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On curvature, volume growth and uniqueness of steady Ricci solitons

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Contents

Abstract

This thesis contains my work during my Ph.D. studies at Lehigh University under the guidance of my advisor Huai-Dong Cao. The work is related to objects called Ricci solitons which serve as singularity models of Ricci flow. We are going to study Ricci solitons in this thesis from the following aspects:

1. Curvature properties.
2. Volume growth properties.
3. Uniqueness under constraints of the asymptotic geometry.

We first explore the curvature estimate for four dimensional steady Ricci solitons. The main result is about control of the full curvature tensor Rm by scalar curvature R.

We are then going to study curvature and volume growth properties of complete steady Kahler Ricci solitons with positive Ricci curvature. The main result is that volume growth is at least half dimensional and scalar curvature behaves like $\frac{1}{r}$ in average where r is the geodesic distance to some point.

In the third part, we are going to study the uniqueness of the steady Kahler Ricci soliton constructed by Huai-Dong Cao under constraints of the asymptotic geometry. The main result says that it is unique if we ask that the metric tensor be C^1 close in some sense to the model.

Chapter 1

Preliminary

1.1 Definition of Ricci solitons

The Ricci flow is a geometric PDE introduced by R. Hamilton in 1982 [37]. It is a nonlinear weakly parabolic system which evolves the metric tensor by its Ricci tensor,

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

It is a powerful tool in the study of the geometry of the underlying manifold where this PDE system evolves. For example, it is the primary tool used in G. Perelman's solution of the Poincaré conjecture[51]. It has also been applied by R. Schoen and S. Brendle[5] in the proof of the differentiable sphere theorem.

Singularity analysis is one of the main parts of studying the Ricci flow. Self similar solutions, called Ricci solitons arise during singularity analysis.

Definition 1.1.1. A complete Riemannian manifold (M, g) is called a Ricci soliton, if there exists a complete vector field V, such that

$$R_{ij} + \frac{1}{2}\mathcal{L}_V g_{ij} = \lambda g_{ij}$$

for some constant $\lambda \in \mathbb{R}$.

Based on the sign of λ, they divide into three types, namely shrinking ($\lambda > 0$), steady ($\lambda = 0$) and expanding ($\lambda < 0$).

Moreover, if V is a gradient vector field, i.e., $V = \nabla f$, then we say it is a gradient Ricci soliton with potential function f. In the these we are going to focus on gradient steady solitons ($V = \nabla f, \lambda = 0$)

$$R_{ij} + \nabla_i \nabla_j f = 0.$$

Ricci solitons are natural generalization of Einstein manifolds ($V = 0$). The self similar solution generated by a Ricci soliton often appears as a singularity model, i.e., the parabolic dilation limit of Ricci flow near a singularity. Therefore, the structure of Ricci soliton helps us know more about the Ricci flow near itsp singularity.

1.2 Curvature equations and inequalities

Lemma 1.2.1. (Hamilton [39]) *Let (M^n, g_{ij}, f) be a complete gradient steady soliton satisfying Eq. (1.1). Then*

$$R = -\Delta f, \tag{1.1}$$

$$\nabla_i R = 2R_{ij}\nabla_j f, \tag{1.2}$$

$$R + |\nabla f|^2 = C_0 \tag{1.3}$$

$$\nabla_l R_{ijkl} = R_{ijkl}\nabla_l f \tag{1.4}$$

We also collect several equations and inequalities of R, Ric and Rm (cf. [39],[54]).

Lemma 1.2.2. *Let (M^n, g_{ij}, f) be a complete gradient steady soliton satisfying Eq.*

(1.1). Then, we have

$$\begin{aligned}
\Delta_f R &= -2|Ric|^2, \\
\Delta_f Ric &= -2R_{ijkl}R_{jl}, \\
\Delta_f Rm &= Rm * Rm,
\end{aligned}$$

where $*$ means linear combinations of contractions between tensors, Δ_f is the f-Laplacian operatorp $\Delta - \nabla f \cdot \nabla$.

Lemma 1.2.3. *Let (M^n, g_{ij}, f) be a complete gradient steady soliton satisfying Eq. (1.1). Then*

$$\begin{aligned}
\Delta_f |Ric|^2 &\geq 2|\nabla Ric|^2 - 4|Rm||Ric|^2, \\
\Delta_f |Rm| &\geq -c|Rm|^2, \\
\Delta_f |Rm|^2 &\geq 2|\nabla Rm|^2 - C|Rm|^3.
\end{aligned}$$

Here $c > 0$ is some universal constant depending only on the dimension n.

Remark 1.2.1. To derive the second differential inequality, one needs to use the Kato inequality $|\nabla|Rm|| \leq |\nabla Rm|$ as shown in [45].

To get nonnegativity of scalar curvature, we will need the following useful result by B.-L. Chen [22].

Proposition 1.2.1. *(B.-L Chen [22]) Let $g_{ij}(t)$ be a complete ancient solution to the Ricci flow on a noncompact manifold M^n. Then the scalar curvature R of $g_{ij}(t)$ is nonnegative for all t.*

Since gradient steady solitons generate self silimar solutions which are not just ancient but eternal, we have,

Lemma 1.2.4. *Let (M^n, g_{ij}, f) be a complete gradient steady soliton. Then it has nonnegative scalar curvature $R \geq 0$.*

Remark 1.2.2. In fact, by Proposition 3.2 in [54], either $R > 0$ or (M^n, g_{ij}) is Ricci flat.

1.3 Basic properties of solitons

Let us compare some previous results for complete gradient shrinking and steady Ricci solitons.

Properties	Shrinking solitons	Steady solitons
Potential function growth	$f \sim \frac{1}{4}r^2$, [15]	(1) If $Rc > 0$, R attains maximum, $c_1 r - c_2 \leq -f \leq \sqrt{R_{max}}\,r + c_3$, [16]
		(2)$\inf_{y \in \partial B_r(x)} f(y) \sim -\sqrt{\Lambda} r$, [63]
Volume growth	(1)$Vol(B_r) \leq cr^n$, [15]	(1)$c \cdot r \leq Vol(B_p(r)) \leq c \cdot e^{a\sqrt{r}}$, [44]
	(2)$Vol(B_r) \geq cr$, [46]	
	(3)If $Rc \geq 0$, then $\lim_{r\to\infty} \frac{Vol(B_r)}{r^n} = 0$, [48]	(2) If f satisfies a uniform condition, $Vol(B_r) \leq r^n$. [61]

1.4 Some examples of steady solitons

Steady solitons arise as certain Type II singularity models of the Ricci flow. Recall a gradient steady Ricci soliton satisfies

$$R_{ij} + f_{ij} = 0.$$

Therefore Ricci flat spaces are steady solitons if we pick our potential function f to be 0. Indeed, by an argument of Hamilton [39], a compact steady soliton has to be trivial (Ricci flat). Therefore a nontrivial steady soliton is noncompact. Many people have constructed nontrivial steady solitons and we list some of them.

Space	$\mathbb{R}^2$	$\mathbb{R}^n (n \geq 3)$	$\mathbb{C}^n (n \geq 2)$
Metric Ansatz	$SO(2)$ or $U(1)$	$SO(n)$	$U(n)$
Potential function	$f = -\ln\cosh(r)$	$f \sim -cr$	$f \sim -cr$
Volume Growth	$Vol(B_r) \sim r$	$Vol(B_r) \sim r^{\frac{n+1}{2}}$	$Vol(B_r) \sim r^n$
Curvature	$R(g) > 0$, $R = O(e^{-2r})$	$sec(g) > 0$, $R = O(\frac{1}{r})$	$sec(g) > 0$, $R = O(\frac{1}{r})$
Found by	R. Hamilton, [38]	R. Bryant, [8]	H.-D. Cao, [11]

Chapter 2

Curvature estimates for four-dimensional steady solitons

2.1 Background

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *gradient steady Ricci soliton* if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = 0. \tag{2.1}$$

The function f is called a *potential function* of the gradient steady soliton. Clearly, when f is a constant the gradient steady Ricci soliton (M^n, g_{ij}, f) is simply a Ricci-flat manifold. Gradient steady solitons play an important role in Hamilton's Ricci flow, as they correspond to translating solutions, and often arise as Type II singularity models. Thus one is interested in possibly classifying them or understanding their geometry.

It turns out that compact steady solitons must be Ricci-flat. In dimension $n = 2$, Hamilton [36] discovered the first example of a complete noncompact gradient steady soliton on $\mathbb{R}^2$, called the *cigar soliton*, where the metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

The cigar soliton has potential function $f = -\log(1 + x^2 + y^2)$, positive curvature $R = 4e^f$, and is asymptotic to a cylinder at infinity. Furthermore, Hamilton [36] showed that the only complete steady soliton on a two-dimensional manifold with bounded (scalar) curvature R which assumes its maximum at an origin is, up to scaling, the cigar soliton. For $n \geq 3$, Bryant [10] proved that there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on $\mathbb{R}^n$; see, e.g., Chow et al. [28] for details. The Bryant soliton has positive sectional curvature, linear curvature decay and volume growth of geodesic balls $B(0, r)$ on the order of $r^{(n+1)/2}$. In the Kähler case, Cao [11] constructed a complete $U(m)$-invariant gradient steady Kähler-Ricci soliton on $\mathbb{C}^m$, for $m \geq 2$, with positive sectional curvature. It has volume growth on the order of r^m and also linear curvature decay. Note that in each of these three examples, the maximum of the scalar curvature is attained at the origin. One can find additional examples of steady solitons, e.g., in [41, 43, 30, 31, 3] etc; see also [14] and the references therein.

In dimension $n = 3$, Perelman [52] claimed that the Bryant soliton is the only complete noncompact, κ-noncollapsed, gradient steady soliton with positive sectional curvature. Recently, Brendle has affirmed this conjecture of Perelman (see [6]; and also [7] for an extension to the higher dimensional case). On the other hand, for $n \geq 4$, Cao-Chen [16] and Catino-Mantegazza [20] proved independently, and using different methods, that any n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton (M^n, g_{ij}, f) is either flat or isometric to the Bryant soliton (the method of Cao-Chen [16] also applies to the case of dimension $n = 3$). In addition, Bach-flat gradient steady solitons (with positive Ricci curvature) for all $n \geq 3$ [19] and half-conformally flat ones for $n = 4$ [25] have been classified respectively.

Inspired by the very recent work of Munteanu-Wang [45], in [17] we studied curvature estimates of four-dimensional complete noncompact gradient steady solitons.

In [45], Munteanu and Wang made an important observation that the curvature tensor of a four-dimensional gradient Ricci soliton (M^4, g_{ij}, f) can be estimated in terms of the potential function f, the Ricci tensor and its first derivates. In addition, the (optimal) asymptotic quadratic growth property of the potential function f proved in [15], as well as a key scalar curvature lower bound $R \geq c/f$ shown in [29] are crucial in their work. Even though gradient steady Ricci solitons in general don't share these two special features (cf. [63, 44, 61] and [29, 34]), some of the arguments in [45] can still be adapted to prove certain curvature estimates for two classes of gradient steady solitons.

2.2 Main Results

Theorem 2.2.1. *Let (M^4, g_{ij}, f) be a complete noncompact 4-dimensional gradient steady Ricci soliton with positive Ricci curvature $Ric > 0$ such that the scalar curvature R attains its maximum at some point $x_0 \in M^4$. Then, (M^4, g_{ij}) has bounded Riemann curvature tensor, i.e.*

$$\sup_{x \in M} |Rm| \leq C$$

for some constant $C > 0$. If in addition R has at most linear decay, then

$$\sup_{x \in M} \frac{|Rm|}{R} \leq C.$$

Theorem 2.2.2. *Let (M^4, g_{ij}, f), which is not Ricci-flat, be a complete noncompact 4-dimensional gradient steady Ricci soliton. If $\lim_{x \to \infty} R(x) = 0$, then, for each $0 < a < 1$, there exists a constant $C > 0$ such that*

$$|Ric|^2 \leq C R^a \qquad \text{and} \qquad \sup_{x \in M} |Rm| \leq C.$$

Suppose in addition R has at most polynomial decay. Then, for each $0 < a < 1$, *there exists a constant $C > 0$ such that*

$$|Rm|^2 \leq CR^a.$$

2.3 Preliminaries

It follows from Remark 1.2.2 that the constant C_0 in (1.3) is positive whenever f is a non-constant function (i.e., the steady soliton is non-trivial). By a suitable scaling of the metric g_{ij}, we can normalize $C_0 = 1$ so that

$$R + |\nabla f|^2 = 1. \tag{2.2}$$

In the rest of this chapter, we shall always assume this normalization (2.4).

