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Hendry's Conjecture on Chordal Graph Subclasses

by

Aydın Gerek

A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
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Aydın Gerek

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Hendry's Conjecture on Chordal Graph Subclasses

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Abstract

A cycle is extendable if there exists another cycle on the same set of vertices plus one more vertex. G.R.T. Hendry conjectured (1990) that every non spanning cycle in a Hamiltonian chordal graph is extendable. This has recently been disproved (2015), but is still open for classes of strongly chordal graphs. Hendry's Conjecture has been shown to hold for the following subclasses of chordal graphs: planar chordal graphs (2002), interval graphs, strongly chordal graphs with (two specific) forbidden subgraphs, split graphs (2006), and spider intersection graphs (2013).

Chapter 1 of this dissertation is an introduction to the subject matter. In chapter 2 we verify that Hendry's Conjecture holds for Ptolemaic graphs which are a subclass of strongly chordal graphs, alongside with a strong result on how smoothly the extension can happen. In chapter 3 we develop tools for working on tree representations of chordal graphs with Hendry's Conjecture in mind. Chapter 4 is an application of these tools to interval graphs, another subclass of chordal graphs. Chapter 5 is about manipulating the aformentioned counterexample to Hendry's Conjecture, and applying tools from chapter 3 on it. This yields information on the structure of graphs for which Hendry's conjecture holds.

Chapter 1

Introduction

All graphs described here are finite simple connected graphs. Unless otherwise noted basic terminology follows [11]. Also note that when a result is well known we will quote a general text rather than the paper it was originally published in. This difference will be noted by prepending the word 'see' before the citation number.

Definitions 1. We use the notation $G[S]$ when we wish to denote the subgraph of G induced on the set of vertices S. A graph is called *Hamiltonian* if it has a *Hamiltonian cycle*: a cycle that passes through every vertex. A *chord* of a cycle is an edge between two vertices of it, which are not adjacent in the cycle. A graph is called *chordal* if all of its cycles of length 4 or greater have a chord. It follows trivially from this definition that an induced subgraph of a chordal graph is also chordal. A vertex is *simplicial* provided that its neighbors form a clique. A *maxclique* or *maximal clique* is a clique that is not the subset of a larger clique. A *vertex separator* is a set of vertices whose removal *separates* two non-adjacent vertices into distinct connected components.

Theorem 2 (see [3])**.** *A chordal graph is either complete, or has two non-adjacent simplicial vertices.*

The above theorem implies that that any chordal graph except the one on a single vertex will have at least two simplicial vertices. Moreover since being chordal is a hereditary property removing a simplicial vertex will often yield new simplicial vertices in the remaining graph, which may in turn be removed.

Definition 3. A *perfect elimination order (PEO)* of a graph G is a sequence $x_1, x_2, \ldots, x_n$ of its vertices such that x_i is simplicial in $G[x_{i+1}, \ldots, x_n]$.

Chordal graphs are characterized by PEOs.

Theorem 4 (see [3])**.** *A graph is chordal if and only if it has a perfect elimination order.*

Definition 5. We will call a cycle with the vertex sequence $x_1x_2x_3 \ldots x_kx_1$ *reducible* if removing one of the vertices in the sequence yields a vertex sequence for another cycle.

Given any cycle in a chordal graph consider the simplicial vertices of the subgraph induced by the vertices of the cycle. Since the neighbors of a simplicial vertex are adjacent to each other, removing one these simplicial vertices from the cycles will reduce it. In other words every cycle in a chordal graph is reducible.

This process does not in general work in reverse. That is given a cycle in a chordal graph there often is no way to insert a vertex into the cycle to form a larger cycle in which the order of the vertices is preserved. However if one relaxes the requirement to preserve the order of the vertices, it may be possible to extend the cycle.

Definitions 6. Given two cycles C and D with the properties

(i) $V(C) \subset V(D)$ and

(ii) $|V(C)| + 1 = |V(D)|$,

we call D an *extension* of C, and we say that C is *extendable*, or *extends* to D. If $y \in V(D) \setminus V(C)$ is the extra vertex, we can also say C *extends by* y. In the rare case where it is possible to insert a vertex into a cycle without changing the order of the vertices, we call it a *nice extension*. If a graph has at least one cycle, and every cycle of this graph is extendable, then we call that graph *cycle extendable*.

By repeated extension of any one of its cycles, it is trivial to show that every cycle extendable graph is also Hamiltonian. In the 80s George Hendry invented and studied this property as part of his PhD dissertation. Hendry's work was mainly focused on proving that classical sufficient conditions on Hamiltonicity also (up to some exceptions) implied cycle extension. As a consequence to his work he also developed a conjecture, known as Hendry's Conjecture.

Conjecture 7 (Hendry's Conjecture [5])**.** *Every Hamiltonian chordal graph is cycle extendable.*

While Hendry's Conjecture has been recently disproved [9], there have been some advances proving Hendry's Conjecture for various subclasses of chordal graphs. These include interval graphs [1, 4], planar chordal graphs [7], strongly chordal graphs with forbidden subgraphs [1], and spider intersection graphs [2]. Much of this work suggests that Hendry's Conjecture may hold true for strongly chordal graphs, which is a superclass of Ptolemaic graphs. The main result in chapter 2 is that Henry's Conjecture holds for Ptolemaic graphs, and we show that the extensions are to some degree nice. In chapter 3 we develop a method for manipulating tree representations of chordal graphs with reference to pairs of cycles (and paths) on them, and in chapter 4 apply it to the case of certain paths on interval graphs. In chapter 5 we apply the same techniques on the counterexample by Lafond and Seamone in an attempt to classify the trees which guarantee that Hamiltonian chordal graphs hosted on them are cycle extendable.

Fact 8 (see [3])**.** *A vertex is simplicial if and only if it belongs in only one maxclique.*

Proof. We prove the contrapositive. Assume a vertex x belongs to two distinct maxcliques M and N. Let $A = M \cap N, B = M \setminus A, C = N \setminus A$. If every pair of vertices from B and C respectively had an edge between them, then $M \cup N$ would be a clique, thus contradicting maximality of both M and N. Therefore there must exist a pair of vertices $y \in B, z \in C$ that are not adjacent. Since $y \in M, z \in N$ and $x \in M \cap N$, then $y, z \in N(x)$. This shows that x cannot be simplicial, since its neighborhood is not a clique.

Assume a vertex x is not simplicial, then by the definition of simplicial, it must have two neighbors $y, z \in N(x)$ that are not adjacent. $\{x, y\}$ and $\{x, z\}$ are both 2-cliques, but they can never be included in the same maxclique since y and z are not adjacent. So x is in at least one maxclique togather with y, and in at least another maxclique together with z. Therefore, x is in at least two distinct maxcliques. □

Fact 9 (see [1]). *Let G be a Hamiltonian Chordal graph of 4 vertices or more, and x be a simplicial vertex of G. Then $G \setminus x$ is Hamiltonian.*

Proof. Let $H = x_1 x x_2 x_3 \ldots x_{n-1} x_1$ be any Hamiltonian cycle of G. Then since x is simplicial and $x_1, x_2 \in N(x)$, it follows that $x_1 x_2 \in E(G)$, and therefore $x_1 x_2 x_3 \ldots x_{n-1} x_1$ is a Hamiltonian cycle in $G \setminus x$. □

Chapter 2

Ptolemaic Graphs

2.1 Basic Facts and Definitions

Definition 10 (see [3])**.** A graph G is *distance hereditary* provided that for any two vertices $x, y \in V(G)$ and for every induced subgraph H of G containing those two vertices, x and y are either disconnected in H or have the same distance between them in H as they did in G.

Ptolemaic graphs have several equivalent definitions. We will use the following definition.

Definition 11. A graph is *Ptolemaic* provided that it is both chordal and distance hereditary.

Fact 12 (see [3])**.** *The Ptolemaic graph property is hereditary, in other words every induced subgraph of a Ptolemaic graph is Ptolemaic.*

We combine facts 12 and 9 for convenience.

Fact 13. *Let G be a Hamiltonian Ptolemaic graph of 4 vertices or more, and x a simplicial vertex of G. Then $G \setminus x$ is a Hamiltonian Ptolemaic graph.*

We will use the following well known fact in the proof of the structural theorem to follow.

Fact 14. *Given two intersecting maxcliques M_1, M_2 in Ptolemaic G, $M_1 \cap M_2$ separates $M_1 \setminus M_2$ from $M_2 \setminus M_1$.*

2.2 Structural Theorem

We will now build towards a structural theorem which really is a special case of the structure developed in [10], but we build it in a way which is both simpler and more intuitive. It is also more useful for our proofs about cycle extendability.

Two sets A, B are *overlapping* provided that, $A \cap B, A \setminus B$, and $B \setminus A$ are all nonempty.

