - \frac{\lambda - R_{1\bar{1}}}{|\nabla f|^4} R_{1\bar{1}} |\nabla f|^2 - \frac{1}{n+1} (\lambda - R_{1\bar{1}}) \\ &= \frac{n+4}{n+1} R_{1\bar{1}} - \frac{1}{n+1} \lambda. \end{aligned}$$

Thus, we have $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4} |\nabla f|^2$ whenever $Rc(\nabla f, \nabla f) \neq 0$.

□

Now we are ready to prove the main theorems.

First, we may assume that f is not a constant function, otherwise M is Kähler-Einstein from the soliton equation.

Proof of theorem 3.2.1 (Steady Case): For steady Kähler-Ricci solitons, we have $\lambda = 0$. From Proposition 3.2.2, we know that $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4} |\nabla f|^2 = 0$

whenever $Rc(\nabla f, \nabla f) \neq 0$, which is a contradiction. Therefore, we always have $Rc(\nabla f, \nabla f) = 0$. Then (3.2.5) implies that $Rc = 0$ in the set $\{p \in M | \nabla f(p) \neq 0\}$. On the other hand, by the soliton equation, it is easy to see that we also have $Rc = 0$ in the interior of the set $\{p \in M | \nabla f(p) = 0\}$. Thus the steady soliton M must be Kähler-Ricci flat. □

Proof of theorem 3.2.1(Shrinking and Expanding Case): For shrinking and expanding Kähler-Ricci solitons, we have $\lambda \neq 0$.

In this case, from Proposition 3.2.2 and the continuity of $\frac{Rc(\nabla f, \nabla f)}{|\nabla f|^2}$, we conclude that in each component of the open set $A = \{p \in M | \nabla f(p) \neq 0\}$, we have either $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2$ or $Rc(\nabla f, \nabla f) = 0$.

If $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2$ in some component Ω of A , then at any point $p \in \Omega$ we have $R_{1\bar{1}} = \frac{\lambda}{n+4}$ and $\nabla R(p) = \frac{\lambda}{n+4}\nabla f(p)$. Therefore, we have $\nabla R = \frac{\lambda}{n+4}\nabla f$ in Ω . It then follows that $R = \frac{\lambda}{n+4}f + C$ in Ω . Thus (3.2.6) implies that $|\nabla f|^2 = \frac{n+1}{n+4}\lambda f + C$ in Ω . Since $R + |\nabla f|^2 - \lambda f = C_0$, we have $f = C_1$ in Ω , which contradicts the fact that $\nabla f \neq 0$ in Ω .

Therefore, we must have $Rc(\nabla f, \nabla f) = 0$ in A . Since f is a constant in the interior of $M \setminus A$, we have $Rc(\nabla f, \nabla f) = 0$ on the whole manifold M . It follows that $\nabla R = 0$ on M . Then (3.2.1) implies that the Ricci curvature tensor is parallel on M . Therefore, by the de Rham decomposition theorem, the universal cover of M is isometric to $N^{n-1} \times \mathbb{C}$, where N is again an $n - 1$ dimensional Kähler-Ricci soliton with harmonic Bochner tensor. Thus by induction, we can finally see that M is isometric to a quotient of the product of a Kähler-Einstein manifold and the complex Euclidean space. □

Chapter 4

Bach-flat Quasi-Einstein Manifolds

With some modification, quasi-Einstein manifolds can also be studied in a similar way to Ricci solitons.

Definition 4.0.1. *An n -dimensional Riemannian manifold (M^n, g, f) is called $(\lambda, n+m)$ -Einstein if the Ricci curvature satisfies:*

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{m} \nabla_i f \nabla_j f = \lambda g_{ij} \quad (4.0.1)$$

The reason to study this equation is that in [1], it is shown that when m is a positive integer, these $(\lambda, n+m)$ -Einstein metrics are exactly those n -dimensional manifolds which are the warped product base of an $(n+m)$ -dimensional Einstein metrics. More precisely, $(M \times F^m, g + e^{-2f/m} g_F)$ is an Einstein manifold with Einstein constant λ , and F^m is another Einstein manifold with some proper Einstein constant. Therefore it is important to understand this equation in order to understand the geometry of Einstein manifolds.

