2012

On stability and rigidity of gradient Ricci solitons

Meng Zhu
Lehigh University

ON STABILITY AND RIGIDITY OF GRADIENT RICCI SOLITONS

by

MENG ZHU

A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

Lehigh University
May 2012

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Meng Zhu

On stability and rigidity of gradient Ricci solitons

4/10/2012

Date

Prof. Huai-Dong Cao, Dissertation Director, Chair

4/17/2012

Accepted Date

Committee Members

Prof. David Johnson

Prof. Terrence Napier

Prof. Justin Corvino

To my wife, Yangmei, and my son, Zhiyuan

Acknowledgments

The author would like to express his sincere gratitude to his advisor, Professor Huai-Dong Cao. This work could not have been done without his invaluable guidance, constant encouragement and support.

Many thanks to Professor Justin Corvino, Professor David Johnson, Professor Terrence Napier, Professor Xiaofeng Sun, Professor Don Davis, Professor Linghai Zhang, Dr. Chenxu He and my fellow graduate students Qiang Chen, Yingying Zhang and Xin Cui for their understanding and many helpful discussions and advice during my stay at Lehigh.

There are many people that I owe thank to. In particular, I want to thank Professor Wei-Min Huang, Mary Ann Dent and Ms. Belinda (Yingbi) Zhang for their help and encouragement, and thank the Department of Mathematics at Lehigh University for the wonderful environment that it has provided. I also would like to thank MaryAnn Haller, the Manager of Graduate Program of the College of Arts and Sciences, for her assistance in the process of my graduation.

Finally, let me thank my wife, Yangmei, for her constant help and encouragement.

Contents

Abstract

Perelman's Ricci steady and shrinker entropies, $\lambda(g)$ and $\nu(g)$, and the Ricci expander entropy, $\nu_+(g)$, introduced by Feldman-Ilmanen-Ni are nondecreasing along the Ricci flow and their critical points are exactly compact gradient steady (i.e., Ricci flat), shrinking and expanding (i.e., negative Einstein) Ricci solitons, respectively.

In [1], Cao-Hamilton-Ilmanen presented the second variations of $\lambda(g)$ and $\nu(g)$ and investigated the entropy stability of compact Ricci flat and positive Einstein manifolds. In this paper, we first compute the second variation of $\nu_+(g)$ and briefly discuss the entropy stability of compact hyperbolic space forms. Next, we calculate the second variation of $\nu(g)$ for general compact gradient shrinking solitons which was essentially due to Cao-Hamilton-Ilmanen (first stated in [2], see also [3]). Our main contributions are that we give all the computational detail which Cao-Hamilton-Ilmanen did not show, and the last term in their formula was corrected. As an application of this formula, we obtain a necessary condition for entropy stable shrinkers in terms of the least eigenvalue and its multiplicity of certain Lichnerowicz type operator associated to the second variation.

Finally, we study the rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor. In particular, we prove that complete gradient steady Kähler-Ricci solitons with harmonic Bochner tensor are Kähler-Ricci flat, i.e., Calabi-Yau, and that complete gradient shrinking (respectively, expanding) Kähler-Ricci solitons with harmonic Bochner tensor must be isometric to a quotient of $N^k \times \mathbb{C}^{n-k}$, where N is a Kähler-Einstein manifold with positive (respectively, negative) scalar curvature.

Chapter 1

Preliminaries on Ricci Solitons

The concept of Ricci solitons was introduced by R. Hamilton [4] in the mid 1980's. The importance of Ricci solitons to the Ricci flow can be illustrated as follows:

- Ricci solitons are natural generalizations of Einstein metrics.
- Ricci solitons correspond to self-similar solutions to the Ricci flow.
- The Li-Yau-Hamilton inequality reaches equality on expanding solitons.
- Ricci solitons often appear as singularity models, i.e., the dilation limits, of singular solutions to the Ricci flow. For instance, under certain conditions, type II and type III singularity models are steady and expanding solitons. Particularly, shrinking solitons are possible type I singularity models in the Ricci flow.
- Ricci solitons are critical points of the entropy functionals. For example, compact gradient steady solitons and shrinking solitons are the critical points of Perelman's λ and ν entropies, respectively.

In this chapter, we will give the definition and introduce some well-known results on Ricci solitons.

1.1 Definition and Basic Identities

Let (M^n, g) be a Riemannian manifold with metric $g = g_{ij}dx^i \otimes dx^j$ in local coordinates $\{x^1, x^2, \cdots, x^n\}$. In the Ricci flow, we study the following degenerate parabolic equation

$$\begin{cases} \dfrac{\partial \tilde{g}_{ij}(t)}{\partial t} = -2\tilde{R}_{ij}(t), \\ \tilde{g}_{ij}(0) = g_{ij}, \end{cases} \tag{1.1.1}$$

where $\tilde{R}_{ij}(t)$ is the Ricci curvature tensor of $\tilde{g}$ at time t.

A very important part of studying the Ricci flow is the study of the geometry and classification of the Ricci solitons.

Definition 1.1.1. *A complete Riemannian manifold (M^n, g) is called a complete Ricci soliton if there exists a vector field V such that the following equation is satisfied*

$$R_{ij} + \frac{1}{2}\mathcal{L}_V g_{ij} = \lambda g_{ij}, \tag{1.1.2}$$

where $\mathcal{L}_V$ is the Lie derivative in the direction of V and λ is a constant. The cases in which $\lambda > 0$, $= 0$, or < 0 correspond to shrinking, steady or expanding solitons, respectively.

Moreover, if V is a gradient vector field, i.e., $V = \nabla f$ for some smooth function f, then we say that (M, g) is a gradient Ricci soliton with potential function f. In this case, the soliton equation (1.1.2) *becomes*

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \tag{1.1.3}$$

Remark 1.1.1. *Given a Ricci soliton (M, g_0) with vector field V satisfying* (1.1.2)*, it is easy to check that we can get the following self-similar solution to the Ricci flow with initial metric g_0:*

$$g(t) = (1 - 2\lambda t)\phi_t^* g_0,$$

where ϕ_t is the one-parameter family of diffeomorphisms generated by $\frac{1}{1-2\lambda t}V$.

Moreover, by a result of Z.-H. Zhang [5], for a complete gradient steady or shrinking Ricci soliton, the family of diffeomorphisms $\{\phi_t\}$ exists on $(-\infty, T)$ for some T.

When the underlying manifold is a Kähler manifold, we have the corresponding notion of a Kähler-Ricci soliton.

Definition 1.1.2. *A Kähler manifold $(X^n, g_{i\bar{j}})$ of complex dimension n is called a Kähler-Ricci soliton if there exists a holomorphic vector field V on X such that the following equation is satisfied*

$$R_{i\bar{j}} + \frac{1}{2}(\nabla_{\bar{j}} V_i + \nabla_i V_{\bar{j}}) = \lambda g_{i\bar{j}}, \tag{1.1.4}$$

for some real constant λ. It is a gradient Kähler-Ricci soliton if $V = \nabla f$ for some real-valued function f, i.e.,

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \lambda g_{i\bar{j}}, \quad and \quad \nabla_i \nabla_j f = 0. \tag{1.1.5}$$

Again, the cases where $\lambda = 0$, > 0 and < 0 correspond to steady, shrinking and expanding solitons, respectively.

Before we start any computation, let us clarify the notations and conventions used in this paper.

Let (M^n, g_{ij}) be a Riemannian manifold of dimension n and denote by Γ_{ij}^k, R_{ijkl}, R_{ij} and R the Christoffel symbol, Riemannian curvature tensor, Ricci curvature tensor and scalar curvature, respectively, in local coordinates $\{x^1, x^2, \cdots, x^n\}$. Thus we have

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

$$R_{ijl}^k = \frac{\partial \Gamma_{jl}^k}{\partial x^i} - \frac{\partial \Gamma_{il}^k}{\partial x^j} + \Gamma_{ip}^k \Gamma_{jl}^p - \Gamma_{jp}^k \Gamma_{il}^p,$$

$$R_{ijkl} = g_{kp} R_{ijl}^p,$$

$$R_{jl} = g^{ik} R_{ijkl} = R_{ijl}^i,$$

and

$$R = g^{jl} R_{jl}.$$

Here, we have also used Einstein convention which means that we take sum over any repeated index. For example, $g^{kl} \frac{\partial g_{il}}{\partial x^j} = \sum_{l=1}^n g^{kl} \frac{\partial g_{il}}{\partial x^j}$.

When $(X^n, g_{i\bar{j}})$ is a Kähler manifold of complex dimension n, we denote by Γ_{ij}^k, $R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j}}$ and R the Christoffel symbol, Riemannian curvature tensor, Ricci curvature tensor and scalar curvature, respectively, in holomorphic coordinates $\{z^1, z^2, \cdots, z^n\}$. Then

$$\Gamma_{ij}^k = g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j},$$

$$R_{i\bar{j}k\bar{l}} = g_{p\bar{l}} R_{i\bar{j}k}^p = \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l},$$

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log(\det(g_{i\bar{j}})),$$

and

$$R = g^{i\bar{j}} R_{i\bar{j}}.$$

According to the conventions above, we have the following Ricci identities:

$$\nabla_i \nabla_j V_k - \nabla_j \nabla_i V_k = -R_{ijk}^l V_l,$$

and

$$\nabla_i \nabla_j T_{kl} - \nabla_j \nabla_i T_{kl} = -R_{ijk}^p T_{pl} - R_{ijl}^p T_{kp},$$

where V_k and T_{kl} are $(0,1)$ and $(0,2)$ tensors, respectively.

Throughout the rest of the paper, we will always use normal coordinates near a given point to perform pointwise computations. This means that for any point p, we choose local coordinates $\{x^1, x^2, \cdots, x^n\}$ near p such that $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ for $k = 1, \cdots, n$. Therefore, we may lower all of the indices and the Ricci identities above become

$$\nabla_i \nabla_j V_k - \nabla_j \nabla_i V_k = R_{ijkl} V_l,$$

and

$$\nabla_i \nabla_j T_{kl} - \nabla_j \nabla_i T_{kl} = R_{ijkp} T_{pl} + R_{ijlp} T_{kp}.$$

Correspondingly, in the Kähler case, we have

$$\nabla_i \nabla_{\bar{j}} V_k - \nabla_{\bar{j}} \nabla_i V_k = -R_{i\bar{j}k\bar{l}} V_l,$$

$$\nabla_i \nabla_{\bar{j}} V_{\bar{l}} - \nabla_{\bar{j}} \nabla_i V_{\bar{l}} = R_{i\bar{j}k\bar{l}} V_{\bar{k}},$$

and

$$\nabla_i \nabla_{\bar{j}} T_{k\bar{l}} - \nabla_{\bar{j}} \nabla_i T_{k\bar{l}} = -R_{i\bar{j}k\bar{p}} T_{p\bar{l}} + R_{i\bar{j}p\bar{l}} T_{k\bar{p}}.$$

Lemma 1.1.1. *Let (M^n, g_{ij}) be a complete gradient Ricci soliton with potential function f satisfying* (1.1.3). *Then we have*

$$R + \Delta f = n\lambda, \tag{1.1.6}$$

$$\nabla_j R_{ik} - \nabla_i R_{jk} = R_{ijkl} \nabla_l f, \tag{1.1.7}$$

$$\nabla_i R = 2R_{ij} \nabla_j f, \tag{1.1.8}$$

and

$$R + |\nabla f|^2 - 2\lambda f = C \tag{1.1.9}$$

for some constant C.

Proof. Equation (1.1.6) is obtained by taking traces on both sides of (1.1.3). From the soliton equation (1.1.3) and the Ricci identity, we have

$$\nabla_j R_{ik} - \nabla_i R_{jk} = \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl} \nabla_l f. \tag{1.1.10}$$

This is equation (1.1.7).

Taking the trace in (1.1.7) with respect to the indices i and k and using the contracted second Bianchi identity

$$\nabla_j R_{ij} = \frac{1}{2} \nabla_i R$$

gives us (1.1.8).

The above Bianchi identity also implies that

$$\begin{aligned}
\nabla_j (R + |\nabla f|^2 - 2\lambda f) &= \nabla_j R + 2\nabla_j \nabla_l f \nabla_l f - 2\lambda \nabla_j f \\
&= 2(R_{jl} + \nabla_j \nabla_l f - \lambda g_{jl}) \nabla_l f \\
&= 0.
\end{aligned}$$

Therefore, we have for some constant C

$$R + |\nabla f|^2 - 2\lambda f = C.$$

$\square$

For a gradient Kähler-Ricci soliton, similarly we have

Lemma 1.1.2. *On a gradient Kähler-Ricci soliton satisfying* (1.1.5)*, we have*

$$R + \Delta f = n\lambda, \tag{1.1.11}$$

$$\nabla_i R_{k\bar{j}} = R_{i\bar{j}k\bar{l}} \nabla_l f, \tag{1.1.12}$$

$$\nabla_i R = R_{i\bar{j}} \nabla_j f, \tag{1.1.13}$$

and

$$R + |\nabla f|^2 - \lambda f = C \tag{1.1.14}$$

for some constant C.

Proposition 1.1.1. (Hamilton [6], Ivey [7]) *Any compact gradient steady or expanding Ricci soliton must be Einstein.*

Proof. We only present the proof for the expanding case. The proof for the steady case is similar yet simpler. Let (M^n, g_{ij}) be a compact gradient expanding soliton such that

$$R_{ij} + \nabla_i \nabla_j f = -\rho g_{ij}, \tag{1.1.15}$$

for some constant $\rho > 0$.

From Lemma 1.1.1, we have for some constant C

$$R + |\nabla f|^2 + 2\rho f = C \tag{1.1.16}$$

and

$$R + \Delta f = -n\rho. \tag{1.1.17}$$

By subtracting (1.1.16) from (1.1.17), we obtain

$$\Delta f - |\nabla f|^2 = 2\rho f + C_0,$$

where $C_0 = -n\rho - C$. Thus, at a minimum point x_1 of f, we have

$$2\rho f(x_1) + C_0 = \Delta f(x_1) \geq 0,$$

i.e., $\min_M f(x) \geq \frac{-C_0}{2\rho}$. However, at a maximum point x_2 of f, we have

$$2\rho f(x_2) + C_0 = \Delta f(x_2) \leq 0,$$

i.e., $\max_M f(x) \leq \frac{-C_0}{2\rho}$. Therefore, it must be the case that $\min_M f = \max_M f$, i.e., f is a constant. Hence, (M, g_{ij}) is an Einstein manifold. □

From the Proposition above, we see that in low dimensions (n=2 and 3), there are no compact gradient steady or expanding solitons other than the ones of constant curvature. It turns out that this is also true for compact gradient shrinking solitons.

Proposition 1.1.2. (Hamilton [8] for $n = 2$, Ivey [7] for $n = 3$) *In dimension $n \leq 3$, there are no compact gradient shrinking Ricci solitons other than those of constant positive curvature.*

In the following, we derive the evolution equations of the curvature tensors for gradient Ricci solitons.

Lemma 1.1.3. *Let (M^n, g_{ij}, f) be a gradient Ricci soliton satisfying* (1.1.3). *Then we have*

$$\Delta R_{ijkl} = \nabla_p R_{ijkl} \nabla_p f + 2\lambda R_{ijkl} - 2R_{ipkq}R_{pjql} - R_{ijpq}R_{pqkl} + 2R_{ipql}R_{pjkq}, \quad (1.1.18)$$

$$\Delta R_{ik} = \nabla_p R_{ik} \nabla_p f + 2\lambda R_{ik} - 2R_{pq}R_{ipkq}, \quad (1.1.19)$$

and

$$\Delta R = \nabla_p R \nabla_p f + 2\lambda R - 2|Rc|^2. \quad (1.1.20)$$

Proof. By the second Bianchi identity, the Ricci identity and the soliton equation

(1.1.3) we have

$$
\begin{aligned}
\Delta R_{ijkl} &= \nabla_p \nabla_p R_{ijkl} \\
&= -\nabla_p \nabla_i R_{jpkl} - \nabla_p \nabla_j R_{pikl} \\
&= -\nabla_i \nabla_p R_{jpkl} - R_{pijq} R_{qpkl} - R_{pipq} R_{jqkl} - R_{pikq} R_{jpql} - R_{pilq} R_{jpkq} \\
&\quad - \nabla_j \nabla_p R_{pikl} - R_{pjpq} R_{qikl} - R_{pjiq} R_{pqkl} - R_{pjkq} R_{piql} - R_{pjlq} R_{pikq} \\
&= \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} - \nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik} \\
&\quad - 2R_{ipjq} R_{pqkl} + R_{iq} R_{qjkl} + R_{jq} R_{iqkl} + 2R_{ipql} R_{pjkq} - 2R_{ipkq} R_{pjql} \\
&= -\nabla_i \nabla_k \nabla_l \nabla_j f + \nabla_i \nabla_l \nabla_k \nabla_j f + \nabla_j \nabla_k \nabla_l \nabla_i f - \nabla_j \nabla_l \nabla_k \nabla_i f \\
&\quad - 2R_{ipjq} R_{pqkl} + R_{iq} R_{qjkl} + R_{jq} R_{iqkl} + 2R_{ipql} R_{pjkq} - 2R_{ipkq} R_{pjql} \\
&= -\nabla_i (R_{kljp} \nabla_p f) + \nabla_j (R_{klip} \nabla_p f) \\
&\quad - 2R_{ipjq} R_{pqkl} + R_{iq} R_{qjkl} + R_{jq} R_{iqkl} + 2R_{ipql} R_{pjkq} - 2R_{ipkq} R_{pjql} \\
&= \nabla_p R_{klij} \nabla_p f - R_{kljp} (\lambda g_{ip} - R_{ip}) + R_{klip} (\lambda g_{jp} - R_{jp}) \\
&\quad - 2R_{ipjq} R_{pqkl} + R_{iq} R_{qjkl} + R_{jq} R_{iqkl} + 2R_{ipql} R_{pjkq} - 2R_{ipkq} R_{pjql} \\
&= \nabla_p R_{ijkl} \nabla_p f + 2\lambda R_{ijkl} - 2R_{ipjq} R_{pqkl} + 2R_{ipql} R_{pjkq} - 2R_{ipkq} R_{pjql} \\
&= \nabla_p R_{ijkl} \nabla_p f + 2\lambda R_{ijkl} - 2R_{ipkq} R_{pjql} - R_{ijpq} R_{pqkl} + 2R_{ipql} R_{pjkq}.
\end{aligned}
$$

In the last step above, we used the first Bianchi identity

$$R_{ipjq} = -R_{ijqp} - R_{iqpj} = R_{ijpq} + R_{iqjp}$$

to derive

$$2R_{ipjq} R_{pqkl} = R_{ijpq} R_{pqkl}.$$

By taking traces on both sides of (1.1.18) with respect to the indices j and l, we have

$$\Delta R_{ik} = \nabla_p R_{ik} \nabla_p f + 2\lambda R_{ik} - 2R_{pq} R_{ipkq} - R_{ilpq} R_{pqkl} + 2R_{ipql} R_{plkq}.$$

Notice that

$$\begin{aligned} R_{ilpq}R_{pqkl} &= R_{iplq}R_{lqkp} \\ &= -R_{iplq}(R_{qklp} + R_{klqp}) \\ &= R_{ipql}R_{plkq} - R_{ipql}R_{kqlp} \\ &= 2R_{ipql}R_{plkq}. \end{aligned}$$

Thus, one gets

$$\Delta R_{ik} = \nabla_p R_{ik} \nabla_p f + 2\lambda R_{ik} - 2R_{pq}R_{ipkq}.$$

By taking traces one more time, we have

$$\Delta R = \nabla_p R \nabla_p f + 2\lambda R - 2|Rc|^2.$$

□

1.2 Examples of Ricci Solitons

In the first section, we saw that compact gradient steady and expanding solitons are Einstein, and that it is also the case for low dimensional compact gradient shrinking solitons. However, for $n \geq 4$ there do exist nontrivial compact gradient shrinking solitons. Also, there exist complete noncompact gradient steady, shrinking and expanding Ricci solitons that are not Einstein. In this section, we will present some of these examples.