Lemma 15 (see [10])**.** *Let G be a Ptolemaic graph, M a maxclique of G, and $M_1, M_2, \ldots, M_k$ distinct maxcliques of G (also distinct from M), whose intersections with M are nonempty. Let $C_i = M \cap M_i$. For any $i \neq j$, either $C_i \cap C_j = \emptyset$, $C_i \subseteq C_j$, or $C_j \subseteq C_i$. In other words they are not overlapping.*

Proof. By way of contradiction, assume C_i and C_j are overlapping sets. Let $x \in C_i \setminus C_j$, $y \in C_j \setminus C_i$. Since $x, y \in V(M)$, it follows that x and y are adjacent. Since $\emptyset \neq C_i \cap C_j \subseteq M_i \cap M_j$, we know that M_i, M_j intersect. Therefore, by Fact 14, $M_i \setminus M_j$ and $M_j \setminus M_i$ must be separated by $M_i \cap M_j$. However the edge xy from $C_i \setminus C_j$ to $C_j \setminus C_i$ bypasses this separator. This contradiction proves the statement. $\square$

Definition 16. A *separating partition* of a connected graph G is a pair $(\mathcal{S}, \mathcal{V})$ where

1. $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ is a nontrivial partition of $V(G)$.
2. $\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$ is a family of subsets of $V(G)$.
3. $\emptyset \subset S_i \subseteq V_i$ for all i.
4. for $i \neq j$ the only edges from V_i to V_j are from S_i to S_j.
5. For each i, $G[V_i]$ and $G[S_i]$ are both connected.

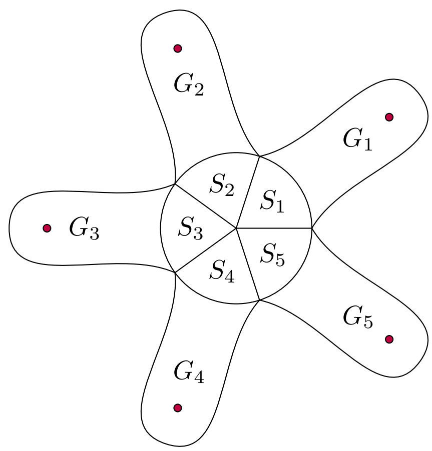

Figure 2.1: An illustration of what a separating partition may look like.

For ease of readability we will denote $G[V_i]$ as G_i.

A visual aid showing what a separating partition may look like is given in Figure 2.1.

We now proceed to the construction of a special type of separating partition that is associated with maxcliques that have no simplicial vertices. The properties of this construction will be shown in a number of lemmas until we conclude with Theorem 24.

Lemma 17. *Let G be a connected Ptolemaic graph, M be a maxclique with no simplicial vertices in G. Then $V(M)$ can be partitioned into $\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$ where each S_i is a vertex separator.*

Proof. Let $\mathcal{F}$ be the family of subsets of $V(M)$, that are nonempty intersections of M with other maxcliques. All members of $\mathcal{F}$ are separators by Fact 14. Let $\mathcal{S}$ be the subfamily of $\mathcal{F}$, composed of its maximal members under set inclusion. To prove that $\mathcal{S}$ partitions $V(M)$ we need to show

(i) $\bigcup_i S_i = V(M)$.

Recall that a vertex is simplicial if and only if it is contained in a unique maxclique

(Fact 8). This means that since M has no simplicial vertices; every vertex of M must also be contained in another maxclique, i.e. must be in the intersection of this maxclique and M. Therefore, each vertex of M is contained in some member of $\mathcal{F}$, and thus in some member of $\mathcal{S}$.

(ii) For $i \neq j$ that $S_i \cap S_j = \emptyset$.

Since S_i and S_j are defined to be maximal under set inclusion, neither is a subset of the other. Then $S_i \cap S_j = \emptyset$ by Lemma 15. We conclude that members of $\mathcal{S}$ partition $V(M)$. $\square$

Lemma 18. *$\mathcal{S}$ defined in Lemma 17 has at least 2 members.*

Proof. If $\mathcal{S}$ is empty, then M has no intersections with other maxcliques. Either G is not connected, or $G = M$ and is a complete graph, making every vertex of M a simplicial vertex. Either possibility is a contradiction of the hypotheses of Lemma 17.

If $\mathcal{S}$ has exactly one member, then that member, which is the intersection of M with another maxclique, would cover all of M. But this would mean the other maxclique is a superset of M, which contradicts M's maximality. $\square$

Given a connected Ptolemaic graph G, a maxclique M with no simplicial vertices in G, and $\mathcal{S}$ a partition of $V(M)$ as defined in Lemma 17, we define $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ in the following way. For a fixed i, let $\mathcal{M}_i = \{M_{i1}, M_{i2}, \ldots\}$ be the set of maxcliques such that $M_{ij} \cap M \subseteq S_i$ for all j. For a given $M_{ij} \in \mathcal{M}_i$ let G_{ij} be the connected component of $G \setminus M$ in which $\{M_{ij} \setminus S_i\}$ resides. Then $V_i = S_i \cup \bigcup_j V(G_{ij})$.

Lemma 19. *$\mathcal{V}$ as defined above partitions $V(G)$.*

Proof. (i) $\bigcup_i V_i = V(G)$.

Since for all i, $V_i \subseteq V(G)$, it follows that $\bigcup_i V_i \subseteq V(G)$. We need to prove that $V(G) \subseteq \bigcup_i V_i$. Pick any $x \in V(G)$. It will suffice to show that $x \in V_i$ for some i. If $x \in M$ then since $\mathcal{S}$ partitions M, $x \in S_i$ for some i. Then $x \in V_i$ for the same i. On the other hand, if $x \notin M$, then x is in some connected component of $G \setminus M$. Let us call this connected component H. It is obvious that, just like any other connected component of $G \setminus M$, H must have some vertex y which is adjacent to some vertex $z \in M$. Since $z \in M$, and $\mathcal{S}$ partitions M, then $z \in S_i$ for some i. Therefore any maxclique containing both y and z will be a member of $\mathcal{M}_i$ for the same i. This in turn means that $H = G_{ij}$ for some j, and therefore $x \in V_i$.

(ii) For all $i \neq j$, $V_i \cap V_j = \emptyset$.

By Lemma 17 for $i \neq j$ we know $S_i \cap S_j = \emptyset$, we need only show that

$$G_{ik} \cap G_{jl} = \emptyset \quad \forall k, l$$

Assume by way of contradiction that this is not so. If there were k, l such that $G_{ik} \cap G_{jl} \neq \emptyset$, firstly it would mean that G_{ik} and G_{jl} are the same connected component of $G \setminus M$. Further it would mean that this connected component intersects with two maxcliques N, N' such that $N \cap M \subseteq S_i$ and $N' \cap M \subseteq S_j$. Then since they are in the same connected component of $G \setminus M$, there is a path P from $N \setminus M$ to $N' \setminus M$ that avoids M. Let x be a vertex of $N' \cap M$. Appending x to P, we get a new path P' from $N \setminus M$ to $N' \cap M \subseteq S_j \subseteq M \setminus S_i \subseteq M \setminus N$ avoiding $M \cap N$, contradicting Fact 14. □

Lemma 20.

$V_i \cap M = S_i$, for all i.

Proof. trivial □

Lemma 21. *Each G_i is connected.*

Proof. Each G_{ij} is adjacent to S_i, and S_i forms a clique. □

Lemma 22. *Each S_i separates $G_i \setminus S_i$ from $G \setminus G_i$.*

Proof. By way of contradiction, and without loss of generality, assume that there's a path P from $G_1 \setminus S_1$ to $G_2 \subseteq G \setminus G_1$ that does not pass through S_1. Let x be the last vertex of P in G_1 and y its first vertex in G_2. Let P' be the subpath of P from x to y inclusive. Let A be a shortest path within G_1 from x to S_1, and likewise let B be a shortest path within G_2 from y to S_2. Shorten A to A', by removing its terminal vertex in S_1. So, the first vertex of A' is not in S_1 but in a maxclique $N \subseteq G_1$ (not necessarily M_1) whose intersection $N \cap M \subseteq S_1$ is a separator. The paths P', A' and B do not intersect since they are all in different $\{G_i\}$ (P' possibly in several). As such we can concatenate $A'P'B$ into a final path P''. P'' is a path from $N \setminus S_1$ to $S_2 \subseteq M \setminus S_1$ that does not pass through S_1. The existance of P'' contradicts that $N \cap M \subseteq S_1$ must be a separator. This proves that each S_i separates $G_i \setminus S_i$ from $G \setminus G_i$. $\square$

Lemma 23. *For $i \neq j$ the only edges between G_i and G_j are between S_i and S_j, and thus in M.*

Proof. Any edge between G_i and G_j that avoids either S_i or S_j, is a path that contradicts Lemma 22. $\square$

Finally we conclude with our main structural theorem.

Theorem 24 (Structure Theorem)**.** *Let G be a connected Ptolemaic graph, M be a maxclique with no vertices that are simplicial in G. Then, as per the construction above, G has a separating partition $(\mathcal{S}, \mathcal{V})$, with the properties*

(i) *$\mathcal{S}$ partitions M,*

(ii) *$S_i = V_i \cap M$ for each i.*

Remark. We know Theorem 24 can be viewed as a special case of the structure developed in [10]. However, in the form presented here it is both shorter and more applicable to what follows.

Remark. The reader may find it of interest to note that Theorem 24 can be extended to maxcliques which have simplicial vertices. To do so, one creates a special part S_0 for the simplicial vertices of the maxclique in question, and sets $V_0 = S_0$. We did not write Theorem 24 in this more general form since doing so detracts from the readability of the proofs that follow.

We note one final lemma that is used in the next section alongside Theorem 24.