Combining (2.1) and (2.4), we obtain $-\Delta f + |\nabla f|^2 = 1$. Thus, setting $F = -f$,

$$\Delta_f F = 1. \tag{2.3}$$

For gradient steady solitons with positive Ricci curvature $Ric > 0$,

Proposition 2.3.1. *(Cao-Chen [16]) Let (M^n, g_{ij}, f) be a complete noncompact gradient steady soliton with positive Ricci curvature $Ric > 0$ such that the scalar curvature R attains its maximum $R_{max} = 1$ at some point $x_0 \in M^n$. Then, there exist some constants $0 < c_1 \leq 1$ and $c_2 > 0$ such that $F = -f$ satisfies the estimates*

$$c_1 r(x) - c_2 \leq F(x) \leq r(x) + |F(x_0)|, \tag{2.4}$$

where $r(x) = d(x_0, x)$ is the distance function from x_0.

Remark 2.3.1. In (2.4), only the lower bound on F requires the assumptions on Ric and R. Note that, under the assumption in Proposition 2.3.1, $F(x)$ is proportional to the distance function $r(x) = d(x_0, x)$ from above and below. Throughout the chapter, we denote

$$\begin{aligned} D(t) &= \{x \in M : F(x) \le t\}, \\ B(t) = B(x_0, t) &= \{x \in M : d(x_0, x) \le t\}. \end{aligned}$$

2.4 Case 1: steady soliton with $Ric > 0$

First of all, we need the following key fact, valid for 4-dimensional gradient steady Ricci solitons in general, due to Munteanu and Wang [45].

Lemma 2.4.1. (Munteanu-Wang [45]) *Let (M^4, g_{ij}, f) be a complete noncompact gradient steady soliton satisfying (1.1). Then there exists some universal constant $c > 0$ such that*

$$|Rm| \le c(\frac{|\nabla Ric|}{|\nabla f|} + \frac{|Ric|^2}{|\nabla f|^2} + |Ric|).$$

Proof. This follows from the same arguments as in the proof of Proposition 1.1 of [45], but without replacing $|\nabla f|^2$ by f in their argument.

□

Proposition 2.4.1. *Let (M^4, g_{ij}, f) be a complete noncompact gradient steady soliton with positive Ricci curvature such that R attains a maximum. Then, there exists some constant $C > 0$, depending on the constant c_1 in (2.7), such that outside a compact set,*

$$|Rm| \le C(|\nabla Ric| + |Ric|^2 + |Ric|).$$

Proof. This easily follows from Lemma 2.4.1 and the following fact shown by Cao-Chen [16]:

$$|\nabla f|^2 \ge c_1 > 0. \tag{2.5}$$

□

Remark 2.4.1. Note that, combining (2.5) with (2.2) and (2.3), we have

$$0 < c_1 \le |\nabla F|^2 = |\nabla f|^2 \le 1. \tag{2.6}$$

Now we are ready to prove our first main result.

Theorem 2.4.1. *Let (M^4, g_{ij}, f) be a complete noncompact gradient steady soliton with positive Ricci curvature $Ric > 0$ such that R attains its maximum at some point $x_0 \in M^4$. Then, there exists some constant $C > 0$, depending on c_1 in (2.5), such that*

$$\sup_{x\in M} |Rm| \leq C.$$

Proof. First of all, from (2.4), we have $R \leq 1$. Hence, since $Ric > 0$, it follows that

$$0 < |Ric| \leq R \leq 1. \tag{2.7}$$

Thus, by Proposition 2.4.1 and (2.7), we see that

$$|\nabla Ric|^2 \geq \frac{1}{2C^2}|Rm|^2 - (|Ric|^2 + |Ric|)^2 \geq \frac{1}{2C^2}|Rm|^2 - 4. \tag{2.8}$$

Using the first two inequalities in Lemma 1.2.3, we obtain

$$\Delta_f(|Rm| + \lambda|Ric|^2) \geq -C|Rm|^2 + 2\lambda(|\nabla Ric|^2 - 2|Rm||Ric|^2). \tag{2.9}$$

By (2.8), (2.9), and picking constant $\lambda > 0$ sufficiently large (depending on the constant C in Proposition 2.4.1, hence on c_1), it follows that

$$\Delta_f(|Rm| + \lambda|Ric|^2) \geq 2|Rm|^2 - 4\lambda|Rm| - C' \geq (|Rm| + \lambda|Ric|^2)^2 - C. \tag{2.10}$$

Next, let $\varphi(t)$ be a smooth function on $\mathbb{R}^+$ so that $0 \leq \varphi(t) \leq 1$, $\varphi(t) = 1$ for $0 \leq t \leq R_0$, $\varphi(t) = 0$ for $t \geq 2R_0$, and

$$t^2\left(|\varphi'(t)|^2 + |\varphi''(t)|\right) \leq c \tag{2.11}$$

for some universal constant c and $R_0 > 0$ arbitrary large. We now take $\varphi = \varphi(F(x))$ as a cut-off function with support in $D(2R_0)$. Note that

$$|\nabla\varphi| = |\varphi'||\nabla F| \leq \frac{c}{R_0} \quad \text{and} \quad |\Delta_f\varphi| \leq |\varphi'\Delta_f F| + |\varphi''||\nabla F|^2 \leq \frac{c}{R_0} \tag{2.12}$$

on $D(2R_0) \setminus D(R_0)$ for some universal constant c.

Setting $u = |Rm| + \lambda |Ric|^2$ and $G = \varphi^2 u$, then direct computations, (2.10) and (2.12) yield

$$\begin{aligned}
\varphi^2 \Delta_f G &= \varphi^4 \Delta_f u + \varphi^2 u \Delta_f(\varphi^2) + 2\varphi^2 \nabla u \cdot \nabla \varphi^2 \\
&\geq \varphi^4 \left(u^2 - C\right) + \varphi^2 u \left(2\varphi \Delta_f \varphi + 2|\nabla \varphi|^2\right) + 2\nabla G \cdot \nabla \varphi^2 - 8|\nabla \varphi|^2 G \\
&\geq G^2 + 2\nabla G \cdot \nabla \varphi^2 - CG - C.
\end{aligned}$$

Now it follows from the maximum principle that $G \leq C$ on $D(2R_0)$ by some constant $C > 0$ depending on c_1 but independent of R_0. Hence $u = |Rm| + \lambda |Ric|^2 \leq C$ on $D(R_0)$. Since $R_0 > 0$ is arbitrary large, we see that

$$\sup_{x \in M} |Rm| \leq \sup_{x \in M} \left(|Rm| + \lambda |Ric|^2\right) \leq C.$$

This completes the proof of Theorem 2.4.1. □

Proposition 2.4.2. *Let (M^4, g_{ij}, f) be a complete noncompact gradient steady soliton with positive Ricci curvature $Ric > 0$ and R attains its maximum at $x_0 \in M^4$. Then the function $u = \frac{|Rm| + \lambda |Ric|^2}{R}$, with $\lambda > 0$ sufficiently large, satisfies the differential inequality*

$$\Delta_f u \geq u^2 R - CR - 2\nabla u \cdot \nabla(\log R)$$

for some constant $C > 0$ outside a compact set.

Proof. First of all, by an argument similar to that of (2.8)-(2.10) in the proof of Theorem 2.4.1, by choosing λ sufficiently large we have

$$\begin{aligned}
\Delta_f(|Rm| + \lambda |Ric|^2) &\geq (|Rm| + \lambda |Ric|^2)^2 - 4\lambda^2 |Ric|^4 - \lambda(|Ric|^4 + |Ric|^2) \\
&\geq (|Rm| + \lambda |Ric|^2)^2 - C|Ric|^2
\end{aligned}$$

for some constant $C > 0$. Here we have also used the fact (2.8).

Thus, by a direct computation,

$$\begin{aligned}
\Delta_f u &= R^{-1}\Delta_f(|Rm| + \lambda|Ric|^2) + (uR)\Delta_f(R^{-1}) + 2\nabla(uR)\cdot\nabla(R^{-1}) \\
&\geq \frac{(|Rm| + \lambda|Ric|^2)^2 - C|Ric|^2}{R} + (uR)\left[2\frac{|Ric|^2}{R^2} + 2\frac{|\nabla R|^2}{R^3}\right] \\
&\quad -\frac{2}{R^2}\left(u|\nabla R|^2 + R\nabla u\cdot\nabla R\right) \\
&\geq Ru^2 - CR - 2\nabla u\cdot\nabla\log R.
\end{aligned}$$

□

Theorem 2.4.2. *Let (M^4, g_{ij}, f) be a complete noncompact gradient steady Ricci soliton with $Ric > 0$ such that R attains its maximum. Suppose R has at most linear decay, i.e. for some $c > 0$, $R(x) \geq c/r(x)$, outside a compact set. Then*

$$\sup_{x\in M}\frac{|Rm|}{R} \leq C.$$

Proof. Fix λ sufficient large so that Proposition 2.4.2 holds and set $u = \frac{|Rm|+\lambda|Ric|^2}{R}$. Next, let $\varphi(t)$ be a Lipschitz function on $\mathbb{R}^+$ so that $\varphi(t) = \frac{d-t}{d}$ for $0 \leq t \leq d$, $\varphi(t) = 0$ for $t \geq d$. Let $\varphi = \varphi(F)$ and $G = \varphi^2 u$. Then on $\mathring{D}(d)\setminus D(1)$ φ satisfies,

$$\begin{aligned}
|\nabla\varphi| = |\varphi'\nabla F| \leq \frac{1}{d}, \\
\Delta_f\varphi = \varphi'\Delta_f F = -\frac{1}{d}.
\end{aligned}$$

Then outside $D(1)$, we have,

$$\begin{aligned}
\varphi^2\triangle_f(G) &= \varphi^4(\triangle_f u) + \varphi^2 u(\triangle_f\varphi^2) + 2\varphi^2\nabla\varphi^2\cdot\nabla u \\
&\geq \varphi^4\left(Ru^2 - cR - 2\nabla u\cdot\nabla\log R\right) \\
&\quad +2\left(\varphi\triangle_f\varphi + |\nabla\varphi|^2\right)G + 2\varphi^2\nabla\varphi^2\cdot\nabla u \\
&\geq RG^2 - cR + 4\varphi\left(\nabla\varphi\cdot\nabla\log R\right)G \\
&\quad -\frac{8}{d}G + \left(2\nabla\varphi^2 - 2\varphi^2\nabla\log R\right)\cdot\nabla G.
\end{aligned}$$

Now by Lemma 1.2.3 and $Ric > 0$, we have $|\nabla \log R| = 2|\frac{Ric(\nabla f)}{R}| \leq 2$. Also, when R has at most linear decay outside some $D(t_0)$ and for $d > t_0$, we have $R \geq \frac{a}{d}$ in $D(d)\backslash D(t_0)$ for some constant $a > 0$ independent d. Therefore there exists c independent of d such that on $\mathring{D}(d) \setminus D(1)$, following inequalities holds,

$$\begin{aligned}\varphi^2 \triangle_f(G) &\geq RG^2 - cR - \frac{c}{d}G + \big(2\nabla\varphi^2 - 2\varphi^2\nabla \log R\big) \cdot \nabla G \\ &\geq \frac{1}{2}RG^2 - cR + \big(2\nabla\varphi^2 - 2\varphi^2\nabla \log R\big) \cdot \nabla G.\end{aligned}$$

Recall $u > 0$, therefore the maximum of G_d must attains in the interior of $D(d)$. Then it follows from maximum principle argument that $u \leq C$ on M^4, hence $|Rm| \leq CR$ on M^4.