- **Examples of Compact Shrinking Solitons**

Example 1.2.1. *The first example of a compact non-Einstein gradient shrinking soliton was found by H.-D. Cao [9] and N. Koiso [10] independently. They showed the existence of a $U(n)$ symmetric gradient shrinking Kähler-Ricci soliton structure on the twisted projective line bundle $\mathbb{P}(L^k \oplus L^{-k})$ over $\mathbb{C}P^{n-1}$ for $n \geq 2$, where L is the hyperplane line bundle over $\mathbb{C}P^{n-1}$ and $1 \leq k \leq n-1$. In particular, in real dimension 4, it implies that there is a shrinking Kähler-Ricci soliton structure on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$, where the negative sign means taking the opposite orientation.*

Example 1.2.2. *In [11], Wang-Zhu proved that there is a unique Kähler-Ricci soliton structure on any toric Kähler manifold with positive first Chern class and nonvanishing Futaki invariant. In particular, in complex dimension 2, this means that a Kähler-Ricci soliton exists on* $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ *with* $U(1)\times U(1)$ *symmetry.*

- **Examples of Noncompact Shrinking Solitons**

Example 1.2.3. *Feldman-Ilmanen-Knopf [12] discovered the first example of a complete noncompact non-Einstein gradient shrinking soliton. They found a family of shrinking Kähler-Ricci solitons with* $U(n)$ *symmetry and a cone-like end at infinity on the twisted line bundle over* $\mathbb{C}P^{n-1}$.

The examples above are all constructed on Kähler manifolds. We point out that so far no example of Non-Kähler Riemannian shrinking soliton has been found.

- **Examples of Noncompact Steady Solitons**

Example 1.2.4. *The first noncompact non-Einstein steady soliton was found by Hamilton [8] on* $\mathbb{R}^2$*, called the* **cigar soliton**. *The metric and the potential function are given by*

$$ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$$

and

$$f = -log(1+x^2+y^2).$$

The cigar has positive curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at infinity.

Example 1.2.5. *R. Bryant [13] (see also [14]) proved the existence and uniqueness of a complete noncompact rotationally symmetric gradient steady soliton with positive curvature on* $\mathbb{R}^n$ *for* $n \geq 3$.

Example 1.2.6. *Examples of noncompact steady solitons on Kähler manifolds were first found by H.-D. Cao [9]. He constructed* $U(n)$ *symmetric gradient steady solitons on both* $\mathbb{C}^n$ *and the blow-up of* $\mathbb{C}^n/\mathbb{Z}_n$ *at the origin.*

• **Examples of Noncompact Expanding Solitons**

Example 1.2.7. *Besides steady solitons, R. Bryant also proved the existence of noncompact rotationally symmetric gradient expanding solitons with positive curvature on* $\mathbb{R}^n$*, see [14].*

Example 1.2.8. *A one-parameter family of gradient Kähler-Ricci expanding solitons was found by H.-D. Cao [9] on* $\mathbb{C}^n$*. These solitons are* $U(n)$ *symmetric and have positive sectional curvature. More examples of noncompact Kähler-Ricci expanding solitons were found by Feldman-Ilmanen-Knopf [12] on the twisted line bundle* L^{-k} *on* $\mathbb{C}P^{n-1}$ *for* $k = n+1, n+2, \cdots$*, where* L *is the hyperplane bundle.*

• **The Gaussian Solitons**

Example 1.2.9. *The Euclidean space* $(\mathbb{R}^n, g_0)$ *with the flat Euclidean metric can be considered as either a gradient shrinking or an expanding soliton, called the Gaussian shrinker or expander, respectively.*
i) The Gaussian shrinker has potential function $f = \frac{|x|^2}{4}$ *satisfying*

$$Rc_0 + Hess(f) = \frac{1}{2} g_0.$$

ii) The Gaussian expander has potential function $f = -\frac{|x|^2}{4}$ *satisfying*

$$Rc_0 + Hess(f) = -\frac{1}{2} g_0.$$

For more examples of Ricci solitons, we refer the readers to the survey paper [3] of H.-D. Cao.

1.3 Geometry of Gradient Ricci Solitons

In this section, we will discuss some important geometric properties and classification results of gradient Ricci solitons.

1.3.1 Geometry of Gradient Shrinking Ricci Solitons

By definition, an **ancient solution** of the Ricci flow is a complete solution existing on the time interval $(-\infty, T)$ for some T. In [5], Z.-H. Zhang proved

Proposition 1.3.1. (Z.-H. Zhang [5]) *Any gradient shrinking or steady Ricci soliton whose underlying manifold is complete must have nonnegative scalar curvature.*

As a corollary, it follows that the gradient vector field of the potential function is complete, and hence one can construct an ancient solution from the shrinking or steady soliton. On the other hand, we have the following nice curvature properties of ancient solutions:

Theorem 1.3.1. (B.-L. Chen [15]) *Any 3-dimensional ancient solution must have nonnegative sectional curvature.*

Proposition 1.3.2. (B.-L. Chen [15]) *Any ancient solution must have nonnegative scalar curvature.*

Proposition 1.3.1 and Theorem 1.3.1 immediately imply the following:

Proposition 1.3.3. *Any 3-dimensional complete gradient shrinking or steady Ricci soliton must have nonnegative sectional curvature.*

When a complete shrinking soliton has bounded nonnegative curvature operator, by a maximum principle of Hamilton [4], it either has positive operator everywhere or its universal cover splits as $N \times \mathbb{R}^k$ with $k \geq 1$ and N a shrinking soliton with positive curvature operator. Moreover, if a shrinking soliton with positive curvature operator is compact, then it must be a finite quotient of the round sphere by the results of Hamilton [4, 16] (for $n = 3, 4$) and Böhm-Wilking [17] (for $n \geq 5$).

Indeed, Perelman [18] showed that, in dimension 3, there is no noncompact gradient shrinking soliton with bounded positive curvature operator.

Proposition 1.3.4. (Perelman [18]) *Any complete 3-dimensional gradient shrinking soliton with bounded positive sectional curvature must be compact.*

Remark 1.3.1. *In the Kähler case, Ni [19] has the following similar result:*

Proposition 1.3.5. (Ni [19]) *In any complex dimension, there is no complete noncompact gradient shrinking Kähler-Ricci soliton with positive holomorphic bisectional curvature.*

Based on Proposition 1.3.4, Perelman [18] obtained the following important classification result:

Theorem 1.3.2. (Perelman [18]) *Any complete 3-dimensional nonflat gradient shrinking soliton with bounded nonnegative sectional curvature must be either a quotient of* $\mathbb{S}^3$ *or a quotient of* $\mathbb{S}^2 \times \mathbb{R}$.

In the past decade, a lot of effort has been made to improve and generalize this result of Perelman. Ni-Wallach [20] and Naber [21] replaced the assumption of nonnegative sectional curvature by nonnegative Ricci curvature. In addition, instead of assuming bounded curvature, Ni-Wallach [20] allows the curvature to grow as fast as $e^{ar(x)}$, where $r(x)$ is the distance function and $a > 0$ is some constant. More specifically, they proved

Proposition 1.3.6. (Ni-Wallach [20]) *Any 3-dimensional complete noncompact non-flat gradient shrinking soliton with* $Rc \geq 0$ *and* $|Rm|(x) \leq e^{ar(x)}$ *must be a quotient of the round cylinder* $\mathbb{S}^2 \times \mathbb{R}$.

Based on Theorem 1.3.1 and Proposition 1.3.6, Cao-Chen-Zhu [22] were able to remove all the assumptions on the curvature.

Theorem 1.3.3. (Cao-Chen-Zhu [22]) *Any 3-dimensional complete noncompact non-flat gradient shrinking soliton must be a quotient of the round cylinder* $\mathbb{S}^2 \times \mathbb{R}$.

For $n = 4$, Ni-Wallach [23] showed that any 4-dimensional gradient shrinking soliton with nonnegative curvature operator and positive isotropic curvature, satisfying certain additional assumptions, is a quotient of $\mathbb{S}^4$ or $\mathbb{S}^3 \times \mathbb{R}$. Using this result, Naber [21] proved

Theorem 1.3.4. (Naber [21]) *Any 4-dimensional complete noncompact shrinking Ricci soliton with bounded nonnegative curvature operator is isometric to* $\mathbb{R}^4$*, or a finite quotient of* $\mathbb{S}^3 \times \mathbb{R}$ *or* $\mathbb{S}^2 \times \mathbb{R}^2$*.*

For higher dimensions, the classification of gradient shrinking solitons was solved under the assumption that the Weyl tensor vanishes by the work of Eminenti-La Nave-Mantegazza [24], Ni-Wallach [20], Z.-H. Zhang [25], Petersen-Wylie [26] and Munteanu-Sesum [27].

Eminenti-La Nave-Mantegazza [24] showed that any compact shrinking Ricci soliton with vanishing Weyl tensor is a quotient of $\mathbb{S}^n$.

In the noncompact case, Ni-Wallach [20] proved

Proposition 1.3.7. (Ni-Wallach [20]) *Let* (M^n, g_{ij}) *be a locally comformally flat gradient shrinking soliton with* $Rc \geq 0$*. Assume that*

$$|Rm|(x) \leq e^{a(r(x)+1)}$$

for some constant $a > 0$*. Then its universal cover is* $\mathbb{R}^n$*,* $\mathbb{S}^n$ *or* $\mathbb{S}^n \times \mathbb{R}$*.*

By showing that locally conformally flat gradient shrinking solitons have non-negative curvature operator and utilizing the above result, Z.-H. Zhang [25] proved

Theorem 1.3.5. (Z.-H. Zhang [25]) *Any gradient shrinking soliton with vanishing Weyl tensor must be a finite quotient of* $\mathbb{R}^n$*,* $\mathbb{S}^n$ *or* $\mathbb{S}^n \times \mathbb{R}$*.*

The work of Petersen-Wylie [26] and of Munteanu-Sesum [27] give another path to get the same classification result. Indeed, Petersen-Wylie first showed

Proposition 1.3.8. (Petersen-Wylie [26]) *Let* (M^n, g_{ij}, f) *be a gradient shrinking soliton with potential function* f*. If the Weyl tensor vanishes and*

$$\int_M |Rc|^2 e^{-f} dV < \infty.$$

Then (M^n, g) *is a finite quotient of* $\mathbb{R}^n$*,* $\mathbb{S}^n$ *or* $\mathbb{S}^n \times \mathbb{R}$*.*

Munteanu-Sesum [27] later proved the L^2 integrability of the Ricci tensor based on the following Cao-Zhou's growth estimate of the potential function [28].

Proposition 1.3.9. (Cao-Zhou [28]) *Let (M^n, g_{ij}, f) be a gradient shrinking soliton with potential function f. Then f satisfies the following estimate*

$$\frac{1}{4}(r(x)-c_1)^2 \le f(x) \le \frac{1}{4}(r(x)+c_2)^2.$$

Here $r(x)$ is the distance from an origin O, c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} in the unit ball $B_O(1)$.

Recently, the same rigidity result has been obtained under two types of weaker assumptions than vanishing of the Weyl tensor.

The first type of weaker assumption is harmonicity of the Weyl tensor, i.e., $\operatorname{div} W = 0$. Fernádez-López and García-Río [29] showed

Proposition 1.3.10. (Fernádez-López and García-Río [29]) *Any complete gradient shrinking soliton (M^n, g_{ij}, f) with harmonic Weyl tensor and*

$$\int_M |\operatorname{div} Rm|^2 e^{-f} dV = \int_M |\nabla Rc|^2 e^{-f} dV \tag{1.3.1}$$

must be rigid, i.e. it is a quotient of $N^{n-k} \times \mathbb{R}^k$, where $0 \le k \le n$, N is an Einstein manifold and $\mathbb{R}^k$ is the Gaussian shrinker.

Again, Muteanu-Sesum [27] used Cao-Zhou's potential function growth estimate to prove (1.3.1). Therefore, we have

Theorem 1.3.6. *Any complete gradient shrinking soliton with harmonic Weyl tensor must rigid.*

The second type of weaker assumption is vanishing of the Bach tensor. This is a result of Cao-Chen [30].

Theorem 1.3.7. (Cao-Chen [30]) *Any complete gradient shrinking soliton (M^n, g_{ij}) with vanishing Bach tensor and $n \ge 4$ is either Einstein, a finite quotient of the Gaussian shrinking soliton or a finite quotient of $N^{n-1} \times \mathbb{R}$, where N is an Einstein manifold with positive scalar curvature.*

1.3.2 Geometry of Gradient Steady and Expanding Ricci Solitons

Since compact steady and expanding solitons are necessarily Einstein, our discussion here only concerns noncompact ones.

Proposition 1.3.11. (Hamilton [6]) *Suppose that we have a noncompact gradient steady soliton (M^n, g_{ij}) so that*

$$R_{ij} = \nabla_i \nabla_j f$$

for some function f. Assume that the Ricci curvature is positive and the scalar curvature attains its maximum R_{max} at some point x_0. Then

$$|\nabla f|^2 + R = R_{max}$$

on M^n. Moreover, the function f is convex and attains its minimum at x_0.

Remark 1.3.2. *Cao-Chen [31] also showed that in this case, the function f is an exhaustion function of linear growth. Hence we have*

Proposition 1.3.12. *A complete noncompact gradient steady soliton with positive Ricci curvature whose scalar curvature attains its maximum at some point must be diffeomorphic to $\mathbb{R}^n$.*

Remark 1.3.3. *If a complete noncompact gradient expanding soliton has nonnegative Ricci curvature, then the potential function f is a convex exhaustion function of quadratic growth. Therefore, we have*

Proposition 1.3.13. *A complete noncompact gradient expanding soliton with nonnegative Ricci curvature must be diffeomorphic to $\mathbb{R}^n$.*

In the Kähler setting, Cao-Hamilton first showed that any noncompact gradient steady Kähler-Ricci soliton with positive Ricci curvature whose scalar curvature attains its maximum at some point is Stein (and also diffeomorphic to $\mathbb{R}^{2n}$). Later, Chau-Tam [32] and Bryant [33] independently improved the result to the following

Theorem 1.3.8. (Chau-Tam [32] and Bryant [33]) *Any noncompact gradient steady Kähler-Ricci soliton with positive Ricci curvature whose scalar curvature attains its maximum at some point is biholomorphic to* $\mathbb{C}^n$*.*

Moreover, Chau-Tam [32] also showed

Theorem 1.3.9. (Chau-Tam [32]) *A complete noncompact gradient expanding soliton with nonnegative Ricci curvature must be biholomorphic to* $\mathbb{C}^n$*.*

The classification of steady Ricci solitons with positive curvature is one of the basic problems in the study of Ricci solitons. In dimension 2, Hamilton [8] proved the following important uniqueness Theorem:

Theorem 1.3.10. (Hamilton [8]) *The only complete steady Ricci soliton on a 2-dimensional manifold with bounded curvature* R *which assumes its maximum* $R_{max} = 1$ *at some point is the cigar soliton on* $\mathbb{R}^2$*.*

Remark 1.3.4. *For a gradient steady soliton, one can remove all the assumptions in the above Theorem. Indeed, by Proposition 1.3.1 and equation* (1.1.9)*, a complete 2-dimensional steady soliton must have bounded nonnegative curvature. Thus by Hamilton's maximum principle [4], we know that it is either flat or has positive curvature. Then a theorem in [14] shows that a 2-dimensional gradient steady soliton with bounded positive curvature must be the cigar soliton.*

In dimension 3, Perelman [18] claimed that any complete noncompact κ noncollapsed gradient steady soliton with bounded positive curvature must be the Bryant soliton. However, he did not provide a proof. The first progress on this problem is made by Cao-Chen [31]. They showed

Theorem 1.3.11. *Let* (M^n, g_{ij}, f)*,* $n \geq 3$*, be an* n*-dimensional complete noncompact locally conformally flat gradient steady soliton. Then* (M^n, g_{ij}, f) *is either flat or isometric to the Bryant soliton.*

Remark 1.3.5. *For* $n \geq 4$*, Catino-Mantegazza [34] independently proved this result by using a different method.*

Remark 1.3.6. *If in the above theorem we only assume that the sectional curvature is positive, then we have uniqueness, i.e.* (M^n, g_{ij}, f) *must be the Bryant soliton.*

Subsequently, Cao-Chen's work has been used by S. Brendle [35] in classifying 3-dimensional steady Ricci solitons satisfying a certain asymptotic condition, and X. X. Chen-Y. Wang [36] in classifying 4-dimensional half-conformally flat steady solitons, respectively.

Recently, Cao-Catino-Chen-Mantegazza-Mazzieri [37] established the uniqueness of gradient steady solitons by assuming Bach-flatness of the manifold, which is a weaker condition than locally conformal flatness and half-conformal flatness.

Proposition 1.3.14. (Cao-Catino-Chen-Mantegazza-Mazzieri [37]) *Any complete Bach-flat gradient steady soliton with positive Ricci curvature such that the scalar curvature attains its maximum at an interior point is isometric to the Bryant soliton.*

Very recently, the claim of Perelman above on the uniqueness of κ-noncollapsed 3 dimensional gradient steady solitons with positive curvature was finally proved by S. Brendle [38].