Lemma 25. *Each $G_i \setminus M$ contains a vertex that is simplicial in G.*

Proof. Given any G_i, if it is a single clique then every vertex of $G_i \setminus M$ is simplicial. If not, then by Fact 2 there are two nonadjacent simplicial vertices of G_i as an induced subgraph of G. At least one of them must be outside M, call it x. Since $x \notin S_i \subseteq M$, it follows that its neighbors are all in G_i, and thus its neighborhood in G_i is exactly the same as its neighborhood in G. Therefore it is still simplicial in G. $\square$

2.3 Main Results

Lemma 26. *Let G be a Ptolemaic graph, C a cycle of G which goes through every simplicial vertex of G, and M any maxclique of G. Then there is an edge of C in M.*

Proof. If G is a complete graph, then the only possible M is all of G. Then every edge of C is in M. From here on assume G is not a complete graph.

Case 1: M has simplicial vertices. Let x_1 be one of them. Let x_k be a simplicial vertex of G, not adjacent to x_1. We know such a vertex exists by Fact 2. Let $P = x_1 x_2 x_3 \ldots x_k$ be one of the two paths from x_1 to x_k induced by C. We note that not all vertices in that sequence may be simplicial vertices of M, let x_{i+1} be the first one which is not. Then x_i is simplicial in M, and so $x_{i+1} \in M$. Therefore, $x_i x_{i+1}$ is an edge of C in M.

Case 2: M does not have simplicial vertices. Then we know by Theorem 24 that M partitions into separators S_i with associated components G_i. And by Lemma 25 each G_i has a simplicial vertex not in M. Without loss of generality let $y_1 \in G_1, y_k \in G_2$ be two such simplicial vertices, and $R = y_1 y_2 \ldots y_k$ be one of the two paths C induces from y_1 to y_k. Let y_{i+1} be the first vertex in that path not on G_1, then y_{i+1} must necessarily be in some G_j (not necessarily G_2). Then $y_i y_{i+1}$ being an edge from G_1 to G_j is an edge in M by Lemma 23. This is an edge of C in M. $\square$

Theorem 27. *Let G be a Ptolemaic graph, $C = x_1x_2 \dots x_kx_1$ a cycle of G which goes through every simplicial vertex of G, and y any vertex of G not on C. Then C can be extended nicely by y.*

Proof. Let M be a maxclique of G containing y. By Lemma 26, there is an edge of C in M, which we may take, without loss of generality, to be x_1x_2. Since $y, x_1, x_2 \in M$, a clique, edges x_1y, and yx_2 exist, thus we can extend C to $x_1yx_2 \dots x_kx_1$. $\square$

Remark. Theorem 27 does not hold in chordal graphs in general. (This does in fact follow from the existance of a counterexample anyways). A Hamiltonian interval graph is given in Figure 2.2. Note that the cycle marked in bold does not extend nicely, even though it passes through every simplicial vertex of the graph.

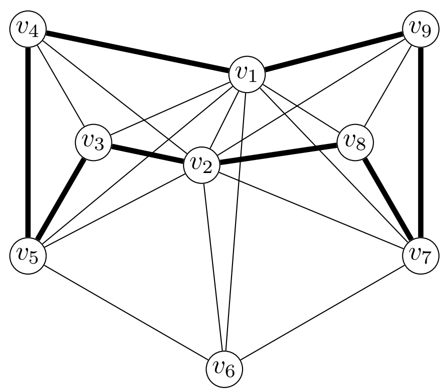

Figure 2.2: A cycle in a Hamiltonian interval graph that passes through every simplicial vertex, and yet has no nice extension

Theorem 27 can be rephrased in terms of a sequence of extensions.

Corollary 28. *Let G be a Ptolemaic graph. If C_1 is a non-Hamiltonian cycle in G which passes through every simplicial vertex, then there exists a sequence of cycles $C_1, C_2, \dots, C_k$, where each C_{i+1} is a nice extension of C_i ($1 \leq i < k$), and the last cycle C_k is Hamiltonian. In fact we can pick any ordering $x_2, x_3, \dots, x_k$ of the vertices of $G \setminus C_1$ such that $V(C_i) = V(C_1) \cup \{x_2, x_3, \dots, x_i\}$.*

Proof. Starting with C_1 we repeatedly extended it via Theorem 27, to any vertex of our choice, until we hit a Hamiltonian cycle. $\square$

Corollary 29. *A Ptolemaic graph which features a cycle that passes through every simplicial vertex is Hamiltonian.*

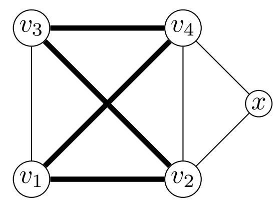

Figure 2.3: Ptolemaic counterexample to nice extension

Remark. The sequence of nice extensions described in Corollary 28 is not always possible if the starting cycle C_1 does not pass through every simplicial vertex of G. For a counter example consider the Ptolemaic graph in Figure 2.3. Let $C_1 = v_1v_2v_3v_4v_1$. The two possible extensions of C_1 do not preserve the order of its vertices. We do, however, have a result that states that this need not happen more than once in the sequence.

Theorem 30. *Let G be a Ptolemaic graph with a Hamiltonian cycle H. Let C_1 be a non-Hamiltonian cycle in G. Then there exists a sequence $C_1, C_2, \ldots, C_k$ of cycles, where $C_k = H$, C_{i+1} is an extension of C_i, and for all but (at most) one i ($1 \leq i < k$) these extensions are nice.*

Proof. We use induction on the number of vertices.

Case 1: If C_1 passes through every simplicial vertex of G, then by Corollary 28 there exists a sequence of cycles $C_1, C_2, \ldots, C_k$, where each C_{i+1} is a nice extension of C_i. Replacing C_k with H, we get the sequence $C_1, C_2, \ldots, C_{k-1}, H$, in which each element of the sequence is a nice extension of the previous element, with the possible exception of the extension from C_{k-1} to H.

Case 2: If there is a simplicial vertex in $G \setminus C_1$, let us call it x. By Fact 13, $G \setminus x$ is a Hamiltonian Ptolemaic graph, and $H \setminus x$ (a reduction of H) is a Hamiltonian cycle of this graph. We use induction on $G \setminus x$ to find a sequence of cycles $C_1, C_2, \ldots, C_{k-2}, H \setminus x$ where every element of the sequence is an extension of the previous, and at most one of these extensions is not nice. Note that since $H \setminus x$ is a reduction of H, then conversely H is a nice extension of $H \setminus x$. Therefore we can append H to this sequence without increasing the number of non-nice extensions. $C_1, C_2, \ldots, C_{k-2}, H \setminus x, H$ is a sequence of cycles that satisfies the hypothesis. □

The proof of Theorem 30 is inductive and hides some ideas about how this sequence can be constructed in a specific instance. What follows is an informal description of how one can construct an extension sequence.

Construction 31.

Case 1: *If C_1 passes through every simplicial vertex of G, refer to the proof (case 1) of Theorem 30.*

Case 2: *Assume C_1 does not pass through every simplicial vertex of G, here's how we construct the sequence $C_1, C_2, \ldots C_k$. We start by coming up with a sequence of vertices $x_1, x_2, \ldots, x_l$ in $G \setminus C_1$ such that for any given i, x_{i+1} is a simplicial vertex in $G \setminus \{x_1, x_2, \ldots, x_i\}$. (This is a partial perfect elimination ordering.) We stop at x_l when no simplicial vertices remain in $G \setminus \{x_1, x_2, \ldots, x_l\}$ that are not also in C_1. We let C_k be a Hamiltonian cycle in G. We let C_{k-i} be the cycle reduced from C_{k-i+1} when x_i is removed and its neighbors connected. In this way we've constructed $C_{k-l}, C_{k-l+1}, \ldots, C_k$, the last $l+1$ cycles in the squence. Note that C_1 passes through every simplicial vertex of $G[V(C_{k-l})]$. So using the same strategy as in Case 1, we can create a sequence of nicely extended cycles $C_1, C_2, \ldots, C_{k-l-1}, D$ in $G[V(C_{k-l})]$. These are also nice extensions in the larger graph G. Note that while $V(D) = V(C_{k-l})$ the order of their vertices need not be the same. Therefore the sequence $C_1, C_2, \ldots, C_k$ of cycles satisfies the theorem, with the one possibly not-nice extension being the one from C_{k-l-1} to C_{k-l}*

A direct corollary of Theorem 30 is that Hendry's Conjecture holds for Ptolemaic graphs.

Corollary 32. *Hamiltonian Ptolemaic graphs are cycle extendable.*

2.4 Further Investigation

Definition 33. A *connected component* of a graph is a subgraph in which there is a path between any two vertices and there is no path from its vertices to the rest of the graph. We denote the number of connected components of a graph G as $c(G)$.

Definition 34. A graph is *tough* provided that it cannot be split into k connected components by the removal of less than k vertices.

A Hamiltonian graph is necessarily tough. We think the converse holds for Ptolemaic graphs. In other words, we think that a Ptolemaic graph is Hamiltonian if and only if tough, and that this can be proved by use of the structure lemma. It is known that the equivalent statement holds for interval graphs, and the larger class of cocomparability graphs.

Conjecture 35. *Tough Ptolemaic graphs are Hamiltonian.*

Provided that Conjecture 35 is true then it might also be possible to transform our proof of Hendry's Conjecture in Ptolemaic graphs into a certifying algorithm which either produces as Hamiltonian path in a given Ptolemaic graph or a set of k vertices whose removal splits the graph into more than k connected components, thereby showing that the graph is not tough.

