2012

Sharpness of exponent bounds for SU(n)

Karen B. McCready
Lehigh University

Sharpness of exponent bounds for SU(n)

by

Karen McCready

Thesis Advisor: Donald M. Davis

A Dissertation
Presented to the Graduate and Research Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

Lehigh University
May 2012

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Karen McCready

Sharpness of exponent bounds for SU(n)

Date

Donald M. Davis, Dissertation Director, Chair

Accepted Date

Committee Members

Martin Bendersky

David Johnson

Susan Szczepanski

Steven Weintraub

Acknowledgments

I would like to thank my advisor, Professor Donald M. Davis, for his excellent guidance, patience, help, and encouragement. Without his knowledge, counsel and kindness, I would not have been able to write this dissertation. I also wish to thank Professor Martin Bendersky, Professor David Johnson, Professor Susan Szczepanski, and Professor Steven Weintraub for their service as committee members.

I am very grateful to my parents, as well as my brother and sisters, for their constant love, support, patience, and encouragement that has helped me to keep going through every challenge. I am also thankful for all of my fellow graduate students and Mary Ann Dent for their encouragement and help.

Finally, I thank the Lord. Without Him I can do nothing, but with Him all things are possible. Soli Deo gloria.

Contents

List of Tables

Abstract

The p-primary v_1-periodic homotopy groups of a topological space X, denoted by $v_1^{-1}\pi_*(X)_{(p)}$, are roughly the parts of the homotopy groups of X localized at a prime p which are detected by K-theory. We will use combinatorial number theory to determine, for p an odd prime, the values of n for which

$$v_1^{-1}\pi_{2(n-1)}(SU(n))_{(p)} \cong \mathbb{Z}/p^{n-1+\nu_p\left(\lfloor \frac{n}{p} \rfloor !\right)}.$$

As a corollary, we obtain new bounds for the p-exponent of $\pi_*(SU(n))$.

Chapter 1

Introduction

For a prime number p, the homotopy p-exponent of a space X, denoted by $\exp_p(X)$, is the largest $e \in \{0, 1, 2, \ldots\}$ such that some homotopy group $\pi_i(X)$ has an element of order p^e. Homotopy groups of spaces are often very difficult to compute. Thus, knowing the p-exponent of a space, for some p, is helpful in understanding more about the structure of the space. We are particularly interested in the special unitary group, $SU(n)$, the group of n-by-n unitary matrices of determinant 1. Much progress has been made in the study of $\exp_p(SU(n))$, and that is the focus of this work.

In [12], Davis and Sun proved a strong lower bound for the homotopy p-exponent of $SU(n)$. Let $\nu_p(n)$ denote the largest power of p that divides n.

Theorem 1.1. *(Davis and Sun, 2007) For any prime p and $n \in \{2, 3, 4, \ldots\}$, some homotopy group $\pi_i(SU(n))$ contains an element of order $p^{n-1+\nu_p(\lfloor \frac{n}{p} \rfloor !)}$, i.e.,*

$$\exp_p(SU(n)) \geq n - 1 + \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

We will study the extent to which this bound might be sharp.

In [11], Davis and Mahowald defined, for any prime p, the p-primary v_1-periodic homotopy groups of a topological space X, denoted by $v_1^{-1}\pi_*(X)_{(p)}$. These are a first approximation to the p-primary homotopy groups, $\pi_*(X)_{(p)}$. For spheres and compact Lie groups, each $v_1^{-1}\pi_i(X)_{(p)}$ group is a direct summand of some homotopy group $\pi_j(X)$. Thus, they provide lower bounds for $\exp_p(X)$. We will use these groups and tools from number theory, in particular, Stirling numbers of the second kind, to find out more about p-divisibility of homotopy groups of $SU(n)$.

For $n, k \in \mathbb{N}$ with $n + k \in \mathbb{Z}^+$, the Stirling number of the second kind, $S(n, k)$, is the number of ways to partition n objects into k nonempty subsets, where $S(0, 0) := 1$. For example, S$(3, 2)$=3. These numbers satisfy the condition

$$S(k, j)j! = (-1)^j \sum_{i=0}^{j} (-1)^i \binom{j}{i} i^k.$$

For p prime and any integer k, we define the partial Stirling numbers,

$$a_p(k, j) = \sum_{i \not\equiv 0(p)} (-1)^i \binom{j}{i} i^k.$$

From these, we define

$$e_p(k, n) = \min(\nu_p(a_p(k, j)) : j \geq n).$$

In [9], Davis showed that $e_p(k, n)$ provides significant information about the groups $v_1^{-1}\pi_{2k}(SU(n))_{(p)}$ and $v_1^{-1}\pi_{2k-1}(SU(n))_{(p)}$.

Theorem 1.2. *(Davis,* 1991*) If p or n is odd, then*

$$v_1^{-1}\pi_{2k}(SU(n))_{(p)} \cong \mathbb{Z}/p^{e_p(k,n)},$$

and $v_1^{-1}\pi_{2k-1}(SU(n))_{(p)}$ is an abelian group of the same order, but not necessarily cyclic.

Thus, for any k, $e_p(k, n)$ gives a lower bound for $\exp_p(SU(n))$. We would like to know the largest value of $e_p(k, n)$ over all possible k. For many n, $e_p(n - 1, n)$ gives the largest v_1-periodic homotopy group of $SU(n)$, or close to it, as discussed in [7]. So this value is of significant interest. Davis and Sun provided a lower bound for $e_p(n-1,n)$ in the following theorem, which was proved in [12], and clearly implies Theorem 1.1.

Theorem 1.3. *(Davis and Sun,* 2007*) Let p be a prime number. Then*

$$e_p(n-1,n) \geq n-1+\nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

We would like to know when this bound is sharp. In [7], Davis gave conditions that tell when equality is obtained for the primes 2 and 3, giving the groups $v_1^{-1}\pi_{2(n-1)}(SU(n))_{(2)}$ and $v_1^{-1}\pi_{2(n-1)}(SU(n))_{(3)}$ for those values of n for which we have equality. The following theorem provides a generalization to all odd primes of Davis' result for $p=3$.

Let $s_p(n)$ denote $n-1+\nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)$.