Theorem 1.3.12. (Brendle [38]) *Any complete 3-dimensional non-flat* κ *noncollapsed gradient steady soliton must be isometric to the Bryant soliton up to scaling.*

Remark 1.3.7. *For higher dimension steady solitons, Brendle [39] also has the following result:*

Proposition 1.3.15. (Brendle [39]) *Let* (M^n, g_{ij}) *be a gradient steady soliton of dimension* $n \geq 4$. *Assume that* M *has positive sectional curvature and is asymptotically cylindrical. Then* (M^n, g_{ij}) *is rotationally symmetric.*

Chapter 2

The Entropy Functionals and Stability of Compact Ricci solitons

2.1 Introduction and Main Results

In [18], Perelman introduced the $\mathcal{F}$ and $\mathcal{W}$ functionals on a compact manifold (M^n, g_{ij}), which are defined as

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2)e^{-f}dV,$$

and

$$\mathcal{W}(g_{ij}, f, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(R + |\nabla f|^2) + f - n]e^{-f}dV,$$

where R is the scalar curvature of M, f is a smooth function and $\tau > 0$ is a constant. The corresponding entropy functionals are the following λ and ν functionals:

$$\lambda(g_{ij}) = \inf\{\mathcal{F}(g_{ij}, f, \tau)|\ f \in C^\infty(M),\ \text{and}\ \int_M e^{-f}dV = 1\},$$

and

$$\nu(g_{ij}) = \inf\{\mathcal{W}(g_{ij}, f, \tau)|\ f \in C^\infty(M),\ \tau > 0,\ \text{and}\ (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}dV = 1\}.$$

He also showed that the λ and ν entropies are monotone increasing under the

Ricci flow, and their critical points are precisely given by gradient steady and shrinking solitons satisfying equations

$$R_{ij} + \nabla_i \nabla_j f = 0, \tag{2.1.1}$$

and

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2\tau} g_{ij}, \tag{2.1.2}$$

respectively. In particular, it follows that all compact steady and shrinking Ricci solitons are gradient Ricci solitons.

To find the corresponding variational structure for the expanding case, M. Feldman, T. Ilmanen and L. Ni [40] introduced the $\mathcal{W}_+$ functional. Let (M^n, g) be a compact Riemannian manifold, f a smooth function on M, and $\sigma > 0$. Define

$$\mathcal{W}_+(g, f, \sigma) = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} [\sigma(|\nabla f|^2 + R) - f + n] dV,$$

and the corresponding entropy

$$\nu_+(g) = \sup_{\sigma > 0} \mu_+(g, \sigma),$$

where

$$\mu_+(g, \sigma) = \inf\{W_+(g, f, \sigma) |\ f \in C^\infty(M), \text{ and } (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} dV = 1\}.$$

They showed that ν_+ is nondecreasing along the Ricci flow and constant precisely on gradient expanding solitons such that

$$R_{ij} + \nabla_i \nabla_j f = -\frac{1}{2\sigma} g_{ij}. \tag{2.1.3}$$

Therefore, compact expanding solitons must also be gradient solitons.

Among all three kinds of compact Ricci solitons, the shrinkers are the ones of the most interest because compact gradient steady or expanding Ricci solitons must be Einstein by Proposition 1.1.1. In Chapter 1, we have seen that many classification and rigidity results have been obtained for gradient shrinking solitons. Moreover, while under many circumstances shrinking solitons are rigid, there do exist non-Einstein non-product gradient shrinking solitons. For example, in dimension 4, the Kähler-Ricci soliton found by Cao [9] and Koiso [10] on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$, and Feldman-Ilmanen-Knopf soliton [12] on the blow-up of $\mathbb{C}^2$ at the origin.

One may notice that most of the classification results for shrinking solitons have assumed either directly or implicitly that the curvature operator is nonnegative. For example, locally conformally flat gradient shrinking solitons have nonnegative curvature operator. However, this condition may not be satisfied in general. For instance, Cao-Koiso's example mentioned above on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ only has positive Ricci curvature, but the curvature operator changes sign. Therefore, shrinking solitons with nonnegative curvature operator may not represent general singularity models.

On the other hand, as far as applications of the Ricci flow to topology are concerned, it is actually more interesting to study stable shrinking solitons, since generic singularity models are expected to be stable. By definition, a compact Ricci soliton (M, g_{ij}) is said to be **entropy stable**, if the second variation of the corresponding entropy functional at g_{ij} is nonpositive, i.e., g_{ij} is a local maximum point. The study of the stability of Ricci solitons was initiated by H.-D. Cao, R. Hamilton and T. Ilmanen [1]. They presented the following second variational formulas of the λ and ν entropies for Ricci flat and positive Einstein manifolds, respectively, and investigated the entropy stability of certain Einstein manifolds.

Theorem 2.1.1. (Cao-Hamilton-Ilmanen [1]) *Let (M, g_{ij}) be a compact Ricci flat manifold and consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then the second variation of the λ entropy is given by*

$$\frac{d^2}{ds^2}\bigg|_{s=0} \lambda(g(s)) = \int_M < Lh, h > dV,$$

where

$$Lh := \frac{1}{2}\Delta h + Rm(h, \cdot) + \mathrm{div}^* \,\mathrm{div}\; h + \frac{1}{2}\nabla^2 v_h,$$

and v_h satisfies

$$\Delta v_h = \mathrm{div}\,\mathrm{div}\; h.$$

Theorem 2.1.2. (Cao-Hamilton-Ilmanen [1]) *Let (M^n, g_{ij}) be an Einstein manifold with $Rc = \frac{1}{2\tau} g_{ij}$ for some $\tau > 0$, and consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then the second variation $\delta_g^2 \nu(h,h)$ of the ν entropy is given by*

$$\frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{\tau}{\mathrm{Vol}(M,g)} \int_M < Nh, h > dV,$$

where

$$Nh := \frac{1}{2}\Delta h + Rm(h,\cdot) + \mathrm{div}^* \, \mathrm{div} \; h + \frac{1}{2}\nabla^2 v_h - \frac{g}{2n\tau \, \mathrm{Vol}(M,g)} \int_M \mathrm{tr}_g \, h \, dV,$$

and v_h is the unique solution of

$$\Delta v_h + \frac{v_h}{2\tau} = \mathrm{div}\,\mathrm{div} \; h \quad \textit{and} \quad \int_M v_h dV = 0.$$

In the above Theorems, we used the notation

$$Rm(h,\cdot)_{ij} = g^{kq} g^{pl} R_{ipjq} h_{kl}.$$

Our first main result is the following second variation of the ν_+ functional for compact expanding solitons, i.e., negative Einstein manifolds.

Theorem 2.1.3. (Z. [41]) *Let (M^n, g) be a compact negative Einstein manifold such that $R_{ij} = -\frac{1}{2\sigma} g_{ij}$ for some $\sigma > 0$. Let h be a symmetric 2-tensor. Consider the variation of metric $g(s) = g + sh$. Then the second variation of ν_+ is*

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} \nu_+(g(s)) = \frac{\sigma}{\mathrm{Vol}(g)} \int_M < N_+ h, h > dV,$$

where

$$N_+ h := \frac{1}{2}\Delta h + \mathrm{div}^* \mathrm{div} h + \frac{1}{2}\nabla^2 v_h + Rm(h,\cdot) + \frac{g}{2n\sigma \mathrm{Vol}(g)} \int_M trh \; dV,$$

and v_h is the unique solution of

$$\Delta v_h - \frac{v_h}{2\sigma} = \mathrm{div}(\mathrm{div} h) \quad \textit{and} \quad \int_M v_h dV = 0.$$

Furthermore, analogous to the work of Cao-Hamilton-Ilmanen in [1], we also investigated the entropy stability of certain negative Einstein manifolds. As an example, we found that compact hyperbolic spaces are entropy stable. But, unlike the positive Einstein and Ricci flat cases, it seems hard to find other examples of negative Einstein manifolds which are either entropy stable or unstable.

Returning to the study of stable shrinking solitons, as part of their examples, Cao-Hamilton-Ilmanen showed that, while S^n and $\mathbb{C}P^n$ are entropy stable, many known positive Einstein manifolds are unstable. In particular, they proved that in complex dimension 2, except for $\mathbb{C}P^2$, there are no stable positive Kähler-Einstein surfaces. Actually, the work of Cao-Hamilton-Ilmanen suggested that most compact gradient shrinking solitons should be unstable, in the sense that they are not the maximal points of the ν-entropy, hence after a small perturbation along certain direction, we can get higher entropy, and the entropy keeps increasing along the Ricci flow so that the solution will never converge back to the original metric. Also for this reason, unstable shrinking solitons may not be able to represent generic singularity models. Thus, it is desirable and important to classify stable shrinking solitons. Certainly, in the study of the stability of shrinking solitons, the second variational formula of the ν-entropy plays an indispensable role.

Other than the second variational formula of the ν-entropy for positive Einstein manifolds, Cao-Hamilton-Ilmanen also announced in [1] (see also [2]) that there was also a second variational formula for general compact shrinking solitons (the formula can be found in [3]). However, the last term in the formula stated in [3] was incorrect. Cao and the author recalculated the second variation of the ν-entropy and gave the detailed computation in [42].

Before stating our next main result, we need to introduce the following notations

$$(\operatorname{div}_f h)_i = g^{jk}(\nabla_j h_{ik} - h_{ik}\nabla_j f),$$

$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

and

$$\int_M < \operatorname{div}_f^{\dagger} w, h > e^{-f} dV = \int_M < w, \operatorname{div}_f h > e^{-f} dV,$$

$$\text{i.e.,}\qquad \operatorname{div}_f^\dagger w = -\frac{1}{2}\mathcal{L}_{w^\#}g.$$

where h is a symmetric 2-tensor, w is a 1-form and $w^\#$ is its dual vector field.

Now we are ready to state the second variation of the ν-entropy for general compact shrinking solitons.

Theorem 2.1.4. (Cao-Hamilton-Ilmanen)(Cao-Z. [42]) *Let (M^n, g_{ij}, f) be a compact gradient Ricci shrinking soliton with potential function f satisfying the Ricci soliton equation* (2.1.2)*. For any symmetric 2-tensor $h = h_{ij}$, consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then the second variation of ν is given by*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \nu(g(s)) = \frac{\tau}{(4\pi\tau)^{n/2}} \int_M < \hat{N}h, h > e^{-f} dV,$$

where the stability operator $\hat{N}$ is given by

$$\hat{N}h := \frac{1}{2}\Delta_f h + Rm(h,\cdot) + \operatorname{div}_f^\dagger \operatorname{div}_f h + \frac{1}{2}\nabla^2 \hat{v}_h - Rc\, \frac{\int_M < Rc, h > e^{-f} dV}{\int_M Re^{-f} dV}, \quad (2.1.4)$$

and $\hat{v}_h$ is the unique solution of

$$\Delta_f \hat{v}_h + \frac{\hat{v}_h}{2\tau} = \operatorname{div}_f \operatorname{div}_f h \quad \textit{and} \quad \int_M \hat{v}_h e^{-f} dV = 0.$$

Remark 2.1.1. *As we pointed out before, Theorem 2.1.4 is essentially due to Cao-Hamilton-Ilmanen (cf. Theorem 6.3 in [3]). However, the coefficient of the last term of the stability operator $\hat{N}$ (which depends on $\delta\tau$, the first variation of the parameter τ) was stated incorrectly in [3]. One of our contributions is deriving an explicit formula for $\delta\tau$ (see Lemma 2.3.2 below), thus obtaining the correct coefficient and hence a complete second variation formula for Ricci shrinkers. We emphasize that, while the stability operator $\hat{N}$ is already quite useful even without knowing the explicit coefficient of the last term, it will be rather crucial to have this explicit and correct coefficient in efforts of trying to classify stable shrinkers. For example, this explicit coefficient is essential in showing that the Ricci tensor is a null eigen-tensor of $\hat{N}$ (see Lemma 2.4.3) which rules out any hope of using the Ricci tensor as a possible unstable direction.*

Remark 2.1.2. *In the very recent work [43], S. Hall and T. Murphy proved that compact Kähler-Ricci shrinking solitons with Hodge number* $h^{1,1} > 1$ *are unstable, thus extended the result of Cao-Hamilton-Ilmanen [1] in the Kähler-Einstein case. In the course of their proof, they also verified the second variation formula stated in [44], though they didn't find out explicitly the coefficient of the last term of* $\hat{N}$ *(which does not affect the proof of their result since they only considered certain special variations orthogonal to Rc).*

Finally, using the second variation formula, we obtain the following necessary condition for stable shrinkers:

Proposition 2.1.1. *Suppose* (M^n, g_{ij}, f) *is a compact stable shrinking soliton satisfying* (2.1.2)*, then* $-\frac{1}{2\tau}$ *is the only negative eigenvalue of the operator* $\mathcal{L}_f$ *(with Rc being an eigen-tensor), defined by*

$$\mathcal{L}_f h = \frac{1}{2}\Delta h + Rm(h, \cdot), \tag{2.1.5}$$

on $\ker \operatorname{div}_f$ *and the multiplicity of* $-\frac{1}{2\tau}$ *is one. In particular,* $-\frac{1}{2\tau}$ *is the least eigenvalue of* $\mathcal{L}_f$ *on* $\ker \operatorname{div}_f$*.*

Remark 2.1.3. *In proving Proposition 2.1.1, the explicit coefficient of the Rc term in* $\hat{N}$ *is not needed.*

Remark 2.1.4. *In the mean curvature flow, Colding and Minicozzi [45] have shown that for any shrinker its mean curvature* H *is an eigenfunction of certain operator involved in the corresponding stability operator, and that for any stable shrinker the mean curvature function* H *belongs to the least eigenvalue of the operator which in turn implies that* H *does not change sign. This fact and a theorem of Huisken allowed them to classify compact stable mean curvature shrinkers. Our Proposition 2.1.1 above can be considered as the Ricci flow analogue of their results.*

2.2 The First Variations of the Entropy functionals

Let (M, g_{ij}) be a compact n-dimensional Riemannian manifold. In this section we will state some basic properties of the entropy functionals and compute their first variations. Recall that the $\mathcal{F}$, $\mathcal{W}$ and $\mathcal{W}_+$ functionals are defined as follows:

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f} dV, \tag{2.2.1}$$

$$\mathcal{W}(g_{ij}, f, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(R + |\nabla f|^2) + f - n] e^{-f} dV, \tag{2.2.2}$$

and

$$\mathcal{W}_+(g, f, \sigma) = (4\pi\sigma)^{-\frac{n}{2}} \int_M [\sigma(|\nabla f|^2 + R) - f + n] e^{-f} dV, \tag{2.2.3}$$

where R is the scalar curvature of M, f is a smooth function and $\tau, \sigma > 0$ are constants.

The corresponding entropy functionals are

$$\lambda(g_{ij}) = \inf\{\mathcal{F}(g_{ij}, f) | f \in C^\infty(M), \text{ and } \int_M e^{-f} dV = 1\}, \tag{2.2.4}$$

$$\nu(g_{ij}) = \inf_{\tau > 0} \mu(g, \tau), \tag{2.2.5}$$

where $\mu(g, \tau) = \inf\{\mathcal{W}(g_{ij}, f, \tau) |\ f \in C^\infty(M), \text{ and } (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} dV = 1\}$, and

$$\nu_+(g) = \sup_{\sigma > 0} \mu_+(g, \sigma), \tag{2.2.6}$$

where $\mu_+(g, \sigma) = \inf\{W_+(g, f, \sigma) |\ f \in C^\infty(M), \text{ and } (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} dV = 1\}$.

First of all, we show that $\lambda(g)$, $\nu(g)$ and $\nu_+(g)$ are actually achievable.

Note that if we set $u = e^{-\frac{f}{2}}$, then the $\mathcal{F}$ functional becomes

$$\mathcal{F}(g_{ij}, u) = \int_M (Ru^2 + 4|\nabla u|^2) dV,$$

and the constraint $\int_M e^{-f} dV = 1$ becomes $\int_M u^2 dV = 1$. Therefore, $\lambda(g_{ij})$ is just the first eigenvalue of the operator $-4\Delta + R$.

Moreover, under the same substitution $u = e^{-\frac{f}{2}}$, the $\mathcal{W}$ functional can be expressed as

$$\mathcal{W}(g_{ij}, u, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2] dV.$$

According to the work of Rothous [46], $\mu(g_{ij}, \tau)$ can be achieved by some positive function u such that $(4\pi\tau)^{-\frac{n}{2}} \int_M u^2 dV = 1$. Then the following Proposition of Perelman [18] shows that $\nu(g_{ij})$, as the infimum of $\mu(g_{ij}, \tau)$, can be further realized by some τ.

Proposition 2.2.1. (Perelman [18]) *For an arbitrary metric g_{ij} on a compact manifold M, the function $\mu(g_{ij}, \tau)$ is negative for small $\tau > 0$ and $\lim_{\tau \to 0^+} \mu(g_{ij}, \tau) = 0$, $\lim_{\tau \to \infty} \mu(g_{ij}, \tau) = \infty$.*

Similarly, for the $\mathcal{W}_+$ functional, if we set $u = (4\pi\sigma)^{-\frac{n}{2}} e^{-f}$, then we have

$$\mathcal{W}_+(g_{ij}, f, \sigma) = \int_M \left[\sigma(\frac{|\nabla u|^2}{u} + Ru) + u \log u\right] dV + \frac{n}{2} \log(4\pi\sigma) + n.$$

Feldman-Ilmanen-Ni [40] proved that $\mu_+(g_{ij}, \sigma)$ and $\nu_+(\sigma)$ can be achieved.

Proposition 2.2.2. (Feldman-Ilmanen-Ni [40]) *Let (M, g_{ij}) be a compact manifold.*
1) $\mu_+(g_{ij}, \sigma)$ can be attained by a unique function u.
2) If $\lambda(g_{ij}) < 0$, then $\nu_+(g_{ij})$ is attained by a unique $\sigma > 0$.

Now, let us compute the first variations of the entropy functionals. Since the computations are similar, we will only present the details for the first variation of the ν functional.