Definition 36. The *minimal path cover* of a graph is the minimal number of paths needed to cover its vertices.

Definition 37. The *scattering number* of a complete graph is $-\infty$, for every other graph it's defined to be

$$sc(G) = max\{c(G[V \setminus S]) - |S| \mid S \subseteq V(G), c(G[V \setminus S] \neq 1)\}.$$

A further question we may consider is that of the minimal path cover of a Ptolemaic graph. Specifically we think it may be equal to the scattering number of the graph. Analogously to the previous paragraph, if this is true there may be an efficient algorithm for finding the minimal path cover of Ptolemaic graphs. It should also be noted that, just as in the case of toughness, similar results for scattering numbers exist for interval and cocomparability graphs.

Another possibility is to investigate the circumference of Hamiltonian Ptolemaic graphs.

Chapter 3

Tree representation

In this chapter we use tree representation of chordal graphs to develop various terms, lemmas and models. There is no main theorem, but the tools we develop are used in chapters 4 and 5. We begin by discussing the characterizations of chordal graphs as intersections of subtrees of a host tree.

Definition 38. Given a tree T and a set $\mathcal{T}$ of subtrees of T, we define the *subtree intersection graph* associated with $(T, \mathcal{T})$ to be the graph whose vertex set is $\mathcal{T}$ and whose edge set is defined by the statement that for every $x, y \in \mathcal{T}$ xy is an edge if and only if x intersects y.

It is well known that the class of subtree intersection graphs corresponds exactly to the class of chordal graphs.

Fact 39. *Every subtree intersection graph is chordal.*

Fact 40. *For every chordal graph G there exists a host tree T and a set of its subtrees $\mathcal{T}$, such that G is isomorphic to the subtree intersection graph associated with $(T, \mathcal{T})$.*

We now formalize this connection between chordal graphs and subtree intersection graphs by defining tree representations.

Definition 41. A *(tree) representation* for a chordal graph G consists of a triple $(T, \mathcal{T}, \rho)$, where T is a (host) tree, $\mathcal{T}$ is a set of subtrees of T, ρ is a map from $V(G)$

to the set of all subtrees of T, with the properties that $uv \in E(G)$ if and only if the subtrees $\rho(u)$ and $\rho(v)$ intersect, and that the range of ρ is exactly $\mathcal{T}$.

Note that since $\mathcal{T} = range(\rho)$, we will sometimes omit $\mathcal{T}$, and refer to (T, ρ) as a representation, and if a specific host tree is unnecessary or irrelevant for purposes of discussion, we will further omit T, and refer to ρ by itself as a representation. Also note that where ρ is irrelevant we will often abuse notation, and refer to the subtree intersection graph associated with $(T, \mathcal{T})$ as a representation.

Since for each chordal graph there is at least one representation, and each representation corresponds to a chordal graph, it is possible to write proofs based on manipulating representations rather than directly manipulating the graphs they represent.

Our proofs will often involve induction on the number of edges. The concept of subrepresentation, which we define in the next paragraph, is eminently relevant to such proofs since taking a subrepresentation removes edges from the graph it represents.

Definitions 42. Given a chordal graph G, with a representation (T, ρ), if ρ' is a map from $V' \subseteq V(G)$ to the subtrees of T such that for every vertex $v \in V'$ we have $\rho'(v) \subseteq \rho(v)$, then we call the pair (T, ρ') a *subrepresentation* of (T, ρ). We will allow the convention $\rho'(v) = \emptyset$ which will mean that the vertex v is deleted. When T is unambiguously specified or a specific T is irrelevant to the discussion, we will call ρ' a subrepresentation of ρ. Note that if (T, ρ) is a representation of a chordal graph G, and (T, ρ') is a subrepresentation of (T, ρ), then not only is (T, ρ') a representation in itself of some graph G' (with $V(G') = V(G)$), but G' is a subgraph of G. We will call this subgraph a *subrepresentation subgraph* and denote it $G_{\rho'}$.

Since Hendry's Conjecture is about cycles, it will be useful for us to consider what paths and cycles look like on a representation. Given a tree T, we will refer to a non-repeating sequence $p_1 p_2 \ldots p_k$ of subtrees of T as a *path*, if it satisfies the property that p_i intersects with p_{i+1} for all $1 \leq i < k$. We call such a sequence a *cycle* if the first and last subtrees are identical. Notice that there is some room for confusion / abuse of notation: Let's say G is a chordal graph for which $(T, \mathcal{T}, \rho)$

is a representation. If $P = p_1 p_2 \ldots p_k$ is a path (or cycle) in G, then $\rho(P) = \rho(p_1)\rho(p_2)\ldots\rho(p_k)$ is a path (or cycle) in the representation and vice versa.

Now consider a graph G with representation (T, ρ) in which we are only interested in studying some path (or cycle) $P = p_1 p_2 \ldots p_k$. If G is meant to be a minimal counterexample, or we will be using induction, it is possible to shorten or trim the subtrees $\rho(p_i)$ such that only subpaths between their intersections with $\rho(p_{i-1})$ and $\rho(p_{i+1})$ remain. Furthermore we can ensure that these intersections are minimal, namely each can be reduced to a single node. This is formally stated as below.

Lemma 43. *Given a chordal graph G with representation (T, ρ) and $P = p_1 p_2 \ldots p_k$ a path (or cycle) in G, there exists a subrepresentation ρ' such that*

1. *Each $\rho'(p_i)$ is a subpath in the host tree T.*
2. *$\rho'(p_i) \cap \rho'(p_{i+1})$ is a single node for all $1 \leq i < k$.*
3. *$\rho'(p_1)\rho'(p_1)\ldots\rho'(p_k)$ is still a path (or cycle).*
4. *$\rho'(v) = \emptyset$ for all vertices v not on P.*

Of course Hendry's conjecture involves at least two cycles: one Hamiltonian cycle and one non-Hamiltonian cycle. So, we now consider how we can shorten subtrees in a representation while preserving two specific paths (or cycles), which we shall name $P = p_1 p_2 \ldots p_k$, and $Q = q_1 q_2 \ldots q_l$.

First, we can shorten the subtrees of P and Q individually as per fact 43. We must now combine this two subrepresentations in such way as to yield a subrepresentation ρ''' of ρ that shortens the subtrees of both P and Q, while preserving both of these paths (or cycles). The obvious solution is (for all $v \in V(G)$) to let $\rho'''(v)$ be the union of $\rho'(v)$ and $\rho''(v)$. Unfortunately, this does not always work, since in some circumstances this union will be a disconnected subforest. In these situations we turn this subforest into a subtree by filling the missing parts in between. We therefore define $\rho'''(v)$ to be the smallest subtree of T containing both $\rho'(v)$ and $\rho''(v)$. We will denote this as $\rho'''(v) = \rho'(v) + \rho''(v)$. Here the $+$ operation takes two subtrees and returns the smallest subtree containing both.

Lemma 44. *Given a chordal graph G with representation (T, ρ) with two paths (or cycles) $P = p_1p_2 \ldots p_k$, and $Q = q_1q_2 \ldots q_l$, there exists a subrepresentation ρ' such that*

1. *For each $v \in V(P) \cup V(Q)$ $\rho'(v)$ is a subtree of at most four leaves.*
2. *$\rho'(p_i) \cap \rho'(p_{i+1})$ is a single node for all $1 \leq i < k$.*
3. *$\rho'(q_i) \cap \rho'(q_{i+1})$ is a single node for all $1 \leq i < l$.*
4. *Both $\rho'(P)$ and $\rho'(Q)$ are still paths (or cycles)*
5. *$\rho'(v) = \rho(v)$ for all vertices v not on P or Q.*

For the sake of future convenience we now name representations that satisfy the above conclusions. We call a tuple $(G, H, C, T, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ a *dual path (or cycle) model* provided that

1. G is a chordal graph,
2. H is a Hamiltonian path (or cycle) of G,
3. C is a non-Hamiltonian path (or cycle) of G,
4. T is a host tree for G, and ρ is a representation,
5. η and γ are subrepresentations of ρ associated with H and C with the properties described in Lemma 43,
6. For each $v \in V(G)$ $\rho(v)$ is the smallest subtree of T containing both $\gamma(v)$ and $\eta(v)$,
7. $\mathcal{H}$ is the sequence of subpaths of T that is the image of $V(H)$ under η,
8. $\mathcal{C}$ is the sequence of subpaths of T that is the image of $V(C)$ under γ,
9. $\phi : \mathcal{C} \to \mathcal{H}$ is the bijective map such that for each $v \in V(C)$ $\phi(\gamma(v)) = \eta(v)$.

Definitions 45. Sometimes we cut corners and refer to the tuple $(T, \mathcal{C}, \mathcal{H}, \phi)$ as a dual path (or cycle) model. We call C (or $\mathcal{C}$ depending on context) an *extendee path (or cycle)*. A vertex in $V(H) \setminus V(C)$ (or a subtree in $\mathcal{H} \setminus im(\phi)$ depending on context) is an *extendee vertex*. We call a maximal sequence of extendee vertices in H an *extendee block*. A vertex in $V(C)$ (or depending on context a subtree in $\mathcal{C}$) is called a *nonextendee vertex*.