There are many examples for quasi-Einstein metrics. Einstein metrics and products of them are trivial examples of quasi-Einstein metrics. The first non-trivial example comes from the Schwarzschild metric, which is a 4-dimensional doubly warped product metric on $\mathbb{S}^2 \times \mathbb{R}^2$, and viewed in two different ways, this will lead to a $(0,2+2)$ -Einstein metric on \mathbb{R}^2 , or a $(0,3+1)$ -Einstein metric on $[0, +\infty) \times \mathbb{S}^2$. Lü-Page-Pope [48] construct non-trivial quasi-Einstein metrics on \mathbb{S}^2 bundles over

Kähler-Einstein bases. C. Böhm [2] constructed interesting quasi-Einstein metrics on spheres and product of spheres. In [42], He-Petersen-Wylie construct nontrivial quasi-Einstein metric examples on solvable Lie groups.

It is easy to observe that when $m = \infty$, the quasi-Einstein equation reduces to the Ricci soliton equation, and thus it is expected that quasi-Einstein manifolds could behave similarly to the Ricci solitons to some extent. However, they are not identical, since Case-Shu-Wei [21] showed that there is no non-trivial Kähler quasi-Einstein metric on a compact manifold, while we do have compact Kähler Ricci-soliton examples. Qian [58] showed that a quasi-Einstein metric must be compact if $\lambda > 0$ and $m > 0$. In [44], D.-S. Kim and Y.-H. Kim showed that the a compact quasi-Einstein metric with $\lambda \leq 0$ is trivial. Analogous to Ricci solitons, G. Catino, C. Mantegazza, L. Mazzieri and M. Rimoldi [22] prove rotational symmetry of locally conformally flat quasi-Einstein manifolds. C. He, P. Petersen and P. Wylie [41] get the same classification result for quasi-Einstein manifolds with slightly weaker condition. Later we found the Bach flat condition fits into the argument as well for the compact case.

Theorem 4.0.2. (**—He [26]**) *Suppose $(M^n, g, f)(n \geq 4)$ is a compact Quasi-Einstein manifold with $m \neq 0, 1, 2 - n$. If we further assume it has flat Bach tensor, then it must have harmonic Weyl tensor and $W(\nabla f, \cdot, \cdot, \cdot) = 0$.*

Then using Theorem 1.5 in [41], we can get the classification result that M is either Einstein or its metric takes the form $g = dt^2 + \psi^2(t)g_L$, where g_L is an Einstein metric with positive Einstein constant.

To begin the computation, first we need to establish the basic formulas for the curvature tensors.

Lemma 4.0.3. *Suppose (M^n, g, f) is $(\lambda, n + m)$ -Einstein with $m \neq 0, 1, 2 - n$, then*

$$R_{ij}\nabla_j f = \frac{m}{2m-2}\nabla_i R + \frac{\lambda(n-1) - R}{m-1}\nabla_i f$$

Proof.

$$\begin{aligned}
\nabla_i R &= 2\nabla_j R_{ij} \\
&= 2\nabla_j(-\nabla_i \nabla_j f + \frac{1}{m} \nabla_i f \nabla_j f) \\
&= 2(\nabla_i \Delta f - R_{il} \nabla_l f + \frac{1}{m} \nabla_j \nabla_i f \nabla_j f + \frac{1}{m} \Delta f \nabla_i f)
\end{aligned}$$

Plug the trace version of (4.0.1), namely $R + \Delta f - \frac{1}{m} |\nabla f|^2 = \lambda n$ into the above expression:

$$-\nabla_i R = 2(-\frac{2}{m} \nabla_i \nabla_j f \nabla_j f - R_{il} \nabla_l f + \frac{1}{m} \nabla_j f \nabla_i \nabla_j f + \frac{1}{m} \nabla_i f (\lambda n + \frac{1}{m} |\nabla f|^2 - R))$$

Plug (4.0.1) into the above expression and simplify the expression as:

$$-\nabla_i R = 2(\frac{1-m}{m} R_{ij} \nabla_j f + \frac{\lambda(n-1)}{m} \nabla_i f - \frac{R}{m} \nabla_i f)$$