□

2.5 Case 2: steady soliton with $\lim_{x\to\infty} R(x) = 0$

In this section, we prove our second main result, Theorem 2.2.2. Throughout the section we assume (M^4, g_{ij}, f) is a complete noncompact, non Ricci-flat, 4-dimensional gradient steady Ricci soliton such that

$$\lim_{x\to\infty} R(x) = 0. \tag{2.13}$$

Note that, by Remark 1.2.2, (M^4, g_{ij}, f) necessarily satisfies $R > 0$.

First of all, we need the following useful Laplacian comparison type result for gradient Ricci solitons.

Lemma 2.5.1. *Let (M^n, g_{ij}, f) be any gradient steady Ricci soliton and let $r(x) = d(x_0, x)$ denote the distance function on M^n from a fixed base point x_0. Suppose that*

$$Ric \leq (n-1)K$$

on the geodesic ball $B(x_0, r_0)$ for some constants $r_0 > 0$ and $K > 0$. Then, for any

$x \in M^n \setminus B(x_0, r_0)$, we have

$$\Delta_f r(x) \leq (n-1)\left(\frac{2}{3}Kr_0 + r_0^{-1}\right).$$

Remark 2.5.1. Lemma 2.5.1 is a special case of a more general result valid for solutions to the Ricci flow due to Perelman [51], see, e.g., Lemma 3.4.1 in [18]. Also see [33] and [62] for a different version.

Theorem 2.5.1. *Let (M^4, g_{ij}, f) be a complete noncompact gradient steady Ricci soliton which is not Ricci-flat,. If $\lim_{x\to\infty} R(x) = 0$, then, for each $0 < a < 1$, there exists a constant $C > 0$ such that*

$$\sup_{x\in M} |Ric|^2 \leq CR^a \qquad \text{and} \qquad \sup_{x\in M} |Rm| \leq C.$$

Proof. The proof is similar to that of Munteanu-Wang [45] except we need to use the distance function to cut-off rather than the potential function since the potential function may not be proper.

Since $\lim_{x\to\infty} R(x) = 0$, it follows from (2.4) that

$$|\nabla f| \geq c_1 > 0$$

for some $0 < c_1 < 1$ outside a compact set. By Lemma 1.2.1 and Lemma 2.4.1, we have

$$\begin{aligned}
\Delta_f |Ric|^2 &\geq 2|\nabla Ric|^2 - C|Rm||Ric|^2 \\
&\geq 2|\nabla Ric|^2 - C\left(|\nabla Ric| + |Ric|^2 + |Ric|\right)|Ric|^2.
\end{aligned}$$

Also, since $R > 0$ on M^4, by using the first identity in Lemma 2.3 we have

$$\Delta_f\left(\frac{1}{R^a}\right) = 2a\frac{|Ric|^2}{R^{a+1}} + a(a+1)\frac{|\nabla R|^2}{R^{a+2}}.$$

Hence,

$$
\begin{aligned}
\Delta_f\left(\frac{|Ric|^2}{R^a}\right) &= \frac{\Delta_f|Ric|^2}{R^a}+|Ric|^2\Delta_f\left(\frac{1}{R^a}\right)+2\nabla|Ric|^2\cdot\nabla\left(\frac{1}{R^a}\right)\\
&\geq \frac{2|\nabla Ric|^2}{R^a}-C\frac{\left(|\nabla Ric|+|Ric|^2+|Ric|\right)|Ric|^2}{R^a}\\
&\quad +|Ric|^2\left[2a\frac{|Ric|^2}{R^{a+1}}+a(a+1)\frac{|\nabla R|^2}{R^{a+2}}\right]-4a\frac{|Ric|\,|\nabla|Ric||\,|\nabla R|}{R^{a+1}}
\end{aligned}
$$

Apply Cauchy's inequality to the last term

$$
\begin{aligned}
-4a\frac{|Ric|\,|\nabla|Ric||\,|\nabla R|}{R^{a+1}} &\geq -4a\frac{|Ric|\,|\nabla Ric|\,|\nabla R|}{R^{a+1}}\\
&\geq -a(a+1)\frac{|Ric|^2|\nabla R|^2}{R^{a+2}}-\frac{4a}{a+1}\frac{|\nabla Ric|^2}{R^a}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\Delta_f\left(|Ric|^2R^{-a}\right) &\geq \frac{2(1-a)}{1+a}\frac{|\nabla Ric|^2}{R^a}-C\frac{|\nabla Ric||Ric|^2}{R^a}\\
&\quad -C\frac{|Ric|^4+|Ric|^3}{R^a}+2a\frac{|Ric|^4}{R^{a+1}}\\
&\geq (2a-\frac{CR}{1-a})\frac{|Ric|^4}{R^{a+1}}-C\frac{|Ric|^3}{R^a}.
\end{aligned}
$$

Therefore, for $u=\frac{|Ric|^2}{R^a}$, we have derived the differential inequality

$$
\Delta_f u\geq(2a-\frac{CR}{1-a})u^2R^{a-1}-Cu^{3/2}R^{a/2}. \qquad (2.14)
$$

Since $R\to 0$, for any $0<a<1$, we can choose a fixed $d_0>0$ depending on a and sufficiently large so that

$$
(2a-\frac{CR}{1-a})\geq a \qquad (2.15)
$$

outside the geodesic ball $B(x_0,d_0)$.

Next, for any $D_0>2d_0$, we choose a function $\varphi(t)$ as follows: $0\leq\varphi(t)\leq 1$ is a smooth function on $\mathbb{R}$ such that

$$\varphi(t) = \begin{cases} 1, & 2d_0 \leq t \leq D_0, \\ 0, & t \leq d_0 \text{ or } t \geq 2D_0. \end{cases}$$

Also,

$$t^2|\varphi''(t)| \leq c \qquad \text{and} \qquad 0 \geq \varphi'(t) \geq -\frac{c}{D_0}, \text{ if } 2d_0 \leq t \leq 2D_0. \tag{2.16}$$

Now we use $\varphi = \varphi(r(x))$ as a cut-off function whose support is in $B(x_0, 2D_0) \setminus B(x_0, d_0)$. Note that by lemma 2.5.1, we get

$$|\nabla\varphi|^2 = |\varphi'|^2 \leq \frac{c}{D_0^2} \quad \text{and} \quad \Delta_f\varphi = \varphi'\Delta_f r(x) + \varphi'' \geq -\frac{C}{D_0}. \tag{2.17}$$

on $B(x_0, 2D_0) \setminus B(x_0, 2d_0)$ respectively.

Setting $G = \varphi^2 u$, then by our choice of φ and (2.17), we see that

$$\begin{aligned} \varphi^2\Delta_f G &= \varphi^4\Delta_f u + \varphi^2 u\Delta_f\varphi^2 + 2\varphi^2(\nabla u \cdot \nabla\varphi^2) \\ &\geq \varphi^4\left(au^2R^{a-1} - Cu^{3/2}R^{a/2}\right) + 2\varphi^2 u(\Delta_f\varphi^2) - 8|\nabla\varphi|^2 G + 2\nabla G \cdot \nabla\varphi^2 \\ &\geq aG^2R^{a-1} - CG^{3/2}R^{a/2} - CG + 2\nabla G \cdot \nabla\varphi^2. \end{aligned}$$

Assume G achieves its maximum at some point $p \in B(x_0, 2D_0)$. If $p \in B(x_0, 2D_0) \setminus B(x_0, 2d_0)$, then it follows from the maximum principle that

$$0 \geq aG^2(p)R^{a-1}(p) - CG^{3/2}(p)R^{a/2}(p) - CG(p).$$

On the other hand, noticing that the fact $0 < a < 1$ and R uniformly bounded from above, implies.

$$G(p) \leq C$$

for some constant C depending on a but independent of D_0.

Thus,

$$\max_{B(x_0,D_0)} u \le \max_{B(x_0,2D_0)} G \le \max\left\{C, \max_{B(2d_0)} u\right\} \le C'$$

for some $C' > 0$ indepedent of D_0. Therefore $|Ric|^2 \le CR^a$ on M^4.

It remains to show $|Rm| \le C$ on M^4. However, once we know $\sup_{x\in M} Ric \le C$, $|Rm| \le C$ follows essentially from the same argument as in the proof of Theorem 2.4.1. We leave the details to the reader.

$\square$

Lemma 2.5.2. *Let (M^4, g_{ij}, f), which is not Ricci-flat, be a complete noncompact gradient steady Ricci soliton with $\lim_{x\to\infty} R(x) = 0$. Then for each $0 < a < 1$ and $\mu > 0$, there exist constants $\lambda > 0$ and $D > 0$ so that function*

$$v = \frac{|Rm|^2 + \lambda|Ric|^2}{R^a}$$

satisfies the differential inequality

$$\Delta_f v \ge \mu v - D.$$

Proof. By Lemma 1.2.2 and Theorem 2.5.1,

$$\begin{aligned} \Delta_f v &= \frac{\Delta_f(|Rm|^2 + \lambda|Ric|^2)}{R^a} + vR^a\Delta_f(\frac{1}{R^a}) + 2\nabla(vR^a)\cdot\nabla(R^{-a}) \\ &\ge \frac{2|\nabla Rm|^2 + 2\lambda|\nabla Ric|^2}{R^a} - c\frac{|Rm|^2 + \lambda|Ric|^2}{R^a} \\ &\quad +(|Rm|^2 + \lambda|Ric|^2)\left[-a\frac{\triangle_f R}{R^{a+1}} + a(a+1)\frac{|\nabla R|^2}{R^{a+2}}\right] \\ &\quad -4a\frac{|Rm||\nabla Rm||\nabla R|}{R^{a+1}} - 4a\lambda\frac{|Ric||\nabla Ric||\nabla R|}{R^{a+1}}. \end{aligned}$$

By applying Cauchy's inequality to terms with $|\nabla R|$,

$$
\begin{aligned}
\Delta_f v &\geq \frac{2|\nabla Rm|^2 + 2\lambda|\nabla Ric|^2}{R^a} - c\frac{|Rm|^2 + \lambda|Ric|^2}{R^a} \\
&\quad - \frac{4a}{a+1}\frac{|\nabla Rm|^2}{R^a} - \frac{4a\lambda}{a+1}\frac{|\nabla Ric|^2}{R^a} \\
&\geq \frac{2\lambda(1-a)}{1+a}\frac{|\nabla Ric|^2}{R^a} - c\frac{|Rm|^2 + \lambda|Ric|^2}{R^a}.
\end{aligned}
$$

Now by Proposition 2.4.1, for some constant $\epsilon > 0$, we have

$$
\begin{aligned}
2\epsilon|Rm|^2 &\leq \left(|\nabla Ric| + |Ric|^2 + |Rc|\right)^2 \\
&\leq 2|\nabla Ric|^2 + 2(|Ric|^2 + |Ric|)^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta_f v &\geq \left[\frac{2\epsilon\lambda(1-a)}{1+a} - c\right]\frac{|Rm|^2}{R^a} - \left[2\lambda\frac{1-a}{1+a}(|Ric|+1)^2 + c\lambda\right]\frac{|Ric|^2}{R^a} \\
&\geq \left[\epsilon\lambda(1-a) - c\right](v - \lambda\frac{|Ric|^2}{R^a}) - \lambda\left[2(1-a)(|Ric|+1)^2 + c\right]\frac{|Ric|^2}{R^a}.
\end{aligned}
$$

