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Two-Player Graph Pebbling

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by

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Abstract

Given a graph G with pebbles on the vertices, we define a *pebbling move* as removing two pebbles from a vertex u, placing one pebble on a neighbor v, and discarding the other pebble, like a toll. The *pebbling number* $\pi(G)$ is the least number of pebbles needed so that every arrangement of $\pi(G)$ pebbles can place a pebble on any vertex through a sequence of pebbling moves. We introduce a new variation on graph pebbling called *two-player pebbling*. In this, players called the *mover* and the *defender* alternate moves, with the stipulation that the defender cannot reverse the previous move. The mover wins only if they can place a pebble on a specified vertex and the defender wins if the mover cannot. We define $\eta(G)$, analogously, as the minimum number of pebbles such that given every configuration of the $\eta(G)$ pebbles and every specified vertex r, the mover has a winning strategy. First, we will investigate upper bounds for $\eta(G)$ on various classes of graphs and find a certain structure for which the defender has a winning strategy, no matter how many pebbles are in a configuration. Then, we characterize winning configurations for both players on a special class of diameter 2 graphs. Finally, we show winning configurations for the mover on paths using a recursive argument.

Chapter 1

Introduction

Graph pebbling was developed by Lagarias and Saks in 1989 as a tool to solve a number theoretic conjecture posed by Erdös. Chung [2] proved the conjecture using graph pebbling. It was also proved independently by number-theoretic methods [10].

Theorem 1.0.1. *[2, 10] Given an integer d and integers $a_1, a_2, \ldots, a_d$, there exists a non-empty set $\mathcal{Q} \subset \{1, 2, \ldots, d\}$ such that $d \Big| \sum_{i \in \mathcal{Q}} a_i$ and $\sum_{i \in \mathcal{Q}} gcd(a_i, d) \leq d$.*

Graph pebbling can be thought of as a type of optimization problem where a utility such as gas, electricity, or computing power travels across a network. While traveling through the network, some amount of the utility may be lost. A natural question that arises is what is the minimum amount of the utility that is needed to travel the network and arrive at a destination.

1.1 Definitions

From this point, all graphs G will be finite and simple (no loops or multiedges). We let $V(G)$ be the set of vertices of G and $|V(G)|$ be the number of vertices in G, otherwise known as the *order* of G. Similarly, we say $E(G)$ is the set of edges of G and $|E(G)|$ is the number of edges in G, known as the *size* of G. In a connected graph, a *path* from u to v is a sequence of distinct edges which connects u to v. For a graph G and two vertices u and v in $V(G)$, the distance between u and v, denoted $dist(u, v)$, is the length of a shortest u, v-path. The *diameter* of a graph G, $diam(G) = \max_{u,v} dist(u, v)$, is the maximum distance over every pair of vertices in G. Label paths of n vertices as $P_n = v_1 v_2 \ldots v_n$. The *open neighborhood* of v, $N(v)$, is the set of vertices adjacent to but not including v. The *closed neighborhood* of v, $N[v] = N(v) \cup \{v\}$, is the set of vertices adjacent to and including v.

We have a definition for neighborhoods that will be useful.

Definition 1.1.1. Let $H \subseteq G$ and $v \in G$. We say the *H-restricted neighborhood of* v, $N_H(v)$, is the set of neighbors of v in H, i.e. $N_H(v) = N(v) - V(G - H)$.

We continue by introducing terms relevant to graph pebbling. Intuitively, we can think of a *pebble* as an indistinguishable discrete object placed on the vertices of a graph G. If a vertex u has a pebble or pebbles in it, then we say u is *pebbled*. If a vertex v has no pebbles on it, then v is *unpebbled* or *pebble-free*.

Definition 1.1.2. Given a graph G, let a *configuration* $C : V(G) \to \mathbb{N}$ be a distribution of pebbles on the vertices of G with $C(v)$ pebbles at vertex v. The *size* of C, $|C| = \sum_{v \in V} C(v)$, is the sum of all $C(v)$'s. We say a vertex is *even* if there is an even number of pebbles distributed on it and a vertex *odd* if there is an odd number of pebbles distributed on it.

It is technically correct to say that a vertex has pebbles distributed on it or there is a configuration on the vertices of G. However, for ease, we will say that a vertex has pebbles on it or that there is a configuration on a graph G.

We need a way to move the pebbles from vertex to vertex.

Definition 1.1.3. A *pebbling move* is a relation $p : \mathcal{C} \to \mathcal{C}$ between the set of all possible configurations $\mathcal{C}$ and itself such that $p(C) = C'$ by the following:

- $|C'| = |C| - 1$
- $\exists$ an edge, uv, where $C'(u) = C(u) - 2$ and $C'(v) = C(v) + 1$

- $C'(x) = C(x), \forall\, x \neq u, v$

i.e., a pebbling move removes two pebbles from a vertex u and adds one pebble to an adjacent vertex v. We look at Figure 1.3 as an example of a configuration and Figure 1.2 as an example of a pebbling move.

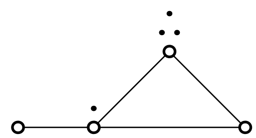

Figure 1.1: A Configuration of Pebbles on G

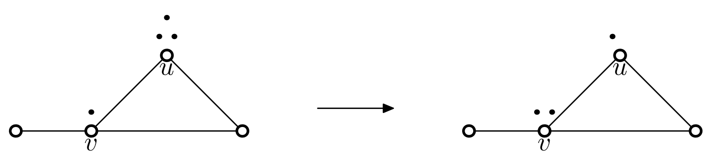

Figure 1.2: An Example of a Pebbling Move on G From u to v

We have the following definition.

Definition 1.1.4. Let C and C' be configurations on G. We say C *contains* C' provided $C(v) \geq C'(v)$ for all $v \in G$. We denote this by $C' \subseteq C$.

It is useful to talk about one configuration being *reachable* from another configuration.

Definition 1.1.5. Let C and C' be configurations on G. We say C' is *reachable* from C provided there is some sequence of pebbling moves on C that results in C'.

The goal of graph pebbling is to use pebbling moves to place at least one pebble on a specified vertex r called the *root*.

Definition 1.1.6. We say a pebbling move from u to v is *greedy* provided $dist(v, r) < dist(u, r)$, and *semi-greedy* provided $dist(v, r) \leq dist(u, r)$.

Definition 1.1.7. Given a configuration C on a graph G and a root $r \in V(G)$, we say C is *r-solvable* provided there exists a reachable configuration C' such that $C'(r) = 1$. If every reachable configuration from C yields $C'(r) = 0$, then we say C is *r-unsolvable*. Given a configuration C on a graph G, we say C is *solvable* provided C is r-solvable for every choice of r. If there exists a choice of r such that C is r-unsolvable, then we say C is *unsolvable*.

Thus, we can think of graph pebbling as a sequence of configurations C, C_1, $\ldots$, C_m, where $C_{i+1} = p(C_i)$, all configurations reachable from C, and C_m has either at least one pebble on the root or no pebbles on the root and no pebbling moves remaining. We can see in Figure 1.3 that C is solvable.

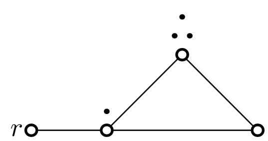

Figure 1.3: A Configuration C is r-solvable

Definition 1.1.8. Given a graph G with root r, the *rooted-pebbling number* $\pi(G, r)$ is the minimum number m such that every configuration of m pebbles is r-solvable.

From this we get the following definition.

Definition 1.1.9. The *pebbling number* $\pi(G)$ is the minimum number m such that every configuration of size m is solvable.

We can see a simple relationship.

Fact 1.1.10. *For any graph G, we have $\pi(G) = \max_{r \in G} \pi(G, r)$.*

We can also characterize $\pi(G)$ in terms of unsolvable configurations.

Fact 1.1.11. *For any graph G, we have $\pi(G) = |C| + 1$ where C is a maximum unsolvable configuration.*

Hence, finding $\pi(G)$ is equivalent to finding a maximum unsolvable configuration. The following fact is useful.

Fact 1.1.12. *Given a graph G with configurations C and C' such that $C' \subseteq C$, if C' is r-solvable, then C is r-solvable.*

Proof. For a graph G with configuration C', any sequence of pebbling moves made in C' can be made in C . $\square$

1.2 Classical Bounds

For graphs H and G, let $H \subseteq G$ denote that H is a subgraph of G. We get the following fact.

Fact 1.2.1. *If H and G are connected graphs with $H \subseteq G$ such that $V(H) = V(G)$, then $\pi(H) \geq \pi(G)$.*

Proof. Any pebbling moves made in H can be made in G. $\square$

Now we move to finding upper and lower bounds for the pebbling number of graphs. The first lower bound is in terms of the order of G.

Fact 1.2.2. *Let $|V(G)| = n$. Then $\pi(G) \geq n$.*

Proof. Let G be a graph and $r \in V(G)$. Consider the configuration C on G described by

$$C(v) = \begin{cases} 0 & \text{if } v = r \\ 1 & \text{if } v \neq r. \end{cases}$$

This has $n-1$ pebbles and no pebbling moves. Thus C is r-unsolvable. Since $|C| = n-1$, we have $\pi(G) \geq n$ by Fact 1.1.11.

$\square$

If equality holds, then we get the following definition.

Definition 1.2.3. A graph G is said to be a *Class 0* graph provided $\pi(G) = |V(G)|$.

There is a necessary condition for G to be a Class 0 graph. Let $\kappa(G)$ be the connectivity of the graph G, i.e. the minimum number of vertices one needs to remove to disconnect the graph.

Theorem 1.2.4. *[4] If $diam(G) = 2$ and $\kappa(G) \geq 3$, then G is of Class 0.*

The following fact shows that if G has a cut vertex, then G is not Class 0.

Fact 1.2.5. *If G has a cut vertex, then $\pi(G) > |V(G)|$.*

Proof. Let G by a graph with a cut vertex x. Let u be a neighbor of x in a different component of $G - x$ than r. Consider the configuration C on G described by

$$C(v) = \begin{cases} 0 & \text{if } v \in \{x, r\} \\ 3 & \text{if } v = u \\ 1 & \text{if } v \notin \{u, r, x\}. \end{cases}$$

The only pebbling move is to x. All vertices except r have one pebble on them. This configuration is r-unsolvable. Since $|C| = |V(G)|$, we have $\pi(G) > |V(G)|$ by Fact 1.1.11. □

If $\pi(G) = |V(G)| + 1$, then G is of Class 1. The next lower bound is in terms of the diameter.

Fact 1.2.6. *Let $diam(G) = d$. Then $\pi(G) \geq 2^d$.*

Proof. Let G be a graph and $r \in V(G)$. Let $u \in G$ be a vertex such that $dist(u, r) = d$. Consider the configuration C on G described by

$$C(v) = \begin{cases} 2^d - 1 & \text{if } v = u \\ 0 & \text{if } v \neq u. \end{cases}$$

It is easy to check that this configuration is r-unsolvable. Since $|C| = 2^d - 1$, we have $\pi(G) \geq 2^d$ by Fact 1.1.11. □

We can show that if the previous configuration has 2^d pebbles on u, then C would be r-solvable.

Fact 1.2.7. *Let $diam(G) = d$. If C is a configuration which has 2^d on $u \in G$, then C is r-solvable to any choice of r.*

Proof. Let G be a graph and $r \in V(G)$. Let $u \in G$. Consider the configuration C on G described by

$$C(v) = \begin{cases} 2^d & \text{if } v = u \\ 0 & \text{if } v \neq u. \end{cases}$$

For any choice of $r \in G$, we know there exists a path almost distance d from u to r, call it $uv_2v_3 \ldots v_kr$. If we use all pebbling from u to v_2, there will be 2^{d-1} pebbles on v_2. Likewise, pebbling from v_2 to v_3 will ensure 2^{d-2} pebbles on v_3. Since $d(u, r) \leq d$, pebbling in a like fashion will place at least $2^{d-d} = 1$ pebbles on r. $\square$

Thus far, we have lower bounds for $\pi(G)$. The next result uses the Pigeonhole Principle for an upper bound for $\pi(G)$.

Fact 1.2.8. *Let $|V(G)| = n$ and $diam(G) = d$. Then $\pi(G) \leq (n-1)(2^d - 1) + 1$.*

Proof. Let G be a graph and $r \in V(G)$. Let C be a configuration on G with $(n-1)(2^d - 1) + 1$ pebbles. If $C(v) \geq 1$ for every v, then C is r-solvable for any choice of r. If, on the other hand, some vertices are pebble-free, then there must be a vertex $x \in V(G)$ such that $C(x) \geq 2^d$. Thus, by Fact 1.2.7, every vertex in G is reachable from x. So C is r-solvable.

$\square$

1.3 Early Results

Let K_n be the complete graph on n vertices.

Fact 1.3.1. *For every positive integer n, we have $\pi(K_n) = n$.*

Proof. Let r be any vertex. Suppose we have a configuration C with $n-1$ pebbles. If every non-root vertex has 1 pebble, then there are no pebbling moves. Now suppose we have a configuration C' with n pebbles. If C' has 1 pebble on r, then we are done. If C' has no pebble on the root, then there must exist at least one vertex v with at least 2 pebbles on it. We can pebble from v to r. $\square$

Next, we have the pebbling number of a path on n vertices, P_n.

Fact 1.3.2. *For every positive integer n, we have $\pi(P_n) = 2^{n-1}$.*

Proof. By Fact 1.2.6, $\pi(P_n) \geq 2^{n-1}$. We now show $\pi(P_n) \leq 2^{n-1}$ by induction on n.

Base: Let $n = 1$. Having 1 pebble on 1 vertex is solvable.

Induction: Let $\pi(P_k) = 2^{k-1}$ for all $k < n$. Suppose we have a configuration C on P_n with 2^{n-1} pebbles. If $C(r) = 1$, we are done. So suppose $C(r) = 0$. First, suppose r is an endpoint and let u be the neighbor to r. By induction, we can place a pebble on u using at most 2^{n-2} pebbles. Since we have at least 2^{n-2} pebbles left, we can place another pebble on u. Since $C'(u) = 2$, we can pebble to r. Now, suppose r is a non-endpoint. Let $dist(v_1, r) = d_1$ and $dist(r, v_n) = d_2$ with $d_1 + d_2 = n - 1$. By the Pigeonhole Principle, either the subpath $v_1 \dots r$ has at least 2^{d_1} pebbles on it or $r \dots v_n$ has at least 2^{d_2} pebbles on it. In either case, we can pebble to r by induction. □

Later, we will have another proof of this result that relies upon a "potential" argument. This result helps find the pebbling number of trees. Let T be a tree and $r \in T$. We build a partition of T into paths as follows. Let P_1 be the longest path in T with r as an endpoint. Let P_2 be the longest path in T with an endpoint in P_1, but otherwise disjoint from P_1. We recursively continue this for every i with P_i being the longest path in T with an endpoint in P_{i-1} until we have an index m such that $T = P_1 \cup P_2 \cup \cdots \cup P_m$. Notice, $|P_i| \geq |P_{i+1}|$ for all $i = 1, 2, \dots, m-1$ and let $|P_i|$ be the length of P_i. We say $\mathcal{P} = (P_1, P_2, \dots, P_m)$ is an *r-maximum path partition* of T.

Theorem 1.3.3. *[2] If T is a tree and $\mathcal{P} = (P_1, P_2, \dots, P_m)$ is an r-maximum path partition of T, then $\pi(T) = \sum_{i=1}^{m} 2^{|P_i|} - m + 1$.*

From [12], we get the pebbling number of cycles.

Theorem 1.3.4. *[12] For every integer $k \geq 2$, we have $\pi(C_{2k}) = 2^k$ and for every integer $k \geq 1$, we have $\pi(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$.*

The following result from [2] gives the pebbling number of hypercubes.

Fact 1.3.5. *[2] If Q_k is the hypercube in dimension k, then $\pi(Q_k) = 2^k$.*

This result from [3] shows complete bipartite and complete multipartite graphs are Class 0 graphs.

Fact 1.3.6. *[3] If $K_{a_1,a_2,\dots,a_m}$ is a complete multipartite graph with $1 < a_1 \leq a_2 \leq \cdots \leq a_m < n$ and $\sum_{i=1}^{m} a_i = n$, then $\pi(K_{a_1,a_2,\dots,a_m}) = n$.*

A famous conjecture by Ronald Graham [2] poses a question about the cartesian product of graphs. First, we need a definition.

Definition 1.3.7. For any two graphs G and H, the *cartesian product*, $G\Box H$, is the graph whose vertex set is $\{(g,h) : g \in G, h \in H\}$ with edges between (g,h) and (g',h') if and only if ($g = g'$ and $\{hh'\} \in E(H)$) or ($h = h'$ and $\{gg'\} \in E(G)$).

Now, we can state the conjecture.

Conjecture 1.3.8 (Graham's Conjecture [2])**.** For any graphs G and H, $\pi(G\Box H) \leq \pi(G)\pi(H)$.

Graham's Conjecture has been verified for certain classes of graphs such as trees with trees [11] and cycles with cycles [6, 7, 12]. Notably, equality was shown for arbitrary products of paths [2].

Theorem 1.3.9. *[2] For positive integers $n_1, n_2, \dots, n_m$,*

$$\pi(P_{n_1+1}\Box P_{n_2+1}\Box \dots \Box P_{n_m+1}) = 2^{n_1+n_2+\cdots+n_m}$$

.

This proof was the foundation for Chung's verification of Theorem 1.0.1.

1.4 Optimal Pebbling

For $\pi(G)$, we are concerned with finding the smallest integer m such that every configuration of m pebbles on G can reach every vertex. In other words, we are looking for the largest configuration that is unsolvable for some choice of root.

Instead, we may want to find the smallest solvable configuration on G. This may be useful for a company trying to determine locations of fuel stations, warehouses, or generators. We have the definition for the *optimal pebbling number*.

Definition 1.4.1. Given a graph G, the *optimal pebbling number* of G, denoted $\pi^*(G)$, is the minimum number k of pebbles such that there exists a solvable configuration of size k.

The key difference between the optimal pebbling number and the pebbling number of G is that for the optimal pebbling number, we only want to find one configuration C of size k that is solvable. It may be true that there is another configuration C' of size k that is not solvable. This gives the first fact for optimal pebbling

Fact 1.4.2. *Given a graph G, we have $\pi^*(G) \leq \pi(G)$.*

We also have a nice upper bound.

