ABSTRACT

Title of dissertation: Mathematical Problems Arising When Connecting Kinetic To Fluid Regimes

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In this dissertation we study two problems that are related to the question of how to obtain appropriate macroscopic descriptions of a gas from its microscopic formulation. Mathematically, fluid equations formulate the macroscopic dynamics of a gas while kinetic equations are used to study the microscopic world. One can derive fluid equations from kinetic equations through formal asymptotic expansions like those of Hilbert or Chapman-Enskog. The first problem we study relates to the justification of the steps in those formal expansions, while the second relates to the well-posedness of a resulting fluid system.

The first problem we study is that of establishing a Fredholm alternative for the linearized Boltzmann collision operator. The Fredholm alternative is used in both the formal asymptotic derivations and the rigorous justifications of fluid approximations to the Boltzmann equation. Results of this type have been obtained for collision kernels satisfying the Grad angular cutoff assumption. However, because DiPerna-Lions' renormalized solutions for the Boltzmann equation are established for more general collision kernels, it is interesting to extend the Fredholm property

of the linearized Boltzmann operator to these collision kernels. We show that under a weak cutoff assumption, the linearized Boltzamnn operator does satisfy the Fredholm alternative.

The second problem we study is the well-posedness of a dispersive fluid system that is formally obtained via an asymptotic expansion of the Boltzmann equation [21] as a first correction to the compressible Navier-Stokes system. This system is degenerate in both dissipation and dispersion. Therefore the theory for strictly dispersive systems does not apply directly. To prove the well-posedness of this degenerate system, we need to study different smoothing effects for different components of the solution. We show that using the regularization effects including dispersion and dissipation, this system has a unique smooth solution for a finite time.

Mathematical Problems Arising When Connecting Kinetic to Fluid Regimes

by

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To my parents, for their endless love.

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Chapter 1

Introduction

In this dissertation we study two problems that are related to the question of how to obtain appropriate macroscopic descriptions of a gas from its microscopic formulation. This has been a central question in kinetic theory since it was founded by Maxwell and Boltzmann [24, 3]. To have any hope of answering this question, we need to gain a good understanding of three things:

- the macroscopic dynamics of a gas as a fluid,
- the microscopic dynamics of a gas,
- the bridge between these two worlds.

Mathematically, fluid-type of equations formulate the macroscopic dynamics of a gas while kinetic equations are used to study the microscopic world. There are various ways to connect these two types of equations. In this dissertation, we focus on problems relevant to asymptotic expansions like Hilbert or Chapman-Enskog type of expansions and the fluid systems derived from these expansions.

In this section, we give an introduction to the two problems studied in this dissertation. The main results will be presented in chapters two and three.

1.1 Fluid Regime

In fluid regimes, we will use the mass density, bulk velocity and temperature, denoted as $(\rho,\, u,\, \theta)$ to describe the state of a gas. These fluid variables $(\rho,\, u,\, \theta)$ depend on the space variable $x \in R^D$ and time $t \geq 0$.

1.1.1 Fluid Systems

If we consider ideal polytropic gases composed of identical monatomic molecules, then according to the conservation laws of mass, momentum, and energy, a fluid system takes the general form

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho\, u \otimes u) + \nabla_x (\rho \theta) &= \nabla_x \cdot S, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho \theta u) &= \nabla_x \cdot (S u) + \nabla_x \cdot q\,,
\end{aligned}
\tag{1.1}
$$

Here $\rho e = \frac{1}{2}\rho\,|u|^2 + \frac{d}{2}\rho\,\theta$ is the total energy density with d being the dimension of the microscopic freedom of the gas molecules; usually $d = 3$. If there is symmetry in the macroscopic motion of the gas, then $D < d$, otherwise $D = d$. Here S and q are the negatives of the stress tensor and heat flux. They are determined by constitutive relations. If $S = 0$, $q = 0$, then (1.1) becomes the Euler system. If we take into account of viscosity and thermal conductivity, then for Newtonian fluids (1.1) becomes the compressible Navier-Stokes system with

$$
S = \mu\, \big(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{d}(\nabla_x \cdot u) I\big) + \lambda\, (\nabla_x \cdot u) I\,, \qquad q = \kappa\, \nabla_x \theta\,, \tag{1.2}
$$

where the scalar quantities $\mu > 0$, $\lambda \geq 0$ are the shear and bulk viscosity coefficients, and $\kappa > 0$ is the thermal conductivity coefficient. These coefficients generally depend on ρ and θ. More complicated systems that include additional terms can be derived from kinetic theory. We will study the well-posedness of one such system in this dissertation.

1.1.2 Entropy

The notion of entropy is an important thermodynamical quantity for a gas. By thermodynamics [7], the specific entropy $\sigma = \sigma(\rho, \theta)$ satisfies the differential relation

$$\mathrm{d}\sigma = \frac{d}{2}\frac{\mathrm{d}\theta}{\theta} - \frac{\mathrm{d}\rho}{\rho}\,, \tag{1.3}$$

that is, $\partial_\theta \sigma = \frac{d}{2\theta}$, $\partial_\rho \sigma = \frac{1}{\rho}$. Thus the physical entropy density for the system (1.1) is given by

$$\rho\sigma = \rho \log\left(\frac{\theta^{d/2}}{\rho}\right) .$$

The second law of thermodynamics states that, in a closed system, the total entropy for a gas in a nonequilibrium state will increase with time until attaining its maximum value when the system reaches equilibrium. The mathematical entropy density η is defined as

$$\eta = -\,\rho \log\left(\frac{(2\pi\,\theta)^{d/2}}{\rho}\right) = \rho \log\left(\frac{\rho}{(2\pi\,\theta)^{d/2}}\right) , \tag{1.4}$$

which is the negative of the physical entropy.

To find the equation for the entropy density η, write (1.1) in terms of the fluid variables $(\rho,\, u,\, \theta)$ in the convective form

$$\begin{aligned}
&\partial_t\rho + u\cdot\nabla_x\rho + \rho\,\nabla_x\cdot u = 0,\\
&\rho\,\left(\partial_t u + u\cdot\nabla_x u\right) + \nabla_x(\rho\,\theta) = \nabla_x\cdot S,\\
&\tfrac{d}{2}\rho\,\left(\partial_t\theta + u\cdot\nabla_x\theta\right) + \rho\,\theta\,\nabla_x\cdot u = S : \nabla_x u + \nabla_x\cdot q.
\end{aligned} \tag{1.5}$$

Then by the differential relation (1.3),

$$\begin{aligned}
\rho\left(\partial_t\sigma + u\cdot\nabla_x\sigma\right) &= \tfrac{d}{2}\tfrac{\rho}{\theta}\left(\partial_t\theta + u\cdot\nabla_x\theta\right) - \left(\partial_t\rho + u\cdot\nabla_x\rho\right),\\
&= -\frac{S}{\theta} : \nabla_x u - \frac{1}{\theta}\nabla_x q.
\end{aligned}$$

This can be put into the divergence form

$$\partial_t\eta + \nabla_x\cdot\left(\eta\,u + \frac{q}{\theta}\right) = -\frac{S}{\theta} : \nabla_x u - \frac{q}{\theta^2}\cdot\nabla_x\theta. \tag{1.6}$$

The local version of the second law of thermodynamics implies the right-hand side of (1.6) must be a divergence plus a nonpositive term. This law is respected by both the compressible Euler and the Navier-Stokes systems. For the compressible Euler system, the right-hand side of (1.6) is zero and the entropy is formally conserved. For the compressible Navier-Stokes system, the right-hand side of (1.6) is computed as

$$-\frac{S}{\theta} : \nabla_x u - \frac{q}{\theta^2}\cdot\nabla_x\theta = -\left(\mu\left|\nabla_x u + (\nabla_x)^T - \tfrac{2}{d}\nabla_x\cdot u\right|^2 + \lambda|\nabla_x\cdot u|^2 + \kappa|\nabla_x\theta|^2\right).$$

Therefore, by the fact that $\mu > 0$, $\lambda \geq 0$, $\kappa > 0$, the right-hand side of (1.6) is nonpositive. Hence, the mathematical entropy is formally dissipated. The fluid system we study later also respect the second law of thermodynamics.

1.2 Kinetic Regime

In kinetic regimes, the phase space of a single particle of a gas is given by its position $x \in R^d$ and velocity $v \in R^d$ at each time $t \geq 0$, and the phase space density function $F(t, x, v)$ is used to describe the gas. The macroscopic mass, momentum, and total energy density functions (ρ, ρu, ρe) can be recovered from F by the following relations:

$$\rho = \int_{R^d} F\, dv, \quad \rho\, u = \int_{R^d} v\, F\, dv, \quad \rho\, e = \int_{R^d} \tfrac{1}{2}|v|^2\, F\, dv. \tag{1.7}$$

1.2.1 General Kinetic Equations

If the gas considered is composed of identical, monatomic particles and is dilute in the sense that the total volume of the gas molecules are negligible compared with the macroscopic volume, then the phase space density function $F(t, x, v)$ is governed by the kinetic equation:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}\,(F), \tag{1.8}$$

where $\mathcal{C}\,(F)$ is the collision term that specifies the type of collisions for the gas molecules. In most cases, this collision term is nonlinear. For example, the classical Boltzmann equation has a quadratic collision term. Because the first problem in this dissertation focuses on the Boltzmann equation, we give a more detailed description of this equation.

1.2.2 Boltzmann Equation

The derivation of the Boltzmann equation is based on the following assumptions due to the rarefaction of the gas:

- there are only binary collisions, that is, multiple collisions are ignored;
- the states of two molecules are independent of each other before they collide.

Under these assumptions, the collision term in the Boltzmann equation is quadratic and we denote it as $\mathcal{B}(F, F)$. The equation has the form

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \tag{1.9}$$

where $\mathcal{B}(F, F)$ is given by

$$\mathcal{B}(F, F) = \iint_{S^{d-1} \times R^d} (F_1' F' - F_1 F) \; b(v_1 - v, \omega)\, d\omega dv_1 \,. \tag{1.10}$$

Notice that the collision term operates only on the velocity variable. Here F_1', F', F_1, and F denotes $F(t, x, \cdot)$ evaluated at the velocities v_1', v', v_1, and v respectively with (v, v_1) and (v', v_1') being two velocity pairs before and after the collision or vice versa. Because we only consider elastic collisions, (v, v_1) and (v', v_1') must conserve both momentum and energy:

$$\begin{aligned} v + v_1 &= v' + v_1', \\ |v|^2 + |v_1|^2 &= |v'|^2 + |v_1'|^2. \end{aligned} \tag{1.11}$$

The unit vector ω is perpendicular to the reflection plane with $d\omega$ being the rotationally invariant unit measure for S^{d-1}. The general solution of (1.11) for (v', v_1') in terms of (v, v_1, ω) is written as

$$v' = v + \omega\,\omega \cdot (v_1 - v), \qquad v_1' = v_1 - \omega\,\omega \cdot (v_1 - v). \tag{1.12}$$

1.2.3 Collision Kernels

The term $b\,(v_1 - v, \omega)$ in (1.10) is called the collision kernel. It determines specific types of interactions among molecules. For example, the collision kernel for the hard sphere [8] model satisfies

$$b(v - v_1, \omega) = c|(v - v_1) \cdot \omega|, \tag{1.13}$$

where $c > 0$ is a constant.

We also consider the case in which the intermolecular potential $V(r)$ is an inverse power law with r being the distance between two molecules. That is, the case in which $V(r)$ is proportional to r^{-k} for some $k > 0$. For this kind of potential,

b has the following form:

$$b(v_1 - v, \omega) = |v_1 - v|^\beta \, \hat{b}\,(\omega \cdot n), \quad n = \frac{v - v_1}{|v - v_1|},$$
$$\beta = 1 - \frac{2(d-1)}{k} < 1. \tag{1.14}$$

Notice that β can be negative which makes b singular when $v = v_1$. We assume that $\beta > -d$ so that $|v - v_1|^\beta$ is locally integrable at the singularity. This assumption is equivalent to $k > 2\frac{d-1}{d+1}$. For $d = 3$, the condition on k becomes $k > 1$, whereby the Coulomb potential is the marginal case.

Notice that the hard sphere case (1.13) also has the form as in (1.14). In genenral β satisfies the bounds

$$-d < \beta \leq 1. \tag{1.15}$$

The range $-d < \beta < 0$ is called the soft potential case, the range $0 < \beta \leq 1$ the hard potential, and $\beta = 0$ the Maxwell molecules where there is no $v - v_1$ dependence for b. The soft potential case is in general harder to deal with than the hard potential due to the singularity.

Another singularity of b occurs when $\omega \cdot n = 0$ since $\hat{b}\,(\omega \cdot n) \sim (\omega \cdot n)^{-(k+1)/(k-1)}$. Notice that this singularity is never integrable. It arises due to the many grazing collisions that occur when two molecules pass far from each other. To avoid this singularity, Grad [16] argued that these collisions can be neglected. He introduced a cutoff assumption that $|\hat{b}(\omega \cdot n)| \leq c|\omega \cdot n|$ near the singular point. This assumption allows him to apply the techniques Hilbert used for the hard sphere case (1.13).

Recently more general types of cutoffs have been introduced. For example, the global existence theory of DiPerna-Lions' renormalized solution to the Boltzmann equation was established for $\hat{b}$ satisfying the weak cutoff assumption:

$$\hat{b}\,(\omega \cdot n) \in L^1(d\omega)\,. \tag{1.16}$$

Many works are based on this global existence result [15, 23]. Therefore, it is interesting to investigate kernels that satisfy this weak cutoff collision kernel.

Under the assumptions (1.14), (1.15) and (1.16) on b, we can separate $\mathcal{B}\,(F, F)$ into a gain and loss part that can be treated individually. Write

$$\mathcal{B}(F, F) = \mathcal{B}^{+}(F, F) - \mathcal{B}^{-}(F, F)\,,$$

where

$$\begin{aligned} \mathcal{B}^{+}(F, F) &= \iint_{S^{d-1} \times R^d} F_1' F' \; b\, \mathrm{d}\omega dv_1, \\ \mathcal{B}^{-}(F, F) &= \iint_{S^{d-1} \times R^d} F_1 F \; b\, \mathrm{d}\omega dv_1, \end{aligned} \tag{1.17}$$

are the gain and loss parts respectively. The gain part denotes the number of molecules turned into velocity v after collisions while the loss part denotes the loss of molecules of velocity v because of collisions. Notice that if those two terms are to be separated, then the weak cutoff assumption (1.16) is a necessary condition for the integral in the loss term to exist.

1.2.4 Conservation Laws

Due to the relations between the velocity pairs (1.11) and the structure of the collision kernel (1.15), the operator $\mathcal{B}(F,F)$ satisfies the conservation properties [8]:

$$\int_{R^d} \mathcal{B}(F,F)\, dv = 0, \quad \int_{R^d} v\, \mathcal{B}(F,F)\, dv = 0, \quad \int_{R^d} |v|^2 \mathcal{B}(F,F)\, dv = 0. \tag{1.18}$$

Therefore, by (1.7), the conservations of macroscopic mass, momentum, and total energy can be formally derived from the Boltzmann equation (1.9). To make notation short, for any integrable $\xi(v)$, let

$$\langle \xi \rangle = \int_{R^D} \xi\, dv.$$

The conservation laws in the local form are

$$\begin{aligned} \partial_t \langle F \rangle + \nabla_x \cdot \langle vF \rangle &= 0, \\ \partial_t \langle vF \rangle + \nabla_x \cdot \langle v \otimes v\, F \rangle &= 0, \\ \partial_t \left\langle \tfrac{1}{2}|v|^2 F \right\rangle + \nabla_x \cdot \left\langle \tfrac{1}{2}|v|^2 vF \right\rangle &= 0. \end{aligned} \tag{1.19}$$

1.2.5 Entropy

The Boltzmann equation has an analogy of the entropy. It derives from symmetries associated with the measure denoted as

$$\mathrm{d}\tilde{\mu} = b\,(v - v_1,\, \omega \cdot n)\, \mathrm{d}\omega\, \mathrm{d}v_1\, \mathrm{d}v, \qquad \langle\!\langle \cdot \rangle\!\rangle = \int \mathrm{d}\tilde{\mu}\,.$$

By the symmetry of b and relations between the velocity pairs (1.11), $d\tilde{\mu}$ is invariant under the changes:

$$(v, v_1) \leftrightarrow (v', v_1'), \qquad (v, v') \leftrightarrow (v_1, v_1').$$

By the symmetry of $d\tilde{\mu}$, Boltzmann observed the following key equality [8] for his fundamental H-theorem :

$$\langle \log F \, \mathcal{B}(F, F) \rangle = \frac{1}{4} \left\langle\!\!\left\langle \log \left(\frac{F_1' \, F'}{F_1 \, F} \right) (F_1 \, F - F_1' \, F') \right\rangle\!\!\right\rangle. \tag{1.20}$$

Notice that for any F, the right-hand side of the above equality is nonpositive. When F is a classical solution of the Boltzmann equation (1.9), one can multiply (1.9) by $\log F$ and obtain the following dissipation law:

$$\begin{aligned} &\partial_t \langle F \log F - F \rangle + \nabla_x \cdot \langle v \, (F \log F - F) \rangle \\ &\qquad = \frac{1}{4} \left\langle\!\!\left\langle \log \left(\frac{F_1' \, F'}{F_1 \, F} \right) \left(F_1 \, F - F_1' \, F' \right) \right\rangle\!\!\right\rangle \leq 0, \end{aligned} \tag{1.21}$$

where $\langle F \log F - F \rangle$ is defined as the entropy density. The above dissipation law of entropy shows the irreversibility of the Boltzmann equation. It is consistent with the second law of thermodynamics.

1.2.6 Equilibrium States

One can see from (1.21) that the equality is true only when

$$F(v')F(v_1') - F(v)F(v_1) = 0, \quad \text{for almost every } (v, v_1) \in R^d \times R^d, \tag{1.22}$$

with v', v_1' satisfying (1.12). Notice that $\mathcal{B}(F, F)$ vanishes for $F(v)$ satisfying (1.22), that is, F is an equilibrium state of $\mathcal{B}(F, F)$. This observation provides a characterization of equilibrium states of the Boltzmann equation through the entropy dissipation. It can be shown that for such an F, we have

$$\log F(v) \in \text{span}\{1,\, v_1,\, v_2,\, \cdots v_d,\, |v|^2\}.$$

Therefore, for any F such that the integrals make sense, the following statements are equivalent:

- $\mathcal{B}(F, F) = 0\,,$
- $\langle \log(F)\, \mathcal{B}(F, F) \rangle = 0\,,$
- $\log F(v) \in \text{span}\{1,\, v_1,\, v_2,\, \cdots v_d,\, |v|^2\}.$

Together with the entropy dissipation law (1.21) it is called the Boltzmann H-theorem. This is the most fundamental property of the Boltzmann equation.

Use $\mathcal{M}$ to denote these equilibrium states and rewrite them as

$$\mathcal{M} = \frac{\rho}{(2\pi\,\theta)^{d/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right), \tag{1.23}$$

with $\rho, \theta > 0$. Note that the operations so far are only on the velocity v. Therefore ρ, u, θ and $\mathcal{M}$ can also depend on (t, x), that is, $(\rho, u, \theta) = (\rho, u, \theta)(t, x)$ and $\mathcal{M} = \mathcal{M}(t, x, v)$. These $\mathcal{M}$'s are called the local Maxwellians. By the definition of $\mathcal{M}$, it can be verified that

$$\langle \mathcal{M} \rangle = \rho, \quad \langle v\mathcal{M} \rangle = \rho\, u, \quad \left\langle \tfrac{1}{2}|v|^2\, \mathcal{M} \right\rangle = \tfrac{1}{2}\rho\, |u|^2 + \tfrac{d}{2}\rho\, \theta\,, \tag{1.24}$$

and the Euler entropy density is given by

$$\langle \mathcal{M} \log \mathcal{M} - \mathcal{M} \rangle = \rho\, \log \left(\frac{\rho}{(2\pi\theta)^{d/2}} \right) - \tfrac{d+2}{2}\rho\,,$$

which is essentially the same as the fluid entropy density (1.4) since they differ only by the term $-\left(\frac{d}{2}\log(2\pi) + \frac{d+2}{2}\right)\rho$. To emphasize the dependence of $\mathcal{M}$ on (ρ, u, θ), we also write it as $\mathcal{M}_{\rho,u,\theta}$.

1.3 Asymptotic Expansions

In this section we are going to connect the kinetic and fluid regimes via the method of asymptotic expansions and give a statement of the first problem studied in this dissertation. We use the Chapman-Enskog expansion as an illustration. There are other kinds of expansions such as the Hilbert expansion [8] and the balance argument used by Maxwell [24] and Boltzmann [4]. We restrict ourselves to the Boltzmann equation.

1.3.1 Knudsen Number

By the dimension analysis (see, for example, [1]), the resulting dimensionless Boltzmann equation has the form :

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon}\mathcal{B}(F, F), \tag{1.25}$$

The parameter ϵ is called the Knudsen number. If we define the mean free path as the scale of distances that molecules travel between collisions when the gas is in its equilibrium state, then the Knudsen number is the ratio of the mean free path with the macroscopic length in consideration. It provides a measurement of how close a gas is to its equilibrium state. Fluid systems give good approximations to the kinetic equation when the Knudsen number becomes small enough.

1.3.2 Chapman-Enskog Expansion

Denote

$$\begin{aligned}
&\mathbf{e} = \left(1, v, \tfrac{1}{2}|v|^2\right)^T, \\
&\boldsymbol{\rho} = \langle \mathbf{e}F \rangle = \left(\rho,\ \rho u,\ \tfrac{1}{2}\rho|u|^2 + \tfrac{d}{2}\rho\theta\right)^T, \\
&\mathcal{E}[\boldsymbol{\rho}] = \mathcal{M}_{\rho,u,\theta}\,.
\end{aligned} \tag{1.26}$$

The formal conservation law of the Boltzmann equation (1.19) is now written as

$$\partial_t \boldsymbol{\rho} + \nabla_x \cdot \langle v\mathbf{e}F \rangle = 0. \tag{1.27}$$

Suppose the space-time dependence of F is governed by $\boldsymbol{\rho}$ through an operator

$\mathfrak{F}$:

$$F(t,x,v) = \mathfrak{F}[\boldsymbol{\rho}(t,x)](v), \quad \text{such that} \quad \boldsymbol{\rho} = \langle \mathbf{e}\, \mathfrak{F}[\boldsymbol{\rho}] \rangle. \tag{1.28}$$

The idea of the Chapman-Enskog expansion is to approximately solve an equation for $\mathfrak{F}$ in terms of $\boldsymbol{\rho}$. The first step is to express $\partial_t F$ in terms of the spatial derivatives of F using the conservation law (1.27):

$$\partial_t F = D_{\boldsymbol{\rho}} \mathfrak{F}[\boldsymbol{\rho}] \partial_t \boldsymbol{\rho} = -D_{\boldsymbol{\rho}} \mathfrak{F}[\boldsymbol{\rho}] \langle \mathbf{e} v \cdot \nabla_x \mathfrak{F}[\boldsymbol{\rho}] \rangle, \tag{1.29}$$

where $D_{\boldsymbol{\rho}} \mathfrak{F}[\boldsymbol{\rho}]$ is the functional derivative of $\mathfrak{F}[\boldsymbol{\rho}]$ defined formally as

$$D_{\boldsymbol{\rho}} \mathfrak{F}[\boldsymbol{\rho}]\, \mathbf{f} = \lim_{\delta \to 0} \frac{\mathfrak{F}[\boldsymbol{\rho} + \delta \mathbf{f}] - \mathfrak{F}[\boldsymbol{\rho}]}{\delta}.$$

Eliminate $\partial_t F$ in the Boltzmann equation using (1.29). We obtain

$$\Big(I - \mathcal{P}_{\mathfrak{F}}[\boldsymbol{\rho}] \Big) v \cdot \nabla_x \mathfrak{F}[\boldsymbol{\rho}] = \frac{1}{\epsilon} \mathcal{B}(\mathfrak{F}[\boldsymbol{\rho}], \mathfrak{F}[\boldsymbol{\rho}]),$$

where

$$\mathcal{P}_{\mathfrak{F}} f = D_{\boldsymbol{\rho}} \mathfrak{F}[\boldsymbol{\rho}]\, \langle \mathbf{e} f \rangle.$$

Suppress the variable $\boldsymbol{\rho}$ in the above equation and write it as an equation for operators as follows:

$$\Big(I - \mathcal{P}_{\mathfrak{F}} \Big) v \cdot \nabla_x \mathfrak{F} = \frac{1}{\epsilon} \mathcal{B}(\mathfrak{F}, \mathfrak{F}). \tag{1.30}$$

By (1.28), it is clear that $I = \langle \mathbf{e} \otimes D_{\boldsymbol{\rho}} \mathfrak{F}[\boldsymbol{\rho}] \rangle$. Therefore $\mathcal{P}_{\mathfrak{F}}^2 = \mathcal{P}_{\mathfrak{F}}$, that is, $\mathcal{P}_{\mathfrak{F}}$

is a projection operator. Define its complement as $\tilde{\mathcal{P}}_{\mathfrak{F}} = I - \mathcal{P}_{\mathfrak{F}}$, which is also a projection. Then we have

$$\text{Range}(\mathcal{P}_{\mathfrak{F}}) = \text{Null}(\tilde{\mathcal{P}}_{\mathfrak{F}}) = \text{span}\{1, v_1, v_2, \cdots, v_d, |v|^2\}, \tag{1.31}$$

Thus, (1.30) is rewritten as:

$$\tilde{\mathcal{P}}_{\mathfrak{F}} v \cdot \nabla_x \mathfrak{F} = \frac{1}{\epsilon} \mathcal{B}(\mathfrak{F}, \mathfrak{F}). \tag{1.32}$$

Expand $\mathfrak{F}$ formally as

$$\mathfrak{F} = \mathfrak{F}_0 + \epsilon \mathfrak{F}_1 + \epsilon^2 \mathfrak{F}_2 \cdots, \tag{1.33}$$

and use this expansion in (1.32).

For order ϵ^{-1}, we have $\mathcal{B}(\mathfrak{F}_0, \mathfrak{F}_0) = 0$. By Boltzmann's H-theorem, $\mathfrak{F}_0[\boldsymbol{\rho}] = \mathcal{E}[\boldsymbol{\rho}]$ for any $\boldsymbol{\rho}$.

For order ϵ^0, we obtain

$$\tilde{\mathcal{P}}_{\mathcal{E}}\, v \cdot \nabla_x \mathcal{M} = -\mathcal{M} \mathcal{L}_{\mathcal{M}} \left(\frac{\mathfrak{F}_1[\boldsymbol{\rho}]}{\mathcal{M}} \right), \tag{1.34}$$

where $\mathcal{M}_{\rho,u,\theta} = \mathcal{E}[\boldsymbol{\rho}]$ with (ρ, u, θ) relating to $\boldsymbol{\rho}$ by (1.26), and

$$\mathcal{L}_{\mathcal{M}} f = \iint_{S^{d-1} \times R^d} (f(v) + f(v_1) - f(v') - f(v_1'))\; b(v - v_1, \omega \cdot n)\, \mathcal{M}(v_1) dv_1, \tag{1.35}$$

for any f in the domain of the operators $\mathcal{L}_{\mathcal{M}}$.

The operator $\mathcal{L}_{\mathcal{M}}$ is the linearized Boltzmann operator around the local Maxwellian $\mathcal{M}_{\rho,u,\theta}$. Following are classical fact about $\mathcal{L}_{\mathcal{M}}$ by a symmetry argument[8]:

- $\mathcal{L}_{\mathcal{M}}$ is self-adjoint and nonnegative over $L^2(\mathcal{M}dv)$;
- Null space of $\mathcal{L}_{\mathcal{M}} = \text{span}\{1, v_1, v_2, \cdots, v_d, |v|^2\}$.

In order to solve the linear equation (1.34), we want $\mathcal{M}\mathcal{L}_{\mathcal{M}}$, and thus $\mathcal{L}_{\mathcal{M}}$ to satisfy the Fredholm alternative property in an appropriate space. Provided this is true, by (1.31), we can solve (1.34) and obtain

$$\mathfrak{F}_1[\boldsymbol{\rho}] = -(\mathcal{M}\mathcal{L}_{\mathcal{M}})^{-1}\tilde{\mathcal{P}}_{\mathcal{E}}\left(v \cdot \nabla_x \mathcal{M}\right), \tag{1.36}$$

where $(\mathcal{M}\mathcal{L}_{\mathcal{M}})^{-1} : \text{Null}(\mathcal{L}_{\mathcal{M}})^{\perp} \to \text{Null}(\mathcal{L}_{\mathcal{M}})^{\perp}$ is the pseudo-inverse of $\mathcal{M}\mathcal{L}_{\mathcal{M}}$. Then the compressible Navier-Stokes system is recovered by using $\mathfrak{F} = \mathcal{E} + \epsilon\mathfrak{F}_1$ in (1.27).