Lemma 2.2.1. (Perelman [18]) *If we have variations $v_{ij} = \delta g_{ij}$, $h = \delta f$, and*

$\eta = \delta\tau$, then the first variation of the $\mathcal{W}$ functional is

$$
\begin{aligned}
&\delta\mathcal{W}(v_{ij}, h, \eta) \\
&= (4\pi\tau)^{-\frac{n}{2}} \int_M -\tau v_{ij}(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}) e^{-f} dV \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M (\frac{1}{2}\operatorname{tr}_g v - h - \frac{n}{2\tau}\eta)[\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1] e^{-f} dV \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M \eta(R + |\nabla f|^2 - \frac{n}{2\tau}) e^{-f} dV.
\end{aligned}
$$

Proof. First, let us recall the formulas of the connection coefficients and the curvatures in local coordinates

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}),$$

$$R_{ijl}^k = \frac{\partial \Gamma_{jl}^k}{\partial x_i} - \frac{\partial \Gamma_{il}^k}{\partial x_j} + \Gamma_{ip}^k \Gamma_{jl}^p - \Gamma_{jp}^k \Gamma_{il}^p,$$

$$R_{jl} = R_{kjl}^k,$$

and

$$R = g^{jl} R_{jl}.$$

In the normal coordinates at a fixed point, we have

$$
\begin{aligned}
\delta\Gamma_{ij}^k &= \frac{1}{2} g^{kl}(\frac{\partial v_{il}}{\partial x_j} + \frac{\partial v_{jl}}{\partial x_i} - \frac{\partial v_{ij}}{\partial x_l}) \\
&= \frac{1}{2} g^{kl}(\nabla_j v_{il} + \nabla_i v_{jl} - \nabla_l v_{ij}),
\end{aligned}
$$

$$
\begin{aligned}
\delta R_{jl} &= \delta R_{kjl}^k \\
&= \frac{\partial \delta\Gamma_{jl}^k}{\partial x_k} - \frac{\partial \delta\Gamma_{kl}^k}{\partial x_j} \\
&= \frac{\partial}{\partial x_k}\left[\frac{1}{2} g^{kp}(\nabla_j v_{pl} + \nabla_l v_{jp} - \nabla_p v_{jl})\right] - \frac{\partial}{\partial x_j}\left[\frac{1}{2} g^{kp} \nabla_l v_{kp}\right] \\
&= \frac{1}{2}(\nabla_p \nabla_l v_{jp} + \nabla_p \nabla_j v_{lp} - \Delta v_{jl} - \nabla_j \nabla_l tr_g v),
\end{aligned}
$$

$$
\begin{aligned}
\delta R &= \delta(g^{jl}R_{jl}) \\
&= -v_{jl}R_{jl} + g^{jl}\delta R_{jl} \\
&= -v_{jl}R_{jl} + \nabla_i\nabla_j v_{ij} - \Delta tr_g v,
\end{aligned}
$$

and

$$
\delta\sqrt{\det(g_{ij})} = \frac{1}{2\sqrt{\det(g_{ij})}} v_{ij}C_{ji} = \frac{1}{2}\sqrt{\det(g_{ij})}v_{ij}g^{ji} = \frac{tr_g v}{2}\sqrt{\det(g_{ij})},
$$

where (C_{ij}) is the matrix of cofactors.

Thus, the first variation of the $\mathcal{W}$ functional is

$$
\begin{aligned}
&\delta\mathcal{W}(v_{ij}, h, \eta) \\
&= \int_M [\eta(R + |\nabla f|^2) + \tau(-\Delta tr_g v + \nabla_i\nabla_j v_{ij} - v_{ij}R_{ij} - v_{ij}\nabla_i f\nabla_j f \\
&\quad + 2\nabla_i f\nabla_i h) + h](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M [\tau(R + |\nabla f|^2) + f - n](-\frac{n}{2}\frac{\eta}{\tau} + \frac{tr_g v}{2} - h)(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&= \int_M [\eta(R + |\nabla f|^2) + h](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M [-\tau v_{ij}(R_{ij} + \nabla_i\nabla_j f) + \tau(tr_g v - 2h)(\Delta f - |\nabla f|^2)](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M [\tau(R + |\nabla f|^2) + f - n](-\frac{n}{2}\frac{\eta}{\tau} + \frac{tr_g v}{2} - h)(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&= -\int_M \tau v_{ij}(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij})(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M (\frac{tr_g v}{2} - h - \frac{n}{2\tau}\eta)[\tau(R + |\nabla f|^2) + f - n + 2\tau(\Delta f - |\nabla f|^2)](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M [\eta(R + |\nabla f|^2 - \frac{n}{2\tau}) + (h - \frac{tr_g v}{2} + \frac{n}{2\tau}\eta)](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&= \int_M -\tau v_{ij}(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij})(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M (\frac{tr_g v}{2} - h - \frac{n}{2\tau}\eta)[\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \\
&\quad + \int_M \eta(R + |\nabla f|^2 - \frac{n}{2\tau})(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV.
\end{aligned}
$$

$\square$

To compute the first variation of the ν functional, we also need the following Lemma:

Lemma 2.2.2. (Cao-Hamilton-Ilmanen [1]) *Assume that $\nu(g)$ is realized by some f and τ. Then it is necessary that the pair (f, τ) solves the following equations:*

$$\tau(-2\Delta f + |\nabla f|^2 - R) - f + n + \nu = 0, \tag{2.2.7}$$

and

$$(4\pi\tau)^{-\frac{n}{2}} \int_M f e^{-f} dV = \frac{n}{2} + \nu. \tag{2.2.8}$$

Proof. For fixed $\tilde{\tau} > 0$, suppose that $\mu(g, \tilde{\tau})$ is attained by some function $\tilde{f}$ such that $(4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}} dV = 1$. According to the Lagrange multiplier method, we consider the following functional

$$\begin{aligned} L(g, \tilde{f}, \tilde{\tau}, \gamma) &= (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}}[\tilde{\tau}(|\nabla \tilde{f}|^2 + R) + \tilde{f} - n] dV \\ &+ \ \gamma[(4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}} dV - 1]. \end{aligned}$$

Denote by $\delta\tilde{f}$ the variation of $\tilde{f}$. Then the variation of L with respect to $\delta\tilde{f}$ is

$$\begin{aligned} \delta L &= (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}}(-\delta\tilde{f})[\tilde{\tau}(|\nabla \tilde{f}|^2 + R) + \tilde{f} - n] dV \\ &+ \ (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}}[2\tilde{\tau}\nabla\tilde{f}\nabla(\delta\tilde{f}) + \delta\tilde{f}] dV \\ &- \ (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M \gamma(\delta\tilde{f}) e^{-\tilde{f}} dV \\ &= (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}}(\delta\tilde{f})[\tilde{\tau}(-2\Delta\tilde{f} + |\nabla\tilde{f}|^2 - R)] dV \\ &+ \ (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}}(\delta\tilde{f})(-\tilde{f} + n + 1 - \gamma) dV. \end{aligned}$$

Since at the minimizer $\tilde{f}$ of $\mu(g, \tilde{\tau})$ we have $\delta L = 0$, it follows that

$$\tilde{\tau}(-2\Delta\tilde{f} + |\nabla\tilde{f}|^2 - R) - \tilde{f} + n + 1 - \gamma = 0. \tag{2.2.9}$$

Integrating both sides with respect to the measure $(4\pi\tilde{\tau})^{-\frac{n}{2}} e^{-\tilde{f}} dV$, we get

$$-\gamma + 1 = (4\pi\tilde{\tau})^{-\frac{n}{2}} \int_M e^{-\tilde{f}}[\tilde{\tau}(|\nabla\tilde{f}|^2 + R) + \tilde{f} - n] dV = \mu(g, \tilde{\tau}).$$

For the minimizer (τ, f) of $\nu(g)$, equation (2.2.9) is just equation (2.2.7).

Now we consider the variations $\delta\tau$ and δf of both τ and f. We have at (f, τ)

$$
\begin{aligned}
0 = (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}(-\frac{n}{2\tau}\delta\tau - \delta f)[\tau(|\nabla f|^2 + R) + f - n]dV \\
+ \ (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}[\delta\tau(|\nabla f|^2 + R) + 2\tau\nabla f\nabla(\delta f) + \delta f]dV. \qquad (2.2.10)
\end{aligned}
$$

and

$$
(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}(-\frac{n}{2\tau}\delta\tau - \delta f)dV = 0. \qquad (2.2.11)
$$

Using (2.2.7) and (2.2.11), we can write (2.2.10) as

$$
\begin{aligned}
0 &= (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}[\delta\tau(|\nabla f|^2 + R) + \delta f]dV \\
&= (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}[\frac{1}{\tau}\delta\tau(\nu - f + n) - \frac{n}{2\tau}\delta\tau]dV \\
&= (\delta\tau)\frac{1}{\tau}(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}(\nu - f + \frac{n}{2})dV.
\end{aligned}
$$

Hence, we obtain equation (2.2.8).

□

One can get the similar necessary condition on the minimizers of the λ and the ν_+ functionals.

Lemma 2.2.3. *The infimum $\lambda(g)$ is achieved by some function f satisfying*

$$
-2\Delta f + |\nabla f|^2 - R = \lambda(g).
$$

Lemma 2.2.4. *The minimizer (f, σ) of the $\nu_+(g)$ functional must satisfy the following equations:*

$$
\sigma(-2\Delta f + |\nabla f|^2 - R) + f - n + \nu_+ = 0, \qquad (2.2.12)
$$

and

$$
(4\pi\sigma)^{-\frac{n}{2}} \int_M f e^{-f} dV = \frac{n}{2} - \nu_+. \qquad (2.2.13)
$$

Combining Lemma 2.2.1 and Lemma 2.2.2, it is easy to get the first variation of the ν entropy.

Proposition 2.2.3. *On a compact manifold (M, g_{ij}), consider the variation $g_{ij}(s)$ of the metric g_{ij} with $g_{ij}(0) = g_{ij}$ and $h_{ij}(s) = \frac{d}{ds}g_{ij}(s)$. Then the first variational formula of the $\nu(g_{ij}(s))$ functional is*

$$\begin{aligned}\frac{d}{ds}\nu(g_{ij}(s)) =&(4\pi\tau)^{-\frac{n}{2}}\int -\tau < h, Rc + \nabla^2 f - \frac{1}{2\tau}g > e^{-f}dV\\ =&(4\pi\tau)^{-\frac{n}{2}}\int -\tau h_{ij}(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij})e^{-f}dV,\end{aligned}$$

where $(f, \tau) = (f(s), \tau(s))$ is a minimizer of $\nu(g_{ij}(s))$.

Proof. Let us denote by δf and $\delta\tau$ the variation of $f(s)$ and $\tau(s)$ separately. From Lemma 2.2.1, we have

$$\begin{aligned}&\frac{d}{ds}\nu(g_{ij}(s))\\ &= (4\pi\tau)^{-\frac{n}{2}}\int_M -\tau h_{ij}(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij})e^{-f}dV\\ &\quad + (4\pi\tau)^{-\frac{n}{2}}\int_M (\frac{1}{2}\operatorname{tr}_g h - \delta f - \frac{n}{2\tau}\delta\tau)[\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1]e^{-f}dV\\ &\quad + (4\pi\tau)^{-\frac{n}{2}}\int_M \delta\tau(R + |\nabla f|^2 - \frac{n}{2\tau})e^{-f}dV.\end{aligned}$$

Using (2.2.7), we may rewrite the second integral on the right hand side of the above formula as

$$(4\pi\tau)^{-\frac{n}{2}}\int_M(-\frac{n}{2\tau}\delta\tau - \delta f + \frac{tr_g h}{2})(\nu(g(s)) - 1)e^{-f}dV.$$

But differentiating $(4\pi\tau)^{-\frac{n}{2}}\int_M e^{-f}dV = 1$ implies

$$(4\pi\tau)^{-\frac{n}{2}}\int_M(-\frac{n}{2\tau}\delta\tau - \delta f + \frac{tr_g h}{2})e^{-f}dV = 0. \qquad (2.2.14)$$

Therefore, the second integral on the right hand side vanishes.

Moreover, integrating (2.2.7) against the measure $(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV$ gives us

$$(4\pi\tau)^{-\frac{n}{2}}\int_M[\tau(R + |\nabla f|^2) + f - n - \nu]e^{-f}dV = 0.$$

Hence, by (2.2.8), we have

$$(4\pi\tau)^{-\frac{n}{2}}\int_M (R+|\nabla f|^2-\frac{n}{2\tau})e^{-f}dV=0.$$

This implies that the third integral on the right hand side of the first variation formula of $\nu(g(s))$ also vanishes and the proof is completed. $\square$

By a similar computation, one can also get the following first variations of the λ and ν_+ functionals.

Proposition 2.2.4. *On a compact manifold (M, g_{ij}), consider the variation $g_{ij}(s)$ of the metric g_{ij} with $g_{ij}(0)=g_{ij}$ and $h_{ij}(s)=\frac{d}{ds}g_{ij}(s)$. Then the first variational formula of the $\lambda(g_{ij}(s))$ functional is*

$$\frac{d}{ds}\lambda(g(s))=\int_M h_{ij}(-R_{ij}-\nabla_i\nabla_j f)e^{-f}dV,$$

where $f=f(s)$ is a minimizer of $\lambda(g(s))$.

Proposition 2.2.5. *On a compact manifold (M, g_{ij}), consider the variation $g_{ij}(s)$ of the metric g_{ij} with $g_{ij}(0)=g_{ij}$ and $h_{ij}=\frac{d}{ds}g_{ij}(s)$. Then the first variational formula of the $\nu_+(g_{ij}(s))$ functional is*

$$\frac{d}{ds}\nu_+(g(s))=(4\pi\sigma)^{-\frac{n}{2}}\int_M \sigma h_{ij}(-R_{ij}-\nabla_i\nabla_j f-\frac{1}{2\sigma}g_{ij})e^{-f}dV,$$

where $(f,\sigma)=(f(s),\sigma(s))$ is the minimizer of $\nu_+(g(s))$.

First of all, we can conclude from Propositions 2.2.3, 2.2.4 and 2.2.5 that the critical points of the λ, ν and ν_+ entropies are exactly the compact gradient steady, shrinking solitons and expanding solitons (M, g_{ij}, f) satisfying

$$R_{ij}+\nabla_i\nabla_j f=\frac{\rho}{2\tau}g_{ij}, \tag{2.2.15}$$

where $\rho=0,1,$ and -1, respectively. In particular, it follows that any compact Ricci soliton is necessarily a gradient soliton.

Moreover, from the definition of the entropy functionals, we can see that they are invariant under diffeomorphisms. This implies that their first variations vanish when $h_{ij} = \nabla_i \nabla_j f = \frac{1}{2}\mathcal{L}_{\nabla f} g_{ij}$, since in this case the metrics vary along the family of diffeomorphisms generated by the vector field ∇f. Given the fact that the ν and ν_+ functionals are also invariant under scaling of the metric, the first variations of the ν and ν_+ functionals also vanish when $h_{ij} = \rho g_{ij}$. Therefore, if $g_{ij}(s)$ varies along the Ricci flow, i.e., $h_{ij} = -2R_{ij}$, we will have

$$\frac{d}{ds}\lambda(g_{ij}(s)) = \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} dV \geq 0,$$

$$\frac{d}{ds}\nu(g_{ij}(s)) = (4\pi\tau)^{-\frac{n}{2}} \int 2\tau |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 e^{-f} dV \geq 0,$$

and

$$\frac{d}{ds}\nu_+(g_{ij}(s)) = (4\pi\sigma)^{-\frac{n}{2}} \int 2\sigma |R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\sigma} g_{ij}|^2 e^{-f} dV \geq 0,$$

which show that the entropy functionals are all nondecreasing along the Ricci flow. Hence we have recovered the Theorem proved in [18] by Perelman.

2.3 The Second Variation of the Entropy Functionals

In the previous section, we have seen that the critical points of the entropy functionals are compact gradient Ricci solitons. In this section, we will calculate the second variation of these functionals at their critical points.

According to Proposition 1.1.1, for the steady and expanding case, we only need to compute the second variations of the λ and ν_+ functionals at Ricci flat and negative Einstein metrics, respectively. The second variation of the λ functional was obtained by Cao-Hamilton-Ilmanen in [1]. In the following, we will present the second variation of the ν_+ entropy calculated by the author in [41].

First, as in [1], we denote $Rm(h,h) = R_{ijkl}h_{ik}h_{jl}$, $\mathrm{div}\omega = \nabla_i \omega_i$, $(\mathrm{div} h)_i = \nabla_j h_{ji}$, $(\mathrm{div}^* \omega)_{ij} = -(\nabla_i \omega_j + \nabla_j \omega_i) = -\frac{1}{2} L_{\omega^\#} g_{ij}$, where h is a symmetric 2-tensor, ω is a 1-tensor, $\omega^\#$ is the dual vector field of ω, and $L_{\omega^\#}$ is the Lie derivative.