Fact 1.4.3. *Given a graph G, let diam$(G) = d$ and $|V(G)| = n$. Then $\pi^*(G) \leq \min\{2^d, n\}$.*

Proof. Any configuration C which places 2^d on a single vertex can reach every other vertex, by Fact 1.2.7. Any configuration C' which places one pebble on every vertex is reachable to every other vertex, vacuously. $\square$

The following gives us a far less trivial bound.

Theorem 1.4.4. *[1] Given a graph G with $|V(G)| = n$, we have $\pi^*(G) \leq \left\lceil \frac{2n}{3} \right\rceil$.*

This upper bound has been shown to be tight for paths [12] and cycles [1].

1.5 Pebbling as a Two-Player Game

There are many other variations of graph pebbling [8, 9]. We introduce a new variation that extends pebbling to a two-person game called Two-Player Pebbling. We will differentiate this variation from $\pi(G)$ by referring to the latter as classical pebbling. The first player, called the *mover*, uses pebbling moves to try to obtain a configuration C' such that $C'(r) = 1$. The second player, called the *defender*, uses pebbling moves to ensure a configuration C'' that admits no pebbling moves and $C''(r) = 0$. The mover wins if there is a pebble on r. The defender wins if, during any player's turn, there are no more pebbling moves possible and $C'(r) = 0$.

We have the following defintion.

Definition 1.5.1. We say a *round* consists of two pebbling moves; the initial move made by the mover and the final move made by the defender. A *turn* will be an individual player's pebbling move.

Every game needs rules; ours is no different. Given an initial configuration C on a graph G, we begin playing round 1 with the following rules:

1. Each player must take their turn.
2. If the mover pebbles from u to v, then the defender cannot pebble from v to u in the same round.
3. If $C'(r) > 0$ at any time, then the mover wins.
4. If $C'(r) = 0$ and there are no more pebbling moves, then the defender wins.

We have considered what would happen if we ignore Rule 1, i.e. if the defender was allowed to forfeit their turn. We will comment later on as to why this variation was not studied in depth. Rule 2 is very important. We can play a quick game to demonstrate why this rule is imperative. Consider P_4 with a configuration C which places 10 pebbles on v_4, 1 pebble on v_3 and v_2, and 0 pebbles on v_1, which will be the root. If we ignore rule 2 and play this game, then we get Figure 1.4.

We can see that the defender will win. However if we include rule 2 again, we will get Figure 1.5.

Of course, these show only one outcome of the game. Because there are two players, we need to consider possible pebbling moves of each player.

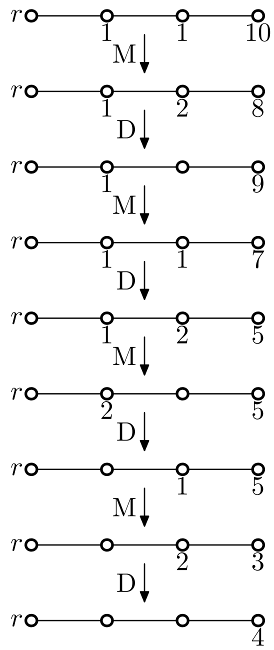

Figure 1.4: A Game on P_4 Without Rule 2.

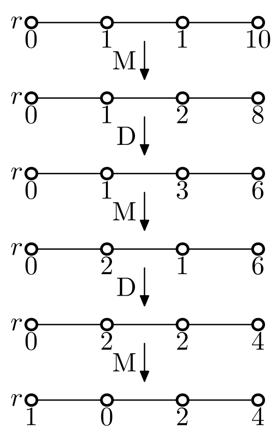

Figure 1.5: A Game on P_4 With Rule 2.

Definition 1.5.2. A *game tree* is a directed graph whose vertices are the possible outcomes for each player's move at each turn and edges are the turns from one configuration to the next based on the previous player's move.

These two figures only show one path of the game tree for simplicity. No matter what moves the mover makes, without rule 2, the defender has a way to win.

This brings up one of the main differences between two-player pebbling and classical pebbling – choice. There are two players with opposite objectives competing; we begin to step in the realm of combinatorial games. So, how should each player play the game? There needs to be a way to measure how well the players play not only against each other, but also against other ways they themselves could play.

Definition 1.5.3. A *strategy* for either player is a choice function $\mathcal{S} : \mathcal{C} \to \mathcal{P}$ from the set of all possible configurations $\mathcal{C}$ to the list of all possible legal pebbling moves $\mathcal{P}$. A strategy $\mathcal{S}$ is *winning* for the mover (or defender) on a configuration C provided the mover (or defender) wins playing $\mathcal{S}$ no matter what the defender (or mover) does.

By this, of course, we mean a strategy is a method of playing the game based on the possible outcomes of any move. The defender also needs to be aware of the mover's previous move so the defender does not make a pebbling move that violates rule 2.

1.6 The Two-Player Pebbling Number

Now we can introduce the values for two-player pebbling.

Definition 1.6.1. For a graph G with root r, the *rooted-two-player-pebbling number*, $\eta(G, r)$, is the minimum number m such that for every configuration of m pebbles, the mover has a winning strategy.

From this we get the following.

Definition 1.6.2. For a graph G, we say the *two-player pebbling number*, $\eta(G)$, is the minimum number m such that for every configuration of m pebbles and every choice of r, the mover has a winning strategy. If for a graph G and a root r there exists configurations of arbitrarily large size for which the defender wins, then $\eta(G, r) = \infty$.

The following definition is useful when considering configurations for which the mover has a winning strategy.

Definition 1.6.3. Given a graph G with root r, we say a *trivial* configuration C on the vertices of G will have $C(r) \geq 1$ or for some $v \in N(r)$, $C(v) \geq 2$. A configuration is *nontrivial* otherwise.

A trivial configuration will be won by the mover in 0 or 1 turns. We move on with some basic statements about $\eta(G)$.

Fact 1.6.4. *For every graph G, $\eta(G) = \max_{r \in G} \eta(G, r)$.*

Proposition 1.6.5. *$\pi(G) \leq \eta(G)$.*

Proof. The mover cannot win with less than the original pebbling number. □

Notice if the defender is not forced to pebble in a winning pebbling move sequence for classical pebbling, then equality fails. Thus far, we have found that equality holds only for complete graphs and paths of 5 or less vertices. Details will follow in later chapters.

Fact 1.6.6. *Let C be a configuration on G with m pebbles. After t rounds, there $m - 2t$ pebbles on G*

Proof. Every pebbling move removes 1 pebble from the graph. □

Here, we find a result if a vertex is adjacent to all other vertices.

Proposition 1.6.7. *If $deg(r) = |V(G)| - 1$, then $\eta(G, r) = |V(G)|$.*

Proof. Let r be a vertex with degree $|V(G)| - 1$. Suppose we have $|V(G)| - 1$ pebbles. If every non-root vertex has 1 pebble, then the defender wins. So suppose we have $|V(G)|$ pebbles. If we have a configuration with 1 pebble on r, then the mover wins. Suppose we have a configuration with no pebbles on the root. Then there must exist at least one vertex with at least 2 pebbles on it. Since the mover begins the game, they will pebble to the root. □

From this, we get a corollary about the complete graph on n vertices, K_n.

Corollary 1.6.8. $\eta(K_n) = n$.

The proof for Proposition 1.6.7 and Corollary 1.6.8 is the same proof for classical pebbling [2].

Chapter 2

General Upper & Lower Bounds

2.1 Paths & Cycles

In this section, we will show that the mover has a winning strategy for paths and cycles. First, we can find an upper bound for the number of pebbles needed anywhere on a path for the mover to have a winning strategy. We also describe the strategy.

Now we can find an upper bound for the *two-player pebbling number* of paths. Lemma 4.2.1 describes why we will only consider the case when v_1 is the root.

Lemma 2.1.1. *For $n \geq 2$, $\eta(P_n) \leq 2^n$.*

Proof. By induction on k.

Base: Let $n = 2$. Any configuration of 4 pebbles on P_2 can be won by the mover in 0 or 1 turns.

Induction: Suppose $\eta(P_k) \leq 2^k$ for all $k < n$. Let C be a non-trivial configuration of 2^n pebbles on the vertices of P_n with $r = v_1$. Suppose $C(v_2) = 1$. Then there are $2^n - 1$ pebbles on the vertices $v_3 v_4 \dots v_n$. Since $2^n - 1 \geq 2^{n-1}$, by induction, the mover can eventually place a pebble on v_2. No matter which player pebbles to v_2, the mover will still pebble to r on their next turn and win. Now, suppose $C(v_2) = 0$. It takes 2^{n-2} pebbles and $2^{n-2} - 1$ pebbling moves for the mover to place one pebble on v_2. The mover will use at most 2^{n-2} pebbles and the defender will use at most 2^{n-2} pebbles. Now our resulting configuration C' has at least $|C'| \geq 2^n - (2^{n-2} + 2^{n-2} - 1) = 2^n - 2^{n-1} + 1 = 2^{n-1} + 1 > 2^{n-1}$ pebbles. By induction, the mover can place another pebble on v_2. No matter which player pebbles to v_2, the mover will still pebble to r on their next turn and win. $\square$

Recalling Fact 1.2.6, we have very nice upper and lower bounds for $\eta(P_n)$.

Corollary 2.1.2. *For $n \geq 1$, we have $2^{n-1} \leq \eta(P_n) \leq 2^n$*

We move on to the upper bounds for cycles. We can consider a cycle on n vertices as a path on n vertices, adding an edge from v_1 to v_n.

Theorem 2.1.3. $\eta(C_n) \leq 2^n$.

Proof. Let r be any vertex in C_n. Label the vertices of $C_n = a_1 a_2 \ldots a_{n-1} r$, i.e. one P_n beginning at a_1 and ending at r. Let C be a configuration with 2^n pebbles on the vertices of C_n. If a_1 or a_{n-1} have 2 pebbles on them, then the mover will pebble to r and win. Suppose a_1 or a_{n-1} have at most 1 pebble on them. If the defender ever pebbles from a_1 to r, then the mover wins. Thus the mover has a winning strategy using at most 2^n pebbles on the vertices by Lemma 2.1.1. $\square$

These are very nice upper bounds. The classical pebbling number of P_n is 2^{n-1}. The upper bound for paths and cycles are in $\mathcal{O}(2 \cdot \pi(P_n))$ and $\mathcal{O}(\pi(C_n)^2)$. Chapter 4 focuses further on pebbling in paths and the difficulty that arises. With further consideration on these graphs, we hope we can refine these upper bounds.

2.2 Fan Graphs, $F_{m,n}$

In this section, we find an upper bound for the *Two-Player Pebbling Number* of a fan graph.

Definition 2.2.1. A *fan graph*, $F_{m,n} = K'_m \vee P_n$, is the join of a independent set and a path.

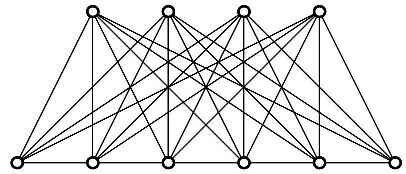

Figure 2.1: A Fan Graph $F_{4,6}$

Figure 2.1 is an example of a fan graph.

First, we have results on classical pebbling of fan graphs.

Theorem 2.2.2. *[5] The Fan Graph $F_{1,n}$ is class 0, i.e. $\pi(F_{1,n}) = n + 1$.*

We can extend this result to all fan graphs. For notation, when we refer to $F_{m,n}$, we will let $u_1, u_2, \ldots, u_m$ be the vertices in the independent set and $v_1, v_2, \ldots v_n$ be the vertices in the path

Theorem 2.2.3. *The Fan Graph $F_{m,n}$ is class 0, i.e. $\pi(F_{m,n}) = m + n$.*

Proof. By Lemma 1.2.2, $\pi(F_{m,n}) \geq m + n$. Let C is a configuration with $m + n$ pebbles.

Let $r = u_k$ for some $k \in \{1, 2, \ldots m\}$.

Case 1: If $C(u_i) \geq 4$ or $C(v_j) \geq 2$ for some i, j, then pebbling from u_i to v_ℓ to u_k or from v_j to u_k places a pebble on the root.

Case 2: If $2 \leq C(u_i) \leq 3$ for all $i \neq k$ and $C(v_j) \leq 1$ for all j, then pebbling from u_i to v_ℓ to u_k, where $C(v_\ell) = 1$, places a pebble on the root.

Now let $r = v_k$ for some $k \in \{1, 2, \ldots n\}$.

Case 1: If $C(u_i) \geq 2$ or $C(v_j) \geq 4$ for some i, j, then pebbling from u_i to v_k or from v_j to u_ℓ to v_k places a pebble on the root.

Case 2: If $C(u_i) \leq 1$ for all i, $C(u_\ell) = 1$ for some ℓ, and $C(v_j) \leq 3$ for all j, then there must exist a v_s such that $C(v_s) \geq 2$. Pebbling from v_s to u_ℓ to v_k places a pebble on the root.

Case 3: If $C(u_i) = 0$ for all i and $C(v_j) \leq 3$ for all j, then there must exist v_ℓ and v_s such that $C(v_\ell) \geq 2$ and $C(v_s) \geq 2$. Pebbling from v_ℓ and v_s to u_i to v_k places a pebble on the root.

□

Noting that $F_{m,n}$ is a class 0 graph for classical pebbling, one would hope that $\eta(F_{m,n}) < \infty$ as well. One thing to note is that if $m \geq 2$ and $r = u_i$ for some i, then this case is exactly one of the diameter-2 graphs described in Chapter 3, for which we get exact values for $\eta(G, r)$. We continue to show that, in fact, $\eta(F_{m,n}) < \infty$

Theorem 2.2.4. *For $m \geq 1$ and $n \geq 2$, we have $\eta(F_{m,n}) \leq \eta(P_n) + 3m$.*

Proof. Let $m = 1$.

Case 1: Suppose $r = u_1$. Let C be a configuration with $n + 1$ pebbles on P_n. Then, by the Pigeonhole Principle, there is at least one vertex v_j such that $C(v_j) \geq 2$. The mover can pebble from v_j to r and win.

Case 2: Suppose $r = v_i$ for some i. Let C be a configuration with $\eta(P_n) + 3$ pebbles on the vertices of $F_{1,n}$. If $C(u_1) \geq 2$, then the mover can pebble from u_1 to r and win. If $C(u_1) = 1$, then the mover will not pebble there. If the defender ever pebbles to u_1, then the mover will pebble from u_1 to r and win. Since there are $\eta(P_n) + 2$ pebbles on P_n, the mover has a winning strategy. If $C(u_1) = 0$, then the mover's first move is to pebble to u_1. If the defender pebbles to u_1, then the mover will pebble from u_1 to r and win. If the defender makes a pebbling move on the vertices of P_n, then there are $\eta(P_n)$ pebbles on the vertices of P_n. Thus the mover has a winning strategy.

Now, let $m \geq 2$.

Case 1: Suppose $r = u_k$ for some $k \in \{1, 2, \ldots, m\}$. Then $\eta(F_{m,n}, r) = \eta(G, r) \leq m + 2n + 3$ for $G \in \mathcal{G}_{n,m-1}$ by Theorem 3.7.7

Case 2: Suppose $r = v_k$ for some $k \in \{1, 2, \ldots, n\}$. Let C be a configuration with $\eta(P_n) + 3m$ pebbles on the vertices of $F_{m,n}$. If $C(u_i) \geq 2$ for any i, then the mover can pebble from u_i to r and win. If $C(u_i) \leq 1$ for all i and there exists some k such that $C(u_k) = 1$, then the mover will only pebble to the pebble-free vertices of $u_1, u_2, \ldots, u_m$. If the defender ever pebbles to u_k, then the mover will pebble from u_k to r and win. Since there are fewer than m unpebbled vertices of $u_1, u_2, \ldots, u_m$, there will be at least $\eta(P_n) + 3$ pebbles on P_n once the mover has placed 1 pebble on every vertex of $u_1, u_2, \ldots, u_m$. Thus the mover has a winning strategy. If $C(u_i) = 0$ for all i, then the mover's first moves are to pebble to $u_1, u_2, \ldots, u_m$. If the defender places a second pebble on u_j for some j, then the mover will pebble from u_j to r and win. If the defender places one pebble on u_k for some k, then the mover will place another pebble on an unpebbled vertex of $u_1, u_2, \ldots, u_m$ or make a pebbling move on P_n if there are no more unpebbled vertices. If the defender makes a pebbling move on the vertices of P_n, then the mover will continue to pebble to the pebble-free vertices of $u_1, u_2, \ldots, u_m$. Once each vertex in $u_1, u_2, \ldots, u_m$ has a pebble on them, the vertices in P_n will have at least $\eta(P_n)$ pebbles on them. Thus the mover has a winning strategy.

□

We note that the proof of Theorem 3.7.7 and any proof relating to $\eta(P_n)$ are independent of this result.

2.3 The Powers of Paths, P_n^k

We move on to look at Two-Player Pebbling on the k^{th} power of paths, P_n^k.

Definition 2.3.1. The k^{th} *power* of a graph, G^k is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{uv \mid d_G(u, v) \leq k\}$.

There is an upper limit when raising a graph to a power. The following fact describes the limit.

Fact 2.3.2. *If diam$(G) = d$, then G^d is complete.*

Also, we notice that P_n^1 is just a path on n vertices and P_n^{n-1} is a complete graph. Hence, we will consider $k \in \{2, 3, \ldots, n-2\}$ when dealing with P_n^k.

First, we see the classical pebbling value for P_n^2.

Theorem 2.3.3. *[12] Let $0 \leq r \leq 1$. Then $\pi(P_{2k+r}^2) = 2^k + r$.*

Now, we can determine whether $\eta(P_n^k)$ is finite or not.

Theorem 2.3.4. *Let $k \geq 2$ and $n \geq 3$. If $n \leq k+3$, then $\eta(P_n^k) \leq 4n-8$.*

Proof. Let $n = k+1$. Then $k = n-1$. Since P_n^{n-1} is a complete graph, we have $\eta(P_n^{n-1}) = n$.

Let $n = k+2$. Then $k = n-2$ and $P_n^{n-2} = K_n - e$ for some edge e. This edge missing is $v_1 v_n$.

Case 1: Suppose $r = v_1$ (the case of $r = v_n$ is completed by symmetry). This graph is a member of $\mathcal{G}_{n-2,1}$, the diameter-2 graphs described in Chapter 3. Thus, by Theorem 3.7.7, $\eta(P_n^{n-2}, r) = 2n-2$.

Case 2: Now suppose $r = v_i$ for $i \neq 1, n$. Since $d(v_i) = n-1$, the mover has a winning strategy using n pebbles by Lemma 1.6.7.