By rearrangement of the terms we get the lemma. □

We can also define a D tensor on a quasi-Einstein manifold:

$$\begin{aligned}
D_{ijk} &= \frac{1}{n-2} (\nabla_i f R_{jk} - \nabla_j f R_{ik}) \\
&\quad + \frac{m}{2(n-1)(n-2)(m-1)} (\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\
&\quad + \frac{\lambda(n-1) - mR}{(n-1)(n-2)(m-1)} (\nabla_i f g_{jk} - \nabla_j f g_{ik})
\end{aligned} \tag{4.0.2}$$

Then we can carry over Lemma 2.2.1 onto quasi-Einstein manifolds as:

Lemma 4.0.4. *Suppose (M^n, g, f) is $(\lambda, n+m)$ -Einstein with $m \neq 0, 1, 2-n$. Then*

$$C_{ijk} + W_{ijkl} \nabla_l f = \frac{m+n-2}{m} D_{ijk},$$

where C_{ijk} is the Cotton tensor defined as (2.1.2) and W_{ijkl} is the Weyl tensor defined as (2.1.1).

Proof.

$$\begin{aligned}
C_{ijk} &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\
&= \nabla_i(-\nabla_j \nabla_k f + \frac{1}{m} \nabla_j f \nabla_k f) - \nabla_j(-\nabla_i \nabla_k f + \frac{1}{m} \nabla_i f \nabla_k f) \\
&\quad - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\
&= -R_{ijkl} \nabla_l f + \frac{1}{m}(\nabla_j f \nabla_i \nabla_k f - \nabla_i f \nabla_j \nabla_k f) \\
&\quad - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\
&= -R_{ijkl} \nabla_l f + \frac{1}{m}(\nabla_i f R_{jk} - \nabla_j f R_{ik}) - \frac{\lambda}{m}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\
&\quad - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}).
\end{aligned}$$

By plugging in the definition of the Weyl tensor (2.1.1), we get:

$$\begin{aligned}
C_{ijk} &= -W_{ijkl} \nabla_l f + \frac{m+n-2}{m(n-2)}(\nabla_i f R_{jk} - \nabla_j f R_{ik}) \\
&\quad + \left(-\frac{R}{(n-1)(n-2)} - \frac{\lambda}{m} \right) (\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\
&\quad - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) + \frac{1}{n-2}(g_{jk} R_{il} \nabla_l f - g_{ik} R_{jl} \nabla_l f).
\end{aligned}$$

Applying Lemma 4.0.3, we will have:

$$\begin{aligned}
C_{ijk} + W_{ijkl} \nabla_l f &= \frac{m+n-2}{m(n-2)}(\nabla_i f R_{jk} - \nabla_j f R_{ik}) \\
&\quad + \frac{m+n-2}{2(n-1)(n-2)(m-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\
&\quad + \frac{(m+n-2)(\lambda(n-1) - mR)}{m(m-1)(n-1)(n-2)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\
&= \frac{m+n-2}{m} D_{ijk}.
\end{aligned}$$

□

As in Proposition 2.2.1, we also have:

Proposition 4.0.3. *Suppose (M^n, g, f) is a $(\lambda, n+m)$ -Einstein with $m \neq 0, 1, 2-n$, then for any regular value c of f , define $\Sigma = \{x \in M \mid f(x) = c\}$, we have:*

$$|D|^2 = \frac{2|\nabla f|^4}{(n-2)^2} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 + \frac{m^2}{2(n-1)(n-2)(m-1)^2} |\nabla^\Sigma R|^2,$$

where h_{ab} denotes the second fundamental form of Σ .