Theorem 2.3.1. (Z. [41]) *Let (M^n, g) be a compact negative Einstein manifold such that $R_{ij} = -\frac{1}{2\sigma} g_{ij}$. Let h_{ij} be a symmetric 2-tensor. Consider the variation of metric $g(s) = g + sh$. Then the second variation of ν_+ is*

$$\frac{\mathrm{d}^2 \nu_+(g(s))}{\mathrm{d}s^2}|_{s=0} = \frac{\sigma}{\mathrm{Vol}(g)} \int_M < N_+ h, h >,$$

where

$$N_+ h := \frac{1}{2}\Delta h + \mathrm{div}^* \mathrm{div} h + \frac{1}{2}\nabla^2 v_h + Rm(h, \cdot) + \frac{g}{2n\sigma \,\mathrm{Vol}(g)} \int_M trh,$$

and v_h is the unique solution of

$$\Delta v_h - \frac{v_h}{2\sigma} = \mathrm{div}(\mathrm{div} h), \quad \text{and} \quad \int_M v_h = 0.$$

Proof. By Proposition 2.2.5, we have the following first variation of the ν_+ functional:

$$\frac{d}{ds}\nu_+(g(s)) = (4\pi\sigma)^{-\frac{n}{2}} \int_M \sigma h_{ij}(-R_{ij} - \nabla_i \nabla_j f - \frac{1}{2\sigma} g_{ij}) e^{-f} dV.$$

Since at $s = 0$, we have $f = C$, $(4\pi\sigma)^{-\frac{n}{2}} e^{-f} = \frac{1}{\mathrm{Vol}(M)}$ and $R_{ij} = -\frac{1}{2\sigma} g_{ij}$, the second variation of ν_+ at g_{ij} is

$$\begin{aligned}
&\frac{d^2}{ds^2}\nu_+(g(s))|_{s=0} \\
&= \frac{1}{\mathrm{Vol}(M)} \int_M \sigma h_{ij}[-\frac{\partial}{\partial s} R_{ij}|_{s=0} - \nabla_i \nabla_j \frac{\partial f}{\partial s}|_{s=0} + \frac{1}{2\sigma^2}\frac{\partial \sigma}{\partial s}|_{s=0} g_{ij} - \frac{1}{2\sigma} h_{ij}] dV.
\end{aligned}$$

Now in Lemma 2.2.1, we have seen that

$$\frac{\partial}{\partial s} R_{ij}|_{s=0} = \frac{1}{2}(\nabla_p \nabla_i h_{jp} + \nabla_p \nabla_j h_{ip} - \Delta h_{ij} - \nabla_i \nabla_j tr_g h).$$

Thus, we have

$$\begin{aligned}
&\int_M h_{ij} \frac{\partial}{\partial s} R_{ij}|_{s=0} dV \\
&= \int_M \frac{1}{2} h_{ij}(\nabla_p \nabla_i h_{jp} + \nabla_p \nabla_j h_{ip} - \Delta h_{ij} - \nabla_i \nabla_j tr_g h) dV \\
&= \int_M [h_{ij} \nabla_k \nabla_i h_{jk} - \frac{1}{2} h_{ij} \Delta h_{ij} - \frac{1}{2} h_{ij} \nabla_i \nabla_j tr_g h] dV \\
&= \int_M [h_{ij}(\nabla_i \nabla_k h_{jk} + R_{kijl} h_{kl} + R_{il} h_{jl}) - \frac{1}{2} h_{ij} \Delta h_{ij} - \frac{1}{2} tr_g h (\mathrm{div}\,\mathrm{div}\, h)] dV \\
&= \int_M [-h_{ij}(\mathrm{div}^* \,\mathrm{div}\, h)_{ij} - h_{ij} R_{ikjl} h_{kl} - \frac{1}{2\sigma}|h_{ij}|^2 - \frac{1}{2} h_{ij} \Delta h_{ij} - \frac{1}{2} tr_g h (\mathrm{div}\,\mathrm{div}\, h)] dV
\end{aligned}$$

Since $(4\pi\sigma)^{-\frac{n}{2}}\int_M e^{-f}dV = 1$, $(4\pi\sigma)^{-\frac{n}{2}}\int_M fe^{-f}dV = \frac{n}{2} - \nu_+$ and $\frac{d}{ds}\nu_+|_{s=0} = 0$, after differentiation, we have

$$(4\pi\sigma)^{-\frac{n}{2}}\int_M e^{-f}(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}|_{s=0} - \frac{\partial f}{\partial s}|_{s=0} + \frac{1}{2}tr_g h)dV = 0,$$

and

$$(4\pi\sigma)^{-\frac{n}{2}}\int_M e^{-f}[f(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}|_{s=0} - \frac{\partial f}{\partial s}|_{s=0} + \frac{1}{2}tr_g h) + \frac{\partial f}{\partial s}|_{s=0}]dV = 0.$$

Hence,

$$\int_M \frac{\partial f}{\partial s}|_{s=0}dV = 0,$$

and

$$\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}|_{s=0} = \frac{1}{\mathrm{Vol}(M)}\int_M \frac{1}{2}tr_g h dV.$$

Therefore, it follows

$$\begin{aligned}
&\frac{d^2}{ds^2}\nu_+(g(s))|_{s=0} \\
&= \frac{\sigma}{\mathrm{Vol}(M)}\int_M [h_{ij}(\mathrm{div}^*\,\mathrm{div}\,h)_{ij} + h_{ij}R_{ikjl}h_{kl} + \frac{1}{2}h_{ij}\Delta h_{ij} \\
&\quad + \frac{1}{2}tr_g h(\mathrm{div}\,\mathrm{div}\,h) - \frac{\partial f}{\partial s}|_{s=0}\,\mathrm{div}\,\mathrm{div}\,h]dV + \frac{1}{2n}\left(\frac{1}{\mathrm{Vol}(M)}\int_M tr_g h dV\right)^2.
\end{aligned}$$

Suppose that v_h is the unique solution to the equation

$$\Delta v_h - \frac{v_h}{2\sigma} = \mathrm{div}(\mathrm{div}\,h),$$

then it is easy to check by differentiating (2.2.12) that

$$v_h = -2\frac{\partial f}{\partial s}|_{s=0} + tr_g h - \frac{n}{\sigma}\frac{\partial\sigma}{\partial s}|_{s=0}.$$

Thus,

$$
\begin{aligned}
&\frac{d^2}{ds^2}\nu_+(g(s))|_{s=0} \\
&= \frac{\sigma}{\mathrm{Vol}(M)}\int_M [h_{ij}(\mathrm{div}^*\,\mathrm{div}\,h)_{ij} + h_{ij}R_{ikjl}h_{kl} + \frac{1}{2}h_{ij}\Delta h_{ij} \\
&\quad + \frac{1}{2}v_h(\mathrm{div}\,\mathrm{div}\,h)]dV + \frac{1}{2n}\left(\frac{1}{\mathrm{Vol}(M)}\int_M tr_g h dV\right)^2 \\
&= \frac{\sigma}{\mathrm{Vol}(M)}\int_M [h_{ij}(\mathrm{div}^*\,\mathrm{div}\,h)_{ij} + h_{ij}R_{ikjl}h_{kl} + \frac{1}{2}h_{ij}\Delta h_{ij} \\
&\quad + \frac{1}{2}h_{ij}\nabla_i\nabla_j v_h)]dV + \frac{1}{2n}\left(\frac{1}{\mathrm{Vol}(M)}\int_M tr_g h dV\right)^2 \\
&= \frac{\sigma}{\mathrm{Vol}(M)}\int_M < N_+h, h > dV.
\end{aligned}
$$

□

By a similar computation, one can also get the second variation of the λ entropy for compact Ricci flat metrics in Theorem 2.1.1.

For the shrinking case, Cao-Hamilton-Ilmanen [1] first derived the second variation of the ν entropy for positive Einstein metrics in Theorem 2.1.2. And as we have mentioned, they also announced in [1] that there is a second variational formula of the ν entropy for general gradient shrinking solitons (the formula can be found in [3]). However, in both references, no detailed computation has been shown. Therefore, Cao and the author recalculated the second variation of the ν entropy for gradient shrinking solitons and found that the last term stated in [3] was actually incorrect.

In the following, we give a detailed computation of the second variational formula stated in Theorem 2.1.4. Throughout the rest of this section, we assume that (M^n, g_{ij}) is a compact gradient shrinking Ricci soliton, and the pair (f, τ) is a minimizer of the functional $\nu(g_{ij})$. Hence, (f, τ) satisfies

$$R_{ij} + \nabla_i\nabla_j f = \frac{1}{2\tau}g_{ij}. \tag{2.3.1}$$

Let us first recall the notations that will be used,

$$(\mathrm{div}_f\, h)_i = g^{jk}(\nabla_j h_{ik} - h_{ik}\nabla_j f),$$
$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

and
$$\int_M < \mathrm{div}_f^\dagger\, w, h > e^{-f} dV = \int_M < w, \mathrm{div}_f\, h > e^{-f} dV,$$
$$\text{i.e.,} \qquad \mathrm{div}_f^\dagger\, w = -\frac{1}{2}\mathcal{L}_{w^\#} g,$$
where h is a symmetric 2-tensor, w is a 1-form and $w^\#$ is its dual vector field.

Lemma 2.3.1. *Suppose that $g_{ij}(s)$ is a variation of g_{ij} such that $g_{ij}(0) = g_{ij}$ and $\delta g_{ij} = h_{ij}$. Then at $s = 0$ we have*
$$\delta Rc + \delta\nabla^2 f - \frac{1}{2\tau}h = -\frac{1}{2}\Delta_f h - Rm(h,\cdot) - \mathrm{div}_f^\dagger\, \mathrm{div}_f\, h - \nabla^2(-\delta f + \frac{1}{2}\,\mathrm{tr}_g\, h).$$

Proof. First of all, in the proof of Lemma 2.2.1, we see that the variation δRc of the Ricci tensor is given by
$$\begin{aligned}(\delta Rc)_{ij} &= \frac{1}{2}(\nabla_k\nabla_i h_{jk} + \nabla_k\nabla_j h_{ik} - \Delta h_{ij} - \nabla_i\nabla_j tr_g h)\\ &= -R_{ikjl}h_{kl} + \frac{1}{2}(\nabla_i\nabla_k h_{jk} + \nabla_j\nabla_k h_{ik} + R_{ik}h_{jk}\\ &\qquad +R_{jk}h_{ik} - \Delta h_{ij} - \nabla_i\nabla_j\,\mathrm{tr}_g\, h), \end{aligned} \tag{2.3.2}$$
and, by direct computation (see, e.g., [41]),
$$(\delta\nabla^2 f)_{ij} = \nabla_i\nabla_j(\delta f) - \frac{1}{2}(\nabla_i h_{jk} + \nabla_j h_{ik} - \nabla_k h_{ij})\nabla_k f. \tag{2.3.3}$$
On the other hand, by the definition of div_f and $\mathrm{div}_f^\dagger$ and using the shrinking soliton equation (2.3.1), we have
$$\begin{aligned}\mathrm{div}_f^\dagger\, \mathrm{div}_f\, h &= -\frac{1}{2}[\nabla_i(\mathrm{div}_f\, h)_j + \nabla_j(\mathrm{div}_f\, h)_i]\\ &= -\frac{1}{2}[\nabla_i(\nabla_k h_{jk} - h_{jk}\nabla_k f) + \nabla_j(\nabla_k h_{ik} - h_{ik}\nabla_k f)]\\ &= -\frac{1}{2}(\nabla_i\nabla_k h_{jk} + \nabla_j\nabla_k h_{ik} - \nabla_k f\nabla_i h_{jk} - \nabla_k f\nabla_j h_{ik})\\ &\quad -\frac{1}{2}(R_{ik}h_{kj} + R_{jk}h_{ki}) + \frac{1}{2\tau}h_{ij}.\end{aligned}$$

Now, combining the above computations, we arrive at

$$\delta Rc + \delta \nabla^2 f = -\frac{1}{2}\Delta_f h - Rm(h, \cdot) - \operatorname{div}_f^\dagger \operatorname{div}_f h - \nabla^2(-\delta f + \frac{1}{2}\operatorname{tr}_g h) + \frac{1}{2\tau}h.$$

$\square$

To compute the second variation of the ν functional, we also need the following key Lemma:

Lemma 2.3.2. *Under the same assumptions as in Lemma 2.3.1, we have at* $s = 0$

$$\delta\tau = \tau\frac{\int_M < Rc, h > e^{-f}}{\int_M Re^{-f}}.$$

Proof. First of all, by taking the trace in (2.3.1) we get

$$R + \Delta f = \frac{n}{2\tau}. \tag{2.3.4}$$

By substituting (2.3.4) in (2.2.7), we have

$$R + |\nabla f|^2 = \frac{f-\nu}{\tau}. \tag{2.3.5}$$

From (2.3.4) and (2.3.5), it follows that

$$-\Delta_f f = |\nabla f|^2 - \Delta f = \frac{f - \nu - n/2}{\tau}. \tag{2.3.6}$$

Moreover, from (2.3.2), (2.3.3) and using (2.3.1), we get

$$\delta R = -\frac{1}{2\tau}\operatorname{tr}_g h + h_{ij}\nabla_i\nabla_j f + \nabla_i\nabla_j h_{ij} - \Delta \operatorname{tr}_g h, \tag{2.3.7}$$

and

$$\delta(\Delta f) = \Delta(\delta f) - h_{ij}\nabla_i\nabla_j f - \nabla_i h_{ij}\nabla_j f + \frac{1}{2}\nabla_i \operatorname{tr}_g h \nabla_i f, \tag{2.3.8}$$

respectively. Moreover,

$$\delta|\nabla f|^2 = 2\nabla_i f\nabla_j(\delta f) - h_{ij}\nabla_i f\nabla_j f. \tag{2.3.9}$$

When we integrate (2.3.5) against the measure $(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV$ and use (2.2.8), we obtain

$$(4\pi\tau)^{-\frac{n}{2}}\int_M \tau(|\nabla f|^2 + R)e^{-f}\mathrm{d}V = \frac{n}{2}. \tag{2.3.10}$$

On the other hand, by differentiating $(4\pi\tau)^{-\frac{n}{2}}\int_M e^{-f}dV = 1$ and (2.2.8), we have

$$(4\pi\tau)^{-\frac{n}{2}}\int_M(-\frac{n}{2\tau}\delta\tau - \delta f + \frac{1}{2}\operatorname{tr}_g h)e^{-f}dV = 0, \tag{2.3.11}$$

and

$$(4\pi\tau)^{-\frac{n}{2}}\int_M f(-\frac{n}{2\tau}\delta\tau - \delta f + \frac{1}{2}\operatorname{tr}_g h)e^{-f}dV + (4\pi\tau)^{-\frac{n}{2}}\int_M \delta f e^{-f}dV = 0. \tag{2.3.12}$$

Now, differentiating (2.2.7) and using (2.3.4), (2.3.8) and (2.3.9), we obtain

$$\begin{aligned}
0 =&\delta\tau(-\frac{n}{2\tau} + |\nabla f|^2 - \Delta f) - \delta f \\
&+ \tau(-2\Delta(\delta f) + 2h_{ij}\nabla_i\nabla_j f + 2\nabla_i h_{ij}\nabla_j f - \nabla_i(\operatorname{tr}_g h)\nabla_i f \\
&+ 2\nabla_i f\nabla_i(\delta f) - h_{ij}\nabla_i f\nabla_j f - \delta R).
\end{aligned}$$

Substituting (2.3.7) in the above identity, we get

$$\begin{aligned}
0 =& -\frac{n}{2\tau}\delta\tau - 2\tau\Delta(\delta f) + 2\tau\nabla(\delta f)\nabla f - \delta f + \delta\tau(|\nabla f|^2 - \Delta f) \\
&+ \tau(2h_{ij}\nabla_i\nabla_j f + 2\nabla_i h_{ij}\nabla_j f - \nabla_i(\operatorname{tr}_g h)\nabla_i f - h_{ij}\nabla_i f\nabla_j f) \\
&+ \tau(\frac{1}{2\tau}\operatorname{tr}_g h - h_{ij}\nabla_i\nabla_j f - \nabla_i\nabla_j h_{ij} + \Delta\operatorname{tr}_g h).
\end{aligned}$$

But, by the definition of div_f, we compute that

$$\begin{aligned}
\operatorname{div}_f\operatorname{div}_f h =&\nabla_i(\nabla_j h_{ij} - h_{ij}\nabla_j f) - \nabla_i f(\nabla_j h_{ij} - h_{ij}\nabla_j f) \\
=&\nabla_i\nabla_j h_{ij} - h_{ij}\nabla_i\nabla_j f - 2\nabla_i f\nabla_j h_{ij} + h_{ij}\nabla_i f\nabla_j f.
\end{aligned}$$

Hence, we get

$$0 = (-\frac{n\delta\tau}{2\tau} - \delta f + \frac{1}{2}\operatorname{tr}_g h) + \delta\tau(-\Delta_f f) + \tau\Delta_f(-2\delta f + \operatorname{tr}_g h) - \tau\operatorname{div}_f\operatorname{div}_f h.$$

Multiplying the above identity by f and integrating against the measure $(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV$, we get

$$
\begin{aligned}
0 &= (4\pi\tau)^{-\frac{n}{2}} \int_M f\left(-\frac{n}{2\tau}\delta\tau - \delta f + \frac{1}{2}\operatorname{tr}_g h\right)e^{-f}dV \\
&\quad + (4\pi\tau)^{-\frac{n}{2}}\delta\tau \int_M f(-\Delta_f f)e^{-f}\mathrm{d}V \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M \tau f \Delta_f(-2\delta f + \operatorname{tr}_g h)e^{-f}dV \\
&\quad - (4\pi\tau)^{-\frac{n}{2}} \int_M \tau f(\operatorname{div}_f \operatorname{div}_f h)e^{-f}dV.
\end{aligned}
$$

By (2.3.12) and integration by parts, the above identity becomes

$$
\begin{aligned}
0 &= (4\pi\tau)^{-\frac{n}{2}} \int_M -\delta f e^{-f}\mathrm{d}V + \delta\tau(4\pi\tau)^{-\frac{n}{2}} \int_M |\nabla f|^2 e^{-f}\mathrm{d}V \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M \tau(-2\delta f + \operatorname{tr}_g h)\Delta_f f e^{-f}\mathrm{d}V \\
&\quad - (4\pi\tau)^{-\frac{n}{2}} \int_M \tau < h, \nabla^2 f > e^{-f}\mathrm{d}V.
\end{aligned}
$$

Using (2.3.1), (2.3.6) and (2.3.10), we obtain

$$
\begin{aligned}
0 &= -(4\pi\tau)^{-\frac{n}{2}} \int_M \delta f e^{-f}\mathrm{d}V + \frac{n}{2\tau}\delta\tau - \delta\tau(4\pi\tau)^{-\frac{n}{2}} \int_M R e^{-f}\mathrm{d}V \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M 2\tau\left(\frac{n}{2\tau}\delta\tau + \delta f - \frac{1}{2}\operatorname{tr}_g h\right)\left(\frac{1}{\tau}f - \frac{\nu}{\tau} - \frac{n}{2\tau}\right)e^{-f}\mathrm{d}V \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M \left(-\frac{1}{2}\operatorname{tr}_g h + \tau h_{ij}R_{ij}\right)e^{-f}\mathrm{d}V.
\end{aligned}
$$

By using (2.3.11) and (2.3.12), we arrive at

$$
\begin{aligned}
0 &= (4\pi\tau)^{-\frac{n}{2}} \int_M \left(\frac{n}{2\tau}\delta\tau + \delta f - \frac{1}{2}\operatorname{tr}_g h\right)e^{-f}\mathrm{d}V - \delta\tau(4\pi\tau)^{-\frac{n}{2}} \int_M R e^{-f}\mathrm{d}V \\
&\quad + (4\pi\tau)^{-\frac{n}{2}} \int_M \tau R_{ij}h_{ij}e^{-f}\mathrm{d}V \\
&= -\delta\tau(4\pi\tau)^{-\frac{n}{2}} \int_M R e^{-f}\mathrm{d}V + (4\pi\tau)^{-\frac{n}{2}} \int_M \tau R_{ij}h_{ij}e^{-f}\mathrm{d}V.
\end{aligned}
$$

Therefore,

$$
\delta\tau = \tau\frac{\int_M R_{ij}h_{ij}e^{-f}\mathrm{d}V}{\int_M Re^{-f}}.
$$

□

Remark 2.3.1. *The denominator in the above Lemma is positive for compact gradient shrinking Ricci solitons, because by Proposition 1.3.1, the scalar curvature R of a compact gradient shrinking soliton must be nonnegative. Then we may use the strong maximum principle on equation* (1.1.20) *to derive the positivity of the scalar curvature.*

Now we are ready to prove Theorem 2.1.4.