Let $n - k = 3$. Then $|N(r)| = n-3$.

Case 1: Suppose $r = v_1$. Let C be a configuration with $4(n-3)+4 = 4n-8$ on the vertices of P_n^{n-3}. If there exists $v_i \in N(r)$ such that $C(v_i) \geq 2$, then the mover will pebble to r and win. If for every $v_i \in N(r)$ we have $C(v_i) \leq 1$, then the mover's strategy will be to pebble to $N(r)$ so all vertices have exactly one pebble on them. If the defender places a second pebble on a vertex of $N(r)$, then the mover can pebble to r and win. If the defender pebbles to an unpebbled vertex in $N(r)$, then the mover will also pebble to a pebble-free vertex of $N(r)$, if no more exist, pebble from v_{n-1} to v_2, or if there is no pebbling move on v_{n-1}, pebble to v_{n-1}. Otherwise, the defender will pebble from v_n to v_{n-1}, from v_{n-1} to v_n, or lose. Once $N(r)$ is pebbled, there are at least 4 pebbles on v_{n-1} and v_n. If $C(v_{n-1}) \geq 2$, then the mover will pebble to v_2. If $C(v_{n-1}) \leq 1$, then the mover will pebble from v_n to v_{n-1}. The defender will either pebble to a vertex in $N(r)$, in which case the mover wins, or pebble to v_{n-1} as well. Now $C(v_{n-1}) \geq 2$ and the mover will pebble to v_2. If the defender pebbles from v_2 to r, then the mover wins. If the defender pebbles from v_2 to a vertex $v_k \in N(r)$, then the mover will pebble from v_k to r and win. If the defender pebbles from v_n to any vertex in $N(v_n)$, then the mover will pebble from v_2 to r and win.

Case 2: Now suppose $r = v_2$ (the case of $r = v_{n-1}$ is completed by symmetry). Let C' be a configuration with $4n-8$ pebbles on the vertices of P_n^{n-3}. If there exists

$v_i \in N(r)$ such that $C'(v_i) \geq 2$, then the mover will pebble to r and win. If for every $v_i \in N(r)$ we have $C'(v_i) \leq 1$, then the mover's strategy will be to pebble to $N(r)$ so all pebbles have exactly one pebble on them. If the defender places a second pebble on a vertex of $N(r)$, then the mover can pebble to r and win. If the defender pebbles to an unpebbled vertex in $N(r)$, then the mover will also pebble to a pebble-free vertex of $N(r)$ or, if no more exist, pebble to from v_n to v_{n-1}. Once $N(r)$ is pebbled, if $C'(v_{n-1}) \geq 2$, then the mover will pebble to r and win, because $v_{n-1} \in N(r)$. If $C'(v_{n-1}) \leq 1$, then the mover will pebble from v_n to v_{n-1}. The defender will either pebble to r or to a vertex in $N(r)$, in either case the mover wins.

Case 3: Lastly, suppose $r = v_i$ for $i \neq 1, 2, n-1, n$. Since $d(v_i) = n-1$, the mover has a winning strategy using n pebbles by Lemma 1.6.7. □

We note that the proof of Theorem 3.7.7 is independent of this result. Unfortunately, not all powers of paths have a finite value for η. There is a subset for which the defender has a winning strategy.

Theorem 2.3.5. *Let $k \geq 2$ and $n \geq 3$. If $n \geq 2k+4$, then $\eta(P_n^k) = \infty$.*

Proof. Let $n \geq 2k+4$. Let $r = v_1$. Notice that $N(r) = \{v_2, v_3, \ldots, v_{k+1}\}$ and $N(N(r)) = \{v_2, v_3, \ldots, v_{k+1}\} \cup \{v_{k+2}, v_{k+3}, \ldots, v_{2k+1}\}$ and there are at least 3 vertices not in $N(N(r))$. Let C be the configuration with all pebbles on v_n for any number of pebbles. The defender's strategy is to pebble from v_n to v_{n-1} or to undo a pebbling move from the mover. If the mover places a second pebble on a vertex in $\{v_{k+3}, v_{k+4}, \ldots, v_{2k+1}\}$, then the defender has at least two pebbles out of $v_{2k+2}, v_{2k+3}, \ldots v_n$ that they can pebble back to. Suppose the mover makes a pebbling move from v_{2k+2} and places a second pebble on v_{k+2}. If any of $v_{k+3}, v_{k+4}, \ldots, v_{2k+1}$ are pebble-free, then the defender will pebble to that vertex. Suppose none of $v_{k+3}, v_{k+4}, \ldots, v_{2k+1}$ are unpebbled. Since the mover pebbles from v_{2k+2}, then on the defender's previous turn, they must have pebbled to v_{2k+2}. Because the defender pebbles from v_n to v_{n-1} or undoes a pebbling move from the mover, the pebbling move must have come from $v_{k+3}, v_{k+4}, \ldots, v_{2k+1}$, leaving one of them pebble-free. This contradicts the assumption that they were not pebble-free. Thus, the mover will not be able to pebble to $N(r)$ and cannot win. □

We note that when $n \geq 2k+5$, then P_n^k satisfies the conditions of Theorem 2.4.1. But when $n = 2k+4$, P_n^k does not satisfy those same conditions and $\eta(P_n^k) = \infty$.

For $k \geq 2$, it is unknown at this time whether $\eta(P_n^k)$ is finite or not for $k+4 \leq n \leq 2k+3$.

Conjecture 2.3.6. Let $k \geq 2$. If $k+4 \leq n \leq 2k+3$, then $\eta(P_n^k) < \infty$.

2.4 Sufficient Condition for Infinite η

In this section, we show there exists a graph structure for which the defender always has a winning strategy. In fact, the condition below will show that "most" graphs yield a configuration giving a winning strategy for the defender. Later, we will show more structured classes of graphs which have winning strategies for the mover.

Theorem 2.4.1. *For a graph G, let S be a cut set of G. Label the components of $G - S$ as $G_0, G_1, \ldots G_k$ with $r \in G_0$. If for every $v \in S$, $|N(v) - V(G_0) - S| \geq 2$ and for every $x \in N(v) - V(G_0) - S$, $|N(x) - S| \geq 2$, then $\eta(G) = \infty$.*

Proof. Let G be described as above. Let m be an arbitrary natural number and $\mathcal{C}$ be the family of configurations with all pebbles m pebbles on the vertices of $N(x) - S$. The only way the mover can win is if the defender is forced to place a second pebble on a vertex in S. To see this, suppose the mover puts a second pebble on a vertex $v \in S$. Because $|N(v) - V(G_0) - S| \geq 2$, the defender can pebble to another vertex in $N(v) - V(G_0) - S$. Let $y \in N(v) - V(G_0) - S$ and suppose the defender must pebble from y. Because $|N(y) - S| \geq 2$, the defender can pebble to a vertex in $N(y) - S$. Therefore, the defender is never forced to place a second pebble on a vertex in S and can exhaust the use of all m pebbles. $\square$

Note that Figure 2.2 satisfies the conditions for Theorem 2.4.1. We see that Figure 2.2 is a tree and a bipartite graph. Therefore, trees and bipartite graphs will have an infinite two-player pebbling number, even though both classes of graphs graphs have a known classical pebbling number [3, 11]. Figure 2.3 has diameter 2. Thus, a graph G having diameter 2 is not a sufficient condition for a winning

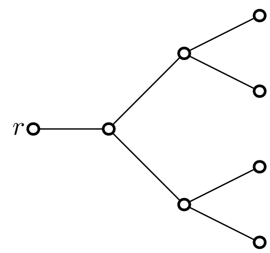

Figure 2.2: A Small Example for Theorem 2.4.1

strategy for the mover, whereas diameter-2 graphs have classical pebbling number of at most $|V(G)| + 1$ [12]. In fact, we are finding that the defender has a winning strategy on the configurations for many classes of graphs. So, we must have more restrictions on graphs to find $\eta(G) < \infty$.

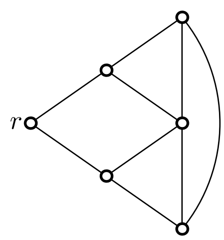

Figure 2.3: A Graph With Diameter 2 for Theorem 2.4.1

We have also found that grids, $P_n \square P_m$ for $m, n \geq 4$ have infinite η because they satisfy the conditions for Theorem 2.4.1. Consider Figure 2.4.

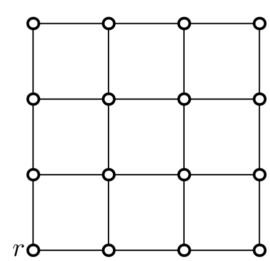

Figure 2.4: $P_4 \square P_4$

It is easily verified that $P_4 \square P_4$ satisfies the conditions of Theorem 2.4.1, so $\eta(P_4 \square P_4) = \infty$. However, we will show in Chaper 3 that $\eta(P_4)$ is finite. This is in direct contrast to Graham's Conjecture [2]. So even for a simple cartesian product of graphs, a two-player pebbling analog of Graham's Conjecture will not hold.

We do wonder if there is an upper bound to the number of pebbles in a configuration one must check to determine if the mover has a winning strategy. If the classical pebbling number of a graph is $\pi(G)$, then it takes at most $\pi(G)-1$ pebbling moves to place a pebble on the root. So if the defender had enough pebbles to never 'use' the pebbles needed by the mover and the defender still had a winning strategy, then $\eta(G, r) = \infty$.

Conjecture 2.4.2. If there exists a configuration C on a graph G and choice of root r with $3 \cdot \pi(G, r) - 1$ pebbles for which the defender has a winning strategy, then $\eta(G, r) = \infty$.

2.5 Removal of Edges

While working through some of the graphs for which the defender has a winning strategy, we noticed that removing edges can completely change the outcome of the game. Take Figure 2.3 for example. Thereom 2.4.1 says that $\eta(G) = \infty$. But if we remove one of the edges so Theorem 2.4.1 is no longer satisfied, as in Figure 2.5, then it is easy to check that the mover has a winning strategy. So we see that a two-player analogue result for Fact 1.2.1 will not hold.

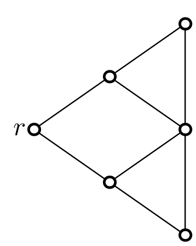

Figure 2.5: Removal of an Edge from Figure 2.3

The removal of edges does not just benefit the mover. Consider Figure 2.6.

It is straightforward to check that the mover has a winning strategy for this

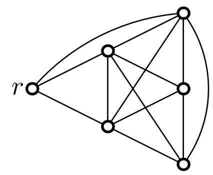

Figure 2.6: Graph for Which Mover has a Winning Strategy

graph. But if we remove one more edge, the game shifts. Figure 2.7 now satisfies the conditions of Theorem 2.4.1

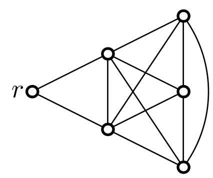

Figure 2.7: Figure 2.6 Minus One Edge

The removal of an edge changed the outcome of the game for either player. The edge removed can determine who is helped. The removal of an edge adjacent to the root will only help the defender.

Proposition 2.5.1. *Let G be a graph and e be an edge adjacent to the root. If the defender has a winning strategy on G, then the defender has a winning strategy on $G - e$.*

Proof. Given a configuration C on the graph G, the defender will never pebble on an edge adjacent to the root unless forced to. So let the defender have a winning strategy on G. Then the defender will play the same strategy on $G - e$ and win. $\square$

Chapter 3

Pebbling on Diameter-2 Graphs

3.1 Construction of $\mathcal{G}_{s,t}$

We move on to the study of two-player pebbling on graphs of diameter 2. Specifically, we characterize the winning player for nearly every configuration for a certain class of diameter-2 graph, we characterize the winning player for every configuration on complete bipartite and complete multipartite graphs, and find exact η values for complete bipartite and complete multipartite graphs. To do this, we define a specific subset of diameter-2 graphs. For a graph G, the *complement* G' is the graph for which $V(G') = V(G)$ and $e \in E(G') \iff e \notin E(G)$. For any two graphs H and G, the *join* of H and G, $H \vee G$, is the graph such that $V(H \vee G) = V(H) \cup V(G)$ and $E(H \vee G)$ contains all edges in H, all edges in G, and edges connecting every vertex in H with every vertex with G.

We define a subset of diameter-2 graphs, $\mathcal{G}_{s,t} = \{(K_1 \cup K_t') \vee S\}$ where S is arbitrary and $|V(S)| = s$. We let the root be K_1, $s \geq 1$ and, $t \geq 2$. We will save the case when $t = 1$ for later, as it is unique. Figure 3.1 gives us an example of a graph in $\mathcal{G}_{s,t}$.

If a starting configuration C has two pebbles on any vertex in S, then C is trivial, i.e. the mover wins with one turn. So we will consider configurations on G with 0 or 1 pebbles on vertices in S. Let k be the number of vertices in S that are pebble-free.

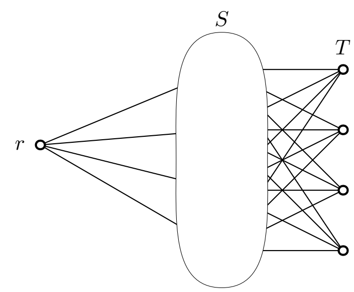

Figure 3.1: The Class $\mathcal{G}_{s,t}$

We develop a condition on the distribution of pebbles on T based on the pebble-free vertices in S. Informally, it appears that we can compare how many pebbling moves are in T to the number of pebble-free vertices in S. If there are many more moves than pebble-free vertices, it would stand to reason that the mover wins. On the other hand, if there are many more pebble-free vertices than pebbling moves in T, the defender should win. We would like a way to count the number of pebbling moves in T. Notice for any vertex $v \in T$ that $\left\lfloor \frac{C(v)}{2} \right\rfloor$ will tell us the number of pebbling moves on v. We have the following definition.

Definition 3.1.1. We say $C_T = \displaystyle\sum_{v \in T} \left\lfloor \frac{C(v)}{2} \right\rfloor$ is the number pebbling moves in T with configuration C.

In fact, if there are k pebble-free vertices in S and $C_T \geq k+3$, then the mover has a winning strategy. If $C_T \leq k$, then the defender has a winning strategy. If $C_T = k+2, k+1$, then it depends on the parity of k and the structure of S to find the winning player.

We can see that C_T will change from configuration to configuration. When a pebbling move is made from T, we can say that the number of pebbling moves in T for the new configuration C' is $C'_T = C_T - 1$ with original configuration C.

We see that for $\mathcal{G}_{s,t}$ the rule that each player must take their turn is important. If the defender is allowed to forfeit their turn, then it is easy to verify that they have a winning strategy for $s \geq 1$ and $t \geq 2$. We want to see a configuration where the mover has a winning strategy and define such a strategy. The winning strategy for the mover is to force the defender to place a second pebble on a vertex in S.

3.2 When k is odd

Lemma 3.2.1 is the base case for induction when k is odd.

Lemma 3.2.1. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with 1 pebble-free vertex in S. The mover has a winning strategy if and only if $C_T \geq 2$.*

Proof. Suppose $C_T \geq 2$. The mover will pebble to the unpebbled vertex. Now there is one more move in T and all vertices in S have a pebble on them. The defender must pebble to a vertex in S, placing a second pebble on a vertex. The mover pebbles to r and wins.

Conversely, suppose $C_T \leq 1$. If $C_T = 0$, then there are no pebbling moves in T and the defender wins. Suppose $C_T = 1$. Since there is 1 pebbling move in T, all the vertices in T without the pebbling move have 0 or 1 pebble on them. The mover has two choices, to pebble to the unpebbled vertex or to place a second pebble on a vertex in S. If the mover pebbles to the pebble-free vertex, then for the new configuration C', $C'_T = 0$. There are no more pebbling moves and the defender wins. So suppose the mover pebbles to a pebbled vertex in S. If they can, then the defender will pebble to the pebble-free vertex in S or T and win. If all vertices in T are pebbled, then the defender will place a second pebble on one vertex in T, yielding an extra pebbling move. The mover has the same two options as earlier. Suppose the mover places a second pebble on a vertex in S, or else they will lose. The vertex in T with the original pebbling move can now have 0 or 1 pebbles on it. The defender will pebble to it. If it is unpebbled, then the defender wins. If it is pebbled, then the defender adds a new pebbling move. The mover will pebble from that vertex to S with the same two options. Again we suppose the mover pebbles to a pebbled vertex. Now there is guaranteed to be an unpebbled vertex from the mover's last two pebbling moves for the defender to pebble to. The defender does so and wins. $\square$

Lemma 3.2.2. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is odd and $C_T \geq k+1$, then the mover has a winning strategy on G.*

Proof. Let k be odd and $C_T \geq k+1$. The mover will pebble to a pebble-free vertex in S. If the defender places a second pebble on a vertex in S, the mover wins. If the defender pebbles to a pebble-free vertex in S, then there are $k-2$ pebble-free vertices in S and the resulting configuration C' has $C'_T = C_T - 2$. Thus $C'_T = k-1$. Hence, by induction, the mover has a winning strategy. □

Next is a result when the defender has a winning strategy.

Lemma 3.2.3. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is odd and $C_T \leq k$, then the defender has a winning strategy on G.*

Proof. By induction on C_t.

Base: Let $C_t = 0 \leq k$. There are no pebbling moves in T so the defender wins.

Induction: Let $C_t \leq k$ for k-pebble-free vertices in S. The mover has two choices, to pebble to a pebble-free vertex in S or to place a second pebble on a vertex in S. If the mover pebbles to a pebble-free vertex and there are no more pebble free vertices, then $k = 1$ and by Lemma 3.2.1 the defender wins. If the mover pebbles to a pebble-free vertex and there is another unpebbled vertex, then the defender will pebble to a pebble-free vertex. We have $C_t \leq k-2$ and by induction, the defender has a winning strategy. If the mover places a second pebble on a vertex in S, then the defender will pebble back to an even vertex in T, if one exists. Now $C_t \leq k+1$ and by induction the defender wins.

□

So for k odd, we have the following:

Initital Value of C_T	Winning Player
$C_T \geq k+1$	Mover
$C_T \leq k$	Defender

Table 3.1: Value of C_T and its Winning Player for k Odd

3.3 When k is even

The section when the number of pebble-free vertices on S is even is a little more difficult. We first show the number of pebbling moves needed in T for the mover to win.