Proof. Compute $|D|^2$ first:

$$\begin{aligned} |D|^2 &= \frac{2}{(n-2)^2} (|Rc|^2 |\nabla f|^2 - R_{jk} \nabla_j f R_{ik} \nabla_i f) \\ &\quad + \frac{m^2}{2(n-1)(n-2)^2(m-1)^2} |\nabla R|^2 \\ &\quad + \frac{2(\lambda(n-1) - mR)^2}{(n-1)(n-2)^2(m-1)^2} |\nabla f|^2 \\ &\quad + \frac{2m}{(n-1)(n-2)^2(m-1)} (R \nabla R \cdot \nabla f - \nabla_j R R_{jk} \nabla_k f) \\ &\quad + \frac{4(\lambda(n-1) - mR)}{(n-1)(n-2)^2(m-1)} (R |\nabla f|^2 - R_{jk} \nabla_j f \nabla_k f) \\ &\quad + \frac{2m(\lambda(n-1) - mR)}{(n-1)(n-2)^2(m-1)^2} \nabla R \cdot \nabla f. \end{aligned}$$

Applying Lemma 4.0.3:

$$\begin{aligned} |D|^2 &= \frac{2}{(n-2)^2} |Rc|^2 |\nabla f|^2 - |\nabla R|^2 \frac{m^2 n}{2(n-1)(n-2)^2(m-1)^2} \\ &\quad + |\nabla f|^2 \frac{-2n[\lambda(n-1)]^2 - 2R^2(m^2 + n-1) + 4R[\lambda(n-1)](m+n-1)}{(n-1)(n-2)^2(m-1)^2} \\ &\quad + \nabla R \cdot \nabla f \frac{-2mn[\lambda(n-1)] + R(2mn - 2m + 2m^2)}{(n-1)(n-2)^2(m-1)^2}. \end{aligned} \tag{4.0.3}$$

To compute the RHS, let's denote e_1, e_2, \dots, e_n as the local orthonormal frame with $e_1 = \nabla f / |\nabla f|$, and whenever we use the indices a, b, c , we refer to the tangent direction of Σ .

$$R_{11} = \frac{R_{ij} \nabla_i f \nabla_j f}{|\nabla f|^2} = \frac{m}{2m-2} \frac{\nabla R \cdot \nabla f}{|\nabla f|^2} + \frac{\lambda(n-1) - R}{m-1}; \tag{4.0.4}$$

$$R_{1a} = \frac{R_{aj} \nabla_j f}{|\nabla f|} = \frac{1}{|\nabla f|} \left(\frac{m}{2m-2} \nabla_a R \right); \quad (4.0.5)$$

$$h_{ab} = \langle e_a, \nabla_b \frac{\nabla f}{|\nabla f|} \rangle = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{\lambda g_{ab} - R_{ab}}{|\nabla f|}. \quad (4.0.6)$$

Thus,

$$H = \frac{\lambda(n-1) - R + R_{11}}{|\nabla f|} = \frac{1}{|\nabla f|} \left(\frac{m}{2m-2} \frac{\nabla R \cdot \nabla f}{|\nabla f|^2} + \frac{m(\lambda(n-1) - R)}{m-1} \right). \quad (4.0.7)$$

Then, we can compute $|h|^2$:

$$\begin{aligned} |h|^2 &= \frac{1}{|\nabla f|^2} |\lambda g_{ab} - R_{ab}|^2 \\ &= \frac{1}{|\nabla f|^2} (\lambda^2(n-1) + |R_{ab}|^2 + 2\lambda(-R + R_{11})) \\ &= \frac{1}{|\nabla f|^2} \left(\lambda^2(n-1) + 2\lambda(R_{11} - R) + |Rc|^2 - |R_{11}|^2 - 2 \sum |R_{1a}|^2 \right). \end{aligned}$$

Plugging in (4.0.4) and (4.0.5):