Similar to the concept of subrepresentation we define a *path (or cycle) submodel* as follows. A path (or cycle) model $(G^2, H^2, C^2, T^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$ is a submodel of a path (or cycle) model $(G^1, H^1, C^1, T^1, \rho^1, \eta^1, \gamma^1, \mathcal{H}^1, \mathcal{C}^1, \phi^1)$ provided that

1. $V(G^2) \subseteq V(G^1)$,
2. T^2 is a subtree of T^1,
3. ρ^2 is a subrepresentation of ρ^1.
4. $V(C^2) \subseteq V(C^1)$

Note that this definition of submodel does not require that γ^2 and η^2 be subrepresentations of γ^1 and η^1 respectively. This is because, often they will not be. Neither is it guaranteed that $C^2 = C^1$ or $H^2 = H^1$.

Chapter 4

Interval Graphs

In general interval graphs are defined as graphs whose vertex set corresponds to a set of intervals of the real number line, and whose edges correspond to the intersections of these intervals. Due to our interest in them as a subclass of chordal graphs we will define them as follows.

Definitions 46. An *interval graph* is a chordal graph with a tree representation whose host tree is a path. Notice that the subtrees of a host tree that is a path are subpaths, which we will name *intervals*. We visualize the host path as drawn horizontally and oriented from left to right with one end picked as the left end, and the other as the right end. Under this orientation each interval will also have a left endpoint and a right endpoint. For a vertex $x \in V(G)$ we will denote its left endpoint under representation ρ as $L_\rho(x)$ and similarly its right endpoint $R_\rho(x)$, or $L(x)$ and $R(x)$ respectively where the representation is irrelevant.

Let G be an interval graph with host path $I = i_1 i_2 \dots i_k$. A path $P = p_1 p_2 \dots p_k$ of G is an *end-to-end path* provided that $L(p_1) = i_1$ and $R(p_k) = i_k$. Moreover we say that an end-to-end path P *extends* to another end-to-end path Q, provided that $V(Q) \setminus V(P)$ is a set of exactly one vertex. If such a Q exists we call P *(path) extendable.*

Definition 47. We call a dual path model $(G, H, C, I, \rho, h, c, \mathcal{H}, \mathcal{C}, \phi)$ an *interval dual path model* provided that the host tree I is a path, and H and C are both end-to-end paths.

The following lemma, which will be central to our proofs, is well known in the literature.

Lemma 48 (Path Straightening Lemma)**.** *Let G be an interval graph with representation ρ and $P = p_1 p_2 \dots p_k$ be an end-to-end path in G. Then there is a subrepresentation ρ' and a permutation $\sigma \in S_k$ such that $P' = p_{\sigma(1)} p_{\sigma(2)} \dots p_{\sigma(k)}$ is an end-to-end path in $G_{\rho'}$ and $R_{\rho'}(p_{\sigma(i)}) = L_{\rho'}(p_{\sigma(i+1)})$*

Note that, given an interval dual path model $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$, we can apply the Path Straightening Lemma to both H and C, resulting in a submodel in which both H and C are straightened paths. Below we give a name to this kind of submodel and prove its existence.

Definition 49. We call an interval dual path model $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ for which

- $R_\eta(h_i) = L_\eta(h_{i+1})$ for $1 \leq i < |V(G)| - 1$
- $R_\gamma(c_i) = L_\gamma(c_{i+1})$ for $1 \leq i < |V(C)| - 1$

straightened.

Lemma 50 (Straightened Submodel Lemma)**.** *Given an interval dual path model $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$, there exists a straightened submodel $(G^2, H^2, C^2, I^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$.*

Proof. We construct the submodel as follows. $I^2 = I$, i.e. the host path remains the same. Let η^2 be the subrepresentation of η given by the path straightenening lemma. Let H^2 be the (permuted) end-to-end path on $V(H)$ given by the same. Simlarly let γ^2 be the subrepresentation of γ, and C^2 be the end-to-end path given by the straightening lemma. We define $\rho^2 = \eta^2 + \gamma^2$. The rest of the submodel can be determined from this much. □

Lemma 51 (Sequential Extension)**.** *Let* $(G, H, C = c_1c_2 \dots c_k, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ *be an interval dual path model. Let* x *be an extendee vertex. If for any* i *we have that* x *is adjacent to both* c_i *and* c_{i+1} *then* C *extends to* $c_1c_2 \dots c_ixc_{i+1} \dots c_k$*.*

Note that sequential extension is a nice extension.

Definition 52. Let $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H} = \mathcal{h}_1\mathcal{h}_2 \dots \mathcal{h}_n, \mathcal{C} = \mathcal{c}_1\mathcal{c}_2 \dots \mathcal{c}_k, \phi)$ be a straightened interval dual path model for which the hypotheses of Lemma 51 hold. This implies that there exists an extendee vertex x and some index i, such that $\rho(x) = \eta(x)$ intersects with both $\mathcal{c}_i$ and $\mathcal{c}_{i+1}$. Since we assume the conclusions of Lemma 50, then it follows that $L_\eta(x) \leq R_\gamma(c_i) = L_\gamma(c_{i+1}) \leq R_\eta(x)$. Conversely for the hypothesis of Lemma 51 to not hold, we must require at the very least for every extendee block $E = e_1e_2 \dots e_t$ and any $c \in V(C)$, which intersects with any one of e_i, that $\gamma(c)$ must completely contain E, that is $L_\gamma(c) \leq L_\eta(e_1) \leq R_\eta(e_t) \leq R_\gamma(c)$, where $L_\gamma(c) = L_\eta(e_1)$ only in cases where $L_\gamma(c)$ is the leftmost node of I, and where $R_\eta(e_t) = R_\gamma(c)$ only in cases where $R_\gamma(c)$ is the rightmost node of I. We will call such a c (or $\gamma(c)$) a *containing vertex (or interval)*

Definition 53. We define a *cross vertex* over an extendee block $E = e_1, e_2, \dots, e_t$ (without loss of generality assumed to be a left to right sequence), to be an nonextendee vertex x such that $\eta(x)$ and $\gamma(x)$ are located at different sides of $\eta(E)$:

1. either $R_\eta(x) \leq L_\eta(e_1)$ and $L_\gamma(x) \geq R_\eta(e_t)$,
2. or $R_\gamma(x) \leq L_\eta(e_1)$ and $L_\eta(x) \geq R_\eta(e_t)$.

Lemma 54 (cross H neighbor reduction)**.** *Let* $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ *be a straightened interval dual path model. Let* $E = e_1e_2 \dots e_t$ *be an extendee block, and* x *a cross vertex over* E*, with* $\eta(x)$ *positioned immediately to the left (or right) of* $\eta(E)$*. Then there exists a submodel* $(G^2, H^2, C^2, I^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$*, such that any extension of* C^2 *in* G^2 *implies an extension of* C *in* G*.*

Proof. Let y be the vertex immediately after e_1 in H. For example if $t \geq 2$ then $y = e_2$. Without loss of generality let us say that $h(x)$ is to the immediate left of

$\eta(E)$, i.e. $R_\eta(x) = L_\eta(e_1)$. Since x is a cross vertex, then $\gamma(x)$ is to the right of $\eta(e_t)$, i.e. $L_\gamma(x) \geq R_\eta(e_t)$. Therefore $\rho(x) = \eta(x) + \gamma(x)$ intersects with $\eta(y)$. This means x is adjacent to y. Consequently we may remove e_1, and induce G^2 on the remaining vertices. Formally $G^2 = G \setminus e_1, H^2 = H \setminus e_1, C^2 = C, I^2 = I$, $\eta^2 = \eta$ everywhere except $\eta^2(x) = \eta(x) + \eta(e_1), \eta^2(e_1) = \emptyset$, and $\gamma^2 = \gamma$ everywhere. The rest of the features of the submodel are deducible from these. Now note that since $C^2 = C$, any extension of C^2 in G^2 is an extension of C in G. $\square$

Lemma 55 (Cross Reduction)**.** *Let $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ be a straightened interval dual path model. Given two vertices x, y with the property that $R_\eta(x) \leq L_\eta(y)$ and $L_\gamma(x) \geq R_\gamma(y)$, if either*

1. *(case 1) $L_\eta(x) \geq L_\gamma(y)$ and $R_\eta(y) \leq R_\gamma(x)$,*
2. *or (case 2) $L_\eta(x) \leq L_\gamma(y)$ and $R_\eta(y) \geq R_\gamma(x)$*

then there exists a submodel $(G^2, H^2, C^2, I^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$, such that any extension of C^2 in G^2 implies an extension of C in G.

Proof. We construct the submodel in question. Let $I^2 = I$.

1. (Case 1) Let $\eta^2(x) = \eta(y), \eta^2(y) = \eta(x)$, with $\eta^2 = \eta$ otherwise, and $\gamma^2 = \gamma$ everywhere.
2. (Case 2) Let $\eta^2 = \eta$ everywhere, and $\gamma^2(x) = \gamma(y), \gamma^2(y) = \gamma(x)$, with $\gamma^2 = \gamma$ otherwise.

Note that these assertions are equivalent to declaring ϕ^2 to be a transposition of ϕ i.e. $\phi^2(\gamma^2(x)) = \eta(y) = \phi(\gamma(y))$ and $\phi^2(\gamma^2(y)) = \eta(x) = \phi(\gamma(x))$.

Together with the hypotheses these give us, $\rho'(x) \subseteq \rho(x)$ and $\rho'(y) \subseteq \rho(y)$, which implies that ρ' is a subrepresentation of ρ.