Lemma 3.3.1. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T \geq k+3$, then the mover has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T \geq 3$. The mover will pebble to S, placing a second pebble on one of the vertices. The defender will pebble back to T or lose. The new configuration C' has $C'_T \geq 2$ and now $k = 1$. By Lemma 3.2.1, the mover wins.

Induction: Let $C_T \geq k+3$ for $k \geq 1$. The mover will pebble to a free vertex. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a free vertex in S, then the new configuration C' has $C'_T = C_T - 2 \geq k+3-2 = k+1$. Since S now has $k-2$ pebble-free vertices, the mover has a wining strategy by induction. □

We will forgo the case when $C_T = k+2$ for now and leave it for its own section.

Lemma 3.3.2. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T \leq k+1$, then the defender has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T \leq 1$. If $C_T = 0$, then there are no pebbling moves in T and the defender wins. If $C_T = 1$, then all but one vertex in T as at most 1 pebble on it. The mover has no choice but to place a second pebble on a vertex in S. The defender will pebble from the vertex in S with two pebbles on it to any vertex in T. For the new configuration C', we have $C'_T \leq 1$ and $k = 1$. So by Lemma 3.2.1, the defender has a winning strategy.

Induction: Let k be even and $C_T \leq k+1$. If the mover pebbles to a pebble-free vertex in S, then the defender will as well. The new configuration C' has $k-2$ pebble-free vertices and $C'_T = C_T - 2 \leq= k-1$. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex in S, the defender will pebble to a vertex in T. The resulting configuration C'' has $k+1$ pebble-free vertex in S and $C''_T \leq C_T \leq k+1$. Since $k+1$ is odd, the defender has a winning strategy by Lemma 3.2.3. □

So for k even, we have the following:

Initital Value of C_T	Winning Player
$C_T \geq k+3$	Mover
$C_T \leq k+1$	Defender

Table 3.2: Value of C_T and its Winning Player for k Wven

3.4 When $C_T = k+2$ with k even

When $C_T = k+2$, the difficulty increases. The number of pebbles in S and how many vertices in T have a non-zero even number of pebbles on them will determine which player has a winning strategy. Each player's strategy changes a little. The mover's goal is to force the defender to pebble to a vertex in T with an odd number of pebbles on it. This will increase the number of pebbling moves in T and yield one of the mover's winning configurations described in an early section. The defender will try to pebble to a vertex in T with an even number of pebbles on it. This adds no new pebbling moves and yields one of the defender's winning configurations.

First we consider the configuration were all the vertices in T have an odd number of pebbles on them.

Lemma 3.4.1. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T = k + 2$ and for all $v \in T$, $C(v)$ is odd, then the mover has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T = 2$ with every vertex in T having an odd number of pebbles on it. The mover will pebble to S, placing a second pebble on one of the vertices. The defender will pebble back to T or lose. Since every vertex in T has an odd number of pebbles, the new configuration C' has $C'_T = 2$ with 1 unpebbled vertex in S. By Lemma 3.2.1 the mover wins.

Induction: Let k be even and $C_T \geq k + 2$ for $k \geq 1$. The mover will pebble to a free vertex. If the defender places a second pebble on a vertex in S, then the mover wins. So the defender will pebble to a free vertex in S. Now for the new configuration C', $C'_T = C_T - 2 \geq k + 2 - 2 = k$. Since S now has $k - 2$ pebble-free vertices, the mover has a wining strategy by induction. □

Now, we look at the case when some vertices in T have an even number of pebbles on them. This becomes more difficult. The strategies for each player depends on how many pebbles on are the vertex with an even number of pebbles.

Lemma 3.4.2. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T = k + 2$ and there is either at least one $x \in T$ such that $C(x) = 0$ or at least two vertices $x, y \in T$ such that $C(x)$ and $C(y)$ are even, then the defender has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T = 2$. The mover will place a second pebble on a vertex in S. The defender will pebble from that vertex in S to the pebble-free vertex in T or to an even vertex in T. For the new configuration C', we have $C'_T = 1$ and $k = 1$. Thus by Lemma 3.2.1, the defender wins.

Induction: Let k be even and $C_T \geq k+2$. The mover can place a second pebble on a vertex in S or pebble to a pebble-free vertex in S. If the mover places a second pebble on a vertex in S, then the defender will pebble to the unpebbled vertex in T or to an even vertex in T, not adding any pebbling moves to T. For our new configuration C', we have $C'_T = k+1$ and k is now odd. Hence, the defender wins by Lemma 3.2.3. If the mover pebbles to a pebble-free vertex in S, then defender will also pebble to a pebble-free vertex in S. Now for our new configuration C', we have $C'_T = k$ and there are $k-2$ pebble-free vertices in S. Since there were no pebbling moves back to T, we can see that T will still have at least one pebble-free vertex or at least two even vertices. Thus, the defender wins by induction. □

So for k even and $C_T = k+2$, we have the following:

Number of Even Vertices in T	Winning Player
None	Mover
At least one pebble-free or at least two even	Defender

Table 3.3: Number of Even Vertices in T and its Winning Player for k Even

3.5 A New Game

In this section, we will characterize the winning player for specific structures on S and certain configurations on $\mathcal{G}_{s,t}$ with an even number of unpebbled vertices in S, one even vertex in T, and the number of pebbling moves from T is two more than the number of pebble-free vertices in S. We will partition S into two subsets.

Definition 3.5.1. Let S_0 be the pebble-free vertices of S and S_1 be the pebbled vertices of S.

We cannot characterize the winning player for all configurations and all structures on S. We will introduce a new game, called the Element Selecting Game (ESG), to help explain why this task is particularly difficult.

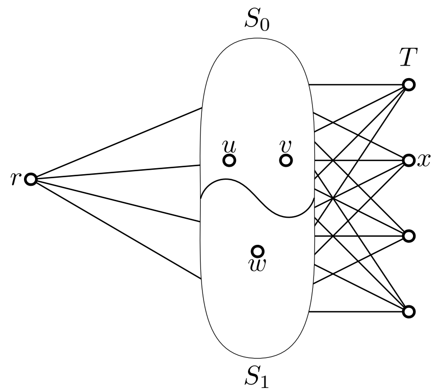

Figure 3.2: Partitioning S

Let $N_1, N_2, \ldots N_k$ be a collection of subsets, possibly empty and intersecting, from a universal set U. There are two players, Mary and Dan. Each player will take turns, Mary beginning and Dan following, selecting one element from U. After a specified number of rounds, we say Mary wins if at least one of the subsets N_i has every one of its elements selected and Dan wins if none of the N_i's has been completely selected. If there exists a subset N_j which is empty, then we say Mary wins vacuously. Which player has a winning strategy?

This game directly relates to this case of exactly one even vertex in T with $C_T = k + 2$ and k pebble-free vertices in S of Two-Player Pebbling by the following definition.

Definition 3.5.2. Given an instance $G \in \mathcal{G}_{s,t}$ with configuration C containing $2j$ pebble-free vertices in S and $C_T = 2j + 2$, we define $\mathcal{E}(G, C)$ as the instance of the Element Selecting Game constructed in the following way: Let $U = S_0$, the set of unpebbled vertices in S. For every vertex $v_i \in S$, let $N_i = N[v_i] \cap U$. For $k = 2j$ pebble-free vertices in S and $C_T = 2j + 2$, Mary and Dan play j rounds of the new game. Mary represents the motives of the mover and Dan represents the motives of the defender.

Here we see two lemmas to illustrate why we want $C_T = 2j + 2$ given we are playing j rounds.

Lemma 3.5.3. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. Suppose there exists a pebbled vertex $v \in S$ such that all its neighbors in S are pebbled. If k is even and $C_T = k + 2$ and there is one $x \in T$ such that $C(x) \geq 2$ and all other vertices in T have an odd number of pebbles, then the mover has a winning strategy.*

Proof. The mover will pebble from x to v. The defender can either pebble to a neighbor of v or pebble to an odd vertex in T. If the defender pebbles to a neighbor of v, then that vertex will have two pebbles on it and the mover wins. If the defender pebbles to an odd vertex in T, then they will add a pebbling move. Now our new configuration C' has $k + 1$ pebble-free vertices in S and $C'_T = k + 2$. By Lemma 3.2.2, the mover has a winning strategy. $\square$

So, we have covered the case when the only even vertex in T has 2 pebbles on it. If S is independent, then the conditions for Lemma 3.5.3 will hold vacuously. Here is a configuration for the defender's winning strategy.

Lemma 3.5.4. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. Suppose that for every pebbled vertex $v \in S$, there exist at least one pebble-free neighbor in $u \in S$. If k is even and $C_T = k + 2$ and there is one $x \in T$ such that $C(x) = 2$ and all other vertices in T have an odd number of pebbles, then the defender has a winning strategy.*

Proof. The mover can pebble to a pebbled vertex or an unpebbled vertex. If the mover pebbles to a pebbled vertex v, then the defender will pebble from v to its pebble-free neighbor, which exists by our hypothesis. Now k is unchanged and our new configuration C' is such that $C'_T = k + 1$. By Lemma 3.3.2, the defender has a winning strategy. If the mover pebbles to an unpebbled vertex, then the defender will pebble from x to another vertex in S which is pebble-free, which exists because k is even and at least 2. By Lemma 3.4.2, the defender has a winning strategy. $\square$

By the time the j rounds are completed, the mover wants to have a pebbled closed neighborhood for some vertex in S and still have at least 2 pebbles on the one even vertex in T.

Now, we can show that the two games are equivalent when we restrict Two-Player Pebbling to this current case.

Lemma 3.5.5. *Let $G \in \mathcal{G}_{s,t}$ and C be a configuration containing $2j$ pebble-free vertices in S and $C_T = 2j + 2$ and $\mathcal{E}(G, C)$ be the instance of the Element Selecting Game constructed from G. Mary has a winning strategy for $\mathcal{E}(G, C)$ if and only if the mover has a winning strategy in G with configuration C.*

Proof. Given $G \in \mathcal{G}_{s,t}$, let C be a non-trivial configuration with $2j$ pebble-free vertices in S, exactly one even vertex in T and $C_T = 2j + 2$. We construct the $\mathcal{E}(G, C)$ as in Definition 3.5.2. Suppose Mary has a winning strategy for the $\mathcal{E}(G, C)$. Then, Mary and Dan have a sequence of elements that they each selected such that at least one of the N_i's has been selected. Every element in U that Mary selects, the mover will pebble from an odd vertex in T to the corresponding vertex in S_0. If the defender ever places a 2nd pebble on a vertex in S, then the mover wins. If the defender places a pebble on a pebble-free vertex, then the mover will pebble to the vertex that corresponds to the next element that Mary selected. Since Mary was able to select every element in one of the N_i's, the mover will be able to have a pebbled closed neighborhood with a new configuration C' such that $C'_T \geq 2$. Thus the mover has a winning strategy.

Conversely, suppose the mover has a winning strategy on G with configuration C. If the mover can not pebble a closed neighborhood after j rounds, then for the new configuration C' every vertex in S will have an unpebbled neighbor and $C'_T = 2$. So the defender wins by Lemma 3.5.4. Thus the mover must be able to pebble a closed neighborhood in S. Mary can select an element in U that corresponds to a pebble-free vertex in S_0 that the mover selects. Because a closed neighborhood is pebbled for some $v_i \in S$, then N_i must be able to have its elements selected. Thus Mary has a winning strategy. $\square$

It will be easier to show cases of $\mathcal{E}(G, C)$ for which Mary has a winning strategy and then show how a case for pebbling can apply.

Lemma 3.5.6. *If there exists an i such that $|N_i| = j$ while playing at least j rounds, then Mary wins the Element Selecting Game.*

Proof. Suppose there exists a set N_i with j elements in it. Suppose Mary and Dan play at least j rounds. Mary can select every element in N_i with her turn and win in at most j rounds. $\square$

Corollary 3.5.7. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even, $C_T = k + 2$ and there is one even vertex $x \in T$ such that $C(x) \geq k + 2$ and all other vertices in T have an odd number of pebbles, then the mover has a winning strategy.*

Proof. Let $k = 2j$. Having k pebble-free vertices in S with $C(x) \geq k+2$ is equivalent to some $|N_i| = j$ and playing j rounds. $\square$

Unfortunately, Lemma 3.5.6 and Corollary 3.5.7 are not necessary conditions for the mover to win in general. There are 'boundary' cases which can violate the conditions of converse Corollary 3.5.7 and the mover still has a winning strategy (Lemma 3.5.3 for example). Specifically, we can have many more pebble-free vertices in S than pebbles on x and the mover has a winning strategy. We see why having exactly one vertex in T with a non-zero even number of pebbles on it is so difficult. It depends on how S is structured. The informal strategy for the mover is to pebble from the even vertex in T to a vertex in S whose neighbors all have pebbles on them. Then the defender must pebble to an odd vertex in T, yielding the odd configuration in Lemma 3.2.2. If the defender can pebble in S, then the mover will lose.

We begin to characterize the winning strategy for each player for the case where $C(x) = 4$ with x as the only even vertex in T. Notice that for the mover to have a winning strategy in the $C(x) = 2$ case we needed a vertex $v \in S_1$ to be such that $N_S(v) \subseteq S_1$. The mover will make a pebbling move from an odd vertex in T to try and force the defender to pebble in such a way that for the next round, the conditions for Lemma 3.5.3 are satisfied.

Lemma 3.5.8. *Suppose Mary and Dan play only 1 round. Then Mary wins the Element Selecting Game if and only if there is an i such that N_i is empty, $N_i = \{y\}$ or there exists an $y \in U$ such that for every $z \in U$, $N_i = \{y, z\}$.*

Proof. Let Mary and Dan play only 1 round.

Suppose that there is some $y \in U$ such that for each $z \in U$, there is a subset such that N_i is empty, $N_i = \{y\}$ or $N_i = \{y, z\}$. If N_i is empty, then Mary wins vacuously. If all N_i's are nonempty, then Mary will select element y. Then Dan will select any other element. By our hypothesis, there must exist a subset of U that is equal to y or equal to y and the element Dan chose. Thus there will be a subset that is selected. Thus Mary wins.

Conversely, suppose for every $y \in U$ there exists a $z \in U$ so that for every N_i is nonempty, $N_i \neq \{y\}$, and $N_i \neq \{y, z\}$. Mary will chose any element y'. By our assumption, there must exist another element z' in U so that for every subset N_i, we have $\{y', z'\}$ is a proper subset of N_i. Thus after 1 round, no subset has been completely selected. Hence, Dan wins. $\square$

Corollary 3.5.9. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. Suppose k is even and $C_T = k + 2$ and there is only one even vertex $x \in T$. The mover has a winning strategy if $C(x) \geq 4$ and there exists a vertex v in S_0 that for every vertex $u \in S_0$ that either:*

a) there is some vertex $w \in S_1$ such that $N_{S_0}(w) = \{v\}$ or $\{u, v\}$, or

b) $N_{S_0}(u) = \{v\}$.

The defender has a winning strategy if $C(x) \leq 4$ and for every vertex v in S_0 there exists a vertex $u \in S_0$ such that there is no vertex $w \in S_1$ with $N_{S_0}(w) = \{v\}$ or $\{u, v\}$ and (b) $N_{S_0}(u) \neq \{v\}$.

Proof. We can consider $C(x) \geq 2(1) + 2$. Thus having $C(x) \geq 4$ is equivalent to playing 1 round in ESG. Let the vertex v in Two-Player Pebbling represent the element y in ESG. Suppose there is some vertex $w \in S_1$ such that $N_{S_0}(w) = \{v\}$ or $\{u, v\}$. Then for the ESG, $N_w = \{v\}$ or $\{u, v\}$. The mover wins by Lemmas 3.5.5

and 3.5.8. Suppose $N_{S_0}(u) = \{v\}$. Then for the ESG, $N_u = \{u, v\}$. The mover wins by Lemmas 3.5.5 and 3.5.8. $\square$

So for k even with $C_T = k+2$ and one even vertex $x \in T$, we have the following:

Structure of S	$C(X)$	Winning Player
Any structure	$C(x) \geq k+2$	Mover
Some pebbled vertex with all pebbled neighbors	$C(x) \geq 2$	Mover
All pebbled vertices have an unpebbled neighbor	$C(x) = 2$	Defender
$\exists v \in S_0$, $\forall u \in S_0$ either $\exists w \in S_1$ such that $N_{S_0}(w) = \{v\}, \{u, v\}$ or $N_{S_0}(u) = \{v\}$	$C(x) \geq 4$	Mover
$\forall v \in S_0$, $\exists u \in S_0$ such that $\forall w \in S_1$, $N_{S_0}(w) \neq \{v\}, \{u, v\}$ and $N_{S_0}(u) \neq \{v\}$	$C(x) = 4$	Defender

Table 3.4: Structure of S and its Winning Player

3.6 Configurations on Complete Multipartite Graphs

We attempted to find a nice necessary condition for Mary to have a winning strategy in the Element Selecting Game while playing 2, 3, etc. rounds. We believe it would be easier to find the winning player for different scenarios in the Element Selecting Game and then translate them to Two-Player Pebbling. However, characterizing scenarios for which Mary has a winning strategy turns out to be very difficult and based on the structure of the subsets $N_1, N_2, \ldots, N_m$. So, we narrow our focus from any $G \in \mathcal{G}_{s,t}$ to G being a complete multipartite graph, and we can characterize the winning player without the aid of the Element Selecting Game.

The goal is to determine the winning player for all configurations on complete bipartite and complete multipartite graphs. Sections 3.2, 3.3, and 3.4 cover all cases except when the number of unpebbled vertices in S, k, is even, $C_T = k + 2$ and there is one even vertex $x \in T$. Notice that for complete bipartite graphs, S is independent so Lemma 3.5.3 and Lemma 3.5.4 finish the argument for complete bipartite graphs. To finish the task for complete multipartite graphs, we need to complete the above argument. If S is a clique, then Corollary 3.5.7 shows when the mover has a winning strategy.

Lemma 3.6.1. *Let G be a complete multipartite graph with partite sets $A_1, A_2, \ldots, A_m$, $r \in A_1$, $|A_1| \geq 3$ and C be a non-trivial configuration with k pebble-free vertices in $G - A_1$. Let A_ℓ have the maximum number of unpebbled vertices in $G - A_1$ and k_ℓ denote the number of unpebbled vertices in A_ℓ. Let k be even, the number of pebbling moves in A_1 be $k+2$, and one even vertex $x \in A_1$. The mover has a winning strategy if and only if $C(x) \geq 2(k - k_\ell) + 2$.*

Proof. By induction on $k - k_\ell$.