$$\begin{aligned} |h|^2 &= \frac{1}{|\nabla f|^2} |Rc|^2 + \frac{1}{|\nabla f|^2} \lambda^2(n-1) \\ &\quad + \frac{2\lambda}{|\nabla f|^2} \left(\frac{m}{2m-2} \frac{\nabla R \cdot \nabla f}{|\nabla f|^2} + \frac{\lambda(n-1) - mR}{m-1} \right) \\ &\quad - \frac{1}{|\nabla f|^2} \left(\frac{m^2}{4(m-1)^2} \frac{(\nabla R \cdot \nabla f)^2}{|\nabla f|^4} + \frac{(\lambda(n-1) - R)^2}{(m-1)^2} + \frac{m(\lambda(n-1) - R)}{(m-1)^2} \frac{\nabla R \cdot \nabla f}{|\nabla f|^2} \right) \\ &\quad - \frac{1}{|\nabla f|^4} \frac{m^2}{2(m-1)^2} |\nabla^\Sigma R|^2. \end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{2|\nabla f|^4}{(n-2)^2} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 = \frac{2|\nabla f|^4}{(n-2)^2} \left(|h|^2 - \frac{H^2}{n-1} \right) \\
& = \frac{2}{(n-2)^2} \left\{ |Rc|^2 |\nabla f|^2 + |\nabla f|^2 \lambda^2 (n-1) + \frac{\lambda m}{m-1} \nabla R \cdot \nabla f \right. \\
& \quad + 2\lambda |\nabla f|^2 \frac{\lambda(n-1) - mR}{m-1} - \frac{m^2}{4(m-1)^2} |\nabla_1 R|^2 \frac{n}{n-1} \\
& \quad - |\nabla f|^2 \frac{(\lambda(n-1) - R)^2}{(m-1)^2} \frac{n + m^2 - 1}{n-1} \\
& \quad - \frac{m(\lambda(n-1) - R)}{(m-1)^2} \frac{n + m - 1}{n-1} \nabla R \cdot \nabla f \\
& \quad \left. - \frac{m^2}{(m-1)^2} |\nabla^\Sigma R|^2 \right\}.
\end{aligned}$$

Adding the term $\frac{m^2}{2(m-1)^2(n-1)(n-2)} |\nabla^\Sigma R|^2$:

$$\begin{aligned}
& \frac{2|\nabla f|^4}{(n-2)^2} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 + \frac{m^2}{2(m-1)^2(n-1)(n-2)} |\nabla^\Sigma R|^2 \\
& = \frac{2}{(n-2)^2} \left\{ |Rc|^2 |\nabla f|^2 + |\nabla f|^2 \lambda^2 (n-1) + \frac{\lambda m}{m-1} \nabla R \cdot \nabla f \right. \\
& \quad + 2\lambda |\nabla f|^2 \frac{\lambda(n-1) - mR}{m-1} - \frac{m^2}{4(m-1)^2} |\nabla R|^2 \frac{n}{n-1} \\
& \quad - |\nabla f|^2 \frac{(\lambda(n-1) - R)^2}{(m-1)^2} \frac{n + m^2 - 1}{n-1} \\
& \quad \left. - \frac{m(\lambda(n-1) - R)}{(m-1)^2} \frac{n + m - 1}{n-1} \nabla R \cdot \nabla f \right\} \\
& = \frac{2}{(n-2)^2} |Rc|^2 |\nabla f|^2 - |\nabla R|^2 \frac{m^2 n}{2(n-1)(n-2)^2(m-1)^2} \\
& \quad + |\nabla f|^2 \frac{-2n[\lambda(n-1)]^2 - 2R^2(m^2 + n - 1) + 4R[\lambda(n-1)](m + n - 1)}{(n-1)(n-2)^2(m-1)^2} \\
& \quad + \nabla R \cdot \nabla f \frac{-2mn[\lambda(n-1)] + R(2mn - 2m + 2m^2)}{(n-1)(n-2)^2(m-1)^2}.
\end{aligned}$$

Comparing with the expression of $|D|^2$ (4.0.3), the proposition follows. \square

Then, analogously to Proposition 2.2.2, we have for the quasi-Einstein case:

Proposition 4.0.4. *Suppose (M^n, g, f) is a $(\lambda, n + m)$ -Einstein manifold with $m \neq 0, 1, 2 - n$ and $D_{ijk} = 0$. Then for any regular value c of f , with $\Sigma = \{x \in M \mid f(x) = c\}$, we have:*

- (a) $|\nabla f|^2$ and the scalar curvature R of (M^n, g_{ij}, f) are constant on Σ ;
- (b) $R_{1abc} = 0$, here $e_1 = \nabla f / |\nabla f|$ is an eigenvector of Rc , and e_a, e_b, e_c are any vectors in the tangent direction of Σ ;
- (c) the second fundamental form h_{ab} of Σ is of the form $h_{ab} = \frac{H}{n-1}g_{ab}$;
- (d) the mean curvature H is constant on Σ ;
- (e) on Σ , the Ricci tensor of (M^n, g_{ij}, f) either has a unique eigenvalue λ , or has two distinct eigenvalues λ and μ of multiplicity 1 and $n - 1$ respectively. In either case, $e_1 = \nabla f / |\nabla f|$ is an eigenvector of λ .