The astute reader will notice that γ^2 and η^2 will most likely not be subrepresentations of γ and η respectively. This raises the question of what will happen to C and H, seeing as $\mathcal{H}^2 = \mathcal{H}$ and $\mathcal{C}^2 = \mathcal{C}$ have not changed, but γ^2 and η^2 have. We note that in

1. (Case 1) H^2 equals H with the positions of x and y transposed, and $C^2 = C$.
2. (Case 2) $H^2 = H$, and C^2 equals C with the positions of x and y transposed.

Finally G^2 is the subgraph of G induced by the subrepresentation ρ^2.

Since $V(C^2) = V(C)$ and G^2 is a subgraph of G, any extension of C^2 in G^2 is an extension of C in G.

□

Lemma 56 (First Lead Reduction (path version)). *Let $(G, H = h_1 h_2 \dots h_n, C = c_1 c_2 \dots c_k, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ be a straightened interval dual path model, in which $\mathcal{h}_1$ is a nonextendee interval. Then there exists a submodel $(G^3, I^3, H^2 = h_1^2 h_2^2 \dots h_n^2, C^2 = c_1^2 c_2^2 \dots c_k^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2 = \mathcal{h}_1^2 \mathcal{h}_2^2 \dots \mathcal{h}_n^2, \mathcal{C}^2 = \mathcal{c}_1^2 \mathcal{c}_2^2 \dots \mathcal{c}_k^2, \phi^2)$ with the properties*

1. *$\phi^2(\mathcal{c}_1^2) = \mathcal{h}_1^2$. (incidentally this implies $\gamma_1^2 = \eta_1^2$)*
2. *Either $L_{\gamma^2}(c_1^2) = R_{\gamma^2}(c_1^2) = L_{\eta^2}(h_1^2)$ or $L_{\eta^2}(h_1^2) = R_{\eta^2}(h_1^2) = L_{\gamma^2}(c_1^2)$*

Furthermore any extension of C^2 in G^2 implies an extension of C in G.

Proof. To start with since H and C are end-to-end paths, we have $L_\gamma(c_1) = L_\eta(h_1)$. This implies that we can cross reduce (Lemma 55) to $\mathcal{c}_1$ and $\mathcal{h}_1$. Consequently we have a submodel $(G^2, H^2 = h_1^2 h_2^2 \dots h_n^2, C^2 = c_1^2 c_2^2 \dots c_k^2, I, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$, in which $c_1^2 = h_1^2$, and an extension of C^2 in G^2 implies extension of C in G. This model satisfies the first property.

Let r represent the leftmost one among the two nodes $R_{\gamma^2}(c_1^2)$ and $R_{\eta^2}(h_1^2)$, and similarly define $l = L_{\gamma^2}(c_1^2) = L_{\eta^2}(h_1^2)$ i.e. the very leftmost point of I^2. Note that the length of I^2 between l and r is completely superfluous, and can be removed. We formally do so by constructing one more submodel $(G^3 = G^2, H^3 = H^2, C^3 = C^2, I^2, \rho^3, \eta^3, \gamma^3, \mathcal{H}^3, \mathcal{C}^3, \phi^3 = \phi^2)$. I^3 is the subpath of I^2 from node r onwards to the right handside. $\eta^3 = \eta^2$ except for $\eta^3(h_1^3)$ which is defined by $L_{\eta^3}(h_1^3) = r$, $R_{\eta^3}(h_1^3) = R_{\eta^2}(h_1^2)$. $\gamma^3 = \gamma^2$ except for $\gamma^3(c_1^3)$ which is defined by $L_{\gamma^3}(c_1^3) = r$, $R_{\gamma^3}(c_1^3) = R_{\gamma^2}(c_1^2)$. $\rho^3 = \eta^3 + \gamma^3$. This model satisfies both properties.

Since $C^3 = C^2$, and $G^3 = G^2$, any extension of C^3 in G^3 is an extension of C^2 in G^2. As per Lemma 55, any extension of C^2 in G^2 implies an extension of C in G. $\square$

Lemma 57 (Second Lead Reduction). *Let $(G, H = h_1h_2 \ldots h_n, C = c_1c_2 \ldots c_k, I, \rho, \eta, \gamma, \mathcal{H}, \mathcal{C}, \phi)$ be a straightened interval dual path model which satisfies the conclusions of Lemma 56 and that c_1 is not a containing interval. Then there exists a submodel $(G^2, H^2 = h_2 \ldots h_n, C^2 = c_2 \ldots c_k, I^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$, such that*

1. *$V(G^2) = V(G) \setminus c_1 = h_1$, and*

2. *any extension of C^2 in G^2 implies an extension of C in G.*

Proof. Firstly, we will manipulate c_1 and h_2 so that their right end points will match. Assume, without loss of generality that $R_\eta(h_1) \geq R_\gamma(c_1)$. Let c_t be the first vertex of C such that $R_\gamma(c_t) > R_\eta(h_1)$.

Since $\mathcal{H}$ is straightened, for all $i \geq 2$ $\eta(c_i)$ will be placed fully to the right of h_1. That means for $i \geq 2$ $L_\eta(c_i) \geq R_\eta(c_1 = h_1)$ This means for $2 \leq i \leq t$ $\rho(c_i)$ includes the node $R_\eta(h_1)$. Then we can lengthen c_1, and shift c_i for $2 \leq i \leq t$ by constructing a submodel. We construct $(G^2, H^2, C^2, I^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$ as follows. $H^2 = H, C^2 = C, \mathcal{H}^2 = \mathcal{H}, c_i^2 = c_i$ for $i > t, \eta^2 = \eta$. We define γ^2 via

- $L_{\gamma^2}(c_1^2) = L_\gamma(c_1),\ R_{\gamma^2}(c_1^2) = R_\eta(h_1)$

- $L_{\gamma^2}(c_i^2) = R_{\gamma^2}(c_i^2) = R_\eta(h_1)$ for $2 \leq i \leq t-1$

- $L_{\gamma^2}(c_t^2) = R_\eta(h_1),\ R_{\gamma^2}(c_t^2) = R_\gamma(c_t)$

- $\gamma^2(c_i^2) = \gamma(c_i)$ for $i > t$

We define $\rho^2 = \eta^2 + \gamma^2$, and G^2 as the subgraph of G induced by ρ^2. We should now demonstrate that ρ^2 is a subrepresentation of ρ. Since $\eta^2 = \eta$ and $\gamma^2(c_i) = \gamma(c_i)$ for $i > t$, we need only show $\rho^2(c_i) \subseteq \rho(c_i)$ for $1 \leq i \leq t$. In the case of c_1 since we have $\phi(c_1) = h_1$, $L_\gamma(c_1) = L_\eta(h_1)$ and, $R_\eta(h_1) \geq R_\gamma(c_1)$ it follows that $\gamma(c_1) \subseteq \eta(h_1) = \rho(c_1 = h_1)$. Therefore shifting the right endpoint of c_1 to the right endpoint of $\mathcal{h}_1$ does not change $\rho^2(c_1^2) = \rho(c_1)$. For c_t we have $\gamma^2(c_t^2) \subseteq \gamma(c_t)$ and $\eta^2(c_t^2) = \eta(c_t)$, therefore $\rho^2(c_t^2) \subseteq \rho(c_t)$. For $2 \leq i \leq t-1$, we have already noted that $L_\eta(c_i) \geq R_\eta(h_1)$ therefore $L_\gamma(c_i) \leq L_{\gamma^2}(c_i^2) = R_\eta(h_1) \leq L_\eta(c_i) = L_{\eta^2}(c_i^2)$. Thus $\rho^2(c_i^2) \subseteq \rho(c_i)$ for $2 \leq i \leq t-1$. This concludes our demonstration that ρ^2 is a subrepresentation of ρ.

Secondly we will remove the vertex $c_1^2 = h_1^2$, and the corresponding intervals $\mathcal{c}_1^2$ and $\mathcal{h}_1^2$. The resulting submodel is $(G^3, H^3 = h_2^2 h_3^2 \ldots h_n^2, C^3 = c_2^2 c_3^2 \ldots c_k^2, I^3, \rho^3, \eta^3, \gamma^3, \mathcal{H}^3 = \mathcal{h}_2^2 \ldots \mathcal{h}_n^2, \mathcal{C}^3 = \mathcal{c}_2^2 \ldots \mathcal{c}_k^2, \phi^3)$, and it is constructed as follows. G^3 is the induced subgraph of G^2 on $V(G^2) \setminus \{c_1^2 = h_1^2\}$. H^3 and C^3 are H^2 and C^2 missing their initial vertex. Likewise $\mathcal{H}^3$ and $\mathcal{C}^3$ are $\mathcal{H}^2$ and $\mathcal{C}^2$ missing their initial intervals. I^3 is the subpath of I^2 from the node $R_{c^2}(c_1^2) = R_{h^2}(h_1^2)$ onwards to the right. η^3 and ρ^3 are η^2 and ρ^2 restricted to $V(H^3)$ respectively. Similarly γ^3 and ϕ^3 are γ^2 and ϕ^2 restricted to $V(C^3)$ respectively. This completes the construction of the submodel.