Theorem 1.4. *Let p be an odd prime and $n \in \{1,2,3,\ldots\}$, with $n = \cdots d_2 d_1 d_0$ in base-p expansion. Then $e_p(n-1,n) = s_p(n)$ if and only if the following condition holds.*

1. *If $d_0 = 0$, then $d_1 = 1$ and $d_{i+1}+d_i < p$ for $i \geq 2$.*
2. *If $d_0 \neq 0$, then $d_1 + d_0 \leq p$ and $d_{i+1}+d_i < p$ for $i \geq 1$.*

We will prove this theorem by showing that, for each n, there is an $N \geq n$ such that $\nu_p(a_p(n-1,N)) = s_p(n)$ if and only if the condition holds.

Corollary 1.5. *If n satisfies the condition of Theorem 1.4, then*

$$v_1^{-1}\pi_{2(n-1)}(SU(n))_{(p)} \cong \mathbb{Z}/p^{s_p(n)}.$$

If n does not satisfy the condition, then $exp_p(SU(n)) > s_p(n)$.

Chapter 2

Historical Background

A great deal of work has been done in homotopy theory during the last 40 years that has led to the work done here. As mentioned before, the p-primary v_1-periodic homotopy groups of a topological space X are a first approximation to the actual homotopy groups of X localized at a prime. They roughly give the part of $\pi_*(X)$ detected by K-theory. They were defined by Davis and Mahowald in [11] as a direct limit of maps of Moore spaces into the space X, using Adams maps. More precisely,

$$v_1^{-1}\pi_i(X)_{(p)} = \varinjlim_{e,N}[M^{i+N\cdot t(e)}(p^e), X],$$

where

$$t(e) = \begin{cases} 2(p-1)p^{e-1} & \text{if } p \text{ is odd,} \\ \max\{8, 2^{e-1}\} & \text{if } p = 2, \end{cases}$$

and $M^n(p^e)$ is the Moore space $S^{n-1} \cup_{p^e} D^n$. The direct system uses Adams maps, introduced in [1], $A : M^{n+t(e)}(p^e) \to M^n(p^e)$, which induce isomorphisms in K-theory for $n \geq 2e + 3$.

In the 1970's, Mahowald developed ideas that led to the introduction of the v_1-periodic homotopy groups. He computed the 2-primary v_1-periodic homotopy groups of odd dimensional spheres, and showed that these are the image of the J-homomorphism and associated unstable elements, in [13]. In 1989, Thompson did analogous work for odd primes p, in [17], showing that the image of the J homomorphism, along with unstable elements, are the only v_1-periodic elements of $\pi_*(S^n)_{(p)}$. A famous result of Cohen, Moore and Neisendorfer, in [6], showed that if $p \neq 2$ then $\exp_p(S^{2n+1}) = n$, and that the image of the J homomorphism gives elements of maximal order. Thus, for odd primes, p-primary v_1-periodic elements give elements of $\pi_*(S^{2n+1})$ of maximal order. This has not yet been proven for the prime 2. Selick's work for the prime 2 can be found in [14]. Davis has conjectured that the v_1-periodic elements of $SU(n)$ also give elements of maximal order in the actual homotopy groups of $SU(n)$.

In the 1980's, Bendersky and others developed the unstable Novikov spectral sequence for the actual homotopy groups of spaces, found in [3] and [2]. It is based on the BP spectrum. He computed the 1-line and unstable elements on the 2-line for spheres, as well as the 1-line for $SU(n)$. Davis observed that his computations for the 1-line and 2-line of spheres, localized at a prime, agree with the p-primary v_1-periodic homotopy groups of spheres. Combining this with a Five Lemma argument, he showed that, localized at a prime, the 1-line and 2-line of the unstable Novikov spectral sequence for $SU(n)$ give its p-primary v_1-periodic homotopy groups. This led to Davis' result that, for p an odd prime, $v_1^{-1}\pi_{2k}(SU(n))_{(p)} \cong \mathbb{Z}/p^{e_p(k,n)}$, in [9]. He then began to study $e_p(n-1,n)$, since the largest values of $e_p(k,n)$ seem to occur when $k = n-1$ or $n-1$ plus a multiple of a large power of p. In [9], Davis showed that $e_p(n-1,n) \geq n-1$. Since any value of $e_p(k,n)$ gives a lower bound for the homotopy p-exponent of the space, this implied that $\exp_p(SU(n)) \geq n-1$ for p odd. Bendersky and Davis then proved an analogous result for the prime 2, in [4]. This case was more complicated than the cases for odd primes because the spectral sequence used to obtain the v_1-periodic homotopy groups has elements in all filtrations at the prime

2, and thus differentials and extensions must be considered. At the odd primes the spectral sequence is nonzero only in filtrations 1 and 2, and hence has no differentials or extensions.

In 1998, in [8], Davis used the unstable Novikov spectral sequence to show that, for odd primes, $\exp_p(SU(n)) \geq n-1+\lfloor\frac{n+2p-3}{p^2}\rfloor+\lfloor\frac{n+p^2-p-1}{p^3}\rfloor$. In 2005, Davis conjectured an improved lower bound for $e_p(n-1,n)$, and thus for $\exp_p(SU(n))$. In [12], he and Sun proved the result, giving the inequality $e_p(n-1,n) \geq n-1+\nu_p(\lfloor\frac{n}{p}\rfloor!)$. Since

$$\nu_p(\lfloor n/p\rfloor!) = \lfloor n/p^2\rfloor + \lfloor n/p^3\rfloor + \lfloor n/p^4\rfloor + \ldots,$$

their result gave a nice generalization of Davis' formula in [8]. Davis then proved number theoretic conditions on n for which this inequality is sharp for the primes 2 and 3, in [7], in 2008. In this dissertation, the condition on n for which the inequalilty $e_p(n-1,n) \geq n-1+\nu_p(\lfloor\frac{n}{p}\rfloor!)$ is sharp is generalized for all odd primes. The method used to obtain this generalization differs greatly from the method used by Davis, in [7], for the prime 3. An attempt to modify that method to show the condition for all odd primes proved to be cumbersome, and less effective. In the last chapter, a chart is included which provides calculations for the prime 5, for n from 2 to 128, to give a sense of how close the inequality $e_p(n-1,n) \geq n-1+\nu_p(\lfloor\frac{n}{p}\rfloor!)$ is to being sharp for various n and whether it gives the largest $e_p(k,n)$.