For ϵ^1, we have

$$\begin{aligned} -\mathcal{M}\mathcal{L}_{\mathcal{M}}(\mathfrak{F}_2[\boldsymbol{\rho}]) &= \tilde{\mathcal{P}}_{\mathcal{E}}\, v \cdot \nabla_x \mathfrak{F}_1[\boldsymbol{\rho}] - \mathcal{B}(\mathfrak{F}_1[\boldsymbol{\rho}], \mathfrak{F}_1[\boldsymbol{\rho}]) - D_{\boldsymbol{\rho}}\mathfrak{F}_1[\boldsymbol{\rho}]\left\langle \mathbf{e}v \cdot \nabla_x \mathcal{E}[\boldsymbol{\rho}]\right\rangle \\ &\triangleq RHS. \end{aligned} \tag{1.37}$$

By the conservation properties of $\mathcal{B}$ (1.18) and the expression of $\mathfrak{F}_1$ (1.36), it is clear that each term on the right hand side of the above equation is in the orthogonal space of $\text{Null}\left(\mathcal{L}_{\mathcal{M}}\right)$. Again, if $\mathcal{L}_{\mathcal{M}}$ satisfies the Fredholm alternative, we can solve for $\mathfrak{F}_2[\boldsymbol{\rho}]$ as

$$\mathfrak{F}_2[\boldsymbol{\rho}] = -(\mathcal{M}\mathcal{L}_{\mathcal{M}})^{-1}(RHS).$$

For $\epsilon^k, k \geq 2$, we always have the equation as:

$$-\mathcal{M}\mathcal{L}_{\mathcal{M}}\Big(\mathfrak{F}_{k+1}[\boldsymbol{\rho}]\Big) = \mathcal{H}_k\Big(\mathfrak{F}_j[\boldsymbol{\rho}]\Big)_{j\leq k},$$

with $\mathcal{H}_k\Big(\mathfrak{F}_j[\boldsymbol{\rho}]\Big)_{j\leq k} \in \mathrm{Null}(\mathcal{L}_{\mathcal{M}})^{\perp}$. Therefore, the Fredholm alternative property of $\mathcal{L}_{\mathcal{M}}$ always guarantees the solvability of the approximated operator equations. It provides a sufficient condition for formally deriving those fluid systems. These are the motivations for the first problem studied in this dissertation that we show $\mathcal{L}_{\mathcal{M}}$ does satisfy the Fredholm alternative as desired.

1.3.3 Fredholm Alternative

There are various results about the Fredholm alternative property of $\mathcal{L}_{\mathcal{M}}$. The differences between them are the assumptions on the collision kernel $b(v - v_1, \omega \cdot n)$. The first result of this kind was given by Hilbert [19] for the hard sphere case as an example to apply his integral theory. After Grad's cutoff assumption was introduced, more general collision kernels with this assumption have been considered. For example, for the 3D case, Grad [17] showed that $\mathcal{L}_{\mathcal{M}}$ has a Fredholm property over $L^2(\mathcal{M}dv)$ for the hard potential case and Caflisch [5] generalized this result to the soft potential case when $-1 < \beta < 1$. Later on, Golse and Poupaud [14] proved it for $-2 \leq \beta < 1$ on a L^2 space with a different weight, and Guo [18] extended Caflisch's result to the full range of the potential where $-3 < \beta < 1$.

Our result extends the previous ones by assuming the weak cutoff condition

(1.16) for b which includes the Grad cutoff case. If we define $a(v)$ as

$$a(v) = \iint_{S^{d-1}\times R^d} b(v - v_1, \omega)\, d\omega\, \mathcal{M}(v_1) dv_1,$$

then the main theorem is roughly stated as

Statement 1. *Assume that the collision kernel $b(v - v_1, \omega)$ satisfies the cutoff assumptions (1.16) and (1.15). Then $\frac{1}{a(v)}\mathcal{L}_{\mathcal{M}}$ is a Fredholm operator on $L^2(a\mathcal{M}dv)$, that is, there exists a compact operator $\mathcal{K}$ on $L^2(a\mathcal{M}dv)$ such that $\frac{1}{a(v)}\mathcal{L}_{\mathcal{M}} = \mathcal{I} - \mathcal{K}$.*

Because a Fredholm operator satisfies the Fredholm alternative, we conclude that $\mathcal{L}_{\mathcal{M}}$ satisfies the Fredholm alternative on the space $L^2(a\mathcal{M}dv)$. The exact theorem is stated in section 2.1.

It will be shown in section 2.1 that $\mathcal{K}$ is a bounded operator on $L^p(a\mathcal{M}dv)$ for any $1 \leq p < \infty$. By interpolation, $\mathcal{K}$ is compact on $L^p(a\mathcal{M}dv)$ for any $1 < p < \infty$. Therefore, we conclude that $\frac{1}{a(v)}\mathcal{L}_{\mathcal{M}}$ satisfies the Fredholm alternative on $L^p(a\mathcal{M}dv)$ for any $1 < p < \infty$.

Once this property is verified for $\mathcal{L}_{\mathcal{M}}$, each step of the Chapman-Enskog expansion can be carried out and fluid systems are formally derived at each order systematically. The Fredholm alternative of $\mathcal{L}_{\mathcal{M}}$ is also used in rigorous justifications of the Navier-Stokes approximation [6].

1.4 Beyond Navier-Stokes

When the Chapman-Enskog expansion is carried out to derive fluid systems beyond Navier-Stokes, we recover the so called Burnett and super-Burnett equations. These equations are known to be linearly ill-posed [2].

To overcome this ill-posedness problem, people introduce various ways to modify the truncations of the Chapman-Enskog expansion. In [21], by respecting the entropy structure of the Boltzmann equation, Levermore proposed a systematic way to construct fluid dynamical systems as corrections to the compressible Navier-Stokes. The formal well-posedness of these fluid systems is given by the entropy dissipation. Among these well-posed systems, the most important one beyond Navier-Stokes is the first correction system. Because the correction is dispersive in nature, we call it the dispersive Naiver-Stokes system, abbreviated as the DNS system.

In order to justify this approximation, we need the well-posedness of this DNS system. The second part of this dissertation is to prove the local well-posedness of this system. To see the structure of the DNS system, here we present a model system that has simpler dispersive corrections to the compressible Navier-Stokes. In spite of the simplification, this model system has all the major structures of the original DNS system. Therefore, we still call it a DNS system.

1.4.1 DNS system (Simplified)

The dispersive Navier-Stokes system has the form

$$\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x (\rho\theta) &= \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{P}, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) + \nabla_x \cdot q &= \nabla_x \cdot (\Sigma u + \tilde{P} u) + \nabla_x \cdot \tilde{q}, \\
(\rho, u, \theta)(x, 0) &= (\rho^{in}, u^{in}, \theta^{in}),
\end{aligned} \tag{1.38}$$

where ρ, u, θ are the density, velocity and temperature of the gas respectively, and the constitutive relations are given as:

- $\rho e = \frac{1}{2}\rho|u|^2 + \frac{d}{2}\rho\theta$ denotes the total energy with $d \geq 2$ being the dimension of the microscopic world.
- $\Sigma = \mu(\theta)\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{d}(\nabla_x \cdot u) I\right)$ with $\mu(\theta) \geq \mu_0 > 0$ being the viscosity.
- $q = \kappa(\theta)\nabla_x \theta$ with $\kappa(\theta) \geq \kappa_0 > 0$ being the thermal conductivity.

In the simplified model, $\tilde{P}$ and $\tilde{q}$ have the forms:

$$\tilde{P} = \theta\left(\nabla_x^2 \theta - \tfrac{1}{d}(\Delta_x \theta) I\right), \quad \tilde{q} = \frac{\theta^2}{2} \nabla_x \cdot \left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{d}(\nabla_x \cdot u) I\right). \tag{1.39}$$

Dispersive effect is introduced by the tensor $\tilde{P}$ and the vector $\tilde{q}$.

1.4.2 Entropy Structure

Observe that $\tilde{P}, \tilde{q}$ satisfy the following relation:

$$\tilde{P} : \frac{\nabla_x u}{\theta} + \tilde{q} \cdot \frac{\nabla_x \theta}{\theta^2} = \nabla_x \cdot \left(\nabla_x \theta \cdot \left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{d} (\nabla_x \cdot u) I \right) \right), \tag{1.40}$$

that is, $\tilde{P} : \frac{\nabla_x \theta}{\theta} + \tilde{q} \cdot \frac{\nabla_x \theta}{\theta^2}$ is a divergence. Accordingly, by the entropy equation (1.6), it is clear that dispersion only contributes to the flux of the entropy. Therefore the entire system formally dissipates the entropy in the same way as the compressible Navier-Stokes. This effect guarantees that the formal well-posedness of the dispersive system (1.38).

1.4.3 Analytic Structure

The DNS system features degeneracies in both dissipation and dispersion. If the system is written in terms of the fluid variables (ρ, u, θ), it is obvious that for the density component of the solution, there is neither dissipation nor dispersion. By the assumptions for the viscosity and heat conductivity, the velocity and temperature equations are strictly dissipative. However, as for the dispersion, we notice another degenerate component other than the density. To see this, calculate $\nabla_x \cdot \tilde{q}$ in the energy equation.

$$\nabla_x \cdot \tilde{q} = \tfrac{d-1}{d}\, \theta^2 \; \Delta_x \nabla_x \cdot u + \theta \nabla_x \cdot \left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{d} (\nabla_x \cdot u) I \right) \cdot \nabla_x \theta, \tag{1.41}$$

where the second term is a lower order term. It is now clear from (3.4) that if we use Hodge decomposition to decompose the velocity field u into the divergence free part and the gradient part, then there is no dispersive regularization for the divergence free part.

To summarize, we have neither dissipative nor dispersive effect for the density function, there is only strict dissipation for the divergence free part of the velocity field, and there are both strict dissipation and strict dispersion for the gradient part of the velocity field and the temperature.

Due to this degeneracy, a well-posedness result for the DNS system is intrinsically interesting. The dispersive systems that have been treated so far are limited to those having strictly and uniformly dispersive effects. Each component of the solution have the same amount of regularization and dispersion alone gives the well-posedness of these systems. In the DNS system, however, to treat the various degeneracies, we need to decouple components with different smoothing effects using tools of pseudodifferential operators. We also need to combine the dispersive regularization with dissipative effect and hyperbolicity to close the energy estimate for the whole system. Using these ideas, we can prove the well-posedness of this system. The main theorem is as follows:

Statement 2. *Let $\langle x\rangle^2 = 1+x^2$. There exists $N = N(d)$ such that given any initial* *data* $(\rho^{in}, u^{in}, \theta^{in})$ *satisfying the non-trapping condition* $\mathcal{A}4$ *and*

$$\rho^{in} - \bar{\rho} \in H^{s+1}(R^d), \quad (u^{in}, \theta^{in} - \bar{\theta}) \in H^s(R^d) \times H^s(R^d),$$
$$\langle x\rangle^2 \partial_x^\beta \rho^{in} \in L^2(R^d), \quad \Big(\langle x\rangle^2 \partial_x^\alpha u^{in}, \langle x\rangle^2 \partial_x^\alpha \theta\Big) \in L^2(R^d) \times L^2(R^d),$$
$$1 \le |\beta| \le s_1 + 1, \quad 1 \le |\alpha| \le s_1$$

where $s_1 \geq \frac{d}{2} + 6$*,* $s \geq \max\{s_1 + 6, N + d/2 + 4\}$*, there exists* $T_0 > 0$ *such that the dispersive system (1.38) has a unique solution* (ρ, u, θ) *with*

$$\rho - \bar{\rho} \in C([0, T_0]; H^s) \cap L^\infty([0, T_0]; H^{s+1}),$$
$$(u, \theta - \bar{\theta}) \in C([0, T_0]; H^{s-1}) \cap L^\infty([0, T_0]; H^s) \cap L^2(0, T_0; H^{s+1}).$$

Notice that initially we need less regularity for u and θ. This is due to the regularization from dispersion and dissipation for these two components. Due to the degeneracy in both effects, ρ does not gain any regularity. However, the dispersive regularization of other components is used to avoid losing regularity for ρ.

Given this existence result, we can now try to justify rigorously the DNS approximation to the Boltzmann equation, as having been done for the compressible Euler and Navier-Stoke system [6, 12]. It is also interesting to compare the DNS system with the Navier-Stokes system to see in which sense could this higher order dispersive system provide a better approximation to the Boltzmann equation.

Chapter 2

Fredholm-Alternative

2.1 Preliminaries

In this chapter we prove that under the weak cutoff assumption (1.16) on the collision kernel, the linearized Boltzmann operator $\mathcal{L}_{\mathcal{M}}$ satisfies a Fredholm alternative. Recall the definition of $\mathcal{L}_{\mathcal{M}}$:

$$\mathcal{L}_{\mathcal{M}} f = \iint_{S^{d-1}\times R^d} \Big(f(v) + f(v_1) - f(v') - f(v_1') \Big) \, b(v - v_1, \omega) \, d\omega \mathcal{M}(v_1) dv_1, \tag{2.1}$$

where $\mathcal{M}$ is a local Maxwellian defined by

$$\mathcal{M}(v) = \mathcal{M}_{\rho,u,\theta}\,(v) = \frac{\rho}{(2\pi\theta)^{d/2}} \exp\left(- \frac{|v-u|^2}{2\theta} \right),$$

The collision kernel $b(v - v_1, \omega \cdot n)$ satisfies

$$\begin{aligned} &b(v_1 - v, \omega) = |v_1 - v|^\beta \, \hat{b}\,(\omega \cdot n), \quad n = \frac{v - v_1}{|v - v_1|}, \\ &- d < \beta \leq 1, \end{aligned} \tag{2.2}$$

where $\hat{b}(\omega \cdot n)$ satisfies the weak cutoff condition

$$\hat{b}\,(\omega \cdot n) \in L^1(d\omega). \tag{2.3}$$

The condition (2.3) is the weakly cutoff assumption for the collision kernel b.

Because $\mathcal{L}_{\mathcal{M}}$ operates only on the velocity variable of $f(t,x,v)$, we only need to consider the case where $(\rho,u,\theta)=(1,0,1)$. This is the equilibrium state of the gas with even density, zero bulk velocity and even temperature. The general case then follows by translating and scaling in v. We call $\mathcal{M}_{1,0,1}$ the absolute Maxwellian and adopt the following notations:

$$M=\mathcal{M}_{1,0,1}=\frac{1}{(2\pi)^{d/2}}e^{-|v|^2/2}, \quad \mathcal{L}=\mathcal{L}_M.$$

The attenuation coefficient $a(v)$ with the absolute Maxwellian is

$$a(v)=C_\beta\int_{R^d}|v_1-v|^\beta\, M_1 dv_1, \tag{2.4}$$

where $C_\beta=\int_{S^{d-1}}\hat{b}\,(\omega\cdot n)d\omega$ is a constant.

From the definition (2.1), it is clear that the first term in $\mathcal{L}$ is just a multiplication of $f(v)$ with the attenuation coefficient $a(v)$. Rewrite $\mathcal{L}$ in the following form:

$$\mathcal{L}f=a(v)f-\hat{\mathcal{K}}f=a(v)\,\left(I-\mathcal{K}f\right)=a(v)\,\left(I+\mathcal{K}^- -\mathcal{K}^+\right)f, \tag{2.5}$$

where

$$\hat{\mathcal{K}}f=\iint_{S^{d-1}\times R^d}\Big(f(v')+f(v_1')-f(v_1)\Big)\,b(v-v_1,\omega)\,d\omega\,M(v_1)dv_1,$$

and

$$\hat{\mathcal{K}} = a(v)\mathcal{K}, \quad \mathcal{K} = \mathcal{K}^+ - \mathcal{K}^-,$$

$$\mathcal{K}^- f = \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} f(v_1)\, b(v - v_1, \omega)\, d\omega\, M_1 dv_1$$

$$\mathcal{K}^+ f = \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} \Big(f(v') + f(v_1') \Big)\, b(v - v_1, \omega)\, d\omega\, M_1 dv_1,$$

Therefore, we have

$$\frac{1}{a(v)}\mathcal{L} = \mathcal{I} - \mathcal{K}.$$

The structure of $\frac{1}{a(v)}\mathcal{L}$ yields the following lemma.

Lemma 2.1.1. $\frac{1}{a(v)}\mathcal{L} : L^p\,(aMdv) \longrightarrow L^p\,(aMdv)$ *is bounded for any* $1 \le p \le \infty$.

Proof. Define the measure

$$d\mu = b(v - v_1, \omega)\, d\omega\, M(v) dv\, M(v_1) dv_1.$$

By the definition of b and the conservations laws for the before and after collision velocity pairs, $d\mu$ is invariant under the changes

$$(v, v_1) \leftrightarrow (v', v_1'), \quad (v, v') \leftrightarrow (v_1, v_1').$$

Therefore, for any $f(v) \in L^p\,(aMdv)$, $\tilde{f}(v) \in L^q\,(aMdv)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left| \left\langle \frac{1}{a(v)}\mathcal{L} f,\, \tilde{f} \right\rangle \right| = |\langle f + f_1 - f' - f_1' \rangle|$$

$$\le \left(\|f\|_{L^p(d\mu)} + \|f_1\|_{L^p(d\mu)} + \|f'\|_{L^p(d\mu)} + \|f_1'\|_{L^p(d\mu)} \right) \|\tilde{f}\|_{L^q(d\mu)}$$

$$\le 4\, \|f\|_{L^p(aMdv)}\, \|\tilde{f}\|_{L^q(aMdv)}.$$

It is clear from the above estimate that $\frac{1}{a(v)}\mathcal{L}$ is a bounded operator with its operator norm equal to 4. $\square$

Because $\frac{1}{a(v)}\mathcal{L} = I - \mathcal{K}$, naturally $\mathcal{K}$ is also bounded. If we can further show that $\mathcal{K} : L^p(aMdv) \to L^p(aMdv)$ is compact, then $\frac{1}{a(v)}\mathcal{L}$ is a Fredholm operator. This is the main theorem we prove in this chapter.

Main Theorem 1. *Assume that the collision kernel $b(v - v_1, \omega)$ satisfies (2.2) and (2.3). Then $\mathcal{K}^{\pm} : L^2(aMdv) \to L^2(aMdv)$ are compact. Therefore, $\frac{1}{a(v)}\mathcal{L}$ is a Fredholm operator on $L^2(aMdv)$ and has a Fredholm alternative.*

There are various results on the compactness of $\hat{\mathcal{K}}$ and $\mathcal{K}$, thus the Fredholm alternative property of the linearized Boltzmann operator. It was first shown for the hard sphere case by Hilbert [19] in 1912 as an application for his integral theory. He showed that the kernel of $\hat{\mathcal{K}}$ decays exponentially and has only first order singularity.

With the Grad angular cutoff assumption for the collision kernel, Grad [17] proved that $\hat{\mathcal{K}}$ is compact on $L^2(Mdv)$ for a hard potential by showing that the kernel of $\hat{\mathcal{K}}$ is Hilbert-Schmidt. Using a similar method, Caflisch [5] generalized Grad's result to soft potential cases with $-1 < \beta < 1$. For the compactness of $\mathcal{K}$, Golse and Poupaud [14] showed that $\mathcal{K}$ is compact on $L^2(aMdv)$ for $-2 < \beta < 1$. In [18] Guo extended Caflisch's result to the full range of β where $-3 < \beta < 1$. Compared with [18], we consider the compactness of $\mathcal{K}$ with a more general assumption for b.

2.2 Outline of Proof

In this section we give an outline of the proof for the compactness of the operator $\mathcal{K}$ defined in (2.5). Because $\mathcal{K} = \mathcal{K}^- - \mathcal{K}^+$, we will show the compactness for these two parts individually. The proof is based on the following theorem in functional analysis:

Theorem 2.2.1. *The space of compact operators is closed under the operator norm.*

We will also use the following basic bound for the proof of a generalized Hilbert-Schmidt theorem.

Theorem 2.2.2. *Let $d\nu$ be a positive Borel measure over R^D. Let $\mathcal{K}$ be defined in the kernel form*

$$\mathcal{K}g(v) = \int_{R^D} K(v, v')g(v')d\nu'. \tag{2.6}$$

with $K(v, v')$ symmetric and $d\nu' = d\nu(v')$. If there exist two constants $r, s \geq 0$ such that $K(v, v')$ satisfies

$$\|K\|_{L^s(d\nu, L^r(d\nu'))} \triangleq \left(\int_{R^D} \left(\int_{R^D} |K(v, v')|^r d\nu' \right)^{\frac{s}{r}} d\nu \right)^{\frac{1}{s}} < \infty, \tag{2.7}$$

where $p, q, r, s \in [0, \infty]$, $r \leq s \leq \infty$, $p, q \in [r, s]$, $\frac{1}{q} + \frac{1}{q^*} = 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2$. Then $\mathcal{K} : L^p(d\nu) \to L^{q^*}(d\nu)$ is bounded and

$$\|\mathcal{K}\|_{\mathcal{BL}(L^p, L^{q^*})} \leq \|K\|_{L^s(d\nu, L^r(d\nu'))}, \tag{2.8}$$

where $\mathcal{BL}(L^p, L^{q^*})$ is the space of all linear bounded operators from $L^p(d\nu)$ to $L^{q^*}(d\nu)$.

Corollary 2.2.1. *If $K(v, v')$ satisfies (2.7) with $s < \infty$ then $\mathcal{K} : L^p(d\nu) \to L^{q^*}(d\nu)$ defined by (2.6) is compact.*

The proof of Corollary 2.2.1 is based on the following facts:

- finite rank operators are compact,
- kernels of finite rank operators are dense in the space $L^s(d\mu, L^r(d\mu'))$,
- if the kernel of an integral operator satisfies (2.7), then there exists a sequence of finite rank operators that converges to this integral operator.

Hence, Theorem 2.2.1 guarantees the compactness of this integral operator.

Therefore, we first try to find the kernels of $\mathcal{K}^{\pm}$ respectively. The kernel of $\mathcal{K}^{-}$ is easy to identify and is in a simple form. We show the compactness proof in section 2.3 using a direct application of Theorem 2.2.1.

To show the compactness of $\mathcal{K}^{+}$, we change variables in the integral in $\mathcal{K}^{+}$ and use the forms introduced by Grad [17] to find its kernel. Due to the singularities in the integral, this kernel is too complicated for a straightforward application of Theorem 2.2.1. The idea is to truncate the operator $\mathcal{K}^{+}$ such that we can avoid the singularities. For the truncated operators, we apply Theorem 2.2.1 to show their compactness. For the remainder we find uniform bounds in the operator norm. By theorem 2.2.1, we conclude that the original $\mathcal{K}^{+}$ is also compact.

2.3 Compactness of the Loss Operator

In this section we show the compactness of the loss operator $\mathcal{K}^-$. By the definition,

$$\begin{aligned}\mathcal{K}^- f &= \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} f(v_1)\, b(v-v_1,\omega)\, d\omega\, M_1 dv_1 \\ &= \frac{C_\beta}{a(v)} \int_{R^d} f(v_1)|v-v_1|^\beta\, M_1 dv_1,\end{aligned}$$

where

$$a(v) = C_\beta \int_{R^d} |v-v_1|^\beta\, M_1 dv_1,$$

is the attenuation coefficient. Therefore, it is clear that $\mathcal{K}^- : L^2(aMdv) \to L^2(aMdv)$ has the kernel

$$K^-(v,v_1) = \frac{C_\beta |v-v_1|^\beta}{a(v)a(v_1)}.$$

Before the compactness proof for $\mathcal{K}^-$, we need the following estimate for $a(v)$. The following lemma shows that $a(v)$ is bounded above and below by $(1+|v|)^\beta$.

Lemma 2.3.1. *Assume that the collision kernel b satisfies (2.2) and (2.3). Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1(1+|v|)^\beta \le a(v) \le c_2(1+|v|)^\beta, \quad \forall v \in R^D.$$

The above inequality is also true when the measure Mdv is changed to $M^\alpha dv$ for any $\alpha > 0$.

Proof. Proof is done by direct estimates over the different regions of v_1.

For $\beta \geq 0$, it is straightforward to see the upper bound for $a(v)$ as

$$a(v) = C \int_{R^d} |v_1 - v|^\beta \, M_1 dv_1 \leq c \int_{R^d} \left(|v_1|^\beta + |v|^\beta \right) \, M_1 dv_1 \leq c_1 \left(1 + |v|\right)^\beta .$$

As for the lower bound, let $\chi(v)$ be the characteristic function such that $\chi(v) = 0$ when $|v| \geq 1$ and $\chi(v) = 1$ when $|v| \leq 1$. Then we have

$$\begin{aligned} a(v) &= C \int_{R^d} |v_1 - v|^\beta \, M_1 dv_1 \\ &\geq c \left(\chi(v) \int_{|v_1| \geq 2} |v_1 - v|^\beta \, M_1 dv_1 + (1 - \chi(v)) \int_{|v_1| \leq \frac{1}{2}|v|} |v_1 - v|^\beta \, M_1 dv_1 \right) \\ &\geq c \left(\chi(v) \int_{|v_1| \geq 2} \frac{1}{2} |v_1|^\beta \, M_1 dv_1 + (1 - \chi(v)) \int_{|v_1| \leq \frac{1}{2}} \frac{1}{2} |v|^\beta \, M_1 dv_1 \right) \\ &\geq c_3 \chi(v) + c_4 (1 - \chi(v)) |v|^\beta \\ &\geq c_2 \left(1 + |v|\right)^\beta . \end{aligned}$$

The estimate for the case $\beta \leq 0$ is done in a similar way. It is now easy to see the lower bound for $a(v)$ since

$$a(v) = C \int_{R^d} |v_1 - v|^\beta \, M_1 dv_1 \geq c \int_{R^d} \left(|v_1|^\beta + |v|^\beta \right) \, M_1 dv_1 \geq c_2 \left(1 + |v|\right)^\beta .$$

Note that $|v_1|^\beta$ is integrable near 0 since we assume that $\beta \geq -d$.

For the upper bound, use the characteristic function $\chi(v)$ again to see that

$$\begin{aligned} a(v) &= C \int_{R^d} |v_1 - v|^\beta \, M_1 dv_1 \\ &= c \left(\chi(v) \int_{R^d} |v_1 - v|^\beta \, M_1 dv_1 + (1 - \chi(v)) \int_{R^d} |v_1 - v|^\beta \, M_1 dv_1 \right) . \end{aligned}$$

where

$$\chi(v)\int_{R^d}|v_1-v|^\beta M_1dv_1 \le \chi(v)\int_{|v_1|\ge 2}|v_1-v|^\beta M_1dv_1+\chi(v)\int_{|v_1|\le 2}|v_1-v|^\beta M_1dv_1$$

$$\le \chi(v)\int_{|v_1|\ge 2}\frac{1}{2}|v_1|^\beta M_1dv_1+\chi(v)\cdot c\int_{|v_1|\le 2}|v_1-v|^\beta dv_1$$

$$\le c\cdot\chi(v)+\chi(v)\cdot c\int_{|v_1|\le 3}|v_1|^\beta dv_1$$

$$\le c_0\cdot\chi(v)\le c_0,$$

and

$$(1-\chi(v))\int_{R^d}|v_1-v|^\beta M_1dv_1$$

$$\le (1-\chi(v))\int_{|v_1-v|\ge 1}|v_1-v|^\beta M_1dv_1+(1-\chi(v))\int_{|v_1-v|\le 1}|v_1-v|^\beta M_1dv_1$$

$$\le (1-\chi(v))\int_{|v_1-v|\ge 1}M_1dv_1+c(1-\chi(v))\cdot M(v)\int_{|v_1-v|\le 1}|v_1-v|^\beta dv_1$$

$$\le (1-\chi(v))\int_{R^d}M_1dv_1+c(1-\chi(v))\cdot M(v)$$

$$\le c(1-\chi(v))(1+M(v))$$

$$\le c(1+|v|)^\beta.$$

Overall for $\beta\le 0$ we also have

$$c_1(1+|v|)^\beta\le a(v)\le c_2(1+|v|)^\beta.$$

It can be seen from the above proof that if M is changed to M^α for any $\alpha>0$, the estimate for $a(v)$ stays the same for the following reasons: there are two places that we use M. One is to guarantee that $aMdv$ is a finite measure. The other is when we can change $M(v_1)$ to $M(v)$ when $|v|$ is bounded by $|v_1|$, and we use the fact that $M(v)$ decays faster than any polynomial. Thus we finish the proof for Lemma 2.3.1. $\square$

The compactness of $\mathcal{K}^-$ is a direct application of Corollary 2.2.1 and Lemma 2.3.1.

Theorem 2.3.1. *$\mathcal{K}^- : L^2(aMdv) \to L^2(aMdv)$ is compact.*

Proof. By the fact that $\beta \in (-d, 1)$, there exists $1 < r < 2$ such that $\beta r \in (-d, 1)$.