Therefore, by Theorem 2.5.1, for each $0 < a < 1$ and $\mu > 0$ one can choose $\lambda \geq C/(1-a)$, with $C > 0$ depending on μ and sufficiently large, so that

$$
\Delta_f v \geq \mu v - D
$$

for some constant $D > 0$ depending on λ. □

Theorem 2.5.2. *Let (M^4, g_{ij}, f), which is not Ricci-flat, be a complete noncompact gradient steady Ricci soliton with $\lim_{r\to\infty} R = 0$. Suppose R has at most polynomial decay, i.e., $R(x) \geq C/r^k(x)$ outside $B(r_0)$ for some fixed $r_0 > 1$, some constant $c > 0$ and positive integer k. Then, for each $0 < a < 1$, there exists a constant C such that*

$$
|Rm| \leq CR^{a/2}.
$$

Proof. Let $p = \frac{k}{2}$. Consider the following function on $\mathbb{R}^+$:

$$\varphi(t) = \begin{cases} \left(\frac{d-t}{d}\right)^p & 0 \le t \le d \\ \\ 0 & t \ge d. \end{cases}$$

Next, let $\varphi = \varphi(r(x_0))$ on M^4. Then on $\mathring{B}(d) \setminus (Cut(x_0) \cup B(r_0))$ we have,

$$\begin{aligned} |\nabla\varphi| &= \frac{p}{d}\left(\frac{d-r}{d}\right)^{p-1}|\nabla r| = \frac{p}{d-r}\varphi, \\ \triangle_f\varphi &= -\frac{p}{d}\left(\frac{d-r}{d}\right)^{p-1}\triangle_f r + \frac{p(p-1)}{d^2}\left(\frac{d-r}{d}\right)^{p-2}|\nabla r|^2 \\ &= \left[-\frac{p}{d-r}\triangle_f r + \frac{p(p-1)}{(d-r)^2}\right]\varphi \end{aligned}$$

Consider $w = v - \frac{D}{\mu}$ with $v = \frac{|Rm|^2+\lambda|Ric|^2}{R^a}$, μ and D as in Lemma 2.5.2. Then, w satisfies

$$\triangle_f w \ge \mu w.$$

Let $G{=}\varphi^2 w$, then on $\mathring{B}(d) \setminus B(r_0)$, we have

$$\begin{aligned} \triangle_f G &= (\Delta_f\varphi^2)w + \varphi^2\Delta_f w + 2(\nabla\varphi^2)\cdot\nabla w \\ &\ge \left(2\varphi\Delta_f\varphi + 2|\nabla\varphi|^2\right)w + \mu\varphi^2 w + 4\varphi\nabla\varphi\cdot\nabla\frac{G}{\varphi^2} \\ &\ge \left(\mu + \frac{2\triangle_f\varphi}{\varphi} - 6\frac{|\nabla\varphi|^2}{\varphi^2}\right)G + \frac{4}{\varphi}\langle\nabla G, \nabla\varphi\rangle. \end{aligned} \tag{2.18}$$

Recall that $G = 0$ outside $B(d)$. Now consider a maximum point q of G.

Case 1. $G(q) \le 0$. Then, $\max\limits_{B(d)} w \le 0$.

Case 2. $G(q) > 0$ and $q \in B(r_0)$. Then, on $\Omega = B((1 - \frac{1}{2^{1/p}})d)$, we have

$$\begin{aligned} \max_{\Omega} w &\leq \max_{\Omega} \frac{1}{\varphi^2} \cdot G(q) \\ &\leq 4G(q) \\ &\leq 4 \max_{B(r_0)} w. \end{aligned}$$

Case 3. $G(q) > 0$ and $q \notin B(r_0)$, $q \notin Cut(x_0)$. Then we could apply by (2.18) and Lemma 2.5.1, at q,

$$\begin{aligned} 0 &\geq \mu + 2\frac{\triangle_f \varphi}{\varphi} - 6\frac{|\nabla \varphi|^2}{\varphi^2} \\ &\geq \mu - 2pK_0 \frac{1}{d-r} - (4p^2 + 2p)\frac{1}{(d-r)^2} \end{aligned}$$

for some constant $K_0 > 0$ depending on r_0 and $\max_{B(r_0)} |Ric|$. Hence $\frac{1}{d - r(q)} > C$ for some constant C depending on μ, $p = k/2$ and K_0. Thus, we have

$$d - r(q) \leq c \tag{2.19}$$

for some constant $c > 0$ independent of d.

Therefore,

$$\begin{aligned} \max_{\Omega} w &\leq \max_{\Omega} \frac{1}{\varphi^2} \cdot G(q) \\ &\leq 4G(q) \\ &\leq 4\frac{(d - r(q))^{2p}}{d^{2p}} \frac{(|Rm|^2 + \lambda |Ric|^2)}{R^a}(q) \\ &\leq C\frac{r^{ak}(q)}{d^{2p}} \\ &\leq Cd^{(a-1)k} \leq C \end{aligned}$$

for some constant $C > 0$ independent of d.

Case 4. $G(q) > 0$, $q \notin B(r_0)$, $q \in Cut(x_0)$. Then we could not apply (2.18) directly since $d(x_0, -)$ is not smooth at p. Now consider the support function G_ϵ

constructed by the following procedure. Firstly, pick any minimal geodesic γ from x_0 to p, then choose a point $x_1 \in \gamma$ very close to x_0. Notice that $x_1 \notin Cut(p)$. Let $\epsilon = d(x_0, x_1)$, consider $r_\epsilon(x) = d(x_1, x)$. Then we have,

- $r_\epsilon(q) + \epsilon = r(q)$
- $r_\epsilon(x) + \epsilon \geq r(q)$
- $r_\epsilon(x)$ is smooth near q

Now consider $G_\epsilon(x)$=$\varphi(r_\epsilon(x) + \epsilon)^2 w$ where φ was defined in the beginning of the section. Then $G_\epsilon(x) \leq G(x) \leq G(q) = G_\epsilon(q)$. Then we could apply maximum principle at q since G_ϵ is smooth at q.

$$\begin{aligned} 0 &\geq \mu + 2\frac{\triangle_f \varphi_\epsilon}{\varphi_\epsilon} - 6\frac{|\nabla \varphi_\epsilon|^2}{\varphi_\epsilon^2} \\ &\geq \mu - 2pK_0(x_1)\frac{1}{d - r_\epsilon - \epsilon} - (4p^2 + 2p)\frac{1}{(d - r_\epsilon - \epsilon)^2} \end{aligned}$$

Hence $\frac{1}{d - r_\epsilon(q) - \epsilon} > C$ for some constant C depending on μ, $p = k/2$ and $K_0(x_1)$. In order to get rid of the dependence of x_1, we let $\epsilon \to 0$, then we have

$$d - r(q) \leq c \tag{2.20}$$

for some constant $c > 0$ independent of d.

Now follow the exact same argument of Case 3, we get an uniform estimate of $\max_\Omega w$ on $\Omega = B((1 - \frac{1}{2^{1/p}})d)$ which is independent of d.

Therefore $\sup_M w \leq C$, and hence $|Rm|^2 \leq CR^a$ on M^4 for each $0 < a < 1$.

$\square$

Chapter 3

Curvature and volume growth of steady Kähler Ricci solitons

3.1 Background

We call a Riemannian metric g_{ij} Kähler if there exists a (1,1) tensor J such that

- $J^2 = -Id_{TM}$
- $g(JX, JY) = g(X, Y)$ for any $X, Y \in TM$
- $\nabla J = 0$

If a steady Ricci soliton is Kähler, we call it a steady Kähler Ricci soliton. For properties of Kähler manifold readers may consult [2], [32]. We are going to list some notations we will use in this chapter.

Firstly consider the complexified tangent bundle $TM^{\mathbb{C}} = TM \otimes \mathbb{C}$. Then extend J by complex linearity. Denote

- $T^{1,0}M = \{X - iJX | X \in TM\}$,
- $T^{0,1}M = \{X + iJX | X \in TM\}$.

Now extend g using complex linearily to $TM^{\mathbb{C}} = TM \otimes \mathbb{C}$. Then $g_{\mathbb{C}}$ satisfies

- $g_{\mathbb{C}}(\overline{X}, \overline{Y}) = \overline{g_{\mathbb{C}}(X, Y)}$
- $g_{\mathbb{C}}(X, \overline{X}) > 0$ for $X \in TM^{\mathbb{C}} - 0$
- $g_{\mathbb{C}}(X, Y) = 0$ for $X, Y \in T^{1,0}M$

$g_{\mathbb{C}}$ becomes a Hermitian metric on $T^{1,0}M$, for simplicity we use $g_{i\bar{j}}$ for this Hermitian metric in this chapter. Since $\nabla J = 0$, Ric also shares the similar symmetry. We denote $Ric_{i\bar{j}}$ for the non vanishing part of the complexified tensor.

In this part we are going to analyze the asymptotic behaviour of steady Kähler Ricci solitons with positive Ricci curvature. We are going to focus on two parts; the first one is volume growth and the second one is curvature decay.

For the volume growth, when the manifold has nonnegative Ricci curvature the classical Bishop comparison theorem implies the volume growth is at most Euclidean. And under the same condition, the volume growth is at least linear by a result of Yau and Calabi [64]. If furthermore the manifold has positive holomorphic bisectional curvature, Bing-Long Chen and Xi-Ping Zhu showed [23] that the volume growth is at least half Euclidean growth and curvature has to decay in the average sense. Applying their method, we showed that if the manifold is a steady Kähler Ricci soliton metric, then similar results hold when the metric has positive Ricci curvature.

3.2 Volume growth

Theorem 3.2.1. *For any Kähler Ricci soliton ($M^{2n}, g_{i\bar{j}}, f$) with positive Ricci curvature and scalar curvature attaining its maximum, volume growth is at least half Euclidean, i.e.,*

$$Vol(B_r) \geq cr^n$$

here r is the geodesic distance to some point x_0.

Proof. Fix $r > 1$, and consider positive function $\varphi_r = e^{-\frac{F}{r}}$ where $F = -f$. Here we consider the Ricci form $\Omega = Ric(JX, Y)$ and Kähler form $\omega = g(JX, Y)$.

$$\begin{aligned}
& \int_{\{\varphi_r > \delta\}} (\varphi_r - \delta)^n (\sqrt{-1})^n (\partial\overline{\partial} F)^n \\
= \quad & -\int_{\{\varphi_r > \delta\}} n(\varphi_r - \delta)^{n-1} (\partial \varphi_r \wedge \overline{\partial} F) \wedge (\sqrt{-1})^n (\partial\overline{\partial} F)^{n-1} \\
= \quad & \int_{\{\varphi_r > \delta\}} n(\varphi_r - \delta)^{n-1} (\frac{\varphi_r}{r} \partial F \wedge \overline{\partial} F) \cdot (\sqrt{-1})^n (\partial\overline{\partial} F)^{n-1} \\
\leq \quad & \int_{\{\varphi_r > \delta\}} n(\varphi_r - \delta)^{n-1} \frac{\varphi_r}{r} |\nabla F|_g^2 \wedge (\sqrt{-1})^{n-1} (\partial\overline{\partial} F)^{n-1} \wedge \omega \\
\leq \quad & \int_{\{\varphi_r > \delta\}} C_1(n, R_0) \frac{(\varphi_r - \delta)^{n-1} \varphi_r}{r} (\sqrt{-1})^{n-1} (\partial\overline{\partial} F)^{n-1} \wedge \omega \\
\leq \quad & \int_{\{\varphi_r > \delta\}} C_2(n, R_0) \frac{(\varphi_r - \delta)^{n-2} \varphi_r^2}{r^2} (\sqrt{-1})^{n-2} (\partial\overline{\partial} F)^{n-2} \wedge \omega^2 \\
\cdots \quad & \\
\leq \quad & \int_{\{\varphi_r > \delta\}} C_n(n, R_0) \frac{\varphi_r^n}{r^n} \omega^n
\end{aligned}$$

Here R_0 is the maximum value of scalar curvature, recall we have $|\nabla F|^2 \leq R_0$.