Chapter 3

Proof of Theorem 1.4

In the remainder of this work we will use Lucas' Theorem to reduce binomial coefficients mod p. This gives, mod p,

$$\binom{n}{k} \equiv \prod_{i=1}^{m} \binom{a_i}{b_i},$$

where $n = \sum_{i=0}^{m} a_i p^i$ and $k = \sum_{i=0}^{m} b_i p^i$ in base-p expansion.

Let $B_n(N) = \frac{1}{\lfloor \frac{n}{p} \rfloor !} \sum_{k \geq 1} (-1)^k \binom{N}{pk} k^{n-1}$. In [7], Davis showed that, for $N \geq n$, $B_n(N)$ can be used to determine exactly when $\nu_p(a_p(n-1, N)) = s_p(n)$.

Proposition 3.1. *(Davis, 2008) For $N \geq n$, $\nu_p(a_p(n-1, N)) = s_p(n)$ if and only if $B_n(N) \not\equiv 0$ mod p.*

We include the proof here.

Proof of Proposition 3.1. Since $N \geq n$, $(-1)^N S(n-1, N)N! = 0$. Thus,

$$\begin{aligned} 0 &= (-1)^N S(n-1, N)N! \\ &= a_p(n-1, N) + \sum_k (-1)^{pk} \binom{N}{pk} (pk)^{n-1} \\ &= a_p(n-1, N) + p^{n-1} \sum_k (-1)^k \binom{N}{pk} k^{n-1}. \end{aligned}$$

Hence, $\nu_p(a_p(n-1, N)) = n - 1 + \nu_p\left(\sum_k (-1)^k \binom{N}{pk} k^{n-1}\right)$, and so $\nu_p(a_p(n-1, N)) = s_p(n) = n - 1 + \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)$ if and only if $\nu_p\left(\sum_k (-1)^k \binom{N}{pk} k^{n-1}\right) = \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)$. This holds if and only if $\nu_p\left(\frac{1}{\left\lfloor \frac{n}{p} \right\rfloor !} \sum_k (-1)^k \binom{N}{pk} k^{n-1}\right) = \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right) - \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right) = 0$, which is true if and only if $\frac{1}{\left\lfloor \frac{n}{p} \right\rfloor !} \sum_{k \geq 1} (-1)^k \binom{N}{pk} k^{n-1} \not\equiv 0 \bmod p$. □

Thus we would like to know whether $B_n(N) \not\equiv 0 \bmod p$ for some $N \geq n$. However, we need not check every possible value of N in order to determine this. In [12], Davis and Sun proved that

$$\nu_p\left(\sum_k (-1)^k \binom{m}{pk} k^l\right) \geq \nu_p\left(\left\lfloor \frac{m}{p} \right\rfloor !\right)$$

for all integers m and l. So,

$$\nu_p\left(\frac{1}{\left\lfloor \frac{n}{p} \right\rfloor !} \sum_k (-1)^k \binom{N}{pk} k^{n-1}\right) \geq \nu_p\left(\left\lfloor \frac{N}{p} \right\rfloor !\right) - \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

Hence, $\nu_p\left(\frac{1}{\left\lfloor \frac{n}{p} \right\rfloor !} \sum_k (-1)^k \binom{N}{pk} k^{n-1}\right) > 0$ unless $\nu_p\left(\left\lfloor \frac{N}{p} \right\rfloor !\right) = \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)$. Since $\nu_p(m!) = \sum_{i \geq 1} \left\lfloor \frac{m}{p^i} \right\rfloor$, this means that we only need to check N such that $\left\lfloor \frac{N}{p^2} \right\rfloor = \left\lfloor \frac{n}{p^2} \right\rfloor$.

From the following lemma, we will see that we can use Stirling numbers of the second kind to determine when $B_n(N) \not\equiv 0 \bmod p$.

Lemma 3.2. *For* $\nu_p\left(\left\lfloor \frac{N}{p}\right\rfloor!\right) = \nu_p\left(\left\lfloor \frac{n}{p}\right\rfloor!\right)$*, we have, mod* p*,*

$$B_n(N) \equiv u \cdot S(n-1, \lfloor N/p \rfloor),$$

for some $u \in \{1, 2, \cdots, p-1\}$*.*

The proof of this lemma uses methods due to Sun in [15]. We will also use the identity $i^k = \sum_l S(k,l) l! \binom{i}{l}$. Let $C_{px+r,l}$ denote $\sum_i (-1)^i \binom{px+r}{pi}\binom{i}{l}$.

Proof. Let $n = px + t$ and $N = p(x+h) + r$, with $0 \le t, r \le p-1$ and $0 \le h < p - \bar{x}$, where $\bar{x}$ denotes the residue of x mod p. We will prove that, mod p,

$$B_{px+t}(p(x+h)+r) \equiv u \cdot S(px+t-1, x+h), \tag{3.1}$$

for some $u \in \{1, 2, \cdots, p-1\}$.

We will use the inequality proved by Sun and Davis in [16, Theorem 1.1], which says that $\nu_p(l! C_{p(x+h)+r,l}) \ge \nu_p((x+h)!) + x + h - l$. This implies that $\nu_p\left(\frac{l! C_{p(x+h)+r,l}}{x!}\right) > 0$ for $l < x + h$.

We have, mod p,

$$\begin{aligned}
B_n(N) &= B_{px+t}(p(x+h)+r) \\
&= \frac{1}{x!}\sum_i (-1)^i \binom{p(x+h)+r}{pi} i^{px+t-1} \\
&= \frac{1}{x!}\sum_i (-1)^i \binom{p(x+h)+r}{pi} \sum_l S(px+t-1,l)l!\binom{i}{l} \\
&= \frac{1}{x!}\sum_l \left(\sum_i (-1)^i \binom{p(x+h)+r}{pi}\binom{i}{l}\right) l!S(px+t-1,l) \\
&= \frac{1}{x!}\sum_{l\leq x+h} C_{p(x+h)+r,l}\, l!S(px+t-1,l) \\
&= \frac{(x+h)!}{x!} C_{p(x+h)+r,x+h} S(px+t-1,x+h) \\
&\quad +\frac{1}{x!}\sum_{l<x+h} C_{p(x+h)+r,l}\, l!S(px+t-1,l) \\
&\equiv (x+h)(x+h-1)\cdots(x+1)C_{p(x+h)+r,x+h}S(px+t-1,x+h).
\end{aligned}$$