First we show that

$$K^-(v_1, v) \in L^\infty(aMdv; L^r(a_1 M_1 dv_1)).$$

By direction calculations,

$$\begin{aligned}
\left\|K^-(v_1, v)\right\|^r_{L^r(a_1 M_1 dv_1)} &= \left(\frac{C}{a(v)}\right)^r \left\|\frac{|v_1 - v|^{\beta r}}{a(v_1)^r}\right\|_{L^1(a_1\, M_1\, dv_1)} \\
&\leq \frac{c}{a(v)^r} \left\|\frac{|v_1 - v|^{\beta r}}{a(v_1)^{r-1}}\right\|_{L^1(M_1\, dv_1)} \\
&\leq \frac{c}{a(v)^r} \int_{R^d} |v_1 - v|^{\beta r}\, M_1\, \mathrm{d}v_1 \\
&\leq \frac{c}{a(v)^r} \left(1 + |v|\right)^{\beta r} \leq \hat{c}\,,
\end{aligned}$$

where $\hat{c}$ is independent of v. Notice that we applied a similar estimate for $\int_{R^d} |v_1 - v|^{\beta r}\, M_1\, \mathrm{d}v_1$ as we have done in Lemma 2.3.1 where $-d < \beta r < 1$ is in the same position as β there.

Because $aMdv$ is a finite measure, for any $1 < s < r$ we have

$$K^{-}(v_1, v) \in L^s(aMdv; L^r(a(v_1)M_1dv_1)).$$

Particularly we choose $s = r^*$ where $\frac{1}{r} + \frac{1}{r^*} = 1$. Using $p = q = 2$ in Corollary 2.2.1, we obtain the compactness of $\mathcal{K}^- : L^2(aMdv) \to L^2(aMdv)$. $\square$

2.4 Compactness of the Gain Operator

What remains is to show the compactness of the gain operator $\mathcal{K}^+$ which is defined as

$$\begin{aligned}
\mathcal{K}^+ g &= \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} (g(v') + g(v_1')) \; b(\omega\cdot n, |v - v_1|)\, d\omega\, M(v_1)dv_1 \\
&= \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} g(v') \; b(\omega\cdot n, |v - v_1|)\, d\omega\, M(v_1)dv_1 \\
&\quad + \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} g(v_1')\, b(\omega\cdot n, |v - v_1|)\, d\omega\, M(v_1)dv_1.
\end{aligned}$$

Noticing that the two term in $\mathcal{K}^+$ depend on different variables v' and v_1', we separate those two terms as

$$\mathcal{K}^+ \triangleq \bar{\mathcal{K}}^+ + \tilde{\mathcal{K}}^+.$$

The basic idea is to apply Corollary 2.2.1 for both $\bar{\mathcal{K}}^+$ and $\tilde{\mathcal{K}}^+$ to show that both $\bar{\mathcal{K}}^+$ and $\tilde{\mathcal{K}}^+$ are compact from $L^2(aMdv)$ to $L^2(aMdv)$. Therefore their sum is also a compact operator from $L^2(aMdv)$ to $L^2(aMdv)$.

To this end, we use their kernels forms introduced by Grad [17]. By the symmetry, we need only to consider the region $\omega \cdot n > 0$. Then kernel of the operator $\bar{\mathcal{K}}^+$ has the form

$$K^+(v,v') = \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')} \int_{y\perp(v-v')} |v-v_1|^\beta \, e^{-\frac{1}{2}|y|^2 - y\cdot v'} \, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) dy,$$

while the kernel of the operator $\tilde{\mathcal{K}}^+$ is

$$\tilde{K}^+(v,v_1') = \frac{2}{a(v)a(v_1')} \int_{z\perp(v-v_1')} \frac{|v-v_1|^\beta}{|z|^{d-1}} \, e^{-\frac{1}{2}|z|^2 - z\cdot v} \, \hat{b}\left(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}}\right) dz.$$

Following are some notations to be used in the following exposition.

$$\langle h_1, h_2\rangle \triangleq \int_{R^d} h_1 \, h_2 \, M dv,$$

$$y = v_1 - v' = v_1' - v, \quad z = v' - v = v_1 - v_1',$$

$$\xi_1 + \xi_2 = v, \quad \xi_1 \parallel (v - v'), \quad \xi_2 \perp (v - v').$$

We want to show the compactness of $\bar{\mathcal{K}}^+$ ($\tilde{\mathcal{K}}^+$) by constructing a sequence of truncated operators which converges in the sup norm of the operator space to $\bar{\mathcal{K}}^+$ ($\tilde{\mathcal{K}}^+$). Because the space of compact operators is closed under this norm, we can conclude that the limit operator $\bar{\mathcal{K}}^+$ ($\tilde{\mathcal{K}}^+$) is also compact. The truncations of the operators are given by the truncations of the kernels, that is, we consider the following approximations of $K^+(v,v'), \tilde{K}^+(v,v_1')$ respectively:

$$\begin{aligned} K^+_{\epsilon,T}(v,v') =& \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')} \, I_{|v|<T} \, I_{|v'|<T} \\ & \times \, I_{|v-v'|>\epsilon|v-v_1|} \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) dy, \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
\tilde{K}^{+}_{\epsilon,T}(v,v_1') =& \frac{2}{a(v)a(v_1')}\, I_{|v|<T}\, I_{|v_1'|<T} \int_{z\perp(v-v_1')} \frac{|v-v_1|^{\beta}}{|z|^{d-1}}\, e^{-\frac{1}{2}|z|^2 - z\cdot v} \\
& \times\, I_{|v-v_1'|>\epsilon|v-v_1|}\, I_{|z|>\epsilon|v-v_1|}\, \hat{b}\left(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}}\right) dz.
\end{aligned} \tag{2.10}
$$

Then we have the following key theorem.

Theorem 2.4.1. *Let $\bar{\mathcal{K}}^{+}_{\epsilon,T}, \tilde{\mathcal{K}}^{+}_{\epsilon,T}$ be the corresponding operators with the kernels $K^{+}_{\epsilon,T}(v,v')$, $\tilde{K}^{+}_{\epsilon,T}(v,v')$ defined in (2.9), (2.10) respectively. Then*

(1) $\bar{\mathcal{K}}^{+}_{\epsilon,T} : L^2(aMdv) \to L^2(aMdv)$ is compact.

(2) $\tilde{\mathcal{K}}^{+}_{\epsilon,T} : L^2(aMdv) \to L^2(aMdv)$ is compact.

Proof. (1). Recall that $|v-v'| = |v-v_1|\cos\theta$, $|y| = |v-v_1'| = |v-v_1'|\cos\theta$. Then

$$
\begin{aligned}
K^{+}_{\epsilon,T}(v,v') =& \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_{y\perp(v-v')} |v-v_1|^{\beta}\, e^{-\frac{1}{2}|y+\xi_2|^2} \\
& \times\, I_{|v-v'|>\epsilon|v-v_1|}\, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) dy \\
\leq & \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_{y\perp(v-v')} |v-v_1|^{\beta}\, e^{-\frac{1}{2}|y+\xi_2|^2} \\
& \times\, I_{|v-v'|>\epsilon|v-v_1|}\, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) dy \\
\leq & \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_{y\perp(v-v')} |v-v_1|^{\beta} \\
& \times\, I_{|v-v'|>\epsilon|v-v_1|}\, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) dy \\
\leq & \frac{c|v-v'|^{-(d-1)}}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_{0}^{c_\epsilon|v-v'|} |v-v_1|^{\beta}\, |y|^{d-2} \\
& \times\, I_{|v-v'|>\epsilon|v-v_1|}\, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) d|y|.
\end{aligned}
$$

Change the variable in the above integral as

$$\cos\theta = \frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}. \tag{2.11}$$

Therefore we have

$$\sin\theta\,d\theta = 2\frac{|v-v'|}{|v-v_1|^3}\,|y|\,d|y|,$$

and the estimate for $K^+_{\epsilon,T}(v,v')$ continues as

$$\begin{aligned}
K^+_{\epsilon,T}(v,v') \leq & \frac{c|v-v'|^{-(d-1)}}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_0^{\pi/2} |v-v_1|^\beta\, |y|^{d-2}\, \frac{|v-v_1|^3}{|v-v'|}\frac{1}{|y|} \\
& \times I_{|v-v'|>\epsilon|v-v_1|}\, \hat{b}(\cos\theta)\, d\theta \\
\leq & \frac{c|v-v'|^{-d}}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_0^{\pi/2} |v-v_1|^{\beta+3}\, |y|^{d-3} \\
& \times I_{|v-v'|>\epsilon|v-v_1|}\, \hat{b}(\cos\theta)\, d\theta.
\end{aligned}$$

By the fact that $\beta > -d$, for $d \geq 3$, in the region $|v-v'| \geq \epsilon|v-v_1|$ we have

$$|v-v_1|^{\beta+3}\,|y|^{D-3} \leq |v-v_1|^{\beta+D}\left(\frac{|y|}{|v-v_1|}\right)^{D-3} \leq |v-v_1|^{\beta+D} \leq c_\epsilon|v-v'|^{\beta+D},$$

and this gives

$$K^+_{\epsilon,T}(v,v') \leq \frac{c_\epsilon|v-v'|^\beta}{a(v)a(v')}\, e^{|\xi_2|^2/2}\, I_{|v|<T}\, I_{|v'|<T} \int_0^{\pi/2} \hat{b}(\cos\theta)\, d\theta.$$

By the definition of ξ_2, we know that $e^{|\xi_2|^2/2} \leq c_T$ because $|\xi_2| \leq |v| \leq T$.

Overall we have

$$|K^+_{\epsilon,T}(v,v')| \le \frac{C_{\epsilon,T}}{a(v)a(v')}\ |v-v'|^{\beta}\, I_{|v|<T}\, I_{|v'|<T}\ \left\|\hat{b}\right\|_{L^1(d\omega)}.$$

For the given β, choose $1<r<2$ such that $r\beta > -d$. Then

$$\begin{aligned}
&\int_{R^d}\left(\int_{R^d} |K^+_{\epsilon,T}(v,v')|^r\, a(v')\, M(v')\, dv'\right)^{\frac{r^*}{r}} a(v)M(v)dv \\
\le &\int_{R^d}\left(\int_{R^d} |v-v'|^{r\beta}\, (a(v'))^{1-r}\, M(v')\, dv'\right)^{\frac{r^*}{r}} I_{|v|<T}\, (a(v))^{1-r}\, M(v)\, dv \\
<&\infty.
\end{aligned}$$

Thus by Theorem 2, $\bar{\mathcal{K}}^+_{\epsilon,T} : L^p\,(aMdv) \to L^p\,(aMdv)$ is compact for any $r \le p \le r^*$, which is particularly true for the case when $p=2$.

(2). The compactness of $\tilde{\mathcal{K}}^+_{\epsilon,T} : L^2(aMdv) \to L^2(aMdv)$ is done in a similar way. Recall that $|z| = |v-v'| = |v-v_1|\cos\theta,\ |v-v_1'| = |v-v_1|\sin\theta$. Make a similar change of variable as in (2.11). Let

$$\cos\theta = \frac{|z|}{\sqrt{|v-v_1'|^2+|z|^2}},$$

which gives

$$\sin\theta\, d\theta = \left(\frac{1}{|v-v_1|} - \frac{|z|^2}{|v-v_1|^3}\right)\ d|z| = \frac{|v-v_1'|^2}{|v-v_1|^3}\ d|z|.$$

Then we have

$$
\begin{aligned}
\tilde{K}^+_{\epsilon,T}(v,v_1') =& \frac{2e^{\frac{1}{2}|\xi_1|^2}}{a(v)a(v_1')} \, I_{|v|<T} \, I_{|v_1'|<T} \int_{z\perp(v-v_1')} \frac{|v-v_1|^\beta}{|z|^{d-1}} e^{-\frac{1}{2}|z-\xi_1|^2} \\
& \times I_{|v-v_1'|>\epsilon|v-v_1|} \, I_{|z|>\epsilon|v-v_1|} \, \hat{b}\left(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}} \right) \, dz \\
\leq & \frac{2e^{\frac{1}{2}|\xi_1|^2}}{a(v)a(v_1')} \, I_{|v|<T} \int_0^{\pi/2} \frac{|v-v_1|^\beta}{|z|^{d-1}} e^{-\frac{1}{2}|z-\xi_1|^2} \frac{|v-v_1|^3}{|v-v_1'|^2} \, |z|^{D-2} \\
& \times I_{|\cos\theta|>\epsilon} \, I_{\sin\theta>\epsilon} \, \hat{b}(\cos\theta) \, \sin\theta \, d\theta \\
\leq & \frac{C_T}{a(v)a(v_1')} I_{|v|<T} \int_0^{\pi/2} \frac{|v-v_1|^{\beta+3}}{|v-v_1'|^2} \frac{1}{|z|} \\
& \times I_{|\cos\theta|>\epsilon} \, I_{\sin\theta>\epsilon} \, \hat{b}(\cos\theta) \, \sin\theta \, d\theta \\
\leq & \frac{C_{\epsilon,T}|v-v_1'|^\beta}{a(v)a(v_1')} I_{|v|<T} \int_0^{\pi/2} \hat{b}(\cos\theta) \, \sin\theta \, d\theta \\
=& C_{\epsilon,T} \frac{|v-v_1'|^\beta}{a(v)a(v_1')} \, I_{|v|<T} \, \|\hat{b}\|_{L^1(d\omega)}.
\end{aligned}
$$

Notice we used in the above proof $|\xi_1| \leq |v| \leq T, \quad |v-v_1'| \leq |v-v_1| \leq C_\epsilon |v-v_1'|$ and $|z| \leq |v-v_1| \leq c_\epsilon |z|$.

Again choose $r > 1$ such that $r\beta > -D$ and by Theorem 2 we know $\tilde{\mathcal{K}}^+_{\epsilon,T} : L^2(aMdv) \to L^2(aMdv)$ is compact and this completes the proof of Theorem 2.4.1. $\square$

Now consider the remainders of the two operators $\bar{\mathcal{K}}^+, \tilde{\mathcal{K}}^+$. Their kernels have the following forms.

$$
\bar{K}^+(v,v') - \bar{K}^+_{\epsilon,T}(v,v') = \bar{K}^+_\epsilon(v,v') + \bar{K}^+_T(v,v') + \bar{K}_{T,T}(v,v'),
$$

$$
\tilde{K}^+(v,v') - \tilde{K}^+_{\epsilon,T}(v,v') = \tilde{K}^+_\epsilon(v,v') + \tilde{K}^+_T(v,v') + \tilde{K}_{T,T}(v,v'),
$$

where

$$\bar{K}^+_\epsilon(v,v') = \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')} \int_{y\perp(v-v')} |v-v_1|^\beta \, e^{-\frac{1}{2}|y|^2 - y\cdot v'} \\ \times \, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) I_{|v-v'|<\epsilon|v-v_1|} \, dy,$$

$$\bar{K}^+_T(v,v') = \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')} I_{|v|>T} \int_{y\perp(v-v')} |v-v_1|^\beta \, e^{-\frac{1}{2}|y|^2 - y\cdot v'} \\ \times \, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) I_{|v-v'|>\epsilon|v-v_1|} \, dy,$$

$$\bar{K}^+_{T,T}(v,v') = \frac{2|v-v'|^{-(d-1)}}{a(v)a(v')} I_{|v|<T} \, I_{|v'|>T} \int_{y\perp(v-v')} |v-v_1|^\beta \, e^{-\frac{1}{2}|y|^2 - y\cdot v'} \\ \times \, \hat{b}\left(\frac{|v-v'|}{\sqrt{|v-v'|^2+|y|^2}}\right) I_{|v-v'|>\epsilon|v-v_1|} \, dy,$$

$$\begin{aligned} \tilde{K}^+_\epsilon(v,v_1') =& \frac{2}{a(v)a(v_1')} \int_{z\perp(v-v_1')} \frac{|v-v_1|^\beta}{|z|^{d-1}} \, e^{-\frac{1}{2}|z|^2 - z\cdot v} \, \hat{b}\Big(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}}\Big) \\ & \times \, \big(1 - I_{|v-v_1'|>\epsilon|v-v_1|} I_{|z|>\epsilon|v-v_1|}\big) \, dz \\ \leq& \frac{2}{a(v)a(v_1')} \int_{z\perp(v-v_1')} \frac{|v-v_1|^\beta}{|z|^{d-1}} \, e^{-\frac{1}{2}|z|^2 - z\cdot v} \, \hat{b}\Big(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}}\Big) \\ & \times \, (I_{|v-v_1'|<\epsilon|v-v_1|} + I_{|z|<\epsilon|v-v_1|}) \, dz, \end{aligned}$$

$$\tilde{K}^+_T(v,v_1') = \frac{2}{a(v)a(v_1')} I_{|v|>T} \int_{z\perp(v-v_1')} \frac{|v-v_1|^\beta}{|z|^{d-1}} e^{-\frac{1}{2}|z|^2 - z\cdot v} \\ \times \, I_{|v-v_1'|>\epsilon|v-v_1|} I_{|z|>\epsilon|v-v_1|} \, \hat{b}\left(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}}\right) \, dz.$$

$$\tilde{K}^+_{T,T}(v,v_1') = \frac{2}{a(v)a(v_1')} I_{|v|<T} \, I_{|v_1'|>T} \int_{z\perp(v-v_1')} \frac{|v-v_1|^\beta}{|z|^{d-1}} e^{-\frac{1}{2}|z|^2 - z\cdot v} \\ \times \, I_{|v-v_1'|>\epsilon|v-v_1|} I_{|z|>\epsilon|v-v_1|} \, \hat{b}\left(\frac{|z|}{\sqrt{|z|^2+|v-v_1'|^2}}\right) \, dz.$$

Use $\bar{\mathcal{K}}^+_\epsilon, \bar{\mathcal{K}}^+_T,\ \bar{\mathcal{K}}^+_{T,T}$ to denote the operators with the kernels $\bar{K}^+_\epsilon(v,v')$, $\bar{K}^+_T(v,v')$, $\bar{K}^+_{T,T}(v,v')$. Similarly $\tilde{\mathcal{K}}^+_\epsilon, \tilde{\mathcal{K}}^+_T, \tilde{\mathcal{K}}^+_{T,T}$ are corresponding operators with kernels $\tilde{K}^+_\epsilon(v,v_1')$,

$\tilde{K}_T^+(v, v_1')$, $\tilde{\mathcal{K}}_{T,T}^+(v, v_1')$ respectively. Let L be the linear operator space endowed with the sup norm $\|\cdot\|_L$. We will show in the following that $\|\bar{\mathcal{K}}_\epsilon^+\|_L \xrightarrow{\epsilon\downarrow 0} 0$, $\|\bar{\mathcal{K}}_T^+\|_L \xrightarrow{T\uparrow\infty} 0$, $\|\bar{\mathcal{K}}_{T,T}^+\|_L \xrightarrow{T\uparrow\infty} 0$, $\|\tilde{\mathcal{K}}_\epsilon^+\|_L \xrightarrow{\epsilon\downarrow 0} 0$, $\|\tilde{\mathcal{K}}_T^+\|_L \xrightarrow{T\uparrow\infty} 0$, $\|\tilde{\mathcal{K}}_{T,T}^+\|_L \xrightarrow{T\uparrow\infty} 0$. Then as L is closed under the sup norm, we get the compactness of $\bar{\mathcal{K}}^+$ and $\tilde{\mathcal{K}}^+$.

The following theorem is to show the smallness of $\|\bar{\mathcal{K}}_\epsilon^+\|_L$, $\|\bar{\mathcal{K}}_T^+\|_L$, $\|\bar{\mathcal{K}}_{T,T}^+\|_L$, $\|\tilde{\mathcal{K}}_\epsilon^+\|_L$, $\|\tilde{\mathcal{K}}_T^+\|_L$ and $\|\tilde{\mathcal{K}}_{T,T}^+\|_L$.

Theorem 2.4.2. *$\forall g, \tilde{g} \in L^2(aMdv)$, we have*

$$\left|\left\langle a(v)\bar{\mathcal{K}}_\epsilon^+ g(v),\ \tilde{g}(v)\right\rangle\right| \le \eta_\epsilon\ \|g\|_{L^2(aMdv)}\ \|\tilde{g}\|_{L^2(aMdv)}\,,$$

$$\left|\left\langle a(v)\bar{\mathcal{K}}_T^+ g(v),\ \tilde{g}(v)\right\rangle\right| \le \eta_T\ \|g\|_{L^2(aMdv)}\ \|\tilde{g}\|_{L^2(aMdv)}\,,$$

$$\left|\left\langle a(v)\bar{\mathcal{K}}_{T,T}^+ g(v),\ \tilde{g}(v)\right\rangle\right| \le \eta_T\ \|g\|_{L^2(aMdv)}\ \|\tilde{g}\|_{L^2(aMdv)}\,,$$

$$\left|\left\langle a(v)\tilde{\mathcal{K}}_\epsilon^+ g(v),\ \tilde{g}(v)\right\rangle\right| \le \eta_\epsilon\ \|g\|_{L^2(aMdv)}\ \|\tilde{g}\|_{L^2(aMdv)}\,,$$

$$\left|\left\langle a(v)\tilde{\mathcal{K}}_T^+ g(v),\ \tilde{g}(v)\right\rangle\right| \le \eta_T\ \|g\|_{L^2(aMdv)}\ \|\tilde{g}\|_{L^2(aMdv)}\,,$$

$$\left|\left\langle a(v)\tilde{\mathcal{K}}_{T,T}^+ g(v),\ \tilde{g}(v)\right\rangle\right| \le \eta_T\ \|g\|_{L^2(aMdv)}\ \|\tilde{g}\|_{L^2(aMdv)}\,,$$

with $\eta_\epsilon \xrightarrow{\epsilon\downarrow 0} 0$, $\eta_T \xrightarrow{T\uparrow\infty} 0$ (for fixed ϵ) independent of $g, \tilde{g}$. Notice here η_T can be dependent on ϵ.

Proof. In what follows, we prove the above six inequalities one by one. First, keeping the various truncations in mind, we write the remainder operators in their original forms instead of the kernel forms, that is, for any $g(v)$,

$$\bar{\mathcal{K}}_{\epsilon}^{+} g = \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} g(v') \; I_{|v-v'|<\epsilon|v-v_1|}$$

$$b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1) dv_1,$$

$$\bar{\mathcal{K}}_{T}^{+} g = \frac{1}{a(v)}\, I_{|v|>T} \iint_{S^{d-1}\times R^d} g(v') \; I_{|v-v'|>\epsilon|v-v_1|}$$

$$b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1) dv_1,$$

$$\bar{\mathcal{K}}_{T,T}^{+} g = \frac{1}{a(v)}\, I_{|v|<T}\, I_{|v'|>T} \iint_{S^{d-1}\times R^d} g(v') \; I_{|v-v'|>\epsilon|v-v_1|}$$

$$b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1) dv_1,$$

$$\tilde{\mathcal{K}}_{\epsilon}^{+} g = \frac{1}{a(v)} \iint_{S^{d-1}\times R^d} g(v_1') \; \Big(1 - I_{|v-v_1'|>\epsilon|v-v_1|\, I_{|v-v'|>\epsilon|v-v_1|}}\Big)$$

$$\times\; b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1) dv_1,$$

$$\tilde{\mathcal{K}}_{T}^{+} g = \frac{1}{a(v)}\, I_{|v|>T} \iint_{S^{d-1}\times R^d} g(v_1')\, I_{|v-v_1'|>\epsilon|v-v_1|}\, I_{|v-v'|>\epsilon|v-v_1|}$$

$$\times\, b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1) dv_1,$$

$$\tilde{\mathcal{K}}_{T,T}^{+} g = \frac{1}{a(v)}\, I_{|v|<T}\, I_{|v_1'|>T} \iint_{S^{d-1}\times R^d} g(v_1')\, I_{|v-v_1'|>\epsilon|v-v_1|}\, I_{|v-v'|>\epsilon|v-v_1|}$$

$$b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1) dv_1,$$

To simplify the notation, let $d\mu = b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1)\, dv_1\, M(v)\, dv$.

To prove the first inequality, we have

$$\begin{aligned}\left|\left\langle a(v)\bar{\mathcal{K}}_\epsilon^+ g(v),\, \tilde{g}(v)\right\rangle\right| &= \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v)\, g(v')\, I_{|v-v'|<\epsilon|v-v_1|}\, d\mu\right| \\ &\le \left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v')|^2\, I_{|v-v'|<\epsilon|v-v_1|}\, d\mu\right)^{\frac{1}{2}} \cdot \|\tilde{g}\|_{L^2(aMdv)}\,.\end{aligned}$$

We need only to check the first factor on the right-hand side of the above inequality.

By changing variables: $(v, v_1) \to (v', v_1')$ and utilizing the symmetric property of the measure $d\mu$,

$$\begin{aligned}&\iiint_{S^{d-1}\times R^d\times R^d} |g(v')|^2\, I_{|v-v'|<\epsilon|v-v_1|}\, d\mu \\ &= \iiint_{S^{d-1}\times R^d\times R^d} |g(v)|^2\, I_{|v-v'|<\epsilon|v-v_1|}\, d\mu \\ &= \int_{R^D} |g(v)|^2 \left(\iint_{S^{d-1}\times R^d} I_{|v-v'|<\epsilon|v-v_1|}\, b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1)\, dv_1\right) M(v)\, dv.\end{aligned}$$

Let

$$J = \iint_{S^{d-1}\times R^d} I_{|v-v'|<\epsilon|v-v_1|}\, b(\omega\cdot n, |v-v_1|)\, d\omega\, M(v_1)\, dv_1.$$

Then by the definition of the collision kernel b,

$$\begin{aligned}J &= \iint_{S^{d-1}\times R^d} |v-v_1|^\beta\ \hat{b}(|\cos\theta|)\, I_{|\cos\theta|<\epsilon}\, d\omega\, M(v_1)\, dv_1 \\ &= \int_{R^D} |v-v_1|^\beta \left(\int_{S^{D-1}} \hat{b}(|\cos\theta|)\, I_{|\cos\theta|<\epsilon}\, d\omega\right) M(v_1) dv_1 \\ &= \int_{R^D} |v-v_1|^\beta M(v_1) dv_1 \cdot \int_{S^{D-1}} \hat{b}(|\cos\theta|)\, I_{|\cos\theta|<\epsilon}\, d\omega.\end{aligned}$$

For any $\eta > 0$, since $\hat{b}(|\cos\theta|) \in L^1(d\omega)$, there exists $\delta_0 > 0$ such that $\forall \delta < \delta_0$, we have

$$\int_{S^{d-1}} \hat{b}(|\cos\theta|)\, I_{|\cos\theta|<\delta}\, d\omega \,<\, \eta.$$

Therefore choosing ϵ small enough, we have

$$J \leq \eta \int_{R^d} |v - v_1|^\beta\, M(v_1)\, dv_1 \,=\, \eta\, a(v),$$

which gives

$$\iiint_{S^{d-1}\times R^d\times R^d} |g(v')|^2\, I_{|v-v'|<\epsilon|v-v_1|}\, d\mu \,\leq\, \eta\, \|g\|_{L^2(aMdv)}\,,$$

when ϵ is small enough. Because $\eta \to 0$ when $\epsilon \to 0$, we use the notation η_ϵ for η. Thus we are done with the first inequality.

Next we show the proof for the second inequality. To this end, we estimate $\left|\left\langle a(v)\bar{\mathcal{K}}_T^+ g, \tilde{g}(v)\right\rangle\right|$ with ϵ fixed.

$$\begin{aligned}
&\left|\left\langle a(v)\bar{\mathcal{K}}_T^+ g, \tilde{g}(v)\right\rangle\right| \\
&= \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v')\, \tilde{g}(v)\ I_{|v|>T}\, I_{|v-v'|>\epsilon|v-v_1|}\ b(\omega\cdot n,\, |v-v_1|)\ M_1\, M\, d\omega\, dv_1\, dv\right| \\
&\leq \iiint_{S^{d-1}\times R^d\times R^d} |g(v')|\ \ |\tilde{g}(v)|\ \ I_{|v_1|>m}\ b(\omega\cdot n,\, |v-v_1|)\, M_1\, M\, d\omega dv_1 dv \\
&\quad + \iiint_{S^{d-1}\times R^d\times R^d} |g(v')|\ \ |\tilde{g}(v)|\ \ I_{|\cos\theta|>\epsilon}\, I_{|v_1|<m}\, I_{|v|>T}\ b\, M_1\, M\, d\omega dv_1 dv \\
&\triangleq I_1 + I_2.
\end{aligned}$$

We are going to estimate I_1 and I_2 individually. First estimate I_1.

$$
\begin{aligned}
I_1 &\leq \left(\iiint_{S^{d-1}\times R^d\times R^d} |\tilde{g}(v)|^2 \; I_{|v_1|>m} \; b \; M_1 \, M \, d\omega \, dv_1 dv \right)^{\frac{1}{2}} \cdot \|g\|_{L^2(aMdv)} \\
&\leq \left(\int_{R^D} |\tilde{g}(v)|^2 \left(\int_{R^D} I_{|v_1|>m} \; |v-v_1|^\beta \, M_1 dv_1 \right) M dv \right)^{\frac{1}{2}} \cdot \left\| \hat{b} \right\|^{\frac{1}{2}}_{L^1(d\omega)} \cdot \|g\|_{L^2(aMdv)} \\
&\leq Ce^{-\frac{1}{4}m^2} \left(\int_{R^d} |\tilde{g}(v)|^2 \; \left(\int_{R^d} I_{|v_1|>m} \; |v-v_1|^\beta \; \sqrt{M_1} dv_1 \right) \; M dv \right)^{\frac{1}{2}} \cdot ||g||_{L^2(aMdv)} \\
&\leq Ce^{-\frac{1}{4}m^2} \left(\int_{R^d} |\tilde{g}(v)|^2 \; \left(\int_{R^d} |v-v_1|^\beta \; \sqrt{M_1} dv_1 \right) \; M dv \right)^{\frac{1}{2}} \cdot ||g||_{L^2(aMdv)} \\
&\leq Ce^{-\frac{1}{4}m^2} \; \|\tilde{g}\|_{L^2(aMdv)} \; \cdot \|g\|_{L^2(aMdv)} \, .
\end{aligned}
$$

Note that the last step is guaranteed by Lemma 2.3.1. Because $m > 0$ is arbitrary in the above estimate, for any $\eta > 0$, we can choose m large enough such that $Ce^{-\frac{1}{4}m^2} < \frac{\eta}{2}$. Then

$$
I_1 \; \leq \; \frac{\eta}{2} \; \|g\|_{L^2(aMdv)} \cdot \|\tilde{g}\|_{L^2(aMdv)} \, .
$$

Now fix m and we prove that for this fixed m, I_2 is arbitrarily small when T is large.