Let $\delta \to 0$ we get

$$\int_M \varphi_r^n \Omega^n \leq \frac{C(n, R_0)}{r^n} \int_M \varphi_r^n \omega^n$$

Left hand side has a strictly lower bound since Ricci form is a positive (1,1) form. Therefore we have,

$$c \leq \frac{C(n, R_0)}{r^n} \int_M \varphi_r^n \omega^n$$

Recall from (2.4) the potential function estimate $c_1 d(x) - c_2 \leq F \leq \sqrt{R_0} d(x) + |F(x_0)|$ which gives the following estimate,

$$\varphi_r \leq e^{-\frac{c_1 d(x) - c_2}{r}} \leq C e^{-c_1 \frac{d}{r}}$$

outside $B(x_0, 1)$. Therefore we have,

$$
\begin{aligned}
\int_M \varphi_r^n \omega^n &\leq C' \int_M e^{-c_1 n \frac{d}{r}} \omega^n \\
&\leq C' \sum_{i=0}^{\infty} \int_{B_{2^{i+1}r} - B_{2^i r}} e^{-c_1 n \frac{d}{r}} \omega^n + C' \int_{B_r} e^{-c_1 n \frac{d}{r}} \omega^n \\
&\leq C' \sum_{i=0}^{\infty} \int_{B_{2^{i+1}r}} e^{-c_1 n \frac{2^i r}{r}} \omega^n + C' \int_{B_r} e^{-c_1 n \frac{r}{r}} \omega^n \\
&= C' \sum_{i=0}^{\infty} e^{-2^i c_1 n} \int_{B_{2^{i+1}r}} \omega^n + C' e^{-c_1 n} Vol(B_r) \\
&\leq \sum_{i=0}^{\infty} e^{-2^i c_1 n} Vol(B_{2^{i+1}r}) + C' e^{-c_1 n} Vol(B_r) \\
&\leq \sum_{i=0}^{\infty} e^{-2^i c_1 n} 2^{(i+1)2n} Vol(B_r) + C' e^{-c_1 n} Vol(B_r) \\
&\leq C''(c1, c2, n) \cdot Vol(B_r)
\end{aligned}
$$

Here c_1, c_2 come from the estimate for φ_r from the previous page.

Therefore
$$c \leq \frac{C(n, R_0)}{r^n} \int_M \varphi_r^n \omega^n \leq C(n, R_0, c_1, c_2) \frac{Vol B_r}{r^n}$$

$\square$

3.3 Curvature estimates

Theorem 3.3.1. *For a steady Kähler Ricci soliton $(M^{2n}, g_{i\bar{j}}, f_{i\bar{j}})$ with positive Ricci curvature such that the scalar curvature attains its maximum, for any x_0 there exists C such that*

$$\frac{1}{Vol(B(r))}\int_{B(r)} R(x) \leq \frac{C}{1+r},$$

here $B(r)$ is a geodesic ball of radius r to any point.

Proof. We are going to use the following theorem by Hörmander. The version we are going to use is in Chapter VIII, Theorem 6.5. [32] With the natural function F our manifold is Stein. Furthermore we have the following inequality,

$$\sqrt{-1}\partial\overline{\partial}(F) + c_1(K_M) + Ric = Ric > 0$$

Now fix a base point x_0 and some cut off function θ near x_0. Let ϵ be some small number such that $\sqrt{-1}\partial\overline{\partial}(F + 2\epsilon\theta\log|z - x_0|)$ is still positive. Then for m large enough such that $[m\epsilon] - n > 0$ we find a nontrivial L^2 section S of K_M^m with prescribed value at x_0, say $S(x_0)$. Furthermore

$$\int_M |S|_h^2 e^{-mF} dV_g < \infty$$

$$\sqrt{-1}\partial\overline{\partial}\log|S|_h^2 = [S = 0] + mRic \geq mRic$$

Since the Ricci curvature is nonnegative, the mean value inequality for subharmonic functions gives,

$$\begin{aligned}
|S|^2(x) &\leq \frac{c(n)}{Vol(B(x, 3d(x_0, x)))}\int_{B(x,3d(x_0,x))} |S|^2(x) \\
&= \frac{c(n)}{Vol(B(x, 3d(x_0, x)))}\int_{B(x,3d(x_0,x))} |S|^2(x)e^{-mF}e^{mF} \\
&\leq C'e^{m\sqrt{R_0}d(x,x_0)}
\end{aligned}$$

Now consider $\widetilde{M} = M \times \mathbb{C}^2$, with the product metric. This space has nonnegative Ricci curvature, therefore parabolicity translates to volume growth.([42] Theorem 5.2). The volume growth of $\widetilde{M}$ is at least $n+4$, therefore there exists a positive Green function $\widetilde{G}(x,y)$ on $\widetilde{M}$. Now consider $\widetilde{G}(x) = \widetilde{G}(x_0', x)$ where $x_0' = (x_0, 0)$ where x_0 is the point that we can prescribe S. Recall that we have,

$$\triangle \log(|S|^2) \geq mR,$$

and $\log(|S|^2)$ has singularity along $S=0$. Therefore we consider $\log(|S|^2 + \delta)$.

$$\triangle \log(|S|^2 + \delta) \geq mR \frac{|S|^2}{|S|^2 + \delta}$$

Now pull back functions $|S|^2$, R on M through map $\widetilde{M} \to M$, we get functions $\widetilde{|S|^2}$, $\widetilde{R}$ such that,

$$\triangle \log(\widetilde{|S|^2} + \delta) \geq m\widetilde{R} \frac{\widetilde{|S|^2}}{\widetilde{|S|^2} + \delta}$$

$$\begin{aligned}
\int_{\beta > \widetilde{G} > \alpha} m\widetilde{R} \frac{\widetilde{|S|^2}}{\widetilde{|S|^2} + \delta} (\widetilde{G} - \alpha)^{1+\epsilon} &\leq \int_{\beta > \widetilde{G} > \alpha} \triangle \log(\widetilde{|S|^2} + \delta)(\widetilde{G} - \alpha)^{1+\epsilon} \\
&= \int_{\beta > \widetilde{G} > \alpha} \log(\widetilde{|S|^2} + \delta) \triangle (\widetilde{G} - \alpha)^{1+\epsilon} \\
&\quad + \int_{\widetilde{G} = \beta} \frac{\partial \log(\widetilde{|S|^2} + \delta)}{\partial n} (\widetilde{G} - \alpha)^{1+\epsilon} \\
&\quad - (1+\epsilon) \int_{\widetilde{G} = \beta} \log(\widetilde{|S|^2} + \delta)(\widetilde{G} - \alpha)^{\epsilon} \frac{\partial \widetilde{G}}{\partial n}
\end{aligned}$$

$$\begin{aligned}
\int_{\beta > \widetilde{G} > \alpha} \log(\widetilde{|S|^2} + \delta) \triangle (\widetilde{G} - \alpha)^{1+\epsilon} &\leq \sup_{\widetilde{G} > \alpha} \log(\widetilde{|S|^2} + \delta) \int_{\beta > \widetilde{G} > \alpha} \triangle (\widetilde{G} - \alpha)^{1+\epsilon} \\
&= \sup_{\widetilde{G} > \alpha} \log(\widetilde{|S|^2} + \delta) \int_{\widetilde{G} = \beta} (1+\epsilon)(\widetilde{G} - \alpha)^{\epsilon} \frac{\partial \widetilde{G}}{\partial n}
\end{aligned}$$

Letting $\epsilon \to 0$, we get

$$\begin{aligned}\int_{\beta>\widetilde{G}>\alpha} m\widetilde{R}\frac{\widetilde{|S|^2}}{\widetilde{|S|^2}+\delta}(\widetilde{G}-\alpha) \quad \leq \quad & \sup_{\widetilde{G}>\alpha}\log(\widetilde{|S|^2}+\delta)\int_{\widetilde{G}=\beta}\frac{\partial\widetilde{G}}{\partial n} \\ & +\int_{\widetilde{G}=\beta}\frac{\partial\log(\widetilde{|S|^2}+\delta)}{\partial n}(\widetilde{G}-\alpha) \\ & -\int_{\widetilde{G}=\beta}\log(\widetilde{|S|^2}+\delta)\frac{\partial\widetilde{G}}{\partial n}\end{aligned}$$

We'll prove later such that $\widetilde{G} \sim \frac{c}{d^{2n+2}}$, $|\frac{\partial\widetilde{G}}{\partial n}| \sim \frac{c'}{d^{2n+3}}$. By using these facts, we obtain,

$$\int_{\widetilde{G}=\beta}\frac{\partial\widetilde{G}}{\partial n} \sim \frac{cd^{2n+3}}{d^{2n+3}} \sim c$$

$$\int_{\widetilde{G}=\beta}|\widetilde{G}-\alpha| \to \frac{cd^{2n+3}}{d^{2n+2}} = cd \to 0$$

$$|\widetilde{S(x_0)}|^2 = 1$$

Letting $\beta \to +\infty$, then letting $\delta \to 0$(furthermore use $\widetilde{S}=0$ has codimension 1)

$$\int_{\widetilde{G}>\alpha} m\widetilde{R}(\widetilde{G}-\alpha) \leq c(n)\sup_{\widetilde{G}>\alpha}\log(\widetilde{|S|^2})$$

On $\widetilde{G}>2\alpha$, we have $(\widetilde{G}-\alpha) \geq \frac{1}{2}\widetilde{G}$ therefore

$$\int_{\widetilde{G}>2\alpha} m\widetilde{R}\widetilde{G} \leq c(n)\sup_{\widetilde{G}>\alpha}\log(\widetilde{|S|^2})$$

Goal: Change $G(x_0,-)$ level set coordinates back to regular geodesic ball.

Tool: Green function estimate

$$\frac{c(n)^{-1}\tilde{d}^2(x,x_0)}{Vol(\tilde{B}(x_0,\tilde{d}(x,x_0))} \leq \tilde{G}(x,x_0) \leq \frac{c(n)\tilde{d}^2(x,x_0)}{Vol(\tilde{B}(x_0,\tilde{d}(x,x_0))}$$

From [42] Thm5.2, we have the following estimate for a space with nonnegative Ricci curvature:

$$c(n)^{-1}\int_{d^2}^{\infty}\frac{dt}{Vol(\sqrt{t})}\leq G(x,x_0)\leq c(n)\int_{d^2}^{\infty}\frac{dt}{Vol(\sqrt{t})}.$$

Lower bound:

$$\int_{d^2}^{\infty}\frac{dt}{Vol(\sqrt{t})}\geq\int_{d^2}^{\infty}\frac{c(n)d^{2n+2}dt}{Vol(\sqrt{d^2})\cdot t^{n+1}}=\frac{c(n)d^{2n+2}dt}{Vol(d)}\int_{d^2}^{\infty}t^{-n-1}=c'\frac{d^2}{V(d)}.$$

Here we use Bishop-Gromov:$\frac{Vol(\sqrt{d^2})}{Vol(\sqrt{t})}\geq(\frac{\sqrt{d^2}}{\sqrt{t}})^{2n+2}$ when $t>d^2$.

Upper bound: This is way back to observation (23) in Shi's construction. Because we have a flat factor which has accurate volume growth information, $\frac{Vol(\sqrt{d^2})}{Vol(\sqrt{t})}\leq C(n)(\frac{\sqrt{d^2}}{\sqrt{t}})^4$.

Since the volume is locally are Euclidean, the above estimates imply $\widetilde{G}\sim\frac{c}{d^{2n+2}}$ when $d\to 0$. By the Cheng-Yau gradient estimate $|\frac{\partial\widetilde{G}}{\partial n}|\sim\frac{c'}{d^{2n+3}}$.