Base: Let $k - k_\ell = 0$. Suppose $C(x) \geq 2$. If $k = 0$, then by Lemma 3.5.3 the mover has a winning strategy. So, suppose $k > 0$. The mover will pebble from a vertex in A_1 other than x, if one exists, to a pebble-free vertex in A_ℓ. If the defender pebbles to a pebbled vertex, then the mover can pebble to r and win. If the defender pebbles to an unpebbled vertex in A_ℓ, then there is at least one pebbled vertex in A_ℓ with all neighbors $G - A_1$ pebbled. Then by Lemma 3.5.3 the mover has a winning strategy.

Conversely, suppose $C(x) = 0$. Then by Lemma 3.4.2, the defender has a winning strategy

Induction: Assume this is true for all $i < k - k_\ell$. First, suppose $C(x) \geq 2(k - k_\ell) + 2$. The mover will pebble from a vertex in A_1 not x, if one exists, to one of the pebble-free vertices in $G - A_1 - A_\ell$. The defender will pebble to any pebble-free vertex in $G - A_1$ (or lose). The resulting configuration C' is such that $C'(x) \geq 2(k - k_\ell)$, $C'_{A_1} = k$ and A_ℓ has at least $k_\ell - 1$ pebble-free vertices. So by induction, the mover has a winning strategy.

Conversely, suppose $C(x) \leq 2(k - k_\ell)$. The mover can either pebble to an unpebbled vertex or to a pebbled vertex of $G - A_1$. If the mover pebbles to an unpebbled vertex of $G - A_1$, then the defender will pebble from x to an unpebbled vertex in A_ℓ. The new configuration C' has $C'(x) \leq 2(k - k_\ell) - 2$ and there are at most $k_\ell - 1$ pebble-free vertices in A_ℓ. By induction, the defender has a winning strategy. If the mover pebbles to a pebbled vertex of $G - A_1$, then the defender will pebble to an unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover chooses, after two rounds the new configuration C'' has $C''(x) \leq 2(k - k_\ell) - 2$ and there are at most $k_\ell - 2$ unpebbled vertices in A_ℓ. By induction, the defender has a winning strategy

□

While exploring the case of complete multipartite graphs, we found a result for a related graph of diameter 2, where S is a disjoint union of cliques.

Lemma 3.6.2. *Let $G \in \mathcal{G}_{s,t}$ and $S = K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_\ell}$ and C be a non-trivial configuration with k pebble-free vertices in S. Let k be even, $C_T = k + 2$, and one even vertex $x \in T$. Let k^* be the number of pebble-free vertices in K_{m_j}, where K_{m_j} has the least number of unpebbled vertices in S The mover has a winning strategy if and only if $C(x) \geq k^* + 2$.*

Proof. By induction on k^*.

Base: The case when $k^* = 0$ is proven in a more general case by Lemma 3.5.3 and Lemma 3.5.4.

Induction: Assume this is true for all $i < k^*$. Let C be a configuration with k^* pebble-free vertices in K_{m_j}, where K_{m_j} has the minimum number of unpebbled vertices in S. First, suppose $C(x) \geq k^* + 2$. The mover will pebble from a vertex in T not x to one of the pebble-free vertices in K_{m_j}. The defender will pebble to any pebble-free vertex in S (or lose). The resulting configuration C' is such that $C'(x) \geq k^*$, $C_T = k$ and K_{m_j} has at least $k^* - 1$ pebble-free vertices. So by induction, the mover has a winning strategy.

Conversely, suppose $C(x) \leq k^*$. The mover can either pebble to an unpebbled vertex or to a pebbled vertex of S. If the mover pebbles to an unpebbled vertex of S, then the defender will pebble from x to an unpebbled vertex not in K_{m_j}. The new configuration C' has $C'(x) \leq k^* - 2$ and there are at most k^* pebble-free vertices in K_{m_j}. By induction, the defender has a winning strategy. If the mover pebbles to a pebbled vertex of S, then the defender will pebble to an unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C'' has $C''(x) \leq k^* - 2$ and there are at most k^* unpebbled vertices in K_{m_j}. By induction, the defender has a winning strategy

□

Lemma 3.6.1, along with Lemma 3.5.7, characterize the winning player for complete multipartite graphs.

So, we have the following:

G is Complete Multipartite	$C(X)$	Winning Player
k_ℓ Pebble-Free Vertices in A_ℓ, Where A_ℓ has Minimum Number of Unpebbled Vertices in $G - A_1$	$C(x) \geq 2(k - k_\ell) + 2$	Mover
k_ℓ Pebble-Free Vertices in A_ℓ, Where A_ℓ has Minimum Number of Unpebbled Vertices in $G - A_1$	$C(x) \leq 2(k - k_\ell)$	Defender

Table 3.5: G Multipartite and its Winning Player

3.7 Determining $\eta(\mathcal{G}_{s,t}, r)$

Now we have the main result of the section which follows from the previous lemmas.

Theorem 3.7.1. *Let G in $\mathcal{G}_{s,t}$ and C be a configuration with k pebble-free vertices in S. If $t \geq 2$, then we have the following:*

The mover has a winning strategy on G	*The defender has a winning strategy on G*
k is odd and $C_T \geq k+1$	*k is odd and $C_T \leq k$*
k is even and $C_T \geq k+3$	*k is even and $C_T \leq k+1$*
k is even and $C_T = k+2$ and all vertices in T are odd	*k is even and $C_T = k+2$ and T has at least one unpebbled vertex or two even vertices*

And if k is even and $C_T = k+2$ and exactly one vertex in T is even, then the game is equivalent to the Element Selecting Game.

There is still one case we have not discussed yet: the case when T is a single vertex, because previous results allowed for a move back to T by the defender. Lemmas 3.7.2 and 3.7.3 are the base case of induction for Lemma 3.7.4, Lemma 3.7.5 and Lemma 3.7.6

Lemma 3.7.2. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$ If there exists a pebbled vertex $v \in S$ such that all of its neighbors in S are pebbled and $C(x) \geq 2$, then the mover has a winning strategy.*

Proof. The mover will pebble to v. The defender can not pebble back to x. So the defender can either pebble to a neighbor of v, which all have pebbles, or to r. In either case, the mover wins. □

Lemma 3.7.3. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. For every $v \in S$, suppose there exists at least one $u \in N_S[v]$ such that u is not pebbled. If $C(x) \leq 2$, then the defender has a winning strategy.*

Proof. If $C(x) < 2$, then there are no pebbling moves in T and the defender wins. If $C(x) = 2$, then the mover will pebble to some vertex $v \in S$. If v is unpebbled, then the defender wins. If v is pebbled, then there must exist an unpebbled neighbor by assumption. The defender will pebble to this vertex and win. □

Lemma 3.7.4. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. For every $v \in S$, suppose there exists at least one $u \in N_S[v]$ such that u is not pebbled and $S \neq N[v]$ for some v. Let k^* be the number of pebble-free vertices in $N[v^*]$ where $N[v^*] \in S$ has the minimum number of unpebbled vertices and $k \geq 2k^*$. Then the mover has a winning strategy if and only if $C(x) \geq 4k^* + 2$.*

Proof. By induction on k^*.

Base: Let $k^* = 0$. This is done by Lemmas 3.7.2 and 3.7.3.

Induction: Let k^* be even. First, suppose $C(x) \geq 4k^* + 2$. The mover will a pebble-free vertex of $N[v^*]$. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a pebble-free vertex in S, then for the new configuration C' we have $C'(x) \geq 4k^* - 2 = 4(k^* - 1) + 2$ and there are $k^* - 2$ unpebbled vertices in $N[v^*]$. By induction, the mover has a winning strategy.

Conversely, suppose $C(x) < 4k^* + 2$. The mover can pebble to any pebble-free vertex or place a second pebble on a vertex in S. If the mover pebbles to an unpebbled vertex in S, then the defender will pebble to an unpebbled vertex not in $N[v^*]$. The new configuration C' has $C'(x) < 4k^* - 2 = 4(k^* - 1) + 2$ and there are at most k^* unpebbled vertices in $N[v^*]$. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex, then the defender will pebble to its unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C'' has $C''(x) < 4k^* - 2 = 4(k^* - 1) + 2$ and there are at most k^* unpebbled vertices in $N[v^*]$. The defender wins by induction. $\square$

Lemma 3.7.5. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. For every $v \in S$, suppose there exists at least one $u \in N_S[v]$ such that u is not pebbled and $S \neq N[v]$ for some v. Let k^* be the number of pebble-free vertices in $N[v^*]$ where $N[v^*] \in S$ has the minimum number of unpebbled vertices and $k < 2k^*$. Then the mover has a winning strategy if and only if $C(x) \geq 2k + 2$.*

Proof. By induction on k.

Base: Let $k = 0$. This is done by Lemmas 3.7.2 and 3.7.3.

Induction: Let k be even. First, suppose $C(x) \geq 2k + 2$. The mover will a pebble-free vertex of S. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a pebble-free vertex in S, then for the new configuration C' we have $C'(x) \geq 2k - 2$ and there are $k - 2$ unpebbled vertices in S. By induction, the mover has a winning strategy.

Conversely, suppose $C(x) < 2k + 2$. The mover can pebble to any pebble-free vertex or place a second pebble on a vertex in S. If the mover pebbles to an unpebbled vertex in S, then the defender will pebble to an unpebbled vertex not in S. The new configuration C' has $C'(x) < 2k - 2$ and there are at most $|S_0|$ unpebbled vertices in S. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex, then the defender will pebble to its unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C' has $C'(x) < 2k - 2$ and there are at most k unpebbled vertices in S. The defender wins by induction. □

Lemma 3.7.6. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. Suppose $S = N[v]$ for some v. Then the mover has a winning strategy if and only if $C(x) \geq 2k + 2$.*

Proof. By induction on k.

Base: Let $k = 0$. This is done by Lemmas 3.7.2 and 3.7.3.

Induction: Let k be even. First, suppose $C(x) \geq 2k + 2$. The mover will a pebble-free vertex of S. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a pebble-free vertex in S, then for the new configuration C' we have $C'(x) \geq 2k - 2 = 2(k - 2) + 2$ and there are $k - 2$ unpebbled vertices in S. By induction, the mover has a winning strategy.

Conversely, suppose $C(x) < 2k + 2$. The mover can pebble to any pebble-free vertex or place a second pebble on a vertex in S. If the mover pebbles to an unpebbled vertex in S, then the defender will pebble to an unpebbled vertex in S. The new configuration C' has $C'(x) < 2k - 2 = 2(k - 2) + 2$ and there are $k - 2$ unpebbled vertices in $N[v]$. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex, then the defender will pebble to its unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C'' has $C''(x) < 2k - 2 = 4(k - 2) + 2$ and there are at most

k unpebbled vertices in S. The defender wins by induction. $\square$

Obtaining the winning configurations for the mover allow us to get $\eta(G, r)$ for $G \in \mathcal{G}_{s,t}$.

Theorem 3.7.7. *If $G \in \mathcal{G}_{s,t}$, then* $\eta(G, r) = \begin{cases} t + 2s + 4, & s \text{ is even} \\ t + 2s + 3, & s \text{ is odd.} \end{cases}$

Proof. Case 1: Let s be even. A configuration of $t + 2s + 3$ pebbles on the vertices of G which gives the defender a winning strategy is the following: in T, leave one vertex pebble-free, put one pebble on $t-2$ vertices and the remaining $2s+5$ pebbles on one vertex and keep S pebble-free. With this configuration, $C_T = s+2$ with one vertex in T having no pebbles on it. By Lemma 3.4.2, the defender wins.

Now suppose there are $m \geq t + 2s + 4$ pebbles on the vertices in G. Let k of the vertices in S be pebble-free. Thus there are $(s-k)$ pebbles in S. Now there are $m - (s-k) \geq t + 2s + 4 - s + k = t + s + k + 4$ pebbles on the vertices in T. To show the mover has a winning strategy, we show any configuration of the remaining pebbles on T, C_T and the configuration satisfies the condition of one of the previous lemmas.

If all of the vertices in T are pebbled, then at most t pebbles can be placed on the vertices and $C_T = 0$. There are $s + k + 4$ pebbles left to arrange. First, let k be even. Then no matter how the rest are arranged, $C_T \geq \frac{s+k}{2} + 2 \geq k + 2$. If there are all distributed evenly, then all vertices have an odd number of pebbles on them. So the mover wins. If they are not distributed evenly, then $C_T \geq k + 3$. So the mover has a winning strategy by Lemma 3.7.1. Now let k be odd. No matter how the $s + k + 4$ pebbles are broken up, $C_T \geq \frac{s+k}{2} + 2 \geq k + 2$. Since k is odd, the mover has a winning strategy by Theorem 3.7.1.

Now suppose not all of the vertices of T have pebbles on them. Let ℓ of the vertices in T be pebble-free. Then at most $t - \ell$ pebbles can be placed on T so $C_T = 0$. There are $s + k + \ell + 4$ pebbles left. Let k be even. If the pebbles are broken up in piles of even numbers, then $C_T = \frac{s+k}{2} + \frac{\ell}{2} + 2 \geq k + 2$. The mover wins. If the pebbles are broken up with some odd piles, then $C_T \geq k + 3$

and the mover wins. Now let k be odd. No matter how the pebbles are arranged, $C_T \geq \frac{s+k}{2}\frac{\ell}{2} + 2 \geq k+2$. Since k is odd, the mover has a winning strategy.

Case 2: Let s be odd. The configuration of $t+2s+2$ pebbles on the vertices of G which give the defender a winning strategy is the following: place 1 pebble on any vertex in S, place 1 pebble on $t-1$ vertices and the remaining $2s+1$ pebbles on one vertex. With this configuration, $C_T = s$ and there are $s-1$ pebble-free vertices in S, with $s-1$ even. By Lemma 3.3.2, the defender has a winning strategy.

A similar argument holds from above for $m \geq t+2s+3$ pebbles on the vertices of G. □

3.8 Complete Bipartite & Complete Multipartite Graphs

Now we get η for complete bipartite and multipartite graphs. We notice that complete bipartite graphs and complete multipartite graphs fall into the class $\mathcal{G}_{s,t}$, with the root in one partite set begin equivalent to $T \cup r$. Since $K_{u,v} \in \mathcal{G}_{s,t}$ with partite sets U and V, we have $u = s$ and $v = t+1$ if $r \in V$ or $u = t+1$ and $v = s$ if $r \in U$

Corollary 3.8.1. *Let* $3 \leq u \leq v$. *Then* $\eta(K_{u,v}) = \begin{cases} v+2u+3, & u \textit{ is even} \\ v+2u+2, & u \textit{ is odd.} \end{cases}$

Proof. We need to check which placement of the root yields a larger configuration to be r-solvable.

Let $u = v+i$.

If $r \in V$, then by Theorem 3.7.7, $\eta(K_{v+i,v}, r) = \begin{cases} v+2v+2i+3, & v+i \text{ is even} \\ v+2v+2i+2, & v+i \text{ is odd.} \end{cases}$

If $r \in U$, then by Theorem 3.7.7, $\eta(K_{v,v+i}, r) = \begin{cases} v+i+2v+3, & v \text{ is even} \\ v+i+2v+2, & v \text{ is odd.} \end{cases}$

We can see for every value of $i \geq 0$, the maximum configurations will be when $r \in V$.

□

Theorem 3.8.2. *If $u = 2$, then $\eta(K_{2,v}) = v + 7$.*

Proof. If $r \in U$, then Lemma 3.7.4 says we need at least 6 pebbles in U with no pebble in V so the mover has a winning strategy. By the Pigeonhole Principle, we need $v+1$ pebbles in V and no pebbles in U for the mover to have a winning strategy. So we need a total of $\max\{v+1, 6\}$ pebbles for the mover to have a winning strategy. If $r \in v$, then Theorem 3.7.7 says we need $v - 1 + 2u + 4 = v - 1 + 4 + 4 = v + 7$ pebbles for the mover to have a winning strategy. $\square$

Corollary 3.8.3. *Let $v \geq 3$. If $u = 1$, then $\eta(K_{1,v}) = v + 4$.*

Proof. If $U = \{r\}$, then by the Pigeonhole Principle the mover has a winning strategy with $v + 1$ pebbles. If $r \in V$, then Theorem 3.7.7 says $v - 1 + 2(1) + 3 = v + 4$ pebbles gives the mover a winning strategy. $\square$

Corollary 3.8.4. *If $3 \leq a_1 \leq a_2 \leq \cdots \leq a_m < n$ and $\displaystyle\sum_{i=1}^{m} a_i = n$, then*

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_1 + 3, & \displaystyle\sum_{i=2}^{m} a_i \text{ is even} \\ 2n - a_1 + 2, & \displaystyle\sum_{i=2}^{m} a_i \text{ is odd.} \end{cases}$$

Proof. If $r \in A_k$ for $k \neq 1$, then by Theorem 3.7.7,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = \begin{cases} a_k + 2\sum_{i \neq k} a_i + 3, & \sum_{i \neq k} a_i \text{ is even} \\ a_k + 2\sum_{i \neq k} a_i + 2, & \sum_{i \neq k} a_i \text{ is odd.} \end{cases}$$

Hence, in this case we have the following:

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_k + 3, & \displaystyle\sum_{i \neq k} a_i \text{ is even} \\ 2n - a_k + 2, & \displaystyle\sum_{i \neq k} a_i \text{ is odd.} \end{cases}$$

If $r \in A_1$, then by Theorem 3.7.7,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = \begin{cases} a_1 + 2\sum_{i=2}^{n} a_i + 3, & \sum_{i=2}^{n} a_i \text{ is even} \\ a_k + 2\sum_{i=2}^{n} a_i + 2, & \sum_{i=2}^{n} a_i \text{ is odd.} \end{cases}$$

So, in this case we have

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_1 + 3, & \sum_{i=2}^{m} a_i \text{ is even} \\ 2n - a_1 + 2, & \sum_{i=2}^{m} a_i \text{ is odd.} \end{cases}$$

Since $a_1 \leq a_k$ for all $k \geq 2$,

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_1 + 3, & \sum_{i=2}^{m} a_i \text{ is even} \\ 2n - a_1 + 2, & \sum_{i=2}^{m} a_i \text{ is odd.} \end{cases}$$

□

Corollary 3.8.5. *If* $2 = a_1 \leq a_2 \leq \cdots \leq a_m < n$ *and* $\sum_{i=1}^{m} a_i = n$, *then*

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 4n - 4a_m - 3a_1, & a_m \geq \sum_{i=2}^{m-1} a_i \\ 2n - a_1, & a_m < \sum_{i=2}^{m-1} a_i. \end{cases}$$

Proof. If $r \in A_k$ for $a_k \geq 3$, then by Theorem 3.7.7,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = \begin{cases} a_k + 2\sum_{i \neq k} a_i + 3, & \sum_{i \neq k} a_i \text{ is even} \\ a_k + 2\sum_{i \neq k} a_i + 2, & \sum_{i \neq k} a_i \text{ is odd.} \end{cases}$$

If $r \in A_1$ and $a_m \geq \sum_{i=2}^{m-1} a_i$, then by Lemma 3.7.4,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = 4\sum_{i=2}^{m-1} a_i + 2$$

.