Proof. The proof is similar to that of Proposition 2.2.2. □

Again, similarly to Lemma 2.3.2, we can establish the analogue for quasi-Einstein manifolds:

Lemma 4.0.5. *Suppose (M^n, g, f) is a $(\lambda, n + m)$ -Einstein with $m \neq 0, 1, 2 - n$ and $D_{ijk} = 0$. Then $C_{ijk} = 0$ and $W_{ijkl}\nabla_l f = 0$.*

Proof. An argument similar to that of Lemma 2.3.2 will work. □

Now, we are ready to conclude the proof for Theorem 4.0.2:
From equation (2.2.2) and Lemma 4.0.4:

$$\begin{aligned} (n-2)B_{ij} &= \nabla_k C_{kij} + R_{kl}W_{ikjl} \\ &= \frac{m+n-2}{m}\nabla_k D_{kij} + \frac{n-3}{n-2}C_{lij}\nabla_l f + \frac{1}{m}W_{ikjl}\nabla_k f\nabla_l f. \end{aligned}$$

Then we have:

$$\begin{aligned}
\frac{m(n-2)}{m+n-2} \int_M B_{ij} \nabla_i f \nabla_j f dV &= \int_M \nabla_k D_{kij} \nabla_i f \nabla_j f dV \\
&= - \int_M D_{lij} \nabla_i f \nabla_l \nabla_j f dV \\
&= \int_M D_{lij} \nabla_i f R_{lj} dV \\
&= \frac{1}{2} \int_M D_{lij} (R_{lj} \nabla_i f - R_{ij} \nabla_l f) dV \\
&= - \frac{n-2}{2} \int_M |D|^2 dV
\end{aligned}$$

Then it is easy to see that the vanishing of the Bach tensor will imply the vanishing of the D tensor, and hence Lemma 4.0.5 will complete the proof of Theorem 4.0.2.

□

Bibliography

- [1] Besse, A.-L., *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete(3) [Results in Mathematics and Related Areas(3)], 10. Springer-Verlag, Berlin, 1987.
- [2] Böhm, C., *Non-compact cohomogeneity one Einstein manifolds*, Bull. Soc. math. France (1999) vol. 127, 135-177.
- [3] Böhm, C. and Wilking, B., *Manifolds with positive curvature operators are spaces forms*, Ann. of Math. (2), 167(3):1079-1097, 2008.
- [4] Brendle, S., *Rotational Symmetry of Self-similar Solutions to the Ricci Flow*, arXiv: 1202.1264v3, [math.DG].
- [5] Brendle, S., *Rotational Symmetry of Ricci Solitons in higher dimensions*, arXiv: 1203.0270v2, [math.DG].
- [6] Brendle, S., *A general convergence result for the Ricci flow in higher dimensions*, Duke Math. J. 145 (2008), no. 3, 585-601.
- [7] Brendle, S. and Schoen, R. *Manifolds with 1/4-pinched curvature are Space Forms*, J. Amer. Math. Soc. 22 (2009), no.1, 287-307.
- [8] Bryant, R., *Gradient Kähler Ricci solitons*, Géométrie différentielle, physique mathématique, mathématiques et société. I. Astérisque No.321 (2008), 51-97.
- [9] Cao, H.-D., *Existence of Gradient Kähler-Ricci Solitons*, Elliptic and Parabolic