Now since $C_{p(x+h)+r,x+h} = (-1)^{x+h}\binom{p(x+h)+r}{p(x+h)} \equiv (-1)^{x+h} \bmod p$, we see that

$$C_{p(x+h)+r,x+h}S(px+t-1,x+h) \equiv (-1)^{x+h}S(px+t-1,x+h)$$

mod p. Also, for $1 \leq h < p - \bar{x}$, $(x+h)(x+h-1)\cdots(x+1) \not\equiv 0 \bmod p$, and so

$$B_{px+t}(p(x+h)+r) \equiv u \cdot S(px+t-1,x+h)$$

mod p, for some $u \in \{1, 2, \cdots, p-1\}$, as desired. $\square$

Therefore, we would like to know the mod p values of $S(px+t-1,x+h)$. Let p be prime and $1 \leq r \leq p$. Let $q_r(x) = \displaystyle\prod_{j=r+1}^{p}(1-jx) = \sum_{k=0}^{p-2} b_{k,r}x^k \in \mathbb{Z}/p[X]$. Note that

$b_{0,r} = 1$ and

$$b_{k,r} = 0 \quad \text{if } k > p - 1 - r. \tag{3.2}$$

Proposition 3.3. *Let p be prime and $x = pa + \Delta$, where $1 \leq \Delta \leq p$. Let $0 \leq h \leq p - 1 - \bar{\Delta}$. If $h \leq t - 1$, then, mod p,*

$$S(px + t - 1, x + h) \equiv \begin{cases} b_{t-1-h,\Delta+h}\binom{(p+1)x}{x} & \text{if } \Delta < p \text{ or } h = 0, \\ b_{t-1-h,h}\binom{(p+1)x}{x} & \text{if } \Delta = p \text{ and } h > 0. \end{cases}$$

If $h \geq t$, then, mod p,

$$S(px + t - 1, x + h) \equiv \begin{cases} b_{p-2+t-h,\Delta+h}\binom{(p+1)x-p}{x} & \text{if } \Delta < p \text{ or } h = 0, \\ b_{p-2+t-h,h}\binom{(p+1)x-p}{x} & \text{if } \Delta = p \text{ and } h > 0, \end{cases}$$

where $\bar{\Delta}$ denotes the residue of Δ mod p.

In the special case where $t = 1$ and $h = 0$, this gives a nice new identity for the Stirling numbers of the second kind.

Corollary 3.4. *For p prime and any nonnegative integer n, mod p, $S(pn, n) \equiv \binom{(p+1)n}{n}$.*

In order to prove Proposition 3.3, we will first prove the following lemma, which provides formulas for $S(pa+r+(p-1)i+k, pa+r)$, where $1 \leq r \leq p$ and $0 \leq k \leq p-2$. The methods used to prove the lemma are similar to those used in [5]. We will also use the following identity, obtained from [5]. For a fixed $k \geq 0$, we have

$$\sum_{n \geq 0} S(n, k)x^n = \prod_{i=1}^{k} \frac{x}{1 - ix}.$$

Lemma 3.5. *Let p be prime, a and i nonnegative integers, and $1 \le r \le p$. Then $S(pa+r+(p-1)i+k, pa+r) \equiv b_{k,r}\binom{a+i}{a}$ mod p, for $0 \le k \le p-2$.*

Proof. We have, mod p,

$$\begin{aligned}
\sum_{n\ge 0} S(n, pa+r)x^n &\equiv x^{pa+r}\cdot\frac{q_r(x)}{(1-x^{p-1})^{a+1}} \\
&= x^{pa+r}\left(\sum_{k=0}^{p-2} b_{k,r}x^k\right)\left(\sum_{i\ge 0}(-1)^i x^{(p-1)i}\binom{-a-1}{i}\right) \\
&= x^{pa+r}\left(\sum_{k=0}^{p-2} b_{k,r}x^k\right)\left(\sum_{i\ge 0} x^{(p-1)i}\binom{a+i}{a}\right).
\end{aligned}$$

Matching coefficients of powers of x on either side of the equation finishes the proof of the lemma. □

Proof of Proposition 3.3. Let $0 \le h \le p-1-\bar{\Delta}$. First suppose that $h \le t-1$. Then, mod p,

$$\begin{aligned}
& S(px+t-1, x+h) \\
= \; & S(p(pa+\Delta)+t-1, pa+\Delta+h) \\
= \; & S(pa+\Delta+h+(p-1)(pa+\Delta)+t-1-h, pa+\Delta+h) \\
\equiv \; & \begin{cases} b_{t-1-h,\Delta+h}\binom{(p+1)a+\Delta}{a} & \text{if } \Delta < p \text{ or } h = 0, \\ b_{t-1-h,h}\binom{(p+1)(a+1)}{a+1} & \text{if } \Delta = p \text{ and } h > 0, \end{cases} \\
\equiv \; & \begin{cases} b_{t-1-h,\Delta+h}\binom{(p+1)pa+p\Delta+\Delta}{pa+\Delta} & \text{if } \Delta < p \text{ or } h = 0, \\ b_{t-1-h,h}\binom{(p+1)p(a+1)}{p(a+1)} & \text{if } \Delta = p \text{ and } h > 0, \end{cases} \\
\equiv \; & \begin{cases} b_{t-1-h,\Delta+h}\binom{(p+1)x}{x} & \text{if } \Delta < p \text{ or } h = 0, \\ b_{t-1-h,h}\binom{(p+1)x}{x} & \text{if } \Delta = p \text{ and } h > 0. \end{cases}
\end{aligned}$$