Let $r(\alpha)$ be the largest number such that $\widetilde{B}(x_0,r(\alpha))\subset\{\tilde{G}>\alpha\}$. Because of the Green function estimate,

$$\tilde{B}(x_0,r(\alpha))\subset\{\tilde{G}>\alpha\}\subset\tilde{B}(x_0,c(n)r(\alpha)).$$

Recall we have,

$$\int_{\widetilde{G}>\alpha}m\widetilde{R}\widetilde{G}\leq c(n)\sup_{\widetilde{G}>\frac{1}{2}\alpha}\log(\widetilde{|S|^2}),$$

together with the lower bound $G\geq\frac{r(\alpha)^2}{Vol(r(\alpha))}$ inside $B(r(\alpha))$,

$$\begin{aligned}\int_{\widetilde{G}>\alpha}m\widetilde{R}\widetilde{G} &\geq \int_{\widetilde{B}(r(\alpha))}m\widetilde{R}\widetilde{G}\\ &\geq m\cdot C\frac{r(\alpha)^2}{Vol(r(\alpha))}\int_{\widetilde{B}(r(\alpha))}\widetilde{R}.\end{aligned}$$

By Green function estimate, and the estimate for $\log(\widetilde{|S|^2})$,

$$\sup_{\widetilde{G}>\frac{1}{2}\alpha} \log(\widetilde{|S|^2}) \leq \sup_{B(c(n)r(\alpha))} \log(\widetilde{|S|^2}) \leq c(n)(\widetilde{r}+c').$$

Therefore on $\widetilde{M}$ we have,

$$\frac{1}{Vol(\widetilde{r})}\int_{\widetilde{B}(r(\alpha))} \widetilde{R} \leq c\frac{\widetilde{r}+c}{\widetilde{r}^2} \leq c\frac{1}{\widetilde{r}+1}.$$

The above estimates also hold for M since

$$B_M(\frac{1}{2}r)\times B_{\mathbb{C}^2}(\frac{1}{2}r) \subset B(\widetilde{r}) \subset B_M(r)\times B_{\mathbb{C}^2}(r).$$

$\square$

Chapter 4

Uniqueness under constraints of the asymptotic geometry.

4.1 Background

Recall a steady Ricci soliton (M, g) is a Riemannian metric which satisfies the equation $2Ric = \mathcal{L}_X(g)$. For a steady Ricci soliton (M, g) if in addition the metric is Kähler and X is the gradient of some real valued function, then we call it steady gradient Kähler Ricci soliton.

H.D. Cao constructed a family of steady gradient Kähler Ricci solitons on $\mathbb{C}^n$ with positive holomorphic bisectional curvature in [11]. This is the first noncompact (nontrivial) example of a steady Kähler Ricci soliton. There are also many important examples of Kähler Ricci solitons constructed by Koiso in [49], M. Feldman, T. Ilmanen, and D. Knopf in [43] and Akito Futaki and Mu-Tao Wang in [1].

In [11], Cao asked a question on symmetry of steady Kähler Ricci solitons with positive holomorphic bisectional curvature on $\mathbb{C}^n$. O. Schnürer, and A. Chau's result in [24] gave a partial answer to this question.

Recently S. Brendle showed $O(n)-$symmetry of certain steady solitons in [57] [58]. His work solved an open problem proposed by Perelman in [51]. Otis Chodosh extended the argument to expanding solitons in [26]. Otis Chodosh and Frederick

Tsz-Ho Fong showed $U(n)-$symmetry of certain gradient expanding Kähler Ricci Solitons in [27].

In this chapter we will give a partial answer to the question proposed by Cao in [11] . The argument is similar to [27] which comes from [57] [58] [26].

We are going to compute norms, gradients and distance with respect to the model metric constructed by Cao in [11] in this Chapter.

4.2 Main Theorem

Main Theorem For $n \geq 2$. Let $(\mathbb{C}^n, g_m, X_m)$ be a steady gradient Kähler Ricci solitons constructed by Cao in [11]. Let $(\mathbb{C}^n, \widetilde{g}, \widetilde{X})$ be some steady gradient Kähler Ricci soliton with following properties.

1. $r^{2+\frac{j}{2}}|\nabla^j(\widetilde{g} - g_m)| = o(1) \qquad \text{for } j = 0, 1$
2. $\widetilde{g}$ has positive holomorphic bisectional curvature,

here r is the geodesic distance to the origin with respect to g_m.

Then there exists a point $p \in \mathbb{C}^n$ and a map $\Phi_p : z \to z + p$ such that $g = \Phi_p^*(\widetilde{g})$ satisfies standard $U(n)-$ symmetry.

4.3 Preliminary

In this section we are going to present expression and basic properties of the metric construct by H.-D. Cao in [11]. Let $(z_1, z_2, ..., z_n)$ be standard holomorphic coordinate on $\mathbb{C}^n$. Let $t = \log(|z|^2)$. Then the U(n)-symmetric steady Kähler Ricci soliton in [11] is given by

$$(g_m)_{i\bar{j}} = e^{-t}\phi(t)\delta_{i\bar{j}} + e^{-2t}\overline{z_i}z_j(\phi'(t) - \phi(t)) \tag{4.1}$$

$$(g_m)^{i\bar{j}} = e^t\frac{\delta^{i\bar{j}}}{\phi(t)} + z_i\overline{z_j}(\frac{1}{\phi'(t)} - \frac{1}{\phi(t)}) \tag{4.2}$$

where $\phi(t)$ satisfies $\phi^{n-1}\phi' e^{\phi} = e^{nt}$ after normalization. From computations in [11] we have following properties.

$$\begin{aligned}
\phi(t) &\to nt \\
\phi'(t) &\to n \\
\phi''(t) &\to 0 \\
\phi'''(t) &\to 0 \\
r(t) &= O(t)
\end{aligned}$$

Here r is the distance to origin with respect to g_m. We'll always use r, t for above purpose.

4.4 Calculations

In the calculation part, we are going to analyze, under the asymptotic constraint, how much could various quantities differ from the original model. From now on g is some steady gradient Kahler Ricci soliton metric satisfies assumptions 1,2. g_m is the model metric constructed in [11].

4.4.1 Killing vectors of the model metric

We are going to show $|U_a| = O(r^{\frac{1}{2}})$, $|\nabla U_a| = O(1)$, $|X| = O(r^{\frac{1}{2}})$, $|\nabla X| = O(1)$ by straightforward computations. Here U_a are killing vectors coming from the unitary symmetry of g_m, X is the soliton vector of the model metric. r is the distance to origin with respect to g_m. Notice that JX is Killing, therefore we only need to do explicit computation for Killing vector fields.

We pick following explicit $\mathbb{R}-$basis of Killing vectors of g_m.

1. $U_k^{1,0} = iz_k \frac{\partial}{\partial z_k}$,

2. $U_{u,v}^{1,0} = z_u \frac{\partial}{\partial z_v} - z_v \frac{\partial}{\partial z_u}$ where $u \neq v$,

3. $\widetilde{U}^{1,0}_{u,v} = i(z_u \frac{\partial}{\partial z_v} + z_v \frac{\partial}{\partial z_u})$ where $u \neq v$.

Our goal is to show $|U_a| = O(r^{\frac{1}{2}})$, $|\nabla U_a| = O(1)$. We can restrict the computation to the direction $(z_1, 0, ..., 0) \in \mathbb{C}^n$ by symmetry.

$\bullet |U_a| = O(r^{\frac{1}{2}})$

By expression 4.1 , at $(z_1, 0, ..., 0)$, metric looks like

$$g_{i\bar{j}} = e^{-t} diag\{\phi'(t), \phi(t), ..., \phi(t)\} \tag{4.3}$$

From the expression of Killing vectors above and the relationship between the Kähler metric and its associated Riemannian metric. It's sufficient to calculate the length of $z_1 \frac{\partial}{\partial z_1}$, $z_1 \frac{\partial}{\partial z_k}$ using the Kähler metric .

(a) $|z_1 \frac{\partial}{\partial z_1}|^2 = e^{-t}|z_1|^2 \phi'(t) = \frac{1}{|z_1|^2}|z_1|^2 \phi'(t) = \phi'(t) \to n \sim O(1)$

(b) $|z_1 \frac{\partial}{\partial z_k}|^2 = e^{-t}|z_1|^2 \phi(t) = \frac{1}{|z_1|^2}|z_1|^2 \phi(t) = \phi(t) \to nt \sim O(r)$

Here we have used the asymptotic behaviour of $\phi(t)$ in Section 4.3.

$\bullet |\nabla U_a| = O(1)$

Since all U_a are real holomorphic, when we calculate $|\nabla U_a|$, we can restrict all discussion to $T^{1,0}_{\mathbb{C}}$. From 4.1, Christoffel symbol Γ^i_{jk} is

$$\frac{g^{\bar{l}i}}{e^{2t}} \left\{ \overline{z_j} \Big[(-\phi(t) + \phi'(t))\delta_{k\bar{l}} + e^{-t}(2\phi(t) - 3\phi'(t) + \phi''(t))\overline{z_k} z_l \Big] + (\phi'(t) - \phi(t))\overline{z_k}\delta_{j\bar{l}} \right\} \tag{4.4}$$

At $(z, 0, ..., 0)$ we have the following 4 cases

(A) $j \neq 1$ $\quad \Gamma^i_{jk} = g^{\bar{j}i}\overline{z_k} e^{-2t}\big(\phi'(t) - \phi(t)\big)$

(B) $j = 1$, $k \neq 1$ $\quad \Gamma^i_{1k} = g^{\bar{k}i}\overline{z_1} e^{-2t}\big(\phi'(t) - \phi(t)\big)$

(C) $j = 1$, $k = 1$, $i \neq 1$ $\quad \Gamma^i_{11} = 0$

(D) $j=1,k=1,i=1 \quad \Gamma^1_{11} = g^{\bar{1}1}\overline{z_1}e^{-2t}\big(\phi''(t)-\phi'(t)\big)$

From the expression of a basis of Killing vectors we pick at the beginning of 4.4.1, we only need to estimate the length of $\nabla(z^u\frac{\partial}{\partial z^v})$ along $(z,0,...0)$.

Case I $\quad u=v=k$ where $k\neq 1$

I.1) $\nabla_{\frac{\partial}{\partial z^l}}(z^k\frac{\partial}{\partial z^k})=0$ for $l\neq k$

I.2) $\nabla_{\frac{\partial}{\partial z^k}}(z^k\frac{\partial}{\partial z^k})=\frac{\partial}{\partial z^k}$

Therefore $|\nabla(z^k\frac{\partial}{\partial z^k})|_{g_m}=\frac{1}{|\frac{\partial}{\partial z^k}|}|\frac{\partial}{\partial z^k}|=1$

Case II $\quad u=v=1$

II.1) Taking the derivative along $\frac{\partial}{\partial z^l}$ where $l\neq 1$

$\nabla_{\frac{\partial}{\partial z^l}}(z^1\frac{\partial}{\partial z^1})=z^1\Gamma^m_{l1}\frac{\partial}{\partial z^m}=\frac{1}{\phi(t)}(\phi'(t)-\phi(t))\frac{\partial}{\partial z^l}$

$\frac{1}{\phi(t)}(\phi'(t)-\phi(t))$ is bounded by the properties of $\phi(t)$ listed in Section 4.3.

II.2) Taking the derivative along $\frac{\partial}{\partial z^1}$

$\nabla_{\frac{\partial}{\partial z^1}}(z^1\frac{\partial}{\partial z^1})=\frac{\partial}{\partial z^1}+z^1\Gamma^m_{l1}\frac{\partial}{\partial z^m}=\Big[1+\frac{1}{\phi'(t)}(\phi''(t)-\phi'(t))\Big]\frac{\partial}{\partial z^1}$

$\frac{1}{\phi(t)}(\phi'(t)-\phi(t))$ is bounded by Section 4.3.