- Methods in Geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, (1996) 1-16.
- [10] Cao, H.-D., *Limits of solutions to the Kähler-Ricci flow*, J. Diff. Geom. 45 (1997) 257-272.
- [11] Cao, H.-D., *Recent progress on Ricci solitons*, Recent advances in geometric analysis, 1–38, Adv. Lect. Math. (ALM), **11** Int. Press, Somerville, MA, 2010.
- [12] Cao, H.-D., Catino, G., Chen, Q., Mantegazza, C., and Mazzieri, L., *Bach-flat gradient steady Ricci solitons*, Calc. Var. Partial Differential Equations, to appear, arXiv:1107.4591v2, [math.DG].
- [13] Cao, H.-D., Chen, B.-L. and Zhu, X.-P., *Recent developments on Hamilton's Ricci flow*, Surveys in differential geometry, Vol. XII, 47–112, Surv. Differ. Geom., **XII**, Int. Press, Somerville, MA, 2008.
- [14] Cao, H.-D. and Chen, Q., *On locally conformally flat gradient steady solitons*, Trans. Amer. Math. Soc. 364 (2012) 2377-2391.
- [15] Cao, H.-D. and Chen, Q., *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. Vol. **162** (2013), No. 6, 1149-1169.
- [16] Cao, H.-D. and Hamilton, R., *Gradient Kähler-Ricci solitons and periodic orbits*, Comm. Anal. Geom. 8 (2000), no.3, 517-529.
- [17] Cao, H.-D., Hamilton, R. S. and Ilmanen, T., *Gaussian densities and stability for some Ricci solitons*, arXiv:0404165, [math.DG].
- [18] Cao, H.-D. and Zhou, D., *On complete gradient shrinking solitons*, J. Differential Geom. **85** (2010), 175–185.
- [19] Cao, H.-D. and Zhu M., *On second variation of Perelman's Ricci shrinker entropy*, Math. Ann., Vol. 353(2012), no. 3, 747-763.

- [20] Cao, X., Wang, B. and Zhang, Z., *On locally conformally flat gradient shrinking Ricci solitons*, Commun. Contemp. Math. 13 (2011), no.2, 269-282.
- [21] Case, J., Shu, Y.-J. and Wei, G.-F., *Rigidity of quasi-Einstein metrics*, Differential Geom. Appl. 29 (2011), no.1, 93-100.
- [22] Catino G., Mantegazza C., Mazzieri L., Rimoldi M., *Locally conformally flat quasi-Einstein manifolds*, J. Reine Angew. Math. (Crelle), to appear, arXiv:1010.1418v3, [math.DG].
- [23] Catino, G. and Mantegazza, C., *Evolution of the Weyl tensor under the Ricci flow*, Ann. Inst. Fourier (Grenoble) 61 (2011), no.4, 1407-1435.
- [24] Chau, A. and Tam, L.-F., *A note on the uniformization of gradient Kähler Ricci solitons*, Math. Res. Lett. 12 (2005), no.1, 19-21.
- [25] Chen, B.-L., *Strong uniqueness of the Ricci flow*, J. Differential Geom. **82** (2009), 363-382.
- [26] Chen, Q., and He, C., *On Bach flat warped product Einstein manifolds*, accepted by Pacific Journal of Mathematics, arXiv:1105.3957v2, [math.DG].
- [27] Chen, Q., and Zhu, M., *On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor*, Proc. Amer. Math. Soc. 140 (2012) 4017-4025.
- [28] Chen, X.X. and Wang, Y., *On four-dimensional anti-self-dual gradient Ricci solitons*, arXiv: 1102.0358, [math.DG].
- [29] Chu, S.-C., *Geometry of 3-dimensional gradient Ricci solitons with positive curvature*, Comm. Anal. Geom., 13 (2005), 129-150.
- [30] Dancer, A. and Wang, M., *On Ricci solitons of cohomogeneity one*, Annals of Global Analysis and Geometry, 39, (2011), 259-292.
- [31] Eminenti, M., La Nave, G. and Mantegazza, C., *Ricci solitons: the equation point of view*, Manuscripta Math. **127**, 345-367 (2008).