Any extension of C^3 in G^3 can be prefixed with c_1^2 to become an extension of C^2 in G^2. Since $C^2 = C$ and G^2 is a subgraph of G, then it follows that this is an extension of C in G as well. □

Theorem 58. *Every end-to-end path in a traceable interval graph is extendable.*

Proof. For any given Hamiltonian interval graph G, any end-to-end Hamiltonian path H in it, and any end-to-end non-Hamiltonian path C in it, we show that C can be extended in G. By way of induction, assume that any smaller end-to-end path in G can be extended, and in any end-to-end path in any Hamiltonian interval graph smaller than G can be extended. Formally, the induction invariant is the sum $|V(G)| + |E(G)| + |V(C)|$. Let $(G, H, C, I, \rho, \eta, \gamma, \mathcal{H} = \mathcal{h}_1 \mathcal{h}_2 \ldots \mathcal{h}_n, \mathcal{C} = \mathcal{c}_1 \mathcal{c}_2 \ldots \mathcal{c}_k, \phi)$ be an interval dual path model.

We first apply the straightening lemma (Lemma 50), and thus have a straightened interval dual path model $(G^2, H^2, C^2, I^2, \rho^2, \eta^2, \gamma^2, \mathcal{H}^2, \mathcal{C}^2, \phi^2)$, where any extension of C^2 implies an extension of C.

If sequential extension (Lemma 51) is possible we extend C^2 sequentially. For the rest of this proof we may assume sequential extension is not possible. One consequence of this is that all extendee blocks are contained in containing intervals.

If c_1^2 is not a containing interval, we can apply 1st Lead Reduction (Lemma 56) to construct submodel $(G^3, H^3, C^3, I^3, \rho^3, \eta^3, \gamma^3, \mathcal{H}^3, \mathcal{C}^3, \phi^3)$. We follow up with 2nd Lead Reduction (Lemma 57) to further construct submodel $(G^4, H^4, C^4, I^4, \rho^4, \eta^4, \gamma^4, \mathcal{H}^4, \mathcal{C}^4, \phi^4)$. Since G^4 is a proper subgraph of G we know by induction hypothesis that C^4 extends in G^4. This implies extension of C^3 in G^3, which implies extension of C^2 in G^2, which finally implies extension of C in G. For the rest of the proof we assume c_1^2 is a containing interval and consider three cases.

1. First extendee vertex of H^2 is h_1^2.

2. First extendee vertex of H^2 is h_2^2.

3. First extendee vertex of H^2 is h_3^2 or a later vertex.

Firstly, if h_1^2 is an extendee vertex, $h_1^2 c_1^2 c_2^2 \dots c_k^2$ is an extension of C^2. This implies an extension of C in G.

Secondly, if the first extendee vertex of H^2 is h_2^2, then we apply Cross Reduction (Lemma 55) on c_1^2 and h_1^2 to construct submodel $(G^3, H^3, C^3, I^3, \rho^3, h^3, c^3, \mathcal{H}^3, \mathcal{C}^3, \phi^3)$. Note that since $h_1^3 = c_1^3$ is a containing vertex, it follows that h_1^3 is a cross vertex over the extendee block containing h_2^3. We apply Cross H-neighbor Reduction removing h_2^3, and thus construct submodel $(G^4, H^4, C^4, I^4, \rho^4, \eta^4, \gamma^4, \mathcal{H}^4, \mathcal{C}^4, \phi^4)$. Since G^4 is a proper subgraph of G induction hypothesis applies. As before, extension of C^4 in G^4 implies extension of C in G.

Thirdly, if the first extendee vertex of H^2 is h_t^2 with $t \geq 3$, again we apply Cross Reduction (Lemma 55) on c_1^2 and h_1^2, to construct submodel $(G^3, H^3, C^3, I^3, \rho^3, \eta^3, \gamma^3, \mathcal{H}^3, \mathcal{C}^3, \phi^3)$. In this submodel since $(\phi^3)^{-1}(\mathcal{h}_1^3) = c_1^3$, it follows that h_2^3 is necessarily a cross vertex over the extendee block containing h_t^3. We apply Cross H-neighbor Reduction removing h_t^3, and thus construct submodel $(G^4, H^4, C^4, I^4, \rho^4, \eta^4, \gamma^4, \mathcal{H}^4, \mathcal{C}^4, \phi^4)$. Since G^4 is a proper subgraph of G induction hypothesis applies. Again, extension of C^4 in G^4 implies extension of C in G. $\square$

4.1 Further Investigation

While there are already two different proofs that Hendry's Conjecture holds for interval graphs, it would be interesting to see if our work in this chapter yields yet another. This should be relatively easy, seing that a straightened Hamiltonian cycle is equivalent to two end-to-end paths. Non-Hamiltonian cycles need not be straightened into two end-to-end paths, but for the ones that do not it is easy to show extension.

In short, interval dual cycle models similar to our interval dual path models can be built to work towards another proof of Hendry's Conjecture for interval graphs. Almost all of our lemmas can be easily modified to work for interval dual cycle models, with the notable exception of the Second Lead Reduction Lemma (Lemma 57). Any work towards this third proof of Hendry's Conjecture for interval graphs must start there.

Chapter 5

Counterexample

We start this chapter with some definitions we'll need in the introductory paragraphs.

Definitions 59. A *subdivision* of a graph G is another graph that can be attained by replacing some (or none) of G's edges with paths. A *star* also denoted $K_{1,n}$ is tree on $n+1$ vertices formed by adding edges between one vertex and every other vertex. A *spider* is a subdivision of a star. A *spider intersection graph* is a chordal graph which can be hosted on a spider. The *leafage* (see [8]) of a chordal graph is the minimum number of leaves a host tree representing that graph may have.

In [1, 4] it is shown that Hendry's conjecture holds for interval graphs. In [2] it is shown that Henry's Conjecture holds for spider intersection graphs. Both of these graph classes are based on the shape a host tree for the graph may take. Since in [9] Lafond and Seamone have shown a family of counterexamples to Henry's Conjecture, it becomes an interesting question to resolve exactly for which host tree shapes it holds. For us the questions takes two specific forms.

Question 60. *For what leafages of chordal graphs is Hendry's Conjecture guaranteed to hold, for which can we find counterexamples? Leafage 2 chordal graphs are exactly interval graphs, and any leafage 3 chordal graph is a spider intersection graph. So the question is already solved for leafage 2 and 3, but what about leafage 4 and higher?*

Question 61. *Given that paths are subdivisions of an edge, and spiders are subdivisions of stars, another way to state the results on interval and spider intersection graphs is: Any Hamiltonian chordal graph hosted on a subdivision of T where T is an edge or a star is cycle extendable. Is there any other such tree T which can guarantee cycle extendability for Hamiltonian chordal graphs hosted on its subdivisions?*

We arrive at our answers for these two questions by modifying Lafond and Seamone's counterexample using tools developed in chapter 3. In Figure 5.1 we present a drawing of the smallest graph of this family which we will refer to as G^1. G^1 is chordal, has a Hamiltonian cycle H^1 and a non-Hamiltonian non-extendable cycle C^1 where

$$H^1 = w_4dcw_2hw_3gz_2z_1bagw_5ew_4,$$

$$C^1 = w_4daw_1bcw_2hw_3gfw_5ew_4.$$

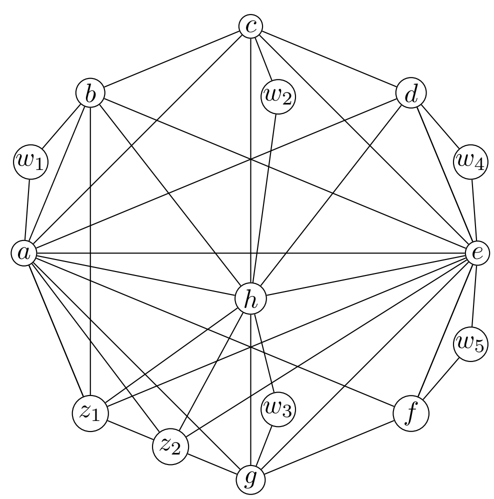

Figure 5.1: G^1, the original counterexample

By selecting a maximal spanning tree of the weighted clique graph of G^1, we produce a representation ρ^1 for G^1. This representation is drawn in Figure 5.2.

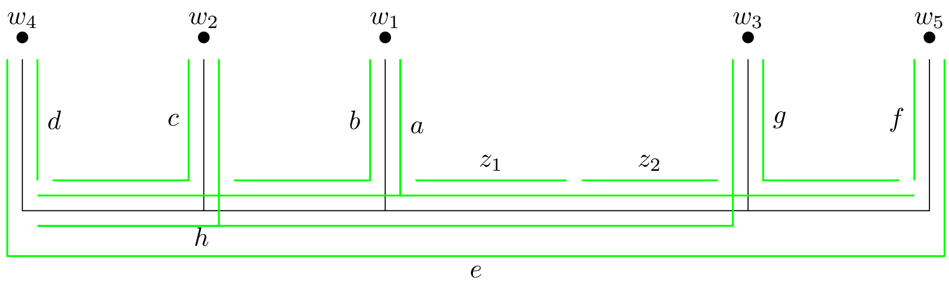

Figure 5.2: ρ^1: a tree representation for G^1

Note that this representation is on a host tree with 5 leaves. It is relatively easy to extend it to host trees with more leaves, so this answers our question on what leafages guarantee Hendry's Conjecture for leafages ≥ 5. This leaves us to ponder the case for leafage 4.