To estimate I_2, notice that when m and ϵ are fixed, $|v_1'-v_1| = |v-v'| > \epsilon|v-v_1|$. So $|v_1'| > \epsilon|v-v_1| - |v_1|$. We can choose T large enough such that $|v_1'| > \dfrac{\epsilon T}{2}$. This can be done because $|v_1| < m$ where m is fixed. By using this T we have

$$
\begin{aligned}
I_2 = & \iiint_{S^{d-1}\times R^d\times R^d} \left(\sqrt{M(v')}\,|g(v')|\right)\cdot\left(\sqrt{M(v)}\,|\tilde{g}(v)|\right) \\
& \times\ I_{|\cos\theta|>\epsilon}\, I_{|v_1|<m}\, I_{|v|>T}\, b\, \frac{M(v_1)M(v)}{\sqrt{M(v')M(v)}}\, d\omega dv_1 dv \\
\leq & \iiint_{S^{d-1}\times R^d\times R^d} \left(\sqrt{M(v')}\,|g(v')|\right)\cdot\left(\sqrt{M(v)}\,|\tilde{g}(v)|\right) \\
& \times\ I_{|v_1'|>\frac{\epsilon T}{2}}\, I_{|v_1|<m}\, b\,\sqrt{M(v_1)}\sqrt{M(v_1')}\, d\omega dv_1 dv \\
\leq & C_m e^{-\frac{(\epsilon T)^2}{32}} \iiint_{S^{d-1}\times R^d\times R^d} \left(\sqrt{M(v')}\,|g(v')|\right)\cdot\left(\sqrt{M}|\,\tilde{g}(v)|\right) \\
& \times\ I_{|v_1|<m}\ e^{-\frac{1}{8}|v_1'|^2}\, b\, d\omega dv_1 dv \\
\leq & C_m e^{-\frac{(\epsilon T)^2}{32}} \left(\iiint_{S^{d-1}\times R^d\times R^d} |\tilde{g}(v)|^2\, I_{|v_1|<m}\, b\, M\, d\omega dv_1 dv\right)^{\frac{1}{2}} \\
& \times\ \left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v')|^2\, M(v')\, \sqrt{M(v_1')}\, b d\omega dv_1 dv\right)^{\frac{1}{2}} \\
= & C_m e^{-\frac{(\epsilon T)^2}{32}} \left(\iiint_{S^{d-1}\times R^d\times R^d} |\tilde{g}(v)|^2\, I_{|v_1|<m}\, b\, M\, d\omega dv_1 dv\right)^{\frac{1}{2}} \\
& \times\ \left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v)|^2\, \sqrt{M_1}\, b\, d\omega dv_1 M\, dv\right)^{\frac{1}{2}} \\
\leq & C_m e^{-\frac{(\epsilon T)^2}{32}}\ \|g\|_{L^2(aMdv)}\ \cdot \|\tilde{g}\|_{L^2(aMdv)}\,.
\end{aligned}
$$

It is clear from the above inequality that for fixed ϵ, $\forall\eta > 0$, we can choose T large enough such that $C_m e^{-\frac{(\epsilon T)^2}{32}} < \frac{\eta}{2}$. Therefore, for $T > 0$ large, we have shown that

$$
I_2 \leq \frac{\eta}{2}\, \|g\|_{L^2(aMdv)}\ \cdot \|\tilde{g}\|_{L^2(aMdv)}\,.
$$

Together with the bound on I_1 we complete the proof of the second inequality. Again, since $\eta \to 0$ as $T \to \infty$, we use η_T for η.

For the third inequality we use symmetry to show that actually it is bounded by the term of the left side of the second inequality.

$$\left|\left\langle a(v)\bar{\mathcal{K}}^{+}_{T,T}g,\ \tilde{g}(v)\right\rangle\right| = \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v')\,\tilde{g}(v)\,I_{|v|<T}\,I_{|v'|>T}\,I_{|v-v'|>\epsilon|v-v_1|}\,d\mu\right|.$$

If we change the variable in the above integral $(v, v_1) \to (v', v_1')$, then

$$\begin{aligned}\left|\left\langle a(v)\bar{\mathcal{K}}^{+}_{T,T}g,\ \tilde{g}(v)\right\rangle\right| &= \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v)\,\tilde{g}(v')\,I_{|v|>T}\,I_{|v'|<T}\,I_{|v-v'|>\epsilon|v-v_1|}\,d\mu\right| \\ &\le \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v)\,\tilde{g}(v')\,I_{|v|>T}\,I_{|v-v'|>\epsilon|v-v_1|}\,d\mu\right|\end{aligned}$$

by applying the symmetry of the measure $d\mu = b\,M_1\,M\,d\omega dv_1 dv$. Therefore it's clear that

$$\left|\left\langle a(v)\bar{\mathcal{K}}^{+}_{T,T}g,\ \tilde{g}(v)\right\rangle\right| \le \left|\left\langle a(v)\bar{\mathcal{K}}^{+}_{T}\tilde{g},\ g(v)\right\rangle\right| \le \eta_T\ \|g\|_{L^2(aMdv)}\ \cdot \|\tilde{g}\|_{L^2(aMdv)}\,.$$

By showing this inequality we have proved that all the remainders of $\bar{\mathcal{K}}^{+}$ are arbitrarily small under the operator norm. Thus, by the comment before Theorem 2.4.2, the first part of the gain operator $\bar{\mathcal{K}}^{+}$ is compact from $L^2(aMdv)$ to $L^2(aMdv)$.

The proofs for the remaining three inequalities related to $\tilde{\mathcal{K}}^{+}$ are similar.

For the fourth inequality, for any $g(v), \tilde{g}(v)$ we have

$$
\begin{aligned}
&\left|\left\langle a(v)\tilde{\mathcal{K}}_\epsilon^+ g(v),\ \tilde{g}(v)\right\rangle\right| \\
\leq &\iiint_{S^{d-1}\times R^d\times R^d} |g(v_1')|\ |g(v)|\ \left(1 - I_{|v-v_1'|>\epsilon|v-v_1|}I_{|v-v'|>\epsilon|v-v_1|}\right)\, d\mu \\
\leq &\iiint_{S^{d-1}\times R^d\times R^d} |g(v_1')|\ |g(v)|\ \left(I_{|v-v_1'|<\epsilon|v-v_1|} + I_{|v-v'|<\epsilon|v-v_1|}\right)\, d\mu \\
\leq & c\left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v_1')|^2\ \ I_{|v-v_1'|<\epsilon|v-v_1|}\, d\mu\right)^{\frac{1}{2}} \cdot \|g(v)\|_{L^2(aMdv)} \\
&+ c\left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v_1')|^2\ \ I_{|v-v'|<\epsilon|v-v_1|}\, d\mu\right)^{\frac{1}{2}} \cdot \|g(v)\|_{L^2(aMdv)} \\
\leq & c\left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v)|^2\ \ I_{|\sin\theta|\leq\epsilon}\, d\mu\right)^{\frac{1}{2}} \cdot \|g(v)\|_{L^2(aMdv)} \\
&+ c\left(\iiint_{S^{d-1}\times R^d\times R^d} |g(v)|^2\ \ I_{|\cos\theta|\leq\epsilon}\, d\mu\right)^{\frac{1}{2}} \cdot \|g(v)\|_{L^2(aMdv)}.
\end{aligned}
$$

It is now evident that the smallness comes from L^1 integrability of $\hat{b}(\omega)$ and the length of the integration interval $\{\theta : |\cos\theta| < \epsilon\} \cup \{\theta : |\sin\theta| < \epsilon\}$ is arbitrarily small, the same as in the proof for the first inequality.

For the fifth inequality, again divide the region into two subregions: $\{v_1 : |v_1| > m\}$ and $\{v_1 : |v_1| < m\}$ for some m large. Over the first region the smallness comes from M_1 when $|v_1|$ is large. Over the second region, for a fixed m, using the fact that $|v| < m$, it can be deduced that $|v'|$ is large because $|v'-v_1| = |v-v_1|\sin\theta > \epsilon|v-v_1|$. Then by $MM_1 = M'M_1'$, use the same method as we estimate I_2 we can get the smallness of $\tilde{\mathcal{K}}_T^+$.

For the last inequality we again change the variables from (v, v_1) to (v_1', v')

and by using the symmetry of the measure $d\mu$. Then

$$
\begin{aligned}
&\left|\left\langle a(v)\tilde{\mathcal{K}}^{+}_{T,T} g,\, \tilde{g}\right\rangle\right| \\
&= \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v_1')\, \tilde{g}(v)\, I_{|v|>T}\, I_{|v_1'|<T}\, I_{|v-v_1'|>\epsilon|v-v_1|}\, I_{|v-v'|>\epsilon|v-v_1|}\, d\mu\right| \\
&= \left|\iiint_{S^{d-1}\times R^d\times R^d} g(v)\, \tilde{g}(v_1')\, I_{|v|<T}\, I_{|v_1'|>T}\, I_{|v-v_1'|>\epsilon|v-v_1|}\, I_{|v-v'|>\epsilon|v-v_1|}\, d\mu\right| \\
&\leq \iiint_{S^{d-1}\times R^d\times R^d} |g(v)|\; |\tilde{g}(v_1')|\; I_{|v|<T}\, I_{|v-v_1'|>\epsilon|v-v_1|}\, I_{|v-v'|>\epsilon|v-v_1|}\, d\mu.
\end{aligned}
$$

As before, the rest of the steps follow from those for the fifth inequality. Thus we finish the proof for Theorem 2.4.2. □

Combining Theorem 2.4.1 and Theorem 2.4.2 we are done with the proof for compactness of $\mathcal{K}^+ : L^2(aMdv) \to L^2(aMdv)$. Together with Theorem 2.3.1, the proof for the main theorem is now completed.

2.5 Conclusion

Based on this compactness result, it is interesting to investigate whether what have been done using more restricted cutoff assumptions can be extended to this weakly cutoff case. For example, whether the machinery developed by Guo [18] to prove the global existence of classical solutions to the Boltzmann equation can be applied to this more general collision kernel; in the setting of DiPerna-Lions renormalized solutions, many rigorous proofs of the hydrodynamic limits are done for more restricted collision kernels such as bounded kernels [15] due to the lack of Fredholm alternative of the linearized Boltzmann operator. In a recent work [23] by

Levermore and Masmoudi, applying the result we prove here, they can generalize the incompressible Navier-Stokes limit from the Boltzmann equation to collision kernel with merely the integrability assumption. We also want to mention that later on Mouhot and Strain [25] generalized the Fredholm alternative to non-cutoff kernels.

Chapter 3

Dispersive Navier-Stokes System

3.1 Introduction

In this chapter we are going to prove a local well-posedness result for the dispersive Navier-Stokes (DNS) system

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ &\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho\theta) = \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{P}, \\ &\partial_t(\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) + \nabla_x \cdot q = \nabla_x \cdot (\Sigma u + \tilde{P} u) + \nabla_x \cdot \tilde{q}, \\ &(\rho, u, \theta)(x, 0) = (\rho^{in}, u^{in}, \theta^{in}), \end{aligned} \tag{3.1}$$

where $(\rho, u, \theta)(t, x)$ are the mass density, bulk velocity and temperature of the gas respectively for $(t, x) \in [0, \infty) \times R^d$, and the constitutive relations are given as:

- $\rho e = \frac{1}{2}\rho|u|^2 + \frac{d}{2}\rho\theta$ denotes the total energy with $d \geq 2$ being the dimension of the microscopic world.
- $\Sigma = \mu(\theta)\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{d}(\nabla_x \cdot u)I\right)$ with $\mu(\theta) \geq \mu_0 > 0$ being the viscosity.
- $q = \kappa(\theta)\nabla_x\theta$ with $\kappa(\theta) \geq \kappa_0 > 0$ being the thermal conductivity.

The dispersive effect is introduced by the tensor $\tilde{P}$ and the vector $\tilde{q}$ and the structures of those two terms are shown below as

$$
\begin{aligned}
\tilde{P} \;&=\; \tau_1(\rho,\theta)\left(\nabla_x^2\theta - \left(\frac{1}{d}\Delta_x\theta\right) I\right) + \tau_2(\rho,\theta)\left(\nabla_x\theta\otimes\nabla_x\theta - \frac{1}{d}|\nabla_x\theta|^2 I\right) \\
&\quad +\tau_3(\rho,\theta)\left(\nabla_x\rho\otimes\nabla_x\theta + \nabla_x\theta\otimes\nabla_x\rho - \frac{2}{d}\nabla_x\rho\cdot\nabla_x\theta I\right), \\
\tilde{q} \;&=\; \tau_4(\rho,\theta)\left(\Delta_x u + \frac{d-2}{d}\nabla_x\nabla_x\cdot u\right) + \tau_5(\rho,\theta)\nabla_x\theta\cdot\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{d}\left(\nabla_x\cdot u\right) I\right) \\
&\quad +\tau_6(\rho,\theta)\nabla_x\rho\cdot\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{d}\left(\nabla_x\cdot u\right) I\right),
\end{aligned}
$$

where from kinetic theory $\tau_1, \tau_2, \cdots, \tau_6$ satisfy the following relations:

$$
\tau_4 = \frac{\theta}{2}\tau_1, \quad \frac{\tau_2}{\theta} + \frac{2\tau_5}{\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{\tau_4}{\theta^2}\right), \quad \theta\tau_3 + \tau_6 = 2\frac{\partial\tau_4}{\partial\rho}. \tag{3.2}
$$

The simplified model (1.39) given in the introduction corresponds to $\tau_1 = \theta$, $\tau_4 = \frac{\theta^2}{2}$, $\tau_i = 0$ for $i = 2, 3, 5, 6$. Recall for this simplified model, the dispersion has an entropy structure

$$
\tilde{P} : \frac{\nabla_x u}{\theta} + \tilde{q}\cdot\frac{\nabla_x\theta}{\theta^2} = \text{divergence}. \tag{3.3}
$$

This entropy structure is also satisfied by $\tilde{P}, \tilde{q}$ for the original DNS system by the relation (3.2). Therefore, the Euler entropy

$$
\eta = \rho\,\log\left(\frac{\rho}{\theta^{d/2}}\right)
$$

is also formally dissipated by the system (3.1).

Same as in the model system, dispersive effect is degenerate for (3.1). Calculation of $\nabla_x \cdot \tilde{q}$ in the energy equation shows that

$$\nabla_x \cdot \tilde{q} = \tfrac{2(d-1)}{d}\ \tau_4(\rho, \theta)\ \Delta_x \nabla_x \cdot u + \text{remainder}, \tag{3.4}$$

where the remainder is given by lower order terms. It is now clear from (3.4) that if we use Hodge decomposition to decompose the velocity field u into the divergence free part and the gradient part, then there is no dispersive regularization for the divergence free part.

Therefore, we have neither dissipative nor dispersive effect for the density function, there is only strict dissipation for the divergence free part of the velocity field, and there are both strict dissipation and strict dispersion for the gradient part of the velocity field and the temperature.

The above observation provides us the whole framework of the proof. The dispersive systems been treated so far in the literature are limited to those having strictly or uniformly dispersive effects. To treat the various degeneracies in the DNS system, we first use the tool of pseudodifferential operators to decouple this system into components with different smoothing effects. For the strictly dispersive part, we apply the strategy from [22] to show the local regularization. Together with the dissipation and hyperbolicity for other components, we can close the energy estimate for the whole system.

3.1.1 Dispersive Regularization

Because the dispersive regularization plays a central role in our proof of well-posedness, we give a short literature review of it.

Dispersion, by the name, means in the propagation of waves, different wave numbers will lead to different phase speeds [28]. Particularly, the dispersive relation in terms of the wave number is real with its Hessian matrix being nonsingular. For simplicity, if we consider the 1D case, then the group velocity (take the magnitude if necessary) is increasing in the wave number. Thus the waves with higher frequencies will travel to infinity faster than those with lower frequencies. Intuitively, this means if we let the initial state evolve, then for any $t > 0$, there is only slower waves left in the local region which gives a local smoothing effect for the overall profile.

Mathematically, the first result for the local smoothing effect for solutions to dispersive equations was shown by Strichartz in his seminal paper [26]. By applying the Fourier restriction theorem, he showed that for the free Schrödinger equation, there is a gain of integrability in space-time topology than the initial data. The theorem states

Theorem 3.1.1. *Let $u(t,x)$ be the solution of the free Schrödinger equation with the initial data $u_0(x) \in L^2(R^n)$,*

$$\partial_t u - i\Delta_x u = 0, \quad x \in R^n,\ t \in R$$

$$u(x,0) = u_0(x).$$

Then

$$\left(\int_{-\infty}^{\infty}\int_{R^n}|u(t,x)|^{2(n+2)/n}\,dxdt\right)^{\frac{n}{2(n+2)}} \leq C\left(\int_{R^n}|u_0(x)|^2\,dx\right)^{1/2}. \qquad (3.5)$$

The Strichartz inequality (3.5) can be generalized to $u(t,x) \in L_t^q L_x^r(R \times R^n)$ for an L^2 initial data where $2 \leq q, r \leq \infty$ satisfies the admissible condition $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q,r,n) \neq (2,\infty,2)$. Similar type of estimates are also been set up for other dispersive equations such as the Airy equation and the wave equation as well [27]. Strichartz type of estimates are crucial in studying the behavior of solutions to nonlinear dispersive equations [27].

There is another type of regularization due to dispersion that was first noticed by Kato when he was studying the 1D KdV equation [20]. By the algebraic properties of the symbol for the KdV equation and the fact that the spacial dimension is one, he showed that locally the solution of the KdV equation is one derivative smoother than the intial data. The theorem is as follows.

Theorem 3.1.2. *Let $u(t,x)$ be the solution of the Korteweg-de Vries (KdV) equation with the smooth initial data $u_0(x)$:*

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad , x, t \in R$$

$$u(x,0) = u_0(x).$$

Then for any $T, R > 0$,

$$\int_{-T}^{T} \int_{-R}^{R} |\partial_x u(t,x)|^2 \, dxdt \leq c\left(T, R, \|u_0(x)\|_{L^2(R)}\right).$$

Later on, it was shown that this is not a property restricted to the KdV equation. Various works generalized Kato's result to general dispersive equations with constant coefficients. For example, in [9], Constantin and Saut showed that if the dispersive equation is of order m, then the solution gains $\frac{m-1}{2}$ derivatives. They also showed in this paper that similar result holds for systems with constant coefficients that are strictly dispersive.

Variable coefficients and nonlinear dispersive equations are also studies by using the tool of pseudodifferential operators. For example, Craig, Kappeler, and Strauss considered a generalization of the Schrödinger equation with variable coefficients in [11]. Assuming the ellipticity of the principle operator, they can quantify the relation between the increase in smoothness of the solution with the moment property of the initial data. Similar result was gained in [10] for a 1D fully nonlinear dispersive equation. Again the strictness of the dispersion was assumed. In [22], Kenig, Ponce, and Vega showed the local regularization for a quasilinear Schrödinger equation with an elliptic operator. Based on this smoothing effect, they derived the local well-posedness of this equation. The treatment of the strictly dispersive part of the DNS system in my dissertation follows from their strategy. The main theorem states that

Statement 3. *Let $\langle x\rangle^2 = 1 + x^2$. Let $\bar{\rho}$, $\bar{\theta} > 0$ be two constants. There exists*

$N = N(d)$ *such that given any initial data* $(\rho^{in}, u^{in}, \theta^{in})$ *satisfying the non-trapping condition* $\mathcal{A}4$ *and*

$$\begin{aligned}
&\rho^{in} - \bar{\rho} \in H^{s+1}(R^d), \qquad (u^{in}, \theta^{in} - \bar{\theta}) \in H^s(R^d) \times H^s(R^d), \\
&\langle x \rangle^2 \partial_x^\beta \rho^{in} \in L^2(R^d), \quad \Big(\langle x \rangle^2 \partial_x^\alpha u^{in}, \langle x \rangle^2 \partial_x^\alpha \theta \Big) \in L^2(R^d) \times L^2(R^d), \\
&1 \leq |\beta| \leq s_1 + 1, \quad 1 \leq |\alpha| \leq s_1
\end{aligned}$$

where $s_1 \geq \frac{d}{2} + 6$, $s \geq \max\{s_1 + 6, N + d/2 + 4\}$, *there exists* $T_0 > 0$ *such that the dispersive system (1.38) has a unique solution* (ρ, u, θ) *with*

$$\begin{gathered}
\rho - \bar{\rho} \in C([0, T_0]; H^s) \cap L^\infty([0, T_0]; H^{s+1}), \\
(u,\, \theta - \bar{\theta}) \in C([0, T_0]; H^{s-1}) \cap L^\infty([0, T_0]; H^s) \cap L^2(0, T_0; H^{s+1})\,.
\end{gathered}$$

3.2 Notations and Outline of Proof

In this short section we state the general strategy for the proof in the following sections and give notations used there.

3.2.1 Outline of Proof

As standard for proving well-posedness for nonlinear PDEs, we need an energy estimate plus compactness in some appropriate space.

Approximating Sequence. To find a solution to the DNS system, first of all, we construct an sequence of approximate solutions to it. To achieve this, we regularize the system by adding an extra high order viscous term. Namely, if we use U to denote the solution, then the regularizing term is chosen as $-\epsilon\Delta_x^2 U$. Here $\epsilon > 0$ is small. For this regularized system, via the contraction mapping theorem, it's not hard to show the existence of a unique solution to this regularized system. Thus, for all $\epsilon > 0$, we obtain a sequence of solutions denoted as U^ϵ.

A Priori Estimate. In order to obtain the compactness of this approximating sequence, we establish an energy estimate which is independent of the regularization. This key estimate is done in two steps. First, we linearize the regularized DNS system and prove an L^2-type of estimate for the linear system. Next, to obtain the a priori estimate for the nonlinear system, we assume the existence of a smooth enough solution. For this solution, we show that its higher order derivatives with or without weights satisfy equations with similar structures as in the linear case. Due to the nonlinearity, the coefficients will depend on the solution itself. But it is shown that this dependence will not change with the order of derivatives. Thus Sobolev inequalities can be applied to close the energy estimate. Finally applying the linear estimate we obtain the a priori estimate needed for the nonlinear system.

Passing to the Limit. To construct a solution to the original DNS system, we let the artificial viscous term go to zero, that is, let $\epsilon \to 0$. Since the compactness of the approximating sequence has been guaranteed by the a priori estimate, this sequence will converge to a solution to the DNS system. Uniqueness is also given by the a priori estimate. Thus we finish the proof for the local well-posedness.

3.2.2 Notations

Except the dissipative term, we do not distinguish among the second order $\psi.d.o$'s and always denote them as Ψ_2. Similarly, we write Ψ_1, Ψ_0 for all first order and zeroth order operators respectively. For constants appearing in the context, we always use c with 0 in the subscript to denote those depending on the initial data, and with A in the subscript for those depending on the assumptions.

3.3 Linear Estimate

In this section and the following ones, we are going to show in detail the proof of the local well-posedness.

As mention in the outline of proof, in order to construct an approximating sequence to the DNS system, first we regularize the DNS system by adding an artificial high order viscous term to it. Because both the dissipation and the dispersion are explicitly in terms of the fluid variables (u, θ), we write the regularization is in (ρ, u, θ).

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= -\epsilon \Delta_x^2 \rho, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x (\rho \theta) &= -\epsilon \Delta_x^2 u + \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{P}, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho \theta u) + \nabla_x \cdot q &= -\epsilon \Delta_x^2 \theta + \nabla_x \cdot (\Sigma u + \tilde{P} u) + \nabla_x \cdot \tilde{q}, \\
(\rho, u, \theta)(x, 0) &= (\rho^{in}, u^{in}, \theta^{in}),
\end{aligned}
\tag{3.6}
$$

Then rewrite (3.6) in terms of (ρ, u, θ) as

$$
\begin{aligned}
&\partial_t \rho = -\epsilon \Delta_x^2 \rho - u \cdot \nabla_x \rho - \rho \nabla_x \cdot u, \\
&\partial_t u = -\epsilon \Delta_x^2 u + \frac{1}{\rho} \nabla_x \cdot \Sigma + \frac{d-1}{\rho d} \tau_1 \Delta_x \nabla_x \theta + \Psi_2(\rho, \theta) + \Psi_1(\rho, u, \theta), \\
&\partial_t \theta = -\epsilon \Delta_x^2 \theta + \frac{2}{\rho d} \nabla_x \cdot q + \frac{4(d-1)}{d^2 \rho} \tau_4 \Delta_x \nabla_x \cdot u + \Psi_2(\rho, u) + \Psi_1(\rho, u, \theta), \\
&(\rho, u, \theta)(x, 0) = (\rho^{in}, u^{in}, \theta^{in}).
\end{aligned}
\tag{3.7}
$$

Because $(\rho - \bar{\rho}, u, \theta - \bar{\theta})$ satisfies the same system as (3.7), we use (ρ, u, θ) to denote $(\rho - \bar{\rho}, u, \theta - \bar{\theta})$ and refer to (3.7) as the system for $(\rho - \bar{\rho}, u, \theta - \bar{\theta})$. In the equation for u, Ψ_2 is given by the second order terms in $\nabla_x \cdot \tilde{P}$. Therefore this is a homogenous $\psi.d.o$ of order 2. By the definition of $\tilde{P}$, the coefficients of Ψ_2 depend on $(\rho, \theta, \nabla_x \rho, \nabla_x \theta)$ and are proportional to $\nabla_x(\rho, \theta)$. Meanwhile, Ψ_1 in the equation for u is given by the convection term, the pressure term and the first order terms in $\tilde{P}$. It is homogeneous of order 1 with coefficients depending on (ρ, u, θ) and $\nabla_x(\rho, \theta)$. These coefficients can be quadratic in $\nabla_x(\rho, \theta)$.

Similarly, in the equation for θ, the homogeneous operator Ψ_2 is given by the second order terms in $\nabla_x \cdot \tilde{q}$. The coefficients of Ψ_2 depend on (ρ, θ) and $\nabla_x(\rho, u, \theta)$. Again these coefficients are proportional to $\nabla_x(\rho, u, \theta)$. Finally, Ψ_1 is given by the convection term, the pressure term, the viscous term and the first order terms in $\tilde{q}$. It is homogeneous of order 1 with coefficients depending nonlinearly on (ρ, u, θ) and $\nabla_x(\rho, u, \theta)$.

Let $V = (\varrho, v, \vartheta)$, $U = (\rho, u, \theta)$. Replace (ρ, u, θ) in the coefficients of Ψ_2, Ψ_1 by (ϱ, v, ϑ) to make these operators linear in U. Define the linear operator with variable coefficients $\mathcal{L}(V)U$ as follows.

$$\begin{aligned}\mathcal{L}(V)U &= \begin{pmatrix} \mathcal{L}_1(V)U \\ \\ \mathcal{L}_2(V)U \\ \\ \mathcal{L}_3(V)U \end{pmatrix} \\ &= \begin{pmatrix} -v \cdot \nabla_x \rho - \varrho \nabla_x \cdot u \\ \\ \frac{1}{\varrho}\nabla_x \cdot \Sigma + \frac{d-1}{\varrho d}\tau_1 \Delta_x \nabla_x \theta + \Psi_2(\rho,\theta) + \Psi_1(\rho,u,\theta) \\ \\ \frac{2}{d\varrho}\nabla_x \cdot q + \frac{4(d-1)}{d^2\varrho}\tau_4 \Delta_x \nabla_x \cdot u + \Psi_2(\rho,u) + \Psi_1(\rho,u,\theta)\,. \end{pmatrix}\end{aligned} \tag{3.8}$$

Note here we also replace (ρ, u, θ) in the transport coefficients μ, κ and in τ_1, τ_4 by (ϱ, v, ϑ). Therefore system (3.7) can be written in an abstract way as

$$\partial_t U = -\epsilon \Delta_x^2 U + \mathcal{L}(U)U, \quad U(x,0) = U^{in}(x). \tag{3.9}$$

In the first subsection we are going to state the assumptions on the initial data U^{in} and the coefficients of the linear operator $\mathcal{L}(V)$ (hence on V). The theorem for the linear estimate is also stated in this subsection. In the second subsection we are going to establish this estimate for the linear system (3.9).