Theorem 2.3.2. (Cao-Hamilton-Ilmanen)(Cao-Z. [42]) *Let (M^n, g_{ij}, f) be a compact gradient Ricci shrinking soliton with potential function f satisfying the Ricci soliton equation* (2.1.2)*. For any symmetric 2-tensor $h = h_{ij}$, consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then the second variation of the ν functional is given by*

$$\frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{\tau}{(4\pi\tau)^{n/2}} \int_M < \hat{N}h, h > e^{-f} dV,$$

where the stability operator $\hat{N}$ is given by

$$\hat{N}h := \frac{1}{2}\Delta_f h + Rm(h, \cdot) + \operatorname{div}_f^\dagger \operatorname{div}_f h + \frac{1}{2}\nabla^2 \hat{v}_h - Rc\, \frac{\int_M < Rc, h > e^{-f} dV}{\int_M Re^{-f} dV}, \quad (2.3.13)$$

and $\hat{v}_h$ is the unique solution of

$$\Delta_f \hat{v}_h + \frac{\hat{v}_h}{2\tau} = \operatorname{div}_f \operatorname{div}_f h, \qquad \int_M \hat{v}_h e^{-f} = 0.$$

Proof. From the first variation formula in Proposition 2.2.3, we see that the second variation at a gradient shrinker (M^n, g_{ij}, f) is given by

$$\begin{aligned}
\delta^2 \nu_g(h, h) =& (4\pi\tau)^{-\frac{n}{2}} \int -\tau < h, \delta(Rc + \nabla^2 f - \frac{1}{2\tau} g) > e^{-f} \\
=& (4\pi\tau)^{-\frac{n}{2}} \int -\tau < h, \delta Rc + \delta\nabla^2 f - \frac{1}{2\tau} h > e^{-f} \\
&+ (4\pi\tau)^{-\frac{n}{2}} (-\frac{\delta\tau}{2\tau}) \int_M \operatorname{tr}_g h e^{-f}.
\end{aligned}$$

By Lemma 2.3.1 and Lemma 2.3.2, the second variation becomes

$$
\begin{aligned}
\delta^2\nu_g(h,h) =&(4\pi\tau)^{-\frac{n}{2}}\int_M -\tau < h, \delta Rc + \delta\nabla^2 f - \frac{1}{2\tau}h > e^{-f}\\
&+ (4\pi\tau)^{-\frac{n}{2}}(-\frac{\delta\tau}{2\tau})\int_M \operatorname{tr}_g he^{-f}\\
=&(4\pi\tau)^{-\frac{n}{2}}\int_M \tau < h, \frac{1}{2}\Delta_f h + Rm(h,\cdot) + \operatorname{div}_f^\dagger \operatorname{div}_f h > e^{-f}\\
&+ (4\pi\tau)^{-\frac{n}{2}}\int_M \tau < h, \nabla^2(-\delta f + \frac{1}{2}\operatorname{tr}_g h) > e^{-f}\\
&+ (4\pi\tau)^{-\frac{n}{2}}(-\frac{\delta\tau}{2\tau})\int_M \operatorname{tr}_g he^{-f}\\
=&\tau(4\pi\tau)^{-\frac{n}{2}}\int_M < h, \frac{1}{2}\Delta_f h + Rm(h,\cdot) + \operatorname{div}_f^\dagger \operatorname{div}_f h + \frac{1}{2}\nabla^2\hat{v}_h > e^{-f}\\
&+ \tau(4\pi\tau)^{-\frac{n}{2}}\frac{\delta\tau}{\tau}\int_M < h, \nabla^2 f - \frac{1}{2\tau}g > e^{-f}\\
=&\tau(4\pi\tau)^{-\frac{n}{2}}\int_M < h, \frac{1}{2}\Delta_f h + Rm(h,\cdot) + \operatorname{div}_f^\dagger \operatorname{div}_f h + \frac{1}{2}\nabla^2\hat{v}_h > e^{-f}\\
&- \tau(4\pi\tau)^{-\frac{n}{2}}\frac{\int_M < Rc, h > e^{-f}dV}{\int_M Re^{-f}dV}\int_M < h, Rc > e^{-f}dV.
\end{aligned}
$$

Here,

$$\hat{v}_h = -2\delta f + \operatorname{tr}_g h - \frac{2\delta\tau}{\tau}(f-\nu),$$

and it is straightforward to check by using (2.2.7) that

$$\Delta_f\hat{v}_h + \frac{\hat{v}_h}{2\tau} = \operatorname{div}_f \operatorname{div}_f h \quad \text{and} \quad \int_M \hat{v}_h e^{-f}dV = 0. \tag{2.3.14}$$

To see the uniqueness of the solution to (2.3.14), it suffices to show that $\lambda_1 > \frac{1}{2\tau}$, where $\lambda_1 = \lambda_1(\Delta_f)$ denotes the first eigenvalue of Δ_f. Let u be a (non-constant) first eigenfunction so that

$$\Delta_f u = -\lambda_1 u.$$

Then by direct computation, we get

$$
\begin{aligned}
\frac{1}{2}\Delta_f|\nabla u|^2 =&|\nabla^2 u|^2 + \nabla(\Delta_f u)\cdot\nabla u + (Rc + \nabla^2 f)(\nabla u, \nabla u)\\
\geq&\frac{1}{n}|\Delta u|^2 + (\frac{1}{2\tau} - \lambda_1)|\nabla u|^2.
\end{aligned}
$$

Thus,

$$0=\int_M \frac{1}{2}\Delta_f|\nabla u|^2 e^{-f}dV \geq \frac{1}{n}\int_M |\Delta u|^2 e^{-f}dV+(\frac{1}{2\tau}-\lambda_1)\int_M |\nabla u|^2 e^{-f}dV.$$

Since u is non-constant, we obtain

$$\lambda_1>\frac{1}{2\tau}.$$

This completes the proof of theorem. □

2.4 Stability of Compact Ricci Solitons

In this section, we will discuss the entropy stability of compact gradient Ricci solitons. According to the definition, a compact gradient Ricci soliton is entropy stable if the second variation of the corresponding entropy functional is nonpositive. This is equivalent to saying that the stability operators L, N, N_+ and $\hat{N}$ in Theorems 2.1.1, 2.1.2, 2.1.3 and 2.1.4 are nonpositive on the space of symmetric 2-tensors.

- **Stability of compact Einstein manifolds**

Let (M, g_{ij}) be a compact Einstein manifold which is either Ricci flat, positive Einstein with $R_{ij}=\frac{1}{2\tau}g_{ij}$, or negative Einstein with $R_{ij}=-\frac{1}{2\sigma}g_{ij}$. Denote by $C^\infty(Symm^2((T^*M))$ the space of symmetric 2-tensors. To study the nonpositivity of the operators L, N and N_+, we may use the following decomposition

$$C^\infty(Symm^2((T^*M))=\ker\operatorname{div}\oplus\operatorname{im}\operatorname{div}^*.$$

It is easy to see that L, N and N_+ all vanish on $\operatorname{im}\operatorname{div}^*$, because 2-tensors in this subspace are the Lie derivatives of the metric g_{ij} along certain directions, and thus the metric varies by a family of diffeomorphisms, but the λ, ν and ν_+ functionals are invariant under diffeomorphisms.

On $\ker\operatorname{div}$, we have

$$L=\frac{1}{2}\Delta_L,$$

where $\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ikjl}h_{kl} - R_{ik}h_{kj} - h_{ik}R_{kj}$ is the Lichnerowicz Laplacian on symmetric 2-tensors. Therefore, the stability of compact Ricci flat manifolds is equivalent to the nonpositivity of Δ_L on $\ker \operatorname{div}$.

Example 2.4.1. *The flat torus T^n is entropy stable, since according to the positive mass theorem, it does not admit any metric with positive scalar curvature.*

Example 2.4.2. *Calabi-Yau K3 surfaces are entropy stable, since $\Delta_L \leq 0$ by Guenther-Isenberg-Knopf [47]. More generally, any manifold with a parallel spinor is stable, according to Dai-Wang-Wei [48].*

Question:(Cao-Hamilton-Ilmanen) Is there any compact unstable Ricci flat manifold?

For positive and negative Einstein manifolds, we may further decompose $\ker \operatorname{div}$ as

$$\ker \operatorname{div} = (\ker \operatorname{div})_0 \oplus \mathbb{R}g,$$

where $(\ker \operatorname{div})_0 = \{h_{ij} \in \ker \operatorname{div} \mid \int_M tr_g h dV = 0\}$.

Since the ν and ν_+ functionals are also scaling invariant, the operators N and N_+ vanish on the $\mathbb{R}g$ part. On $(\ker \operatorname{div})_0$, it is easy to see that

$$N = \frac{1}{2}(\Delta_L + \frac{1}{\tau}),$$

and

$$N_+ = \frac{1}{2}(\Delta_L - \frac{1}{\sigma}).$$

Let us denote by μ_L, μ_N and μ_{N_+} the largest eigenvalue of Δ_L, N and N_+ on $(\ker \operatorname{div})_0$. Firstly, one may notice that we have the following further decomposition of $(\ker \operatorname{div})_0$ on negative Einstein manifolds (M, g_{ij}) with $R_{ij} = -\frac{1}{2\sigma}g_{ij}$:

$$(\ker \operatorname{div})_0 = S_0 + S_1,$$

where $S_1 = \{h_{ij} \in (\ker \operatorname{div})_0 \mid h_{ij} = (-\frac{1}{2\sigma}u + \Delta u)g_{ij} - \nabla_i\nabla_j u,\ u \in C^\infty(M) \text{ and } \int_M u = 0\}$ and $S_0 = \{h_{ij} \in (\ker \operatorname{div})_0 \mid tr_g h = 0\}$ is the space of transverse traceless 2-tensors.

Define

$$Tu := (-\frac{1}{2\sigma}u + \Delta u)g_{ij} - \nabla_i\nabla_j u$$

for a smooth function u. Then one can check that $\Delta_L(Tu) = T(\Delta u)$ and $\ker T = \{0\}$. Thus, the Lichnerowicz Laplacian on S_1 and the Laplacian on functions have the same eigenvalues. It follows that N_+ is always negative on S_1. Therefore, to check the stability of compact negative Einstein manifolds, one needs only to compute μ_{N_+} on S_0.

In the following we present several examples in [1] and [41] for positive and negative Einstein manifolds:

Example 2.4.3. *The round sphere S^n is entropy stable with $\mu_N = -\frac{2}{n-1}\tau < 0$. In fact, it is geometrically stable in the sense that the solution to the Ricci flow converges to it starting from any nearby metric by the results of R. Hamilton [4, 8, 16] and G. Huisken [49].*

Example 2.4.4. *The complex projective space $\mathbb{C}P^n$ is neutrally stable with $\mu_L = -\frac{1}{\tau}$ and $\mu_N = 0$ (see e.g. Boucetta [50]).*

Example 2.4.5. *Any product of two Einstein manifolds $M = M_1^{n_1} \times M_2^{n_2}$ is unstable with $\mu_N = \frac{1}{2\tau}$ and eigen-tensor $h = \frac{1}{n_1}g_1 - \frac{1}{n_2}g_2$.*

Example 2.4.6. *Any compact Kähler-Einstein manifold M^n of positive scalar curvature with $\dim H^{1,1}(M) \geq 2$ is unstable. Indeed, in this case we may choose a harmonic 2-form η perpendicular to the Kähler form so that if h is the corresponding variational 2-tensor, we have $\Delta_L h = 0$ and $\mu_N = \frac{1}{2\tau}$.*

Example 2.4.7. *Let Q^n denote the complex hyperquadric in $\mathbb{C}P^{n+1}$ defined by*

$$\sum_{i=0}^{n} z_i^2 = 0.$$

Then Q^2 is $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is unstable by the example above. For $n = 3$, we have $\dim H^{1,1}(Q^3) = 1$, so the method in the example above does not apply. But according to Gasqui and Goldschmidt [51], Q^3 is unstable with $\mu_L = -\frac{2}{3}\tau$ and $\mu_N = \frac{1}{6}\tau$. When $n = 4$, we have that Q^4 is entropy stable by the work of Gasqui and Goldschmidt [52] with $\mu_L = -\frac{1}{\tau}$ and $\mu_N = 0$.

Example 2.4.8. *Suppose that M is an n-dimensional compact real hyperbolic space form with $n \geq 3$. By [53] or [54], the biggest eigenvalue of Δ_L on transverse traceless symmetric 2-tensors on real hyperbolic space is $-\frac{(n-1)(n-9)}{4}$. Since on M we have $Rc = -(n-1)g$, $\frac{1}{\sigma} = 2(n-1)$, thus the biggest eigenvalue of N_+ on S_0 is not greater than $-\frac{(n-1)^2}{8}$. It implies that M is entropy stable for $n \geq 3$.*

Remark 2.4.1. *When $n = 3$, D. Knopf and A. Young [55] proved that closed 3-folds with constant negative curvature are geometrically stable under a certain normalized Ricci flow. R. Ye obtained a more powerful stability result earlier in [56].*

Remark 2.4.2. *For n=2, R. Hamilton [8] proved that when the average scalar curvature is negative, the solution of the normalized Ricci flow with any initial metric converges to a metric with constant negative curvature. In particular, they are entropy stable. On the other hand, in [57] we see that the biggest eigenvalue of the Lichnerowicz Laplacian on trace free symmetric 2-tensors is 2. Thus N_+ is nonpositive definite on $(\ker \operatorname{div})_0$, which also implies the entropy stability.*

Remark 2.4.3. *For the noncompact case, in [58], V. Suneeta proved a certain geometric stability of $\mathbb{H}^n$ using different methods.*

• **Stability of Compact Shrinking Solitons**

Let (M, g_{ij}, f) be a compact shrinking soliton. Without loss of generality, we may assume that the shrinking soliton (M^n, g_{ij}, f) satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \tag{2.4.1}$$

We also normalize f so that $(4\pi)^{-\frac{n}{2}} \int_M e^{-f} dV = 1$.

In this case, the entropy stability is equivalent to the nonpositivity of the operator $\hat{N}$ in Theorem 2.1.4.

Example 2.4.9. *In [43], S. Hall and T. Murphy showed that any compact Kähler-Ricci soliton with $h^{1,1} \geq 2$ must be unstable. Indeed, the condition that $h^{1,1} \geq 2$ allows one to construct a $(1,1)$-form which is Δ_f harmonic and perpendicular to the Ricci form. Moreover, the corresponding 2-tensor provides us an unstable direction.*

In particular, the Cao-Koiso soliton on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ and the Wang-Zhu soliton on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ are unstable as compact non-Einstein shrinking solitons.

However, there are no other known examples of either stable or unstable compact non-Einstein shrinking solitons. In the following, we explore some nice properties of $\hat{N}$ which may help us study the classification of compact stable shrinking solitons.

Since the potential function f is not a constant in general, we need to decompose $C^\infty(Symm^2(M))$ as

$$C^\infty(Symm^2(M)) = \operatorname{im}\operatorname{div}_f^\dagger \oplus \ker\operatorname{div}_f\,.$$

It is not hard to verify that $\operatorname{im}\operatorname{div}_f^\dagger = \operatorname{im}\operatorname{div}^*$. Thus, we have $\hat{N}=0$ on $\operatorname{im}\operatorname{div}_f^\dagger$. On $\ker\operatorname{div}_f$, one can simplify $\hat{N}$ to

$$\hat{N}h = \mathcal{L}_f h - Rc\,\frac{\int_M < Rc, h > e^{-f}dV}{\int_M Re^{-f}dV}, \tag{2.4.2}$$

where

$$\mathcal{L}_f h = \frac{1}{2}\Delta_f h + Rm(h,\cdot).$$

It is easy to see that $\mathcal{L}_f$ is a self-adjoint operator with respect to the weighted L^2-inner product $(\cdot,\cdot)_f = \int_M < \cdot,\cdot > e^{-f}dV$.

Lemma 2.4.1. *For any complete shrinking Ricci soliton satisfying* (2.4.1)*, we have*

$$Rc \in \ker\operatorname{div}_f\,.$$

Proof. By definition and the second contracted Bianchi identity,

$$(\operatorname{div}_f Rc)_i = \nabla_j R_{ij} - R_{ij}\nabla_j f = \frac{1}{2}\nabla_i R - R_{ij}\nabla_j f.$$

On the other hand, from (1.1.8), we have

$$\nabla_i R = 2R_{ij}\nabla_j f.$$

Therefore, $\operatorname{div}_f(Rc) = 0$. □

Lemma 2.4.2. *For any complete shrinking soliton satisfying* (2.4.1)*, its Ricci tensor is an eigen-tensor of the operator* $\mathcal{L}_f$*:*

$$\mathcal{L}_f(Rc) = \frac{1}{2}Rc.$$

Proof. From (1.1.19), we have

$$\Delta R_{ij} = \nabla_l R_{ij} \nabla_l f + 2R_{kijl}R_{kl} + R_{ij}, \tag{2.4.3}$$

i.e., $2\mathcal{L}_f(R_{ij}) = R_{ij}$. □

From (2.4.3) we obtain

$$\Delta_f R = R - 2|Rc|^2, \tag{2.4.4}$$

from which it follows that

$$2\int_M |Rc|^2 e^{-f} = \int_M Re^{-f}. \tag{2.4.5}$$

Therefore, by Lemma 2.4.2, (2.4.4) and (2.4.5), we have

Lemma 2.4.3.

$$\hat{N}(Rc) = 0.$$

Now we are ready to prove

Proposition 2.4.1. *Suppose that* (M^n, g_{ij}, f) *is an entropy stable compact shrinking soliton satisfying* (2.4.1)*, then* $-1/2$ *is the only negative eigenvalue of the operator* $\mathcal{L}_f$ *on* $\ker \operatorname{div}_f$*, and the multiplicity of* $-1/2$ *is one. In particular,* $-1/2$ *is the least eigenvalue of* $\mathcal{L}_f$ *on* $\ker \operatorname{div}_f$*.*

Proof. By Lemma 2.4.1 and Lemma 2.4.2, we know that $Rc \in \ker \operatorname{div}_f$, and is an eigen-tensor of $\mathcal{L}_f$ with eigenvalue $-1/2$. Suppose that there exists a (non-zero) symmetric 2-tensor $h \in \ker \operatorname{div}_f$ such that

$$\mathcal{L}_f h = \alpha h,$$

with $\alpha > 0$, and

$$(Rc, h)_f =: \int_M < Rc, h > e^{-f} = 0.$$

Then, by (2.4.2), we have

$$\begin{aligned}\delta^2\nu_g(h,h) =& \frac{1}{(4\pi)^{n/2}}\int_M < \hat{N}h, h > e^{-f}\\ =& \frac{1}{(4\pi)^{n/2}}\int_M < \mathcal{L}_f h, h > e^{-f}\\ =& \frac{\alpha}{(4\pi)^{n/2}}\int_M |h|^2 e^{-f} > 0,\end{aligned}$$

a contradiction to the entropy stability of (M^n, g_{ij}, f). Thus $-1/2$ is the only negative eigenvalue of $\mathcal{L}_f$ on $\ker \mathrm{div}_f$, with multiplicity one. □

Remark 2.4.4. *In [43], S. Hall and T. Murphy have given a very nice interpretation of their proof in terms of the multiplicity of the eigenvalue $-1/2$: for any compact shrinking Kähler-Ricci soliton satisfying* (2.4.1)*, the eigen-space of eigenvalue $-1/2$ has multiplicity at least $h^{1,1}$. Hence a compact shrinking Kähler-Ricci soliton with $h^{1,1} > 1$ is unstable.*

Chapter 3

Rigidity of Gradient Kähler-Ricci Soliton with Harmonic Bochner Tensor

3.1 Introduction

We have seen in Chapter 1 that many results have been obtained on the classification of Riemannian Ricci solitons. However, very few results are known for Kähler-Ricci solitons. In [59], H.-D. Cao and R. Hamilton observed that complete noncompact gradient steady Kähler-Ricci solitons with positive Ricci curvature such that the scalar curvature attains its maximum must be Stein (and also diffeomorphic to $\mathbb{R}^{2n}$). Later, under the same assumptions, A. Chau and L.-F. Tam [32], and R. Bryant [33] independently proved that such steady Kähler-Ricci solitons are actually biholomorphic to $\mathbb{C}^n$. Moreover, Chau and Tam [32] showed that complete noncompact expanding Kähler-Ricci solitons with nonnegative Ricci curvature are also biholomorphic to $\mathbb{C}^n$.