Also note that there is a part of $\rho^1(h)$ that is unnecessary for either H or C. We remove that part and construct subrepresentation $\rho^2(h)$ as drawn in Figure 5.3. In Figure 5.4 we see G^2, the graph associated with ρ^2. As can be seen in the figure, $G^2 = G^1 \setminus hd$.

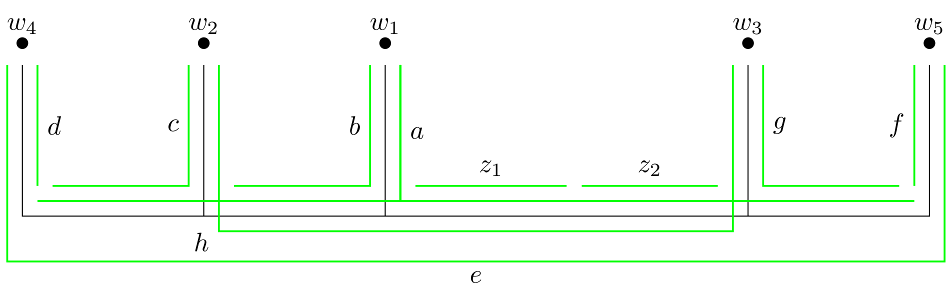

Figure 5.3: ρ^2: reduced from ρ^1

We then build a model from ρ^2 as seen in Figure 5.5. We name this model $\mathcal{M}^2$

We can manipulate the model by shortening the middle branch as can be seen in Figure 5.6. We shall name this new model $\mathcal{M}^3$. It is important to note that $\mathcal{M}^3$ is not a submodel of $\mathcal{M}^2$, since it adds two edges: w_1h and w_1e. We can see

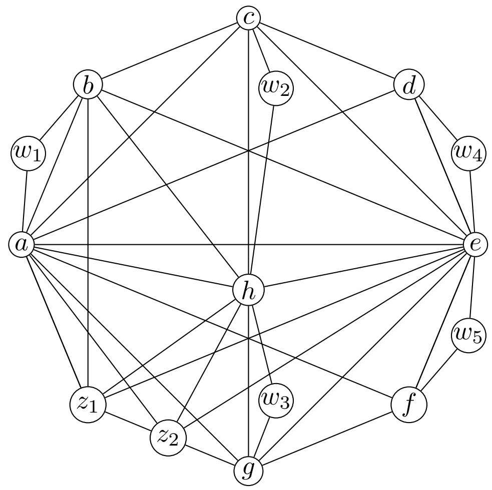

Figure 5.4: G^2: graph corresponding to ρ^2

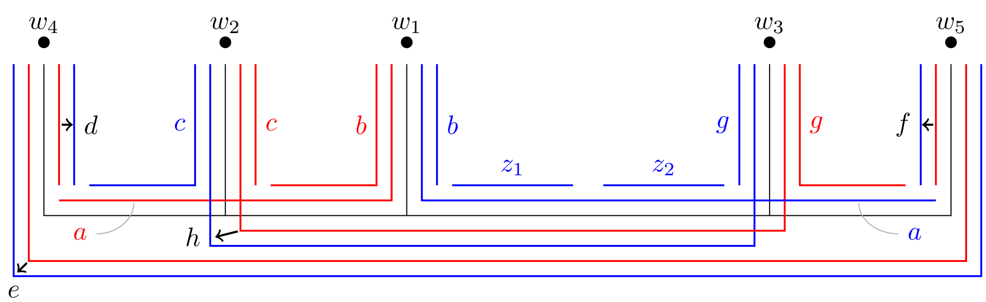

Figure 5.5: $\mathcal{M}^2$: a model for ρ^2

the extra edges in Figure 5.7 where we draw G^3, the graph associated with M^3. However any extension of C in G^3 cannot use these extra edges, since in order to include the vertices w_2, w_3, w_4, w_5 in the cycle, the edges w_2h, hw_3, w_4e, and ew_5 are forced, accounting for both pair of edges h and e may have. As such any extension of C in G^3 would imply an extension of C in G^2 and therefore in G^1, implying that G^3 is also a counterexample to Hendry's Conjecture.

$\mathcal{M}^3$ (and G^3) shows that there's a counterexample to Hendry's Conjecture with leafage 4.

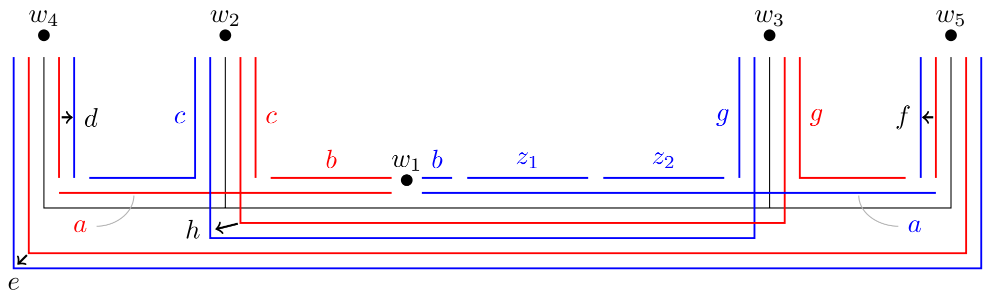

Figure 5.6: M^3: modified from model M^2

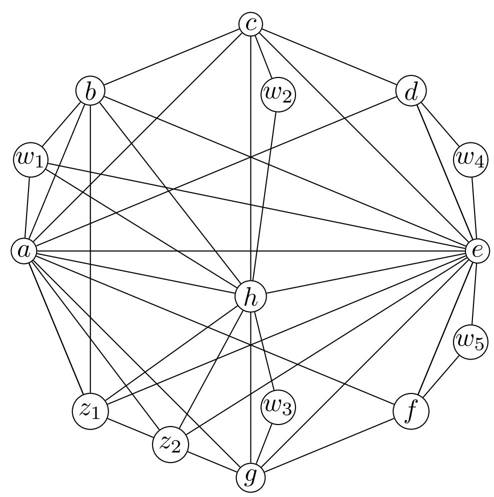

Figure 5.7: G^3: the graph associated with M^3

Theorem 62. *Hendry's Conjecture holds for chordal graphs of leafage 3 or less, and this result is sharp in that there are counterexamples for chordal graphs of leafage* ≥ 4.

That answers question 60. Using the same counterexample we can also answer question 61.

Definitions 63. We define T_H to be the tree on 6 vertices with exactly two of them of degree 3, and the rest degree 1. An *H-shape tree* [6] is a subdivision of T_H.

Theorem 64. *Hamiltonian chordal graphs hosted on spiders (which are subdivisions of stars) are cycle extendable. Any tree which is not a spider has a subdivision on which a counterexample to Hendry's Conjecture may be hosted. In fact one needs to subdivide at most five times to arrive at such a subdivision.*

Proof. Let T be a tree that is not a spider. T must have an induced H-shape tree. As can be seen in model $\mathcal{M}^3$, G^3 can be hosted on an H-shape tree, that can be attained by subdividing T_H five times. This means we can build a representation for G^3 on T just using the induced H-shape tree and subdividing at most five times to add the necessary vertices. □

G^3 has one more advantage over G^1 (the original Lafond, Seamone counter example). Looking at $\mathcal{M}^3$ we note that (unlike in ρ^1) the subtrees representing the vertices of G^3 are all paths. This answers the question of whether Hendry's Conjecture holds for another subclass of chordal graphs.

Definition 65. A *VPT graph* is a chordal graph which has a tree representation in which the subtrees representing its vertices are paths.

Theorem 66. *Hendry's Conjecture does not hold for VPT graphs.*

5.1 Further Investigation

Definition 67. A *double star* is a tree which may be obtained by joining the central vertices of two stars. We will name this added edge *the central edge* of the double star, and denote the central vertices *the left central vertex* and *the right central vertex* arbitrarily. We will name the remaining edges *side edges* and divide them into two sets, *left side edges* and *right side edges*, depending on which central vertex they are incident to.

Since we know Hamiltonian chordal graphs hostable on subdivisions of stars (i.e. spider intersection graphs) are cycle extendable, it makes sense to ask whether Hendry's Conjecture holds for chordal graphs hostable on subdivisions of double stars. We will ignore the double stars for which the one of the central vertices has degree two or less since that describes a subdivision of a star (i.e. spider).

Theorem 64 tells us that Hendry's Conjecture does not necessarily hold for all chordal graphs hosted on all subdivisions of double stars. Specifically we know that that if a double star has at least two left side edges, two right side edges, and that at least one edge on both sides is subdivided at least once, and the central edge is subdivided at least three times, then on the resultant tree we can host a counter example to Hendry's Conjecture. That does not cover all subdivisions of double stars. There are in fact two classes of trees that merit further investigation.

Question 68. *Start with a double star that has two or more left side edges and two or more right side edges. Subdivide the central edge as many times as you like. Subdivide the left side edges as many times as you like. Are Hamiltonian chordal graphs hosted on the resultant tree guaranteed to be cycle extendable?*

Question 69. *Start with a double star that has two or more left side edges and two or more right side edges. Subdivide the central edge at most twice. Subdivide the side edges as many times as you like. Are Hamiltonian chordal graphs hosted on the resultant tree guaranteed to be cycle extendable?*

These two classes constitute exactly the subdivisions of double stars not covered by Theorem 64. It should be noted that the smallest subdivisions of double stars that are not spiders are H-shape trees. In other words, H-shape trees are the class of graphs on which these two questions should be initially studied.

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