3.3.1 Assumptions and Statement of the Linear Estimate

To derive the linear estimate, we need to make the following assumptions on $V = (\varrho, v, \vartheta)$, which in turn give the assumptions on the coefficients of the linear operator $\mathcal{L}(V)$. These assumptions suggest the proper functional spaces in which we can find a unique solution to system (3.1).

Assumptions.

$\mathcal{A}_1$. *Decay of the coefficients of the second order symbols.* Assume that there exist constants $c_A, T_1 > 0$ such that $\forall (x,t) \in R^d \times [0, T_1]$,

$$|\partial_t(\varrho, v, \vartheta)(t,x)| + |\nabla_x(\varrho, v, \vartheta)(t,x)| + |\partial_t \nabla_x(\varrho, v, \vartheta)(t,x)| \leq \frac{c_A}{\langle x \rangle^2}, \tag{3.10}$$

with $\langle x \rangle \triangleq (1 + |x|^2)^{\frac{1}{2}}$.

$\mathcal{A}_2$. *Regularity of the coefficients.* Assume that there exists $T_2 > 0$ such that for each $0 \leq t \leq T_2$, $(\varrho, v, \vartheta)(t,x) \in C_b^{N+4}(R^d)$ for N sufficiently large. This guarantees that the proofs involving $\psi.d.o$'s can be carried out. Again use c_A to denote the uniform bound (in t) of (ϱ, v, ϑ) in $C_b^N(R^D)$. Choose $T_3 = \min(T_1, T_2)$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are both satisfied for any $(x,t) \in R^d \times [0, T_3]$.

$\mathcal{A}_3$. *Lower bounds of the coefficients.* Assume that there exists a constant τ_0 such that $\varrho, \vartheta \geq \tau_0 > 0$.

$\mathcal{A}_4$. *Non-trapping condition.* Let $h(x,\xi) = \sqrt{\hat{\tau}_1(x,0)\hat{\tau}_4(x,0)}|\xi|^3$ and H_h be the corresponding Hamiltonian flow. Assume that H_h satisfies the non-trapping condition.

Remark 3.3.1. Notice that by inequality (3.10) , if we assume that there exists $c_0 > 0$ such that

$$|\nabla_x(\varrho^{in}, v^{in}, \vartheta^{in})| = |\nabla_x(\rho^{in}, u^{in}, \theta^{in})| \leq \frac{c_0}{2} \frac{1}{\langle x \rangle^2}, \quad \forall\, x \in R^d,$$

then we can choose $T_4 > 0$ small enough such that

$$|\nabla_x(\varrho, v, \vartheta)(t,x)| \le \frac{c_0}{\langle x\rangle^2}, \quad \forall\, (t,x) \in [0,T_2] \times R^d, \tag{3.11}$$

where T_4 depends on both c_0 and c_A, but the bound on $\nabla_x(\varrho, v, \vartheta)$ depends only on the initial data.

The same observation holds for the constant in assumption $\mathcal{A}_2$. Since essentially (ϱ, v, ϑ) is the solution of system (1.1), the time derivative of $\partial_x^\alpha(\varrho, v, \vartheta)$ for any $|\alpha| \le N+1$ is bounded by $\|(\varrho, v, \vartheta)\|_{C_b^{N+4}}$. Therefore if we assume that

$$\|(\varrho^{in}, v^{in}, \vartheta^{in})\|_{C_b^{N+1}} \le \frac{c_0}{2},$$

then there exists $T_5 > 0$ depending on c_0, c_A such that

$$\|(\varrho, v, \vartheta)(t,x)\|_{C_b^{N+1}} \le c_0, \qquad \forall\, (t,x) \in [0,T_5] \times R^d. \tag{3.12}$$

Remark 3.3.2. Let $p(t,x,\xi)$ be the symbol of Ψ_2 in the equations for u and for θ. Then p is a homogeneous second order polynomial in ξ. By $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and the above two remarks, there exists a constant $c_{0,1} > 0$ depending on c_0, τ_0 and $T_6 > 0$ depending on c_A such that

$$|p(t,x,\xi)| \le \frac{c_{0,1}}{\langle x\rangle^2}|\xi|^2, \quad \forall\, (x,t) \in R^d \times [0,T_6],$$

and every coefficient of p is in $C_b^N(R^d)$ uniformly for $t \in [0,T_6]$.

In what follows, use $T > 0$ to denote the time interval on which all the assumptions $\mathcal{A}_1 - \mathcal{A}_4$ are true.

We can now state the theorem for the linear estimate based on the above assumptions.

Theorem 3.3.1. *Suppose the coefficients of the linear operator $\mathcal{L}(V)$ defined in (3.8) satisfy the assumptions $\mathcal{A}_1 - \mathcal{A}_4$. Let $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ be a smooth solution to the system (3.9). Then there exist $c_0, T_0 > 0$ such that*

$$\begin{aligned}
&\sup_{[0,T_0]} \left(\|\rho^\epsilon - \bar{\rho}\|^2_{H^1_x} + \|(u^\epsilon, \theta^\epsilon - \bar{\theta})\|^2_{L^2_x} \right)(t) + \int_0^{T_0} \|\nabla_x (u^\epsilon, \theta^\epsilon)\|^2_{L^2_x}(s)\, ds \\
\leq\, & c_0 \left(\|\rho^{in} - \rho\|^2_{H^1_x} + \|(u^{in}, \theta^{in} - \bar{\theta})\|^2_{L^2_x} \right),
\end{aligned}$$

where c_0 depends only on the initial data and τ_0, while T_0 depends on c_A, τ_0.

3.3.2 Linear Estimate

In this section we establish the linear estimate for solutions to the DNS system with artificial viscosities. As shown in Theorem 1, this estimate is performed in the space $(\rho, u, \theta)(t, \cdot) \in H^1 \times L^2 \times L^2$ for each t. The local smoothing effect from the dispersion is illustrated by this estimate. It will be clear that the artificial viscosities do not contribute to the proof. Therefore the same linear estimate holds for the original DNS system.

The strategy for proving the linear estimate is that we first study the subsystem given by the uniformly non-degenerate dispersive components of the solution. Combining the dispersive regularization effect gained for these components with dissipation and hyperbolicity for other components, we derive the linear energy estimate for the regularized DNS system. The proof is divided into six steps.

Step1. Decomposition of u. Since Qu is the gradient part of the velocity field u, there exists a scalar function ϕ such that $Qu = \nabla_x \phi$. It will be shown below that in the Fourier space essentially it is the component of $\hat{u}(\xi)$ along the ξ-direction that has a non-degenerate dispersive effect. Therefore we are interested in studying the regularity of ϕ.

To this end, let $\theta_R(\xi) \in C^\infty(R^d)$ be a cutoff function such that $\theta_R(\xi) = 1$ for $|\xi| > 2R$, $\theta_R(\xi) = 0$ for $|\xi| < R$ and $0 \le \theta_R(\xi) \le 1$ otherwise. Let

$$p_{0,k}(\xi) = -i\frac{\xi_k}{|\xi|}, \quad \psi_R(\xi) = 1 - \theta_R(\xi), \quad k = 1, 2, \ldots, d.$$

Let $\Psi_{p_{0,k}}$, Ψ_{ψ_R} be the corresponding $\psi.d.o$'s. Write $\sum_{k=1}^d \Psi_{p_{0,k}} u_k$ in short as $\Psi_{p_0} Qu$.

Decompose the velocity field u as

$$u = \Psi_{\theta_R} Qu + \Psi_{\theta_R} Pu + \Psi_{\psi_R} u.$$

By Plancherel's theorem, it is clear that for any real number s,

$$\|\Psi_{\theta_R} Qu\|_{H^s(R^d)} = \|\Psi_{\theta_R} \Psi_{p_0} u\|_{H^s(R^d)}.$$

In addition, it is obvious that

$$\begin{aligned}\|u\|_{H^s(R^d)} \leq &\|\Psi_{\theta_R}\Psi_{p_0}u\|_{H^s(R^d)} + \|\Psi_{\theta_R}Pu\|_{H^s(R^d)} + \|\Psi_{\psi_R}u\|_{H^s(R^d)}\\ \leq &3\|u\|_{H^s(R^d)}.\end{aligned}$$

Furthermore, for any real number s the following inequality is true for the weighted Sobolev spaces $H^s\left(\langle x\rangle^2 dx\right)$, that is, there exists a $c_s > 0$ depending only on s such that

$$\begin{aligned}\|u\|_{H^s(\langle x\rangle^2 dx)} \leq &\|\Psi_{\theta_R}\Psi_{p_0}u\|_{H^s(\langle x\rangle^2 dx)} + \|\Psi_{\theta_R}Pu\|_{H^s(\langle x\rangle^2 dx)} + \|\Psi_{\psi_R}u\|_{H^s(\langle x\rangle^2 dx)}\\ \leq &c_s\|u\|_{H^s(\langle x\rangle^2 dx)},\end{aligned}$$

by the fact that in the Fourier space the weight $\langle x\rangle^2$ becomes the second derivative on ξ.

Since $H^s(dx)$ and $H^s\left(\langle x\rangle^2 dx\right)$ are all the functional spaces we consider, these equivalences of the norms justify that we need only to study the behavior of the components of u, that is, $\Psi_{\theta_0}\Psi_{p_0}u$, $\Psi_{\theta_R}Pu$ and $\Psi_{\psi_R}u$.

Apply these three operators to the equation for u, and work out the equations for these components respectively.

The equation for $\Psi_{\theta_R}\Psi_{p_0}u$ shows that

$$\begin{aligned}\partial_t(\Psi_{\theta_R}\Psi_{p_0}u) = &-\epsilon\Delta_x^2(\Psi_{\theta_R}\Psi_{p_0}u) + \Psi_{\theta_R}\Psi_{p_0}\left(\tfrac{1}{\varrho}\nabla_x\cdot\Sigma\right)\\ &+ \Psi_{\theta_R}\Psi_{p_0}\left(\tfrac{1}{\varrho}\nabla_x\cdot\tilde{P}\right) + \Psi_1(\rho,u,\theta),\end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
\Psi_1(\rho,u,\theta) = & -\Psi_{\theta_0}\Psi_{p_0}\left(v\cdot\nabla_x u + \tfrac{\vartheta}{\varrho}\nabla_x\rho + \nabla_x\theta\right) \\
& = \Psi_1\rho + \Psi_1 u + \Psi_1\theta \\
& = \Psi_1\rho + \Psi_1\Psi_{\theta_R}Qu + \Psi_1\Psi_{\theta_R}Pu + \Psi_1\Psi_{\psi_R}u + \Psi_1\theta \\
& = \Psi_1\rho + \Psi_1\left(\Psi_{\theta_R}\Psi_{p_0}u\right) + \Psi_1\left(\Psi_{\theta_R}Pu\right) + \Psi_1\theta + \Psi_0\left(\Psi_{\psi_R}u\right).
\end{aligned}
$$

Notice that we use the following fact in the above calculation:

$$
\text{the symbol of } \Psi_{\theta_R}Q = \theta_R\frac{\xi\otimes\xi}{|\xi|^2}\cdot = \theta_R\frac{i\xi}{|\xi|}\left(\theta_R p_0\right),
$$

that is,

$$
\Psi_{\theta_R}Qu = \Psi_{E^0}\left(\Psi_{\theta_R}\Psi_{p_0}u\right).
$$

Compute the second and third terms on the right hand side of (3.13). We have

$$
\begin{aligned}
& \Psi_{\theta_R}\Psi_{p_0}\left(\tfrac{1}{\varrho}\nabla_x\cdot\Sigma\right) \\
= & \Psi_{\theta_R}\Psi_{p_0}\left(\tfrac{\mu}{\varrho}\left(\Delta_x + \tfrac{d-2}{d}\nabla_x\nabla_x\cdot\right)u\right) + \Psi_{\theta_R}\Psi_{p_0}\Psi_1(u) \\
= & \frac{\mu}{\varrho}\Psi_{\theta_R}\Psi_{p_0}\left(\Delta_x + \tfrac{d-2}{d}\nabla_x\nabla_x\cdot\right)u + \Psi_1 u \\
= & \tfrac{2(d-1)}{d}\tfrac{\mu}{\varrho}\Delta_x\left(\Psi_{\theta_R}\Psi_{p_0}u\right) + \Psi_1\left(\Psi_{\theta_R}\Psi_{p_0}u\right) + \Psi_1\left(\Psi_{\theta_R}Pu\right) + \Psi_0\left(\Psi_{\psi_R}u\right),
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
&\Psi_{\theta_R}\Psi_{p_0}\left(\tfrac{1}{\varrho}\nabla_x\cdot\tilde{P}\right)\\
&=\Psi_{\theta_R}\Psi_{p_0}\left(\tfrac{d-1}{d}\tfrac{\tau_1}{\varrho}\nabla_x\Delta_x\theta\right)+\Psi_2(\rho,\theta)+\Psi_1(\rho,\theta)\\
&=-\tfrac{d-1}{d}\tfrac{\tau_1}{\varrho}\Psi_{\theta_R}(-\Delta_x)^{\frac{3}{2}}\theta+\Psi_{E^2}(\rho,\theta)+\Psi_1(\rho,\theta)\\
&=-\tfrac{d-1}{d}\tfrac{\tau_1}{\varrho}(-\Delta_x)^{\frac{3}{2}}\theta+\Psi_2(\rho,\theta)+\Psi_1(\rho,\theta)+\Psi_0\theta.
\end{aligned}
$$

Overall, the equation for $\Psi_{\theta_R}\Psi_{p_0}u$ is written as

$$
\begin{aligned}
&\partial_t\left(\Psi_{\theta_R}\Psi_{p_0}u\right)\\
&=-\epsilon\Delta_x^2(\Psi_{\theta_R}\Psi_{p_0}u)+\tfrac{2(d-1)}{d}\tfrac{\mu}{\varrho}\Delta_x\left(\Psi_{\theta_R}\Psi_{p_0}u\right)-\tfrac{d-1}{d}\tfrac{\tau_1}{\varrho}(-\Delta_x)^{\frac{3}{2}}\theta\\
&\quad+\Psi_2(\rho,\theta)+\Psi_1\left(\rho,\ \Psi_{\theta_R}\Psi_{p_0}u,\ \Psi_{\theta_R}Pu,\ \Psi_{\psi_R}u,\theta\right).
\end{aligned}
\tag{3.15}
$$

where as stated before, the symbols of the second order operators $\Psi_2(\varrho,\vartheta)$ are homogeneous of order two and proportional to $\nabla_x(\varrho,\vartheta)$, while the norms of the first and zeroth order operators depend on finitely many derivatives of (ϱ,v,ϑ).

Similarly, apply the operator $\Psi_{\theta_R}P$ to the equation for the velocity field u to obtain the equation for the component $\Psi_{\theta_R}Pu$.

$$
\begin{aligned}
\partial_t(\Psi_{\theta_R}Pu)=&-\epsilon\Delta_x^2(\Psi_{\theta_R}Pu)+\Psi_{\theta_R}P\left(\frac{1}{\varrho}\nabla_x\cdot\Sigma\right)\\
&+\Psi_{\theta_R}P\left(\frac{1}{\varrho}\nabla_x\cdot\tilde{P}\right)+\Psi_1(\rho,u,\theta),
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
\Psi_1(\rho, u, \theta) = &- \Psi_{\theta_0} P \left(v \cdot \nabla_x u + \frac{\vartheta}{\varrho} \nabla_x \rho + \nabla_x \theta \right) \\
&= \Psi_1 u + \Psi_0(\rho, \theta) \\
&= \Psi_1 \left(\Psi_{\theta_R} \Psi_{p_0} u \right) + \Psi_1 \left(\Psi_{\theta_R} P u \right) + \Psi_0 \left(\rho,\ \Psi_{\psi_R} u,\ \theta \right).
\end{aligned}
$$

Similar calculations show that

$$
\begin{aligned}
\Psi_{\theta_R} P \left(\frac{1}{\varrho} \nabla_x \cdot \Sigma \right) &= \Psi_{\theta_R} P \left(\frac{\mu}{\varrho} \left(\Delta_x + \frac{d-2}{d} \nabla_x \nabla_x \cdot \right) u \right) + \Psi_{\theta_R} P \Psi_1 u \\
&= \frac{\mu}{\varrho} \Psi_{\theta_R} P \left(\Delta_x + \frac{d-2}{d} \nabla_x \nabla_x \cdot \right) u + \Psi_1 u \\
&= \frac{\mu}{\varrho} \Delta_x \left(\Psi_{\theta_R} P u \right) + \Psi_1 \left(\Psi_{\theta_R} \Psi_{p_0} u, \Psi_{\theta_R} P u \right) + \Psi_0 \left(\Psi_{\psi_R} u \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_{\theta_R} P \left(\frac{1}{\varrho} \nabla_x \cdot \tilde{P} \right) &= \Psi_{\theta_R} P \left(\frac{d-1}{d} \frac{\tau_1}{\varrho} \nabla_x \Delta_x \theta \right) + \Psi_2 \theta + \Psi_2 \rho + \Psi_1 \theta + \Psi_1 \rho \\
&= \Psi_2(\rho, \theta) + \Psi_1(\rho, \theta)
\end{aligned}
$$

Notice that in the equality for $\Psi_{\theta_R} P \left(\frac{1}{\varrho} \nabla_x \cdot \tilde{P} \right)$, we use the fact

$$
\Psi_{\theta_R} P \left(\nabla_x \left(\frac{d-1}{d} \frac{\tau_1}{\varrho} \Delta_x \theta \right) \right) = 0.
$$

Therefore the equation for $\Psi_{\theta_R} P u$ is written as

$$
\begin{aligned}
\partial_t \left(\Psi_{\theta_R} P u \right) = &-\epsilon \Delta_x^2 \left(\Psi_{\theta_R} P u \right) + \frac{\mu}{\varrho} \Delta_x \left(\Psi_{\theta_R} P u \right) + \Psi_2(\rho, \theta) \\
&+ \Psi_1 \left(\rho,\ \Psi_{\theta_R} \Psi_{p_0} u,\ \Psi_{\theta_R} P u,\ \Psi_{\psi_R} u, \theta \right) + \Psi_0(\rho, \Psi_{\psi_R} u, \theta).
\end{aligned}
\tag{3.17}
$$

Finally, by applying the operator Ψ_{ψ_R} to the equation for u, we obtain the equation for the third component $\Psi_{\psi_R} u$ as

$$\partial_t \left(\Psi_{\psi_R} u\right) = \Psi_0 \left(\rho,\ \Psi_{\theta_R} \Psi_{p_0} u,\ \Psi_{\theta_R} P u,\ \Psi_{\psi_R} u, \theta\right). \tag{3.18}$$

Combine the mass equation, (3.15), (3.17), (3.18) and the equation for the temperature θ.

The complete system is written as

$$\begin{aligned}
\partial_t \rho =& - \epsilon \Delta_x^2 \rho - \nabla_x \cdot (\rho u) \\
\partial_t \left(\Psi_{\theta_R} P u\right) =& - \epsilon \Delta_x^2 \left(\Psi_{\theta_R} P u\right) + \frac{\mu}{\varrho} \Delta_x \left(\Psi_{\theta_R} P u\right) + \Psi_2(\rho, \theta) \\
& + \Psi_1 \left(\rho,\ \Psi_{\theta_R} P u,\ \Psi_{\theta_R} \Psi_{p_0} u,\ \theta\right) + \Psi_0(\rho, \Psi_{\psi_R} u, \theta) \\
\partial_t \left(\Psi_{\theta_R} \Psi_{p_0} u\right) =& - \epsilon \Delta_x^2 \left(\Psi_{\theta_R} \Psi_{p_0} u\right) + \frac{2(d-1)}{d} \frac{\mu}{\varrho} \Delta_x \left(\Psi_{\theta_R} \Psi_{p_0} u\right) - \frac{d-1}{d} \frac{\tau_1}{\varrho} (-\Delta_x)^{\frac{3}{2}} \theta \\
& + \Psi_2(\rho, \theta) + \Psi_1 \left(\rho, \Psi_{\theta_R} P u,\ \Psi_{\theta_R} \Psi_{p_0} u,\ \theta\right) + \Psi_0 \left(\Psi_{\psi_R} u\right) \\
\partial_t \left(\Psi_{\psi_R} u\right) =& \Psi_0 \left(\rho,\ \Psi_{\theta_R} \Psi_{p_0} u,\ \Psi_{\theta_R} P u,\ \Psi_{\psi_R} u, \theta\right) \\
\partial_t \theta =& - \epsilon \Delta_x^2 \theta + \frac{2}{d} \frac{\kappa}{\varrho} \Delta_x \theta + \frac{4(d-1)}{d^2} \frac{\tau_4}{\varrho} (-\Delta_x)^{\frac{3}{2}} \left(\Psi_{\theta_R} \Psi_{p_0} u\right) \\
& + \Psi_2 \left(\rho,\ \Psi_{\theta_R} P u,\ \Psi_{\theta_R} \Psi_{p_0} u\right) \\
& + \Psi_1 \left(\rho,\ \Psi_{\theta_R} P u,\ \Psi_{\theta_R} \Psi_{p_0} u,\ \theta\right) + \Psi_0 \left(\Psi_{\psi_R} u\right) \\
(\rho, u, \theta)(x, 0) =& (\rho^{in}, u^{in}, \theta^{in}),
\end{aligned} \tag{3.19}$$

where the symbols of the second order operators $\Psi_2(\varrho, \theta)$ are homogeneous of order two and proportional to $\nabla_x(\varrho, v, \vartheta)$, while the norms of the first and zeroth order operators depend on finitely many derivatives of (ϱ, v, ϑ).

System (3.19) shows that dispersion occurs only for $\Psi_{\theta_R}\Psi_{p_0}u$ and θ. This validates the statement that the DNS system is degenerate in dispersion. Compared with other components, it is reasonable to expect extra smoothing effect for those two strictly dispersed terms. To see this dispersive smoothing effect, we study the subsystem for $(\Psi_{\theta_R}\Psi_{p_0}u,\ \theta)$ first.

To make notations simple, let

$$\begin{aligned}
&\vec{\omega} = (\omega_1, \omega_2)^T \triangleq \left(\Psi_{\theta_R}\Psi_{p_0}u,\ \theta\right)^T, \\
&\vec{\eta} = (\eta_1, \vec{\eta_2})^T \triangleq \left(\rho,\ \Psi_{\theta_R}Pu\right)^T, \quad \vec{\zeta} \triangleq \Psi_{\psi_R}u, \\
&\hat{\tau}_1 = \frac{d-1}{d}\frac{\tau_1}{\varrho}, \quad \hat{\tau}_4 = \frac{4(d-1)}{d^2}\frac{\tau_4}{\varrho}
\end{aligned}$$

Therefore, the system for $\vec{\omega}$ is written as

$$\partial_t \vec{\omega} = -\epsilon\Delta_x^2 \vec{\omega} + \Psi_D \vec{\omega} + \Psi_{L_0} \vec{\omega} + \Psi_{B_0} \vec{\omega} + \Psi_2 \vec{\eta} + \Psi_1 \left(\vec{\eta}, \vec{\omega}\right) + \Psi_0 \vec{\zeta}, \tag{3.20}$$

where

$$\Psi_D \vec{\omega} = \begin{pmatrix} \frac{2(d-1)}{d}\frac{\mu}{\varrho}\Delta_x\omega_1 \\ \\ \frac{2}{d}\frac{\kappa}{\varrho}\Delta_x\omega_2 \end{pmatrix},$$

$$\Psi_{L_0} \vec{\omega} = \begin{pmatrix} -\frac{d-1}{d}\frac{\tau_1}{\varrho}(-\Delta_x)^{\frac{3}{2}}\omega_2 \\ \\ \frac{4(d-1)}{d^2}\frac{\tau_4}{\varrho}(-\Delta_x)^{\frac{3}{2}}\omega_1 \end{pmatrix}, \quad \Psi_{B_0} \vec{\omega} = \begin{pmatrix} \Psi_2\omega_2 \\ \\ \Psi_2 \vec{\omega} \end{pmatrix}.$$

The corresponding symbols of Ψ_D and Ψ_{L_0} are

$$D = \begin{pmatrix} -\frac{2(d-1)}{d}\frac{\mu}{\varrho}|\xi|^2 & 0 \\ 0 & -\frac{2}{d}\frac{\kappa}{\varrho}|\xi|^2 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & -\hat{\tau}_1|\xi|^3 \\ \hat{\tau}_4|\xi|^3 & 0 \end{pmatrix},$$

while repeatedly, B_0, the symbol of Ψ_{B_0}, is homogeneous of order two with coefficients proportional to $\nabla_x(\varrho, v, \vartheta)$. Therefore, by the assumptions for (ϱ, v, ϑ), there exists $c_{0,2} > 0$ depending on c_0, τ_0 such that B_0 satisfies the following condition.

$$|B_0(t,x,\xi)| \leq \frac{c_{0,2}|\xi|^2}{\langle x\rangle^2}, \qquad \forall\,(t,x,\xi) \in [0,T] \times R^d \times R^d. \tag{3.21}$$

The same condition is satisfied by the symbol of Ψ_{E^2} in (3.17), the equation for $\Psi_{\theta_R}Pu$.

Step2. Diagonalization of Ψ_{L_0}. This step is to diagonalize L_0 to make the dispersion explicit. Obviously L_0 has two eigenvalues $\lambda_\pm = \pm i\sqrt{\hat{\tau}_1\hat{\tau}_4}|\xi|^3$.

Introduce the matrix S and its inverse as follows

$$S = \begin{pmatrix} -i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \\ i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \end{pmatrix}, \quad S^{-1} = \frac{1}{2}\begin{pmatrix} i & -i \\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1} \end{pmatrix}.$$

Note that due to the specific structures of $(\hat{\tau}_1, \hat{\tau}_4)$, matrix S (therefore S^{-1}) depend on θ only. Each entry of S and S^{-1} is a zeroth order $\psi.d.o.$ To be specific, since $S = S(t,x)$, the corresponding operator Ψ_S is just a multiplication, that is, $\Psi_S \vec{\omega} = S\vec{\omega}$. Furthermore, as an operator, S is invertible on $H^s(dx), H^s(\langle x\rangle^2 dx)$ and $H^s(\langle x\rangle^{-2}dx)$ for all s by $\mathcal{A}2$ and $\mathcal{A}3$.

By choosing such an S we have

$$SL_0 = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} S \triangleq LS.$$

To diagonalize L_0, multiply S to (1.10). Define $\vec{\beta} \triangleq S\vec{\omega}$.

$$\begin{aligned}\partial_t \vec{\beta} &= \partial_t(S\vec{\omega}) \\ &= -\epsilon S\Delta_x^2 \vec{\omega} + S\Psi_D \vec{\omega} + S\Psi_{L_0} \vec{\omega} + S\Psi_{B_0} \vec{\omega} \\ &\quad + S\Psi_2 \vec{\eta} + S\Psi_1 \left(\vec{\eta}, \vec{\omega}\right) + (\partial_t S) \vec{\omega} + S\Psi_0 \vec{\zeta}\ .\end{aligned} \tag{3.22}$$

To obtain a system for $\vec{\beta}$, we need to study each of the above terms respectively. First,

$$\epsilon S\Delta_x^2 \vec{\omega} = \epsilon\Delta_x^2 S\vec{\omega} + \epsilon(\Psi_{R_1}S^{-1})S\vec{\omega} \triangleq \epsilon\Delta_x^2 \vec{\beta} + \epsilon(\Psi_{R_2})\vec{\beta},$$

where Ψ_{R_1}, hence Ψ_{R_2}, are third order $\psi.d.o$'s with seminorms bounded by the constants c_A, τ_0.