Similarly we have, when $h \geq t$, mod p,

$$
\begin{aligned}
& S(px+t-1, x+h) \\
= \;& S(p(pa+\Delta)+t-1, pa+\Delta+h) \\
= \;& S(pa+\Delta+h+(p-1)(pa+\Delta-1)+p-2+t-h, pa+\Delta+h) \\
\equiv \;& \begin{cases} b_{p-2+t-h,\Delta+h}\binom{(p+1)a+\Delta-1}{a} & \text{if } \Delta < p \text{ or } h=0, \\ b_{p-2+t-h,h}\binom{(p+1)(a+1)-1}{a+1} & \text{if } \Delta = p \text{ and } h>0, \end{cases} \\
\equiv \;& \begin{cases} b_{p-2+t-h,\Delta+h}\binom{(p+1)pa+p\Delta-p+\Delta}{pa+\Delta} & \text{if } \Delta < p \text{ or } h=0, \\ b_{p-2+t-h,h}\binom{(p+1)p(a+1)-p}{p(a+1)} & \text{if } \Delta = p \text{ and } h>0, \end{cases} \\
\equiv \;& \begin{cases} b_{p-2+t-h,\Delta+h}\binom{(p+1)x-p}{x} & \text{if } \Delta < p \text{ or } h=0, \\ b_{p-2+t-h,h}\binom{(p+1)x-p}{x} & \text{if } \Delta = p \text{ and } h>0. \end{cases}
\end{aligned}
$$

$\square$

The following lemma will also be used in the proof of Theorem 1.4.

Lemma 3.6. *Let a be a natural number. Then $\binom{(p+1)a-1}{a} \equiv 0$ mod p.*

Proof. Let $a = pq+t$, $1 \leq t \leq p-1$. We have, mod p,

$$\binom{(p+1)(pq+t)-1}{pq+t} = \binom{p(p+1)q+pt+(t-1)}{pq+t} \equiv \binom{(p+1)q+t}{q}\binom{t-1}{t} \equiv 0.$$

So $\binom{(p+1)a-1}{a} \equiv 0$ mod p for $a \not\equiv 0$ mod p.

Let r be a natural number such that $r \not\equiv 0$ mod p and let $a = p^l r$, $l \geq 1$. We will proceed by induction on l to show that $\binom{(p+1)a-1}{a} \equiv 0$ mod p.

Base case: $l = 1$. So $a = pr$. We have, mod p,

$$\begin{aligned}\binom{(p+1)a-1}{a} &= \binom{p(pr)+p(r-1)+p-1}{pr} \\ &\equiv \binom{pr+r-1}{r} = \binom{(p+1)r-1}{r} \equiv 0.\end{aligned}$$

Now assume that $\binom{(p+1)p^l r-1}{p^l r} \equiv 0 \bmod p$ for $l > 1$. Then, mod p,

$$\begin{aligned}\binom{(p+1)p^{l+1}r-1}{p^{l+1}r} &= \binom{p(p^{l+1}r)+p^{l+1}r-1}{p^{l+1}r} \\ &= \binom{p(p^{l+1}r)+p\cdot p^l(r-1)+p(p^l-1)+p-1}{p(p^l r)} \\ &\equiv \binom{p^{l+1}r+p^l(r-1)+p^l-1}{p^l r} \\ &= \binom{p\cdot p^l r+p^l r-1}{p^l r} = \binom{(p+1)p^l r-1}{p^l r} \equiv 0,\end{aligned}$$

by the induction hypothesis. □

From Proposition 3.1, (3.1), and Proposition 3.3, we see that for $n = px + t$ and $N = p(x+h) + r$, with $0 \le t, r \le p-1$, $0 \le h \le p-1-\bar{x}$ and Δ defined as in Proposition 3.3, we have $\nu_p(a_p(n-1, N)) = s_p(n)$ if and only if the following condition holds: for $h \le t-1$, $b_{t-1-h,\Delta+h}\binom{(p+1)x}{x} \not\equiv 0 \bmod p$ or $b_{t-1-h,h}\binom{(p+1)x}{x} \not\equiv 0 \bmod p$, and, for $h \ge t$, $b_{p-2+t-h,\Delta+h}\binom{(p+1)x-p}{x} \not\equiv 0 \bmod p$ or $b_{p-2+t-h,h}\binom{(p+1)x-p}{x} \not\equiv 0 \bmod p$. We will use this to find the conditions on n for which $e_p(n-1, n) = s_p(n)$.

Proof of Theorem 1.4. Let $n = px + t$ and $N = p(x+h) + r$, as above. Let $t = 0$. Then $h \ge t$, and $\nu_p(a_p(n-1, N)) = s_p(n)$ if and only if $\binom{(p+1)x-p}{x} \not\equiv 0 \bmod p$ and, for $\Delta < p$ or $h = 0$, $b_{p-2-h,\Delta+h} \ne 0$, while for $\Delta = p$ and $h > 0$, $b_{p-2-h,h} \ne 0$. If $\Delta = p$

and $h > 0$, then, mod p,

$$\binom{(p+1)x-p}{x} \equiv \binom{(p+1)(a+1)-1}{a+1} \equiv 0,$$

by Lemma 3.6. If $1 \le \Delta \le p-1$ or $h = 0$, by (3.2), the coefficient $b_{p-2-h,\Delta+h} = 0$ if $\Delta \ge 2$. So $b_{p-2-h,\Delta+h} = 0$ unless $\Delta = 1$. Let $N = n$. Then $h = 0$, and we have $b_{p-2-h,1} = b_{p-2,1} = 1$, by Wilson's Theorem. Thus, $\nu_p(a_p(n-1,n)) = s_p(n)$ if and only if $\binom{(p+1)x-p}{x} \not\equiv 0 \bmod p$ and $\Delta = 1$, where $\Delta = d_1$. Additionally, since $b_{p-2,1} \neq 0$, if $\nu_p(a_p(n-1,n)) > s_p(n)$, then $\nu_p(a_p(n-1,N)) > s_p(n)$ for all $N \ge n$. Thus, this requirement also gives the condition on n for which $e_p(n-1,n) = s_p(n)$. Note that $\binom{(p+1)x-p}{x} \not\equiv 0 \bmod p$ if and only if, when n is written in base-p expansion, the sum of any two consecutive digits, except perhaps the sums involving the last three digits, is less than p. For if $n = \ldots d_2 d_1 d_0$ in base-p expansion, then, mod p,

$$\binom{(p+1)x-p}{x} = \binom{\cdots + (d_3+d_2)p^2 + (d_2+d_1-1)p + d_1}{\cdots + d_3p^2 + d_2p + d_1} \not\equiv 0$$

if and only if $d_{i+1} + d_i < p$ for $i \ge 2$, $d_2 + d_1 \le p$ and $d_1 \neq 0$. So if $n = \ldots d_2 d_1 0$ in base-p expansion, then $e_p(n-1,n) = s_p(n)$ if and only if $d_1 = 1$ and $d_{i+1} + d_i < p$ for $i \ge 2$.