If $r \in A_1$ and $a_m < \sum_{i=2}^{m-1} a_i$, then by Lemma 3.7.5,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = 2\sum_{i=2}^{m} a_i + 2$$

□

Corollary 3.8.6. *If* $1 = a_1 \leq a_2 \leq \cdots \leq a_m < n$ *with* a_k *the size of the smallest partite set not equal to 1 and* $\sum_{i=1}^{m} a_i = n$*, then*

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 4n - 4a_m - 3a_1, & a_k = 2 \text{ and } a_m \geq \sum_{i=2}^{m-1} a_i \\ 2n - a_1, & a_k = 2 \text{ and } a_m < \sum_{i=2}^{m-1} a_i \\ 2n - a_k + 3, & a_k > 2 \text{ and } \sum_{i \neq k} a_i \text{ is even} \\ 2n - a_k + 2, & a_k > 2 \text{ and } \sum_{i \neq k} a_i \text{ is odd.} \end{cases}$$

Proof. If $r \in A_1$, then by the Pigeonhole Principle the mover has a winging strategy with $\sum_{i \neq 1} a_i + 1$ pebbles.

If $r \in A_k$ where a_k the size of the smallest partite set not equal to 1, then see Corollary 3.8.4 and 3.8.5.

□

Chapter 4

Two-Player Pebbling on Paths

4.1 A Result in Classical Pebbling

We begin with a definition.

Definition 4.1.1. For a graph G with configuration C, the *value* of a vertex, $f(v)$, with respect to a given root r is $f(v) = \frac{C(v)}{2^{dist(v,r)}}$. We say the *value* of a configuration C, with respect to a given root r, is $f(C) = \sum_{v \in V(G)} f(v)$.

One thing we notice is that for greedy pebbling moves, from v to u, the value of the configuration is unchanged because $dist(v,r) = dist(u,r) + 1$.

$$\begin{aligned}
\frac{C(v)-2}{2^{dist(v,r)}} + \frac{C(u)+1}{2^{dist(u,r)}} &= \frac{C(v)}{2^{dist(v,r)}} + \frac{C(u)}{2^{dist(u,r)}} - \frac{2}{2^{dist(v,r)}} + \frac{1}{2^{dist(u,r)}} \\
&= \frac{C(v)}{2^{dist(v,r)}} + \frac{C(u)}{2^{dist(u,r)}} - \frac{2}{2^{dist(u,r)+1}} + \frac{1}{2^{dist(u,r)}} \\
&= \frac{C(v)}{2^{dist(v,r)}} + \frac{C(u)}{2^{dist(u,r)}} - \frac{2}{2^{dist(u,r)} \cdot 2} + \frac{1}{2^{dist(u,r)}} \\
&= \frac{C(v)}{2^{dist(v,r)}} + \frac{C(u)}{2^{dist(u,r)}} - \frac{1}{2^{dist(u,r)}} + \frac{1}{2^{dist(u,r)}} \\
&= \frac{C(v)}{2^{dist(v,r)}} + \frac{C(u)}{2^{dist(u,r)}}
\end{aligned}$$

With this we give an alternate method for finding the classical pebbling number of paths.

Lemma 4.1.2. *For any path P_n with $r = v_1$ with an initial configuration C, $f(C) \geq 1 \iff C$ is r-solvable in the classical pebbling sense.*

Proof. Let the path be $v_1v_2 \ldots v_n$ with v_1 as the root. Suppose we have any configuration on P_n that is r-unsolvable. Given the starting configuration, if a vertex has two or more pebbles on it, then make pebbling moves towards the root whenever possible. Once we make all possible pebbling moves, all vertices must have at most one pebble on them. Thus

$$f(C) = \sum_{v \in P} f(v) = \sum_{i=2}^{n} \frac{C(v_i)}{2^{dist(v_i,r)}} \leq \sum_{i=2}^{n} \frac{1}{2^{dist(v_i,r)}} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

Conversely, suppose we have an r-solvable configuration C. For all the vertices with two or more pebbles on them, make pebbling moves towards the root. This will not change the sum of the values. We know that we can place at least one pebble on the root. Thus

$$\begin{aligned} f(C) &= \sum f(v) \\ &= \frac{C(v_n)}{2^{n-1}} + \frac{C(v_{n-1})}{2^{n-2}} + \cdots + \frac{C(v_2)}{2} + \frac{C(v_1)}{1} \\ &\geq \frac{C(v_1)}{1} \\ &\geq \frac{1}{1} \\ &= 1 \end{aligned}$$

□

We can still look at the sum of the values of the vertices if the root is an inner vertex. If this is the case, the we can break up the path into two subpaths, i.e. if $r = v_k$, then we consider $v_1v_2 \ldots v_{k-1}r$ as one subpath and $rv_{k+1} \ldots v_n$ as the other subpath.

Corollary 4.1.3. *For any path P_n with $r = v_k$ for $k \neq 1, n$ and a initial configuration C. Then C is r-solvable in the classical pebbling sense $\iff \sum_{i=1}^{k} f(v_i) \geq 1$ or*

$\sum_{i=k}^{n} f(v_i) \geq 1.$

Proof. We apply Lemma 4.1.2 to the two subpaths. $\square$

Now we can verify the classical pebbling number for paths.

Theorem 4.1.4. *For every positive integer n, we have $\pi(P_n) = 2^{n-1}$.*

Proof. Case 1: Let $r = v_1$. If we have a configuration of $2^{n-1}-1$ pebbles all of which are on v_n, then $\sum_{v \in P_n} f(v) = \frac{2^{n-1}-1}{2^{n-1}}$. By Lemma 4.1.2, the root is not reachable. Suppose C is a configuration with at least 2^{n-1} pebbles. Then,

$$\begin{aligned}
\sum f(v) &= \frac{C(v)}{2^{n-1}} + \frac{C(v_{n-1})}{2^{n-2}} + \cdots + \frac{C(v_2)}{2} \\
&= \frac{C(v) + 2C(v_{n-1}) + 2^2 C(v_{n-2}) + \cdots + 2^{n-2} C(v_2)}{2^{n-1}} \\
&\geq \frac{2^{n-1}}{2^{n-1}} \\
&= 1
\end{aligned}$$

Case 2: Let $r = v_k$ for $k \neq 1, n$. Let $dist(v_1, v_k) = k-1$ and $dist(v_k, v_n) = n-k$. By the Pigeonhole Principle, either the subpath $v_1 \ldots v_k$ has at least 2^{k-1} pebbles on it or $v_k \ldots v_n$ has at least 2^{n-k} pebbles on it. In either case, we can pebble to r by the argument in Case 1. $\square$

4.2 Configurations Winnable for the Mover

One thing we want to discuss is the placement of the root in P_n. When considering paths, we will let $r = v_1$. Here is a general lemma that speaks to why we want $r = v_1$.

Lemma 4.2.1. *If r is a cut vertex of G and $G_1, G_2, \ldots G_k$ are the graphs induced by the components $G - r$ and r, then $\eta(G, r) = 1 + \sum_{i=1}^{k} (\eta(G_i, r) - 1)$.*

Proof. Let r be a cut vertex of G and $G_1, G_2, \ldots G_k$ be the graphs induced by the components $G - r$ and r. Let C be a configuration with $\sum_{i=1}^{k}(\eta(G_i, r) - 1)$ pebbles arranged so that component G_i receives $\eta(G_i, r) - 1$ pebbles in such a way that each component is r-unsolvable. Since each component has less than the number of pebbles needed to place a pebble on the root, the defender has a winning strategy.

Now, suppose C' is a configuration with $\sum_{i=1}^{k}(\eta(G_i, r) - 1) + 1$ pebbles. By the Pigeonhole Principle, at least one component G_k will have at least $\eta(G_k, r)$ pebbles distributed on it. Thus the mover wins. $\square$

Next, we find three configurations on a path for which the mover can always win, one when it is the mover's turn and the other two when it is the defender's turn.

Lemma 4.2.2. *Given it is the mover's turn, a winning configuration on P_n for the mover is 1 pebble each on vertices $v_2, v_3, \ldots v_k$, at least 2 pebbles on v_{k+1} and any number of pebbles on the rest of the path.*

Proof. We show a winning strategy for the mover. The mover will pebble to v_{k-1}. Now we have 1 pebble on $v_2, v_3, \ldots v_{k-2}$ and two pebbles on v_{k-1}. The defender has three options for moves: pebbling from v_{k-1} to v_{k-2}, pebbling from v_k to v_{k-1} or pebbling anywhere after v_k. In any of the three cases, we have 1 pebble on $v_2, v_3, \ldots, v_i$ and at least two pebbles on v_{i+1} for $i < k$. Thus, by induction, the mover can win. $\square$

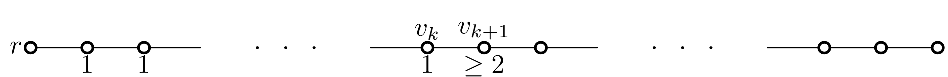

Figure 4.1: Configuration for Lemma 4.2.2

The following Lemma shows configurations that reduce to the one described in Lemma 4.2.2 but with the extra condition of the defender starting play.

Lemma 4.2.3. *Given it is the defender's turn, the following configurations are always winnable for the mover:*

- *1 pebble each on vertices $v_2, v_3, \ldots v_k$, at least 4 pebbles on v_{k+1} and any number of pebbles on the rest of the path,*

- *1 pebble each on vertices $v_2, v_3, \ldots v_k$, at least 3 pebbles on v_{k+1}, at least 2 pebbles on v_{k+2} and any number of pebbles on the rest of the path.*

Proof. For the first configuration, we can let the defender make any pebbling move. Now it's the mover's turn and we have a configuration as in Lemma 4.2.2. Thus the mover wins.

Now for the second configuration, if the defender pebbles v_k to v_{k+1}, it is the mover's turn and we have a configuration as in Lemma 4.2.2 with v_{k+1} as the vertex with at least 2 pebbles. If the defender pebbles v_{k+1} to v_{k+2}, it is the mover's turn and we have a configuration as in Lemma 4.2.2 with v_k as the vertex with at least 2 pebbles. If the defender pebbles on a vertex not v_k or v_{k+1}, it is the mover's turn and we have a configuration as in Lemma 4.2.2 with v_k as the vertex with at least 2 pebbles. In any of the cases, the mover wins. $\square$

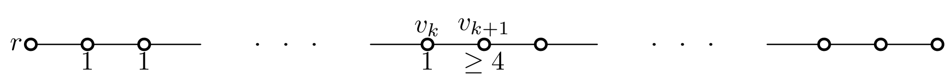

Figure 4.2: First Configuration for Lemma 4.2.3

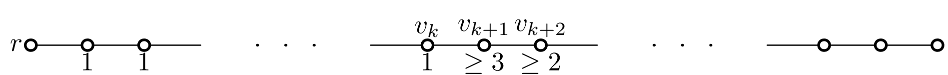

Figure 4.3: Second Configuration for Lemma 4.2.2

Next is a definition similar to a configuration being reachable in classical pebbling

Definition 4.2.4. Given two configurations C and D with $|C| > |D|$, we say configuration C *reduces* to configuration D provided there is a sequence of pebbling moves for both players in C that leads to configuration D.

The lemma below shows the importance of the three configurations always winnable for the mover.

Lemma 4.2.5. *If C is a configuration on any path P_n for which the mover has a winning strategy, then C reduces to one of the three winnable configurations.*

Proof. Suppose we have a configuration C' that is winnable and does not reduce to one of the three configurations in Lemmas 4.2.2 and 4.2.3. Then, while playing, we must have at least one of $v_2, v_3, \ldots v_k$ have 0 pebbles on them (Else, it would be one of the three configurations). Let v_i be the vertex closest to r with no pebbles on it to this point. Since C' is winnable, we must eventually be able to put 1 pebble on v_i. Now we either have a winnable configuration or another vertex farther from the root than v_i has no pebbles on it. Since C' is winnable, we must eventually be able to put 1 pebble this vertex. This can continue to v_n. So after playing, we have a path with all 0's and 1's. Thus the defender wins. A contradiction. □

We can see that is any configuration has at least 2 pebbles on v_2, then the mover can pebble to v_1 and win. So, for paths, a *non-trivial* configuration C will have 0 pebbles on v_1 and 0 or 1 pebbles on v_2.

4.3 Strategies on Paths

An initial study of paths led us to believe that they would be straightforward, having $\eta(P_n) = \pi(P_n)$. Notice that if the defender ever pebbles back towards v_n, then $\eta(P_n) \neq \pi(P_n)$.

One aspect of paths that the mover will take advantage of is the fact that paths are 1-dimensional. Pebbling moves can only move towards the root or away from the root. What makes this useful for the mover is the end of the path. There are many configurations that give the mover the opportunity to force the defender to pebble towards the root. The mover's winning strategies will take advantage of this.

For now, we will consider initial configurations C on paths with all pebbles placed on v_n; we are restricting the configurations because of difficulty. Consider Figure 4.4, classical pebbling on P_5 with 16 pebbles on v_5.

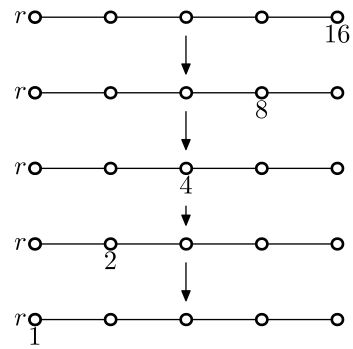

Figure 4.4: $\pi(P_5) = 16$

As long as all of the pebbling moves go towards the root, the configuration is r-solvable.

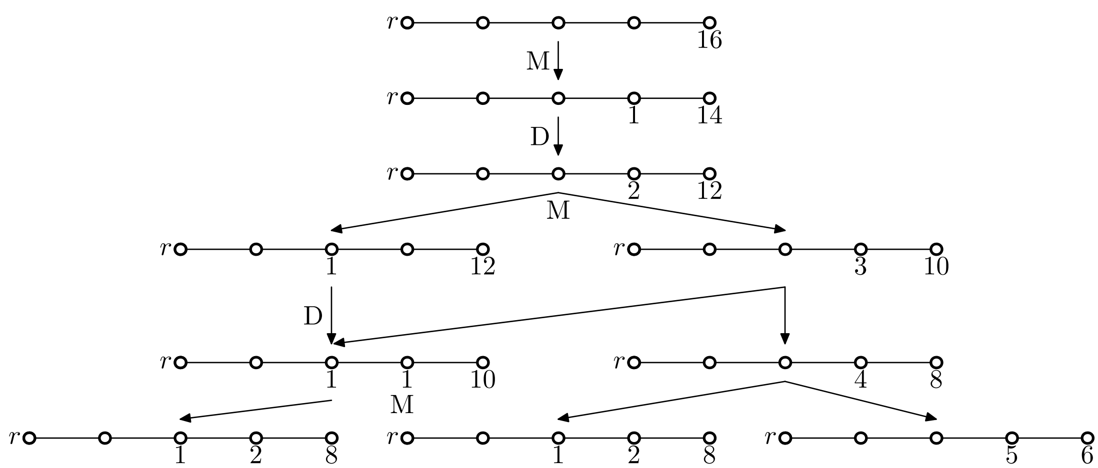

Figure 4.5: Game Tree for P_5

Yet, when we transition to Two-Player Pebbling, it is not so simple. Because there are two players, we need to consider possible pebbling moves of each player. We use the game tree to try to investigate each player's best possible moves. We are able to see the different choices for moves the mover or defender could make. With Figure 4.5, we take a look at the beginning of the game tree of P_5 with 16 pebbles to find $\eta(P_5)$. Notice that on the right side of the game tree, the defender is able to pebble backwards. Thus, the mover loses on those branches of the tree. So, if we only consider the left side, we continue and obtain Figure 4.6.

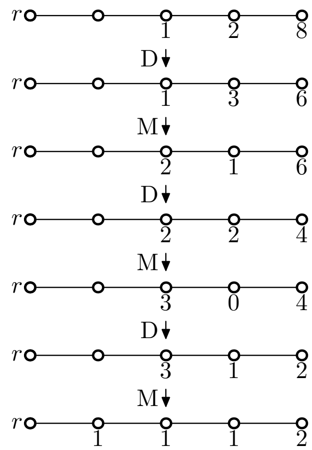

Figure 4.6: The Branch of the Game Tree for P_5

However, when we consider a P_6 a different situation become clear. If we try to use 32 pebbles on v_6 of P_6, as the pattern would suggest, then we come to a problem.

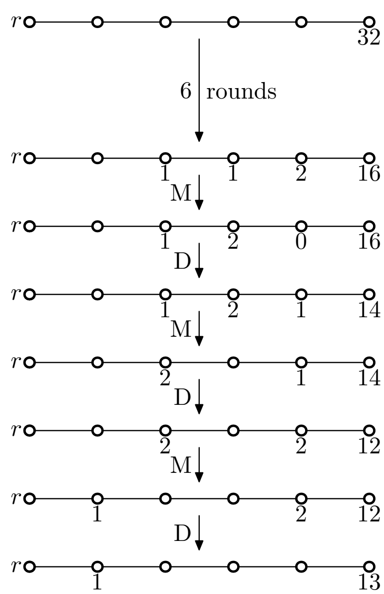

Figure 4.7: Playing on P_6

From Figure 4.7 we can see the resulting configuration when the defender pebbles to v_6. The defender finally has an opportunity to pebble backwards. Thus $\eta(P_6) \neq \pi(P_6)$. Every path after this must account for the choice by the defender. We restrict the strategies each player can use. The most natural strategy for the mover is pebbling towards r as close to r as possible. Let this strategy be S_M. The most natural strategy for the defender is pebbling away from r as close to r as possible and, if forced to move towards r, only pebbling towards r as far from r as possible. Let this be S_D. Both of these strategies are greedy. We define both strategies below:

- Mover: S_M
  - First i such that $C(v_i) > 1$
    - Pebble from v_i to v_{i-1}
- Defender: S_D
  - First i such that $C(v_i) > 1$ and Mover did not pebble to v_i
    - Pebble from v_i to v_{i+1}
  - If only i is at v_n or Mover pebbled to v_i
    - Pebble from v_i to v_{i-1}

We define a variation on η that will aid in finding the two-player pebbling number for paths.