Next we have

$$S\Psi_{B_0} \vec{\omega} = S\Psi_{B_0}S^{-1} \vec{\beta}\ .$$

Since both S and S^{-1} are both of zeroth order, $S\Psi_{B_0}S^{-1}$ is still a second order operator and we use Ψ_{B_1} to denote the highest order part of this operator, that is, $\Psi_{B_1} - S\Psi_{B_0}S^{-1}$ is a first order operator and $B_1(t, x, \xi)$ is homogeneous of order 2 in ξ. Notice that by assumptions $\mathcal{A}_1 - \mathcal{A}_3$, there exists $c_{0,3} > 0$ such that B_1 satisfies the following condition,

$$|B_1(x,t,\xi)| \le \frac{c_{0,3}|\xi|^2}{\langle x\rangle^2}, \qquad \forall\,(x,t,\xi) \in R^d \times [0,T] \times R^d. \tag{3.23}$$

By the same token, we can write $S\Psi_2 \overrightarrow{\eta} + S\Psi_1 \left(\overrightarrow{\eta}, \overrightarrow{\omega}\right)$ as

$$S\Psi_2 \overrightarrow{\eta} + S\Psi_1 \left(\overrightarrow{\eta}, \overrightarrow{\omega}\right) = \Psi_2 \overrightarrow{\eta} + \Psi_1(\overrightarrow{\eta}, \overrightarrow{\beta}), \quad S\Psi_0 \overrightarrow{\zeta} = \Psi_0 \overrightarrow{\zeta},$$

where Ψ_2 is a second order operator with its symbol homogeneous of order 2 in ξ and satisfying the same inequality as (3.23). At the same time, by the assumptions of (ϱ, v, ϑ), $\partial_t S$ is a zeroth order operator. Therefore,

$$(\partial_t S) \overrightarrow{\omega} = (\partial_t S) S^{-1} \overrightarrow{\beta} = \Psi_0 \beta.$$

For the dissipative term, we have

$$\begin{aligned}
S\Psi_D S^{-1} =& \frac{1}{2} \begin{pmatrix} -i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \\ i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \end{pmatrix} \begin{pmatrix} \frac{2(d-1)}{d}\frac{\mu}{\varrho}\Delta_x & 0 \\ 0 & \frac{2}{d}\frac{\kappa}{\varrho}\Delta_x \end{pmatrix} \begin{pmatrix} i & -i \\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1} \end{pmatrix} \\
=& \begin{pmatrix} \left(\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x & 0 \\ 0 & \left(\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x \end{pmatrix} \\
&+ \begin{pmatrix} 0 & \left(-\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x \\ \left(-\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x & 0 \end{pmatrix} + \Psi_{r_1}
\end{aligned}$$

with $r_1 \in S^1$. Therefore,

$$S\Psi_D \vec{\omega} = \left(\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x \vec{\beta}$$
$$+ \begin{pmatrix} 0 & \left(-\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x \\ \left(-\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x & 0 \end{pmatrix} \vec{\beta} + \Psi_{r_1} S^{-1} \vec{\beta}.$$

Notice that although the second term on the right side of the above equation is of second order, there is no contribution from the diagonal. Combine this term with Ψ_{B_1}, and use Ψ_{B_2} to denote this new second order operator. Obviously the diagonal terms of B_2 satisfy the same property as B_1 in (3.23).

Next we study the structure of $S\Psi_{L_0}$. Using the fact that $SL_0 = LS$, we have

$$S\Psi_{L_0} = \Psi_L S + (\Psi_{SL_0} - \Psi_{LS}) + (\Psi_{LS} - \Psi_L S) = \Psi_L S + (\Psi_{LS} - \Psi_L S).$$

We claim that $\Psi_{LS} - \Psi_L S$ is a second order operator. This is shown by the following calculation.

$$\Psi_{LS} - \Psi_L S = \begin{pmatrix} \sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} & i\sqrt{\hat{\tau}_1\hat{\tau}_4}\sqrt{\hat{\tau}_1/\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} \\ \sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} & -i\sqrt{\hat{\tau}_1\hat{\tau}_4}\sqrt{\hat{\tau}_1/\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} \end{pmatrix}$$
$$- \begin{pmatrix} \sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} & i\sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}}\sqrt{\hat{\tau}_1/\hat{\tau}_4} \\ \sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} & -i\sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}}\sqrt{\hat{\tau}_1/\hat{\tau}_4} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & i\sqrt{\hat{\tau}_1\hat{\tau}_4}\sqrt{\hat{\tau}_1/\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} - i\sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}}\sqrt{\hat{\tau}_1/\hat{\tau}_4} \\ 0 & -i\sqrt{\hat{\tau}_1\hat{\tau}_4}\sqrt{\hat{\tau}_1/\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}} + i\sqrt{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{\frac{3}{2}}\sqrt{\hat{\tau}_1/\hat{\tau}_4} \end{pmatrix}.$$

The nonzero terms in the above matrix are commutators of a zeroth order operator and a third order operator. Therefore each entry of $\Psi_{LS} - \Psi_L S$ is of second order. Let B_3 be the symbol matrix corresponding to the leading order of $\Psi_{LS} - \Psi_L S$. By the calculus of $\psi.d.o$'s and assumptions $\mathcal{A}1 - \mathcal{A}3$, there exists $c_{0,4} > 0$ such that B_3 satisfies the following inequality.

$$|B_3(x,t,\xi)| \leq \frac{c_{0,4}|\xi|^2}{\langle x\rangle^2}, \qquad \forall\,(x,t,\xi) \in R^d \times [0,T_2] \times R^d.$$

Combine Ψ_{B_2} with Ψ_{B_3}, and use Ψ_B to denote this second order operator. Then the diagonal of B satisfies that

$$|B_{diag}(x,t,\xi)| \leq \frac{c_{0,5}|\xi|^2}{\langle x\rangle^2}, \qquad \forall\,(x,t,\xi) \in R^d \times [0,T] \times R^d. \tag{3.24}$$

Overall the system for $\vec{\beta}$ is written as

$$\begin{aligned} \partial_t \vec{\beta} =& -\epsilon\Delta_x^2 \vec{\beta} + \epsilon\Psi_{R_2} \vec{\beta} + \left(\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{\kappa}{\varrho d}\right)\Delta_x \vec{\beta} + \Psi_L \vec{\beta} + \Psi_B \vec{\beta} \\ &+ \Psi_2 \vec{\eta} + \Psi_1(\vec{\eta}, \vec{\beta}) + \Psi_0(\vec{\zeta}, \vec{\beta}). \end{aligned} \tag{3.25}$$

<u>*Step3. Diagonalization of Ψ_B.*</u> In this step we continue to diagonalize the main parts of the system (3.25) so that we can decouple β_1, β_2. From (3.25) it is clear that both dissipation and dispersion parts for $\vec{\beta}$ have been diagonalized. The only

term left is Ψ_B. To this end, write

$$\Psi_B = \Psi_{B_{diag}} + \Psi_{B_{anti}} = \begin{pmatrix} \Psi_{B_{11}} & 0 \\ 0 & \Psi_{B_{22}} \end{pmatrix} + \begin{pmatrix} 0 & \Psi_{B_{12}} \\ \Psi_{B_{21}} & 0 \end{pmatrix}.$$

We will show that by suitable transformation, $\Psi_{B_{anti}}$ is essentially canceled out by terms from Ψ_L. The cancelation is based on the observation that the second order off-diagonal terms can be recovered from the dispersion terms for the corresponding variables by multiplying operators of order -1 to these equations. This technique will be used again in estimates for ρ.

Let

$$h(t,x,\xi) = \sqrt{\hat{\tau}_1\hat{\tau}_4}|\xi|^3, \qquad \tilde{h}(t,x,\xi) = h^{-1}(t,x,\xi)\theta_R(\xi),$$

where $0 \leq \theta_R(\xi) \leq 1$ is again the C^∞ cutoff function. Then $\tilde{h} \in S^{-3}$ uniformly in t and $\Psi_{\tilde{h}}\Psi_h = I + \Psi_{r_2}$ with $r_2 \in S^{-1}$ uniformly in t. Define

$$T_{12} = i\frac{1}{2}\Psi_{B_{12}}\Psi_{\tilde{h}}, \quad T_{21} = -i\frac{1}{2}\Psi_{B_{21}}\Psi_{\tilde{h}}, \quad T = \begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix},$$

and the diagonalizing transform Λ of order 0

$$\Lambda = I - T.$$

Note that since T is of order -1, its S^0 seminorm is of order $O(1/R)$. Therefore by taking R large enough one can assume that Λ is invertible on $H^s(dx), H^s(\langle x\rangle^2 dx)$ and $H^s(\langle x\rangle^{-2}dx)$ with operator norm between $1/2$ and 2. Also the inverse of Λ is of order 0 with operator norm between $1/2$ and 2.

The transformation Λ acting on system (3.25) shows the following computation. First,

$$\begin{aligned}\epsilon\Lambda\Delta_x^2 + \epsilon\Lambda\Psi_{R_2} &= \epsilon\Delta_x^2\Lambda + \epsilon\left(\Lambda\Delta_x^2 - \Delta_x^2\Lambda\right)\Lambda^{-1}\Lambda + \epsilon\left(\Lambda\Psi_{R_2}\Lambda^{-1}\right)\Lambda \\ &= \epsilon\Delta_x^2\Lambda + \epsilon\Psi_{R_3}\Lambda,\end{aligned}$$

with $\Psi_{R_3} = \left(\Lambda\Delta_x^2 - \Delta_x^2\Lambda\right)\Lambda^{-1} + \Lambda\Psi_{R_2}\Lambda^{-1}$ being a third order $\psi.d.o$, for it's easy to see that $\left(\Lambda\Delta_x^2 - \Delta_x^2\Lambda\right)\Lambda^{-1}$ is a second order $\psi.d.o$. The seminorms of Ψ_{R_3} depend on the constants c_A, τ_0.

Second,

$$\Lambda\partial_t \vec{\beta} = \partial_t(\Lambda \vec{\beta}) - (\partial_t\Lambda)\Lambda^{-1}\Lambda \vec{\beta},$$

where $(\partial_t\Lambda)\Lambda^{-1}$ is a 0'th order operator.

Next, since the symbol of T is in S^{-1}, and from

$$\Lambda\Psi_{B_{diag}} - \Psi_{B_{diag}}\Lambda = -T\Psi_{B_{diag}} + \Psi_{B_{diag}}T,$$

it's clear that $\Lambda\Psi_{B_{diag}} = \Psi_{B_{diag}}\Lambda + \Psi_{E^1}\Lambda$.

Similarly,

$$\begin{aligned}
&\Lambda\Psi_{B_{anti}} = \Psi_{B_{anti}}\Lambda + \Psi_1\Lambda, \\
&\Lambda\left((1-\tfrac{1}{d})\frac{\mu}{\varrho d} + \frac{2\kappa}{\varrho d}\right)\Delta_x I = \left((1-\frac{1}{d})\frac{\mu}{\varrho d} + \frac{2\kappa}{\varrho d}\right)\Delta_x \Lambda I + \Psi_1\Lambda, \\
&\Lambda\Psi_2 \overrightarrow{\eta} = \Psi_2 \overrightarrow{\eta}, \qquad \Lambda\Psi_1\left(\overrightarrow{\eta}, \overrightarrow{\beta}\right) = \Psi_1\left(\overrightarrow{\eta}, \Lambda\overrightarrow{\beta}\right), \\
&\Lambda\Psi_0\left(\overrightarrow{\zeta}, \overrightarrow{\beta}\right) = \Psi_0\left(\overrightarrow{\zeta}, \Lambda\overrightarrow{\beta}\right) .
\end{aligned}$$

Now check the term $\Lambda\Psi_L - \Psi_L\Lambda$.

$$\begin{aligned}
&\Lambda\Psi_L - \Psi_L\Lambda = \Psi_L T - T\Psi_L \\
&= i\begin{pmatrix} \Psi_h & 0 \\ 0 & -\Psi_h \end{pmatrix}\begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix} - i\begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix}\begin{pmatrix} \Psi_h & 0 \\ 0 & -\Psi_h \end{pmatrix} \\
&= i\begin{pmatrix} 0 & \Psi_h T_{12} + T_{12}\Psi_h \\ -[\Psi_h T_{21} + T_{21}\Psi_h] & 0 \end{pmatrix}.
\end{aligned}$$

By the fact that $\Psi_h T_{12} = T_{12}\Psi_h + \Psi_1$, we have

$$\begin{aligned}
i(\Psi_h T_{12} + T_{12}\Psi_h) &= 2iT_{12}\Psi_h + \Psi_1 = -\Psi_{B_{12}} + \Psi_1, \\
-i(\Psi_h T_{21} + T_{21}\Psi_h) &= -2iT_{21}\Psi_h + \Psi_1 = -\Psi_{B_{21}} + \Psi_1.
\end{aligned}$$

Therefore,

$$\Lambda\Psi_L + \Lambda\Psi_{B_{anti}} = \Psi_L\Lambda + \Psi_1 = \Psi_L\Lambda + \Psi_1\Lambda.$$

Let $\vec{z} = \Lambda \vec{\beta}$. Then the system for $\vec{z}$ is written as

$$\begin{aligned} \partial_t \vec{z} = &- \epsilon \Delta_x^2 \vec{z} + \epsilon \Psi_{R_3} \vec{z} + \left(\frac{d-1}{d} \frac{\mu}{\varrho} + \frac{2}{d} \frac{\kappa}{\varrho} \right) \Delta_x \vec{z} + \Psi_L \vec{z} + \Psi_{B_{diag}} \vec{z} \\ &+ \Psi_2 \vec{\eta} + \Psi_1 (\vec{\eta}, \vec{z}) + \Psi_0 (\vec{\zeta}, \vec{z}), \end{aligned} \tag{3.26}$$

where B_{diag} satisfies (3.24). The same condition holds for the symbol for Ψ_2.

As stated before, the dispersion will bring a local smoothing effect to the system. Besides that, it will be illustrated in the following calculation that the control of the second order term $\Psi_{B_{diag}} \vec{z}$ comes from the dispersion. To achieve both objectives, we need a further transformation.

Step4. Decoupling of the nondispersive and dispersive parts. Since the couplings of the dispersive terms with the nondispersive ones, $(\rho, \Psi_{\theta_R} Pu)$, makes the latter terms uncontrollable, we need to decouple those two parts before we can hope to obtain the energy estimate. We apply the same idea as we did to cancel the B_{anti} in the last step.

First notice that in the continuity equation, the coupling is introduced by the term $\rho \nabla_x \cdot u$, essentially it is the term $\rho \nabla_x \cdot Qu$. This shows the coupling comes solely from the strictly dispersive part. Since this strictly dispersive part (Qu, θ) has been transformed into $\vec{z}$, we rewrite the continuity equation in terms of $\left(\rho, \ \Psi_{\theta_R} Pu, \ \vec{z}\right)$ as follows.

$$\partial_t \rho = -\epsilon \Delta_x^2 \rho - v \cdot \nabla_x \rho + \Psi_{\Gamma_1} z_1 + \Psi_{\Gamma_2} z_2 + \Psi_0 \vec{\zeta} \ . \tag{3.27}$$

Here we have applied the facts that the transformations S, Λ are zeroth order operators and they are invertible so that Qu can be written in terms of $\vec{z}$.

Define

$$T_1 = i\Psi_{\Gamma_1}\Psi_{\tilde{h}}, \qquad T_2 = -i\Psi_{\Gamma_2}\Psi_{\tilde{h}}\,.$$

Then T_1, T_2 are of order -2. By the definition of $\Psi_{\tilde{h}}$ we see that

$$\begin{aligned} T_1\Psi_{ih} + \Psi_{\Gamma_1} &= -\Psi_{\Gamma_1}\Psi_{\tilde{h}}\Psi_h + \Psi_{\Gamma_1}, \\ T_2\Psi_{-ih} + \Psi_{\Gamma_2} &= -\Psi_{\Gamma_2}\Psi_{\tilde{h}}\Psi_h + \Psi_{\Gamma_2}. \end{aligned} \tag{3.28}$$

By the calculus of $\psi.d.o$'s, $T_1\Psi_{ih}+\Psi_{\Gamma_1}$, $T_2\Psi_{-ih}+\Psi_{\Gamma_2}$ are both zeroth order operators.

Now consider the equations for T_1z_1 and T_2z_2. Apply T_1, T_2 to the equations for z_1, z_2 respectively to obtain that

$$\begin{aligned} \partial_t(T_1z_1) &= -\epsilon\Delta_x^2(T_1z_1) + \epsilon\Psi_{R_5}z_1 + T_1\Psi_{ih}z_1 + \Psi_0\left(\vec{\eta}, \vec{\zeta}, \vec{z}\right), \\ \partial_t(T_2z_2) &= -\epsilon\Delta_x^2(T_2z_2) + \epsilon\Psi_{R_6}z_2 + T_2\Psi_{-ih}z_2 + \Psi_0\left(\vec{\eta}, \vec{\zeta}, \vec{z}\right), \end{aligned} \tag{3.29}$$

where $R_5, R_6 \in S^1$. Combine (3.27) and (3.29). Let $\sigma = \rho + T_1z_1 + T_2z_2$. Then the equation for ϱ is written as

$$\partial_t\sigma = -\epsilon\Delta_x^2\sigma + \epsilon\Psi_1\vec{z} - v\cdot\nabla_x\sigma + \Psi_0\left(\sigma, \Psi_{\theta_R}Pu, \vec{\zeta}, \vec{z}\right). \tag{3.30}$$

Similar steps follow for the second order term for θ in the equation for $\Psi_{\theta_R}Pu$. First write θ which is also denoted as ω_2 in terms of $\vec{z} = (z_1, z_2)^T$. Then there exist $\Psi_{\Gamma_3}, \Psi_{\Gamma_4}$ such that $\Psi_{E^2}\omega_2 = \Psi_{\Gamma_3}z_1 + \Psi_{\Gamma_4}z_2$.

Define

$$T_3 = i\Psi_{\Gamma_3}\Psi_{\tilde{h}}, \qquad T_4 = i\Psi_{\Gamma_4}\Psi_{\tilde{h}}.$$

Thus T_3, T_4 are of order -1, and we have

$$
\begin{aligned}
T_3\Psi_{ih} + \Psi_{\Gamma_3} &= -\Psi_{\Gamma_3}\Psi_{\tilde{h}}\Psi_h + \Psi_{\Gamma_3}, \\
T_4\Psi_{-ih} + \Psi_{\Gamma_4} &= -\Psi_{\Gamma_4}\Psi_{\tilde{h}}\Psi_h + \Psi_{\Gamma_4}.
\end{aligned}
\tag{3.31}
$$

By the calculus of ψ.d.o's, $T_3\Psi_{ih} + \Psi_{\Gamma_3}$, $T_4\Psi_{-ih} + \Psi_{\Gamma_4}$ are both first order operators.

Apply T_3, T_4 to the equations for z_1, z_2 respectively to obtain that

$$
\begin{aligned}
\partial_t(T_3 z_1) &= -\epsilon\Delta_x^2(T_3 z_1) + \epsilon\Psi_{R_7} z_1 + T_3\Psi_{ih} z_1 + \Psi_1\left(\vec{\eta}, \vec{\zeta}, \vec{z}\right) + \Psi_0\left(\vec{\eta}, \vec{\zeta}, \vec{z}\right), \\
\partial_t(T_4 z_2) &= -\epsilon\Delta_x^2(T_4 z_2) + \epsilon\Psi_{R_8} z_2 + T_4\Psi_{-ih} z_2 + \Psi_1\left(\vec{\eta}, \vec{\zeta}, \vec{z}\right) + \Psi_0\left(\vec{\eta}, \vec{\zeta}, \vec{z}\right),
\end{aligned}
\tag{3.32}
$$

where $R_7, R_8 \in S^2$.

The equation for $\Psi_{\theta_R} Pu$ is written as

$$
\begin{aligned}
\partial_t\left(\Psi_{\theta_R} Pu\right) = &- \epsilon\Delta_x^2\left(\Psi_{\theta_R} Pu\right) + \frac{\mu}{\varrho}\Delta_x\left(\Psi_{\theta_R} Pu\right) + \Psi_{E^2}\sigma + \Psi_{\Gamma_3} z_1 + \Psi_{\Gamma_4} z_2 \\
&+ \Psi_1\left(\vec{\eta}, \vec{z}\right) + \Psi_0\left(\vec{\eta}, \vec{z}, \vec{\zeta}\right)
\end{aligned}
\tag{3.33}
$$

Therefore the equation for $\Psi_{\theta_R} Pu + T_3\alpha_1 + T_4\alpha_2 \triangleq \vec{v}$ is as follows:

$$
\partial_t \vec{v} = -\epsilon\Delta_x^2 \vec{v} + \epsilon\Psi_2 \vec{z} + \frac{\mu}{\varrho}\Delta_x \vec{v} + \Psi_{E^2}\sigma + \Psi_{E^1}\left(\sigma, \vec{z}, \vec{v}\right) + \Psi_{E^0}\left(\sigma, \vec{z}, \vec{v}, \vec{\zeta}\right).
\tag{3.34}
$$

Overall, the whole system is written as

$$
\begin{aligned}
\partial_t \sigma =& -\epsilon \Delta_x^2 \sigma + \epsilon \Psi_1 \vec{z} - v \cdot \nabla_x \sigma + \Psi_{E^0}\left(\vec{v}, \vec{\zeta}, \vec{z}\right), \\
\partial_t \vec{v} =& -\epsilon \Delta_x^2 \vec{v} + \epsilon \Psi_2 \vec{z} + \frac{\mu}{\varrho} \Delta_x \vec{v} + \Psi_2 \sigma + \Psi_1\left(\sigma, \vec{v}, \vec{z}\right) + \Psi_0\left(\sigma, \vec{v}, \vec{z}, \vec{\zeta}\right), \\
\partial_t \vec{z} =& -\epsilon \Delta_x^2 \vec{z} + \epsilon \Psi_{R_3} \vec{z} + \left(\frac{d-1}{d}\frac{\mu}{\varrho} + \frac{2}{d}\frac{\kappa}{\varrho}\right) \Delta_x \vec{z} + \Psi_L \vec{z} + \Psi_{B_{diag}} \vec{z} \\
& + \Psi_2\left(\sigma, \vec{v}\right) + \Psi_1\left(\sigma, \vec{v}, \vec{z}\right) + \Psi_0\left(\sigma, \vec{v}, \vec{\zeta}, \vec{z}\right) \\
\partial_t \vec{\zeta} =& \Psi_{E^0}\left(\sigma, \vec{v}, \vec{\alpha}, \vec{\zeta}\right).
\end{aligned}
\tag{3.35}
$$

Note that we have changed all the $\vec{\eta}$ by $\vec{v}$ and σ without changing the structure of the system.

Step5. A further transformation. At this step we are going to define a transformation to show the control of the B_{diag} term in the equation for $\vec{z}$ by the commutator of L with this new transformation. Before we give a definition, we state one lemma and prove a slightly more general version of it based on the assumptions $\mathcal{A}_1 - \mathcal{A}_4$.

Lemma 3.3.1. *[13] Let $h_1(x, \xi) = h(x, 0, \xi)$. Under the assumptions $\mathcal{A}_1 - \mathcal{A}_4$ at $t = 0$ there exists $p \in S^0$ real, and constants c_1, c_2 which depend on the constants in $\mathcal{A}_1 - \mathcal{A}_3$ for $t = 0$ and the non-trapping condition $\mathcal{A}_4$ such that*

$$
H_{h_1} p \geq c_1 \frac{|\xi|^2}{\langle x \rangle^2} - c_2, \qquad \forall\, (x, \xi) \in R^d \times R^d .
$$

Moreover, finitely many seminorms of p are bounded. These bounds depend only on $\mathcal{A}_1 - \mathcal{A}_3$ at $t = 0$.

Note that there is no time dependence in Lemma 3.3.1. We can extend Lemma

3.3.1 to the time-dependent case, as stated in the following lemma.

Lemma 3.3.2. *There exists $T^* > 0$, depending on the constants c_A, τ_0 and Lemma 1, such that for every $t \in [0, T^*)$ we have*

$$H_h p = \{h, p\}(x, t, \xi) \geq \frac{c_1}{2} \frac{|\xi|^2}{\langle x \rangle^2} - c_2, \qquad \forall\, (x, \xi) \in R^d \times R^d\,.$$

Proof. By definition, $H_h p = \sum_{j=1}^d \left(\partial_{\xi_j} h\, \partial_{x_j} p - \partial_{x_j} h\, \partial_{\xi_j} p\right)$, while

$$\partial_{\xi_j} h = 3\sqrt{\hat{\tau}_1 \hat{\tau}_4}|\xi|\xi_j, \quad \partial_{x_j} h = \partial_{x_j}\left(\sqrt{\hat{\tau}_1 \hat{\tau}_4}\right)|\xi|^3.$$

Thus it follows from the assumption on (ρ, θ) that

$$|\partial_{\xi_j} h(x, t, \xi) - \partial_{\xi_j} h(x, 0, \xi)| \leq \frac{c_{0,4} T^*}{\langle x \rangle^2}|\xi|^2,$$
$$|\partial_{x_j} h(x, t, \xi) - \partial_{x_j} h(x, 0, \xi)| \leq \frac{c_{0,5} T^*}{\langle x \rangle^2}|\xi|^3.$$

Hence

$$|H_h p - H_{h_1} p| \leq \frac{c_{0,6} T^*}{\langle x \rangle^2}|\xi|^2.$$

Choose T^* small enough to complete the proof. □

Now we are ready to construct the transformation. Let

$$q_1(x, \xi) = \exp\left(M p(x, \xi) \theta_R(\xi)\right), \quad q_2(x, \xi) = \exp\left(-M p(x, \xi) \theta_R(\xi)\right),$$

where $\theta_R(\xi)$ is again the cutoff function and $M > 0$ to be chosen. Then

$$\Psi_{q_1}\Psi_{q_2} = I + \Psi_{r_3}, \qquad \Psi_{q_2}\Psi_{q_1} = I + \Psi_{r_4}\,,$$

with $r_3, r_4 \in S^{-1}$. Thus Ψ_{q_1}, Ψ_{q_2} are invertible and their inverses are of order 0 for large R.

From the calculus of symbols we know that $\Psi_h\Psi_{q_1} - \Psi_{q_1}\Psi_h = \Psi_{-i\{h,q_1\}} + \Psi_1$ with

$$\begin{aligned}
\{h, q_1\} &= \sum_{j=1}^{d} \left(\partial_{\xi_j} h\, \partial_{x_j} q_1 - \partial_{x_j} h\, \partial_{\xi_j} q_1\right) \\
&= \sum_{j=1}^{d} \left(M\,\theta_R\, \partial_{\xi_j} h\, \partial_{x_j} p - M\,\theta_R\, \partial_{x_j} h\, \partial_{\xi_j} p\right) q_1 - \sum_{j=1}^{d} \left(M\, p\, \partial_{x_j} h\, \partial_{\xi_j}\, \theta_R\right)\, q_1\,,
\end{aligned}$$

that is,

$$\{h, q_1\} = M\,\theta_R\,(H_h p)\, q_1 + \Psi_0.$$

Overall we have

$$\Psi_{ih}\Psi_{q_1} - \Psi_{q_1}\Psi_{ih} = \Psi_{M\theta_R H_h p}\Psi_{q_1} + \Psi_1\,.$$

A similar computation shows that

$$\Psi_{ih}\Psi_{q_2} - \Psi_{q_2}\Psi_{ih} = -\Psi_{M\theta_R H_h p}\Psi_{q_2} + \Psi_1\,.$$

Now consider a new system in the variable

$$\vec{\alpha} = (\alpha_1, \alpha_2)^T = \begin{pmatrix} \Psi_{q_1} & 0 \\ 0 & \Psi_{q_2} \end{pmatrix} \vec{z} \triangleq \Psi \vec{z} .$$

Note that Ψ is invertible and $\Psi^{-1} = \begin{pmatrix} \Psi_{q_1}^{-1} & 0 \\ 0 & \Psi_{q_2}^{-1} \end{pmatrix}$ is also a matrix of order 0.

Apply the operator matrix Ψ to the system (3.26). We have

$$\begin{aligned} \partial_t \vec{\alpha} &= \Psi \partial_t \vec{z} \\ &= -\epsilon \Psi \Delta_x^2 \vec{z} + \epsilon \Psi \Psi_{R_3} \vec{z} + \Psi \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \tfrac{\kappa}{\varrho d} \right) \Delta_x \vec{z} + \Psi \Psi_L \vec{z} \\ &\quad + \Psi \Psi_{B_{diag}} \vec{z} + \Psi \Psi_2 \left(\sigma, \vec{v} \right) + \Psi \Psi_1 \left(\sigma, \vec{v}, \vec{z} \right) + \Psi \Psi_0 \left(\sigma, \vec{v}, \vec{\zeta}, \vec{z} \right) . \end{aligned}$$

Evaluate each term on the right as follows. First, there exists $R_4 \in S^3$ such that

$$-\epsilon \Psi \Delta_x^2 \vec{z} + \epsilon \Psi \Psi_{R_3} \vec{z} = -\epsilon \Delta_x^2 \vec{\alpha} + \epsilon \Psi_{R_4} \vec{\alpha} .$$

Second,

$$\begin{aligned} &\Psi \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \frac{2\kappa}{\varrho d} \right) \Delta_x \vec{z} \\ &= \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \tfrac{2\kappa}{\varrho d} \right) \Psi \Delta_x \Psi^{-1} \vec{\alpha} + \Psi_1 \vec{\alpha} \\ &= \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \tfrac{2\kappa}{\varrho d} \right) \begin{pmatrix} \Psi_{q_1} \Delta_x \Psi_{q_1}^{-1} & 0 \\ 0 & \Psi_{q_2} \Delta_x \Psi_{q_2}^{-1} \end{pmatrix} \vec{\alpha} + \Psi_1 \vec{\alpha} \\ &= \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \tfrac{2\kappa}{\varrho d} \right) \Delta_x \vec{\alpha} + \Psi_1 \vec{\alpha} . \end{aligned}$$

Similarly,

$$\Psi\Psi_{B_{diag}} \vec{z} = \Psi_{B_{diag}} \vec{\alpha} + (\Psi\Psi_{B_{diag}} - \Psi_{B_{diag}}\Psi) \vec{z} = \Psi_{B_{diag}} \vec{\alpha} + \Psi_1 \vec{\alpha} \ .$$

It is essential that B_{diag} in the above equation is the same as B_{diag} in (3.29), that is, it does not depend on M.