Recently, Q. Chen and the author [60] showed the rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor, so that more information on all three kinds of Kähler-Ricci solitons has been gathered. To state our result, let us first recall that a complete Kähler manifold $(M^n, g_{i\bar{j}})$ is called a gradient Kähler-Ricci soliton if there is a real valued smooth function f such that

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \lambda g_{i\bar{j}}, \quad \text{and} \quad \nabla_i \nabla_j f = 0.$$

The cases where $\lambda = 0, > 0$ and < 0 correspond to steady, shrinking and expanding solitons, respectively.

Moreover, on a Kähler manifold, there is an object called the Bochner tensor which is similar to the Weyl tensor in the Riemannian case. The Bochner tensor $W_{i\bar{j}k\bar{l}}$ is defined by

$$\begin{aligned} W_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} &- \frac{R}{(n+1)(n+2)}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \\ &+ \frac{1}{(n+2)}(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}). \end{aligned}$$

We also denote the divergence of the Bochner tensor by

$$\begin{aligned} C_{i\bar{j}k} &= g^{l\bar{q}} \nabla_l W_{i\bar{j}k\bar{q}} \\ &= \frac{n}{n+2}\nabla_i R_{k\bar{j}} - \frac{n}{(n+1)(n+2)}(g_{k\bar{j}}\nabla_i R + g_{i\bar{j}}\nabla_k R). \end{aligned}$$

Definition 3.1.1. *A Kähler manifold M^n is said to have harmonic Bochner tensor if $C_{i\bar{j}k} = 0$, i.e.,*

$$\nabla_i R_{k\bar{j}} = \frac{1}{n+1}(g_{k\bar{j}}\nabla_i R + g_{i\bar{j}}\nabla_k R).$$

Very recently, by using an argument similar to that in the paper [31] of Cao-Chen, Y. Su and K. Zhang [61] have shown that any complete noncompact gradient Kähler-Ricci soliton with vanishing Bochner tensor is necessarily Kähler-Einstein, and hence a quotient of the corresponding complex space form.

In the following, we investigate gradient Kähler-Ricci solitons with harmonic Bochner tensor, and extend the classification results of Su and Zhang. Our main results are:

Theorem 3.1.1. (Chen-Z. [60]) *Any complete gradient steady Kähler-Ricci soliton with harmonic Bochner tensor must be Kähler-Ricci flat (i.e., Calabi-Yau).*

Theorem 3.1.2. (Chen-Z. [60]) *Any complete gradient shrinking (respectively, expanding) Kähler-Ricci soliton with harmonic Bochner tensor must be isometric to the quotient of $N^k \times \mathbb{C}^{n-k}$, where N^k is a k-dimensional Kähler-Einstein manifold with positive (respectively, negative) scalar curvature.*

Remark 3.1.1. *It is known that a compact Kähler manifold with vanishing Bochner tensor (also called Bochner-Kähler or Bochner-flat) is necessarily a compact quotient of $M_c^k \times M_{-c}^{n-k}$, where M_c^k and M_{-c}^{n-k} denote the complex space forms of constant holomorphic sectional curvature c and $-c$, respectively (cf., e.g., Corollary 4.17 in [62]). It follows immediately that any compact Kähler-Ricci soliton with vanishing Bochner tensor must be a quotient of a complex space form.*

Remark 3.1.2. *We recall that in the Riemannian case, by using a rigidity result of Petersen and Wylie [26], Fernández-López and García-Río [29], and Munteanu and Sesum [27] proved that Ricci shrinkers with harmonic Weyl tensor must be rigid, i.e., a quotient of the product of an Einstein manifold and $\mathbb{R}^k$.*

3.2 Proof of the Main Theorems

Let $(M^n, g_{i\bar{j}}, f)$ be a gradient Kähler-Ricci soliton, i.e.,

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \lambda g_{i\bar{j}}, \quad \text{and} \quad \nabla_i \nabla_j f = 0. \tag{3.2.1}$$

Recall, Lemma 1.1.2 in Chapter 1 that the following basic identities hold.

$$R + |\nabla f|^2 - \lambda f = C; \tag{3.2.2}$$

$$R + \Delta f = n\lambda; \tag{3.2.3}$$

$$\nabla_i R_{k\bar{j}} = R_{i\bar{j}k\bar{l}} \nabla_l f; \tag{3.2.4}$$

and

$$\nabla_i R = R_{i\bar{j}} \nabla_j f. \tag{3.2.5}$$

From now on, we assume that $(M^n, g_{i\bar{j}}, f)$ is a gradient Kähler-Ricci soliton with harmonic Bochner tensor so that

$$\nabla_i R_{k\bar{j}} = \frac{1}{n+1}(\nabla_i R g_{k\bar{j}} + \nabla_k R g_{i\bar{j}}). \tag{3.2.6}$$

Lemma 3.2.1. *We have*

$$\begin{aligned} &\lambda R_{i\bar{j}} - R_{i\bar{j}k\bar{l}}R_{\bar{k}l} \\ &= \frac{1}{n+1}[\frac{1}{n+1}\nabla_k R\nabla_{\bar{k}} f g_{i\bar{j}} + (\lambda R - |Rc|^2)g_{i\bar{j}} - \frac{n}{n+1}\nabla_i R\nabla_{\bar{j}} f \\ &\qquad + \lambda R_{i\bar{j}} - R_{i\bar{k}}R_{k\bar{j}}], \end{aligned} \tag{3.2.7}$$

and

$$\begin{aligned} &2(n+1)\lambda\nabla_i R - 2R\nabla_i R - 2R_{i\bar{j}}\nabla_j R \\ &= -\frac{1}{n+1}\nabla_i R|\nabla f|^2 - \frac{1}{n+1}\nabla_k R\nabla_{\bar{k}} f\nabla_i f. \end{aligned} \tag{3.2.8}$$

Proof. On one hand, by differentiating (3.2.5) and using the contracted second Bianchi identity, we obtain

$$\Delta R = \nabla_k\nabla_{\bar{k}} R = \nabla_k R\nabla_{\bar{k}} f + R_{k\bar{l}}\nabla_{\bar{k}}\nabla_l f.$$

From (3.2.1) and (3.2.6), we get

$$\begin{aligned} \nabla_k\nabla_{\bar{k}} R_{i\bar{j}} &= \frac{1}{n+1}(\Delta R g_{i\bar{j}} + \nabla_i\nabla_{\bar{j}} R) \\ &= \frac{1}{n+1}(\nabla_k R\nabla_{\bar{k}} f g_{i\bar{j}} + R_{k\bar{l}}\nabla_{\bar{k}}\nabla_l f g_{i\bar{j}} + \nabla_i R_{k\bar{j}}\nabla_{\bar{k}} f + R_{k\bar{j}}\nabla_i\nabla_{\bar{k}} f) \\ &= \frac{1}{n+1}[\nabla_k R\nabla_{\bar{k}} f g_{i\bar{j}} + (\lambda R - |Rc|^2)g_{i\bar{j}} + \frac{1}{n+1}\nabla_i R\nabla_{\bar{j}} f \\ &\qquad + \frac{1}{n+1}\nabla_k R\nabla_{\bar{k}} f g_{i\bar{j}} + \lambda R_{i\bar{j}} - R_{i\bar{k}}R_{k\bar{j}}]. \end{aligned} \tag{3.2.9}$$

On the other hand, by differentiating (3.2.4), we have

$$\begin{aligned} \nabla_k\nabla_{\bar{k}} R_{i\bar{j}} &= \nabla_i R_{\bar{j}l}\nabla_{\bar{l}} f + R_{i\bar{j}k\bar{l}}\nabla_{\bar{k}}\nabla_l f \\ &= \nabla_k R_{i\bar{j}}\nabla_{\bar{k}} f + R_{i\bar{j}k\bar{l}}\nabla_{\bar{k}}\nabla_l f \\ &= \nabla_k R_{i\bar{j}}\nabla_{\bar{k}} f + \lambda R_{i\bar{j}} - R_{i\bar{j}k\bar{l}}R_{\bar{k}l}. \end{aligned}$$

Now, by plugging in formula (3.2.9), we obtain (3.2.7).

Next, by taking the divergence on both sides of (3.2.7), we get

$$
\begin{aligned}
&\lambda\nabla_i R - (\nabla_i R_{k\bar{l}})R_{\bar{k}l} - R_{i\bar{j}k\bar{l}}\nabla_j R_{\bar{k}l}\\
&= \frac{1}{n+1}[\frac{1}{n+1}\nabla_i\nabla_k R\nabla_{\bar{k}} f + \frac{1}{n+1}\nabla_k R\nabla_i\nabla_{\bar{k}} f + \lambda\nabla_i R - \nabla_i|Rc|^2\\
&\qquad - \frac{n}{n+1}\nabla_j\nabla_i R\nabla_{\bar{j}} f - \frac{n}{n+1}\nabla_i R\Delta f + \lambda\nabla_i R - (\nabla_j R_{i\bar{k}})R_{k\bar{j}} - R_{i\bar{k}}\nabla_k R]\\
&= \frac{1}{n+1}[\frac{1}{n+1}\nabla_i R_{k\bar{l}}\nabla_l f\nabla_{\bar{k}} f + \frac{\lambda}{n+1}\nabla_i R - \frac{1}{n+1}R_{i\bar{k}}\nabla_k R + \lambda\nabla_i R - 2R_{k\bar{l}}\nabla_i R_{\bar{k}l}\\
&\qquad - \frac{n}{n+1}\nabla_i R_{j\bar{k}}\nabla_k f\nabla_{\bar{j}} f - \frac{\lambda n^2}{n+1}\nabla_i R + \frac{n}{n+1}R\nabla_i R + \lambda\nabla_i R\\
&\qquad - R_{k\bar{j}}\nabla_i R_{j\bar{k}} - R_{i\bar{k}}\nabla_k R].
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\lambda\nabla_i R - (\nabla_i R_{k\bar{l}})R_{\bar{k}l} - R_{i\bar{j}k\bar{l}}\nabla_j R_{\bar{k}l}\\
&= \frac{1}{n+1}[-\frac{n-1}{(n+1)^2}\nabla_i R|\nabla f|^2 - \frac{n-1}{(n+1)^2}\nabla_k R\nabla_{\bar{k}} f\nabla_i f\\
&\qquad + (3-n)\lambda\nabla_i R - (1+\frac{1}{n+1})R_{i\bar{k}}\nabla_k R - 3R_{k\bar{l}}\nabla_i R_{\bar{k}l} + \frac{n}{n+1}R\nabla_i R].
\end{aligned}
$$

But,

$$
\begin{aligned}
R_{l\bar{k}}\nabla_i R_{k\bar{l}} &= \frac{1}{n+1}R_{l\bar{k}}(\nabla_i R g_{k\bar{l}} + \nabla_k R g_{i\bar{l}})\\
&= \frac{1}{n+1}R\nabla_i R + \frac{1}{n+1}R_{i\bar{j}}\nabla_j R,
\end{aligned}
$$

and

$$
\begin{aligned}
R_{i\bar{j}k\bar{l}}\nabla_j R_{l\bar{k}} &= \frac{1}{n+1}R_{i\bar{j}k\bar{l}}(\nabla_j R g_{l\bar{k}} + \nabla_l R g_{j\bar{k}})\\
&= \frac{1}{n+1}R_{i\bar{j}}\nabla_j R + \frac{1}{n+1}R_{i\bar{l}}\nabla_l R\\
&= \frac{2}{n+1}R_{i\bar{j}}\nabla_j R.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\lambda\nabla_i R - \frac{1}{n+1}R\nabla_i R - \frac{3}{n+1}R_{i\bar{j}}\nabla_j R \\
&= \lambda\nabla_i R - (\nabla_i R_{k\bar{l}})R_{\bar{k}l} - R_{i\bar{j}k\bar{l}}\nabla_j R_{\bar{k}l} \\
&= \frac{1}{n+1}[-\frac{n-1}{(n+1)^2}\nabla_i R|\nabla f|^2 - \frac{n-1}{(n+1)^2}\nabla_k R\nabla_{\bar{k}} f\nabla_i f \\
&\qquad + (3-n)\lambda\nabla_i R - (1+\frac{1}{n+1})R_{i\bar{k}}\nabla_k R - 3R_{k\bar{l}}\nabla_i R_{\bar{k}l} + \frac{n}{n+1}R\nabla_i R].
\end{aligned}
$$

Therefore, formula (3.2.8) follows easily.

□

Now, suppose that $\nabla f \neq 0$ at some point p. Then we may choose an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ of holomorphic vector fields at p such that e_1 is parallel to ∇f. Therefore, we have $|\nabla_1 f| = |\nabla f|$ and $\nabla_k f = 0$ for $k = 2, \cdots, n$ at p.

Lemma 3.2.2. *Suppose $\nabla f \neq 0$ at p. Then, under the frame $\{e_1, e_2, \cdots, e_n\}$ chosen above, we have at p:*

$$R_{k\bar{1}} = R_{1\bar{k}} = 0 \quad \textit{for} \quad k \geq 2.$$

Proof. From (3.2.4)-(3.2.6), we have at p,

$$R_{i\bar{j}k\bar{1}}\nabla_1 f = \frac{1}{n+1}(\nabla_i R g_{k\bar{j}} + \nabla_k R g_{i\bar{j}}) = \frac{1}{n+1}(R_{i\bar{1}}g_{k\bar{j}} + R_{k\bar{1}}g_{i\bar{j}})\nabla_1 f.$$

It follows that

$$R_{i\bar{j}k\bar{1}} = \frac{1}{n+1}(R_{i\bar{1}}g_{k\bar{j}} + R_{k\bar{1}}g_{i\bar{j}}). \tag{3.2.10}$$

In particular, for $k \geq 2$, we have that

$$R_{1\bar{1}k\bar{1}} = \frac{1}{n+1}R_{k\bar{1}} \quad \text{and} \quad R_{1\bar{k}1\bar{1}} = 0.$$

However, on the other hand, it is easy to see that

$$R_{1\bar{1}k\bar{1}} = \overline{R_{\bar{1}1\bar{k}1}} = \overline{R_{1\bar{k}1\bar{1}}} = 0.$$

Therefore, $R_{k\bar{1}} = R_{1\bar{k}} = 0$ for $k \geq 2$. □

Lemma 3.2.2 tells us that ∇f is an eigenvector of the Ricci curvature tensor. Thus we may choose another orthonormal frame $\{w_1 = e_1, w_2, \cdots, w_n\}$ at p such that $|\nabla_1 f| = |\nabla f|$ and the Ricci curvature tensor is diagonalized at p, i.e.,

$$R_{i\bar{j}} = R_{i\bar{i}}\delta_{ij}.$$

Proposition 3.2.1. *Suppose that $\nabla f \neq 0$ at p. Then under the orthonormal frame $\{w_1, w_2, \cdots, w_n\}$ chosen above, we have the following identities at p:*

$$n\lambda R_{1\bar{1}} - RR_{1\bar{1}} = \lambda R - |Rc|^2 - \frac{n-1}{n+1}R_{1\bar{1}}|\nabla f|^2, \tag{3.2.11}$$

and

$$(n+1)\lambda R_{1\bar{1}} - RR_{1\bar{1}} - R_{1\bar{1}}^2 = -\frac{1}{n+1}R_{1\bar{1}}|\nabla f|^2. \tag{3.2.12}$$

Proof. In (3.2.7), setting $i = j = 1$ and using (3.2.10), we have

$$\begin{aligned}
&\lambda R_{1\bar{1}} - \frac{1}{n+1}R_{1\bar{1}}^2 - \frac{1}{n+1}RR_{1\bar{1}} \\
&= \lambda R_{1\bar{1}} - \frac{2}{n+1}R_{1\bar{1}}^2 - \frac{1}{n+1}R_{1\bar{1}}(R - R_{1\bar{1}}) \\
&= \lambda R_{1\bar{1}} - \frac{2}{n+1}R_{1\bar{1}}^2 - \frac{1}{n+1}R_{1\bar{1}}\sum_{k=2}^{n} R_{k\bar{k}} \\
&= \lambda R_{1\bar{1}} - R_{1\bar{1}1\bar{1}}R_{1\bar{1}} - \sum_{k=2}^{n} R_{1\bar{1}k\bar{k}}R_{k\bar{k}} \\
&= \lambda R_{1\bar{1}} - \sum_{k=1}^{n} R_{1\bar{1}k\bar{k}}R_{k\bar{k}} \\
&= \frac{1}{n+1}[\frac{1}{n+1}R_{1\bar{1}}|\nabla f|^2 + \lambda R - |Rc|^2 - \frac{n}{n+1}R_{1\bar{1}}|\nabla f|^2 + \lambda R_{1\bar{1}} - R_{1\bar{1}}^2].
\end{aligned}$$

Thus, formula (3.2.11) follows immediately.

Next, by setting $i = 1$ in (3.2.8) and dividing both sides of the equation by $\nabla_1 f$, we get (3.2.12).