Definition 4.3.1. Given a P_n, let $\eta(P_n, \mathcal{C}, S_M, S_D)$ be the minimum number of pebbles given a collection of configurations $\mathcal{C}$ with the mover playing strategy S_M and the defender playing S_D such that the mover can win.

We restrict our search to configurations with all pebbles on v_n and the mover and defender playing S_M and S_D, respectively. Table 4.1 shows a sample of the results from a computer program we created.

S_M vs S_D	$\eta(P_n, \mathcal{C}, S_M, S_D)$
P_6	38
P_7	79
P_8	164
P_9	331
P_{10}	668
P_{11}	1345

Table 4.1: Mover & Defender Playing Natural Strategies

The next question we tried to answer is, can both players do any better. Is there some way to change their strategy so that they could play better? The answer is yes. The mover has a new strategy, S_M^*. The defender has a new strategy, S_D^*. Below are the strategies:

- Mover: S_M^*
  - First i such that $C(v_i) > 1$.
    - If $C(v_{i-2}) = C(v_{i-1}) = 1$, $C(v_i) = 2$ and $C(v_{i+1}) = 2, 3$, then pebble from v_i to v_{i-1}.
    - Else, if $C(v_i) = 2$ and $C(v_{i+1}) = 2, 3$, then pebble from $v_{i+1}i$ to v_i
    - Else, if $C(v_{i+1}) = 2, 3$ and $i + 1 = n$, then pebble from $v_{i+1}i$ to v_i
    - Else, if $C(v_i) \equiv 0 \pmod 2$ and $C(v_{i+1}) = 2, 3$, then pebble from $v_{i+1}i$ to v_i
    - Else, if $\forall\, k \geq i + 1$, $C(v_k) \leq 1$, then pebble from v_{i+1} to v_i.
    - Else, pebble from v_i to v_{i-1}.
- Defender: S_D^*
  - Look for the first i such that $C(v_i) > 1$ and the Mover has not pebbled to v_i.
    - If $\forall\ k \neq i$, $C(v_k) \leq 1$, then pebble from v_i to v_{i-1}.

* Else, if $C(v_{i-1}) = 1$ and $\exists\, k < i$ such that $C(v_k) > 1$, then pebble from v_i to v_{i-1}.
* Else, if $C(v_{i-1}) = 1$ and $C(v_i) = 2$ and $\exists\, k < i$ such that $C(v_k) > 1$, then pebble from v_i to v_{i-1}.
* Else, if $C(v_{i+1}) = 2$ and $\exists\ k < i$ such that $C(v_k) > 1$, then pebble from v_i to v_{i+1}.
* Else, if $C(v_{i+1}) = 2$ and $\forall\ k < i$, $C(v_k) \leq 1$, then pebble from v_{i+1} to v_{i+2}.
* Else, pebble from v_i to v_{i+1}.

Some finer points of these strategies appear peculiar, however, they are necessary. For instance, consider the mover's instruction to check the parity of the first vertex with more than one pebble on it. Figure 4.8 is an example of such an instance.

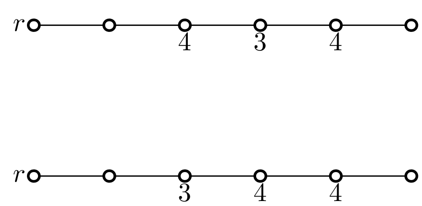

Figure 4.8: Parity of First Playable Vertex

It can verified that if the mover pebbles from v_3 to v_2, then they would lose on the first configuration and win on the second. However, if the mover initially pebbles from v_4 to v_3, then they would win on the first configuration and lose on the second. The reason for this is the even and odd parity. Pebbling from v_3 to v_2 would not add any pebbling moves in the first configuration but would add a pebbling move in the second, helping the defender. So the mover needs to be aware of when and how adding a pebbling move can affect the game.

Another example comes from the defender's strategy. It would seem counter-productive for the defender to pebble forward when they are not forced to. Yet, consider Figure 4.9:

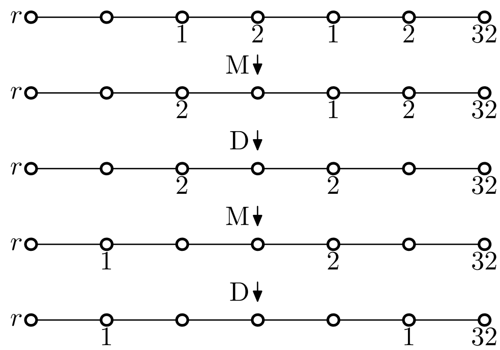

Figure 4.9: Defender Pebbling Forward

We see that, instead of the defender pebbling from v_6 to v_7 on their first move, they pebble from v_6 to v_5. On their next move, they pebble from v_5 back to v_6. If the defender did not play this way, then it can be verified that the mover has a winning strategy. However, we see that the defender can pebble forward to obtain a better configuration later. Table 4.2 shows a sample of the updated results from improving the strategies in our computer program.

P_n	$\eta(P_n, \mathcal{C}, S_M^*, S_D^*)$	$\eta(P_n, \mathcal{C}, S_M, S_D)$
P_6	35	38
P_7	73	79
P_8	152	164
P_9	307	331
P_{10}	620	668
P_{11}	1249	1345

Table 4.2: Mover & Defender Playing Improved Strategies

4.4 Adjusted η values of Paths

The goal is to recursively define $\eta(P_n, \mathcal{C}, S_M^*, S_D^*)$ as a function of $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*)$. The strategies S_M^* and S_D^* were written into a computer program. It seems reasonable that the minimum number of pebbles needed for the mover to win on P_n should be on the order of twice the number of pebbles needed for P_{n-1}. In fact, this is the case.

For some shorter paths, $n \leq 5$, the mover has a winning strategy using $\pi(P_n)$ pebbles.

Lemma 4.4.1. *For $n \leq 5$, we have $\eta(P_n, \mathcal{C}, S_M^*, S_D^*) = 2^{n-1}$.*

Proof. Let $n = 2$. Any configuration C' of 2 pebbles on P_2 is a trivial configuration. So the mover wins.

Let $n = 3$. Let v_3 have all 4 pebbles. The mover will pebble to v_2. The defender's only move is to pebble to v_2 as well. Now our new configuration has 2 pebbles on v_2 and is trivial. So the mover wins.

Let $n = 4$. Let v_4 have all 8 pebbles. The mover and defender must pebble to v_3. The mover will pebble from v_3 to v_2. The defender will pebble from v_4 to v_3. Our new configuration C' has 1 pebble on v_2, 1 pebble on v_3, and 2 pebbles on v_4. By Lemma 4.2.2, the mover has a winning strategy.

Let $n = 5$. Let v_5 have 16 pebbles on it. The mover and defender will pebble to v_4. The mover will pebble from v_4 to v_3. The defender will pebble from v_5 to v_4. Now, the mover and defender will pebble to v_4. The mover will pebble from v_4 to v_3. The defender will pebble from v_5 to v_4. Our new configuration C' has 2 pebbles on v_3, 2 pebble on v_4, and 4 pebbles on v_5. By the strategy S_M^*, the mover will pebble from v_4 to v_3, placing a third pebble on v_3. The defender will pebble from v_5 to v_4. The mover will pebble from v_3 to v_2 and the defender is forced to pebble from v_5 to v_4. Our new configuration C'' has 1 pebble on v_2, 1 pebble on v_3, 2 pebbles on v_4 and 0 pebbles on v_5 . By Lemma 4.2.2, the mover has a winning strategy.

□

Now, we move on to paths with 6 or more vertices. These are unique cases because no matter how the mover plays, the defender will be able to make move away from the root. For a recursion, we need an initial case.

Lemma 4.4.2. $\eta(P_6, \mathcal{C}, S_M^*, S_D^*) = 35.$

Proof. Following the strategies S_M^* and S_D^*, we get the following

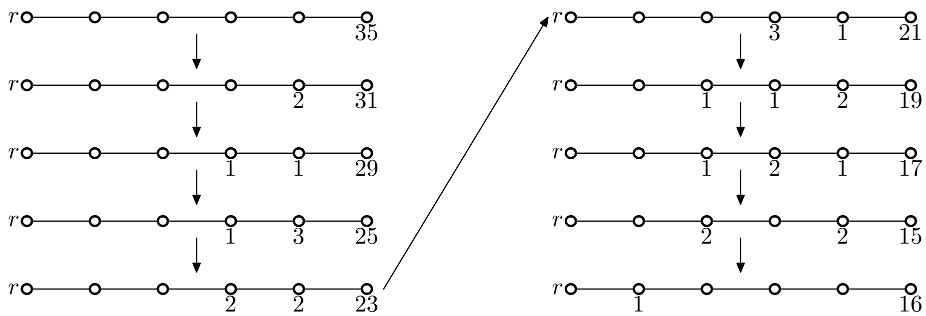

Figure 4.10: Finding $\eta(P_6, \mathcal{C}, S_M^*, S_D^*)$

By Lemma 4.4.1, the mover can places a second pebble on v_2 which the defender can not undo. Thus the mover wins. □

We have the following definition:

Definition 4.4.3. We say two configurations C and D are *equivalent* provided they reduce to the same configuration when playing the same strategy on both configurations.

When playing the game, we noticed that frequently we had situations with a leading 1, followed by 0's, and then a 0, 1, or 2 on v_{n-1} and some surplus of pebbles on v_n. Thus in trying to find the configuration with the largest number of pebbles, it seems that we should see which starting configuration would need the most pebbles.

Lemma 4.4.4. *Given P_7 with a configuration with 0 pebbles on r and v_2 and 1 pebble on v_3, v_4, v_5, 2 pebbles on v_6 and sufficiently large N on v_7. When v_2 has 1 pebble on it, then v_3, v_4, and v_5 will have 0 pebbles on them, v_6 will have* 1 *pebble on it, and v_7 will have $N - 5$ pebbles on it.*

Proof. The strategies S_M^* and S_D^* state that the mover will pebble to v_5. The defender will pebble to v_6, placing 1 pebble on it. Now the mover will make 3 pebbling moves towards v_2 to place a pebble on v_2. The defender will pebble to v_6, placing a second pebble on it, pebble back to v_7, then lastly pebbling to v_6. The current configuration has 1 pebble on v_2, no pebbles on v_i for $i = 3, \ldots, 5$ and 1 pebble on v_6. The defender makes a total of 3 pebbling moves from v_7 and one pebbling move to v_7 for a total of $N - 5$ pebbles left. $\square$

Lemma 4.4.5. *Given P_n with $n \geq 8$ and N sufficiently large, the following configurations are equivalent:*

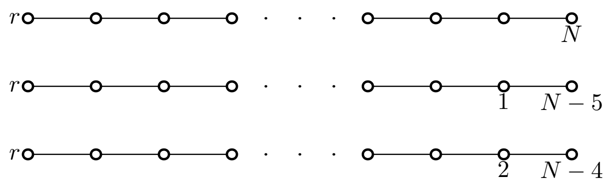

Figure 4.11: Equivalent Configuratoins

Proof. The three configurations will be labeled C_1, C_2, and C_3, respectively. Playing one round of C_1 yields C_3. Playing 40 rounds of C_1 with S_M^* and S_D^* yield the same configuration as playing 38 rounds of C_3 with S_M^* and S_D^*. $\square$

The main difference between Lemma 4.4.4 and Lemma 4.4.5 is the number of rounds played. If the game is played on P_n with $n \leq 7$, then the mover and defender will play under 40 rounds. Thus we need a separate case for when they play more than 40 rounds. Now that we have three equivalent configurations, we would like to know how many pebbles are needed so the mover has a winning configuration.

Lemma 4.4.6. *Given $P_n, n \geq 8$ and a configuration C having 0 pebbles on r, 1 pebble on v_2, and 0 pebble on $v_3, v_4, \ldots, v_{n-2}$. If v_{n-1} has 0 pebbles, 1 pebble or 2 pebbles on it, then v_n needs $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*), \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 5, \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 4$ pebbles, respectively, for the mover to have a winning strategy.*

Proof. By definition, if there are at least $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*)$ pebbles on v_n, then the mover can place on pebble on v_2. Since there is already 1 pebble on v_2, when the second pebble is moved to v_2, the defender will not be able to undo it. Thus, the mover wins.

By Lemma 4.4.5, if v_{n-1} initially had 1 pebble on it, then the mover only needs $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 5$ pebbles on v_n to place a second pebble on v_2 and thus on the root. Likewise, if v_{n-1} initially had 2 pebbles on it, then the mover only needs $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 4$ pebbles on v_n to place a second pebble on v_2

□

We continue with a lemma regarding when we see the configuration described above.

Lemma 4.4.7. *Given $P_n, n \geq 8$ with a configuration with 0 pebbles on r and v_2 and 1 pebble on $v_3, v_4, \ldots, v_{n-1}$ and sufficiently large N on v_n. When v_2 has 1 pebble on it, v_i will have 0 pebbles on it for $i = 3, \ldots, n-2$, v_{n-1} will have $n+1 \pmod 3$ pebbles on it, and v_n will have $N - 5 - 3\left\lfloor \frac{n-5}{3} \right\rfloor - 2(n+1) \pmod 3$ pebbles on it.*

Proof. Given a configuration as in Figure 4.12:

Figure 4.12: Starting Configuration for Lemma 4.4.7

The strategies S_M^* and S_D^* state that the mover will pebble forward through the string of 1's.

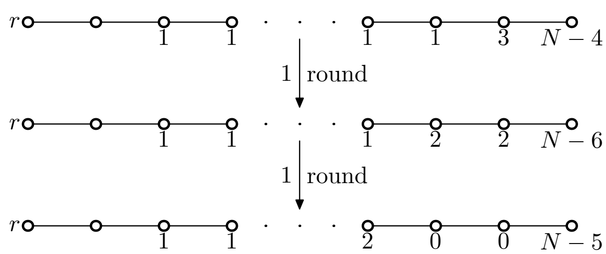

Figure 4.13: Playing S_M^* and S_D^* on Figure 4.12

From this point, the mover will pebble forward to v_2 and pebble across $n-5$ vertices to reach v_2. The defender will now pebble to v_{n-1}, then put a second pebble on v_{n-1}, and finally pebble back to v_n. Since it takes three rounds for the defender to pebble back to v_n, the value of $n+1 \pmod 3$ will determine how many pebbles are left on v_{n-1}. While the mover is pebble to v_2 along the $n-5$ vertices, the defender will be pebbling twice to v_{n-1} and once back to v_n. This uses $3\left\lfloor \frac{n-5}{3} \right\rfloor$ pebbles. However, since v_{n-1} will have $n+1 \pmod 3$ pebbles on it, the defender will use an additional $2(n+1) \pmod 3$ pebbles.

$\square$

Here we find the recursive formula for $\eta(P_n, \mathcal{C}, S_M^*, S_D^*)$, the main result in Chapter 4. Our final goal is to find an explicit, non-recursive, formula for $\eta(P_n, \mathcal{C}, S_M^*, S_D^*)$ that only depends on n.

Theorem 4.4.8. *Given $P_n, n \geq 7$,*

$$\eta(P_n, \mathcal{C}, S_M^*, S_D^*) = \begin{cases} 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 6 & \text{if } n \equiv 0 \pmod 3 \\ 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 4 & \text{if } n \equiv 1 \pmod 3 \\ 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $n = 7$. Suppose there are 73 pebbles on v_7. By Lemma 4.4.2, we will get the configuration seen in Figure 4.14.

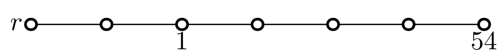

Figure 4.14: First Resulting Configuration on P_7

Then by Lemma 4.4.1, we will have the configuration seen in Figure 4.15.

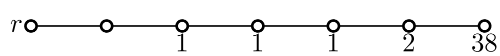

Figure 4.15: Second Resulting Configuration on P_7

The mover will pebble to v_2 and, by Lemma 4.4.4, we will obtain the configuration in Figure 4.16.

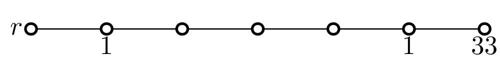

Figure 4.16: Third Resulting Configuration on P_7

Playing S_M^* and S_D^* for 14 rounds yields Figure 4.17. By Lemma 4.2.2, the mover wins.

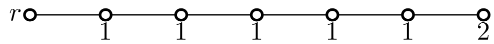

Figure 4.17: Fourth Resulting Configuration on P_7

Let $n \geq 8$. Suppose there are $\eta(P_n, \mathcal{C}, S_M^*, S_D^*)$ pebbles on v_n. By Induction, the mover will need $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 2$ pebbles to have 1 pebble on v_i for $i = 3, 4, \ldots n-1$. Now, the mover will continue to pebble to v_2. By Lemma 4.4.7, when the mover places 1 pebble on v_2, the defender would have used $5+3\left\lfloor \frac{n-5}{3} \right\rfloor + 2(n+13)$ pebbles on v_n. Our new configuration has 1 pebble on v_2, $n+1 \pmod 3$ pebbles on $vn-1$. By Lemma 4.4.6, for the mover to win, there needs to be an additional $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 5, \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 4$, or $\eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*)$ pebbles on v_n for $n \equiv 0, 1, 2 \pmod 3$, respectively.

If $n \equiv 0 \pmod 3$, then

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3 + 3\left\lfloor \frac{n-5}{3} \right\rfloor + 2 + \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 5 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3\left\lfloor \frac{n-5}{3} \right\rfloor \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3\left\lfloor \frac{3k-5}{3} \right\rfloor \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3k - 6 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 6
\end{aligned}$$

If $n \equiv 1 \pmod 3$, then

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3 + 3\left\lfloor \frac{n-5}{3} \right\rfloor + 4 + \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) - 4 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3\left\lfloor \frac{3k+1-5}{3} \right\rfloor + 3 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3\left\lfloor \frac{3k-4}{3} \right\rfloor + 3 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3k - 6 + 3 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3k + 1 - 4 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 4
\end{aligned}$$

If $n \equiv 2 \pmod 3$, then

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3 + 3\left\lfloor \frac{n-5}{3} \right\rfloor + \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3\left\lfloor \frac{3k+2-5}{3} \right\rfloor + 3 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3\left\lfloor \frac{3k-3}{3} \right\rfloor + 3 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3k - 3 + 3 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + 3k + 2 - 2 \\
&= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 2
\end{aligned}$$

□

We can further simplify our recursion to get closer to finding a non-recursive formula by obtaining recursions that only depend on 1 equivalence class modulo 3.