Next,

$$\begin{aligned}
&\Psi\Psi_2 \vec{v} = \Psi_2 \left(\Psi_{q_1} \vec{v},\ \Psi_{q_2} \vec{v}\right) + \Psi_1 \vec{v}, \qquad \Psi\Psi_2\sigma = \Psi_2\sigma,\\
&\Psi\Psi_k \vec{\eta} = \Psi_k \vec{\eta}, \qquad \Psi\Psi_0 \vec{\zeta} = \Psi_0 \vec{\zeta},\\
&\Psi\Psi_k \vec{z} = \Psi\Psi_k\Psi^{-1} \vec{\alpha} = \Psi_k \vec{\alpha}, \qquad k = 1, 0.
\end{aligned}$$

Similarly, the second order operator on $\vec{v}$ does not depend on M. The norms of the other $\psi.d.o$'s depend on finitely many derivatives of (ϱ, v, ϑ) and M.

Now for the dispersive part,

$$\begin{aligned}
\Psi\Psi_L \vec{z} &= \Psi_L \vec{\alpha} + \begin{pmatrix} \Psi_{q_1}\Psi_{ih} - \Psi_{ih}\Psi_{q_1} & 0 \\ 0 & -(\Psi_{q_2}\Psi_{ih} - \Psi_{ih}\Psi_{q_2}) \end{pmatrix}\\
&= \Psi_L \vec{\alpha} + \begin{pmatrix} -\Psi_{M\theta_R H_h p} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} \end{pmatrix} \vec{\alpha} + \Psi_1 \vec{\alpha} \ .
\end{aligned}$$

Overall, the system for $\vec{\alpha}$ is written as

$$
\begin{aligned}
\partial_t \vec{\alpha} &= -\epsilon\Delta_x^2 \vec{\alpha} + \epsilon\Psi_{R_4} \vec{\alpha} + \left((1-\tfrac{1}{d})\tfrac{\mu}{\varrho} + \tfrac{\kappa}{\varrho d}\right) \Delta_x \vec{\alpha} + \Psi_L \vec{\alpha} \\
&\quad + \begin{pmatrix} -\Psi_{M\theta_R H_h p} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} \end{pmatrix} \vec{\alpha} + \Psi_{B_{diag}} \vec{\alpha} \\
&\quad + \Psi_2\left(\Psi_{q_1} \vec{v},\ \Psi_{q_2} \vec{v}\right) + \Psi_2\sigma + \Psi_1(\vec{\eta}, \vec{\alpha}) + \Psi_0(\vec{\zeta}, \vec{\alpha}),
\end{aligned}
\tag{3.36}
$$

where $\Psi_{B_{diag}}$ and the second order operator on $\vec{v}$ do not depend on M.

By performing the transformations Ψ_{q_1}, Ψ_{q_2} on the equation for $\vec{v}$ respectively, it is easy to work out the equations for $\left(\Psi_{q_1} \vec{v},\ \Psi_{q_2} \vec{v}\right) \triangleq \vec{n}$ as

$$
\partial_t \vec{n} = -\epsilon\Delta_x^2 \vec{n} + \epsilon\Psi_2 \vec{\alpha} + \frac{\mu}{\varrho}\Delta_x \vec{n} + \Psi_2\sigma + \Psi_1\left(\sigma, \vec{n}, \vec{z}\right) + \Psi_0\left(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta}\right). \tag{3.37}
$$

Therefore the overall system in terms of $(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta})$ has the form

$$
\begin{aligned}
\partial_t\sigma &= -\epsilon\Delta_x^2\sigma + \epsilon\Psi_1 \vec{\alpha} - v\cdot\nabla_x\sigma + \Psi_0\left(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta}\right), \\
\partial_t \vec{n} &= -\epsilon\Delta_x^2 \vec{n} + \epsilon\Psi_2 \vec{\alpha} + \frac{\mu}{\varrho}\Delta_x \vec{n} + \Psi_2\sigma + \Psi_1\left(\sigma, \vec{n}, \vec{z}\right) + \Psi_0\left(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta}\right), \\
\partial_t \vec{\alpha} &= -\epsilon\Delta_x^2 \vec{\alpha} + \epsilon\Psi_{R_4} \vec{\alpha} + \left((1-\tfrac{1}{d})\tfrac{\mu}{\varrho} + \tfrac{\kappa}{\varrho d}\right) \Delta_x \vec{\alpha} + \Psi_L \vec{\alpha} \\
&\quad + \begin{pmatrix} -\Psi_{M\theta_R H_h p} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} \end{pmatrix} \vec{\alpha} + \Psi_{B_{diag}} \vec{\alpha} \\
&\quad + \Psi_2\left(\sigma, \vec{n}\right) + \Psi_1\left(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta}\right) + \Psi_0\left(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta}\right), \\
\partial_t \vec{\zeta} &= \Psi_0\left(\sigma, \vec{n}, \vec{\alpha}, \vec{\zeta}\right).
\end{aligned}
\tag{3.38}
$$

Step6. Energy estimate. We perform the energy estimate in this step. Since the estimate for the strictly dispersive part $\vec{\alpha}$ is most technical, we start with estimations for $\vec{\alpha}$. By the standard way of doing energy estimate we multiply the equation for $\vec{\alpha}$ in (3.38) by $\vec{\alpha}$ and integrate over R^d. Denote the usual L^2 inner product over R^d as $\langle\cdot\rangle$. Then

$$
\begin{aligned}
&\frac{d}{dt}\left\langle\vec{\alpha},\vec{\alpha}\right\rangle \\
&=\left\langle\partial_t\,\vec{\alpha},\ \vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\ \partial_t\,\vec{\alpha}\right\rangle \\
&=-\epsilon\left\langle\Delta_x^2\,\vec{\alpha},\ \vec{\alpha}\right\rangle-\epsilon\left\langle\vec{\alpha},\ \Delta_x^2\,\vec{\alpha}\right\rangle+\epsilon\left\langle\Psi_{R_4}\,\vec{\alpha},\ \vec{\alpha}\right\rangle+\epsilon\left\langle\vec{\alpha},\ \Psi_{R_4}\,\vec{\alpha}\right\rangle \\
&\quad+\left\langle\left((1-\tfrac{1}{d})\tfrac{\mu}{\varrho}+\tfrac{\kappa}{\varrho d}\right)\Delta_x\,\vec{\alpha},\ \vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\ \left((1-\tfrac{1}{d})\frac{\mu}{\varrho}+\frac{\kappa}{\varrho d}\right)\Delta_x\,\vec{\alpha}\right\rangle \\
&\quad+\left\langle\Psi_L\,\vec{\alpha},\ \vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\ \Psi_L\,\vec{\alpha}\right\rangle+\left\langle\Psi_{B_{diag}}\,\vec{\alpha},\vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\Psi_{B_{diag}}\,\vec{\alpha}\right\rangle \\
&\quad+\left\langle\begin{pmatrix}-\Psi_{M\theta_R H_h p} & 0\\ 0 & -\Psi_{M\theta_R H_h p}\end{pmatrix}\vec{\alpha},\ \vec{\alpha}\right\rangle \\
&\quad+\left\langle\vec{\alpha},\ \begin{pmatrix}-\Psi_{M\theta_R H_h p} & 0\\ 0 & -\Psi_{M\theta_R H_h p}\end{pmatrix}\vec{\alpha}\right\rangle \\
&\quad+\left\langle\Psi_2\,\vec{n},\vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\Psi_2\,\vec{n}\right\rangle+\left\langle\Psi_2\sigma,\vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\Psi_2\sigma\right\rangle \\
&\quad+\left\langle\Psi_1\left(\sigma,\vec{n},\vec{\alpha}\right),\ \vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\ \Psi_1\left(\sigma,\vec{n},\vec{\alpha}\right)\right\rangle \\
&\quad+\left\langle\Psi_0\left(\sigma,\vec{v},\vec{\alpha},\vec{\zeta}\right),\ \vec{\alpha}\right\rangle+\left\langle\vec{\alpha},\ \Psi_0\left(\sigma,\vec{n},\vec{\alpha},\vec{\zeta}\right)\right\rangle.
\end{aligned}
$$

Now estimate each term above. For the terms containing ϵ,

$$
-\epsilon\langle\Delta_x^2\,\vec{\alpha},\vec{\alpha}\rangle-\epsilon\langle\vec{\alpha},\Delta_x^2\,\vec{\alpha}\rangle=-2\epsilon\|\Delta_x\,\vec{\alpha}\,\|_{L^2}^2\,.
$$

Since $R_4 \in S^3$, by interpolation it's clear that

$$\epsilon \left| \left\langle \Psi_{R_4} \vec{\alpha}, \vec{\alpha} \right\rangle \right| + \epsilon \left| \left\langle \vec{\alpha}, \Psi_{R_4} \vec{\alpha} \right\rangle \right| \leq \epsilon c_A \| \vec{\alpha} \|^2_{H^{\frac{3}{2}}} \leq 2\epsilon \|\Delta_x \vec{\alpha} \|^2_{L^2} + \epsilon c_{0,6} \| \vec{\alpha} \|^2_{L^2}$$

This shows that the terms containing Ψ_{R_4} are well-controlled by the biharmonic terms.

For the dissipative term,

$$\begin{aligned} &\left\langle \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \tfrac{\kappa}{\varrho d} \right) \Delta_x \vec{\alpha}, \ \vec{\alpha} \right\rangle + \left\langle \vec{\alpha}, \ \left((1 - \tfrac{1}{d}) \tfrac{\mu}{\varrho} + \tfrac{\kappa}{\varrho d} \right) \Delta_x \vec{\alpha} \right\rangle \\ &\leq -c_3 \|\nabla_x \vec{\alpha} \|^2_{L^2(R^D)} + c_{0,7} \| \vec{\alpha} \|^2_{L^2}, \end{aligned}$$

where $c_3 > 0$ depending only on μ_0, κ_0, D, d and τ_0.

Next we deal with the second order terms for $\vec{n}$ and σ.

For the $\Psi_2 \vec{n}$ term, denote the symbol for this second order term as Γ_5. By assumptions $\mathcal{A}_1$ and $\mathcal{A}_3$, we can choose $T_7 > 0$ small enough such that Γ_5 satisfies the upper bound $|\Gamma_5| \leq \dfrac{c_{0,8}}{\langle x \rangle^2} |\xi|^2$, $\forall (x,t) \in R^d \times [0, T_7]$. Therefore we derive that

$$\begin{aligned} &\left| \langle \Psi_{\Gamma_5} \vec{n}, \vec{\alpha} \rangle \right| + \left| \langle \vec{\alpha}, \Psi_{\Gamma_5} \vec{n} \rangle \right| \\ \leq &\eta \|\nabla_x \vec{n} \|^2_{L^2} + c_{0,\eta} \int_{R^2} \frac{1}{\langle x \rangle^2} |\nabla_x \vec{\alpha} |^2 dx + c_{0,\eta} \| \vec{\alpha} \|^2_{L^2}, \end{aligned}$$

where $c_{0,\eta}$ depends only η and the data. Note that $c_{0,\eta}$ does not depend on M.

To control the first term on the right-hand side of the above inequality, that is, $\eta \|\nabla_x \vec{n} \|_{L^2_x}$, we need to utilize the equation for $\vec{n}$.

From equation (3.37) by multiplying on both sides $\vec{n}$ and integration by parts, it's easy to see that the energy estimate for $\vec{n}$ is as follows.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| \vec{n} \|_{L^2}^2 + c_{0,9}\|\nabla_x \vec{n} \|_{L^2}^2 \\
&\leq \eta\epsilon\|\Delta_x \vec{\alpha} \|_{L^2}^2 + c_{A,M}\left(\| \vec{\alpha} \|_{L^2}^2 + \|\sigma\|_{L^2}^2 + \| \vec{n} \|_{L^2}^2 + \| \vec{\zeta} \|_{L^2}^2\right) \\
&\quad + c_{A,M}\|\nabla_x\sigma\|_{L^2}^2.
\end{aligned} \tag{3.39}$$

The above inequality shows that we need to control the H^1 norm of σ. Control of $\|\sigma\|_{H^1}$ is also needed for the estimate of the second order term in the equation for $\vec{\alpha}$.

First multiply the σ-equation in the system (3.38) by σ to obtain the following L^2 estimate:

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|_{L^2}^2 \leq \eta\epsilon\|\nabla_x \vec{\alpha} \|_{L^2}^2 + c_{A,M}\left(\|\sigma\|_{L^2}^2 + \| \vec{\alpha} \|_{L^2}^2 + \| \vec{n} \|_{L^2}^2\right). \tag{3.40}$$

Differentiate the equation for σ in (3.38) with respect to $x_l, 1 \leq l \leq d$, multiply by $\partial_l\sigma$ and integrate over R^d. Then we gain the following equality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_l \sigma\|_{L^2(R^d)}^2 - \frac{1}{2}\int_{R^d} |\partial_l \sigma|^2 \nabla_x \cdot u \; dx + \int_{R^d} (\partial_l \sigma) \nabla_x\sigma \cdot \partial_l u \; dx \\
&= -\epsilon\left\langle\Delta_x^2 (\partial_l \sigma),\; \partial_l \sigma\right\rangle + \left\langle\epsilon\Psi_2 \vec{\alpha}, \partial_l \sigma\right\rangle \\
&\quad + \left\langle\Psi_1 \vec{n}, \partial_l \sigma\right\rangle + \left\langle\Psi_1 \vec{\alpha}, \partial_l \sigma\right\rangle + \left\langle\Psi_0 \vec{\zeta}, \partial_l \sigma\right\rangle
\end{aligned}$$

Therefore, the energy inequality shows that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_l\,\sigma\|_{L^2}^2 \leq & c_{A,M,\eta}\|\partial_l\,\sigma\|_{L^2}^2 + \eta\epsilon\|\Delta_x\,\vec{\alpha}\,\|_{L^2}^2 \\
& + \eta\left(\|\nabla_x\,\vec{v}\,\|_{L^2}^2 + \|\nabla_x\,\vec{\alpha}\,\|_{L^2}^2\right) + \|\,\vec{\zeta}\,\|_{L^2}^2.
\end{aligned}
\tag{3.41}
$$

Combining (3.40) and (3.41) we obtain the energy estimate for $\|\sigma\|_{H^1}^2$ as

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sigma\|_{H^1}^2 \leq\ & c_{A,M,\eta}\|\sigma\|_{H^1}^2 + \eta\epsilon\|\Delta_x\,\vec{\alpha}\,\|_{L^2}^2 + \eta\left(\|\nabla_x\,\vec{n}\,\|_{L^2}^2 + \|\nabla_x\,\vec{\alpha}\,\|_{L^2}^2\right) \\
& + \|\,\vec{\alpha}\,\|_{L^2}^2 + \|\,\vec{n}\,\|_{L^2}^2 + \|\,\vec{\zeta}\,\|_{L^2}^2.
\end{aligned}
\tag{3.42}
$$

For the first order terms, let $\eta > 0$ small enough. Then

$$
\begin{aligned}
\left|\left\langle \Psi_1\,\vec{\alpha},\vec{\alpha}\right\rangle\right| + \left|\left\langle \vec{\alpha},\Psi_1\,\vec{\alpha}\right\rangle\right| &\leq \eta\|\nabla_x\,\vec{\alpha}\,\|_{L^2(R^2)}^2 + c_{A,M,\eta}\|\,\vec{\alpha}\,\|_{L^2}^2, \\
\left|\left\langle \Psi_1\,\vec{n},\vec{\alpha}\right\rangle\right| + \left|\left\langle \vec{\alpha},\Psi_1\,\vec{n}\right\rangle\right| &\leq \eta\|\nabla_x\,\vec{n}\,\|_{L^2(R^2)}^2 + c_{A,M,\eta}\|\,\vec{\alpha}\,\|_{L^2}^2, \\
\left|\left\langle \Psi_1\sigma,\vec{\alpha}\right\rangle\right| + \left|\left\langle \vec{\alpha},\Psi_1\sigma\right\rangle\right| &\leq \eta\|\nabla_x\,\vec{\alpha}\,\|_{L^2(R^2)}^2 + c_{A,M,\eta}\|\sigma\|_{L^2}^2,
\end{aligned}
\tag{3.43}
$$

with $c_{A,M,\eta}$ depending on M,η and the bounds c_A.

The estimate for $\vec{\zeta}$ is straightforward from the equation for $\vec{\zeta}$ in (3.38), and it shows that

$$
\frac{1}{2}\frac{d}{dt}\|\,\vec{\zeta}\,\|_{L^2}^2 \leq \|\,\vec{\zeta}\,\|_{L^2}^2 + \|\,\vec{\alpha}\,\|_{L^2}^2 + \|\,\vec{v}\,\|_{L^2}^2 + \|\varrho\|_{L^2}^2. \tag{3.44}
$$

Next, by the calculus it can be shown that

$$
\langle \Psi_L\,\vec{\alpha},\vec{\alpha}\rangle + \langle \vec{\alpha},\Psi_L\,\vec{\alpha}\rangle = \langle \vec{\alpha},\Psi_{\hat{B}_{diag}}\,\vec{\alpha}\rangle,
$$

where

$$\hat{B}_{diag} = \begin{pmatrix} \hat{B}_{11} & 0 \\ \\ 0 & \hat{B}_{22} \end{pmatrix}, \quad |\hat{B}_{kk}| \le \frac{c_{0,10}}{\langle x \rangle^2}|\xi|^2, \quad \forall (t,x,\xi) \in [0,T] \times R^d \times R^d, k = 1,2.$$

By the fact that $\hat{B}_{diag}$ is real, we can combine $\frac{1}{2}\hat{B}_{diag}$ with B_{diag} and still denote it as $B_{diag} = \begin{pmatrix} B_{11} & 0 \\ \\ 0 & B_{22} \end{pmatrix}$. This will not change the fact that B_{diag} does not depend on M. Notice that the diagonal entries of B_{diag} satisfy that $|B_{kk}| \le \frac{c_{0,11}|\xi|^2}{\langle x \rangle^2}$ for $k = 1, 2$. By Lemma 3.3.2, taking M large enough we have

$$-M\theta_R H_h p + |B_{kk}| \le c_{0,12} - \frac{1}{2}c_{0,13}\frac{|\xi|^2}{\langle x \rangle^2}, \quad \forall |\xi| > R. \tag{3.45}$$

Obviously the choice of M depends only on c_0.

For a shorter notation, let $c' = c_{0,13}$. Then c' depends only on the data. By the *sharp Gårding* inequality

$$
\begin{aligned}
&\left\langle \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha},\ \vec{\alpha} \right\rangle \\
&+ \left\langle \vec{\alpha},\ \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha} \right\rangle \\
&\leq c\| \vec{\alpha} \|^2_{H^{\frac{1}{2}}} - Re\left\langle \begin{pmatrix} \Psi_{c'|\xi|^2/\langle x\rangle^2} & 0 \\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2} \end{pmatrix} \vec{\alpha},\ \vec{\alpha} \right\rangle \\
&\leq \eta\| \vec{\alpha} \|^2_{H^1} + c_\eta \| \vec{\alpha} \|^2_{L^2} - Re\left\langle \begin{pmatrix} \Psi_{c'|\xi|^2/\langle x\rangle^2} & 0 \\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2} \end{pmatrix} \vec{\alpha},\ \vec{\alpha} \right\rangle.
\end{aligned}
$$

Here $\eta > 0$ is chosen to be small enough so that the first term can be controlled by the dissipation.

For the operator $\Psi_{c'|\xi|^2/\langle x\rangle^2}$, by the calculus of the $\psi.d.o's$,

$$\Psi_{c'|\xi|^2/\langle x\rangle^2} = \frac{1}{\langle x\rangle^2}\Psi_{c'|\xi|^2} + \Psi_1, \quad \Psi_{c'|\xi|^2} = -c'\Delta_x,$$

and

$$-\frac{1}{\langle x\rangle^2}\Delta_x = -\nabla_x \cdot \left(\frac{1}{\langle x\rangle^2}\nabla_x\right) + \Psi_1.$$

Therefore,

$$
\begin{aligned}
&Re\left\langle \begin{pmatrix} \Psi_{c'|\xi|^2/\langle x\rangle^2} & 0 \\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2} \end{pmatrix} \vec{\alpha}, \vec{\alpha} \right\rangle \\
&\geq c' \int_{R^d} \frac{1}{\langle x\rangle^2} |\nabla_x \vec{\alpha} |^2 dx - \eta \|\nabla_x \vec{\alpha} \|^2_{L^2} - c_\eta \| \vec{\alpha} \|^2_{L^2}.
\end{aligned}
$$

Overall we have

$$\begin{aligned}
&\left\langle \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha}, \vec{\alpha} \right\rangle \\
&+ \left\langle \vec{\alpha}, \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha} \right\rangle \\
&\leq \eta \| \vec{\alpha} \|_{H^1}^2 + c_\eta \| \vec{\alpha} \|_{L^2}^2 - c' \int_{R^d} \frac{1}{\langle x \rangle^2} |\nabla_x \vec{\alpha}|^2 dx.
\end{aligned} \tag{3.46}$$

Remark 3.3.3. It is exactly the above inequality that introduces the dispersive regularization. The first term including η on the right-hand side of the above inequality can be controlled by the dissipation. However, we will show below another way which tells us that dissipation is actually not necessary here.

Instead of considering $\vec{\alpha}$ directly, we consider it with a weight, that is, $\frac{\vec{\alpha}}{\langle x \rangle}$. Then we have the following estimate.

$$\begin{aligned}
&\left\langle \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha},\ \vec{\alpha} \right\rangle \\
=&\left\langle \langle x \rangle \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \langle x \rangle \frac{\vec{\alpha}}{\langle x \rangle},\ \frac{\vec{\alpha}}{\langle x \rangle} \right\rangle \\
=&\left\langle \langle x \rangle^2 \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \frac{\vec{\alpha}}{\langle x \rangle},\ \frac{\vec{\alpha}}{\langle x \rangle} \right\rangle \\
+&\left\langle \Psi_1 \left(\frac{\vec{\alpha}}{\langle x \rangle} \right),\ \frac{\vec{\alpha}}{\langle x \rangle} \right\rangle.
\end{aligned} \tag{3.47}$$

By (3.45), the symbol of $\langle x\rangle^2\left(-\Psi_{M\theta_R H_h p}+B_{kk}\right)$ satisfies that

$$\langle x\rangle^2\left(-\Psi_{M\theta_R H_h p}+B_{kk}\right)\le c_{0,10}\langle x\rangle^2-\frac{1}{2}c_{0,11}|\xi|^2. \tag{3.48}$$

We perform the following estimate.

$$\begin{aligned}
&\left\langle\begin{pmatrix}-\Psi_{M\theta_R H_h p}+B_{11} & 0\\ 0 & -\Psi_{M\theta_R H_h p}+B_{22}\end{pmatrix}\vec{\alpha},\ \vec{\alpha}\right\rangle\\
&+\left\langle\vec{\alpha},\ \begin{pmatrix}-\Psi_{M\theta_R H_h p}+B_{11} & 0\\ 0 & -\Psi_{M\theta_R H_h p}+B_{22}\end{pmatrix}\vec{\alpha}\right\rangle\\
=&\left\langle\langle x\rangle^2\begin{pmatrix}-\Psi_{M\theta_R H_h p}+B_{11} & 0\\ 0 & -\Psi_{M\theta_R H_h p}+B_{22}\end{pmatrix}\frac{\vec{\alpha}}{\langle x\rangle},\ \frac{\vec{\alpha}}{\langle x\rangle}\right\rangle\\
&+\left\langle\frac{\vec{\alpha}}{\langle x\rangle},\ \langle x\rangle^2\begin{pmatrix}-\Psi_{M\theta_R H_h p}+B_{11} & 0\\ 0 & -\Psi_{M\theta_R H_h p}+B_{22}\end{pmatrix}\frac{\vec{\alpha}}{\langle x\rangle}\right\rangle+\left\langle\Psi_1\frac{\vec{\alpha}}{\langle x\rangle},\ \frac{\vec{\alpha}}{\langle x\rangle}\right\rangle\\
\le& c\left\|\frac{\vec{\alpha}}{\langle x\rangle}\right\|_{H^{\frac{1}{2}}}^2+c_{0,14}\|\ \vec{\alpha}\ \|_{L^2}^2-c_{0,15}\left\|\frac{\vec{\alpha}}{\langle x\rangle}\right\|_{H^1}^2\\
\le&-c_{0,14}\left\|\frac{\vec{\alpha}}{\langle x\rangle}\right\|_{H^1}^2+c_{0,15}\|\ \vec{\alpha}\ \|_{L^2}^2.
\end{aligned}$$

The last inequality is achieved by interpolation. Accordingly, we have shown that there exist $c_{0,16}, c_{0,17} > 0$ depending only on the data such that

$$
\begin{aligned}
&\left\langle \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha},\ \vec{\alpha} \right\rangle \\
&+ \left\langle \vec{\alpha}, \begin{pmatrix} -\Psi_{M\theta_R H_h p} + B_{11} & 0 \\ 0 & -\Psi_{M\theta_R H_h p} + B_{22} \end{pmatrix} \vec{\alpha} \right\rangle \\
&\leq - c_{0,16} \int_{R^2} \frac{1}{\langle x \rangle^2} |\nabla_x \vec{\alpha}\,|^2 dx + c_{0,17} \| \vec{\alpha} \|_{L^2}^2.
\end{aligned} \tag{3.49}
$$

Overall by adding (3.39), (3.42), (3.44), (3.43) and (3.46) we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\sigma\|_{H^1}^2 + \| \vec{\alpha} \|_{L^2}^2 + \| \vec{n} \|_{L^2}^2 + \| \vec{\zeta} \|_{L^2}^2 \right) + \hat{c} \int_{R^d} \left(\|\nabla_x \vec{\alpha} \|^2 + \|\nabla_x \vec{n} \|^2 \right) dx \\
&\leq \tilde{c} \left(\|\sigma\|_{H^1}^2 + \| \vec{\alpha} \|_{L^2}^2 + \| \vec{n} \|_{L^2}^2 + \| \vec{\zeta} \|_{L^2}^2 \right),
\end{aligned} \tag{3.50}
$$

where $\tilde{c}$ depends on c_A, θ_0 and $\hat{c}$ depends on μ, κ, θ_0, d.

By Gronwall's inequality we clearly see that

$$
\begin{aligned}
&\sup_{0 \leq t \leq T} \left(\|\sigma\|_{H^1}^2 + \| \vec{\alpha} \|_{L^2}^2 + \| \vec{n} \|_{L^2}^2 + \| \vec{\zeta} \|_{L^2}^2 \right) + \int_0^T \|\nabla_x (\vec{\alpha},\ \vec{n})\|_{L^2}^2(s)\, ds \\
&\leq c' e^{TK_0} \left(\|\sigma(0)\|_{H^1}^2 + \| \vec{\alpha}\,(0)\|_{L^2}^2 + \| \vec{n}\,(0)\|_{L^2}^2 + \| \vec{\zeta}\,(0)\|_{L^2}^2 \right) \\
&\leq c' \left(\|\varrho(0)\|_{H^1}^2 + \| \vec{\alpha}\,(0)\|_{L^2}^2 + \| \vec{n}\,(0)\|_{L^2}^2 + \| \vec{\zeta}\,(0)\|_{L^2}^2 \right)
\end{aligned} \tag{3.51}
$$

where c' depends only on the initial data and θ_0 which is the lower bound of ρ, θ, K_0 depends on c_A, θ_0 and $T > 0$ is chosen to be small enough such that the second inequality is true.