Now let $t = 1$. If $h = 0$, then $b_{t-1-h,\Delta+h} = b_{0,\Delta} = 1$. So $\nu_p(a_p(n-1,n)) = s_p(n)$ if and only if $\binom{(p+1)x}{x} \not\equiv 0 \bmod p$. For $h > 0$ and $\Delta = p$, we have $h \ge t$, and $\binom{(p+1)x-p}{x} \equiv 0 \bmod p$, by Lemma 3.6, as above. For $h > 0$ and $\Delta < p$, we have $b_{p-2+t-h,\Delta+h} = b_{p-1-h,\Delta+h}$, which is 0, by (3.2), since $\Delta \ge 1$. Thus, if $\nu_p(a_p(n-1,n)) > s_p(n)$, then $\nu_p(a_p(n-1,N)) > s_p(n)$ for $N \ge n$. Hence, $e_p(n-1,n) = s_p(n)$ if and only if $\binom{(p+1)x}{x} \not\equiv 0 \bmod p$. As noted by Davis in [7], for $n = \ldots d_2 d_1 d_0$ in base-p expansion, we have, mod p,

$$\binom{(p+1)x}{x} = \binom{\cdots + (d_3+d_2)p^2 + (d_2+d_1)p + d_1}{\cdots + d_3p^2 + d_2p + d_1} \not\equiv 0$$

if and only if $d_{i+1} + d_i < p$ for $i \geq 1$. Therefore, if $n = \dots d_2 d_1 1$ in base-p expansion, then $e_p(n-1, n) = s_p(n)$ if and only if $d_{i+1} + d_i < p$ for $i \geq 1$.

Finally, let $2 \leq t \leq p-1$. If $h \leq t-1$, then $\nu_p(a_p(n-1, N)) = s_p(n)$ if and only if $b_{t-1-h,\Delta+h} \neq 0$ or $b_{t-1-h,h} \neq 0$ and $\binom{(p+1)x}{x} \not\equiv 0 \bmod p$. Note that, for $\Delta < p$ or $h = 0$, we have $b_{t-1-h,\Delta+h} = 0$ if $t + \Delta \geq p+1$, by (3.2). For $t + \Delta \leq p$, or for $\Delta = p$ and $h > 0$, consider $h = t-1$. This gives $b_{t-1-h,\Delta+h} = b_{0,\Delta+h} = 1$ and $b_{t-1-h,h} = b_{0,h} = 1$. So for $N = p(x+t-1)$, we have $\nu_p(a_p(n-1, N)) = s_p(n)$ if and only if $\binom{(p+1)x}{x} \not\equiv 0 \bmod p$ and $t + \bar{\Delta} \leq p$, where $t = d_0$ and $\bar{\Delta} = d_1$.

For $h \geq t$, when $\Delta = p$ and $h > 0$, the binomial coefficient $\binom{(p+1)x-p}{x} \equiv 0 \bmod p$ by Lemma 3.6, as seen earlier. For $\Delta < p$ or $h = 0$, the coefficient $b_{p-2+t-h,\Delta+h} = 0$ if $2 - t \leq \Delta$, by (3.2), which is true since $\Delta \geq 1$ and $t \geq 2$. Thus, $\nu_p(a_p(n-1, N)) > s_p(n)$ when $h \geq t$. So if $\nu_p(a_p(n-1, N)) > s_p(n)$ for $N = p(x+t-1)$, then $\nu_p(a_p(n-1, N)) > s_p(n)$ for all $N \geq n$. Hence, $e_p(n-1, n) = s_p(n)$ if and only if $\binom{(p+1)x}{x} \not\equiv 0 \bmod p$ and $t + \bar{\Delta} \leq p$. As we saw before, this condition is equivalent to requiring that if $n = \dots d_2 d_1 d_0$ in base-p expansion, then $d_{i+1} + d_i < p$ for $i \geq 1$ and $d_1 + d_0 \leq p$. $\square$

Chapter 4

Sharpness of Inequalities

In this section we compare the values of $s_p(n)$, $e_p(n-1,n)$ and $\bar{e}_p(n)$ for the prime 5. The term $\bar{e}_5(n)$ is the maximum value of $e_5(k,n)$ over all k, and we use $k_{\max}$ to denote the least positive k for which $e_5(k,n) = \bar{e}_5(n)$. The chapter ends with a table that lists each of these values for $2 \le n \le 128$. These were computed using Maple. Similar tables by Davis can be found in [7] for the prime 2 and in [12] for the prime 3. The purpose of the table is to study how close each of the following inequalities,

$$s_5(n) \le e_5(n-1,n) \le \bar{e}_5(n),$$

is to being sharp for various n.

The value of $s_5(n)$, for each n, was computed using the formula $s_p(n) = n - 1 + \nu_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)$. The value of $e_5(n-1,n)$ was determined using the definition, $e_5(k,n) = \min(\nu_5(\sum_{i \not\equiv 0(5)} (-1)^i \binom{N}{i} i^k) : N \ge n)$. Letting $N = n - 1 + j$, Maple was used to compute the value of

$$\nu_5(\sum_{i \not\equiv 0(5)} (-1)^i \binom{n-1+j}{i} i^k), \tag{4.1}$$

where $k = n - 1$ and $1 \le j \le 25$, since we only need to check $N \ge n$ such that $\left\lfloor \frac{N}{p^2} \right\rfloor = \left\lfloor \frac{n}{p^2} \right\rfloor$. For each n, the smallest value in this array gives $e_5(n-1,n)$.