Corollary 4.4.9. *Given $P_n, n \geq 9$,*

$$\eta(P_n, \mathcal{C}, S_M^*, S_D^*) = \begin{cases} 8 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + 7n - 36 & \text{if } n \equiv 0 \pmod 3 \\ 8 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + 7n - 34 & \text{if } n \equiv 1 \pmod 3 \\ 8 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + 7n - 44 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $n \equiv 0 \pmod 3$.

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 6 \\
&= 2 \cdot [2 \cdot \eta(P_{n-2}, \mathcal{C}, S_M^*, S_D^*) + (n-1) - 2] + n - 6 \\
&= 4 \cdot \eta(P_{n-2}, \mathcal{C}, S_M^*, S_D^*) + 3n - 12 \\
&= 4 \cdot [2 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + (n-2) - 4] + 3n - 12 \\
&= 8 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + 7n - 36
\end{aligned}$$

Let $n \equiv 1 \pmod 3$.

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 4 \\
&= 2 \cdot [2 \cdot \eta(P_{n-2}, \mathcal{C}, S_M^*, S_D^*) + (n-1) - 6] + n - 4 \\
&= 4 \cdot \eta(P_{n-2}, \mathcal{C}, S_M^*, S_D^*) + 3n - 18 \\
&= 4 \cdot [2 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + (n-2) - 2] + 3n - 18 \\
&= 8 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + 7n - 34
\end{aligned}$$

Let $n \equiv 2 \pmod 3$.

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= 2 \cdot \eta(P_{n-1}, \mathcal{C}, S_M^*, S_D^*) + n - 2 \\
&= 2 \cdot [2 \cdot \eta(P_{n-2}, \mathcal{C}, S_M^*, S_D^*) + (n-1) - 4] + n - 2 \\
&= 4 \cdot \eta(P_{n-2}, \mathcal{C}, S_M^*, S_D^*) + 3n - 12 \\
&= 4 \cdot [2 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + (n-2) - 6] + 3n - 12 \\
&= 8 \cdot \eta(P_{n-3}, \mathcal{C}, S_M^*, S_D^*) + 7n - 44
\end{aligned}$$

□

Finally, we come to an explicit formula for $\eta(P_n, \mathcal{C}, S_M^*, S_D^*)$ that only depends on n.

Corollary 4.4.10. *Given $P_n, n \geq 9$,*

$$\eta(P_n, \mathcal{C}, S_M^*, S_D^*) = \begin{cases} \frac{275}{224} \cdot 2^{n-1} - n + \frac{12}{7} & \text{if } n \equiv 0 \pmod 3 \\ \frac{275}{224} \cdot 2^{n-1} - n + \frac{10}{7} & \text{if } n \equiv 1 \pmod 3 \\ \frac{275}{224} \cdot 2^{n-1} - n + \frac{20}{7} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $n \equiv 0 \pmod 3$.

$$\begin{aligned}
\eta(P_{3k}, \mathcal{C}, S_M^*, S_D^*) &= 8 \cdot \eta(P_{3(k-1)}, \mathcal{C}, S_M^*, S_D^*) + 21k - 36 \\
&= 8 \cdot [8 \cdot \eta(P_{3(k-2)}, \mathcal{C}, S_M^*, S_D^*) + 21(k-1) - 36] + 21k - 36 \\
&= 8^2 \cdot \eta(P_{3(k-2)}, \mathcal{C}, S_M^*, S_D^*) + 21[8(k-1) + k] - 36(8+1) \\
&= 8^2 \cdot [8 \cdot \eta(P_{3(k-3)}, \mathcal{C}, S_M^*, S_D^*) + 21(k-2) - 36] + 21[8(k-1) + k] - 36(8+1) \\
&= 8^3 \cdot \eta(P_{3(k-2)}, \mathcal{C}, S_M^*, S_D^*) + 21[8^2(k-2) + 8(k-1) + k] - 36(8^2 + 8 + 1) \\
&= 8^m \cdot \eta(P_{3(k-m)}, \mathcal{C}, S_M^*, S_D^*) + 21 \sum_{i=0}^{m-1} 8^i (k-i) - 36 \sum_{i=0}^{m-1} 8^i
\end{aligned}$$

The base case for our recursion is $\eta(P_6, \mathcal{C}, S_M^*, S_D^*) = 35$. So if $k - m = 2$, then $m = k - 2$. So,

$$\begin{aligned}
\eta(P_{3k}, \mathcal{C}, S_M^*, S_D^*) &= 8^m \cdot \eta(P_{3(k-m)}, \mathcal{C}, S_M^*, S_D^*) + 21 \sum_{i=0}^{m-1} 8^i (k-i) - 36 \sum_{i=0}^{m-1} 8^i \\
&= 8^{k-2} \cdot \eta(P_6, \mathcal{C}, S_M^*, S_D^*) + 21 \sum_{i=0}^{k-3} 8^i (k-i) - 36 \sum_{i=0}^{k-3} 8^i \\
&= 8^{k-2} \cdot \eta(P_6, \mathcal{C}, S_M^*, S_D^*) + (21k - 36) \sum_{i=0}^{k-3} 8^i - 21 \sum_{i=0}^{k-3} i 8^i
\end{aligned}$$

We will use the known formulas $\sum_{i=0}^{N} x^i = \frac{x^{N+1} - 1}{x - 1}$ and $\sum_{i=0}^{N} i x^i = \frac{(N+1)x^{N+1}}{x-1} - \frac{x(x^{N+1} - 1)}{(x-1)^2}$ for $N = k - 3$ and $x = 8$ to solve the recursion. Thus,

$$\begin{aligned}
\eta(P_{3k}, \mathcal{C}, S_M^*, S_D^*) &= 8^{k-2} \cdot \eta(P_6, \mathcal{C}, S_M^*, S_D^*) + (21k - 36) \sum_{i=0}^{k-3} 8^i - 21 \sum_{i=0}^{k-3} i 8^i \\
&= 8^{k-2} \cdot \eta(P_6, \mathcal{C}, S_M^*, S_D^*) + (21k - 36) \frac{8^{k-2} - 1}{7} \\
&\quad - 21 \left[\frac{(k-2)8^{k-2}}{7} - \frac{8(8^{k-2} - 1)}{49} \right] \\
&= 35 \cdot 8^{k-2} + \frac{30}{7} 8^{k-2} - 3k + \frac{12}{7} \\
&= \frac{275}{7} 8^{k-2} - 3k + \frac{12}{7} \\
&= \frac{275}{7} 2^{3k-6} - 3k + \frac{12}{7}
\end{aligned}$$

Substituting n for $3k$, we get

$$\begin{aligned}\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= \frac{275}{7}2^{n-6} - n + \frac{12}{7} \\ &= \frac{275}{224}2^{n-1} - n + \frac{12}{7}\end{aligned}$$

Let $n \equiv 1 \pmod 3$.

$$\begin{aligned}\eta(P_{3k+1}, \mathcal{C}, S_M^*, S_D^*) &= 8 \cdot \eta(P_{3(k-1)+1}, \mathcal{C}, S_M^*, S_D^*) + 7(3k+1) - 34 \\ &= 8 \cdot \eta(P_{3(k-1)+1}, \mathcal{C}, S_M^*, S_D^*) + 21k - 27 \\ &= 8^m \cdot \eta(P_{3(k-m)+1}, \mathcal{C}, S_M^*, S_D^*) + 21\sum_{i=0}^{m-1} 8^i(k-i) - 27\sum_{i=0}^{m-1} 8^i\end{aligned}$$

The base case for our recursion is $\eta(P_7, \mathcal{C}, S_M^*, S_D^*) = 73$. So if $k - m = 2$, then $m = k - 2$. So,

$$\begin{aligned}\eta(P_{3k+1}, \mathcal{C}, S_M^*, S_D^*) &= 8^m \cdot \eta(P_{3(k-m)+1}, \mathcal{C}, S_M^*, S_D^*) + 21\sum_{i=0}^{m-1} 8^i(k-i) - 27\sum_{i=0}^{m-1} 8^i \\ &= 8^{k-2} \cdot \eta(P_7, \mathcal{C}, S_M^*, S_D^*) + (21k - 27)\sum_{i=0}^{k-3} 8^i - 21\sum_{i=0}^{k-3} i8^i \\ &= 8^{k-2} \cdot \eta(P_7, \mathcal{C}, S_M^*, S_D^*) + (21k - 27)\frac{8^{k-2} - 1}{7} \\ &\quad - 21\left[\frac{(k-2)8^{k-2}}{7} - \frac{8(8^{k-2} - 1)}{49}\right] \\ &= 73 \cdot 8^{k-2} + \frac{39}{7}8^{k-2} - 3k + \frac{3}{7} \\ &= \frac{550}{7}8^{k-2} - 3k + \frac{3}{7} \\ &= \frac{550}{7}2^{3k-6} - 3k + \frac{3}{7} \\ &= \frac{550}{7}2^{3k+1-7} - (3k+1) + \frac{10}{7}\end{aligned}$$

Substituting n for $3k + 1$, we get

$$\begin{aligned}\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= \frac{550}{7}2^{n-7} - n + \frac{10}{7} \\ &= \frac{275}{224}2^{n-1} - n + \frac{10}{7}\end{aligned}$$

Finally, let $n \equiv 2 \pmod 3$.

$$\begin{aligned}
\eta(P_{3k+2}, \mathcal{C}, S_M^*, S_D^*) &= 8 \cdot \eta(P_{3(k-1)+2}, \mathcal{C}, S_M^*, S_D^*) + 7(3k+2) - 44 \\
&= 8 \cdot \eta(P_{3(k-1)+2}, \mathcal{C}, S_M^*, S_D^*) + 21k - 30 \\
&= 8^m \cdot \eta(P_{3(k-m)+2}, \mathcal{C}, S_M^*, S_D^*) + 21 \sum_{i=0}^{m-1} 8^i (k-i) - 30 \sum_{i=0}^{m-1} 8^i
\end{aligned}$$

The base case for our recursion is $\eta(P_8, \mathcal{C}, S_M^*, S_D^*) = 152$. So if $k - m = 2$, then $m = k - 2$. So,

$$\begin{aligned}
\eta(P_{3k+2}, \mathcal{C}, S_M^*, S_D^*) &= 8^m \cdot \eta(P_{3(k-m)+1}, \mathcal{C}, S_M^*, S_D^*) + 21 \sum_{i=0}^{m-1} 8^i (k-i) - 30 \sum_{i=0}^{m-1} 8^i \\
&= 8^{k-2} \cdot \eta(P_8, \mathcal{C}, S_M^*, S_D^*) + (21k - 30) \sum_{i=0}^{k-3} 8^i - 21 \sum_{i=0}^{k-3} i 8^i \\
&= 8^{k-2} \cdot \eta(P_8, \mathcal{C}, S_M^*, S_D^*) + (21k - 30) \frac{8^{k-2} - 1}{7} \\
&\quad - 21 \left[\frac{(k-2)8^{k-2}}{7} - \frac{8(8^{k-2} - 1)}{49} \right] \\
&= 152 \cdot 8^{k-2} + \frac{36}{7} 8^{k-2} - 3k + \frac{6}{7} \\
&= \frac{1100}{7} 8^{k-2} - 3k + \frac{6}{7} \\
&= \frac{1100}{7} 2^{3k-6} - 3k + \frac{6}{7} \\
&= \frac{1100}{7} 2^{3k+2-8} - (3k+2) + \frac{20}{7}
\end{aligned}$$

Substituting n for $3k+1$, we get

$$\begin{aligned}
\eta(P_n, \mathcal{C}, S_M^*, S_D^*) &= \frac{1100}{7} 2^{n-8} - n + \frac{20}{7} \\
&= \frac{275}{224} 2^{n-1} - n + \frac{20}{7}
\end{aligned}$$

□

Chapter 5

Conclusion

In addition to presenting the basics of graph pebbling, this dissertation introduces a new two-player game played in the context of graph pebbling and determines the winning player for certain classes of graphs. In Chapter 2, we found various upper bounds for path, cycles, and fan graphs. We note that the study of Fan Graphs in Chapter 2 is an extension of the diameter-2 graphs, $\mathcal{G}_{s,t}$, described in Chapter 3. We can see that if for some graph G the set T is a cycle, then $G_{s,t}$ satisfies the conditions of Theorem 2.4.1, hence $\eta(G_{s,t}, r) = \infty$. Determining the value of $\eta(G_{s,t}, r)$ when the set T is a path seems an interesting problem and natural extension of the results completed in Chapters 2 and 3.

To determine whether the k^{th} powers of paths were finite or not, it was necessary to partition the class based on the relationship between n and k. Some values of n and k were found to yield an infinite value for $\eta(P_n^k)$. Conjecture 2.3.6 deals with the values of n and k for which there is not an answer currently.

In determining these upper bounds, we found that, for some configurations, it may not be possible for the mover to win. Theorem 2.4.1 characterizes a structure and configuration for which the defender has a winning strategy. Specifically, some classes of graphs that fit this structure are bipartite graphs, line graphs, trees, and $n \times m$ grids where $n, m \geq 4$. We saw in Chapter 2 that $\eta(P_m \square P_n) = \infty$ for $m, n \geq 4$.

We found that when $n = 2k + 4$, P_n^k does not satisfy the conditions of Theorem 2.4.1 and $\eta(P_n^k) = \infty$. We would like to find a more complete result to classify the structure of graphs for which the defender has a winning strategy. It would be interesting to determine $\eta(P_2\square P_n)$, $\eta(P_3\square P_n)$, and $\eta(P_2\square G)$ for various graphs G. In order to find more graphs for which the mover has a winning strategy, we had to restrict the classes we considered. Because there is no necessary and sufficient condition for a graph to have an infinite value for η, it would be helpful to know that there is a limit to the size of configurations one must check in hopes of finding a finite η value for a graph. Conjecture 2.4.2 poses such a bound.

A constructed class of diameter-2 graphs was studied in Chapter 3. Comparing the number of unpebbled vertices in one subset to the number of pebbling moves in another subset yielded the results necessary to find the two-player pebbling number for complete bipartite and complete multipartite graphs. However, there was one case in the more general constructed class of diameter-2 graphs that is still open. It was found to be equivalent to a new Element Selecting Game, also played with two players. We conjecture that the task of finding the winning player in the Element Selecting Game is NP-Complete.

While Chapter 2 found the upper bound for $\eta(P_n)$, Chapter 4 aimed to find an exact value for $\eta(P_n)$. Three configurations were found for which the mover has a winning strategy. To this point, the strategies S_M^* and S_D^* are the best strategies for the mover and defender that we have found. Corollary 4.4.9 does not establish $\eta(P_n)$ exactly, but instead establishes a value for stacking everything on the last vertex with the mover playing S_M^* and the defender playing S_D^*. A computer program was used to accomplish some of the larger, more cumbersome cases. We believe that this value for $\eta(P_n)$ will hold if we allow any configuration of the pebbles on P_n. We also believe that this value of $\eta(P_n)$ will hold if we allow the defender to play any strategy. This will allow us to be able to find an exact value for $\eta(P_n)$.

Finding $\eta(G)$ for a graph appears to be more difficult that determining $\pi(G)$. When another player is added with an opposite objective, each player's strategy needs to be considered. An example of this is $\pi(P_n)$ versus $\eta(P_n)$. The only consideration for determining $\pi(P_n)$ is how many pebbles are needed to pebble towards the root. Specifically, when trying to narrow down $\eta(P_n)$, we found that the current best strategy for the defender is not intuitive. There is a configuration for which pebbling towards the root, while not being forced to, turns out to put the defender in a better position than if they had not. With classical pebbling, paths are greedy. With two-player pebbling, they are not.

In conclusion, it is our belief that Two-Player Graph Pebbling is a very interesting area of research. Many problems have proven to be challenging to solve or are still waiting to be solved. The techniques developed here can be used to find the two-player pebbling number of other classes of graphs and have applications in other discrete mathematical games.

Bibliography

[1] David P Bunde, Erin W Chambers, Daniel Cranston, Kevin Milans, and Douglas B West, *Pebbling and optimal pebbling in graphs*, Journal of Graph Theory **57** (2008), no. 3, 215–238.

[2] Fan R. K. Chung, *Pebbling in hypercubes*, SIAM J. Discrete Math. **2** (1989), no. 4, 467–472. MR 1018531 (91a:05095)

[3] T. Clarke, *Pebbling on graphs*, Master's thesis, Arizona State University, 1996.

[4] TA Clarke, Robert A Hochberg, and Glenn H Hurlbert, *Pebbling in diameter two graphs and products of paths*, Journal of Graph Theory **25** (1997), no. 2, 119–128.

[5] Rongquan Feng and Ju Young Kim, *Pebbling numbers of some graphs*, Science in China Series A: Mathematics **45** (2002), no. 4, 470–478.

[6] David S Herscovici, *Graham's pebbling conjecture on products of cycles*, Journal of Graph Theory **42** (2003), no. 2, 141–154.

[7] David S Herscovici and Aparna W Higgins, *The pebbling number of* $c_5 \square c_5$, Discrete mathematics **187** (1998), no. 1, 123–135.

[8] Glenn Hurlbert, *General graph pebbling*, Discrete Applied Mathematics **161** (2013), no. 9, 1221–1231.

[9] Glenn H. Hurlbert, *A survey of graph pebbling*, Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and

Computing (Boca Raton, FL, 1999), vol. 139, 1999, pp. 41–64. MR 1744229 (2000k:05261)

[10] Paul Lemke and Daniel Kleitman, *An addition theorem on the integers modulo n*, J. Number Theory **31** (1989), no. 3, 335–345. MR 993907 (90a:11004)

[11] David Moews, *Pebbling graphs*, J. Combin. Theory Ser. B **55** (1992), no. 2, 244–252. MR 1168965 (94e:05151)

[12] Lior Pachter, Hunter S Snevily, and Bill Voxman, *On pebbling graphs*, Congressus Numerantium **107** (1995), 65–80.

[13] Luke Postle, Noah Streib, and Carl Yerger, *Pebbling graphs of diameter three and four*, Journal of Graph Theory **72** (2013), no. 4, 398–417.

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