Therefore $|\nabla(z^1\frac{\partial}{\partial z^1})|_{g_m}=O(1)$

Case III $\quad u\neq v$ and $u\neq 1$

$\nabla_{\frac{\partial}{\partial z^l}}(z^u\frac{\partial}{\partial z^v})=\delta_{lu}\frac{\partial}{\partial z^v}$

Therefore $|\nabla(z^u\frac{\partial}{\partial z^v})|_{g_m}\leq\frac{1}{|\frac{\partial}{\partial z^u}|}C|\frac{\partial}{\partial z^u}|\leq C$ ($\frac{\partial}{\partial z_u}$ direction is longer by 4.3 and properties of $\phi(t)$ in Section 4.3)

Case IV $u \neq v$ and $u = 1$

VI.1)Taking the derivative along $\frac{\partial}{\partial z^l}$ where $l \neq 1$

$\nabla_{\frac{\partial}{\partial z^l}}(z^1 \frac{\partial}{\partial z^v}) = \delta_{1v} \frac{1}{\phi(t)}\big(\phi'(t) - \phi(t)\big)\frac{\partial}{\partial z^l}$

VI.2)Taking the derivative along $\frac{\partial}{\partial z^1}$

$\nabla_{\frac{\partial}{\partial z^1}}(z^1 \frac{\partial}{\partial z^v}) = \frac{\partial}{\partial z^v} + z^1 \Gamma^s_{lv} \frac{\partial}{\partial z^s} = \frac{\phi'(t)}{\phi(t)}\big(\frac{\partial}{\partial z^v}\big)$

From the expression of the metric in 4.3 and the properties of $\phi(t)$ in Section 4.3 we see that $|\nabla(z^1 \frac{\partial}{\partial z^v})|_{g_m}$ is also $O(1)$

4.4.2 Shifting preserves the assumption 1

We are going to show for $\widetilde{g}$ satisfies Assumption 1 , 2 , and any point $p \in \mathbb{C}^n$, $\Phi_p^*(\widetilde{g})$ also satisfies Assumption 1 , 2.

We just need to check for Assumption 1 still holds for $\Phi_p^*(g)$. This is equivalent to say that $r^{2+\frac{j}{2}}|\nabla^j(\Phi_p^*(\widetilde{g}) - g_m)|_{g_m} = o(1)$ for $j = 0, 1$.

Directly pull back Assumption 1 by Φ_p^*, we get.

$r^{2+\frac{j}{2}}|\nabla^j(\widetilde{g} - g_m)|_{g_m} = o(1) \Longrightarrow r^{2+\frac{j}{2}}|\Phi_p^*(\nabla)^j(\Phi_p^*(\widetilde{g}) - \Phi_p^*(g_m))|_{\Phi_p^*(g_m)} = o(1)$

After this, it's sufficient to check following facts..

1. $|\Phi_p^*(g_m) - g_m|_{g_m} = O(\frac{\log(|z|^2)}{|z|})$
2. $|\nabla(\Phi_p^*(g_m) - g_m)|_{g_m} = O(\frac{(\log(|z|^2))^2}{|z|})$
3. $r^{2+\frac{j}{2}}|(\Phi_p^*\nabla)^j(k)|_{\Phi_p^* g_m} = o(1)$ for $j = 0, 1$ implies $r^{2+\frac{j}{2}}|\nabla^j(k)|_{g_m} = o(1)$ for $j = 0, 1$

STEP 1 $|\Phi_p^*(g_m) - g_m|_{g_m} = O(\frac{\log(|z|^2)}{|z|})$

Plug 4.1 into $(\Phi_p^*(g_m) - g_m)_{i\bar{j}}$, we get

$$
\begin{aligned}
(\Phi_p^*(g_m) - g_m)_{i\bar{j}} = & \left[\frac{\phi(\log(|z+p|^2))}{|z+p|^2} - \frac{\phi(\log(|z|^2))}{|z|^2} \right] \\
& + (\overline{z_i} + \overline{p_i})(z_j + p_j) \left[\frac{\phi'(\log(|z+p|^2)) - \phi(\log(|z+p|^2))}{|z+p|^4} - \frac{\phi'(\log(|z|^2)) - \phi(\log(|z|^2))}{|z|^4} \right] \\
& + \frac{\phi'(\log(|z|^2)) - \phi(\log(|z|^2))}{|z|^4} \left[(\overline{z_i} + \overline{p_i})(z_j + p_j) - \overline{z_i} z_j \right]
\end{aligned} \tag{4.5}
$$

For the norm of the first two terms, apply mean value theorem to real valued function $f(x) = \frac{\phi(\log(x^2))}{x^2}$, we get $\left| f(|z+p|^2) - f(|z|^2) \right| \leq |p| \left| f'(|z|+\xi) \right|, \xi \in (-|p|, |p|)$ $\left| f'(|z| + \xi) \right| = 2 \left| \frac{\phi'\left(\log((|z|+\xi)^2 \right) - \phi\left(\log((|z|+\xi)^2) \right)}{\left(|z|+\xi \right)^3} \right| \leq C \frac{\log(|z|^2)}{|z|^3}$. The last step uses the asymptotic of ϕ in Section 4.3. Therefore the norms of the first two terms is bounded by $C \frac{\log(|z|^2)}{|z|^3}$. This bound works for other 2 coupled terms by similar arguments.

Together with the coarse estimate $\left| g^{u\bar{v}} \right| \leq \widetilde{C} e^t = \widetilde{C} |z|^2$ from expression 4.2 , we have $|\Phi_p^*(g_m) - g_m|_{g_m} = O(\frac{\log(|z|^2)}{|z|})$

STEP 2 $|\nabla(\Phi_p^*(g_m) - g_m)|_{g_m} = O(\frac{(\log(|z|^2))^2}{|z|})$

Let $\delta g = \Phi_p^*(g_m) - g_m$. Then for any i, j, $(\delta g)_{i\bar{j}} = O(\frac{\log(|z|^2)}{|z|^3})$ by STEP 1.

$(\nabla_m \delta g)_{i\bar{j}} = \frac{\partial}{\partial z_m}(\delta g_{i\bar{j}}) - (\delta g)(\nabla_m \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}})$

1. $\frac{\partial}{\partial z_m}(\delta g_{i\bar{j}})$

Expression of $\delta g_{m\bar{t}}$ has been separated into three lines in 4.5

When $\frac{\partial}{\partial z_m}$ hits the first line of 4.5

$$
\begin{aligned}
&\frac{\partial}{\partial z_m}\left[\frac{\phi(\log(|z+p|^2))}{|z+p|^2}-\frac{\phi(\log(|z|^2))}{|z|^2}\right] \\
&=(\overline{z_m+p_m})\left[\frac{(\phi'-\phi)(\log(|z+p|^2))}{|z+p|^4}-\frac{(\phi'-\phi)(\log(|z|^2))}{|z|^4}\right] \\
&+\frac{(\phi'-\phi)(\log(|z|^2))}{|z|^4}\left[\overline{z_m+p_m}-\overline{z_m}\right]
\end{aligned}
$$

By applying the mean value theorem to $f(x)=\frac{\phi'(\log(x^2))-\phi(\log(x^2))}{x^4}$ and asymptotic of $\phi(x)$, we get the order of the sum of these two terms is $O(\frac{\log(|z|^2)}{|z|^4})$.

When $\frac{\partial}{\partial z_m}$ hits the second and third line of 4.5, the order is also $O(\frac{\log(|z|^2)}{|z|^4})$ by the asymptotic of $\phi(t)$ up to third order and a similar discussion.

2. $(\delta g)(\nabla_m\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z_j}})=O(\frac{(\log(|z|^2))^2}{|z|^4})$ This is done by using the result from STEP 1 and the coarse bound $|\Gamma^r_{pq}|=O(\frac{\log(|z|^2)}{|z|})$ which is directly from its expression 4.4

Therefore $|\nabla(\Phi_p^*(g_m)-g_m)|_{g_m}=O(\frac{(\log(|z|^2))^2}{|z|})$

STEP 3 $r^{2+\frac{j}{2}}|(\Phi_p^*\nabla)^j(k)|_{\Phi_p^*g_m}=o(1)$ $j=0,1$ implies $r^{2+\frac{j}{2}}|\nabla^j(k)|_{g_m}=o(1)$

For $j=0$, this comes from the equivalence of $\Phi_p^*g_m$ and g_m by STEP 1.

For $j=1$, it's sufficient to show $|(\Phi_p^*\nabla-\nabla)k|_{g_m}=O(\frac{\log(|z|^2)}{|z|}))$

$\big((\Phi_p^*\nabla_i-\nabla_i)k\big)_{j\bar{l}}=k_{s\bar{l}}(\widetilde{\Gamma^s_{ij}}-\Gamma^s_{ij})=k_{s\bar{l}}\big(\widetilde{g}^{s\overline{v}}\partial_i\widetilde{g}_{j\overline{v}}-g^{s\overline{v}}\partial_i g_{j\overline{v}}\big)$

1.As a complex number, $|k_{s\bar{l}}|=O(\frac{1}{|z|^2})$ by properties of k and g.

2.$|\widetilde{g}^{s\overline{v}}\partial_i\widetilde{g}_{j\overline{v}}-g^{s\overline{v}}\partial_i g_{j\overline{v}}|=|(\widetilde{g}^{s\overline{v}}-g^{s\overline{v}})\partial_i\widetilde{g}_{j\overline{v}}-g^{s\overline{v}}\partial_i(\widetilde{g}_{j\overline{v}}-g_{j\overline{v}})|=O(\frac{\log(|z|^2)}{|z|^2})$

$|(\Phi_p^*\nabla-\nabla)k|_{g_m}=O(\frac{\log(|z|^2)}{|z|}))$ holds by 1,2 together with $|g^{u\overline{v}}|\le|z|^2$.

4.4.3 Rigidity of the soliton vector

In this subsection we are going to see the soliton vector $\widetilde{X}$ of g must satisfies $\widetilde{X}^{1,0}=(\lambda z^i+b^i)\frac{\partial}{\partial z_i}$ where $Re(\lambda)\neq 0$. Therefore there exists an shifting map $(\Phi_p^*(\widetilde{X}))^{1,0}=\lambda z^i\frac{\partial}{\partial z_i}$. In other words, the soliton vector field is rigid suppose we know information about the asymptotic geometry.

Write $\widetilde{X}^{1,0}$ as $u^i(z)\frac{\partial}{\partial z_i}$. Then $u^i(z)$ is a holomorphic function on $\mathbb{C}^n$.

By [12] $|\widetilde{X}|_g^2 + R(g)$ is constant. $R(g) > 0$ by Assumption 2 . Therefore we have $|\widetilde{X}|_g^2 < \infty$. By Assumption 1 , we see that $|\widetilde{X}|_{g_m}^2 < \infty$

From properties of $\phi(t)$ and $\phi'(t)$ in section 4.3 . We see that there exists a positive $C(t_0)$ s.t. $\phi(t), \phi'(t)$ is greater than C for all $t > t_0$. Recall the expression of g_m at $(z, 0, ...0)$ is $diag\{\frac{\phi'(t)}{e^t}, \frac{\phi(t)}{e^t} ... \frac{\phi(t)}{e^t}\}$. Therefore $|\widetilde{X}|_{g_m}^2$ is finite implies for $|z| > e^{t_0}$, $+\infty > g_{i\bar{j}} u^i(z)\overline{u^j(z)} \geq \sum_i C\frac{|u^i(z)|^2}{|z|^2}$. Therefore $|u^i(z)|^2$ is at most linear growth. $u^i(z) = a^i_j z^j + b^i$. Let $\widetilde{X_L} = b^i\frac{\partial}{\partial z_i}$ Then $|\widetilde{X_L}|_{g_m} \to 0$ uniformly.