Using the fact that the equivalence of $\|\varrho\|_{H^1}^2 + \| \vec{\alpha} \|_{L^2}^2 + \| \vec{v} \|_{L^2}^2 + \| \vec{\zeta} \|_{L^2}^2$

and $\|\rho\|_{H^1}^2 + \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2$ depends only on the data, we conclude that there exist $T > 0$ depending on c_A, θ_0, while $c > 0$ depending only on the data and θ_0 such that

$$\begin{aligned}&\sup_{[0,T]} \left(\|\rho\|_{H^1}^2 + \|(u,\theta)\|_{L^2}^2\right)(t) + \int_0^T \|\nabla_x (u,\theta)\|_{L^2}^2(s)\, ds \\ \le\, & c\left(\|\rho^{in}\|_{H^1}^2 + \|(u^{in},\theta^{in})\|_{L^2}^2\right).\end{aligned} \tag{3.52}$$

3.4 A Priori Estimate

Based on the linear estimate, we establish the a priori estimate for the nonlinear system (3.1). Before the statement of the theorem, we define the functional space in which system (3.1) is locally well-posed. Let

$$\begin{aligned}&s_1 \ge \frac{d}{2} + 6, \quad s \ge \max\{s_1 + 6, N + d/2 + 4\}, \\ &\lambda = \|\rho^{in} - \bar\rho\|_{H^{s+1}} + \|u^{in}\|_{H^s} + \|\theta^{in} - \bar\theta\|_{H^s} + \sum_{1\le|\alpha|\le s_1} \|\langle x\rangle^2 \partial_x^\alpha (\rho^{in}, u^{in}, \theta^{in})\|_{L^2}, \\ &M_0 = 100 c\lambda,\end{aligned}$$

with c being the constant in (3.52). Therefore M_0 depends only on the data.

For functions $(\rho, u, \theta) : R^d \times [0,T] \to R^1 \times R^d \times R^1$ satisfying

$$\begin{aligned}&\rho - \bar\rho \in C([0,T]; H^{s+1}), \qquad (u,\, \theta - \bar\theta) \in C([0,T]; H^s), \\ &\langle x\rangle^2 \partial_x^\alpha (\rho, u, \theta) \in C([0,T]; L^2), \quad \forall 1 \le |\alpha| \le s_1,\end{aligned}$$

define the space $X_{T,M}$ as follows:

$$X_{T,M} = \left\{(\rho,u,\theta) : |||(\rho,u,\theta)|||_T \le M, \quad (\rho,u,\theta)(0) = (\rho^{in}, u^{in}, \theta^{in})\right\},$$

with the norm

$$
\begin{aligned}
&|||(\rho, u, \theta)|||_T \\
&= \sup_{[0,T]} \left(\|\rho(t) - \bar{\rho}\|_{H^{s+1}} + \|u(t)\|_{H^s} + \|\theta(t) - \bar{\theta}\|_{H^s} + \sum_{1 \le |\alpha| \le s_1} \|\langle x \rangle^2 \partial_x^\alpha (\rho, u, \theta)(t)\|_{L^2} \right).
\end{aligned}
$$

The a priori estimate of system (3.1) is stated in the following theorem.

Theorem 3.4.1. *Let $(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \in X_{T_\epsilon, M_0}$ be a solution to the regularized DNS system (1.1). Then there exists $T_0 > 0$ independent of ϵ such that $|||(\rho^\epsilon, u^\epsilon, \theta^\epsilon)|||_{T_0} \le M_0$.*

Proof. First for the L^2 bound, we see that if we use $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ for (ϱ, v, ϑ), then by Theorem 3.3.1, there will be a $T_1 > 0$ independent of ϵ such that

$$
\sup_{[0,T_1]} \left(\|\rho^\epsilon - \bar{\rho}\|_{H^1} + \|u^\epsilon\|_{L^2} + \|\theta^\epsilon - \bar{\theta}\|_{L^2} \right) \le M_0. \tag{3.53}
$$

We need to check this for higher order terms and terms with weights. For the time being, write (ρ, u, θ) for $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$.

To obtain the estimate for high order terms apply ∂_x^α to the nonlinear system. The resulting system for $(\partial_x^\alpha \rho,\ \partial_x^\alpha u,\ \partial_x^\alpha \theta)$ shows that

$$
\begin{aligned}
\partial_t (\partial_x^\alpha \rho) &= -\epsilon \Delta_x^2 (\partial_x^\alpha \rho) + \tilde{\mathcal{L}}_1 (\rho, u, \theta)\ (\partial_x^\alpha \rho,\ \partial_x^\alpha u) + \Psi_0 (\partial_x^\alpha \rho,\ \partial_x^\alpha u) + f_{\alpha,0} \\
\partial_t (\partial_x^\alpha u) &= -\epsilon \Delta_x^2 (\partial_x^\alpha u) + \tilde{\mathcal{L}}_2 (\rho, u, \theta)\ (\partial_x^\alpha \rho,\ \partial_x^\alpha u,\ \partial_x^\alpha \theta) + \Psi_0 (\partial_x^\alpha \rho,\ \partial_x^\alpha u,\ \partial_x^\alpha \theta) + f_{\alpha,1} \\
\partial_t (\partial_x^\alpha \theta) &= -\epsilon \Delta_x^2 (\partial_x^\alpha \theta) + \tilde{\mathcal{L}}_3 (\rho, u, \theta)\ (\partial_x^\alpha \rho, \partial_x^\alpha u, \partial_x^\alpha \theta) + \Psi_0 (\partial_x^\alpha \rho,\ \partial_x^\alpha u,\ \partial_x^\alpha \theta) + f_{\alpha,2},
\end{aligned}
$$

where $f_{\alpha,k}, k = 0, 1, 2$ are functions dependent on $(\partial_x^\gamma \rho, \partial_x^\gamma u, \partial_x^\gamma \theta)_{|\gamma| \le |\alpha| - 1}$ and since these lower derivatives of (ρ, u, θ) have been estimated we treat them as forcing terms.

Here $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\mathcal{L}}_3$ are different linear operators from $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ but they have the same structures. It's easy to see that the assumptions $\mathcal{A}_1 - \mathcal{A}_4$ are satisfied for $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\mathcal{L}}_3$. We have an extra zero-order operator in each equation. By Hölder's inequality it's obvious that they won't hurt either the H^1 estimate for $\partial_x^{\alpha+1}\rho$ or the L^2 estimates for $(\partial_x^\alpha u,\, \partial_x^\alpha \theta)$. Together with the fact that $f_{\alpha,0}, f_{\alpha,1}, f_{\alpha,2}$ have already been estimated in the $L_T^\infty L_x^2$-norm, we conclude that the same linear estimate (1.45) applies for $(\partial_x^\alpha \rho, \partial_x^\alpha u, \partial_x^\alpha \theta)$ for all $1 \leq |\alpha| \leq s$, that is, there exists $T_0 > 0$ depending on M_0, τ_0 such that

$$\begin{aligned}
&\sup_{[0,T_0]} \left(\|\partial_x^\alpha \rho\|_{H^1}^2 + \|\partial_x^\alpha u\|_{L^2}^2 + \|\partial_x^\alpha \theta\|_{L^2}^2 \right) + \int_0^{T_0} \|\nabla_x \left(\partial_x^\alpha u, \partial_x^\alpha \theta\right)\|_{L^2}^2(s)ds \\
\leq\, & c\left(\|\partial_x^\alpha \rho^{in}\|_{H^1}^2 + \|\partial_x^\alpha u^{in}\|_{L^2}^2 + \|\partial_x^\alpha \theta^{in}\|_{L^2}^2 + \int_0^{T_0} \|(f_{\alpha,0}, f_{\alpha,1}, f_{\alpha,2})\|_{L^2}^2(s)ds \right).
\end{aligned}$$

Notice that since $(f_{\alpha,0}, f_{\alpha,1}, f_{\alpha,2}) \in L^\infty(0, T_0; L^2(R^d))$, the last term including the forcings can be made arbitrarily small by taking T_0 small. Thus, there exists a time $T_0 > 0$ independent of ϵ such that

$$\sup_{[0,T_0]} \left(\|\rho^\epsilon - \bar{\rho}\|_{H^{s+1}} + \| \left(u^\epsilon, \theta^\epsilon - \bar{\theta}\right) \|_{H^s} \right) \leq M_0.$$

Now check the bounds for $\langle x\rangle^2(\rho, u, \theta)$ and $\langle x\rangle^2\partial_x^\alpha(\rho, u, \theta)$ with $1 \leq |\alpha| \leq s_1$. We show that the system for $\langle x\rangle^2\partial_x^\alpha(\rho, u, \theta)$ has the same structure as those for (ρ, u, θ) and $\partial_x^\alpha(\rho, u, \theta)$.

The system satisfied by $x_l(\rho, u, \theta)$ has the form:

$$\begin{aligned}\partial_t(x_l\rho, x_l u, x_l\theta) =& -\epsilon\Delta_x^2(x_l\rho, x_l u, x_l\theta) + \mathcal{L}\,(\rho, u, \theta)(x_l\rho, x_l u, x_l\theta)\\ &+ f_l\bigg((\partial_x^\beta(\rho, u, \theta))_{|\beta|\le 3}\bigg),\end{aligned}$$

where f_l term depends only on the H^3 norm of (ρ, u, θ) and thus is well-controlled.

Clearly the linear estimate holds for $(x_l\rho, x_l u, x_l\theta)$ for each $l = 1, 2, \ldots, d$.

Similar situations hold for $x_l\partial_x^\alpha(\rho, u, \theta)$ and for $\langle x\rangle^2\partial_x^\alpha(\rho, u, \theta)$. This can be seen from the equivalence of norms of $\langle x\rangle^2\partial_x^\alpha(\rho, u, \theta)$ with norms of $\partial_x^\alpha\left(\langle x\rangle^2(\rho, u, \theta)\right)$. Consequently we conclude that there exists $T_0 > 0$ independent of ϵ such that

$$|||(\rho^\epsilon, u^\epsilon, \theta^\epsilon)|||_{T_0} \le M_0.$$

Therefore $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ can be extended to the time interval $[0, T_0]$, that is, $(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \in X_{T_0,M_0}$. □

This a priori estimate shows that for the approximating sequence $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$, there exists a common time interval $[0, T_0]$ such that they are uniformly bounded in the norm $|||\cdot|||_{T_0}$ by $M_0 > 0$ with M_0 depending only on the initial data.

3.5 Local Existence Proof

Using the a priori estimate in the last section, we can now establish the proof for the local well-posedness of the nonlinear system (1.1). To construct the solution to the DNS system, we first show the existence of solutions to the regularized DNS system using the standard contraction mapping argument. These solutions yield an approximating sequence. By the compactness from the a priori estimate we can pass to the limit to find a solution. Uniqueness is also shown.

For each given U, let $\Gamma(U)$ be the solution operator to the system

$$\partial_t \Gamma(U) = -\epsilon \Delta_x^2\, \Gamma(U) + \mathcal{L}(U)U, \quad \Gamma(U)(x,0) = U^{in}, \tag{3.54}$$

where $\mathcal{L}(U)$ is defined as the operator in (1.2).

For U smooth enough, (3.1) is equivalent to the following integral form for $\Gamma(U)$, that is,

$$\Gamma(U)(t) = e^{-\epsilon t \Delta_x^2} U^{in} + \int_0^t e^{-\epsilon(t-t')\Delta_x^2}\, \mathcal{L}(U)U(t')dt'. \tag{3.55}$$

To show that the regularized DNS system has a solution, it's enough to show that Γ has a fixed point in an appropriate space. By studying the semigroup generated by $-\epsilon\Delta_x^2$, we can prove that

Theorem 3.5.1. *For each $\epsilon \in (0,1)$ there exists $T_\epsilon = O(\epsilon)$ such that the operator Γ defined in (3.1) is a contraction mapping on X_{T_ϵ, M_0}.*

Proof. To show Γ is a contraction mapping, we first show that for each $\epsilon > 0$, there exists $T_\epsilon > 0$ such that Γ maps X_{T_ϵ, M_0} into itself. Observe that

$$\|(-\Delta_x)^{\frac{3}{2}} e^{-\epsilon t \Delta_x^2} U\|_{L^2} \leq \frac{1}{(\epsilon t)^{3/4}} \|U\|_{L^2}. \tag{3.56}$$

Take α derivatives for $\alpha \leq s$ on both sides of (3.1). It shows

$$\partial_x^\alpha \Gamma(U) = e^{-\epsilon t \Delta_x^2} \partial_x^\alpha U^{in} + \int_0^t e^{-\epsilon(t-t')\Delta_x^2} \, \partial_x^\alpha \left(\mathcal{L}(U)U\right)(t')dt'.$$

By the definition of $\mathcal{L}(U)U$ and (3.3),

$$\begin{aligned}
&\sup_{[0,T_1]} \sum_{\alpha\leq s} \|\partial_x^\alpha \Gamma(U)\|_{L^2} \leq \sum_{\alpha\leq s} \|\partial_x^\alpha U^{in}\|_{L^2} + \sum_{\alpha\leq s} \int_0^{T_1} \left\| e^{-\epsilon(t-t')\Delta_x^2} \, \partial_x^\alpha \left(\mathcal{L}(U)U\right)(t') \right\|_{L^2} dt' \\
&\leq \sum_{\alpha\leq s} \|\partial_x^\alpha U^{in}\|_{L^2} + \frac{c_s}{\epsilon^{3/4}} Q_1(M_0) \int_0^{T_1} \frac{1}{(t-t')^{3/4}} dt' \\
&\leq \sum_{\alpha\leq s} \|\partial_x^\alpha U^{in}\|_{L^2} + \frac{c_s T_1^{1/4}}{\epsilon^{3/4}} Q_1(M_0) \leq \|U^{in}\|_{H^s} + M_0/4 \leq M_0/2
\end{aligned}$$

by taking $T_1 = O(\epsilon^3)$ small enough, while $Q_1(M_0)$ is an increasing function in M_0 given by

$$Q_1(M_0) = M_0^2 \left(1 + ||\tau_1(\cdot), \tau_4(\cdot)\|_{L^\infty[0,M_0]}\right),$$

and c_s is a constant depending only on s. Therefore we have verified that

$$\sup_{[0,T_1]} \|\Gamma(U)\|_{H^s} \leq M_0/2,$$

for T_1 sufficiently small.

Next, check the weighted Sobolev norm $\|\Gamma(U)\|_{H^{s_1}(\langle x\rangle^2 dx)}$. For each $1 \leq l \leq d$, the equation for $x_l \partial_x^\alpha \Gamma(U)$ is written as

$$\partial_t \left(x_l \partial_x^\alpha \Gamma(U)\right) = -\epsilon \Delta_x^2 \left(x_l \partial_x^\alpha \Gamma(U)\right) + \partial_x^\alpha \Big(\mathcal{L}(U)(x_l U) \Big) + F, \tag{3.57}$$

where

$$F = \epsilon c_{s_1} \Delta_x \partial_{x_l} \partial_x^\alpha \Gamma(U) + \partial_x^{\alpha-1}\Big(\mathcal{L}(U)U\Big) + \partial_x^\alpha\Big(x_l\mathcal{L}(U) - \mathcal{L}(U)x_l\Big)U,$$

and $x_l\mathcal{L}(U) - \mathcal{L}(U)x_l$ is a second order operator that does not depend on x_l. Therefore F depends on only the $\|U\|_{H^s}$ provided $s_1 \leq s-3$. By the quadratic structure of $\mathcal{L}(U)$ and the fact that $U \in X_{T_\epsilon, M_0}$, we have $\sup_{[0,T_\epsilon]} \|F\|_{L^2} \leq c_{s_1} M_0^2$.

Overall, by the integral form of (3.4), the estimate of $x_l\partial_x^\alpha\Gamma(U)$ shows

$$\begin{aligned}
&\sup_{[0,T_2]} \sum_{\alpha\leq s_1} \|x_l\partial_x^\alpha\Gamma(U)\|_{L^2} \\
&\leq \sum_{\alpha\leq s_1} \|x_l\partial_x^\alpha U^{in}\|_{L^2} + \sum_{\alpha\leq s_1} \int_0^{T_2} \left\| e^{-\epsilon(t-t')\Delta_x^2} \Big(\partial_x^\alpha\, \mathcal{L}(U)(x_l U) + F\Big)(t')\right\|_{L^2} dt' \\
&\leq \sum_{\alpha\leq s_1} \|x_l\partial_x^\alpha U^{in}\|_{L^2} + \frac{c_s}{\epsilon^{3/4}}(Q_2(M_0) + M_0^2)\int_0^{T_1} \frac{1}{(t-t')^{3/4}} dt' \\
&\leq \sum_{\alpha\leq s_1} \|x_l\partial_x^\alpha U^{in}\|_{L^2} + \frac{c_s T_2^{1/4}}{\epsilon^{3/4}} Q_1(M_0) \leq \|U^{in}\|_{H^{s_1}(\langle x\rangle^2 dx)} + M_0/4 \leq M_0/2,
\end{aligned} \tag{3.58}$$

where Q_2 is an increasing function depending on M_0, τ_1, τ_4 and T_2 is chose to be sufficiently small.

Similarly, the equation for $x_l^2\partial_x^\alpha\Gamma(U)$ shows that

$$\partial_t\left(x_l^2\partial_x^\alpha\Gamma(U)\right) = -\epsilon\Delta_x^2\left(x_l^2\partial_x^\alpha\Gamma(U)\right) + \partial_x^\alpha\left(\mathcal{L}(U)\left(x_l^2 U\right)\right) + \tilde{F},$$

where $\tilde{F}$ depends on $\|x_l\partial_x^\alpha\Gamma(U)\|_{L^2}$ for $\alpha \leq s_1$, $\|U\|_{H^s}$ provided $s \geq s_1 + 6$ and $\|\Gamma(U)\|_{H^s}$. Since these quantities have all been shown bounded by M_0, we deduce that $\|\tilde{F}\|_{H^{s_1}}$ is bounded by a constant multiple of M_0.

By the same estimate as in (3.5), we conclude that there exists $T_3 > 0$ sufficiently small which depends on ϵ such that

$$\sup_{[0,T_3]} \sum_{\alpha \le s_1} \|x_l^2 \partial_x^\alpha \Gamma(U)\|_{L^2} \le M_0/2 \tag{3.59}$$

for each $\alpha \le s_1$. And this finishes the proof for the claim that there exists $T_\epsilon > 0$ such that for each $U \in X_{T_\epsilon, M_0}$, $\Gamma(U)$ is also in the space X_{T_ϵ, M_0}.

Next we show that Γ is a contraction mapping. Let $U, U \in X_{T_\epsilon, M_0}$. Then

$$\Gamma(U) - \Gamma(W) = \int_0^t e^{-\epsilon(t-t')\Delta_x^2} \left(\mathcal{L}(U)U - \mathcal{L}(W)W\right)(t')dt'.$$

By similar arguments we can show that

$$\sup_{[0,T]} \|\Gamma(U) - \Gamma(W)\|_{H^s} \le \frac{c_s T^{1/4}}{\epsilon^{3/4}} Q_3(M_0) \sup_{[0,T]} \|U - W\|_{H^s},$$

and

$$\sup_{[0,T]} \sum_{\alpha \le s_1} \|\langle x\rangle^2 \partial_x^\alpha (\Gamma(U) - \Gamma(W))\|_{L^2} \le \frac{c_s T^{1/4}}{\epsilon^{3/4}} Q_4(M_0) \sup_{[0,T]} \sum_{\alpha \le s_1} \|\langle x\rangle^2 (\partial_x^\alpha U - \partial_x^\alpha W)\|_{L^2}$$
$$+ T Q_5(M_0) \left(\sup_{[0,T]} \|\Gamma(U) - \Gamma(W)\|_{H^s} + \sup_{[0,T]} \|U - W\|_{H^s} \right). \tag{3.60}$$

Consequently, by taking $T > 0$ small enough we can guarantee that $\Gamma : X_{T,M_0} \longrightarrow X_{T,M_0}$ is a contraction mapping. Thus we finish the proof for Theorem 3.5.1. $\square$

By Theorem 3.5.1, there exists a solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \in X_{T_\epsilon, M_0}$ to the nonlinear system with an artificial viscosity. In the following theorem we show that this sequence of approximated solutions converges to the solution of the original DNS system. Uniqueness is also established in the following theorem.

Theorem 3.5.2. *Given the initial data $(\rho^{in}, u^{in}, \theta^{in})$ satisfying the condition that*

$$\|\rho^{in}\|_{H^{s+1}} + \|(u^{in}, \theta^{in})\|_{H^s} + \sum_{1 \le |\alpha| \le s_1} \left(\|\langle x \rangle^2 \partial_x^\alpha \rho^{in}\|_{H^1} + \|\langle x \rangle^2 \partial_x^\alpha (u^{in}, \theta^{in}\|_{L^2} \right) < \infty,$$

there exists $T_0 > 0$ independent of ϵ such thata system (2.2) has a unique solution in X_{T_0, M_0}. Moreover, there exists $\rho \in C([0, T_0]; H^s) \cap L^\infty([0, T_0]; H^{s+1}), (u, \theta) \in C([0, T_0]; H^{s-1}) \cap L^\infty([0, T_0]; H^s) \cap L^2(0, T_0; H^{s+1})$ such that for any $\alpha \le s_1$ we have

$$\begin{aligned}
&\rho^\epsilon - \bar{\rho} \longrightarrow \rho - \bar{\rho} \quad in \quad C([0, T_0]; H^s), \\
&(u^\epsilon, \theta^\epsilon - \bar{\theta}) \longrightarrow (u, \theta - \bar{\theta}) \quad in \quad C([0, T_0]; H^{s-1}), \\
&\langle x \rangle^2 \partial_x^\alpha \rho^\epsilon \longrightarrow \langle x \rangle^2 \partial_x^\alpha \rho \quad in \quad C([0, T_0]; H^1), \\
&\langle x \rangle^2 \partial_x^\alpha (u^\epsilon, \theta^\epsilon) \longrightarrow \langle x \rangle^2 \partial_x^\alpha (u, \theta) \quad in \quad C([0, T_0]; L^2) \quad as \quad \epsilon \to 0,
\end{aligned}$$

and (ρ, u, θ) solves the original system (3.1).

Proof. The first part has been shown in Theorem 3.5.1. To show the convergence, we apply the standard high-low technique. Basically, we will show that $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ converges in $C(0, T_0; L^2(R^d))$. Then by using the interpolation and the uniform bounds on $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ we can show that $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ converges in $C(0, T_0; H^{s-1}(R^d))$.

For $\epsilon, \epsilon' > 0$, let $\varrho = \rho^\epsilon - \rho^{\epsilon'}, v = u^\epsilon - u^{\epsilon'}, \eta = \theta^\epsilon - \theta^{\epsilon'}$ and study the system for (ϱ, v, η).

$$\partial_t \varrho = -\epsilon \Delta_x^2 \varrho - (\epsilon - \epsilon')\Delta_x^2 \rho^{\epsilon'} + \mathcal{L}_1(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v) + \Psi_{E^0}(\varrho, v)$$

$$\partial_t v = -\epsilon \Delta_x^2 v - (\epsilon - \epsilon')\Delta_x^2 u^{\epsilon'} + \mathcal{L}_2(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v, \eta) + \Psi_{E^0}(\varrho, v, \eta)$$

$$\partial_t \eta = -\epsilon \Delta_x^2 \eta - (\epsilon - \epsilon')\Delta_x^2 \theta^{\epsilon'} + \mathcal{L}_3(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v, \eta) + \Psi_{E^0}(\varrho, v, \eta).$$

It is clear that given $(\rho^\epsilon, u^\epsilon, \theta^\epsilon), (\rho^{\epsilon'}, u^{\epsilon'}, \theta^{\epsilon'}) \in X_{T_0, M_0}$, the linear estimate applies to the above system. Therefore we have

$$\begin{aligned}
&\sup_{[0,T_0]} \left(\|\varrho\|_{H^1}^2 + \|(v,\eta)\|_{L^2}^2\right) \\
&\leq c(\epsilon - \epsilon') \int_0^{T_0} \left(\|\Delta_x^2 \rho^\epsilon(\cdot, s)\|_{H^1}^2 + \|\Delta_x^2 u^\epsilon(\cdot, s)\|_{L^2}^2 + \|\Delta_x^2 \theta^\epsilon(\cdot, s)\|_{L^2}^2\right) ds \\
&\leq c(\epsilon - \epsilon') T_0 M_0.
\end{aligned}$$

This shows $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ is a Cauchy sequence in $C([0,T_0]; L^2(R^d))$. Since this is also a bounded sequence in $L^\infty(0, T_0; H^s(R^d))$ we conclude that it's a Cauchy sequence in $C([0,T_0]; H^{s-1}(R^d))$. Thus, there exists $(\rho, u, \theta) \in C([0,T_0]; H^{s-1}(R^d))$ such that $(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \longrightarrow (\rho, u, \theta)$ in $C([0,T_0]; H^{s-1}(R^d))$. By the weak compactness of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ in $L^\infty(0, T_0; H^s(R^d))$ we also have that $(\rho, u, \theta) \in L^\infty(0, T_0; H^s(R^d))$.

By interpolation it's clear that for each $1 \leq l \leq d$,

$$x_l \partial_x^\alpha \rho^\epsilon \longrightarrow x_l \partial_x^\alpha \rho \quad \text{in} \quad C([0,T_0]; H^1(R^d)), \quad \forall |\alpha| \leq s_1,$$

$$x_l \partial_x^\alpha u^\epsilon \longrightarrow x_l \partial_x^\alpha u \quad \text{in} \quad C([0,T_0]; L^2(R^d)), \quad \forall |\alpha| \leq s_1,$$

$$x_l \partial_x^\alpha \theta^\epsilon \longrightarrow x_l \partial_x^\alpha \theta \quad \text{in} \quad C([0,T_0]; L^2(R^d)), \quad \forall |\alpha| \leq s_1.$$

Now apply ∂_x^α to the system for $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ and multiply the result by $\langle x\rangle^2$, using similar arguments we can show that

$$\langle x\rangle^2\partial_x^\alpha\rho^\epsilon \longrightarrow \langle x\rangle^2\partial_x^\alpha\rho \quad \text{in} \quad C([0,T_0];H^1(R^d)), \quad \forall|\alpha|\le s_1,$$

$$\langle x\rangle^2\partial_x^\alpha u^\epsilon \longrightarrow \langle x\rangle^2\partial_x^\alpha u \quad \text{in} \quad C([0,T_0];L^2(R^d)), \quad \forall|\alpha|\le s_1,$$

$$\langle x\rangle^2\partial_x^\alpha\theta^\epsilon \longrightarrow \langle x\rangle^2\partial_x^\alpha\theta \quad \text{in} \quad C([0,T_0];L^2(R^d)), \quad \forall|\alpha|\le s_1.$$

Based on the above results, we see that if we let $\epsilon \longrightarrow 0$ then (ρ, u, θ) will be a classical solution to the nonlinear system with $(\rho, u, \theta) \in C([0,T_0];H^{s-1}(R^d)) \cap C^1((0,T_0];H^{s-4}(R^d))$.

By considering the system for $(\rho_1-\rho_2, u_1-u_2, \theta_1-\theta_2)$ provided (ρ_1, u_1, θ_1) and (ρ_2, u_2, θ_2) are two solutions and using the linear estimate with $\epsilon = \epsilon' = 0$ for the difference as before, we show the uniqueness of the classical solution.

Overall, there exists a unique solution (ρ, u, θ) such that

$$\rho-\bar{\rho} \in C([0,T_0];H^s(R^d)) \cap C^1((0,T_0];H^{s-1}(R^d)) \cap C([0,T_0];H^{s_1+1}(\langle x\rangle^2dx)),$$

$$(u, \theta-\bar{\theta}) \in C([0,T_0];H^{s-1}(R^d)) \cap C^1((0,T_0];H^{s-4}(R^d)) \cap C([0,T_0];H^{s_1}(\langle x\rangle^2dx))$$

with $s_1 \ge d/2+6, s \ge \max\{s_1+6, N+d/2+4\}$ to the nonlinear system. $\square$

3.6 Conclusion

Given the local existence result for the dispersive Navier-Stokes system, we can begin to work on the following problems:

1. how to justify the approximation to the kinetic equation by the DNS system;

2. compared with the Navier-Stokes system, is the DNS system a better approximation to the kinetic equation?

As for the DNS system itself, we can also think about the following problems:

1. Notice that the entropy structure shows a global L^2 bound for the solution. This is a fact we haven't utilized in the above proof. Although the L^2 bound may not be strong enough for the global time existence of classical solutions, we can still ask whether there is any global existence result for solutions in a weaker sense, and whether Lions' theory for the compressible Navier-Stokes system can be applied.

2. As for the regularization effect, we want to know whether dispersion or the coupling of dispersion and dissipation give more regularity to the solution of the DNS system compared with the compressible Navier-Stokes. Using dispersive corrections provides a perspective to study the compressible Navier-Stokes system too.

3. In the assumptions for the proof, we assume that the density and temperature are both bounded away from zero. This assumption is made mainly for the dispersive estimate. Since τ_1, τ_4 in the dispersion are more dependent on the temperature than the density, it is natural to ask whether the well-posedness can be generalized to the case when there is appearance of vacuum.

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