Table 4.1: Values of $\nu_5(a_5(k,N))$ relevant to $e_5(k,25)$

$\nu_5(k-24) \setminus N$	25	26	27	28	29	30
17	26	22	**22**	22	24	24
18	27	23	**23**	23	25	25
19	≥ 28	24	**24**	24	26	≥ 26
20	28	25	**25**	25	27	26
21	28	≥ 26	≥ 26	≥ 26	28	**26**
≥ 22	28	26	26	26	≥ 29	**26**

Table 4.2: Values of $\nu_5(a_5(k,N))$ relevant to $e_5(k,26)$

$\nu_5(k-25) \setminus N$	26	27	28	29	30	31
17	22	22	**22**	22	27	23
18	23	23	**23**	23	27	24
19	24	24	**24**	24	27	25
20	25	25	**25**	25	27	26
21	≥ 26	≥ 26	≥ 26	≥ 26	27	≥ 27
≥ 22	26	26	**26**	26	27	27

The three tables on this page provide the relevant values of $e_5(k,n)$ for $n = 25$ and $n = 26$, to illustrate how the value of $\bar{e}_5(n)$ was determined when $k_{\max} = n-1$ and $k_{\max} > n-1$. The value $\bar{e}_5(n)$ was determined, using Maple, by first computing the values of (4.1), with $n \leq N \leq n+24$, for increasing values of $\nu_5(k-(n-1))$. The smallest value in each row is $e_5(k,n)$, which is shown in boldface. Values of N larger than those shown yield larger values of $\nu_5(a_5(k,N))$, and so they do not determine the value of $e_5(k,n)$. The charts display the largest values of $e_5(k,n)$, and show when these values stabilize. The largest value of $e_5(k,n)$ is $\bar{e}_5(n)$. In the case $n = 26$ and $\nu_5(k-25) = 21$, the value of $e_5(k,n)$ is not clear initally. The third table gives more information, providing the values of $e_5(k,n)$ for this case.

Table 4.3: Values of $\nu_5(a_5(k,N))$ relevant to $e_5(k,26)$ when $\nu_5(k-25) = 21$

$\nu_5(\frac{k-25}{4} - 5^{21}) \setminus N$	26	27	28	29	30	31
21	26	26	**26**	26	27	≥ 27
≥ 22	≥ 27	≥ 27	≥ 27	≥ 27	**27**	27

From Table 4.4 we see that when $n \equiv 1$ or 2 mod 25, and perhaps 3 mod 5^3, the inequality $e_5(n-1,n) \leq \bar{e}_5(n)$ fails by 1 to be sharp. Here the maximum value of $e_5(k,n)$ first occurs at a value of k equal to $n-1$ plus a multiple of a power of 5. Determining whether this pattern continues requires further study. Additionally, the inequality $s_5(n) \leq e_5(n-1,n)$ appears to fail by one to be sharp when n is a multiple of 25, but in these cases the inequality $e_5(n-1,n) \leq \bar{e}_5(n)$ seems to be sharp. Since the above observations suggest agreement with the work of Davis for the primes 2 and 3, we have the following analogous conjecture for the prime 5.

Conjecture 4.2. *If $n = 5^t$, where $t \geq 2$, then $\bar{e}_5(n) = e_5(n-1,n) = s_5(n) + 1$. If $n = 5^t + 1$, with $t \geq 1$, then $\bar{e}_5(n) = e_5(n-1,n) + 1 = s_5(n) + 1$.*

This conjecture claims that the inequality $s_5(n) \leq e_5(n-1,n)$ fails to be sharp by 1 when $n = 5^t$, for $t > 1$. As we already know from Theorem 1.4, that inequality is sharp for $n = 5^t + 1$, but in this case $k_{\max} \neq n-1$. It appears that $k_{\max}$ occurs at $5^t + 4 \cdot 5^{5^{t-1}+t-1}$ in these cases. Davis has proven a result similar to Conjecture 4.2 for the prime 2 in [10] and has conjectured the analogous statement to hold for the prime 3.

A goal for future work would be to prove Conjecture 4.2 and then to prove an analogous statement for all odd primes. Eventually we would like to have a sharp lower bound for $\exp(SU(n))$ for each n. One aspect of accomplishing this is to determine exactly when the inequality $e_p(n-1,n) \leq \bar{e}_p(n)$ is not sharp, and then to find $\bar{e}_p(n)$ for those n. Then we would still need to find $\bar{e}_p(n)$ for those n such that $k_{\max} = n-1$ where the inequality $s_p(n) \leq e_p(n-1,n)$ is not sharp.

Table 4.4: Comparison of values

n	$s_5(n)$	$e_5(n-1,n)$	$\bar{e}_5(n)$	$k_{\max}$
2	1	1	1	1
3	2	2	2	2
4	3	3	3	3
5	4	4	4	4
6	5	5	6	$5+4\cdot 5^4$
7	6	6	6	6
8	7	7	7	7
9	8	8	8	8
10	9	10	10	9
11	10	10	11	$10+4\cdot 3\cdot 5^8$
12	11	11	11	11
13	12	12	12	12
14	13	14	14	13
15	14	15	15	14
16	15	15	16	$15+4\cdot 3\cdot 5^{12}$
17	16	16	16	16
18	17	18	18	17
19	18	19	19	18
20	19	20	20	19
21	20	20	21	$20+4\cdot 5^{16}$
22	21	22	22	21
23	22	24	24	22
24	23	25	25	23
25	25	26	26	24

n	$s_5(n)$	$e_5(n-1,n)$	$\bar{e}_5(n)$	$k_{\max}$
26	26	26	27	$25+4\cdot 5^{21}$
27	27	27	28	$26+4\cdot 5^{21}$
28	28	28	28	27
29	29	29	29	28
30	30	30	30	29
31	31	31	32	$30+4\cdot 3\cdot 5^{24}$
32	32	32	32	31
33	33	33	33	32
34	34	34	34	33
35	35	36	36	34
36	36	36	37	$35+4\cdot 5^{28}$
37	37	37	37	36
38	38	38	38	37
39	39	40	40	38
40	40	41	41	39
41	41	41	42	$40+4\cdot 3\cdot 5^{32}$
42	42	42	42	41
43	43	44	44	42
44	44	46	46	43
45	45	47	47	44
46	46	47	48	$45+4\cdot 3\cdot 5^{37}$
47	47	49	49	46
48	48	50	50	47
49	49	51	51	48
50	51	52	52	49