□

Proposition 3.2.2. *At a point p where $\nabla f \neq 0$, we have either*

$$Rc(\nabla f, \nabla f) = 0,$$

or

$$Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2.$$

Proof. Since at point p, $\nabla f \neq 0$, formula (3.2.12) implies that in a neighborhood of p we have

$$\left[(n+1)\lambda - R - \frac{R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} + \frac{1}{n+1}|\nabla f|^2\right]\frac{R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} = 0. \tag{3.2.13}$$

Therefore, there are two possibilities, either

I) $R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f = 0$ at p,

or

II) $R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f \neq 0$ at p. In this case, near p we have

$$-(n+1)\lambda + R + \frac{R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} - \frac{1}{n+1}|\nabla f|^2 = 0.$$

Taking the covariant derivative on both sides and using (3.2.1) gives us

$$\begin{aligned}
0 &= \nabla_k R + \frac{1}{|\nabla f|^2}(\nabla_i f \nabla_{\bar{j}} f \nabla_k R_{j\bar{i}} + R_{j\bar{i}}\nabla_i f \nabla_k \nabla_{\bar{j}} f) - \frac{\nabla_j f \nabla_k \nabla_{\bar{j}} f}{|\nabla f|^4} R_{l\bar{i}}\nabla_i f \nabla_{\bar{l}} f \\
&\quad - \frac{1}{n+1}(\nabla_j f \nabla_k \nabla_{\bar{j}} f) \\
&= \nabla_k R + \frac{1}{(n+1)|\nabla f|^2}\nabla_i f \nabla_{\bar{j}} f(\nabla_k R g_{j\bar{i}} + \nabla_j R g_{k\bar{i}}) + \frac{1}{|\nabla f|^2}(\lambda \nabla_k R - R_{k\bar{j}}\nabla_j R) \\
&\quad - \frac{\lambda \nabla_k f - \nabla_k R}{|\nabla f|^4}\nabla_i R \nabla_{\bar{i}} f - \frac{1}{n+1}(\lambda \nabla_k f - \nabla_k R).
\end{aligned}$$

Evaluating the identity above at p under the orthonormal frame $\{w_1, w_2, \cdots, w_n\}$ yields

$$\begin{aligned}
0 &= R_{1\bar{1}} + \frac{2}{(n+1)|\nabla f|^2}R_{1\bar{1}}|\nabla f|^2 + \frac{1}{|\nabla f|^2}(\lambda R_{1\bar{1}} - R_{1\bar{1}}^2) \\
&\quad - \frac{\lambda - R_{1\bar{1}}}{|\nabla f|^4}R_{1\bar{1}}|\nabla f|^2 - \frac{1}{n+1}(\lambda - R_{1\bar{1}}) \\
&= \frac{n+4}{n+1}R_{1\bar{1}} - \frac{1}{n+1}\lambda.
\end{aligned}$$

Thus, we have $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2$ whenever $Rc(\nabla f, \nabla f) \neq 0$.

$\square$

Now we are ready to prove the main Theorems.

First, we may assume that f is not constant, for otherwise we get that M is Kähler-Einstein from the soliton equation.

Proof of theorem 3.1.1: For steady Kähler-Ricci solitons, we have $\lambda = 0$. From Proposition 3.2.2, we know that in case of $\lambda = 0$, we always have $Rc(\nabla f, \nabla f) = 0$. Then (3.2.2) and (3.2.11) imply that $Rc = 0$ on the set $\{p \in M | \nabla f(p) \neq 0\}$. On the other hand, by the soliton equation, it is easy to see that we also have $Rc = 0$ in the interior of the set $\{p \in M | \nabla f(p) = 0\}$. Thus the steady soliton M must be Kähler-Ricci flat. $\square$

Proof of theorem 3.1.2: For shrinking and expanding Kähler-Ricci solitons, we have $\lambda \neq 0$.

In this case, from Proposition 3.2.2 and the continuity of $\frac{Rc(\nabla f, \nabla f)}{|\nabla f|^2}$, we conclude that in each component of the open set $A = \{p \in M | \nabla f(p) \neq 0\}$, we have either $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4} |\nabla f|^2$ or $Rc(\nabla f, \nabla f) = 0$.

If $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4} |\nabla f|^2$ in some component Ω of A, then at any point $p \in \Omega$ we have $R_{1\bar{1}} = \frac{\lambda}{n+4}$ and $\nabla R(p) = \frac{\lambda}{n+4} \nabla f(p)$ from formula (3.2.5). Therefore, we have $\nabla R = \frac{\lambda}{n+4} \nabla f$ in Ω. It then follows that $R = \frac{\lambda}{n+4} f + C$ in Ω. Thus (3.2.12) implies that $|\nabla f|^2 = \frac{n+1}{n+4} \lambda f + C'$ in Ω. Since $R + |\nabla f|^2 - \lambda f = C_0$, we have $f = C_1$ in Ω, which contradicts the fact that $\nabla f \neq 0$ in Ω.

Therefore, we must have $Rc(\nabla f, \nabla f) = 0$ in A. Since locally f is a constant in the interior of $M \backslash A$, we have $Rc(\nabla f, \nabla f) = 0$ on the whole manifold M. It follows by (3.2.5) that $\nabla R = 0$ on M. Then (3.2.6) implies that the Ricci curvature tensor is parallel on M. Therefore, by the de Rahm decomposition theorem, the universal cover of M is isometric to $N^{n-1} \times \mathbb{C}$, where N is again an $(n-1)$ dimensional Kähler-Ricci soliton with harmonic Bochner tensor. Thus by induction, we can finally get that M is isometric to a quotient of the product of a Kähler-Einstein manifold and the complex Euclidean space. $\square$

Bibliography

[1] Cao, H.-D., Hamilton, R. & Ilmanen, T. Gaussian densities and stability for some Ricci solitons. *arXiv:math/0404165v1* (2004).

[2] Cao, H.-D. Geometry of Ricci solitons. *Chinese Ann. Math. Ser. B* **27**, 121–142 (2006).

[3] Cao, H.-D. Recent progress on Ricci solitons. In *Recent advances in geometric analysis*, vol. 11 of *Adv. Lect. Math. (ALM)*, 1–38 (Int. Press, Somerville, MA, 2010).

[4] Hamilton, R. S. Four–manifolds with positive curvature operator. *J. Differential Geom.* **24**, 153–179 (1986).

[5] Zhang, Z.-H. On the completeness of gradient Ricci solitons. *Proc. Amer. Math. Soc.* **137**, 2755–2759 (2009).

[6] Hamilton, R. S. The formation of singularities in the Ricci flow. In *Surveys in Differential Geometry*, vol. 2, 7–136 (International Press, Combridge, MA, 1995).

[7] Ivey, T. Ricci solitons on compact three-manifolds. *Diff. Geom. Appl.* **3**, 301–307 (1993).

[8] Hamilton, R. S. The Ricci flow on surfaces. In *Mathematics and general relativity, Contemp. Math.,* ***71***, 237–262 (Amer. Math. Soc., Providence, RI, 1988).

[9] Cao, H.-D. Existence of gradient Kähler-Ricci solitons. In *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, 1–16 (A K Peters, Wellesley, MA, 1996).

[10] Koiso, N. On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. In *Recent topics in differential and analytic geometry*, vol. 18 of *Adv. Stud. Pure Math.*, 327–337 (Academic Press, Boston, MA, 1990).

[11] Wang, X.-J. & Zhu, X. Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.* **188**, 87–103 (2004).

[12] Feldman, M., Ilmanen, T. & Knopf, D. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geom.* **65**, 169–209 (2003).

[13] Rryant, R. Local existence of gradient Ricci solitons. *unpublished* .

[14] Chow, B. *et al.* *The Ricci flow: techniques and applications. Part I*, vol. 135 of *Mathematical Surveys and Monographs* (American Mathematical Society, Providence, RI, 2007). Geometric aspects.

[15] Chen, B.-L. Strong uniqueness of the Ricci flow. *J. Differential Geom.* **82**, 363–382 (2009).

[16] Hamilton, R. S. Three manifolds with positive Ricci curvature. *J. Differential Geom.* **17**, 255–306 (1982).

[17] Böhm, C. & Wilking, B. Manifolds with positive curvature operator are space forms. *Ann. of Math.* **167**, 1079–1097 (2008).

[18] Perelman, G. The entropy formula for the Ricci flow and its geoemtric applicaitions. *arXiv: math.DG/0211159* (2002).

[19] Ni, L. Ancient solutions to Kähler-Ricci flow. *Math. Res. Lett.* **12**, 633–653 (2005).

[20] Ni, L. & Wallach, N. On a classification of gradient shrinking solitons. *Math. Res. Lett.* **15**, 941–955 (2008).

[21] Naber, A. Noncompact shrinking four solitons with nonnegative curvature. *J. Reine Angew. Math.* **645**, 125–153 (2010).

[22] Cao, H.-D., Chen, B.-L. & Zhu, X.-P. Recent developments on Hamilton's Ricci flow. In *Surveys in differential geometry. Vol. XII. Geometric flows*, vol. 12 of *Surv. Differ. Geom.*, 47–112 (Int. Press, Somerville, MA, 2008).

[23] Ni, L. & Wallach, N. On four-dimensional gradient shrinking solitons. *Int. Math. Res. Not. IMRN* Art. ID rnm152, 13 (2008).

[24] Eminenti, M., La Nave, G. & Mantegazza, C. Ricci solitons: the equation point of view. *Manuscripta Math.* **127**, 345–367 (2008).

[25] Zhang, Z.-H. Gradient shrinking solitons with vanishing Weyl tensor. *Pacific J. Math.* **242**, 189–200 (2009).

[26] Petersen, P. & Wylie, W. On the classification of gradient Ricci solitons. *Geom. Topo.* **14**, 2277–2300 (2010).

[27] Munteanu, O. & Sesum, N. On gradient Ricci solitons. *to appear in J. Geom. Anal.* .

[28] Cao, H.-D. & Zhou, D. On complete gradient shrinking Ricci solitons. *J. Differential Geom.* **85**, 175–185 (2010).

[29] Fernández-López, M. & García-Río, E. Rigidity of shrinking Ricci solitions. *Math. Z.* **269**, 461–466 (2011).

[30] Cao, H.-D. & Chen, Q. On Bach-flat gradient shrinking Ricci solitons. *arXiv:1105.3163* .

[31] Cao, H.-D. & Chen, Q. On locally conformally flat gradient steady Ricci solitons. *Trans. Amer. Math. Soc.* **364**, 2377–2391 (2012).

[32] Chau, A. & Tam, L.-F. A note on the uniformization of gradient Kähler Ricci solitons. *Math. Res. Lett.* **12**, 19–21 (2005).

[33] Bryant, R. L. Gradient Kähler Ricci solitons. *Astérisque* 51–97 (2008). Géométrie différentielle, physique mathématique, mathématiques et société. I.

[34] Catino, G. & Mantegazza, C. The evolution of the Weyl tensor under the Ricci flow. *Ann. Inst. Fourier* **61**, 1407–1435 (2011).

[35] Brendle, S. Uniqueness of gradient Ricci solitons. *Math. Res. Lett.* **18**, 531–538 (2011).

[36] Chen, X. & Wang, Y. On four-dimensional anti-self-dual gradient Ricci solitons. *arXiv:1102.0358* .

[37] Cao, H.-D., Catino, G., Chen, Q., Mantegazze, C. & Mazzieri, L. Bach-flat gradient steady Ricci solitons. *arXiv:1107.4591* .

[38] Brendle, S. Rotational symmetry of self-similar solutions to the Ricci flow. *arXiv:1202.1264* .

[39] Brendle, S. Rotational symmetry of Ricci solitons in higher dimensions. *arXiv:1203.0270* .

[40] Feldman, M., Ilmanen, T. & Ni, L. Entropy and reduced distance for Ricci expanders. *J. Geom. Anal.* **15**, 49–62 (2005).

[41] Zhu, M. The second variation of the Ricci expander entropy. *Pacific J. Math.* **251**, 499–510 (2011).

[42] Cao, H.-D. & Zhu, M. On second variation of Perelman's Ricci shrinker entropy. *to appear in Math. Ann.* .

[43] Hall, S. & Murphy, T. On the linear stability of Kähler-Ricci solitons. *Proc. Amer. Math. Soc.* **139**, 3327–3337 (2011).

[44] Cao, H.-D. Recent progress on Ricci solitons. In *Recent advances in geometric analysis*, vol. 11 of *Adv. Lect. Math. (ALM)*, 1–38 (Int. Press, Somerville, MA, 2010).

[45] Colding, T. & Minicozzi, W. Generic mean curvature flow I; generic singularities. *Ann. of Math.* **175**, 755–833 (2012).

[46] Rothaus, O. S. Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators. *J. Funct. Anal.* **42**, 110–120 (1981).

[47] Guenther, C., Isenberg, J. & Knopf, D. Stability of the Ricci flow at Ricci-flat metrics. *Comm. Anal. Geom.* **10**, 741–777 (2002).

[48] Dai, X., Wang, X. & Wei, G. On the stability of Riemannian manifold with parallel spinors. *Invent. Math.* **161**, 151–176 (2005).

[49] Huisken, G. Ricci deformation of the metric on a Riemannian manifold. *J. Differential Geom.* **21**, 47–62 (1985).

[50] Boucetta, M. Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on $\mathrm{P}^n(\mathbb{C})$. *J. Geom. Phys.* **60**, 1352–1369 (2010).

[51] Gasqui, J. & Goldschmidt, H. Radon transforms and spectral rigidity on the complex quadrics and the real Grassmannians of rank two. *J. Reine Angew. Math.* **480**, 1–69 (1996).

[52] Gasqui, J. & Goldschmidt, H. On the geometry of the complex quadric. *Hokkaido Math. J.* **20**, 279–312 (1991).

[53] Delay, E. Essential spectrum of the Lichnerowicz Laplacian on two tensor on asymptotically hyperbolic manifolds. *J. Geom. and Physics* **43**, 33–44 (2002).

[54] Lee, J. Fredholm operators and Einstein metrics on conformally compact manifolds. *Mem. Amer. Math. Soc.* **183**, vi+83pp (2006).

[55] Knopf, D. & Young, A. Asymptotic stability of the cross curvature flow at a hyperbolic metric. *Proc. Amer. Math. Soc.* **137**, 699–709 (2009).

[56] Ye, R. Ricci flow, einstein metrics and space forms. *Trans. Amer. Math. Soc.* **338**, 871–896 (1993).

[57] Delay, E. Spectrum of the Lichnerowicz Laplacian on asymptotically hyperbolic surfaces. *Math. Phys. Anal. Geom.* **11**, 365–379 (2008).

[58] Suneeta, V. Investigating off-shell stability of anti-de sitter space in string theory. *Classical Quantum Gravity* **26** (2009).

[59] Cao, H.-D. & Hamilton, R. S. Gradient Kähler-Ricci solitons and periodic orbits. *Comm. Anal. Geom.* **8**, 517–529 (2000).

[60] Chen, Q. & Zhu, M. On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor. *to appear in Proc. Amer. Math. Soc., arXiv:1105.1152* .

[61] Su, Y. & Zhang, K. On the Kähler-Ricci solitons with vanishing Bochner-Weyl tensor. *preprint* .

[62] Bryant, R. L. Bochner-Kähler metrics. *J. Amer. Math. Soc.* **14**, 623–715 (electronic) (2001).

CURRICULUM VITA

Meng Zhu

Department of Mathematics
Lehigh University
Bethlehem, PA 18015

Email: *mez206@lehigh.edu*
Phone: 610-974-1751
Fax: 610-758-3767

Education

- **Lehigh University,** Bethlehem, PA
  Aug 2006 - Present
  Ph.D. (expected May 2012)
  Advisor: Prof. Huai-Dong Cao.

- **Zhejiang University,** Hangzhou, China
  Sep 2003 - Jun 2006
  Graduate Student.

- **Zhejiang University,** Hangzhou, China
  Sep 1999 - Jun 2003
  B.S., Mathematics, Jun 2003.

Employment

- Teaching Assistant,
  Aug 2008 - Present,
  Lehigh University, Bethlehem, PA

- Research Assistant,
  Aug 2006 - May 2008,
  Lehigh University, Bethlehem, PA

Awards and Honors

- 2006 - 2008 University fellowship, Lehigh University.
- Jun 2010 - Jul 2010 Lehigh University College of Arts and Sciences summer research fellowship.

Research Interests

Differential Geometry and Geometric Analysis, especially Geometric Flows.

Publications

- *A note on compact Kähler-Ricci flow with positive bisectional curvature* (joint with Huai-Dong Cao), Math. Res. Lett. **16** (2009), No. 6, 935-939.
- *The second variation of the Ricci expander entropy*, Pacific J. Math. **251** (2011), No. 2, 499-510.
- *On second variation of Perelman's Ricci shrinker entropy* (joint with Huai-Dong Cao), to appear in Math. Ann. (arXiv:1008.0842).
- *On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor* (joint with Qiang Chen), arXiv:1105.1152, to appear in Proc. Amer. Math. Soc..
- *Ricci solitons on Sasakian manifolds* (joint with Chenxu He), arXiv:1109.4407.

Presentations

◇ *On the stability of compact Ricci solitons*, AMS Eastern Sectional Meeting, special session on Topics in PDEs and Geometric Analysis, College of the Holy Cross, 2011.

◇ *Introduction to Ricci solitons*, Graduate Students Intercollegiate Mathematics Seminar, Lehigh University, 2010.

◇ *The second variation of the Ricci expander entropy*, Geometry Seminar, Lehigh University, 2009.

◇ *3-manifolds with positive Ricci curvature*, Graduate Student Colloquium, Lehigh University, 2007.

Other Presentations

- A series of presentations on Hamilton-Perelman's theory on the Ricci flow and its application, Lehigh University, Bethlehem, PA, Spring 2007 - Fall 2007.

- A series of presentations on Algebraic Geometry and Kähler-Ricci flow, Lehigh University, Bethlehem, PA, Spring 2008.

Teaching Experience

As Teaching Assistant:
- Math 22: Calculus II, Fall 2011.
- Math 21: Calculus I, Fall 2011.
- Math 205: Linear Models, Spring 2011.
- Math 23: Calculus III, Fall 2010.
- Math 205: Linear Models, Spring 2010.
- Math 75: Calculus I Part A, Fall 2009.
- Math 75: Calculus I Part A, Fall 2008.

Conferences Participated:

◇ *Geometry Festival - In honor of Eugenio Calabi, on the occasion of his 88th birthday*, University of Pennsylvania, Philadelphia, PA, Apr 2011.

◇ *AMS Eastern Sectional Meeting*, College of the Holy Cross, Worcester, MA, Apr 2011.

- ◇ *International Symposium on Geometry and Topology, In memory of Chuan-Chih Hsiung*, Lehigh University, Bethlehem, PA, May 2010.
- ◇ *Geometry Festival*, New York University, New York City, Apr 2010.
- ◇ *Geometry and Topology Conference*, Lehigh University, Bethlehem, PA, Apr 2009.
- ◇ *Geometry, Topology, and Physics 2008 - A conference celebrating the 70th birthday of Herman Gluck*, Rutgers University-Newark, Newark, NJ, Nov 2008.
- ◇ *Current Developments in Mathematics*, Harvard University, Cambridge, MA, Nov 2007.
- ◇ *Geometry and Topology Conference*, Lehigh University, Bethlehem, PA, Apr 2007.
- ◇ *Geometry Festival*, University of Maryland, College Park, MD, Apr 2007.

Professional Membership

Member of American Mathematical Society.