- [32] Feldman, M., Ilmanen, T. and Knopf, D. *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Diff. Geom. 65. 169-209 (2003).
- [33] Fernández-López, M. and García-Río, E., *Rigidity of shrinking Ricci solitons*, Math. Z. **269** (2011), no. 1-2, 461-466.
- [34] Futaki, A. and Wang, M., *Constructing Kähler-Ricci solitons from Sasaki-Einstein manifolds*, Asian J. Math. Vol 15, 1, (2011), 33-52.
- [35] Gromoll, D. and Meyer, W. *On complete open manifolds of positive curvature*, Ann. of Math., 90 (1969), 75-90.
- [36] Guo, H., *Area growth rate of the level surface of the potential function on the 3-dimensional steady Ricci soliton*, Proc. Amer. Math. Soc. 137 (2009), no. 6, 2093-2097.
- [37] Hamilton, R. S., *Three manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255–306.
- [38] Hamilton, R. S., *Four-manifolds with positive curvature operator*, J. Diff. Geom. **24**, 153-179 (1986).
- [39] Hamilton, R. S., *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry (Cambridge, MA, 1993), **2**, 7-136, International Press, Combridge, MA, 1995.
- [40] Hamilton, R. S., *The Ricci flow on surfaces*, Mathematics and General Relativity, Contemporary Mathematics 71 (1998), 237-261.
- [41] He, C., Petersen, P. and Wylie, P., *On the classification of Warped product Einstein metrics*, arXiv: 1010.5488v2 [math.DG].
- [42] He, C., Petersen, P. and Wylie, P., *Warped Product Einstein metrics over spaces with constant scalar curvature*, arXiv: 1012.3446v1 [math.DG].

- [43] Ivey, T., *Ricci solitons on compact three-manifolds*, J. Diff. Geom., 21, 1985, 47-62.
- [44] Kim, D.-S. and Kim, Y.-H., *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Amer. Math. Soc. 131(2003), no.8, 2573-2576.
- [45] Koiso, N., *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math. 18 (1990), 327-337.
- [46] Kotschwar, B., *On rotationally invariant shrinking Ricci solitons*, Pacific J. Math. **236** (2008), 73–88.
- [47] LeBrun, C. *On Einstein, Hermitian 4-Manifolds*, J. Differential Geom. 90 (2012), no.2, 277-302.
- [48] Lü, H., Page, Don N. and Pope, C.N., *New inhomogeneous Einstein metrics on sphere bundles over Einstein-Kähler manifolds*, Phys. Lett. B 593 (2004), no. 1-4, 218-226.
- [49] Munteanu, O. and Sesum, N., *On gradient Ricci solitons*, J. Geom. Anal. **23** (2013), no. 2, 539-561.
- [50] Naber, A., *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645**, 125-153 (2010).
- [51] Ni, L., *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. **12**, 633-653 (2005).
- [52] Ni, L. and Wallach, N., *On a classification of the gradient shrinking solitons*, Math. Res. Lett. **15** (2008), 941–955
- [53] Ni, L. and Wallach, N., *On four-dimensional gradient shrinking solitons*, Int. Math. Res. Not. IMRN **4** (2008), Art. ID rnm152, 13pp.

- [54] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:0211159, [math.DG].
- [55] Perelman, G., *Ricci flow with surgery on three manifolds* arXiv:0303109, [math.DG].
- [56] Petersen, P. and Wylie, P., *On the classification of gradient Ricci solitons*, *Geom. Topol.* **14** (2010), 2277–2300.
- [57] Pigola, S., Rimoldi, M. and Setti, A. G., *Remarks on non-compact gradient Ricci solitons*, *Math. Z.* **268** (2011), no. 3-4, 777790.
- [58] Qian Z., *Estimates for weighted volumes and applications*, *Quart. J. Math. Oxford Ser. (2)* **48** (1997), 235-242.
- [59] Su Y., Zhang K., *On the Kähler-Ricci Solitons with vanishing Bochner-Weyl tensor*, *Acta Math. Sci. Ser. B Engl. Ed.* **32** (2012), no.3, 1239-1244.
- [60] Wang, X.-J. and Zhu X.-H., *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, *Adv. Math.*, **188:1** (2004), pp. 87-103.
- [61] Zhang, Z.-H., *Gradient shrinking solitons with vanishing Weyl tensor*, *Pacific J. Math.* **242** (2009), 189–200.

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- Chen, Q. and Zhu, M., *On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor*, Proc. Amer. Math. Soc. **140** (2012), 4017-4025.
- Cao, H.-D. and Chen, Q., *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. Vol. **162** (2013), No. 6, 1149-1169.
- Cao, H.-D., Catino, G., Chen, Q., Mantegazza, C. and Mazzieri, L., *Bach-flat gradient steady Ricci solitons*, accepted by Calc. Var. & PDE.
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