n	$s_5(n)$	$e_5(n-1,n)$	$\bar{e}_5(n)$	$k_{\max}$
51	52	52	53	$50+4\cdot 3\cdot 5^{41}$
52	53	53	54	$51+4\cdot 3\cdot 5^{41}$
53	54	54	54	52
54	55	55	55	53
55	56	56	56	54
56	57	57	58	$55+4\cdot 3\cdot 5^{44}$
57	58	58	58	56
58	59	59	59	57
59	60	60	60	58
60	61	62	62	59
61	62	62	63	$60+4\cdot 3\cdot 5^{48}$
62	63	63	63	61
63	64	64	64	62
64	65	66	66	63
65	66	68	68	64
66	68	68	69	$65+4\cdot 3\cdot 5^{53}$
67	68	69	69	66
68	69	71	71	67
69	70	72	72	68
70	71	73	73	69
71	72	73	74	$70+4\cdot 4\cdot 5^{57}$
72	73	75	75	71
73	74	76	76	72
74	75	77	77	73
75	77	78	78	74

n	$s_5(n)$	$e_5(n-1,n)$	$\bar{e}_5(n)$	$k_{\max}$
76	78	78	79	$75+4\cdot 3\cdot 5^{61}$
77	79	79	80	$76+4\cdot 3\cdot 5^{61}$
78	80	80	80	77
79	81	81	81	78
80	82	82	82	79
81	83	83	84	$80+4\cdot 5^{64}$
82	84	84	84	81
83	85	85	85	82
84	86	86	86	83
85	87	88	88	84
86	88	89	90	$85+4\cdot 5^{69}$
87	89	90	90	86
88	90	91	91	87
89	91	93	93	88
90	92	94	94	89
91	93	94	95	$90+4\cdot 2\cdot 5^{73}$
92	94	95	95	91
93	95	97	97	92
94	96	98	98	93
95	97	99	99	94
96	98	99	100	$95+4\cdot 2\cdot 5^{77}$
97	99	101	101	96
98	100	102	102	97
99	101	103	103	98
100	103	104	104	99

n	$s_5(n)$	$e_5(n-1,n)$	$\bar{e}_5(n)$	$k_{\max}$
101	104	104	105	$100 + 4 \cdot 5^{81}$
102	105	105	106	$101 + 4 \cdot 5^{81}$
103	106	106	106	102
104	107	107	107	103
105	108	108	108	104
106	109	111	111	105
107	110	112	112	106
108	111	113	113	107
109	112	114	114	108
110	113	116	116	109
111	114	116	117	$110 + 4 \cdot 4 \cdot 5^{90}$
112	115	117	117	111
113	116	118	118	112
114	117	120	120	113
115	118	121	121	114
116	119	121	122	$115 + 4 \cdot 5^{94}$
117	120	122	122	116
118	121	124	124	117
119	122	125	125	118
120	123	126	126	119
121	124	126	127	$120 + 4 \cdot 4 \cdot 5^{98}$
122	125	128	128	121
123	126	129	129	122
124	127	130	130	123
125	130	131	131	124
126	131	131	132	$125 + 4 \cdot 5^{102}$
127	132	132	134	$126 + 4 \cdot (5^{103} + 5^{102})$
128	133	133	134	$127 + 4 \cdot 5^{102}$

Bibliography

[1] J. F. Adams, On the groups J(X), IV, *Topology*, **5**, (1966) 21-71.

[2] M. Bendersky, Some calculations in the unstable Adams-Novikov spectral sequence, *RIMS*, **16** (1980) 529-542.

[3] M. Bendersky, and E. B. Curtis, and H. R. Miller, The unstable Adams spectral sequence for generalized homology, *Topology*, **17** (1978) 229-248.

[4] M. Bendersky, and D. M. Davis, 2-primary v_1-periodic homotopy groups of $SU(n)$, *Amer Jour Math*, **114** (1991) 465-494.

[5] O-Yeat Chan, and Dante Manna, Congruences for Stirling numbers of the second kind, in *Gems in Experimental Mathematics*, Contemporary Mathematics **517**, Amer. Math. Soc., (2010) 97-111.

[6] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, The double suspension and exponents of the homotopy groups of spheres, *Annals of Math.*, **110** (1979) 549-565.

[7] D. M. Davis, Divisibility by 2 and 3 of certain Stirling numbers, *Integers*, **8** (2008) A56.

[8] D. M. Davis, Elements of large order in $\pi_*(SU(n))$, *Topology*, **37** (1998) 293-327.

[9] D. M. Davis, v_1-periodic homotopy groups of $SU(n)$ at odd primes, *Proc. London Math. Soc.*, **43** (1991) 529-544.

[10] D. M. Davis, v_1-periodic 2-exponents of $SU(2^e)$ and $SU(2^e+1)$, *Jour. Pure Appl. Alg.*, **216** (2012) 1268-1272.

[11] D. M. Davis, and M. Mahowald, Some remarks on v_1-periodic homotopy groups, *London Math. Society Lecture Notes*, **176** (1992) 55-72.

[12] D. M. Davis, and Z. W. Sun, A number-theoretic approach to homotopy exponents of $SU(n)$, *Jour. Pure Appl. Alg.*, **209** (2007) 57-69.

[13] M. Mahowald, The image of J in the EHP sequence, *Annals of Math*, **116** (1982) 65-112.

[14] P. Selick, Mod 2 exponents for the homotopy groups of spheres, *Topology*, **23** (1984) 97-99.

[15] Z. W. Sun, Combinatorial congruences and Stirling numbers, *Acta Arith.*, **126** (2007) 387-398.

[16] Z. W. Sun, and D. M. Davis, Combinatorial congruences modulo prime powers, *Trans. Amer. Math. Soc.*, **359** (2007) 5525-5553.

[17] R. Thompson, The v_1-periodic homotopy groups of an unstable sphere at odd primes, *Trans. Amer. Math. Soc.*, **319** (1990) 535-559.

Vita

Karen McCready was born in Abington, Pennsylvania to Douglas and Dianne McCready. She attended The College of New Jersey from 2002 to 2006, receiving a Bachelor of Arts in Mathematics and graduating Summa Cum Laude in May of 2006. She has attended Lehigh University from 2006 to 2012. She received a Master of Science in Mathematics in 2008 and anticipates receiving a Doctor of Philosophy in Mathematics in May 2012. From 2008 to 2012 she has been a PhD student of Donald M. Davis in mathematics, in the area of algebraic topology. Her professional experience includes being a teaching assistant and an instructor of undergraduate mathematics courses at Lehigh University and at a local college.