1. $a^i_j = \delta^i_j \widetilde{a^j}$. To see this we use $|\widetilde{X} - \widetilde{X_L}|_{g_m}$ is finite and make a calculation at $(0, ..., z, ..., 0)$ where the j-th place is not 0. By 4.1 , the metric is $\frac{1}{|z|^2} diag\{\phi, ..., \phi', ..., \phi\}$ at this point.

$$\begin{aligned} \infty > |X - X_L|_{g_m} &= \left\langle a^q_p z^p \frac{\partial}{\partial z_q}, \overline{a^s_r z^r \frac{\partial}{\partial z_s}} \right\rangle \\ &= a^q_j \overline{a^s_j} |z^j|^2 g_{q\bar{s}} \\ &= \sum_{q \neq j} \phi(t)|a^q_j|^2 + |a^j_j|^2 \phi'(t) \end{aligned}$$

From the properties of $\phi(t)$ in Section 4.3. We must have $a^q_j = 0$ for $q \neq j$

2. $\widetilde{a^1} = \widetilde{a^2} = ... = \widetilde{a^n}$ We still use $|\widetilde{X} - \widetilde{X_L}|_{g_m}$ is finite and make a calculation at $(0, ..., z, ..., z, ..., 0)$. Here the nonzero places, the ith and jth ones, are equal. By 4.1, at this point, we have

$$\begin{aligned} (g_m)_{p\bar{p}} &= (g_m)_{q\bar{q}} = e^{-t}\phi(t) + e^{-2t}|z|^2(\phi'(t) - \phi(t)) \\ (g_m)_{p\bar{q}} &= (g_m)_{q\bar{p}} = e^{-2t}|z|^2(\phi'(t) - \phi(t)) \end{aligned}$$

Therefore $\infty > |\widetilde{X} - \widetilde{X_L}|_{g_m} = \frac{1}{4}\left(|\widetilde{a^i}|^2 + |\widetilde{a^j}|^2\right)\phi'(t) + \frac{1}{4}|\widetilde{a^i} - \widetilde{a^j}|^2 \phi(t)$. From Section 4.3, we see that $\widetilde{a^i} = \widetilde{a^j}, \forall i, j$

3. $Re(\lambda) \neq 0$ Suppose λ is purely imaginary. Then there exists a closed circle as integral curve for vector field $\widetilde{X}$. This contradicts with $\widetilde{X}$ being the gradient of some real function.

4.4.4 Decay rate of Ricci of the model metric

We are going to show there exists a $C > 0$ such that $Ric_m - \frac{C}{t^2} g_m \geq 0$ by straightforward computations. Since both Ric_m and g_m are $U(n)-$symmetric, we can restrict our discussion along $(z, 0, ..., 0)$. Along this direction, we have

$$\begin{aligned} g_{i\bar{j}} &= e^{-t} diag\left\{\phi'(t), \phi(t), ..., \phi(t)\right\} \\ R_{i\bar{j}} &= e^{-t} diag\left\{v''(t), v'(t), ..., v'(t)\right\} \end{aligned}$$

where $v(t) = nt - (n-1)\log(\phi(t)) - \log(\phi'(t))$. As we assume g_m have the same normalization as in [11]. The equation of ϕ is $\phi^{n-1}\phi' e^{\phi} = e^{nt}$.

Let's consider $\lambda_1(t) = \frac{v''(t)}{\phi'(t)}$, $\lambda_k(t) = \frac{v'(t)}{\phi(t)}$ where $k \geq 2$

1. λ_k, $k \geq 2$

$$\lambda_k = \frac{n - (n-1)\frac{\phi'(t)}{\phi(t)} - \frac{\phi''(t)}{\phi'(t)}}{\phi(t)} > \frac{C_k}{t}$$

2. λ_1

$$\lambda_1 = \frac{1}{\phi'(t)} \left[(n-1)\frac{(\phi'(t))^2 - \phi''(t)\phi(t)}{(\phi(t))^2} + \frac{(\phi''(t))^2 - \phi'''(t)\phi'(t)}{(\phi'(t))^2} \right]$$

We have the following identities from Cao's paper [11]

$$\begin{aligned}\phi'' =& n\phi' - (\phi')^2 - (n-1)\frac{(\phi')^2}{\phi}\\ \phi''' =& n^2\phi' - 3n(\phi')^2 + 2(\phi')^3 - 3n(n-1)\frac{(\phi')^2}{\phi}\\ &+ 4(n-1)\frac{(\phi')^3}{\phi} + (n-1)(2n-1)\frac{(\phi')^2}{\phi^2}\end{aligned}$$

Plugging them into expression of λ_1, we have

$$\lambda_1 = \frac{1}{\phi'}\left[\phi'' + \frac{(2n-1)(n-1)\phi'}{\phi^2}(\phi'-1)\right]$$

By the asymptotic of ϕ in Section 4.3, we see that $\frac{(2n-1)(n-1)\phi'}{\phi^2}(\phi'-1) \sim \frac{n(2n-1)(n-1)^2}{t^2}$

Therefore we only need to show $\phi'' > 0$ if $n > 1$.

Since $\phi' > 0$, we can write t as a function of ϕ. From the soliton equation, we have $\phi' = \frac{e^{nt}}{\phi^{n-1}e^{\phi}}$ as a function of ϕ. Plug it into the expression of ϕ'', we have

$$\phi'' = \frac{e^{nt}}{\phi^{2n-2}e^{2\phi}\phi}\left(\phi^n e^{\phi} n - e^{nt}\phi - (n-1)e^{nt}\right).$$

Let $f_k(\phi) = \frac{d^k}{d\phi^k}\left(\phi^n e^{\phi} n - e^{nt}\phi - (n-1)e^{nt}\right)$. By expansion of ϕ at zero from [11], $lim_{\phi\to 0} f_0(\phi) = 0$. Now take the derivative on f_0 using $\frac{d}{d\phi}e^{nt} = n\phi^{n-1}e^{\phi}$

$$f_1(\phi) = \frac{d}{d\phi}f_0 = n\phi^{n-1}e^{\phi} - e^{nt}.$$

Then we have $lim_{\phi\to 0} f_1(\phi) = 0$. Take the derivative again

$$f_2(\phi) = \frac{d}{d\phi}f_1 = n\phi^{n-2}e^{\phi}(n-1) > 0.$$

Therefore we have $\phi'' > 0$. Hence $\exists\ C_1 > 0$ s.t. $\lambda_1 > \frac{C}{t^2}$.

4.4.5 Main Argument

Proof We can assume that g_m has the same normalization($\alpha = 1, \beta = 1$) as in [11], since scaling on metric and dilation on coordinates does not affect our assumption. Pick an $\mathbb{R}$–basis of Killing vectors of g_m, $\{U_a\}_{a=1}^{n^2}$.

We have seen in 4.4.3 that there exist a $p \in \mathbb{C}^n$ s.t. $\widetilde{X_p} = 0$. For this specific p, let $g = \Phi_p^*(\widetilde{g})$, $X = \Phi_p^*(\widetilde{X})$. We'll show U_a is also Killing for g.

Now let $h = \mathcal{L}_{U_a} g$, $Z = \triangle_{\mathbb{R}} U_a + Ric(U_a) = 0$, then by Proposition 2.3.7 in[53]

$$\begin{aligned}\triangle_L(h) &= -2\mathcal{L}_{U_a}(Ric) + \mathcal{L}_Z(g) \\ &= -\mathcal{L}_{U_a}(\mathcal{L}_X(g)) \\ &= -\mathcal{L}_X(\mathcal{L}_{U_a}(g)) \\ &= -\mathcal{L}_X(h)\end{aligned}$$

Here we use a fact in 4.4.3 that X is a radial vector. Therefore $[X, U_a] = 0$. Furthermore U_a is real holomorphic, hence $Z = \triangle_{\mathbb{R}} U_a + Ric(U_a) = 0$

In 4.4.1 , we have seen that $|X| = O(1), |\nabla X| = O(1)$. Since assumption 1 is preserved by shifting by 4.4.2, we get $|\mathcal{L}_X(g - g_m)| = o(r^{-2})$. By the argument in 4.4.3, $X^{1,0} = \lambda X_m^{1,0}$ where $Re(\lambda) \neq 0$. Hence we have $|Ric_g - Re(\lambda) Ric_{g_m}| = o(r^{-2})$. In 4.4.4 we saw that $Ric_{g_m} \geq \frac{c}{r^2} g_m$ for $n \geq 2$. Together with positivity of Ric_g, we get $Ric_g \geq \frac{\widetilde{c}}{r^2} g_m$.

By 4.4.1 , $|U_a| = O(r^{\frac{1}{2}})$, $|\nabla U_a| = O(1)$. These combined with assumption 1 give us $|h| = o(r^{-2})$. Therefore for sufficient large θ, $\theta(Ric_g) > h$. The following argument is quite similar to the analysis in Prop 4.4 in [27].

Consider $\theta_0 = inf\{\,\theta \,|\, \theta(Ric_g) > h\}$. And let $w = \theta_0 Ric_g - h$.

We'll see that $\theta_0 > 0$ leads to a contradiction. If $\theta_0 > 0$, by $Ric_g \geq \frac{\widetilde{c}}{r^2} g_m$, $|h| = o(r^{-2})$ and the positivity of Ric_g, $\exists\, p \in M, e_1 \in T_pM$ s.t. $w(e_1, e_1) = 0$. Parallel translating e_1 in a neighbourhood of p, then we have $(\triangle w)(e_1, e_1) > 0$, $(D_X w)(e_1, e_1) = 0$ at p.

Now the discussion goes to the complexified tangent bundle. Extend w by $\mathbb{C}-$linearity. The discussion separates into two parts. The first part is to show

$Tr\,w = 0$ at p. The second part is to show $Tr\,w$ satisfies $\triangle(Tr\,w) + D_X(Tr\,w) \leq 0$.

For the first part, let $\eta_1 = \frac{1}{2}(e_1 - iJe_1) \in T^{1,0}_{p,\mathbb{C}}$, then $w(\eta_1, \overline{\eta_1}) = 0$. Together with $w \geq 0$, this implies we can extend η_1 into unitary basis η_1 ... $\eta_n \in T^{1,0}_{q,\mathbb{C}}$ such that $w(\eta_i, \overline{\eta_j})$ is diagonal. Also we can parallel extend η_i like η_1.

Since $\triangle_L w + \mathcal{L}_X w = 0$, plug in η_i,$\overline{\eta_i} \in T^{1,0}_{p,\mathbb{C}}M$ (Extend J-invariant (0,2)-tensor w by $\mathbb{C}$-linearity.)

$$
\begin{aligned}
0 = (\triangle w)(\eta_i, \overline{\eta_i}) + 2\Sigma_k Rm(\eta_i, \overline{\eta_k}, \overline{\eta_i}, \eta_k) w(\eta_k, \overline{\eta_k}) - 2w(Ric(\eta_i), \overline{\eta_i}) \\
+(D_X w)(\eta_i, \overline{\eta_i}) + w(D_{\eta_i} X, \overline{\eta_i}) + w(\eta_i, D_{\overline{\eta_i}} X) \\
w(D_{\eta_i} X, \overline{\eta_i}) = \eta_i(\eta_{\bar{i}} f) w(\eta_i, \eta_{\bar{i}}) = \eta_{\bar{i}}(\eta_i f) w(\eta_i, \eta_{\bar{i}}) = w(\eta_i, D_{\overline{\eta_i}} X)
\end{aligned}
$$

From the soliton equation, we have $Ric(\eta_i) = D_{\eta_i} X$, therefore

$$
0 = (\triangle w)(\eta_i, \overline{\eta_i}) + 2\Sigma_k Rm(\eta_i, \overline{\eta_k}, \overline{\eta_i}, \eta_k) w(\eta_k, \overline{\eta_k}) + (D_X w)(\eta_i, \overline{\eta_i}). \tag{4.6}
$$

Now take $i = 1$,we see that $0 \geq 2\Sigma_k Rm(\eta_1, \overline{\eta_k}, \overline{\eta_1}, \eta_k) w(\eta_k, \overline{\eta_k})$. That g has positive holomorphic bisectional curvature implies $Rm(\eta_1, \overline{\eta_k}, \overline{\eta_1}, \eta_k) > 0$. Therefore $w(\eta_k, \overline{\eta_k}) = 0$ for any k at p. The nonnegative function $Tr\,w = 0$ at p.

Equation (4.6) only uses $\triangle_L w + \mathcal{L}_X w = 0$, the soliton equation, $w(\eta_i, \overline{\eta_j})$ is diagonal and the extension is parallel. Now sum (4.6) for all i at q give us $\triangle(Tr\,w) + D_X(Tr\,w) \leq 0$.

By Hopf's strong maximum principle, $Tr\,w = 0$. Therefore w is 0. This violates the asymptotic of Ric_g and h. Therefore $\theta_0 = 0$, $h \leq 0$. Now apply a similar argument to $-h$ implies $h = 0$. Therefore g is $U(n)-$symmetric. Therefore it must be in the family of steady solitons in [11].

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